A New Design of Membership-Function-Dependent Controller for T-S Fuzzy Systems Under Imperfect Premise Matching

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Abstract—This paper is concerned with the problem of membership-function-dependent controller design for a class of discrete-time T-S fuzzy systems. Based on the partition method of premise variable space, the original T-S fuzzy model is equivalently converted into a kind of piecewise-fuzzy system. Then by employing some staircase functions, the continuous membership functions are approximated by a series of discrete values, via which the information of membership functions is brought into the stability analysis to reduce the design conservatism. With piecewise Lyapunov functions, the approaches to the piecewise-fuzzy state feedback and observer-based output feedback controller design are proposed, respectively, in terms of linear matrix inequalities such that the closed-loop system is asymptotically stable with a prescribed $H_{\infty}$ performance level. It is shown that the membership functions of the fuzzy model and fuzzy controllers are not necessarily the same, which allows more design flexibility. Finally, two illustrative examples are provided to show the effectiveness of the developed methods.

Index Terms—Takagi-Sugeno fuzzy systems, membership-function-dependent, Lyapunov stability, imperfect premise matching, controller design

I. INTRODUCTION

It is known that nearly all the physical plants and industrial processes in practice are nonlinear, and these different kinds of nonlinearities impose great difficulties on stability analysis and controller synthesis. During the past few decades, as an effective way to tackle these complex nonlinear systems, fuzzy logic control has received much attention in control community. The fuzzy logic control provides a means of converting a linguistic control strategy based on expert knowledge into an automatic control strategy, and many significant results reported in the literature have shown that it offers an effective method for the control problem of complex nonlinear systems or even nonanalytic systems (please refer to [1]–[7] and references therein). To mention a few, in [5] a fuzzy logic controller was designed and implemented on a test vehicle for the lateral motion control, which is modularized as a feedback, preview, and gain scheduling rule bases. The results on stability analysis of continuous-time fuzzy-model-based control systems were reviewed in [2], where some fundamental and essential aspects related to membership-function-dependent analysis methods were summarized. In [6], the problem of robust adaptive fuzzy cooperative tracking control for a class of uncertain nonlinear multi-agent systems was studied, which are subject to time delays and dead-zone nonlinearities. For a class of chaotic systems with unknown functions and disturbances, the problem of asymptotic stabilization for such systems were investigated in [7] based on the adaptive fuzzy logic control theory.

In general, fuzzy control can be classified into model-free approach and model-based approach [2], [8]–[12]. Among various types of model-based fuzzy control approaches, Takagi-Sugeno (T-S) fuzzy models have attracted plenty of research interests and a great deal of research effort has been devoted to the problem of stability analysis and controller synthesis based on T-S fuzzy models in recent years [13]–[17]. With T-S fuzzy models, a nonlinear plant can be approximated in a general form by a series of linear subsystems via fuzzy membership functions, which locally describe the dynamics of the nonlinear plant [18]–[22]. In this case, one can resort to the fruitful linear multivariable system theory for the stability analysis and controller synthesis for these nonlinear systems represented by T-S fuzzy models in a unified framework. In [23], a sufficient condition is firstly provided for the quadratic stability of T-S fuzzy systems by considering a Lyapunov function for all subsystems involved. Then, the problem of stability analysis and stabilization for T-S fuzzy systems are converted into the task of searching for a common positive definite Lyapunov matrix, and plenty of significant results on relaxed quadratic stability conditions can be found in [3], [24]–[26] and references therein. However, it should be noted that the stability analysis and design results must be valid for all subsystems with a common quadratic Lyapunov matrix, which are very conservative. In some cases, it is very difficult to find such a common matrix, especially for some highly nonlinear complex fuzzy systems. Moreover, it is often the case that a common quadratic Lyapunov matrix does not exist for some
fuzzy systems even if they are asymptotically stable [27], [28]. In order to release the conservatism of the conditions based on common Lyapunov functions (CLFs), a great number of results are reported in the literature on the problem of stability analysis and controller synthesis for fuzzy systems in the framework of piecewise- or fuzzy-Lyapunov-function-based approaches [28]–[33]. It has been demonstrated that piecewise or fuzzy Lyapunov functions (PLFs or FLFs) could offer much richer classes of Lyapunov function candidates than common quadratic Lyapunov functions. Moreover, further relaxations can be expected by introducing some slack variables in stability analysis and controller synthesis in different ways [34], [35].

On the other hand, most control design techniques via linear matrix inequalities are based on determining some matrix variables to guarantee the positiveness of a double fuzzy summation, and nearly all the stability conditions aforementioned are independent of the “shape” of the membership functions of fuzzy systems. However, these results tend to be conservative as some important properties of membership functions are not used in the stability analysis. Sometimes it might be the case that the analysis strategies on the original nonlinear model could provide better solutions than those on its equivalent fuzzy T-S one [36]. Therefore, the information of membership functions plays an important role for the relaxation of the results on stability analysis and controller synthesis, and some related membership-function-dependent analysis strategies were revealed in the existing literature [2], [36]–[40]. To mention a few, with the known minimum and maximum grades of membership, new relaxed stability conditions were derived by employing the information of the uncertain membership functions [39]. In [36], a series of relaxed linear matrix inequality (LMI) conditions were presented for fuzzy control that incorporates the membership function shape information in the form of polynomial constraints. By employing the staircase membership functions and some slack matrices, the LMI-based stability conditions were obtained, which include the membership functions of both fuzzy model and fuzzy controller, and then an approach to the state feedback controller design was proposed in [40]. For the most-recent advances on this topic, please refer to the survey paper [2] and references therein.

It is also worthy to mention that almost all the aforementioned results are about stability analysis or state feedback controller design for T-S fuzzy systems under consideration of the information of membership functions. However, in some cases it is difficult or even impossible to obtain all the state information of the physical plant to be controlled for subsequent controller design. Thus, some other control strategies deserve further investigation with membership-function-dependent analysis approaches, such as, observer-based output feedback control. In the existing membership-function-dependent stability analysis conditions were derived with the common quadratic Lyapunov functions, which is a source of conservatism as mentioned above. To the best of our knowledge, few research effort has been devoted to the membership-function-dependent stability analysis and controller synthesis for T-S fuzzy systems subject to external disturbances within the piecewise- or fuzzy-Lyapunov-function-based framework. All these motivate our present work.

In this paper, the problem of membership-function-dependent controller design is considered for a class of discrete-time T-S fuzzy systems within the piecewise Lyapunov-function-based framework. To overcome the difficulty in using the continuous information of membership functions, staircase membership functions are employed to approximate the original membership functions of fuzzy plant and fuzzy controllers. With the piecewise Lyapunov functions, the approaches to the piecewise-fuzzy state feedback controller and the observer-based output feedback controller designs are studied, respectively, which guarantee the closed-loop fuzzy system to be asymptotically stable with a predefined $H_\infty$ performance level. It is shown that all the synthesis conditions are formulated in terms of linear matrix inequalities, and two illustrative examples are provided to demonstrate the effectiveness and advantages of the proposed methods.

The remainder of this paper is organized as follows. Section II is devoted to problem formulation. In Section III, the main results of stability analysis and controller design of piecewise-fuzzy state feedback controller and observer-based output feedback controller are presented, respectively. The simulation results are given in Section IV. Finally, we conclude the paper in Section V.

**Notations:** The notations used throughout the paper are fairly standard. The superscript “$T$” stands for matrix transpose; $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space; $\mathbb{R}^{m \times n}$ is the set of all real $m \times n$ matrices and $\mathbb{S}^n$ denotes the set of $n \times n$ real symmetric matrices; $I_n$ and $0_{m \times n}$ represent $n$-dimensional identity matrix and $m \times n$ zero matrix, respectively. We use an asterisk (*) to represent a term that is induced by symmetry in symmetric matrices. \text{diag}\{\cdots\} stands for a block-diagonal matrix. The notation $P > 0$ means that $P$ is real symmetric and positive definite. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

**II. Model Description and Problem Formulation**

In this section, we study the control problem for a class of discrete-time T-S fuzzy dynamic systems, which are used to describe a complex nonlinear physical plant by the smooth combination of local analytic linear models with fuzzy membership functions. To facilitate the controller design, we first re-formulate the T-S fuzzy model into a piecewise form. And then, two piecewise-fuzzy control laws, i.e., state feedback control and observer-based output feedback control, are proposed.

**A. Discrete-time T-S Fuzzy Dynamic Models**

The discrete-time T-S fuzzy system with external disturbance is given as follows:

**Plant Rule** \( \mathcal{R}_P^m \) : IF \( f_1(x(t)) \) is \( \mathcal{M}_1^m \), AND \( \cdots \), AND \( f_\Psi(x(t)) \) is \( \mathcal{M}_\Psi^m \), THEN

\[
\begin{align*}
  x(t+1) &= A_m x(t) + B_m u(t) + D_m w(t) \\
  z(t) &= L_m x(t) + N_m u(t) \\
  y(t) &= C x(t), m \in \mathcal{R}_P \triangleq \{1, 2, \ldots, p\}
\end{align*}
\]

(1)
where $M_m$ is a fuzzy term of rule $m$ corresponding to the known function $f_m(x(t))(\alpha = 1, 2, \ldots, \Psi)$ and $m \in \Omega$. $f(x(t)) = [f_1(x(t)), \ldots, f_\Psi(x(t))]^T$ is the premise variable vector. $x(t) \in \mathbb{R}^{n_x}$ stands for the state vector and $u(t) \in \mathbb{R}^{n_u}$ is the control input; $y(t) \in \mathbb{R}^{n_y}$ is the measured output. Without loss of generality, assuming the system output matrix $C$ to be of full-row rank, $z(t) \in \mathbb{R}^{n_z}$ is the regulated output, and $w(t) \in \mathbb{R}^{n_w}$ is the external disturbance belonging to $l_2[0, \infty)$. $A_m, B_m, D_m, L_m,$ and $N_m$ are known constant matrices.

Let $\mu_m(x(t))$ be the normalized fuzzy membership function given as

$$\mu_m(x(t)) = \frac{\prod_{j=1}^{\Psi} \mu_{m,j}^\Psi(f_j(x(t)))}{\sum_{j=1}^{\Psi} \mu_{m,j}^\Psi(f_j(x(t)))} \geq 0, \quad \sum_{m=1}^{P} \mu_m(x(t)) = 1$$

(2)

where $\mu_{m,j}^\Psi(f_j(x(t)))$ represents the grade of membership of $f_j(x(t))$ in $M_m$. For brevity, $\mu_m(x(t))$ is denoted as $\mu_m$ without ambiguity. Following a standard fuzzy inference method, the compact form of T-S fuzzy system in (1) can be re-expressed as

$$\begin{align*}
\begin{cases}
  x(t+1) = A(\mu)x(t) + B(\mu)u(t) + D(\mu)w(t) \\
  z(t) = L(\mu)x(t) + N(\mu)u(t) \\
  y(t) = Cx(t)
\end{cases}
\end{align*}$$

(3)

where

$$\begin{align*}
A(\mu) & := \sum_{m=1}^{P} \mu_m A_m, \\
B(\mu) & := \sum_{m=1}^{P} \mu_m B_m, \\
D(\mu) & := \sum_{m=1}^{P} \mu_m D_m, \\
N(\mu) & := \sum_{m=1}^{P} \mu_m N_m.
\end{align*}$$

In this paper, an approach to the robust stability analysis and controller synthesis in the piecewise-Lyapunov-function-based framework is developed, and to facilitate the subsequent controller design, the original T-S fuzzy dynamic model is firstly converted into the equivalent piecewise-fuzzy system. To this end, we partition the premise variable space $\Omega$ into two kinds of polyhedral regions, i.e., crisp regions and fuzzy regions [28]. Denote those regions as $\Omega_l^P (l \in \Lambda_P)$ with $\Lambda_P$ being the set of polyhedral region indices.

Define $\mathcal{I}_P(l)$ as the index set of the fired rules in each region $\Omega_l^P$, that is,

$$\mathcal{I}_P(l) := \{m | \mu_m(x(t)) > 0, f(x(t)) \in \Omega_l^P, m \in \Omega, l \in \Lambda_P\},$$

and then the crisp and fuzzy regions can be given as

$$\begin{align*}
\Omega_l^P & := \{f(x(t)) | \mu_l(x(t)) = 1, \text{for all } k \in \mathcal{I}_P(l)\} \\
\Omega_l^F & := \{f(x(t)) | 0 < \mu_l(x(t)) < 1, \text{for all } k \in \mathcal{I}_P(l)\}.
\end{align*}$$

With the above partition method, in the $l$th region $\Omega_l^P$, one has $\sum_{m \in \mathcal{I}_P(l)} \mu_m = 1$, and the fuzzy system can be further expressed as

$$\begin{align*}
\begin{cases}
  x(t+1) = A_l x(t) + B_l u(t) + D_l w(t) \\
  z(t) = L_l x(t) + N_l u(t) \\
  y(t) = C x(t)
\end{cases}
\end{align*}$$

(4)

where

$$\begin{align*}
A_l & := \sum_{m \in \mathcal{I}_P(l)} \mu_m A_m, \\
B_l & := \sum_{m \in \mathcal{I}_P(l)} \mu_m B_m, \\
D_l & := \sum_{m \in \mathcal{I}_P(l)} \mu_m D_m, \\
L_l & := \sum_{m \in \mathcal{I}_P(l)} \mu_m L_m, \\
N_l & := \sum_{m \in \mathcal{I}_P(l)} \mu_m N_m.
\end{align*}$$

For subsequent use, let a set $\mathcal{T}_P$ denote all possible region transitions during the evolution of the system dynamics

$$\mathcal{T}_P := \{(l, i) | f(x(t)) \in \Omega_l^P, f(x(t+1)) \in \Omega_i^P, l, i \in \Lambda_P\}.$$  

B. Piecewise-fuzzy State Feedback Controller

A fuzzy controller with $q$ fuzzy rules is to be designed for the T-S fuzzy system in (1). Following the similar partition method, we assume that the premise variable space of the controller is divided into several regions $\Omega_l^C (s \in \Lambda_C)$ and let $\Lambda_C$ and $\mathcal{I}_C(s)$ denote the set of region indices and the index of fired rules in the $s$th region $\Omega_l^C$, respectively. The piecewise-fuzzy state feedback controller is constructed as follows:

Region Rule $s$: If $g(x(t)) \in \Omega_l^C, s \in \Lambda_C$,

Controller Rule $\mathcal{I}_C(s)$: If $g_l(x(t)) \in \mathcal{M}_l^C, AND \ldots, AND$ $g_{\Psi}(x(t)) \in \mathcal{M}_\Psi^C$, THEN

$$u(t) = K_s x(t), n \in \{1, 2, \ldots, q\},$$

(5)

where $g(x(t)) = [g_1(x(t)), \ldots, g_{\Psi}(x(t))]^T$ is the premise variable vector and $K_s = (n \in \mathcal{I}_C(s), s \in \Lambda_C)$ are the state feedback controller gains of each local model in each region to be determined later. Thus, in the $s$th region, the controller can be given by

$$u(t) = K_s x(t), s \in \Lambda_C,$$

(6)

where $K_s = \sum_{n \in \mathcal{I}_C(s)} \nu_n K_n$ and $\nu_n$ or $\nu_n(x(t))$ is the normalized fuzzy membership function for the controller given by

$$\nu_n = \frac{\prod_{j=1}^{\Psi} \nu_{n,j}^\Psi(g_j(x(t)))}{\sum_{s=1}^{q} \prod_{j=1}^{\Psi} \nu_{s,j}^\Psi(g_j(x(t)))} \geq 0, \sum_{n=1}^{q} \nu_n = 1.$$

Similarly, we define a set $\mathcal{T}_C$ to denote all possible region transitions from $\Omega_l^C$ to $\Omega_i^C, s, j \in \Lambda_C$

$$\mathcal{T}_C := \{(s, j) | g(x(t)) \in \Omega_s^C, g(x(t+1)) \in \Omega_i^C, s, j \in \Lambda_C\}.$$

Based on the mathematical expressions of physical plant and state feedback controller in piecewise-fuzzy form in (4) and (6), respectively, the closed-loop system can be obtained as:

$$\begin{align*}
\begin{cases}
  x(t+1) = (A_l + B_l K_s)x(t) + D_l w(t) \\
  z(t) = (L_l + N_l K_s)x(t) \\
  y(t) = C x(t), (l \in \Lambda_P, s \in \Lambda_C).
\end{cases}
\end{align*}$$

Remark 2.1: From the expression of state feedback controller in (5), it is worth noting that: 1) a fuzzy-weighted state feedback controller gain $K_m$ is proposed in each region, that is, there are several controller gains in fuzzy regions instead of one single controller gain in one region. In this case, it offers a greater design flexibility for the state feedback controller. 2) Moreover, in the controller formulation, it is not required that the T-S fuzzy physical plant and fuzzy state feedback controller share the same membership functions, which also offers the design flexibility for the membership functions of fuzzy controller as the membership functions for fuzzy controller can be selected to be less complex than those of the physical plant.
C. Piecewise-fuzzy Observer-based Output Feedback Controller

In the previous subsection, we consider the piecewise-fuzzy state feedback controller for T-S fuzzy systems by taking the unmatched premise variables into account. However, it might not be the case in practice that all the state information is available at the controller side as usually some states cannot be directly measured. Thus, in this section, we consider the problem of piecewise-fuzzy observer-based output feedback control for the T-S fuzzy systems under the piecewise-Lyapunov-function-based framework.

Similarly, we define \( \Omega_{C}^{s} \) \( (s \in \Lambda_{C}) \), \( \Lambda_{C} \), and \( I_{C}(s) \) as the partitioned regions, the set of region indices, and the index of fired rules in the \( s \)-th region \( \Omega_{C}^{s} \), respectively. For the discrete-time T-S fuzzy system in (1) and its piecewise form in (4), we consider the following piecewise-fuzzy observer-based output feedback controller with unmatched premise variables:

**Region Rule s:** IF \( g(\hat{x}(t)) \in \Omega_{C}^{s} \), \( s \in \Lambda_{C} \), AND \( \ldots \), AND \( g_{\Phi}(\hat{x}(t)) \) is \( A_{P}^{\Phi} \), THEN

\[
\begin{aligned}
\dot{\hat{x}}(t+1) &= A_{n}\hat{x}(t) + B_{n}u(t) + R_{s}(y(t) - \hat{y}(t)) \\
\dot{\hat{y}}(t) &= C\hat{x}(t) \\
u(t) &= K_{s}\hat{x}(t), \quad n \in \{1, 2, \ldots, q\}
\end{aligned}
\]  

(8)

where \( g(\hat{x}(t)) = [g_{1}(\hat{x}(t)), \ldots, g_{q}(\hat{x}(t))]^{T} \) is the premise variable depending on the estimated states \( \hat{x}(t) \in \mathbb{R}^{n} \), and \( R_{s} \) and \( K_{s} \) \( (n \in I_{C}(s), s \in \Lambda_{C}) \) are the observer and controller gains of each local model in each region.

With the partition approach, the fuzzy observer-based controller in piecewise-fuzzy form can be given as

\[
\begin{aligned}
\dot{\hat{x}}(t+1) &= \tilde{A}_{n}\hat{x}(t) + \tilde{B}_{n}u(t) + \tilde{R}_{s}(y(t) - \hat{y}(t)) \\
\dot{\hat{y}}(t) &= C\hat{x}(t) \\
u(t) &= \tilde{K}_{s}\hat{x}(t)
\end{aligned}
\]

(9)

where

\[
\begin{aligned}
\tilde{A}_{n} &= \sum_{n \in I_{C}(s)} \nu_{n} A_{n}, \quad \tilde{B}_{n} = \sum_{n \in I_{C}(s)} \nu_{n} B_{n}, \\
\tilde{R}_{s} &= \sum_{n \in I_{C}(s)} \nu_{n} R_{s}, \quad \tilde{K}_{s} = \sum_{n \in I_{C}(s)} \nu_{n} K_{s}
\end{aligned}
\]

and \( \nu_{n} \) or \( \nu_{\alpha}(\hat{x}(t)) \) is the normalized fuzzy membership function given as

\[
\nu_{n} = \frac{\prod_{j=1}^{q} \nu_{j}^{\alpha} \left[ g_{j}(\hat{x}(t)) \right]}{\sum_{\alpha=1}^{q} \prod_{j=1}^{q} \nu_{j}^{\alpha} \left[ g_{j}(\hat{x}(t)) \right]} \geq 0, \quad \sum_{n=1}^{q} \nu_{n} = 1.
\]

For subsequent use, we also define a set \( T_{C} \) to denote all possible region transitions of the observer dynamics from \( \Omega_{C}^{s} \) to \( \Omega_{C}^{j} \), \( s, j \in \Lambda_{C} \)

\[
T_{C} := \{(s, j) | g(\hat{x}(t)) \in \Omega_{C}^{s}, g(\hat{x}(t+1)) \in \Omega_{C}^{j}, s, j \in \Lambda_{C}\}
\]

Defining the estimation error as \( e(t) = x(t) - \hat{x}(t) \), and by combining (4) and (9), the augmented closed-loop system can be rewritten as

\[
\begin{aligned}
\xi(t+1) &= \tilde{A}_{ls}\xi(t) + \tilde{D}_{l}w(t) \\
z(t) &= \tilde{L}_{ls}\xi(t)
\end{aligned}
\]

(10)

where

\[
\begin{bmatrix}
\xi(t) = \left[ x^{T}(t) \ e^{T}(t) \right]^{T}, \quad \tilde{D}_{l} = \left[ \begin{array}{c} D_{l} \\
\end{array} \right] \\
\tilde{A}_{ls} = \left[ \begin{array}{cc} A_{l} + \tilde{B}_{l}\tilde{K}_{s} & (\tilde{B}_{l} - \tilde{B}_{s})\tilde{K}_{s} \\
-B_{l}\tilde{K}_{s} & \tilde{A}_{s} - (\tilde{B}_{l} - \tilde{B}_{s})\tilde{K}_{s} - \tilde{R}_{s}C \\
\end{array} \right] \\
\tilde{L}_{ls} = \left[ \begin{array}{c} L_{l} + N_{l}\tilde{K}_{s} - N_{l}\tilde{K}_{s} \\
\end{array} \right]
\end{bmatrix}
\]

For the given discrete-time T-S fuzzy dynamic system in (1) and piecewise form in (4), the objective of this paper is to design the admissible piecewise-fuzzy state feedback controller and observer-based output feedback controller in the form of (5) and (8), respectively, such that the closed-loop systems in (7) and (10) are asymptotically stable with guaranteed \( H_{\infty} \) performance, that is, the \( l_{2} \)-norm from the disturbance \( w(t) \) to the regulated output \( z(t) \) is less than \( \gamma \) under zero initial conditions

\[
\sum_{t=0}^{\infty} z^{T}(t)z(t) \leq \gamma^{2} \sum_{t=0}^{\infty} w^{T}(t)w(t)
\]

for all \( w(t) \in l_{2}[0, \infty) \).

III. MAIN RESULTS

In this section, the results on piecewise-fuzzy controller design are presented based on piecewise Lyapunov functions for the discrete-time T-S fuzzy systems. By considering the shape of membership functions of the physical plant and controllers, the membership-function-dependent approaches to the controller design are proposed.

A. State Feedback Controller Design

**Theorem 3.1:** Considering the discrete-time T-S fuzzy model given in (1), the piecewise-fuzzy state feedback controller in (5) with unmatched premise variables, and a series of predefined scalars \( \tilde{\mu}_{m} \) and \( \tilde{\nu}_{m} \) utilized to respectively approximate the membership functions \( \mu_{m} \) and \( \nu_{m} \), and \( \theta_{mn} \) such that \( \mu_{m}\tilde{\nu}_{m} - \tilde{\mu}_{m}\nu_{m} - \theta_{mn} \geq 0 \), the closed-loop system in (7) is asymptotically stable with a guaranteed \( H_{\infty} \) performance level \( \gamma \) if there exist \( M_{lsij}, W_{lsij} \in \mathbb{S}^{n_{x}+n_{w}+n_{z}}, P_{ls} \in \mathbb{S}^{n_{x}}, G_{s} \in \mathbb{R}^{n_{x}x}, \) and \( \tilde{K}_{s} \in \mathbb{R}^{n_{x}x} \), such that the following conditions hold:

\[
P_{ls} > 0, W_{lsij} \geq 0
\]

(11)

\[
\sum_{m \in I_{C}(l)} \sum_{n \in I_{C}(s)} \left[ \tilde{\theta}_{mn} M_{lsij} \right] < 0
\]

(12)

\[
\sum_{m \in I_{C}(l)} \sum_{n \in I_{C}(s)} \left[ \tilde{\theta}_{mn} M_{lsij} + W_{lsij} \right] < 0
\]

(13)

where

\[
\begin{bmatrix}
P_{ls} - G_{s}^{T} \quad 0_{n_{w}x} \quad 0_{n_{w}x} \\
0_{n_{w}x} \quad D_{m} \quad -P_{ij} \\
0_{n_{w}x} \quad D_{m} \quad L_{m}G_{s} + N_{m}\tilde{K}_{s} - N_{m}\tilde{K}_{s}
\end{bmatrix}
\]
Moreover, the admissible piecewise-fuzzy state feedback controller gains in the form of (5) are given by
\[
K_{sn} = \hat{K}_{sn}G_{s}^{-1}.
\] (14)

**Proof:** Consider the following piecewise Lyapunov function:
\[
V(t) = x^T(t)P_{ls}^{-1}x(t), f(x(t)) \in \Omega_{l}^{P},
g(x(t)) \in \Omega_{s}^{F}, l \in \Lambda_{P}, s \in \Lambda_{C},
\] (15)
where \(0 < P_{ls} \) are Lyapunov matrices to be determined.

Define \(\eta(t) = \left[ x^T(t)w^T(t) \right]^{T} \) and \(\Delta V(t) = V(t+1, x(t+1)) - V(t, x(t)). \) With the given piecewise Lyapunov function, it is sufficient to show the system (7) satisfying the aforementioned requirements in last section if the following inequality holds:
\[
\mathcal{L}(t) \triangleq V(t+1, x(t+1)) - V(t, x(t)) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) < 0. \] (16)

Without loss of generality, we assume that \(f(x(t+1)) \in \Omega_{l}^{P}, g(x(t+1)) \in \Omega_{s}^{F} \quad \forall \quad l \in \Lambda_{P}, j \in \Lambda_{C}, \) and \((l, i) \in \mathcal{T}_{P}, (s, j) \in \mathcal{T}_{C}. \) Along the trajectory of (7), the left-hand-side of equation (16) is reformulated as:
\[
\mathcal{L}(t) = \left[ (A_{l} + B_{l}K_{s})x(t) + D_{l}w(t) \right]^T P_{lj}^{-1} \left[ (A_{l} + B_{l}K_{s})x(t) + D_{l}w(t) \right] - x^T(t)P_{ls}^{-1}x(t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t)
\] (17)
where
\[
\tilde{\xi}_{lsij}(\mu, \nu) = \sum_{m \in \mathcal{I}_{P}(l)} \sum_{n \in \mathcal{I}_{C}(s)} \mu_{mn}^{\nu} \tilde{\xi}_{mn}^{\nu n}_{lsij},
\] (18)

Obviously, if \(\tilde{\xi}_{lsij}(\mu, \nu) < 0,\) one can conclude that the closed-loop system in (7) is asymptotically stable with a guaranteed \(\mathcal{H}_{\infty}\) performance level \(\gamma.\)

On the other hand, considering the definitions of system matrices and controller gain in piecewise-fuzzy form, we have
\[
\tilde{\xi}_{lsij}(\mu, \nu) = \sum_{m \in \mathcal{I}_{P}(l)} \sum_{n \in \mathcal{I}_{C}(s)} \mu_{mn}^{\nu} \tilde{\xi}_{mn}^{\nu n}_{lsij},
\] (19)

which yields
\[
-G_{s}P_{ls}^{-1}G_{s}^{T} \leq P_{ls} - G_{s} - G_{s}^{T},
\] (23)

Now, performing a congruence transformation to (21) and (22) with \(G_{s} = \text{diag} \{G_{s}; I_{n_{1}}; I_{n_{2}}; I_{n_{3}} \},\) and considering the inequality in (24), conditions (21) and (22) hold if
\[
\sum_{m \in \mathcal{I}_{P}(l)} \sum_{n \in \mathcal{I}_{C}(s)} \left[ \mu_{mn}^{\nu} \tilde{\xi}_{mn}^{\nu n}_{lsij} \right] < 0,
\] (25)

from which we can see that the information of memory functions \(\mu_{mn}(x(t))\) and \(\nu_{n}(x(t))\) is involved.

To further reduce the design conservatism, we introduce a series of discrete values \(\bar{\mu}_{m}\) and \(\bar{\nu}_{n}\) to approximate the membership functions \(\mu_{mn}\) and \(\nu_{n},\) which can be determined according to the membership functions with respect to some points on the abscissa axis. In this case, it can be easily known that \(\sum_{m \in \mathcal{I}_{P}(l)} \bar{\mu}_{m} = 1\) and \(\sum_{n \in \mathcal{I}_{C}(s)} \bar{\nu}_{n} = 1. \) To facilitate the subsequent controller design, some slack matrices 0 \(\leq \tilde{W}_{lsij} \in \mathbb{S}^{n_{s} + n_{w} + n_{s}},\) are introduced, and thus we have
\[
\tilde{\xi}_{lsij}(\mu, \nu) \leq \sum_{m \in \mathcal{I}_{P}(l)} \sum_{n \in \mathcal{I}_{C}(s)} \left[ \bar{\mu}_{mn}^{\nu} \tilde{\xi}_{mn}^{\nu n}_{lsij} \right]
\] (19)

Finally, we are in the position to derive the controller gains based on the stability conditions given above. It should be noted that the matrix variables \(P_{ls}^{-1}\) and \(P_{lj}\) co-exist in (21) and (22), which leads to some difficulties in the solution of optimization problem. Thus, we adopt the widely used inequality as follows by introducing some nonsingular slack matrices \(G_{s} \in \mathbb{R}^{n_{s} \times n_{s}} \quad (s \in \Lambda_{C}):\)
\[
P_{ls} - G_{s} - G_{s}^{T} + G_{s}P_{ls}^{-1}G_{s}^{T} = (P_{ls} - G_{s})P_{ls}^{-1}(P_{ls} - G_{s})^{T} \geq 0,
\] (23)

which yields
\[
-G_{s}P_{ls}^{-1}G_{s}^{T} \leq P_{ls} - G_{s} - G_{s}^{T}.
\] (24)
with
\[
\Xi_{mn}^{ij} = \begin{bmatrix}
P_{ls} - G_s - G_s^T & 0_{n_u \times n_x} & 0_{n_u \times n_x} & 0_{n_u \times n_x} & - \gamma_I_{n_w} & \ast & \ast & \ast \\
A_mG_s + B_mK_s & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
L_m + N_mK_s & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\end{bmatrix} < 0
\]
which further implies the stability condition with a guaranteed \(H_\infty\) performance level \(\gamma\) in (16).

By defining \(K_s = K_mG_s, W_{lsij} = \hat{G}_{l}^T \hat{W}_{lsij} \hat{G}_s,\) and \(M_{lsij} = \hat{G}_s^T \hat{M}_{lsij} \hat{G}_s\), we readily have (25) and (26) from (11)-(13) in Theorem 3.1. The proof is thus completed.

Remark 3.1: In Theorem 3.1, the stability analysis and controller synthesis are investigated for discrete-time T-S fuzzy systems under piecewise-Lyapunov-function-based framework. It should be noted that, to further reduce the design conservatism, some information of the membership functions is incorporated by choosing finite discrete values of these membership functions. To one extreme, if \(\mu_m = \nu_n\) can be well approximated by a large number of \(\mu_m = \nu_n\) variables, the stability condition with a guaranteed \(H_\infty\) performance level \(\gamma\) in (11)-(13) will reduce to \(\sum_{m \in T_P(l)} \sum_{n \in N_C(s)} \theta_{mn}^{ij} \Xi_{mn}^{ij} < 0\) by letting \(M_{lsij} = 0, W_{lsij} = 0\), and \(\theta_{mn} = 0\). On the other hand, with the introduction of slack matrices \(M_{lsij}, W_{lsij}\), and the increasing number of partitioned regions, the computational load will become higher.

In Theorem 3.1, a membership-function-dependent approach to the state feedback controller design is proposed under piecewise-Lyapunov-function-based framework. However, if the information of membership functions is not considered in the stability analysis as in some existing literatures, such as [31], we will directly derive the following conditions on state feedback controller design.

Corollary 3.1: For the discrete-time T-S fuzzy model given in (1) and the fuzzy-piecewise state feedback controller in (5) with unmatched premise variables, the closed-loop system in (7) is asymptotically stable with a guaranteed \(H_\infty\) performance level \(\gamma\) if there exist \(P_{ls} \in S^{n_x}\), \(G_s \in R^{n_u \times n_x}\), and \(\bar{K}_s \in R^{n_u \times n_x}\), such that the following conditions hold:
\[
\Xi_{mn}^{ij} = \begin{bmatrix}
P_{ls} - G_s - G_s^T & 0_{n_u \times n_x} & 0_{n_u \times n_x} & 0_{n_u \times n_x} & - \gamma_I_{n_w} & \ast & \ast & \ast \\
A_mG_s + B_mK_s & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
L_m + N_mK_s & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\end{bmatrix} < 0
\]

and moreover, the admissible piecewise-fuzzy state feedback controller in the form of (5) can be derived by (14).

The proof of Corollary 3.1 can be easily completed based on (19) by extracting the membership functions.

Remark 3.2: In the aforementioned results on controller design, it is not required that the piecewise-fuzzy state feedback controller shares the same membership functions with physical plant. However, if the \(H_\infty\) performance analysis for the closed-loop system in (7) and the corresponding state feedback controller design are conducted with parallel distributed compensation (PDC) method, i.e., \(p = c\) and \(\mu_m(x(t)) = \nu_m(x(t))\) for all \(m \in \{1, 2, \ldots , p\}\), the conservativeness of the controller design results in Theorem 3.1 can be further relaxed under the membership-function-dependent analysis strategy with some approaches reported in the existing literatures [15], [25], [26], [40].

B. Observer-based Output Feedback Controller Design

In this subsection, we present the main result on the piecewise-fuzzy observer-based output feedback controller design.

Theorem 3.2: Consider the T-S fuzzy discrete-time system in (4) and the observer-based output feedback controller in the form of (8) with unmatched premise variables. For the given scalars \(\bar{\mu}_m = \bar{\nu}_n\) and \(\bar{\theta}_{mn}\) satisfying \(\mu_m = \mu_m = \nu_n = \nu_n = 0\), the closed-loop system (9) is asymptotically stable with a guaranteed \(H_\infty\) performance level \(\gamma\) if there exists a set of matrices \(P_{ls} \in S^{n_x}, S_{ls1} \in M^{(n_u \times n_y)}\), \(S_{ls2} \in M^{(n_x \times (n_x - n_y))}, \hat{S}_{ls3} \in M^{(n_x \times (n_x - n_y))}\) such that the following linear matrix inequality conditions are satisfied
\[
P_{ls} > 0, W_{lsij} \geq 0,
\sum_{m \in T_P(l)} \sum_{n \in N_C(s)} (\theta_{mn}^{ij} M_{lsij}) < 0,
\]

and \(J_1, J_2, J_3\) are obtained via the singular value decomposition (SVD) of the output matrix \(C\), i.e., \(C = J_1 J_2^T C^T S_{ls1}^1 0_{n_u \times (n_x - n_y)} J_2^T\). Moreover, the observer and controller gains can be calculated via
\[
R_s = R_{sn} S_{ls1}^1, K_s = K_{sn} S_{ls1}^1, s \in \Lambda, n \in N_C(s).
\]

Proof: We consider the following piecewise quadratic Lyapunov function as in state feedback case:
\[
V(t) = \xi^T(t) P_{ls}^{-1} \xi(t), f(x(t)) \in \Omega_P^T
\]
are known scalars satisfying 

\[ g(\hat{x}(t)) \in \Omega_s^l, l \in \Lambda_s, s \in \Lambda_s, \]  

where \(0 < P_s \in \mathbb{S}^{2n_x} \) are Lyapunov matrices to be determined, and the following inequality will be used to guarantee the closed-loop system in (10) to be asymptotically stable with a prescribed \( H_\infty \) performance level \( \gamma \)

\[ V(t + 1) - V(t) < -z^T(t)z(t) + \gamma^2 w^T(t)w(t). \]  

(33)

Following the similar line in Theorem 3.1, it can be easily shown that the aforementioned inequality holds if along the trajectories of system (10) the following inequality is satisfied:

\[ \sum_{m \in \mathcal{T}_1(l)} \sum_{n \in \mathcal{T}_1(s)} \mu_{mn} \bar{v}_n \Pi_{mn}^{\text{ss}} \Pi_{mn}^{\text{lsij}} < 0, \]  

(34)

where

\[
\begin{align*}
\Pi_{mn}^{\text{ss}} &= \begin{bmatrix}
-P_{1s}^{-1} & \ast & \ast & \ast \\
0_{n_x \times 2n_x} & -\gamma^2 I_{n_x} & \ast & \ast \\
A_m & D_m & -P_{ij} & \ast \\
\bar{C}_m & 0_{n_x \times 2n_x} & 0_{n_x \times 2n_x} & -I_{n_x}
\end{bmatrix}, \\
\Pi_{mn}^{\text{lsij}} &= \begin{bmatrix}
-A_n - (B_m - B_n)K_{sn} & \ast & \ast & \ast \\
A_m + B_mK_{sn} & \ast & \ast & \ast \\
-A_m - (B_m - B_n)K_{sn} & \ast & \ast & \ast \\
D_m & \ast & \ast & \ast
\end{bmatrix}, \\
A_m &= [1], \\
D_m &= [1], \\
F_m &= [1].
\end{align*}
\]

By introducing the slack matrices \( M_{mn} \in \mathbb{S}^{4n_x+n_x+n_w}, \) and \( 0 \leq W_{mn} \in \mathbb{S}^{4n_x+n_x+n_w}, \) the inequality (34) holds if

\[
\sum_{m \in \mathcal{T}_1(l)} \sum_{n \in \mathcal{T}_1(s)} \left[ \theta_{mn} M_{mn} + (\mu_{mn} \bar{v}_n + \theta_{mn})(\Pi_{mn}^{\text{lsij}} + W_{mn}) \sum_{m \in \mathcal{T}_1(l)} \sum_{n \in \mathcal{T}_1(s)} (\mu_{mn} \bar{v}_n - \mu_{mn} \bar{v}_n - \theta_{mn}) \right] < 0, \]  

(35)

where \( \mu_{mn} \) and \( \bar{v}_n \) are a series of discrete values to approximate the membership functions \( \mu_m \) and \( \bar{v}_n \), respectively, and \( \theta_{mn} \) is known scalars satisfying \( \mu_{mn} \bar{v}_n - \mu_{mn} \bar{v}_n - \theta_{mn} \geq 0 \) \((m \in \mathcal{T}_1(l), n \in \mathcal{T}_1(s))\).

By considering the matrix inequality in (23), and performing a congruence transformation to (35) with \( S_s = \text{diag}\{S_{s1}, I_{n_x}, I_{n_x}, I_{n_x}\} \) and its transpose, we have the following two conditions such that the closed-loop is asymptotically stable with \( H_\infty \) performance level \( \gamma \):

\[
\sum_{m \in \mathcal{T}_1(l)} \sum_{n \in \mathcal{T}_1(s)} [\theta_{mn} M_{mn} + (\mu_{mn} \bar{v}_n + \theta_{mn})(\Pi_{mn}^{\text{ss}} + W_{mn})] < 0, \]  

(36)

\[
(\Pi_{mn}^{\text{ss}} + M_{mn} + W_{mn}) < 0, \]  

(37)

for \( m \in \mathcal{T}_1(l), n \in \mathcal{T}_1(s), l, i \in \Lambda_s, s, j \in \Lambda_s, \) \((i, i) \in \mathcal{T}_1, (s, j) \in \mathcal{T}_1, \) where \( W_{mn} = S_m^T W_{mn} S_m, \) \( M_{mn} = S_m^T M_{mn} S_m, \) and

\[
\Pi_{mn}^{\text{ss}} = \begin{bmatrix}
P_{1s} - S_m & 0_{n_x \times 2n_x} & \ast & \ast & \ast \\
0_{n_x \times 2n_x} & \gamma^2 I_{n_x} & \ast & \ast & \ast \\
A_m & D_m & -P_{ij} & \ast & \ast \\
\bar{C}_m & 0_{n_x \times 2n_x} & 0_{n_x \times 2n_x} & -I_{n_x}
\end{bmatrix}.
\]

Without loss of generality, we assume the output matrix \( C \) to be full rank row, and then its SVD can be given as follows:

\[
C = J_1 \begin{bmatrix} J_2 & 0_{n_y \times (n_x-n_y)} \end{bmatrix} J_3^T, \]

\[
J_1 J_1^T = I_{n_y}, J_2 J_2^T = I_{n_x}, \]  

(38)

where \( J_1 \in \mathbb{R}^{n_y \times n_y}, J_2 \in \mathbb{R}^{n_x \times n_y}, \) and \( J_3 \in \mathbb{R}^{n_x \times n_x} \) are all nonsingular matrices.

To facilitate the calculations of observer and controller gains, we specify the matrix \( S_s \) to be with the structure as follows:

\[
S_s = \text{diag}\{S_1, S_1, \}
\]

\[
S_1 = J_3 \begin{bmatrix} J_2 J_2^T & 0_{n_y \times (n_x-n_y)} \end{bmatrix} J_3^T, \]

where \( S_1(1) \in \mathbb{R}^{n_y \times n_y}, S_1(2) \in \mathbb{R}^{(n_x-n_y) \times n_y}, S_1(3) \in \mathbb{R}^{(n_x-n_y) \times (n_x-n_y)}, \) and \( S_1(1) \) and \( S_1(3) \) are nonsingular.

Then, with (38) and (39) one can obtain

\[
R_{sn} C S_1 = R_{sn} J_1 \begin{bmatrix} J_2 & 0_{n_y \times (n_x-n_y)} \end{bmatrix} J_3^T \times J_3 \begin{bmatrix} J_2 J_2^T & 0_{n_y \times (n_x-n_y)} \end{bmatrix} J_3^T \]

\[
= [R_{sn}, 0_{n_y \times (n_x-n_y)}] J_3 \mapsto R_{sn} J_3^T. \]  

(39)

By defining \( \check{K}_{sn} = K_{sn} S_1, \) and \( \check{R}_{sn} = R_{sn} S_1(1) \), we readily have (36)-(37), and (29)-(30), which further implies (34) and (33). Thus, the proof is completed.

Remark 3.3: From the inequalities given in Theorems 3.1 and 3.2, it can be observed that the information of membership functions is incorporated in the stability and \( H_\infty \) performance analysis within the piecewise-Lyapunov-function-based framework, by which the design conservatism is further reduced (please see Table II in Example 4.1). However, it should be noted that two different types of space separations are introduced, that is, the partition on premise variable space for system conversion from T-S fuzzy model to equivalent piecewise-fuzzy form and the division on x-axis of membership functions for approximation of these functions with a series of discrete values. In this case, with increase of the partitions, the number of stability conditions in terms of matrix inequalities will increase accordingly, and thus, a computational problem might arise for some systems with more complex membership functions.

Remark 3.4: With the introduction of slack matrices \( S_s \) in Theorem 3.2, a sufficient condition in terms of linear matrix inequality for the observer and controller design is derived, which can be directly solved via some commercially available softwares, such as, MatLab. However, it is noted that, to facilitate the linearization procedure of inequality constraints in (36) and (37), \( S_s \) is specified with a structural constraints.
in (38), which inevitably introduced some degree of design conservatism as expected. To further reduce the conservatism, the cone complementarity linearization (CCL) design strategy can be adopted [41]. For more details about CCL, please refer to [42].

IV. SIMULATION RESULTS

In this section, two illustrative examples are given to show the effectiveness of the proposed approaches to the design of piecewise-fuzzy state feedback controller and observer-based output feedback controller.

Example 4.1: Consider a discrete-time T-S fuzzy system in the form of (1) with three local models, and the system parameters are given as follows:

\[
\begin{align*}
A_1 &= \begin{bmatrix} 0.9000 & -1.1800 \\ -0.0900 & -0.6300 \end{bmatrix}, & A_2 &= \begin{bmatrix} 1.0500 & -0.8000 \\ 0.0500 & -0.2000 \end{bmatrix} \\
A_3 &= \begin{bmatrix} 1.3440 & -1.4280 \\ 0.3360 & -1.1200 \end{bmatrix}, & B_1 &= \begin{bmatrix} -1 \\ 0 \end{bmatrix}, & D_1 &= \begin{bmatrix} -1 \\ 0.5 \end{bmatrix} \\
B_2 &= B_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, & D_2 &= D_3 &= \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \\
L_m &= \begin{bmatrix} 0 & 0.2 \end{bmatrix}, & N_m &= 0.1, & C &= \begin{bmatrix} 1 & 1 \end{bmatrix}
\end{align*}
\]

for \( m \in \{1, 2, 3\} \).

The membership functions of the physical plant are depicted in Figure 1, from which it can be seen that we have three regions \( \Omega^P_1, \Omega^P_2, \text{ and } \Omega^P_3 \) according to the space partition method mentioned in Section II. Moreover, with the partition, one has \( \mathcal{V}_P(1) = \{1, 2\}, \mathcal{V}_P(2) = \{2\}, \text{ and } \mathcal{V}_P(3) = \{2, 3\} \).

Now, a piecewise-fuzzy state feedback controller in the form of (5) with three fuzzy rules is employed to stabilize the fuzzy system and guarantee a certain level of \( \mathcal{H}_\infty \) performance of the closed-loop system. To facilitate the design of the piecewise-fuzzy controller, we assume that the membership functions of the state feedback controller are given by trapezoid as shown in Figure 2, which are not same as those of physical plant.

Now, we consider using the staircase functions to approximate the original membership functions of both fuzzy model and piecewise-fuzzy controller. To this end, the \( x \)-axis is uniformly divided into different intervals in each region

\[\Omega^P_s (\Omega^P_s, l, s \in 1, 2, 3).\]

Please see Figure 1 in \( \Omega^P_s \). We choose the membership function values at the middle point of each interval to approximate the membership functions. It can be easily observed that, with the increasing number of the partitioned intervals, approximation errors among the discrete values \( \bar{\mu}_m, \bar{\nu}_n \) and the corresponding \( \mu_m \) and \( \nu_n \) become small. To guarantee \( \mu_m \nu_n - \bar{\mu}_m \bar{\nu}_n - \theta_{mn} \geq 0 \), let \( \theta_{mn} = \mu_m \nu_n - \bar{\mu}_m \bar{\nu}_n - 10^{-4} \). To demonstrate the effectiveness of the proposed method, we select the step length as 1.0, and the obtained values of \( \theta_{mn} \) are listed in Table I. Now, by Theorem 3.1, we obtain an admissible piecewise-fuzzy state feedback controller for the system in Example 4.1, which are given as follows

\[
\begin{align*}
K_{11} &= \begin{bmatrix} -0.0155 & 0.0324 \end{bmatrix}, & K_{12} &= \begin{bmatrix} -0.0297 & 0.0198 \end{bmatrix} \\
K_{13} &= \begin{bmatrix} -0.9502 & 0.4583 \end{bmatrix}, & K_{15} &= \begin{bmatrix} -1.0103 & 0.4416 \end{bmatrix} \\
K_{22} &= \begin{bmatrix} -0.9052 & 0.3700 \end{bmatrix}
\end{align*}
\]

and the optimal \( \mathcal{H}_\infty \) performance index is \( \gamma_{\text{min}} = 2.0475 \).

To further illustrate the advantage of the proposed method over some existing fuzzy controller design methods, such as, piecewise-Lyapunov-function-based framework and common-Lyapunov-based framework, we consider different cases with step length 1.0, 0.4, and 0.2, that is, the definition range of membership functions \([-10, 10]\) is uniformly divided into 20, 50, 100 intervals, respectively. A detailed comparison of the obtained minimum \( \mathcal{H}_\infty \) performance indices \( \gamma_{\text{min}} \) under different scenarios and design methods is summarized in Table II. The results clearly show that better performance can be obtained via the developed method in this paper over the existing ones under piecewise-Lyapunov-function-based and common-Lyapunov-based framework, and moreover, with the increasing number of partitioned intervals, the \( \mathcal{H}_\infty \) performance can be further improved.

Example 4.2: In this example, the truck-trailer model formulated in [43] is used to demonstrate the effectiveness of the piecewise-fuzzy observer-based output feedback controller design in this paper. As the control purpose of this system is to back up a truck-trailer along a desired trajectory, a simplified

\[
\begin{align*}
\end{align*}
\]
model is given as follows:

\[
\begin{align*}
\dot{x}_1(t+1) &= (1 - \frac{v_0 t_0}{L_0}) x_1(t) + \frac{v_0 t_0}{L_0} u(t) + 0.1 w(t) \\
\dot{x}_2(t+1) &= x_2(t) + \frac{v_0 t_0}{L_0} x_1(t) + 0.1 u(t) \\
\dot{x}_3(t+1) &= x_3(t) + v_0 t_0 \sin(x_2(t) + \frac{v_0 t_0}{L_0} x_1(t)) + 0.1 w(t)
\end{align*}
\]

where \(x_1(t)\) is the angle difference between truck and trailer, and \(x_2(t)\) is the angle of the trailer; \(x_3(t)\) is the vertical position of the rear end of the trailer; \(u(t)\) is the control input of steering angle; \(w(t)\) is the external disturbance and \(L_0\) is the length of the trailer. The parameters in this model are given as \(L_0 = 5.5m, v_0 = -1.0m/s, t_0 = 2.0s,\) and the measurement and regulated output are, respectively, given by

\[
\begin{align*}
y(t) &= \begin{bmatrix} 7 & -2 & 0.03 \\ 0.8 & 2.1 & 0 \end{bmatrix} x(t), \\
z(t) &= \begin{bmatrix} 0.5 & 0 & 0 \end{bmatrix} x(t) + 0.2 u(t).
\end{align*}
\]

By the fuzzy modeling approach in [43], one can obtain the corresponding T-S fuzzy approximation of the nonlinear system, and the Gaussian-type membership functions are employed as the fuzzy basis functions [44], which are given in Figure 3.

Then the T-S fuzzy dynamics of the truck-trailer model is given as follows:

**Plant Rule** \( \mathcal{R}_p^2 \): IF \( x_2(t) + \frac{v_0 t_0}{L_0} x_1(t) \) is \( M_1^n \), THEN

\[
\begin{align*}
\dot{x}(t+1) &= A_n x(t) + B_n u(t) + D_n w(t) \\
\dot{z}(t) &= L_m x(t) + N_m u(t) \\
y(t) &= C x(t), m \in \mathcal{R}_p \triangleq \{1, 2\},
\end{align*}
\]

and the system matrices are

\[
\begin{align*}
A_1 &= \begin{bmatrix} 1 - \frac{v_0 t_0}{L_0} & 0 & 0 \\ \frac{v_0 t_0}{L_0} & 1 & 0 \\ \frac{v_0 t_0}{2L_0} & \frac{v_0 t_0}{2L_0} & 1 \end{bmatrix}, \\
A_2 &= \begin{bmatrix} 1 - \frac{v_0 t_0}{L_0} & 0 & 0 \\ \frac{v_0 t_0}{L_0} & 1 & 0 \\ \frac{v_0 t_0}{L_0} & \frac{v_0 t_0}{L_0} & 1 \end{bmatrix}, \\
B_1 &= B_2 = \begin{bmatrix} \frac{v_0 t_0}{L_0} \\ \frac{v_0 t_0}{L_0} \\ \frac{v_0 t_0}{L_0} \end{bmatrix}^T, \\
D_1 &= D_2 = \begin{bmatrix} 0.1 & 0.1 & 0.1 \end{bmatrix}^T,
\end{align*}
\]

where \(d\) equals to \(0.01\) to guarantee the system to be theoretically controllable.

---

**TABLE I**

<table>
<thead>
<tr>
<th>( \Omega_{11}^i ), ( \Omega_{12}^i )</th>
<th>( \Omega_{21}^i ), ( \Omega_{22}^i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta_{11} )</td>
<td>(-0.0629)</td>
</tr>
<tr>
<td>( \theta_{12} )</td>
<td>(-0.0623)</td>
</tr>
<tr>
<td>( \theta_{21} )</td>
<td>(-2.0292 \times 10^{-4})</td>
</tr>
<tr>
<td>( \theta_{22} )</td>
<td>(-1.0989 \times 10^{-4})</td>
</tr>
</tbody>
</table>

**TABLE II**

<table>
<thead>
<tr>
<th>( \gamma_{\infty, \text{min}} )</th>
<th>( \gamma_{\infty, \text{step}=0.2} )</th>
<th>( \gamma_{\infty, \text{step}=0.4} )</th>
<th>( \gamma_{\infty, \text{step}=1.0} )</th>
<th>( \text{PLF-based method} )</th>
<th>( \text{CLF-based method} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{Theorem 3.1} )</td>
<td>( \text{Theorem 3.1} )</td>
<td>( \text{Theorem 3.1} )</td>
<td>( \text{PLF-based method} )</td>
<td>( \text{CLF-based method} )</td>
<td></td>
</tr>
<tr>
<td>( \gamma_{\infty, \text{min}} )</td>
<td>1.1538</td>
<td>1.5021</td>
<td>2.0475</td>
<td>2.8350</td>
<td>5.3660</td>
</tr>
</tbody>
</table>

Fig. 3. The membership functions in Example 4.2

Consider the piecewise-fuzzy output feedback controller in the form of (8) with the same membership functions in Figure 3. It is ready for the design of observer-based output feedback controller according to Theorem 3.2 with the step length \( \pi/24 \). A feasible solution is obtained with the optimal \( \mathcal{H}_\infty \) performance index \( \gamma_{\infty, \text{min}} = 0.2747 \), and the corresponding observer and controller gains are given as follows:

\[
\begin{align*}
K_{11} &= \begin{bmatrix} 1.9409 \\ -1.0700 \end{bmatrix}, \\
K_{21} &= \begin{bmatrix} 1.9409 \\ -1.0700 \end{bmatrix}, \\
K_{22} &= \begin{bmatrix} 1.9314 \\ -0.9790 \end{bmatrix}, \\
R_{11} &= \begin{bmatrix} 0.2235 \\ 0.1941 \end{bmatrix}, \\
R_{21} &= \begin{bmatrix} -0.0822 \end{bmatrix}, \\
R_{22} &= \begin{bmatrix} -0.0827 \end{bmatrix}.
\end{align*}
\]
and \((K_{11}, R_{11})\) and \((K_{21}, K_{22}, R_{21}, R_{22})\) are the controller and observer gains when the dynamics of the piecewise-fuzzy observer evolutes in region \(\Omega_1^C\) and \(\Omega_2^C\), respectively.

To illustrate the asymptotical stability of the closed-loop system with the obtained observer-based output feedback controller, we show some simulation results under different conditions with \(w(t) = 0\). In Figure 4, the state responses under \(x(0) = \begin{bmatrix} \frac{2}{\pi} & \frac{1}{\pi} & 0 \\ \end{bmatrix}^T\), \(\hat{x}(0) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T\), and \(w(t) = 0\) are shown. Now, we consider the \(\mathcal{H}_\infty\) performance of the closed-loop system with the external disturbance given by \(w(t) = e^{-0.05\sin(0.4\pi t)}\) under zero initial condition. In Figure 5 and Figure 6, the curves of state response of the closed-loop system and the ratio \(\sqrt{\sum_{i=0}^{\infty} z^T(i)z(i)}/\sqrt{\sum_{i=0}^{\infty} w^T(i)w(i)}\) are plotted, respectively, from which we can see that the ratio is obviously less than \(\gamma_{\text{min}} = 0.2747\), further implying the effectiveness of the proposed method.

V. CONCLUSION

In this paper, the problem of membership-function-dependent controller design for T-S fuzzy system is considered under piecewise-Lyapunov-function-based framework. By employing the staircase functions, the continuous information of original membership functions is approximated and introduced into the stability analysis and controller synthesis. Via piecewise Lyapunov functions, two approaches to the piecewise-fuzzy state feedback controller and observer-based output feedback controller design are proposed in terms of linear matrix inequalities, which guarantee the closed-loop system to be asymptotically stable with a prescribed \(\mathcal{H}_\infty\) performance level. Simulation examples are presented to demonstrate the effectiveness and advantages of the developed methods. However, it should be noted that the controller design for type-1 fuzzy-model-based systems is considered in present work. As an extension, the interval type-2 T-S fuzzy model, which effectively captures the system nonlinearities and parameter uncertainties, has attracted plenty of research interests in recent years. Thus, the problem of stability analysis and controller synthesis for interval type-2 fuzzy systems deserves further investigation under piecewise-Lyapunov-function-based framework.

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REFERENCES


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