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# Accepted Manuscript

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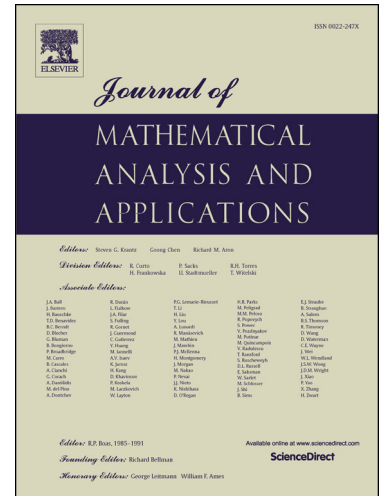
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# The Brown-Halmos Theorem for a Pair of Abstract Hardy Spaces

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## Abstract

Let  $H[X]$  and  $H[Y]$  be abstract Hardy spaces built upon Banach function spaces  $X$  and  $Y$  over the unit circle  $\mathbb{T}$ . We prove an analogue of the Brown-Halmos theorem for Toeplitz operators  $T_a$  acting from  $H[X]$  to  $H[Y]$  under the only assumption that the space  $X$  is separable and the Riesz projection  $P$  is bounded on the space  $Y$ . We specify our results to the case of variable Lebesgue spaces  $X = L^{p(\cdot)}$  and  $Y = L^{q(\cdot)}$  and to the case of Lorentz spaces  $X = Y = L^{p,q}(w)$ ,  $1 < p < \infty$ ,  $1 \leq q < \infty$  with Muckenhoupt weights  $w \in A_p(\mathbb{T})$ .

*Keywords:* Toeplitz operator, Banach function space, pointwise multiplier, Brown-Halmos theorem, variable Lebesgue space, Lorentz space, Muckenhoupt weight.

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## 1. Introduction

For  $1 \leq p \leq \infty$ , let  $L^p := L^p(\mathbb{T})$  represent the standard Lebesgue space on the unit circle  $\mathbb{T}$  in the complex plane  $\mathbb{C}$  with respect to the normalized Lebesgue measure  $dm(t) = |dt|/(2\pi)$ . For  $f \in L^1$ , let

$$\widehat{f}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\varphi}) e^{-in\varphi} d\varphi, \quad n \in \mathbb{Z},$$

be the sequence of the Fourier coefficients of  $f$ . For  $1 \leq p \leq \infty$ , the classical Hardy spaces  $H^p$  are defined by

$$H^p := \{f \in L^p : \widehat{f}(n) = 0 \text{ for all } n < 0\}.$$

Consider the operators  $S$  and  $P$ , defined for a function  $f \in L^1$  and an a.e. point  $t \in \mathbb{T}$  by

$$(Sf)(t) := \frac{1}{\pi i} \text{p.v.} \int_{\mathbb{T}} \frac{f(\tau)}{\tau - t} d\tau, \quad (Pf)(t) := \frac{f(t) + (Sf)(t)}{2}, \quad (1.1)$$

respectively, where the integral is understood in the Cauchy principal value sense. The operator  $S$  is called the Cauchy singular integral operator. It is well known that the operators  $P$  and  $S$  are bounded on  $L^p$  if

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$p \in (1, \infty)$  and are not bounded on  $L^p$  if  $p \in \{1, \infty\}$  (see, e.g., [5, Section 4.4] or [6, Section 1.42]). Note that using the elementary equality

$$\frac{e^{i\theta}}{e^{i\theta} - e^{i\vartheta}} = \frac{1}{2} \left( 1 + i \cot \frac{\vartheta - \theta}{2} \right), \quad \theta, \vartheta \in [-\pi, \pi],$$

one can write for  $f \in L^1$  and  $\vartheta \in [-\pi, \pi]$ ,

$$(Sf)(e^{i\vartheta}) = \frac{1}{\pi} \text{p.v.} \int_{-\pi}^{\pi} \frac{f(e^{i\theta})e^{i\theta}}{e^{i\theta} - e^{i\vartheta}} d\theta = \widehat{f}(0) + i(\mathcal{C}f)(e^{i\vartheta}),$$

where the operator  $\mathcal{C}$ , called the Hilbert transform, is defined for  $f \in L^1$  by

$$(\mathcal{C}f)(e^{i\vartheta}) := \frac{1}{2\pi} \text{p.v.} \int_{-\pi}^{\pi} f(e^{i\theta}) \cot \frac{\vartheta - \theta}{2} d\theta, \quad \vartheta \in [-\pi, \pi]. \quad (1.2)$$

Hence the definition of  $Pf$  for  $f \in L^1$  in terms of the Cauchy singular integral operator given by the second equality in (1.1) is equivalent to the following definition in terms of the Hilbert transform and the zeroth Fourier coefficient of  $f$  (cf. [13, p. 104] and [6, Section 1.43]):

$$Pf := \frac{1}{2}(f + i\mathcal{C}f) + \frac{1}{2}\widehat{f}(0). \quad (1.3)$$

If  $f \in L^1$  is such that  $Pf \in L^1$ , then

$$\widehat{Pf}(n) = \widehat{f}(n) \quad \text{for } n \geq 0, \quad \widehat{Pf}(n) = 0 \quad \text{for } n < 0. \quad (1.4)$$

Since we are not able to provide a precise reference to this well known fact, we will give its proof in Subsection 2.6. Note that definitions (1.1) can be extended to more general Jordan curves in place of  $\mathbb{T}$  (see, e.g., [5] and also [16, 17, 21]), while definitions (1.2) and (1.3) are used only in the case of the unit circle. If  $1 < p < \infty$ , then the operator  $P$  projects  $L^p$  onto  $H^p$ . In view of this fact, the operator  $P$  is usually called the Riesz projection.

For  $a \in L^\infty$ , the Toeplitz operator  $T_a$  with symbol  $a$  on  $H^p$ ,  $1 < p < \infty$ , is defined by

$$T_a f = P(af), \quad f \in H^p.$$

The theory of Toeplitz operators has its origins in the classical paper by Otto Toeplitz [39]. Brown and Halmos [7, Theorem 4] proved that an operator on  $H^2$  is a Toeplitz operator if and only if its matrix with respect to the standard basis is a Toeplitz matrix, that is, an infinite matrix of the form  $(a_{j-k})_{j,k=0}^\infty$  (see also [34, Part B, Theorem 4.1.4] and [36, Theorem 1.8]). An analogue of this result is true for Toeplitz operators acting on  $H^p$ ,  $1 < p < \infty$  (see [6, Theorem 2.7]). Tolokonnikov [40] was the first to study Toeplitz operators acting between different Hardy spaces  $H^p$  and  $H^q$ . In particular, [40, Theorem 4] contains a description of all symbols generating bounded Toeplitz operators from  $H^p$  to  $H^q$  for  $0 < p, q \leq \infty$ .

Let  $X$  be a Banach function space. We postpone the precise definition until Subsection 2.1. For the moment, we observe only that it is continuously embedded in  $L^1$ . Following [41, p. 877], we consider the

abstract Hardy space  $H[X]$  built upon the space  $X$ , which is defined by

$$H[X] := \{f \in X : \widehat{f}(n) = 0 \text{ for all } n < 0\}.$$

It is clear that if  $1 \leq p \leq \infty$ , then  $H[L^p]$  is the classical Hardy space  $H^p$ .

**Lemma 1.1.** *If the operator  $P$  defined by (1.1) is bounded on a Banach function space  $X$  over the unit circle  $\mathbb{T}$ , then its image  $P(X)$  coincides with the abstract Hardy space  $H[X]$  built upon  $X$ .*

Since  $X \subset L^1$ , this lemma follows immediately from formula (1.4) and the uniqueness theorem for Fourier series (see, e.g., [24, Chap. 1, Theorem 2.7]).

Thus, the operator  $P$  projects the Banach function space  $X$  onto the abstract Hardy space  $H[X]$ . We will call  $P$  the Riesz projection as in the case of the spaces  $L^p$  with  $1 < p < \infty$ .

The Brown-Halmos theorem was extended by the first author to abstract Hardy spaces  $H[X]$  built upon reflexive rearrangement-invariant Banach function spaces  $X$  with non-trivial Boyd indices [18, Theorem 4.5]. Under this assumption, the Riesz projection  $P$  is bounded on  $X$ . Further, it was shown in [19, Theorem 1] that the Brown-Halmos theorem remains true for abstract Hardy spaces built upon arbitrarily, not necessarily rearrangement-invariant, reflexive Banach function spaces  $X$  under the assumption that the Riesz projection is bounded on  $X$ . In particular, it is true for the weighted Hardy spaces  $H^p(w)$ ,  $1 < p < \infty$ , with Muckenhoupt weights  $w \in A_p(\mathbb{T})$  [19, Corollary 9].

The space of all bounded linear operators from a Banach space  $E$  to a Banach space  $F$  is denoted by  $\mathcal{B}(E, F)$ . We adopt the standard abbreviation  $\mathcal{B}(E)$  for  $\mathcal{B}(E, E)$ . We will write  $E = F$  if  $E$  and  $F$  coincide as sets and there are constants  $c_1, c_2 \in (0, \infty)$  such that  $c_1\|f\|_E \leq \|f\|_F \leq c_2\|f\|_E$  for all  $f \in E$ , and  $E \equiv F$  if  $E$  and  $F$  coincide as sets and  $\|f\|_E = \|f\|_F$  for all  $f \in E$ .

The aim of this paper is to study Toeplitz operators acting between abstract Hardy spaces  $H[X]$  and  $H[Y]$  built upon different Banach function spaces  $X$  and  $Y$  over the unit circle  $\mathbb{T}$ . We extend further the results by Leśnik [29], who additionally assumed that the Banach function spaces  $X$  and  $Y$  are rearrangement-invariant. Let  $L^0$  be the space of all measurable complex-valued functions on  $\mathbb{T}$ . Following [32], let  $M(X, Y)$  denote the space of pointwise multipliers from  $X$  to  $Y$  defined by

$$M(X, Y) := \{f \in L^0 : fg \in Y \text{ for all } g \in X\}$$

and equipped with the natural operator norm

$$\|f\|_{M(X, Y)} = \|M_f\|_{\mathcal{B}(X, Y)} = \sup_{\|g\|_X \leq 1} \|fg\|_Y.$$

Here  $M_f$  stands for the operator of multiplication by  $f$  defined by  $(M_f g)(t) = f(t)g(t)$  for  $t \in \mathbb{T}$ .

In particular,  $M(X, X) \equiv L^\infty$ . Note that it may happen that the space  $M(X, Y)$  contains only the zero function. For instance, if  $1 \leq p < q \leq \infty$ , then  $M(L^p, L^q) = \{0\}$ . The continuous embedding  $L^\infty \subset M(X, Y)$

holds if and only if  $X \subset Y$  continuously. For example, if  $1 \leq q \leq p \leq \infty$ , then  $L^p \subset L^q$  and  $M(L^p, L^q) \equiv L^r$ , where  $1/r = 1/q - 1/p$ . For these and many other properties and examples, we refer to [26, 30, 32, 33] (see also references therein).

If the Riesz projection  $P$  is bounded on the space  $Y$ , then one can define the Toeplitz operator  $T_a$  with symbol  $a \in M(X, Y)$  by

$$T_a f = P(af), \quad f \in H[X]$$

(cf. [29]). It follows from Lemma 1.1 that  $T_a f \in H[Y]$  and, clearly,

$$\|T_a\|_{\mathcal{B}(H[X], H[Y])} \leq \|P\|_{\mathcal{B}(Y)} \|a\|_{M(X, Y)}.$$

Let  $X'$  be the associate space of  $X$  (see Subsection 2.1). For  $f \in X$  and  $g \in X'$ , put

$$\langle f, g \rangle := \int_{\mathbb{T}} f(t) \overline{g(t)} dm(t).$$

For  $n \in \mathbb{Z}$  and  $\tau \in \mathbb{T}$ , put  $\chi_n(\tau) := \tau^n$ . Then the Fourier coefficients of a function  $f \in L^1$  can be expressed by  $\widehat{f}(n) = \langle f, \chi_n \rangle$  for  $n \in \mathbb{Z}$ . With these notation, our main result reads as follows.

**Theorem 1.2 (à la Brown-Halmos).** *Let  $X, Y$  be two Banach function spaces over the unit circle  $\mathbb{T}$ . Suppose that  $X$  is separable and the Riesz projection  $P$  is bounded on the space  $Y$ . If  $A \in \mathcal{B}(H[X], H[Y])$  and there exists a sequence  $\{a_n\}_{n \in \mathbb{Z}}$  of complex numbers such that*

$$\langle A\chi_j, \chi_k \rangle = a_{k-j} \quad \text{for all } j, k \geq 0, \quad (1.5)$$

then there is a function  $a \in M(X, Y)$  such that  $A = T_a$  and  $\widehat{a}(n) = a_n$  for all  $n \in \mathbb{Z}$ . Moreover,

$$\|a\|_{M(X, Y)} \leq \|T_a\|_{\mathcal{B}(H[X], H[Y])} \leq \|P\|_{\mathcal{B}(Y)} \|a\|_{M(X, Y)}. \quad (1.6)$$

Under the additional assumption that the Banach function spaces  $X$  and  $Y$  are rearrangement-invariant, this result was recently obtained by Leśnik [29, Theorem 4.2].

The above theorem and the fact that  $M(X, X) \equiv L^\infty$  (see [32, Theorem 1]) immediately imply the following.

**Corollary 1.3.** *Let  $X$  be a separable Banach function spaces over the unit circle  $\mathbb{T}$  and let the Riesz projection  $P$  be bounded on  $X$ . If  $A \in \mathcal{B}(H[X])$  and there is a sequence  $\{a_n\}_{n \in \mathbb{Z}}$  of complex numbers satisfying (1.5), then there exists a function  $a \in L^\infty$  such that  $A = T_a$  and  $\widehat{a}(n) = a_n$  for all  $n \in \mathbb{Z}$ . Moreover,*

$$\|a\|_{L^\infty} \leq \|T_a\|_{\mathcal{B}(H[X])} \leq \|P\|_{\mathcal{B}(X)} \|a\|_{L^\infty}.$$

Note that Corollary 1.3 is also new. Under the additional assumption that the Banach function space  $X$  is reflexive, it was proved by the first author in [19, Theorem 1]. On the other hand, under the additional hypothesis that  $X$  is rearrangement-invariant, it is established in [29, Corollary 4.4].

The paper is organized as follows. In Section 2, we collect preliminary facts on Banach function spaces  $X$ , including results on the density of the set of all trigonometric polynomials  $\mathcal{P}$  in  $X$  and the density of the set of all analytic polynomials  $\mathcal{P}_A$  in the abstract Hardy space  $H[X]$  built upon  $X$ . Further, we show that if each function in the closure  $(X')_b$  of all simple functions in the associate space  $X'$  has absolutely continuous norm, then the norm of any function  $f \in X$  can be expressed as follows:

$$\|f\|_X = \sup\{|\langle f, p \rangle| : p \in \mathcal{P}, \|p\|_{X'} \leq 1\}. \quad (1.7)$$

We conclude Section 2 with several facts from complex analysis on the Hilbert transform and inner functions. In particular, we recall a result by Qiu [38, Lemma 5.1] (see also [8, Theorem 7.2]) saying that, for every measurable set  $E \subset \mathbb{T}$  and an arc  $\gamma \subset \mathbb{T}$  of the same measure, there exists an inner function  $u$  such that  $u^{-1}(\gamma)$  and  $E$  coincide almost everywhere.

We start Section 3 on the consequences of the boundedness of the operator  $P$  defined by (1.1) with a discussion of operators of weak type. It is easy to see that if the Riesz projection  $P$  is bounded on  $X$ , then the Hilbert transform  $\mathcal{C}$  is of weak types  $(L^\infty, X)$  and  $(L^\infty, X')$ . Using the existence of the inner function  $u$  mentioned above and properties of the Hilbert transform, we show that if  $\mathcal{C}$  is of weak types  $(L^\infty, X)$  and  $(L^\infty, X')$ , then each function in the closures  $X_b$  and  $(X')_b$  of the simple functions in  $X$  and  $X'$ , respectively, has absolutely continuous norm. Thus, for every  $f \in X$ , formula (1.7) holds under the only assumption that  $P \in \mathcal{B}(X)$ .

In Section 4, we present a proof of Theorem 1.2. Armed with the density of the set of analytic polynomials  $\mathcal{P}_A$  in the abstract Hardy space  $H[X]$  built upon a separable Banach function space  $X$  and formula (1.7) with  $Y$  such that  $P \in \mathcal{B}(Y)$  in place of  $X$ , we can adapt the proofs given in [6, Theorem 2.7] (for  $X = Y = L^p$  with  $1 < p < \infty$ ) and in [29, Theorem 4.2] (for the case of separable rearrangement-invariant spaces  $X \subset Y$  such that  $Y$  has non-trivial Boyd indices) to our setting.

In Section 5, we specify the result of Theorem 1.2 to the case of variable Lebesgue spaces (also known as Nakano spaces)  $X = L^{p(\cdot)}$  and  $Y = L^{q(\cdot)}$ . It is known that if  $1/q(t) = 1/p(t) + 1/r(t)$  for  $t \in \mathbb{T}$ , then  $M(L^{p(\cdot)}, L^{q(\cdot)}) = L^{r(\cdot)}$  and that the Riesz projection  $P$  is bounded on  $L^{q(\cdot)}$  if the variable exponent  $q$  is sufficiently smooth and bounded away from 1 and  $\infty$ . Since the spaces  $L^{p(\cdot)}$  and  $L^{q(\cdot)}$  are not rearrangement-invariant, in general, the main result of Section 5 cannot be obtained from [29, Theorem 4.2].

In Section 6, we apply Corollary 1.3 to the case of Lorentz spaces  $L^{p,q}(w)$ ,  $1 < p < \infty$ ,  $1 \leq q < \infty$ , with Muckenhoupt weights  $w \in A_p(\mathbb{T})$ . Under these assumptions,  $L^{p,q}(w)$  is a separable Banach function space and the Riesz projection  $P$  is bounded on  $L^{p,q}(w)$ . The space  $L^{p,1}(w)$  is not reflexive and not rearrangement-invariant. Hence the earlier results of [19, Theorem 1] and [29, Corollary 4.4] are not applicable to the space  $L^{p,1}(w)$ , while Corollary 1.3 is.

## 2. Preliminaries

### 2.1. Banach function spaces

Let  $L_+^0$  be the subset of functions in  $L^0$  whose values lie in  $[0, \infty]$ . The characteristic (indicator) function of a measurable set  $E \subset \mathbb{T}$  is denoted by  $\mathbb{I}_E$ .

Following [1, Chap. 1, Definition 1.1], a mapping  $\rho : L_+^0 \rightarrow [0, \infty]$  is called a Banach function norm if, for all functions  $f, g, f_n \in L_+^0$  with  $n \in \mathbb{N}$ , for all constants  $a \geq 0$ , and for all measurable subsets  $E$  of  $\mathbb{T}$ , the following properties hold:

- (A1)  $\rho(f) = 0 \Leftrightarrow f = 0$  a.e.,  $\rho(af) = a\rho(f)$ ,  $\rho(f + g) \leq \rho(f) + \rho(g)$ ,
- (A2)  $0 \leq g \leq f$  a.e.  $\Rightarrow \rho(g) \leq \rho(f)$  (the lattice property),
- (A3)  $0 \leq f_n \uparrow f$  a.e.  $\Rightarrow \rho(f_n) \uparrow \rho(f)$  (the Fatou property),
- (A4)  $m(E) < \infty \Rightarrow \rho(\mathbb{I}_E) < \infty$ ,
- (A5)  $\int_E f(t) dm(t) \leq C_E \rho(f)$

with a constant  $C_E \in (0, \infty)$  that may depend on  $E$  and  $\rho$ , but is independent of  $f$ . When functions differing only on a set of measure zero are identified, the set  $X$  of all functions  $f \in L^0$  for which  $\rho(|f|) < \infty$  is called a Banach function space. For each  $f \in X$ , the norm of  $f$  is defined by  $\|f\|_X := \rho(|f|)$ . The set  $X$  under the natural linear space operations and under this norm becomes a Banach space (see [1, Chap. 1, Theorems 1.4 and 1.6]). If  $\rho$  is a Banach function norm, its associate norm  $\rho'$  is defined on  $L_+^0$  by

$$\rho'(g) := \sup \left\{ \int_{\mathbb{T}} f(t)g(t) dm(t) : f \in L_+^0, \rho(f) \leq 1 \right\}, g \in L_+^0.$$

It is a Banach function norm itself [1, Chap. 1, Theorem 2.2]. The Banach function space  $X'$  determined by the Banach function norm  $\rho'$  is called the associate space (Köthe dual) of  $X$ . The associate space  $X'$  can be viewed as a subspace of the (Banach) dual space  $X^*$ .

### 2.2. Density of polynomials

For  $n \in \mathbb{Z}_+ := \{0, 1, 2, \dots\}$ , a function of the form  $\sum_{k=-n}^n \alpha_k \chi_k$ , where  $\alpha_k \in \mathbb{C}$  for all  $k \in \{-n, \dots, n\}$ , is called a trigonometric polynomial of order  $n$ . The set of all trigonometric polynomials is denoted by  $\mathcal{P}$ . Further, a function of the form  $\sum_{k=0}^n \alpha_k \chi_k$  with  $\alpha_k \in \mathbb{C}$  for  $k \in \{0, \dots, n\}$  is called an analytic polynomial of order  $n$ . The set of all analytic polynomials is denoted by  $\mathcal{P}_A$ .

Following [1, Chap. 1, Definition 3.1], a function  $f$  in a Banach function space  $X$  is said to have absolutely continuous norm in  $X$  if  $\|f\mathbb{I}_{\gamma_n}\|_X \rightarrow 0$  for every sequence  $\{\gamma_n\}_{n \in \mathbb{N}}$  of measurable sets such that  $\mathbb{I}_{\gamma_n} \rightarrow 0$  almost everywhere as  $n \rightarrow \infty$ . The set of all functions of absolutely continuous norm in  $X$  is denoted by  $X_a$ . If  $X_a = X$ , then one says that  $X$  has absolutely continuous norm. Let  $S_0$  be the set of all simple functions on  $\mathbb{T}$ . Following [1, Chap. 1, Definition 3.9], let  $X_b$  denote the closure of  $S_0$  in the norm of  $X$ .



**Lemma 2.1.** *Let  $X$  be a Banach function space over the unit circle  $\mathbb{T}$ . If  $X_a = X_b$ , then the set of trigonometric polynomials  $\mathcal{P}$  is dense in  $X_b$ .*

PROOF. The proof is analogous to the proof of [20, Lemma 2.2.1]. Assume that  $\mathcal{P}$  is not dense in  $X_b$ . Then, by a corollary of the Hahn-Banach theorem (see, e.g., [2, Chap. 7, Theorem 4.2]), there exists a nonzero functional  $\Lambda \in (X_b)^*$  such that  $\Lambda(p) = 0$  for all  $p \in \mathcal{P}$ . It follows from [1, Chap. 1, Theorems 3.10 and 4.1] that if  $X_a = X_b$ , then  $(X_b)^* = X'$ . Hence there exists a nonzero function  $h \in X' \subset L^1$  such that

$$\int_{\mathbb{T}} p(t)h(t) dm(t) = 0 \quad \text{for all } p \in \mathcal{P}.$$

Taking  $p(t) = t^n$  for  $n \in \mathbb{Z}$ , we obtain that all Fourier coefficients of  $h \in L^1$  vanish, which implies that  $h = 0$  a.e. on  $\mathbb{T}$  by the uniqueness theorem of the Fourier series (see, e.g., [24, Chap. I, Theorem 2.7]). This contradiction proves that  $\mathcal{P}$  is dense in  $X_b$ .  $\square$

Combining the above lemma with [1, Chap. 1, Corollary 5.6 and Theorem 3.11], we arrive at the following well known result.

**Corollary 2.2.** *A Banach function space  $X$  over the unit circle  $\mathbb{T}$  is separable if and only if the set of trigonometric polynomials  $\mathcal{P}$  is dense in  $X$ .*

The analytic counterpart of the above result had a hard birth. First, observe that under the additional assumption that the Riesz projection  $P$  is bounded on  $X$ , the density of the set of analytic polynomials  $\mathcal{P}_A$  in the abstract Hardy space  $H[X]$  trivially follows from (1.4), Lemma 1.1, and Corollary 2.2 (see [19, Lemma 4]). Leśnik [28] conjectured that the boundedness of  $P$  is superfluous here and  $\mathcal{P}_A$  must be dense in the abstract Hardy space  $H[X]$  under the hypothesis that  $X$  is merely separable.

If  $X$  is a separable rearrangement-invariant Banach function space, then

$$\|f * F_n - f\|_X \rightarrow 0 \quad \text{for every } f \in X \quad \text{as } n \rightarrow \infty, \quad (2.1)$$

where  $\{F_n\}$  is the sequence of the Fejér kernels on the unit circle  $\mathbb{T}$ . The property in (2.1) implies the density of  $\mathcal{P}_A$  in  $H[X]$  (see, e.g., [29, Lemma 3.1(c)] or [20, Theorem 1.0.1]). If  $X$  is an arbitrary separable Banach function space, then (2.1) is true under the assumption that the Hardy-Littlewood maximal operator  $M$  is bounded on its associate space  $X'$  [20, Theorem 3.2.1], whence  $\mathcal{P}_A$  is dense in  $H[X]$  (see [20, Theorem 1.0.2]). Finally, in [22, Theorem 1.4] we constructed a separable weighted  $L^1$  space  $X$  such that (2.1) does not hold. On the other hand, we proved Leśnik's conjecture.

**Lemma 2.3 ([22, Theorem 1.5]).** *If  $X$  is a separable Banach function space over the unit circle  $\mathbb{T}$ , then the set of analytic polynomials  $\mathcal{P}_A$  is dense in the abstract Hardy space  $H[X]$  built upon the space  $X$ .*

### 2.3. Formulae for the norm in a Banach function space

Let  $X$  be a Banach function space over the unit circle  $\mathbb{T}$  and  $X'$  be its associate space. Then for every  $f \in X$  and  $h \in X'$ , one has the following well known formulae:

$$\|f\|_X = \sup\{|\langle f, g \rangle| : g \in X', \|g\|_{X'} \leq 1\}, \quad (2.2)$$

$$\|f\|_X = \sup\{|\langle f, s \rangle| : s \in S_0, \|s\|_{X'} \leq 1\}, \quad (2.3)$$

$$\|h\|_{X'} = \sup\{|\langle h, s \rangle| : s \in S_0, \|s\|_X \leq 1\}. \quad (2.4)$$

Equality (2.2) follows from [1, Chap. 1, Theorem 2.7 and Lemma 2.8]. Equality (2.3) can be proved by a literal repetition of the proof of [23, Lemma 2.10]. Equality (2.4) is obtained by applying formula (2.3) to  $h \in X'$  and recalling that  $X \equiv X''$  in view of the Lorentz-Luxemburg theorem (see [1, Chap. 1, Theorem 2.7]).

**Lemma 2.4.** *Let  $X$  be a Banach function space over the unit circle  $\mathbb{T}$ . If  $(X')_a = (X')_b$ , then for every  $f \in X$ ,*

$$\|f\|_X = \sup\{|\langle f, p \rangle| : p \in \mathcal{P}, \|p\|_{X'} \leq 1\}. \quad (2.5)$$

PROOF. Since  $\mathcal{P} \subset X'$ , equality (2.2) immediately implies that

$$\|f\|_X \geq \sup\{|\langle f, p \rangle| : p \in \mathcal{P}, \|p\|_{X'} \leq 1\}. \quad (2.6)$$

Take any  $g \in (X')_b$  such that  $0 < \|g\|_{X'} \leq 1$ . Since  $(X')_a = (X')_b$ , it follows from Lemma 2.1 that there is a sequence  $q_n \in \mathcal{P} \setminus \{0\}$  such that  $\|q_n - g\|_{X'} \rightarrow 0$  as  $n \rightarrow \infty$ . For  $n \in \mathbb{N}$ , put  $p_n := (\|g\|_{X'} / \|q_n\|_{X'}) q_n \in \mathcal{P}$ . Then, arguing as in [19, Lemma 5], one can show that

$$|\langle f, g \rangle| = \lim_{n \rightarrow \infty} |\langle f, p_n \rangle| \leq \sup_{n \in \mathbb{N}} |\langle f, p_n \rangle| \leq \sup\{|\langle f, p \rangle| : p \in \mathcal{P}, \|p\|_{X'} \leq 1\}.$$

This inequality and equality (2.2) imply that

$$\|f\|_X \leq \sup\{|\langle f, p \rangle| : p \in \mathcal{P}, \|p\|_{X'} \leq 1\}. \quad (2.7)$$

Combining inequalities (2.6) and (2.7), we arrive at equality (2.5).  $\square$

Note that Lešnik proved formula (2.5) for arbitrary rearrangement-invariant Banach function spaces  $X$  (see [29, Lemma 3.2]). His proof relies on the interpolation theorem of Calderón (see [1, Chap. 3, Theorem 2.2]), which allows one to prove that for  $f \in X'$ , the sequence  $p_n = f * F_n \in \mathcal{P}$  satisfies  $\|p_n\|_{X'} \leq \|f\|_{X'}$  for all  $n \in \mathbb{N}$ . In the setting of arbitrary Banach function spaces, the tools based on interpolation are not available, but one can prove (2.5) for translation-invariant Banach function spaces and their weighted generalizations with positive continuous weights (cf. [23, Corollary 2.13]). In the next section, we show that if the Riesz projection  $P$  is bounded on a Banach function space  $X$ , then  $(X')_a = (X')_b$ , whence formula (2.5) holds.

#### 2.4. Hardy spaces on the unit disk and inner functions

Let  $\mathbb{D}$  denote the open unit disk in the complex plane  $\mathbb{C}$ . Recall that a function  $F$  analytic in  $\mathbb{D}$  is said to belong to the Hardy space  $H^p(\mathbb{D})$ ,  $0 < p \leq \infty$ , if

$$\|F\|_{H^p(\mathbb{D})} := \sup_{0 \leq r < 1} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |F(re^{i\theta})|^p d\theta \right)^{1/p} < \infty, \quad 0 < p < \infty, \quad \|F\|_{H^\infty(\mathbb{D})} := \sup_{z \in \mathbb{D}} |F(z)| < \infty.$$

Recall that an inner function is a function  $u \in H^\infty(\mathbb{D})$  such that  $|u(e^{i\theta})| = 1$  for a.e.  $\theta \in [-\pi, \pi]$ .

The following important fact was observed by Nordgren (see corollary to [35, Lemma 1] and also [9, Remark 9.4.6]).

**Lemma 2.5.** *If  $u$  is an inner function such that  $u(0) = 0$ , then  $u$  is a measure-preserving transformation from  $\mathbb{T}$  onto itself.*

PROOF. We include a sketch of the proof for the readers' convenience. Let  $G$  be an arbitrary measurable subset of  $\mathbb{T}$  and let  $h$  be the bounded harmonic function on  $\mathbb{D}$  with the boundary values equal to  $\mathbb{I}_G$ . Then  $h \circ u$  is the bounded harmonic function on  $\mathbb{D}$  with the boundary values equal to  $\mathbb{I}_{u^{-1}(G)}$ , and

$$m(G) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbb{I}_G(e^{i\theta}) d\theta = h(0) = h(u(0)) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbb{I}_{u^{-1}(G)}(e^{i\theta}) d\theta = m(u^{-1}(G)),$$

which completes the proof.  $\square$

The next result is one of the most important ingredients in our proof. It appeared in [38, Lemma 5.1] and [8, Theorem 7.2].

**Theorem 2.6.** *If  $E \subset \mathbb{T}$  is a measurable set and  $\gamma \subset \mathbb{T}$  is an arc such that  $m(E) = m(\gamma)$ , then there exists an inner function  $u$  satisfying  $u(0) = 0$  and such that the sets  $u^{-1}(\gamma)$  and  $E$  are equal almost everywhere.*

#### 2.5. The Hilbert transform and Poisson integrals

For  $\vartheta \in [-\pi, \pi]$  and  $r \in [0, 1)$ , let

$$P_r(\vartheta) := \frac{1 - r^2}{1 - 2r \cos \vartheta + r^2}, \quad Q_r(\vartheta) := \frac{2r \sin \vartheta}{1 - 2r \cos \vartheta + r^2}$$

be the Poisson kernel and the conjugate Poisson kernel, respectively.

**Theorem 2.7.** *Let  $1 < p < \infty$ .*

(a) *If  $f \in L^p$  is a real-valued function, then the function defined by*

$$u(re^{i\vartheta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) (P_r + iQ_r)(\vartheta - \theta) d\theta, \quad \vartheta \in [-\pi, \pi], \quad r \in [0, 1), \quad (2.8)$$

*belongs to the Hardy space  $H^p(\mathbb{D})$ . Its nontangential boundary values  $u(e^{i\vartheta})$  as  $z \rightarrow e^{i\vartheta}$  exist for a.e.  $\vartheta \in [-\pi, \pi]$  and*

$$\operatorname{Re} u(e^{i\vartheta}) = f(e^{i\vartheta}), \quad \operatorname{Im} u(e^{i\vartheta}) = (\mathcal{C}f)(e^{i\vartheta}) \quad \text{for a.e. } \vartheta \in [-\pi, \pi], \quad (2.9)$$

*where  $\mathcal{C}$  is the Hilbert transform defined by (1.2).*

(b) If  $u \in H^p(\mathbb{D})$  and  $\text{Im } u(0) = 0$ , then there is a real-valued function  $f \in L^p$  such that (2.8) holds.

This statement is well known (see, e.g., [27, Chap. I, Section D and Chap. V, Section B.2°]).

### 2.6. Fourier coefficients of $Pf$ : proof of formula (1.4)

Since  $f \in L^1$ , the Cauchy integral

$$F(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(\tau)}{\tau - z} d\tau, \quad z \in \mathbb{D},$$

belongs to  $H^p(\mathbb{D})$  for all  $0 < p < 1$  (see, e.g., [12, Theorem 3.5]). By Privalov's theorem (see, e.g., [14, Chap. X, §3, Theorem 1]), the nontangential limit of  $F(z)$  as  $z \rightarrow e^{i\vartheta}$  coincides with  $(Pf)(e^{i\vartheta})$  for a.e.  $\vartheta \in [-\pi, \pi]$ . Hence, taking into account that  $Pf \in L^1$ , by Smirnov's theorem (see, e.g., [14, Chap. IX, §4, Theorem 4] or [12, Theorem 3.4]),  $F \in H^1(\mathbb{D})$ . Then (1.4) follows from [12, Theorem 3.4] and the formula for the Taylor coefficients of  $F$ :

$$\frac{1}{n!} F^{(n)}(0) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(\tau)}{\tau^{n+1}} d\tau = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\varphi}) e^{-in\varphi} d\varphi = \widehat{f}(n), \quad n \geq 0,$$

which completes the proof.  $\square$

## 3. Consequences of the boundedness of the Riesz projection

### 3.1. Operators of weak type

Let  $X$  and  $Y$  be Banach function spaces over the unit circle. Following [3], we say that a linear operator  $A : X \rightarrow L^0$  is of weak type  $(X, Y)$  if there exists a constant  $C > 0$  such that for all  $\lambda > 0$  and  $f \in X$ ,

$$\left\| \mathbb{I}_{\{\zeta \in \mathbb{T} : |(Af)(\zeta)| > \lambda\}} \right\|_Y \leq C \frac{\|f\|_X}{\lambda}. \quad (3.1)$$

We denote the infimum of the constants  $C$  satisfying (3.1) by  $\|A\|_{\mathcal{W}(Y, X)}$  and the set of all operators of weak type  $(X, Y)$  by  $\mathcal{W}(X, Y)$ .

**Lemma 3.1.** *Let  $X, Y$  be Banach function spaces over the unit circle  $\mathbb{T}$ . If  $A \in \mathcal{B}(X, Y)$ , then  $A \in \mathcal{W}(X, Y)$  and  $\|A\|_{\mathcal{W}(X, Y)} \leq \|A\|_{\mathcal{B}(X, Y)}$ .*

PROOF. For all  $\lambda > 0$ ,  $f \in X$  and almost all  $\tau \in \mathbb{T}$ , one has

$$\mathbb{I}_{\{\zeta \in \mathbb{T} : |(Af)(\zeta)| > \lambda\}}(\tau) \leq \mathbb{I}_{\{\zeta \in \mathbb{T} : |(Af)(\zeta)| > \lambda\}}(\tau) \frac{|(Af)(\tau)|}{\lambda} \leq \frac{|(Af)(\tau)|}{\lambda}.$$

It follows from the above inequality, the lattice property, and the boundedness of the operator  $A$  that

$$\left\| \mathbb{I}_{\{\zeta \in \mathbb{T} : |(Af)(\zeta)| > \lambda\}} \right\|_Y \leq \left\| \frac{Af}{\lambda} \right\|_Y \leq \|A\|_{\mathcal{B}(X, Y)} \frac{\|f\|_X}{\lambda},$$

which completes the proof.  $\square$

### 3.2. Pointwise estimate for the Hilbert transform

For a set  $G \subset [-\pi, \pi]$ , we use the following notation

$$\mathbb{I}_G^*(e^{i\theta}) := \begin{cases} 1, & \theta \in G, \\ 0, & \theta \in [-\pi, \pi] \setminus G. \end{cases}$$

Let  $|G|$  denote the Lebesgue measure of  $G$ .

**Lemma 3.2.** *For every measurable set  $E \subset [-\pi, \pi]$  with  $0 < |E| \leq \pi/2$ , there exists a measurable set  $F \subset [-\pi, \pi]$  with  $|F| = \pi$  such that*

$$|(\mathcal{C}\mathbb{I}_F^*)(e^{i\vartheta})| > \frac{1}{\pi} \left| \log \left( \sqrt{2} \sin \frac{|E|}{2} \right) \right| \quad \text{for a.e. } \vartheta \in E. \quad (3.2)$$

PROOF. Let  $\ell := \{e^{i\eta} \in \mathbb{T} : \pi - |E| < \eta < \pi\}$ . By Theorem 2.6, there exists an inner function  $V$  such that  $V(0) = 0$  and

$$V(e^{i\vartheta}) \in \begin{cases} \ell & \text{for a.e. } \vartheta \in E, \\ \mathbb{T} \setminus \ell & \text{for a.e. } \vartheta \in [-\pi, \pi] \setminus E. \end{cases} \quad (3.3)$$

Consider the set

$$F := \{\theta \in [-\pi, \pi] : \operatorname{Im} V(e^{i\theta}) \leq 0\}. \quad (3.4)$$

Since  $V(0) = 0$  and  $V$  is inner, it defines a measure-preserving transformation of  $\mathbb{T}$  onto itself due to Lemma 2.5. Therefore,

$$|F| = |\{\vartheta \in [-\pi, \pi] : \operatorname{Im} e^{i\vartheta} \leq 0\}| = \pi.$$

For  $\eta \in [-\pi, \pi]$  and  $r \in [0, 1)$ , let

$$w(re^{i\eta}) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbb{I}_{[-\pi, 0]}^*(e^{i\zeta}) (P_r + iQ_r)(\eta - \zeta) d\zeta.$$

By Theorem 2.7, the function  $w \in H^2(\mathbb{D})$  has nontangential boundary values  $w(e^{i\eta})$  as  $z \rightarrow e^{i\eta}$  for a.e.  $\eta \in [-\pi, \pi]$  and

$$\operatorname{Re} w(e^{i\eta}) = \mathbb{I}_{[-\pi, 0]}^*(e^{i\eta}) \quad \text{for a.e. } \eta \in [-\pi, \pi], \quad (3.5)$$

$$\operatorname{Im} w(e^{i\eta}) = (\mathcal{C}\mathbb{I}_{[-\pi, 0]}^*)(e^{i\eta}) \quad \text{for a.e. } \eta \in [-\pi, \pi]. \quad (3.6)$$

It is clear that for  $\eta \in (\pi - |E|, \pi)$ ,

$$(\mathcal{C}\mathbb{I}_{[-\pi, 0]}^*)(e^{i\eta}) = \frac{1}{2\pi} \int_{-\pi}^0 \cot \frac{\eta - \zeta}{2} d\zeta = \frac{1}{\pi} \log \sin \frac{\eta}{2} - \frac{1}{\pi} \log \sin \frac{\eta + \pi}{2}. \quad (3.7)$$

Since  $|E| \in (0, \pi/2]$ , we have for all  $\eta \in (\pi - |E|, \pi)$ ,

$$\log \sin \frac{\eta}{2} > \log \sin \frac{\pi}{4} = -\log \sqrt{2} \geq \log \sin \frac{|E|}{2} > \log \sin \frac{\eta + \pi}{2}. \quad (3.8)$$

It follows from (3.6)–(3.8) that for a.e.  $\eta \in (\pi - |E|, \pi)$ ,

$$|\operatorname{Im} w(e^{i\eta})| > \frac{1}{\pi} \left( -\log \sqrt{2} - \log \sin \frac{|E|}{2} \right) = \frac{1}{\pi} \left| \log \left( \sqrt{2} \sin \frac{|E|}{2} \right) \right|. \quad (3.9)$$

Consider now the function  $W = w \circ V$ , which belongs to  $H^2(\mathbb{D})$  (see, e.g., [12, Section 2.6]). In view of (3.4) and (3.5), we have

$$\operatorname{Re} W(e^{i\vartheta}) = \begin{cases} 1 & \text{if } \operatorname{Im} V(e^{i\vartheta}) \leq 0, \\ 0 & \text{if } \operatorname{Im} V(e^{i\vartheta}) > 0 \end{cases} = \mathbb{I}_F^*(e^{i\vartheta}) \quad \text{for a.e. } \vartheta \in [-\pi, \pi].$$

Then, by Theorem 2.7,

$$\operatorname{Im} W(e^{i\vartheta}) = (\mathcal{C}\mathbb{I}_F^*)(e^{i\vartheta}) \quad \text{for a.e. } \vartheta \in [-\pi, \pi], \quad (3.10)$$

If  $\vartheta \in E$ , then it follows from (3.3) that  $V(e^{i\vartheta}) \in \ell$ . In this case inequality (3.9) implies that for a.e.  $\vartheta \in E$ ,

$$|\operatorname{Im} W(e^{i\vartheta})| = |\operatorname{Im} w(V(e^{i\vartheta}))| > \frac{1}{\pi} \left| \log \left( \sqrt{2} \sin \frac{|E|}{2} \right) \right|. \quad (3.11)$$

Combining equality (3.10) and inequality (3.11), we arrive at (3.2).  $\square$

### 3.3. Equality $X_a = X_b$ if $\mathcal{C} \in \mathcal{W}(L^\infty, X)$

**Lemma 3.3.** *Let  $X$  be a Banach function space over the unit circle  $\mathbb{T}$ . If the Hilbert transform  $\mathcal{C}$  is of weak type  $(L^\infty, X)$ , then for every measurable set  $E \subset [-\pi, \pi]$  with  $0 < |E| \leq \pi/2$ , one has*

$$\|\mathbb{I}_E^*\|_X \leq \frac{\pi \|\mathcal{C}\|_{\mathcal{W}(L^\infty, X)}}{\left| \log \left( \sqrt{2} \sin \frac{|E|}{2} \right) \right|}. \quad (3.12)$$

PROOF. Let

$$\lambda = \frac{1}{\pi} \left| \log \left( \sqrt{2} \sin \frac{|E|}{2} \right) \right|.$$

By Lemma 3.2, there exists a measurable set  $F \subset [-\pi, \pi]$  with  $|F| = \pi$  such that for a.e.  $\tau \in \mathbb{T}$ ,

$$\mathbb{I}_E^*(\tau) \leq \mathbb{I}_{\{\zeta \in \mathbb{T} : |(\mathcal{C}\mathbb{I}_F^*)(\zeta)| > \lambda\}}(\tau).$$

Therefore, by the lattice property, taking into account that  $\mathcal{C} \in \mathcal{W}(L^\infty, X)$ , we obtain

$$\|\mathbb{I}_E^*\|_X \leq \left\| \mathbb{I}_{\{\zeta \in \mathbb{T} : |(\mathcal{C}\mathbb{I}_F^*)(\zeta)| > \lambda\}} \right\|_X \leq \frac{1}{\lambda} \|\mathcal{C}\|_{\mathcal{W}(L^\infty, X)} \|\mathbb{I}_F^*\|_{L^\infty} = \frac{\pi \|\mathcal{C}\|_{\mathcal{W}(L^\infty, X)}}{\left| \log \left( \sqrt{2} \sin \frac{|E|}{2} \right) \right|},$$

which completes the proof.  $\square$

**Theorem 3.4.** *Let  $X$  be a Banach function space over the unit circle  $\mathbb{T}$ . If the Hilbert transform  $\mathcal{C}$  is of weak type  $(L^\infty, X)$ , then  $X_a = X_b$ .*

PROOF. Let  $\Gamma \subset \mathbb{T}$  be a measurable set. Consider a sequence of measurable subsets  $\{\gamma_n\}_{n \in \mathbb{N}}$  of  $\mathbb{T}$  such that  $\mathbb{I}_{\gamma_n} \rightarrow 0$  a.e. on  $\mathbb{T}$ . By the dominated convergence theorem,

$$m(\gamma_n) = \int_{\mathbb{T}} \mathbb{I}_{\gamma_n}(\tau) dm(\tau) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Without loss of generality, one can assume that  $0 < m(\gamma_n) \leq 1/4$  for all  $n \in \mathbb{N}$ . For every  $n \in \mathbb{N}$ , there exists a measurable set  $E_n \subset [-\pi, \pi]$  such that  $\mathbb{I}_{\gamma_n}(\tau) = \mathbb{I}_{E_n}^*(\tau)$  for all  $\tau \in \mathbb{T}$ . It is clear that  $|E_n| = 2\pi m(\gamma_n) \leq \pi/2$  for  $n \in \mathbb{N}$ . By Lemma 3.3, for every  $n \in \mathbb{N}$ ,

$$\|\mathbb{I}_{\Gamma} \mathbb{I}_{\gamma_n}\|_X \leq \|\mathbb{I}_{\gamma_n}\|_X = \|\mathbb{I}_{E_n}^*\|_X \leq \frac{\pi \|\mathcal{C}\|_{\mathcal{W}(L^\infty, Y)}}{\left| \log \left( \sqrt{2} \sin \frac{|E_n|}{2} \right) \right|} = \frac{\pi \|\mathcal{C}\|_{\mathcal{W}(L^\infty, Y)}}{\left| \log \left( \sqrt{2} \sin(\pi m(\gamma_n)) \right) \right|}.$$

Since  $m(\gamma_n) \rightarrow 0$  as  $n \rightarrow \infty$ , the above estimate implies that  $\|\mathbb{I}_{\Gamma} \mathbb{I}_{\gamma_n}\|_X \rightarrow 0$  as  $n \rightarrow \infty$ . Thus the function  $\mathbb{I}_{\Gamma}$  has absolutely continuous norm. By [1, Chap. 1, Theorem 3.13],  $X_a = X_b$ .  $\square$

### 3.4. Weak types $(L^\infty, X)$ and $(L^\infty, X')$ of the Hilbert transform if $P \in \mathcal{B}(X)$

**Lemma 3.5.** *Let  $X$  be a Banach function space over the unit circle  $\mathbb{T}$  and  $X'$  be its associate space. If  $\mathcal{C} \in \mathcal{B}(X_b, X)$ , then  $\mathcal{C} \in \mathcal{B}((X')_b, X')$  and*

$$\|\mathcal{C}\|_{\mathcal{B}((X')_b, X')} \leq \|\mathcal{C}\|_{\mathcal{B}(X_b, X)}. \quad (3.13)$$

PROOF. It is well known that the operator  $i\mathcal{C}$  is a self-adjoint operator on the space  $L^2$  (see, e.g., [34, Section 5.7.3(a)]). Therefore, for all  $s, v \in S_0 \subset L^2$ , one has

$$\langle \mathcal{C}v, s \rangle = -\langle v, \mathcal{C}s \rangle. \quad (3.14)$$

It follows from equalities (2.4), (3.14), and Hölder's inequality (see [1, Chap. 1, Theorem 2.4]) that for every  $v \in S_0$ ,

$$\begin{aligned} \|\mathcal{C}v\|_{X'} &= \sup\{|\langle \mathcal{C}v, s \rangle| : s \in S_0, \|s\|_X \leq 1\} = \sup\{|\langle v, \mathcal{C}s \rangle| : s \in S_0, \|s\|_X \leq 1\} \\ &\leq \sup\{\|v\|_{X'} \|\mathcal{C}s\|_X : s \in S_0, \|s\|_X \leq 1\} \leq \|\mathcal{C}\|_{\mathcal{B}(X_b, X)} \|v\|_{X'}. \end{aligned}$$

Since  $S_0$  is dense in  $(X')_b$ , we conclude that  $\mathcal{C} \in \mathcal{B}((X')_b, X')$  and (3.13) holds.  $\square$

**Lemma 3.6.** *Let  $X$  be a Banach function space over the unit circle  $\mathbb{T}$  and  $X'$  be its associate space. If the Riesz projection  $P$  is bounded on  $X$ , then  $\mathcal{C} \in \mathcal{W}(L^\infty, X)$  and  $\mathcal{C} \in \mathcal{W}(L^\infty, X')$ .*

PROOF. Since  $X$  is continuously embedded into  $L^1$ , the functional  $f \mapsto \widehat{f}(0)$  is continuous on the space  $X$ . Then it follows from (1.3) that  $P \in \mathcal{B}(X)$  if and only if  $\mathcal{C} \in \mathcal{B}(X)$ . Since  $L^\infty$  is continuously embedded into  $X$ , one has  $\mathcal{B}(X) \subset \mathcal{B}(L^\infty, X)$ . By Lemma 3.1,  $\mathcal{B}(L^\infty, X) \subset \mathcal{W}(L^\infty, X)$ . These observations imply that  $\mathcal{C} \in \mathcal{W}(L^\infty, X)$  if  $P \in \mathcal{B}(X)$ . Since  $X_b$  is a Banach space isometrically embedded into  $X$  (see [1, Chap. 1, Theorem 3.1]), we see that  $\mathcal{C} \in \mathcal{B}(X) \subset \mathcal{B}(X_b, X)$  if  $P \in \mathcal{B}(X)$ . Then, by Lemma 3.5,  $\mathcal{C} \in \mathcal{B}((X')_b, X')$ . Taking into account that  $L^\infty$  is continuously embedded into  $(X')_b$  (see, e.g., [1, Chap. 1, Proposition 3.10]), we get  $\mathcal{C} \in \mathcal{B}((X')_b, X') \subset \mathcal{B}(L^\infty, X')$ , which implies that  $\mathcal{C} \in \mathcal{W}(L^\infty, X')$  in view of Lemma 3.1.  $\square$

3.5. Equalities  $X_a = X_b$  and  $(X')_a = (X')_b$  if  $P \in \mathcal{B}(X)$

Now we are in a position to formulate the main result of this section.

**Theorem 3.7.** *Let  $X$  be a Banach function space over the unit circle  $\mathbb{T}$ . If the Riesz projection  $P$  is bounded on  $X$ , then  $X_a = X_b$  and  $(X')_a = (X')_b$ .*

PROOF. If the Riesz projection  $P$  is bounded on a Banach function space  $X$ , then the Hilbert transform  $\mathcal{C}$  is of weak types  $(L^\infty, X)$  and  $(L^\infty, X')$  in view of Lemma 3.6. In turn,  $\mathcal{C} \in \mathcal{W}(L^\infty, X)$  implies that  $X_a = X_b$  and  $\mathcal{C} \in \mathcal{W}(L^\infty, X')$  implies that  $(X')_a = (X')_b$  due to Theorem 3.4.  $\square$

Combining Theorem 3.7 and Lemma 2.4, we immediately arrive at the following.

**Corollary 3.8.** *Let  $X$  be a Banach function space over the unit circle  $\mathbb{T}$ . If the Riesz projection  $P$  is bounded on  $X$ , then for every  $f \in X$ ,*

$$\|f\|_X = \sup\{|\langle f, p \rangle| : p \in \mathcal{P}, \|p\|_{X'} \leq 1\}.$$

## 4. Proof of the main result

### 4.1. Multiplication operators

**Lemma 4.1.** *Let  $X, Y$  be Banach functions spaces over the unit circle  $\mathbb{T}$ . Suppose  $X$  is separable and  $A \in \mathcal{B}(X, Y)$ . If there exists a sequence  $\{a_n\}_{n \in \mathbb{Z}}$  of complex numbers such that*

$$\langle A\chi_j, \chi_k \rangle = a_{k-j} \quad \text{for all } j, k \in \mathbb{Z}, \quad (4.1)$$

*then there exists a function  $a \in M(X, Y)$  such that  $A = M_a$  and  $\widehat{a}(n) = a_n$  for all  $n \in \mathbb{Z}$ .*

PROOF. This statement was proved in [29, Lemma 4.1] under the additional hypothesis that  $X$  and  $Y$  are rearrangement-invariant Banach function spaces. Put  $a := A\chi_0 \in Y$ . Then, one can show exactly as in [29] that  $(af)^\wedge(j) = (Af)^\wedge(j)$  for all  $j \in \mathbb{Z}$  and  $f \in \mathcal{P}$ . Therefore,  $Af = af$  for all  $f \in \mathcal{P}$  in view of the uniqueness theorem for Fourier series (see, e.g., [24, Chap. 1, Theorem 2.7]).

Now let  $f \in X$ . Since the space  $X$  is separable, the set  $\mathcal{P}$  is dense in  $X$  by Corollary 2.2. Then there exists a sequence  $p_n \in \mathcal{P}$  such that  $p_n \rightarrow f$  in  $X$  and, whence,  $Ap_n \rightarrow Af$  in  $X$  as  $n \rightarrow \infty$ . By [1, Chap. 1, Theorem 1.4],  $p_n \rightarrow f$  and  $Ap_n \rightarrow af$  in measure as  $n \rightarrow \infty$ . Then  $ap_n \rightarrow af$  in measure as  $n \rightarrow \infty$  (see, e.g., [4, Corollary 2.2.6]). Hence, the sequence  $Ap_n = ap_n$  converges in measure to the functions  $Af$  and  $af$  as  $n \rightarrow \infty$ . This implies that  $Af$  and  $af$  coincide a.e. on  $\mathbb{T}$  (see, e.g., the discussion preceding [4, Theorem 2.2.3]). Thus  $Af = af$  for all  $f \in X$ . This means that  $A = M_a$  and  $a \in M(X, Y)$  by the definition of  $M(X, Y)$ .  $\square$



#### 4.2. Proof of Theorem 1.2

The aim of this subsection is to present a proof of our extension of the Brown-Halmos theorem. Although it follows the scheme of the proof of [6, Theorem 2.7] with modifications that are necessary in the setting of different spaces  $X$  and  $Y$  (cf. [29, Theorem 4.2]), it uses results obtained in this paper (e.g., Theorem 3.7 and Corollary 3.8) and in [22] (see Lemma 2.3 above). We provide details for the sake of completeness.

Since  $P \in \mathcal{B}(Y)$ , it follows from Theorem 3.7 that  $(Y')_a = (Y')_b$ . Then, by Lemma 2.1, the set of trigonometric polynomials  $\mathcal{P}$  is dense in  $(Y')_b$ . Therefore,  $(Y')_b$  is separable. It follows from [1, Chap. 1, Theorems 3.11 and 4.1] that  $((Y')_b)^* = Y''$ . On the other hand, by the Lorentz-Luxemburg theorem (see [1, Chap. 1, Theorem 2.7]),  $Y'' \equiv Y$ . Thus, the Banach function space  $Y$  is canonically isometrically isomorphic to the dual space  $((Y')_b)^*$  of the separable Banach space  $(Y')_b$ .

For  $n \geq 0$ , put  $b_n := \chi_{-n} A \chi_n$ . Then  $b_n \in Y$  and

$$\|b_n\|_Y = \|A \chi_n\|_Y = \|A \chi_n\|_{H[Y]} \leq \|A\|_{\mathcal{B}(H[X], H[Y])} \|\chi_n\|_X = \|A\|_{\mathcal{B}(H[X], H[Y])} \|1\|_X. \quad (4.2)$$

Put

$$V = \left\{ y \in (Y')_b : \|y\|_{Y'} < \frac{1}{\|A\|_{\mathcal{B}(H[X], H[Y])} \|1\|_X} \right\}.$$

It follows from the Hölder inequality (see [1, Chap. 1, Theorem 2.4]) and (4.2) that

$$|\langle b_n, y \rangle| \leq \|b_n\|_Y \|y\|_{Y'} < 1 \quad \text{for all } y \in V, n \geq 0.$$

Applying a corollary of the Banach-Alaoglu theorem (see, e.g., [37, Theorem 3.17]) to the neighborhood  $V$  of zero in the separable Banach space  $(Y')_b$  and the sequence  $\{b_n\}_{n \in \mathbb{N}} \subset Y = ((Y')_b)^*$ , we deduce that there exists a function  $b \in Y$  such that some subsequence  $\{b_{n_k}\}_{k \in \mathbb{N}}$  of  $\{b_n\}_{n \in \mathbb{N}}$  converges to  $b$  in the weak-\* topology of  $((Y')_b)^*$ . It follows from [1, Chap. 1, Proposition 3.10] that  $\chi_j \in (Y')_b$  for all  $j \in \mathbb{Z}$ . Hence

$$\lim_{k \rightarrow +\infty} \langle b_{n_k}, \chi_j \rangle = \langle b, \chi_j \rangle \quad \text{for all } j \in \mathbb{Z}. \quad (4.3)$$

On the other hand, we get from the definition of  $b_n$  and (1.5) for  $n_k + j \geq 0$ ,

$$\langle b_{n_k}, \chi_j \rangle = \langle \chi_{-n_k} A \chi_{n_k}, \chi_j \rangle = \langle A \chi_{n_k}, \chi_{n_k+j} \rangle = a_j. \quad (4.4)$$

It follows from (4.3) and (4.4) that

$$\langle b, \chi_j \rangle = a_j \quad \text{for all } j \in \mathbb{Z}. \quad (4.5)$$

Now define the mapping  $B$  by

$$B : \mathcal{P} \rightarrow Y, \quad f \mapsto bf. \quad (4.6)$$

Assume that  $f$  and  $g$  are trigonometric polynomials of order  $m$  and  $r$ , respectively. Using equalities (1.5) and (4.5) and definition (4.6), one can show that for  $n \geq \max\{m, r\}$ ,

$$\langle Bf, g \rangle = \langle \chi_{-n} A(\chi_n f), g \rangle. \quad (4.7)$$

It is clear that for those  $n$ , one has  $\chi_n f \in H[X]$ . Since  $A \in \mathcal{B}(H[X], H[Y])$ , we obtain

$$\|A(\chi_n f)\|_Y = \|A(\chi_n f)\|_{H[Y]} \leq \|A\|_{\mathcal{B}(H[X], H[Y])} \|\chi_n f\|_{H[X]} = \|A\|_{\mathcal{B}(H[X], H[Y])} \|f\|_X. \quad (4.8)$$

Hence, by the Hölder inequality (see [1, Chap. 1, Theorem 2.4]), we deduce from (4.8) that

$$|\langle \chi_{-n} A(\chi_n f), g \rangle| \leq \|\chi_{-n} A(\chi_n f)\|_Y \|g\|_{Y'} = \|A(\chi_n f)\|_Y \|g\|_{Y'} \leq \|A\|_{\mathcal{B}(H[X], H[Y])} \|f\|_X \|g\|_{Y'}. \quad (4.9)$$

It follows from (4.7) and (4.9) that

$$|\langle Bf, g \rangle| \leq \limsup_{n \rightarrow \infty} |\langle \chi_{-n} A(\chi_n f), g \rangle| \leq \|A\|_{\mathcal{B}(H[X], H[Y])} \|f\|_X \|g\|_{Y'}. \quad (4.10)$$

Since the Riesz projection  $P$  is bounded on  $Y$ , inequality (4.10) and Corolary 3.8 imply that for every  $f \in \mathcal{P}$ ,

$$\|Bf\|_Y = \sup\{|\langle Bf, g \rangle| : g \in \mathcal{P}, \|g\|_{Y'} \leq 1\} \leq \|A\|_{\mathcal{B}(H[X], H[Y])} \|f\|_X.$$

Since  $X$  is separable, the set  $\mathcal{P}$  is dense in  $X$  in view of Corollary 2.2. Hence the above inequality shows that the linear mapping defined in (4.6) extends to an operator  $B \in \mathcal{B}(X, Y)$  with

$$\|B\|_{\mathcal{B}(X, Y)} \leq \|A\|_{\mathcal{B}(H[X], H[Y])}. \quad (4.11)$$

We deduce from (4.5)–(4.6) that

$$\langle B\chi_j, \chi_k \rangle = \langle b\chi_j, \chi_k \rangle = \langle b, \chi_{k-j} \rangle = a_{k-j} \quad \text{for all } j, k \in \mathbb{Z}.$$

Then, by Lemma 4.1, there exists a function  $a \in M(X, Y)$  such that  $B = M_a$  and  $\widehat{a}(n) = a_n$  for all  $n \in \mathbb{Z}$ . Moreover,

$$\|B\|_{\mathcal{B}(X, Y)} = \|M_a\|_{\mathcal{B}(X, Y)} = \|a\|_{M(X, Y)}. \quad (4.12)$$

It follows from the definition of the Toeplitz operator  $T_a$  that

$$\langle T_a \chi_j, \chi_k \rangle = \widehat{a}(k - j) \quad \text{for all } j, k \geq 0.$$

Combining this identity with (1.5), we obtain

$$\langle T_a \chi_j, \chi_k \rangle = a_{k-j} = \langle A\chi_j, \chi_k \rangle \quad \text{for all } j, k \geq 0. \quad (4.13)$$

Since  $T_a \chi_j, A\chi_j \in H[Y] \subset H^1$ , it follows from (4.13) and the uniqueness theorem for Fourier series (see, e.g., [24, Chap. 1, Theorem 2.7]) that  $T_a \chi_j = A\chi_j$  for all  $j \geq 0$ . Therefore,

$$T_a f = Af \quad \text{for all } f \in \mathcal{P}_A. \quad (4.14)$$

By Lemma 2.3,  $\mathcal{P}_A$  is dense in  $H[X]$ . This observation and (4.14) imply that  $T_a = A$  on  $H[X]$  and

$$\|T_a\|_{\mathcal{B}(H[X], H[Y])} = \|A\|_{\mathcal{B}(H[X], H[Y])}. \quad (4.15)$$

Combining inequality (4.11) with equalities (4.12) and (4.15), we arrive at the first inequality in (1.6). The second inequality in (1.6) is obvious.  $\square$

**Remark 4.2.** Let the functions  $a \in M(X, Y)$  and  $b \in Y$  be as in the above proof. Since  $\chi_0 \in L^\infty \subset X$  and  $a \in M(X, Y)$ , we have  $a = a\chi_0 \in Y \subset L^1$ . On the other hand,  $b \in Y \subset L^1$ . Note that the functions  $a, b \in L^1$  have equal Fourier coefficients (see (4.5) and Lemma 4.1) and hence coincide (see, e.g., [24, Chap. 1, Theorem 2.7]).

## 5. Applications to variable Lebesgue spaces

### 5.1. Variable Lebesgue spaces

Let  $\mathfrak{P}(\mathbb{T})$  be the set of all measurable functions  $p : \mathbb{T} \rightarrow [1, \infty]$ . For  $p \in \mathfrak{P}(\mathbb{T})$ , put

$$T_\infty^{p(\cdot)} := \{t \in \mathbb{T} : p(t) = \infty\}.$$

For a function  $f \in L^0$ , consider

$$\varrho_{p(\cdot)}(f) := \int_{\mathbb{T} \setminus T_\infty^{p(\cdot)}} |f(t)|^{p(t)} dm(t) + \|f\|_{L^\infty(\mathbb{T}_\infty^{p(\cdot)})}.$$

The variable Lebesgue space  $L^{p(\cdot)}$  is defined (see, e.g., [10, Definition 2.9]) as the set of all measurable functions  $f \in L^0$  such that  $\varrho_{p(\cdot)}(f/\lambda) < \infty$  for some  $\lambda > 0$ . This space is a Banach function space with respect to the Luxemburg-Nakano norm given by

$$\|f\|_{L^{p(\cdot)}} := \inf\{\lambda > 0 : \varrho_{p(\cdot)}(f/\lambda) \leq 1\}$$

(see [10, Theorems 2.17, 2.71 and Section 2.10.3]). If  $p \in \mathfrak{P}(\mathbb{T})$  is constant, then  $L^{p(\cdot)}$  is nothing but the standard Lebesgue space  $L^p$ . Variable Lebesgue spaces are often called Nakano spaces. We refer to Maligranda's paper [31] for the role of Hidegoro Nakano in the study of variable Lebesgue spaces.

For  $p \in \mathfrak{P}(\mathbb{T})$ , put

$$p_- := \operatorname{ess\,inf}_{t \in \mathbb{T}} p(t), \quad p_+ := \operatorname{ess\,sup}_{t \in \mathbb{T}} p(t).$$

It is well known that the variable Lebesgue space  $L^{p(\cdot)}(\mathbb{T})$  is separable if and only if  $p_+ < \infty$  and is reflexive if and only if  $1 < p_-, p_+ < \infty$  (see, e.g., [10, Theorem 2.78 and Corollary 2.79]).

The following result was obtained by Nakai [33, Example 4.1] under the additional hypothesis

$$\sup_{t \in \mathbb{T} \setminus T_\infty^{r(\cdot)}} r(t) < \infty$$

(and in the more general setting of quasi-Banach variable Lebesgue spaces over arbitrary measure spaces). Nakai also mentioned in [33, Remark 4.2] (without proof) that this hypothesis is superfluous. One can find its proof in the present form in [21, Theorem 4.8].

**Theorem 5.1.** Let  $p, q, r \in \mathfrak{P}(\mathbb{T})$  be related by

$$\frac{1}{q(t)} = \frac{1}{p(t)} + \frac{1}{r(t)}, \quad t \in \mathbb{T}. \quad (5.1)$$

Then  $M(L^{p(\cdot)}, L^{q(\cdot)}) = L^{r(\cdot)}$ .

### 5.2. The Riesz projection on variable Lebesgue spaces

We say that an exponent  $q \in \mathfrak{P}(\mathbb{T})$  is log-Hölder continuous (cf. [10, Definition 2.2]) if  $1 < q_- \leq q_+ < \infty$  and there exists a constant  $C_{q(\cdot)} \in (0, \infty)$  such that

$$|q(t) - q(\tau)| \leq \frac{C_{q(\cdot)}}{-\log|t - \tau|} \quad \text{for all } t, \tau \in \mathbb{T} \text{ satisfying } |t - \tau| < 1/2.$$

The class of all log-Hölder continuous exponent will be denoted by  $LH(\mathbb{T})$ . Some authors denote this class by  $\mathbb{P}^{\log}(\mathbb{T})$  (see, e.g., [25, Section 1.1.4]). The following result is well known (see, e.g., [25, Section 10.1] or [19, Lemma 12]).

**Theorem 5.2.** *If  $q \in LH(\mathbb{T})$ , then the Riesz projection  $P$  is bounded on  $L^{q(\cdot)}$ .*

### 5.3. Toeplitz operators between abstract Hardy space built upon variable Lebesgue spaces

Applying Theorems 1.2, 5.1, and 5.2, we arrive at the following.

**Theorem 5.3.** *Let  $p, q, r \in \mathfrak{P}(\mathbb{T})$  be related by (5.1). Suppose  $q \in LH(\mathbb{T})$  and  $p_+ < \infty$ . If a linear operator  $A$  is bounded form  $H[L^{p(\cdot)}]$  to  $H[L^{q(\cdot)}]$  and there exists a sequence  $\{a_n\}_{n \in \mathbb{Z}}$  of complex numbers such that*

$$\langle A\chi_j, \chi_k \rangle = a_{k-j} \quad \text{for all } j, k \geq 0,$$

*then there is a function  $a \in L^{r(\cdot)}$  such that  $A = T_a$  and  $\hat{a}(n) = a_n$  for all  $n \in \mathbb{Z}$ . Moreover, there exist constants  $c_{p,q}, C_{p,q} \in (0, \infty)$  depending only on  $p$  and  $q$  such that*

$$c_{p,q} \|a\|_{L^{r(\cdot)}} \leq \|T_a\|_{\mathcal{B}(H[L^{p(\cdot)}], H[L^{q(\cdot)}])} \leq C_{p,q} \|P\|_{\mathcal{B}(L^{q(\cdot)})} \|a\|_{L^{r(\cdot)}}.$$

Note that if  $p, q \in LH(\mathbb{T})$  coincide, then the constants  $c_{p,q}$  and  $C_{p,q}$  in the above inequality are equal to one (cf. [19, Corollary 13]).

## 6. Applications to Lorentz spaces with Muckenhoupt weights

### 6.1. Rearrangement-invariant Banach function spaces

The distribution function  $m_f$  of an a.e. finite function  $f \in L^0$  is given by

$$m_f(\lambda) := m\{t \in \mathbb{T} : |f(t)| > \lambda\}, \quad \lambda \geq 0.$$

The non-increasing rearrangement of an a.e. finite function  $f \in L^0$  is defined by

$$f^*(x) := \inf\{\lambda : m_f(\lambda) \leq x\}, \quad x \in [0, 1].$$

We refer to [1, Chap. 2, Section 1] for properties of distribution functions and non-increasing rearrangements.

Two a.e. finite functions  $f, g \in L^0$  are said to be equimeasurable if their distribution functions coincide:  $m_f(\lambda) = m_g(\lambda)$  for all  $\lambda \geq 0$ . A Banach function space  $X$  over the unit circle  $\mathbb{T}$  is called rearrangement-invariant if for every pair of equimeasurable functions  $f, g \in L^0$ ,  $f \in X$  implies that  $g \in X$  and the equality  $\|f\|_X = \|g\|_X$  holds. For a rearrangement-invariant Banach function space  $X$ , its associate space  $X'$  is also rearrangement-invariant (see [1, Chap. 2, Proposition 4.2]).

### 6.2. Lorentz spaces $L^{p,q}$

Let  $f$  be an a.e. finite function in  $L^0$ . For  $x \in (0, 1]$ , put

$$f^{**}(x) = \frac{1}{x} \int_0^x f^*(y) dy.$$

Suppose  $1 < p < \infty$  and  $1 \leq q \leq \infty$ . The Lorentz space  $L^{p,q}$  consists of all a.e. finite functions  $f \in L^0$  for which the quantity

$$\|f\|_{L^{p,q}} = \begin{cases} \left( \int_0^1 \left( x^{1/p} f^{**}(x) \right)^q \frac{dx}{x} \right)^{1/q}, & \text{if } 1 \leq q < \infty, \\ \sup_{0 < x < 1} \left( x^{1/p} f^{**}(x) \right), & \text{if } q = \infty, \end{cases}$$

is finite. It is well known that  $L^{p,q}$  is a rearrangement-invariant Banach function space with respect to the norm  $\|\cdot\|_{L^{p,q}}$  (see, e.g., [1, Chap. 4, Theorem 4.6], where the case of spaces of infinite measure is considered; in the case of spaces of finite measure, the proof is the same). It follows from [1, Chap. 2, Proposition 1.8 and Chap. 4, Lemma 4.5] that  $L^{p,p} = L^p$  (with equivalent norms).

### 6.3. Weighted Lorentz spaces $L^{p,q}(w)$

For  $q \in [1, \infty]$ , put  $q' = q/(q-1)$  with the usual conventions  $1/0 = \infty$  and  $1/\infty = 0$ . A function  $w \in L^0_+$  is referred to as a weight if  $0 < w(\tau) < \infty$  for a.e.  $\tau \in \mathbb{T}$ .

Let  $1 < p < \infty$  and  $1 \leq q \leq \infty$ . Suppose  $w : \mathbb{T} \rightarrow [0, \infty]$  is a weight such that  $w \in L^{p,q}$  and  $1/w \in L^{p',q'}$ . The weighted Lorentz space  $L^{p,q}(w)$  is defined as the set of all a.e. finite functions  $f \in L^0$  such that  $fw \in L^{p,q}$ .

The next lemma follows directly from well known results on Lorentz spaces.

**Lemma 6.1.** *Let  $1 < p < \infty$ ,  $1 \leq q \leq \infty$  and  $w : \mathbb{T} \rightarrow [0, \infty]$  be a weight such that  $w \in L^{p,q}$ ,  $1/w \in L^{p',q'}$ .*

- (a) *The space  $L^{p,q}(w)$  is a Banach function space with respect to the norm  $\|f\|_{L^{p,q}(w)} = \|fw\|_{L^{p,q}}$  and  $L^{p',q'}(1/w)$  is its associate space.*
- (b) *If  $1 < q < \infty$ , then the space  $L^{p,q}(w)$  is reflexive.*
- (c) *The space  $L^{p,1}(w)$  is separable and non-reflexive.*

PROOF. (a) In view of [1, Chap. 4, Theorem 4.7], the associate space of the Lorentz space  $L^{p,q}$ , up to equivalence of norms, is the Lorentz space  $L^{p',q'}$ . It is easy to check that  $L^{p,q}(w)$  is a Banach function space and  $L^{p',q'}(1/w)$  is its associate space.

(b) Note that  $L^{p,q}(w) \ni f \mapsto wf \in L^{p,q}$  is an isometric isomorphism of  $L^{p,q}(w)$  and  $L^{p,q}$ . Hence these spaces have the same Banach space theory properties, e.g., reflexivity and separability. If  $1 < p, q < \infty$ , then  $L^{p,q}$  is reflexive in view of [1, Chap. 4, Corollary 4.8]. Then the weighted Lorentz space  $L^{p,q}(w)$  is reflexive too.

(c) If  $1 < p < \infty$ , then  $L^{p,1}$  has absolutely continuous norm and  $(L^{p,1})^* = L^{p',\infty}$  (see [1, Chap. 4, Corollary 4.8]). Then  $L^{p,1}$  is separable in view of [1, Chap. 1, Corollary 5.6]. It is known that

$$L^{p,1} \subsetneq (L^{p',\infty})^* = (L^{p,1})^{**}$$

(see [11, p. 83]). Hence  $L^{p,1}$  is non-reflexive. Therefore,  $L^{p,1}(w)$  is also separable and nonreflexive.  $\square$

6.4. *The Riesz projection on  $L^{p,q}(w)$  with  $1 < p < \infty$ ,  $1 \leq q < \infty$  and  $w \in A_p(\mathbb{T})$*

Let  $1 < p < \infty$  and  $w$  be a weight. It is well known that the Riesz projection  $P$  is bounded on the weighted Lebesgue space  $L^p(w) := \{f \in L^0 : fw \in L^p\}$  if and only if the weight  $w$  satisfies the Muckenhoupt  $A_p$ -condition, that is,

$$\sup_{\gamma \subset \mathbb{T}} \left( \frac{1}{m(\gamma)} \int_{\gamma} w^p(\tau) dm(\tau) \right)^{1/p} \left( \frac{1}{m(\gamma)} \int_{\gamma} w^{-p'}(\tau) dm(\tau) \right)^{1/p'} < \infty,$$

where the supremum is taken over all subarcs  $\gamma$  of the unit circle  $\mathbb{T}$  (see [15] and also [5, Section 6.2], [6, Section 1.46], [34, Section 5.7.3(h)]). In this case, we will write  $w \in A_p(\mathbb{T})$ .

**Lemma 6.2.** *Let  $1 < p < \infty$  and  $1 \leq q \leq \infty$ . If  $w \in A_p(\mathbb{T})$ , then  $w \in L^{p,q}$  and  $1/w \in L^{p',q'}$ .*

PROOF. By the stability property of Muckenhoupt weights (see, e.g., [5, Theorem 2.31]), there exists  $\varepsilon > 0$  such that  $w \in A_s(\mathbb{T})$  for all  $s \in (p - \varepsilon, p + \varepsilon)$ . Therefore,  $w \in L^s$  and  $1/w \in L^{s'}$  for all  $s \in (p - \varepsilon, p + \varepsilon)$ . In particular, if  $s_1, s_2$  are such that  $p - \varepsilon < s_1 < p < s_2 < p + \varepsilon$ , then  $w \in L^{s_2} = L^{s_2, s_2} \subset L^{p,q}$  and  $1/w \in L^{s_1'} = L^{s_1', s_1'} \subset L^{p', q'}$  in view of the embeddings of Lorentz spaces (see, e.g., [1, Chap. 4, remark after Proposition 4.2]).  $\square$

Lemmas 6.1(a) and 6.2 imply that if  $w \in A_p(\mathbb{T})$ , then  $L^{p,q}(w)$  is a Banach function space.

**Theorem 6.3.** *Let  $1 < p < \infty$  and  $1 \leq q \leq \infty$ . If  $w \in A_p(\mathbb{T})$ , then the Riesz projection  $P$  is bounded on the weighted Lorentz space  $L^{p,q}(w)$ .*

PROOF. It follows from [1, Chap. 4, Theorem 4.6] and [16, Theorem 4.5] that the Cauchy singular integral operator  $S$  is bounded on  $L^{p,q}(w)$ . Thus, the Riesz projection  $P$  is bounded on  $L^{p,q}(w)$  in view of (1.1).  $\square$

6.5. *Toeplitz operators on abstract Hardy spaces built upon  $L^{p,q}(w)$  with  $1 < p < \infty$ ,  $1 \leq q < \infty$ ,  $w \in A_p(\mathbb{T})$*

The next theorem is an immediate consequence of Corollary 1.3, Lemmas 6.1 and 6.2, and Theorem 6.3.

**Theorem 6.4.** *Let  $1 < p < \infty$ ,  $1 \leq q < \infty$ , and  $w \in A_p(\mathbb{T})$ . If an operator  $A$  is bounded on the abstract Hardy space  $H[L^{p,q}(w)]$  and there exists a sequence  $\{a_n\}_{n \in \mathbb{Z}}$  of complex numbers such that*

$$\langle A\chi_j, \chi_k \rangle = a_{k-j} \quad \text{for all } j, k \geq 0,$$

*then there is a function  $a \in L^\infty$  such that  $A = T_a$  and  $\widehat{a}(n) = a_n$  for all  $n \in \mathbb{Z}$ . Moreover,*

$$\|a\|_{L^\infty} \leq \|T_a\|_{\mathcal{B}(H[L^{p,q}(w)])} \leq \|P\|_{\mathcal{B}(L^{p,q}(w))} \|a\|_{L^\infty}.$$

For  $p = q$  this result is contained in [19, Corollary 9]. For  $1 < q < \infty$ , this result as well follows from [19, Theorem 1]. The most interesting case is when  $q = 1$  because in this case the weighted Lorentz space  $L^{p,1}(w)$  is separable and non-reflexive. Moreover, it is not rearrangement-invariant. Therefore [19, Theorem 1] and [29, Corollary 4.4] are not applicable, while Corollary 1.3 works in this case.

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