Risk managing tail-risk seekers:
VaR and expected shortfall vs S-shaped utility

Abstract

We consider market players with tail-risk-seeking behaviour modelled by S-shaped utility, as introduced by Kahneman and Tversky. We argue that risk measures such as value at risk (VaR) and expected shortfall (ES) are ineffective in constraining such players, as such measures cannot reduce the traders expected S-shaped utilities. Indeed, when designing payoffs aiming to maximize utility under a VaR or ES risk limit, the players will attain the same supremum of expected utility with or without VaR or ES limits. By contrast, we show that risk management constraints based on a second more conventional concave utility function can reduce the maximum S-shaped utility that can be achieved by the investor. Indeed, product designs leading to progressively larger S-shaped utilities will lead to progressively lower expected constraining conventional utilities, violating the related risk limit. These results hold in a variety of market models, including the Black Scholes options model, and are particularly relevant for risk managers given the historical role of VaR and the endorsement of ES by the Basel committee in 2012–2013.

value at risk constraints; expected shortfall constraints; concave utility constraints; S-shaped utility maximization; limited liability investors; tail-risk-seeking investors; effective risk constraints; concave utility risk constraints.

JEL classification codes: D81, G11, G13.

1. Introduction

We consider the effects of imposing risk constraints on a trader in the case when that trader exhibits risk-seeking behaviour over losses or has limited liability.

When making investment decisions, a trader is constrained in how much funds they can use in setting up the investment (giving rise to a budget or price constraint). In addition a trader is typically limited on how much risk of future losses the investment may entail. Risk limits are common in banking and represent a tool for risk control. A natural risk limit to adopt is a risk limit based on classic risk measures such as Value at Risk (VaR) and Expected Shortfall (ES). The trader is then constrained to choose an investment whose VaR or ES is below a given limit, set by the bank risk manager when approving the trade.

The question we seek to answer in this paper is whether such limits are effective when the trader exhibits tail-risk-seeking behaviour or has some form of limited liability. We consider a risk limit to be "effective" if it has an economic impact upon the trader, in the sense that imposing the limit decreases the expected utilities that the trader can achieve. We model the tail-risk-seeking behaviour and/or limited liability by assuming that the utility of the trader is given by an S-shaped function of the payoff of their investments. We assume that the trader acts to maximize their expected utility, subject to the budget constraint and any risk constraints. We

find that in market models including the benchmark Black–Scholes model, neither VaR nor ES constraints are effective in curbing such risk-seeking behaviour. This result is particularly important in the light of the fact that ES has been officially endorsed and suggested as a risk measure by the Basel committee in 2012-2013 (Basel Committee on Banking Supervision, 2013, 2016), partly for its "coherent risk measure" properties (Artzner et al., 1999; Acerbi and Tasche, 2002).

The function which determines the utility from the investment payoff is taken to be S-shaped, in the sense that it is convex on the left and concave on the right. This may occur for two reasons. We suppose that the trader receives a personal reward, such as their pay packet, given by some function of the investment payoff. We also suppose that the trader's preferences over possible personal gains and losses are determined by their personal utility function. The trader's utility as a function of the investment payoff is then given by the composition of their utility function and the reward function. The composition may then be S-shaped if either the trader has an S-shaped personal utility function or the reward function is non-linear.

If the trader's utility function is S-shaped, then this models the case of traders exhibiting a risk-seeking attitude towards losses. S-shaped utility functions were introduced in Kahneman and Tversky (1979) to explain the risk preferences they observed empirically. Their work provides evidence that a risk manager should seriously consider the possibility that traders take a risk-seeking attitude towards losses.

The reward function may be non-linear if, for example, the pay packet of the trader is proportional to the investment returns in the event that the investments

make a profit, but is zero if there is a loss, irrespective of the loss made. This is the case of "limited liability".

As we shall see, if the trader seeks to maximize their expected utility (given as an S-shaped function of the payoff) and is subject to no constraints other than a budget constraint, then, in familiar market models such as the Black–Scholes options market, such traders will pursue highly risky investment strategies. They will seek investments where a large probability of making a profit is subsidized by taking on the risk of a small probability of a catastrophic loss.

In the first part of this paper we will prove that in these same market models such traders are not prevented from pursuing such strategies by the imposition of ES and VaR limits. Moreover, imposing these limits does not decrease the expected value that the trader can achieve, so these limits have no economic impact upon the trader. The trader is unperturbed by the constraint and will continue to pursue the high-risk investment strategies, and, in this sense, the risk constraints can be said to be ineffective.

The key assumptions that we make to obtain this result are that the S-shaped function is sufficiently concave in the left tail, and that the market is a complete one period market model with a non-zero market price of risk. These assumptions apply to the benchmark case of the Black-Scholes model, to traders with S-shaped utilities of the standard functional form proposed in Kahneman and Tversky (1979), and to traders with limited liability.

To understand intuitively why ES may fail to constrain a trader, it is helpful to know that it is possible to manufacture digital options (as shown in Figure 3) that have negative ES and that can be purchased at no cost. This may seem

counter-intuitive as one might expect that a riskless position (as measured with ES), with a non-zero potential upside, should have a positive cost. However, this argument fails because, in complete markets, prices are evaluated using the $\mathbb{Q}$ measure whereas risk is assessed in the $\mathbb{P}$ measure. In the Black-Scholes model the ratio of the $\mathbb{Q}$ measure density to the $\mathbb{P}$ measure density is unbounded, and this unbounded discrepancy between the measures can be leveraged to create such digital positions. An arbitrarily large multiple of such a position will then give a trader with limited liability an arbitrarily large upside, will cost nothing, and will not violate a given ES constraint. The precise result proved in this paper is in fact slightly stronger than implied by this argument. We will show in future work how this intuitive example can be applied to other coherent risk measures.[1]

We note, that the same mathematical model can be applied whether the "trader" is an individual trader working in a bank, an independent investor, or an institution. We have focussed on the relationship between a trader and their risk manager, but similar considerations would apply to a regulator who wishes to control the behaviour of a bank.

In summary, the first part of the paper gives a negative result on the use of ES for curbing excessive tail-risk-seeking behaviour. The natural question is what alternatives could work?

In the second part of the paper we introduce a possible solution. We consider the same optimization problem, but with risk constraints imposed by requiring that the expected value of a concave increasing function, u_R, applied to the payoff must be above some limit. We show that, under rather modest assumptions,

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[1]Reference omitted to preserve double blind

such constraints are effective, i.e. they decrease the excepted utility the trader can achieve. Hence imposing such constraints will have an economic impact upon the trader.

Since u_R has the form of a conventional utility function, we will refer to such constraints as expected utility constraints. We refer to u_R rather loosely as the risk manager's choice of utility function, but this terminology is not intended to imply that u_R represents the personal risk preferences of the risk manager. The risk manager should choose u_R to represent the preferences of whoever bears the losses between different possible payoff distributions. The risk manager should then choose the limit to reflect the maximum acceptable level of risk. Setting risk limits in this way is not standard practice, but our result does demonstrate that, in principle at least, it is possible to set effective risk limits. In practice one might try to implement such limits by choosing a one parameter family of utility functions, u_R, such as the family of exponential utilities. Two parameters would then be required to specify a risk constraint: a risk aversion level and a limit, which could be expressed as a cash equivalent value. Thus the number of parameters required in practice would be the same as the number of parameters required to specify an ES constraint.

We have focussed so far upon the economic impact on the trader. We should also consider the economic impact on whoever bears the cost of any losses. Let us assume that the loss-bearer's preferences are given by the conventional concave increasing utility function u_R applied to the investment returns. Taken together, our first and second results imply that the expected utility of the loss-bearer will be higher if they impose expected utility constraints rather than ES or VaR con-

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straints. Thus our results have economic consequences for the loss-bearer.

We now present a literature review of earlier related work. We keep this short due to space constraints.[2]

Utility optimization under risk measure constraints was considered in Basak and Shapiro (2001), which adopts a framework that is the same as we adopt in this paper, except that they use a standard concave increasing utility function where we use an S-shaped utility function. They show that in their model, in cases where large losses occur, even larger losses occur under VaR based risk management.

In Cuoco et al. (2008), it is shown that VaR constraints behave better when the portfolio VaR is re-evaluated dynamically by incorporating available conditioning information, as is done in practice. Again this is done under standard, rather than S-shaped, utility.

As there has been growing interest in the work of Kahneman and Tversky (Kahneman and Tversky, 1979; Tversky and Kahneman, 1992), and its application, there is now an extensive literature addressing non-concave objective functions. We refer, for example, to Benartzi and Thaler (1995), Barberis et al. (2001), Berkelaar et al. (2004), Gomes (2005), Carassus and Rásonyi (2015), Bernard and Ghossoub (2010), He and Zhou (2011a), and Rieger (2012). S-shaped utility has also been studied in more general contexts than finding optimal payoffs under risk constraints. For example, in Henderson (2012) the disposition effect in the presence of S-shaped utility is considered.

Further related research has appeared in a series of papers by Xunyu Zhou and

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[2]We have a more comprehensive literature review on the online preprint that we omit citing to preserve double blind

co-authors (Jin and Zhou, 2008; Zhou, 2010; He and Zhou, 2011b; He et al., 2015).[3] While we only consider S-shaped utility, these papers deal with prospect theory in full and include probability weighting in the analysis. They anticipate some of the techniques and proofs we consider here, and the notion of X-rearrangement in particular. Where we depart from these papers is in the economic aims of our research. We wish to investigate the effectiveness of commonly imposed risk constraints such as VaR and ES in constraining traders, whereas the existing research is interested in determining how a trader with S-shaped utility should behave in practice. As a result, the specific economic problems we consider of optimization under ES and utility constraints are different from those studied in (Jin and Zhou, 2008; Zhou, 2010; He and Zhou, 2011b; He et al., 2015).

Our contribution is to apply the theory of optimization with S-shaped utility functions to the study of ES and to thereby highlight a potential weakness of ES as a means of setting risk limits. We believe that risk managers, regulators and policy makers should be aware of this potential weakness since ES is the risk measure officially endorsed by the Basel committee. As such this paper should be viewed as a contribution to the literature on the merits and demerits of different risk measures. Let us briefly mention some aspects of this extensive literature. Artzner et al. (1999) proposed that risk measures should encourage diversification and this motivated their definition of coherent risk measures. VaR is not a coherent risk measure, and, partly as a result of this criticism, the Basel committee moved from endorsing VaR as a risk measure to endorsing ES. This line of research and its

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[3]again, in our online preprint we discuss such references more in detail and add others. We do not cite this preprint here to preserve double blind

history is discussed in detail in Föllmer and Schied (2011). Another consideration that has been debated is that it may be desirable for risk measures to be "elicitable", a statistical concept related to backtesting risk measures (Gneiting, 2011; Nolde and Ziegel, 2017). One further, less technical, consideration is how easy it is for a non-specialist to understand the meaning of any given risk figure. This point is discussed in, for example, Acerbi and Scandolo (2008). The paper Emmer et al. (2015) gives a fuller discussion of the pros and cons of different risk measures and the history of this field. They also attempt to evaluate what risk measures work best in practice. Our paper contributes to this literature by proposing that the effectiveness of risk limits in controlling excessive risk-seeking behaviour should also be considered when evaluating the pros and cons of different risk measurement methodologies.

The paper is organized as follows. In Section 2 we introduce S-shaped utility and define the notion of a tail-risk-seeking market player. In Section 3 we formally introduce the optimization problem that will be used in the paper and state a theorem which shows how the optimization problem can be solved by studying the relationship between the pricing measure $\mathbb{Q}$ and the physical measure $\mathbb{P}$. In Section 4 we state the main negative result of the paper that shows that ES is ineffective in curbing the excessive risk taking of tail-risk-seeking investors in markets such as the Black–Scholes options market. Section 5 states the main positive result of this paper, namely that limits based on expected utility constraints can be effective in curbing excessive risk taking. The proofs of all theorems are given in the appendix.
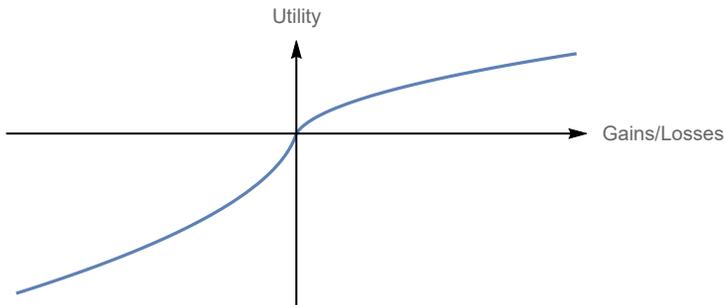
Figure 1: An S-Shaped utility function.

2. S-shaped utility and limited liability

Kahneman and Tversky (1979) observed that individuals appear to have preferences governed by an S-shaped utility function. By "S-shaped" utility Kahneman and Tversky mean the following properties.

(i) The utility is increasing.

(ii) The utility is strictly convex on the left.

(iii) The utility is strictly concave on the right.

(iv) The utility is non-differentiable at the origin.

(v) The utility is asymmetrical: negative events are considered worse than positive events are considered good.

A typical S-shaped utility function is shown in Figure 1. The prototypical example of S-shaped utility (see for example Föllmer and Schied (2011), Formula 2.9) is

$$u(x) = x^\gamma 1_{\{x \geq 0\}} - \lambda(-x)^\gamma 1_{\{x < 0\}}, \tag{1}$$

for a zero benchmark level, with $\lambda > 0$ and $0 < \gamma \le 1$.

It is generally agreed that an individual who is rational, loss-averse and risk-averse should have a utility function which is increasing and concave. Thus Kahneman and Tversky's result appears to give empirical evidence for the hypothesis that either individuals are not risk-averse or they do not behave rationally.

Alternatively, one might argue that the behaviour is due to failing to fully analyse the actual returns experienced by actors. For example, a trader who is not particularly conscientious may be interested in their pay packet and not in the performance of their portfolios. Thus the fact that such a trader may be willing to risk enormous losses is perfectly consistent with risk aversion: they personally only lose their job, and possibly their reputation, even if they bring down the bank they are working for. The utility of such traders should be calculated by applying a conventional concave increasing function to their pay packet. The net effect is that their utility can still be calculated by applying an S-shaped function u to the portfolio value. If one incorrectly interprets u as the trader's personal preferences over gains and losses, one is given a false impression that these traders are not risk-averse. Similarly, it is perfectly rational and risk-averse for a limited liability company to take enormous risks with other people's money.

Whether these S-shaped functions arise due to a lack of risk aversion, irrationality or limited liability, there is certainly good evidence that they are a useful tool for modelling real-world behaviour. A regulator or risk manager should certainly consider the possibility that they must regulate or manage actors who behave as though governed by S-shaped utility. For the sake of readability, in the rest of this paper we will use the term utility function for this S-shaped curve whether it

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arrives through risk preferences or limited liability.

Not all of the characteristics of S-shaped utility functions are important to us in this paper. We are primarily interested in the convexity on the left. Motivated by Kahneman and Tversky's original example (1), we introduce the following definition.

Definition 2.1. *An increasing function* $u : \mathbb{R} \longrightarrow \mathbb{R}$ *(to be thought of as a utility function) is said to be* "risk-seeking in the left tail" *if there exist constants* $N \leq 0$, $\eta \in (0, 1)$ *and* $c > 0$ *such that:*

$$u(x) > -c|x|^{\eta} \qquad \forall x \leq N. \tag{2}$$

Similarly u *is said to be* "risk-averse in the right tail" *if there exists* $N \geq 0$, $\eta \in (0, 1)$ *and* $c > 0$ *such that*

$$u(x) < c|x|^{\eta} \qquad \forall x \geq N. \tag{3}$$

The first definition will be helpful in establishing the main negative result of the paper, namely that ES constraints do not change the maximum expected S-shaped utility of the trader, who will thus not feel constrained by them. In the proof of Theorem 4.1 below, the key property will be the behaviour of the left tail of the utility function, which must be bounded from below by a curve growing like a power function with exponent smaller than one, and thus convex. In other words, the left tail of the utility function is not allowed to decrease too fast for negative losses, and decreases at smaller and smaller speeds as losses increase. Economically, this constrains the utility function to conform to a pattern that is

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reminiscent of "limited liability", where larger and larger losses are weighted less and less in relative terms. We need this pattern only from some point onward, and not on the whole negative axis, which explains the role of the constant N in our definition.

The standard pictures of "S-shaped" utility functions in the literature appear to have the above properties. Furthermore the S-shaped utility functions that arise due to a limited liability are bounded below, and so would certainly be risk-seeking in the left tail.

We give a formal definition of S-shaped for the purposes of this paper.

Definition 2.2. *A function u is said to be "S-shaped" if:*

1. *u is increasing;*

2. *$u(x) \leq 0$ for $x \leq 0$;*

3. *$u(x) \geq 0$ for $x \geq 0$;*

4. *for $x \geq 0$, $u(x)$ is concave;*

5. *u is risk-seeking in the left tail;*

6. *u is risk-averse in the right tail.*

As we hinted above, only the tail behaviour of the utility is needed in the proof of our results. Definition 2.2 thus allows more general shapes of utility function than that illustrated in Figure 1. For example, the trader utility for losses can become negative very fast in proximity to the level of negative wealth that would cause them to lose their job, but then decrease very slowly for losses

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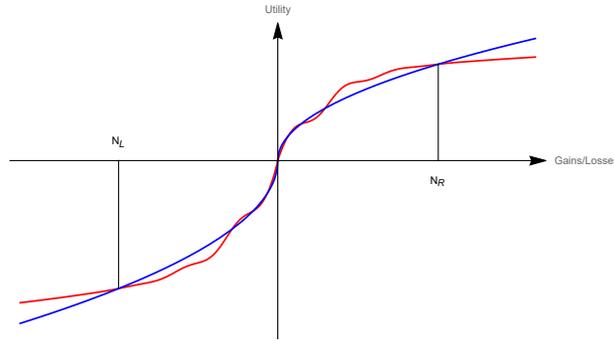
Figure 2: An example of S-shaped curve (red), whose tails are bounded by a classic S-shaped utility (blue), as in Definition 2.2

beyond this point. An example of S-shaped function satisfying Definition 2.2 is given in Figure 2.

Note that, in Kahneman and Tversky's studies of economic behaviour, the horizontal coordinate of the point of inflection of the utility curve is normally very significant. They found that the position of the inflection point depends upon whether the problem is framed in a manner that makes individuals focus upon the potential gains or upon the potential losses. It is for this reason that we have labelled the horizontal axis in Figure 1 as "gains/losses" rather than as "terminal wealth". Nevertheless, for the results given in this paper it is only the tail behaviour that is important, and we make no assumptions about the location of the point of inflection.

3. The problem and the law-invariant portfolio optimization setting

Our problem sees a trader who is trying to maximize their expected utility by designing a suitable payoff, while being subject to budget and risk constraints:

Find the payoff that maximizes   Expected(utility(payoff))   (4)

subject to   Budget constraint: $\text{Price}(\text{payoff}) \leq C$

Risk constraint: $\text{Risk measure}(\text{payoff}) \geq L.$

We now introduce a mathematical framework to formalize this problem. We present the technicalities that will allow us to formulate our portfolio optimization problem as a problem based only on probability distributions of the relevant quantities.

We will work both under the objective probability measure $\mathbb{P}$, to model risk measures, and under the pricing or risk neutral measure $\mathbb{Q}$, needed for prices in the budget constraint. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\frac{d\mathbb{Q}}{d\mathbb{P}}$ be a positive random variable with $\int_\Omega \frac{d\mathbb{Q}}{d\mathbb{P}} \, d\mathbb{P}(\omega) = 1$. The random variable $\frac{d\mathbb{Q}}{d\mathbb{P}}$ encodes the relationship between the two probability measures $\mathbb{P}$ and $\mathbb{Q}$, and is related to the market price of risk and risk premia. We will use this setting to represent a complete financial market as follows:

(i) We assume there is a fixed risk free interest rate, r, assumed to be a deterministic constant.

(ii) Given a payoff, f (a random variable), one can purchase a derivative security with payoff f at time T for the price given by the risk neutral expectation $\mathbb{E}^{\mathbb{Q}}$ of the discounted cash flow,

$$\mathbb{E}^{\mathbb{Q}}[e^{-rT}f] := \int_{\Omega} e^{-rT} f(\omega) \left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\omega) \right) \mathrm{d}\mathbb{P}, \tag{5}$$

assuming that this integral exists.

We note that the properties we require of $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ allow us to define a measure $\mathrm{d}\mathbb{Q} := \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \, \mathrm{d}\mathbb{P}$, justifying our notation.

In Appendix A, we list some technical conventions and details that will be needed in this section, but that are not of immediate interest for the reader. We refer to [arXiv SSRN preprint link removed for double blind] for full details.

We assume that the investor's preferences are encoded by a preference functional, v, that takes as argument probability distribution functions, F, of random variables. This means that an investor will prefer a security with payoff f over a security with payoff g if and only if $v(F_f) > v(F_g)$ (we are writing F_f for the cumulative distribution function of the random variable f). Thus the investor's preferences are law-invariant, in the sense that two payoffs, f and g, with the same law, $F_f = F_g$, will have the same investor preferences, $v(F_f) = v(F_g)$.

We assume that the investor has a fixed budget, C, so that they can only

purchase securities with payoff, f, costing less than C. Hence

$$-\infty \le \text{Price}(f) = \mathbb{E}^{\mathbb{Q}}[e^{-rT}f] = \int_{\Omega} e^{-rT} f(\omega) \left(\frac{d\mathbb{Q}}{d\mathbb{P}}(\omega) \right) d\mathbb{P}(\omega) \le C. \quad (6)$$

We assume that all the other trading constraints, namely the risk constraints, are law-invariant. For example: the investor may have to ensure that the minimum payoff is almost surely above a certain level; they may be operating under ES or VaR constraints; they may be operating under constraints based on a second utility function. We model the combined trading constraints using a set, $\mathcal{A}$, of admissible cumulative distribution functions and requiring that $F_f \in \mathcal{A}$. In other words, a payoff, f, will satisfy our risk (and possibly other) constraints when the cumulative distribution function, F_f, of f is in $\mathcal{A}$. This provides a unifying framework for VaR, ES and constraints based on expected utility.

In summary, our original problem (4) is now written as

$$\sup_{f \in L^0(\Omega, \mathbb{P})} v(F_f)$$

$$\text{subject to a price constraint} \quad -\infty \le \int_{\Omega} e^{-rT} f(\omega) \frac{d\mathbb{Q}}{d\mathbb{P}}(\omega) \, d\mathbb{P}(\omega) \le C$$

$$\text{and risk management constraints} \quad F_f \in \mathcal{A}.$$

$$(7)$$

In Appendix B, we prove the following theorem which shows we may assume f has a particular form when solving (7).

Theorem 3.1. *Suppose that the sample space Ω is non-atomic. Let $F_{\frac{d\mathbb{Q}}{d\mathbb{P}}}$*

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denote the cumulative distribution function of the random variable $\frac{d\mathbb{Q}}{d\mathbb{P}}$. Let F_f denote the cumulative distribution function of the random variable f. There exists a uniformly distributed random variable U such that:

(i) $\frac{d\mathbb{Q}}{d\mathbb{P}} = (1 - F_{\frac{d\mathbb{Q}}{d\mathbb{P}}})^{-1}(U)$ almost surely;

(ii) if f satisfies the price and risk management constraints of (7), then

$$\varphi(U) = F_f^{-1}(U)$$

also satisfies the constraints of (7), and is equal to f in distribution, and hence has the same objective value as f, namely $v(F_f) = v(F_f^{-1}(U))$.

What is remarkable in the above theorem is that we can use the same uniform random variable U for re-scaling both $\frac{d\mathbb{P}}{d\mathbb{Q}}$ and f. It is a general property of random variables that every random variable can be expressed as the inverse of its cumulative distribution function (or inverse of its survival function) applied to a uniform. In this sense it will always be true that

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = (1 - F_{\frac{d\mathbb{Q}}{d\mathbb{P}}})^{-1}(U_1) \quad \text{and} \quad f = F_f^{-1}(U_2)$$

for two suitable uniform random variables U_1 and U_2. What is more surprising is that we can actually put a single uniform U in both transformations and preserve the price (and risk) constraints of the optimization problem. The optimization problem with a single U will be much simpler since the optimization will run over a single random variable rather than two. Proving

our theorems in the next sections will be possible thanks to this simplification.

4. Portfolio optimization with S-shaped utility and ES constraints

Let u be a function which need not necessarily be either concave or increasing. Consider problem (7) where the objective, v, is the expected utility for u, and where we have a single ES constraint. For a definition of VaR and ES, we refer, for example, to McNeil et al. (2015) or Acerbi and Tasche (2002). Suppose our probability model is non-atomic, and let U be the random variable given in Theorem 3.1, and define $q = 1 - F_{\frac{d\mathbb{Q}}{d\mathbb{P}}}$, so q is a decreasing function.

By Theorem 3.1, under an ES risk constraint the optimization problem is equivalent to solving

$$\sup_{\varphi:[0,1]\to\mathbb{R},\varphi \text{ increasing}} \mathcal{F}(\varphi) := \int_0^1 u(\varphi(x))\,\mathrm{d}x \qquad (8)$$

$$\text{subject to the price constraint} \quad \int_0^1 \varphi(x)q(x)\,\mathrm{d}x \le C \qquad (9)$$

$$\text{and the ES constraint} \quad \frac{1}{p}\int_0^p \varphi(x)\,\mathrm{d}x \ge L. \qquad (10)$$

Moreover, this map preserves the objective values and the supremum. Note that the ES representation in the left hand side of (10) comes from (3.3) in Acerbi and Tasche (2002).

We are now ready to state the main negative result of this paper.
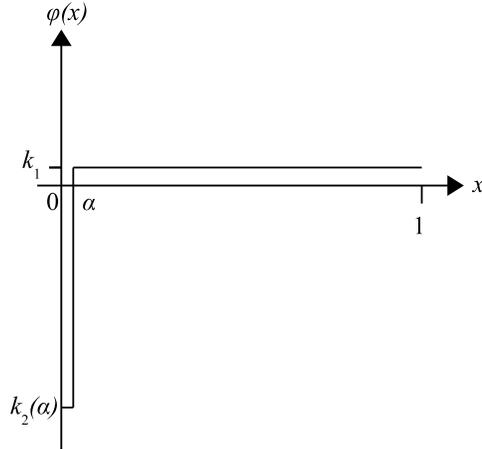
Figure 3: Type of payoff whose limit for $\alpha \to 0$ (implying $k_2(\alpha) \downarrow -\infty$) is used to find the supremum utility value

Theorem 4.1 (Irrelevance of ES constraints for tail-risk-seekers). *Suppose u is risk-seeking in the left tail and*

$$\lim_{x \to 0} q(x) = \infty$$

then the supremum value of the optimization problem (8) (9) *under the ES constraint* (10) *is the same as the supremum value of the unconstrained problem,* $\sup u$.

The proof is presented in Appendix C. We can see the type of payoff leading to the supremum in the proof. A sketch of the form of the payoff is given in Figure 3. In this figure we focus on an example with positive k_1 and negative $k_2 = k_2(\alpha) = k_1 + p(L - k_1)/\alpha$. If we assume L to be negative then for $\alpha \downarrow 0$ we have $k_2 \downarrow -\infty$ for any positive k_1. Essentially the payoff

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we use is a digital option with a very large (in absolute value) negative value k_2 in a very small area of size α near 0, and with a much smaller positive payoff k_1 in the large area $[\alpha, 1]$. This is the type of payoff that satisfies the budget and ES (or VaR) constraints while producing larger and larger expected S-shaped utility as $\alpha \downarrow 0$.

Example. *Consider an investor who wishes to optimize their utility at time T by investing in options on a single underlying, all options with maturity T. This investor will follow a buy and hold strategy, but their portfolio will be an option portfolio. We assume that put and call options can be bought and sold at a wide variety of strikes and so the market can be reasonably well approximated by a complete market. In this market any European derivative whose payoff is a function of the final stock price, otherwise known as simple contingent claim, can be bought or sold at a fixed price.*

We must choose a model for the price of these derivatives. As an example, we will consider European derivatives on a stock which follows the Black–Scholes–Merton model.

In Appendix D, we show that European derivatives at time T in the Black–Scholes–Merton market satisfy the assumptions of Theorem 4.1. We have, therefore, the following

Corollary 4.2 (Irrelevance of ES in reducing tail-risk-seeking behaviour in a Black–Scholes market)**.** *In a Black–Scholes market, the expected utility of an investor who is risk-seeking in the left tail and constrained only by ES or VaR constraints is limited only by the supremum of their utility function.*

Investors can achieve any desired expected utility below this supremum by trading in the bond and a digital option. ES constraints do not impact the supremum of the expected utility.

5. Portfolio optimization with limited liability and utility constraints

We have stated an important but negative result: tail-risk-seeking investors and investors who aim to maximize S-shaped utilities are not impacted by ES constraints. While this tells us that (in this context) ES is not effective in curbing excessive risk taking, what should one do instead? We have reached the point in the paper where we can make a positive proposal for an alternative approach.

Let us return to Problem (7). We suppose that the regulator is indifferent to the precise outcome if the portfolio payoff is positive, and so imposes a risk constraint on the expected utility of the negative part of the payoff. We specialise our analysis to the case where the investor is indifferent to the outcome if the portfolio payoff is negative. This means that we feel free to set the investor utility, u_I, to zero in the negative domain of the utility function. A possible cause for this preference is that the trader has de facto "limited liability" towards the bank, in the sense we explained in the introduction. We model this setup by choosing two utility functions u_R and u_I, representing the regulator and the investor's utility functions respectively. The two utility functions will turn out to be rather different, as the preferences of the trader and of the risk manager will be rather different.

We will now present a theorem that will allow us to solve Problem (7) with a new risk constraint based on u_R. We will need the following definitions for the terms arising in Problem (7). Define the preference functional, $v_I(F_f)$, as the expected u_I-utility, $v_I(F_f) = \mathbb{E}[u_I(f)] = \int_{-\infty}^{\infty} u_I(x) \, dF_f(x)$. Define the set, $\mathcal{A}$, of admissible distribution functions satisfying the risk constraint $\mathcal{A}_R = \left\{ F_f \mid \mathbb{E}[u_R(f)] = \int_{-\infty}^{\infty} u_R(x) \, dF_f(x) \geq L \right\}$ for a given negative level L. Define $q(x) = (1 - F_{\frac{d\mathbb{Q}}{d\mathbb{P}}})^{-1}(x)$.

Theorem 5.1. *Let u_R be a concave increasing function (associated with the risk constraint utility function) equal to 0 when $x \geq 0$, defining the admissible set $\mathcal{A} = \mathcal{A}_R$ above. Let u_I be an increasing function equal to 0 when $x \leq 0$ and concave in the region $x \geq 0$ (associated with the investor utility function).*

Given $p \in [0, 1]$, define $C_1(p) \in \mathbb{R} \cup \{-\infty\}$ to be the infimum of the optimization problem

$$
\inf_{f_1:[0,p] \to (-\infty,0), \text{ with } f_1 \text{ increasing}} \int_0^p f_1(x) q(x) dx
$$
$$
\text{subject to} \quad \int_0^p u_R(f_1(x)) \, dx \geq L. \tag{11}
$$

Define $V(p) \in \mathbb{R} \cup \{\infty\}$ to be the supremum of the optimization problem

$$
\sup_{f_2:[p,1] \to [0,\infty), \text{ with } f_2 \text{ increasing}} \int_p^1 u_I(f_2(x)) dx
$$
$$
\text{subject to} \quad \int_p^1 f_2(x) q(x) dx \leq C_2(p) := e^{rT} C - C_1(p). \tag{12}
$$

The supremum of the problem (7) with $\mathcal{A} = \mathcal{A}_R$ and $v = v_I$ is equal to

$\sup_{p \in [0,1]} V(p)$.

Remark. *The value of the theorem comes from the fact that the problems (11) and (12) are easy to solve, see Lemma F.1 below. One may then compute $\sup_{p \in [0,1]} V(p)$ by line search. Moreover, it is simple to obtain an explicit solution of (7) given solutions to each of these simpler problems. The risk constraint will typically be binding, unlike the case of ES constraints.*

Remark. *Although we have specialised to the case of limited liability, note that the strategies pursued by an investor with limited liability will be at least as aggressive as those pursued by an investor with a more general S-shaped utility. Thus if we can find bounds for an investor with limited liability, we will obtain bounds for more general S-shaped utilities. Finding explicit solutions to the problem (7) for general S-shaped utilities would seem a rather more difficult problem.*

The proof of Theorem 5.1 is given in Appendix E.

The case when the supremum of the optimization problem (12) is equal to the supremum of the investor's utility function u_I is rather uninteresting as the risk constraints clearly will play no role. This motivates the following definition.

Definition 5.2. *In a given market, an investor with utility function u_I is said to be* difficult to satisfy *if the supremum of the optimization problem (12) is less then the supremum of their utility function for any finite cost constraint C_2 and any $p \in (0,1)$.*

We now compute the solution of the problem in Theorem 5.1 in a specific case where both the investor and the risk management u_I and u_R are power functions with exponents smaller (S-shaped) and larger (risk aversion) than one, respectively. The significance of this computation is that it will allow us to immediately write down an upper bound on the solution of the problem in 5.1 for many financially interesting cases.

Theorem 5.3. *Let $\gamma_R \in (1, \infty)$ be given. Let $u_R(x) = -(-x)^{\gamma_R} 1_{\{x \leq 0\}} + 0 \times 1_{\{x > 0\}}$.*

Suppose that we wish to solve the optimization problem of Theorem 5.1 and that u_I is such that the investor is difficult to satisfy. The risk constraint in Theorem 5.1 is binding if and only if the expectation $\mathbb{E}_\mathbb{P} \left(\frac{d\mathbb{Q}}{d\mathbb{P}}^{\frac{\gamma_R}{\gamma_R - 1}} \right)$ is finite.

If the investor's utility function is given by $u_I(x) = x^{\gamma_I} 1_{\{x \geq 0\}} + 0 \times 1_{\{x < 0\}}$ for $\gamma_I \in (0, 1)$, then the investor is difficult to satisfy if the expectation $\mathbb{E}_\mathbb{P} \left(\frac{d\mathbb{Q}}{d\mathbb{P}}^{\frac{\gamma_I}{\gamma_I - 1}} \right)$ is finite.

This theorem is proved in Appendix F.

We now summarize the key findings of the paper in a single theorem. Using power functions for u_I and u_R, we will show that if the investor has an S-shaped u_I, then one can build a sequence of portfolios whose expected u_I grow larger and larger (to infinity) while ES constraints remain satisfied. For any such sequence of portfolios, the expected u_R will meanwhile grow increasingly negative (to minus infinity). Thus to achieve the large values of expected u_I that are possible, the trader must take excessive risks as measured by expected u_R.

Theorem 5.4. *Suppose that* $(\Omega, \mathcal{F}, \mathbb{P})$ *and* $\frac{d\mathbb{Q}}{d\mathbb{P}}$ *define a complete market. Define*

$$e(\gamma) := \mathbb{E}_{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}^{\frac{\gamma}{\gamma-1}}\right). \tag{13}$$

An investor with S-shaped utility function, u_I, *who is subject only to ES constraints can find a sequence of portfolios satisfying these constraints whose expected* u_I-*utility tends to infinity. If, in addition, the investor is difficult to satisfy, and* u_R *is the function* $u_R(x) = -(-x)^{\gamma_R}1_{\{x\leq 0\}} + 0 \times 1_{\{x>0\}}$ *with* $\gamma_R > 1$ *and* $e(\gamma_R)$ *finite, then any sequence of portfolios whose expected* u_I-*utility tends to infinity will have expected* u_R *utility tending to* $-\infty$.

Proof. The result follows from Theorem 5.1, Theorem 5.3 and our formal definition of S-shaped utility. □

Remark. *The conditions that* $e(\gamma_R)$ *is finite and that the investor is difficult to satisfy are always satisfied for the market of derivatives on the Black–Scholes–Merton market when the market price of risk is non-zero and the investor utility function* u_I *is not bounded above. This is proved in Appendix D.1.*

A comment on the choice of u_R is in order. If the risk manager is to choose a u_R, how should they proceed? Suppose that we agree, for practical reasons, to limit ourselves to a parametric form such as the power function above. One would have to find a way to select the exponent γ_R to be used in the risk management constraint. This could be based on trial and error on some test portfolios, and on analyzing how constraints based on different

exponents would have fared through history. The parameter γ_R would have the clear interpretation of expressing how we weight larger and larger losses in our risk constraint. Small γ_R would mean less weight, large γ_R more weight. The choice of γ_R may seem partly arbitrary, however even in measures such as VaR and ES the choice of the confidence level is somewhat arbitrary before one knows the tail structure of the portfolio loss distribution. This is not to say that the choice of γ_R would be straightforward, as this would be a matter for further research, but it is definitely plausible that a calibration of γ_R based on historical simulation on some key portfolios may be possible.

6. Conclusions

We have shown that in typical complete markets with non-zero market price of risk, ES constraints do not affect the supremum of the investor utility that can be achieved by an investor with S-shaped utility, u_I. By contrast, replacing the ES constraints with even very weak expected utility constraints for a concave increasing limiting utility function can reduce the supremum that can be achieved. In these circumstances, if a risk manager with such a concave increasing utility function, u_R, only imposes ES constraints, they should expect that tail-risk-seeking investors and limited liability investors will choose investment strategies with infinitely bad u_R-utilities. These findings were stated in full detail in Theorem 5.4.

We believe that this shows that VaR or ES constraints alone are insufficient to constrain the behaviour of tail-risk-seeking investors and limited

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liability investors, and that concave utility risk constraints could be used instead.

7. Competing interests

We have no competing interests.

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Appendix A Technical assumptions and details for Section 3

In convex analysis it is often convenient to allow infinite values in calculations (see Rockafellar (2015)). We will use the following conventions. Let us write f^+ and f^- for the positive and negative parts of a measurable function f. Suppose that $\int_\Omega f^+(\omega)\,d\mathbb{P}$ is finite but $\int_\Omega f^-(\omega)\,d\mathbb{P}$ is not finite, then we will write $\int_\Omega f(\omega)\,d\mathbb{P} = -\infty$. We can similarly define what it means for an integral to equal $+\infty$.

We will be considering investment under cost constraints as in Eq. (6). In our model we will assume that it is possible to purchase a derivative with payoff $f(\omega)$ whose price as given in (6) is $-\infty$. At an intuitive level, we are simply saying that one can always purchase an asset if one is willing to overpay. Assets where the cost is $+\infty$, or where the price is undefined, cannot be purchased.

Appendix B Proof of Theorem 3.1

First let F_f^{-1} denote generalized inverse of the cumulative distribution function F_f defined by $F_f^{-1}(p) := \inf\{x : F_f(x) \geq p\}$. We similarly write $(1 - F_f)^{-1}$ for the generalized inverse of complementary cumulative distribution function which is defined by $(1 - F_f)^{-1}(p) := \inf\{x \mid 1 - F_f(x) \leq p\}$.

To prove Theorem 3.1, we define the notion of X-rearrangement.

Definition B.1. *Given random variables $X, f \in L^0(\Omega, \mathbb{R})$ with X having a*

continuous distribution we define the X-rearrangement of f, denoted f^X by:

$$f^X(\omega) = F_f^{-1}(\mathbb{P}(X \le X(\omega))) = F_f^{-1}(F_X(X(\omega))).$$

Lemma B.2. *The X-rearrangement has the following properties:*

  (i) If X has a continuous probability distribution then f^X is equal to f in distribution.

  (ii) If $k \in \mathbb{R}$ then $(\max\{f, k\})^X = \max\{f^X, k\}$ and $(\min\{f, k\})^X = \min\{f^X, k\}$

  (iii) $f^X = (f^+)^X + (f^-)^X$.

  (iv) $X^X = X$ almost surely.

  (v) If $g(\omega) = G(X(\omega))$ with G increasing and if X has a continuous probability distribution then $g^X = g$ almost surely.

Proof of (i). We recall that: for any distribution function F with generalized inverse F^{-1}, $F^{-1}(p) \le x$ if and only if $p \le F(x)$; $F_X \circ F_X^{-1} = \mathrm{id}$ if X has a

continuous distribution. Hence if X has a continuous distribution:

$$F_{f^X}(y) = \mathbb{P}(f^X(\omega) \leq y)$$

$$= \mathbb{P}(F_f^{-1}(\mathbb{P}(X \leq X(\omega)) \leq y)$$

$$= \mathbb{P}(\mathbb{P}(X \leq X(\omega)) \leq F_f(y))$$

$$= \mathbb{P}(F_X(X(\omega)) \leq F_f(y))$$

$$= \mathbb{P}(X(\omega) \leq F_X^{-1}F_f(y))$$

$$= F_X(F_X^{-1}(F_f(y)))$$

$$= F_f(y).$$

$\square$

Proof of (ii). The result follows from the definition of f^X and the following identities:

$$F_{\max\{f,k\}}^{-1}(t) = \inf\{z \in \mathbb{R} : \mathbb{P}(\max\{f, k\} \leq z) \geq t\}$$

$$= \inf\{z \in \mathbb{R} : \mathbb{P}(f \leq z \text{ and } k \leq z) \geq t\}$$

$$= \max\{\inf\{z \in \mathbb{R} : \mathbb{P}(f \leq z) \geq t\}, k\}$$

$$= \max\{F_f^{-1}(t), k\}.$$

$$F_{\min\{f,k\}}^{-1}(p) = \inf\{z \in \mathbb{R} : \mathbb{P}(\min\{f, k\} \leq z) \geq p\}$$

$$= \inf\{z \in \mathbb{R} : \mathbb{P}(f \leq z \text{ or } k \leq z) \geq p\}$$

$$= \min\{\inf\{z \in \mathbb{R} : \mathbb{P}(f \leq z) \geq p\}, k\}$$

$$= \min\{F_f^{-1}(p), k\}.$$

$$\square$$

Proof of (iii). We use (ii) to derive the following identity

$$F_f^{-1}(p) = (F_f^{-1})^+(p) + (F_f^{-1})^-(p)$$

$$= \max\{F_f^{-1}(p), 0\} + \min\{F_f^{-1}(p), 0\}$$

$$= F_{\max\{f,0\}}^{-1}(p) + F_{\min\{f,0\}}^{-1}(p)$$

$$= F_{f^+}^{-1}(t) + F_{f^-}^{-1}.$$

The result now follows from the definition of f^X. $\qquad\square$

Proof of (iv). We wish to prove that the set

$$A = \{W \in \Omega : F_X^{-1} F^X X(W) \neq X(W)\}$$

is null.

We recall that

$$F_X^{-1} F_X(x) \leq x \text{ and } F_X F_X^{-1}(p) \geq p \tag{14}$$

for all $x \in \mathbb{R}$ and $p \in [0, 1]$. We note that since F_X is increasing, this first inequality implies that

$$F_X(F_X^{-1} F_X(x)) \leq F_X(x)$$

and the second implies

$$F_X F_X^{-1}(F_X(x)) \geq F_X(x).$$

We deduce

$$F_X F_X^{-1} F_X(x) = F_X(x). \tag{15}$$

Suppose that F_X^{-1} is continuous at $F_X X(W) \in [0,1]$ then

$$F_X^{-1} F_X X(W) = \inf\{F_X^{-1}(q) \mid q > F_X X(W)\}$$

$$= \inf\{\inf\{x \mid F_X(x) \geq q\} \mid q > F_X X(W)\}$$

$$= \inf\{x \mid F_X(x) > F_X X(W)\}$$

$$\geq X(W).$$

But by (14), $F_X^{-1} F_X X(W) \leq X(W)$ for all $W \in \Omega$. So if F_X^{-1} is continuous at $F_X X(W)$ then $F_X^{-1} F_X X(W) = X(W)$, so $W \notin A$. Let P denote the set of discontinuities of F_X^{-1}. We have shown:

$$A \subseteq \bigcup_{p \in P} \{\omega \mid F_X^{-1} F_X X(\omega) \neq X(\omega) \text{ and } F_X X(\omega) = p\}.$$

Since F_X^{-1} is monotone, P is countable. Thus we can find a countable set

$\{\omega_1, \omega_2, \ldots\}$ of elements of Ω such that

$$
\begin{aligned}
A \subseteq & \bigcup_{\omega_i} \{\omega \mid F_X^{-1} F_X X(\omega) \neq X(\omega) \text{ and } F_X X(\omega) = F_X X(\omega_i)\} \\
= & \bigcup_{\omega_i} \{\omega \mid F_X^{-1} F_X X(\omega_i) \neq X(\omega) \text{ and } F_X X(\omega) = F_X X(\omega_i)\} \quad (16) \\
= & \bigcup_{\omega_i} A_i
\end{aligned}
$$

where

$$
\begin{aligned}
A_i := & \{\omega \mid F_X^{-1} F_X X(\omega_i) \neq X(\omega) \text{ and } F_X X(\omega) = F_X X(\omega_i)\} \\
= & \{\omega \mid F_X X(\omega) = F_X X(\omega_i)\} \setminus \{\omega \mid F_X^{-1} F_X X(\omega_i) = X(\omega)\}. \quad (17)
\end{aligned}
$$

We now note that

$$
\{\omega \mid F_X X(\omega) = F_X X(\omega_i)\} =
$$

$$
\{\omega \mid F_X X(\omega) \leq F_X X(\omega_i)\} \setminus \{\omega \mid F_X X(\omega) < F_X X(\omega_i)\} \quad (18)
$$

and

$$
\{\omega \mid F_X^{-1} F_X X(\omega_i) = X(\omega)\} =
$$

$$
\{\omega \mid X(\omega) \leq F_X^{-1} F_X X(\omega_i)\} \setminus \{\omega \mid X(\omega) < F_X^{-1} F_X X(\omega_i)\}. \quad (19)
$$

Now

$$X(\omega) < F_X^{-1} F_X X(\omega_i) \implies X(\omega) < \inf\{x : F_X(x) \geq F_X X(\omega_i)\}$$
$$\implies F_X X(\omega) < F_X X(\omega_i). \tag{20}$$

Conversely we can use (15) to see that

$$F_X X(\omega) < F_x X(\omega_i) \implies F_X X(\omega) < F_X F_X^{-1} F_X X(\omega_i)$$
$$\implies X(\omega) < F_X^{-1} F_X X(\omega_i) \tag{21}$$

since F_X is increasing. Together (20) and (21) imply

$$\{\omega \mid F_X X(\omega) < F_X X(\omega_i)\} = \{\omega \mid X(\omega) < F_X^{-1} F_X X(\omega_i)\}. \tag{22}$$

We use (18), (19) and (22) to rewrite (17) as

$$A_i = \{\omega \mid F_X X(\omega) \leq F_X X(\omega_i)\} \setminus \{\omega \mid X(\omega) \leq F_X^{-1} F_X X(\omega_i)\}. \tag{23}$$

Let $L_i = \{\omega \mid X(\omega) \leq F_X^{-1} F_X X(\omega_i)\}$. We use (15) to compute that

$$\mathbb{P}(\omega \in L_i) = F_X F_X^{-1} F_X X(\omega_i) = F_X X(\omega_i). \tag{24}$$

Let $R_i = \{\omega \in \Omega \mid F_X X(\omega) = F_X X(\omega_i)\}$. We know R_i is non empty since it contains ω_i. Therefore we may choose a sequence v_i^j in R_i such that $X(v_i^j)$ is increasing and has limit equal to $\sup_{v \in R_i} X(v)$. Moreover if this supremum

is obtained we may assume that the sequence $X(v_i^j)$ obtains its limit.

$$
\begin{aligned}
\{\omega \mid F_X X(\omega) \leq F_X X(\omega_i)\} &= \bigcup_x \{\omega \mid X(\omega) \leq x \text{ and } F_X(x) \leq F_X X(\omega_j)\} \\
&= \bigcup_j \{\omega \mid X(\omega) \leq X(v_i^j) \text{ and } F_X X(v_i^j) \leq F_X X(\omega_j)\} \\
&= \bigcup_j \{\omega \mid X(\omega) \leq X(v_i^j)\} \\
&= \bigcup_j V_i^j
\end{aligned}
$$

$$(25)$$

Where $V_i^j := \{\omega \mid X(\omega) \leq X(v_i^j)\}$. We now compute that

$$
\mathbb{P}(\omega \in V_i^j) = F_X X(v_i^j) = F_X X(\omega_i), \tag{26}
$$

since $v_i^j \in R_i$.

Since $F_X^{-1} F_X X(\omega_i) = \inf\{X(\omega) \mid \omega \in R_i\} \leq X(v_i^j)$ we see that $L_i \subseteq V_i^j$. Hence $\mathbb{P}(\omega \in V_i^j \setminus L_i) = \mathbb{P}(\omega \in V_i^j) - \mathbb{P}(\omega \in L_i) = 0$, using (26) and (24). By (16), (23) and (25) we have

$$
A \subseteq \bigcup_i \bigcup_j (V_i^j \setminus L_i).
$$

So A_i is a countable union of null sets and hence is null. $\qquad\square$

Proof of (v). We define a generalized inverse for G by

$$G^{-1}(y) = \sup\{x \in \mathbb{R} \mid G(x) \leq y\}.$$

We define a function $\tilde{G}$ by

$$\tilde{G}(x) = \inf\{G(x') \mid x' \geq x\}.$$

We see that $\tilde{G}(x) = G(x)$ except possibly at the discontinuities of G.

We note that

$$
\begin{aligned}
\tilde{G}(x) \leq y &\iff \inf\{G(x') \mid x' \geq x\} \leq y \\
&\iff \exists x' \text{ with } G(x') \leq y \text{ and } x' \geq x \\
&\iff x \leq \sup\{x' \mid G(x') \leq y\} \\
&\iff x \leq G^{-1}(y).
\end{aligned}
\tag{27}
$$

We define $\tilde{g}(\omega) = \tilde{G}X^X(\omega)$. G is monotone so only has a countable number of discontinuities. Let D denote the set of discontinuities of G. Then $\tilde{G}(x) = G(x)$ unless $x \in D$. So the set of ω for which $\tilde{g}(\omega) \neq g(\omega)$ is contained in $X^{-1}(D) \cup A$ where A is the null set defined in (iv). By the continuity of the distribution of X, $X^{-1}(x)$ is a null set for all x. Hence $\tilde{g} = g$ almost surely.

We now wish to calculate $g_X(W)$ for $W \in \Omega$. In the calculation below, W should be thought of as fixed and ω should be thought of as a random

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scenario. So, for example $\mathbb{P}(X(\omega) \leq X(W)) = F_X(X(W))$.

$$
\begin{aligned}
g^X(W) &= F_g^{-1}(\mathbb{P}(X(\omega) \leq X(W))) \\
&= F_{\tilde{g}}^{-1}(\mathbb{P}(X(\omega) \leq X(W))) \\
&= \inf\{x \mid F_{\tilde{g}}(x) \geq \mathbb{P}(X(\omega) \leq X(W))\} \\
&= \inf\{x \mid F_{\tilde{G}X}(x) \geq \mathbb{P}(X(\omega) \leq X(W))\} \\
&= \inf\{x \mid \mathbb{P}(\tilde{G}X(\omega) \leq x) \geq \mathbb{P}(X(\omega) \leq X(W))\} \\
&= \inf\{x \mid \mathbb{P}(X(\omega) \leq G^{-1}x) \geq \mathbb{P}(X(\omega) \leq X(W))\} \qquad \text{(by (27))} \\
&= \inf\{x \mid F_X(G^{-1}x) \geq F_X(X(W))\} \\
&= \inf\{x \mid G^{-1}x \geq X^X(W)\} \\
&= \inf\{x \mid X^X(W) \leq G^{-1}x\} \qquad \text{(by (27))} \\
&= \inf\{x \mid \tilde{G}(X^X(W)) \leq x\} \\
&= \tilde{G}X^X(W) \\
&= \tilde{g}(W).
\end{aligned}
$$

Hence $g^X = g$ almost surely. $\qquad\square$

Lemma B.3. *If $f, g \in L^0(\Omega; \mathbb{R})$ and:*

(i) $f(\omega) \geq k$ for some $k \in \mathbb{R}$;

(ii) $g \geq 0$;

(iii) $\int_\Omega g \, d\mathbb{P} < \infty$;

(iv) X has a continuous distribution;

then

$$\int_\Omega fg \, \mathrm{d}\mathbb{P} \le \int_\Omega f^X g^X \, \mathrm{d}\mathbb{P} \le \infty.$$

Proof. (Note: this proof is modelled on the proof of the Hardy–Littlewood inequality for "symmetric decreasing rearrangements".)

Since $f \ge k$ we have the "layer-cake" representation of f

$$f(\omega) = k + \int_0^\infty \mathbb{1}_{L(f,x+k)}(\omega) \, \mathrm{d}x$$

where

$$L(f, t) := \{\omega \mid f(\omega) > t\}.$$

We also have

$$g(\omega) = \int_0^\infty \mathbb{1}_{L(g,x)}(\omega) \, \mathrm{d}x.$$

We note that for any random variable h

$$\begin{aligned}
L(h^X, x) &= \{\omega \mid h^X(\omega) > x\} \\
&= \{\omega \mid F_h^{-1}(\mathbb{P}(X \le X(\omega))) > x\} \\
&= \{\omega \mid \mathbb{P}(X \le X(\omega)) > F_h(x)\}.
\end{aligned}$$

Hence for any h_1, h_2, x_1, x_2 either

$$L(h_1^X, x_1) \subseteq L(h_2^X, x_2) \quad \text{or} \quad L(h_2^X, x_2) \subseteq L(h_1^X, x_1). \tag{28}$$

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We also note that

$$\mathbb{P}(L(h,x)) = \mathbb{P}(h(\omega) > x) = 1 - F_h(x).$$

In particular $\mathbb{P}(L(h,x))$ only depends upon the distribution of h and hence $\mathbb{P}(L(h^X,x)) = \mathbb{P}(L(h,x))$ by Lemma B.2.

We now compute:

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}}(\mathbb{1}_L(f^X, x+k)(\omega)\mathbb{1}_L(g^X, y)(\omega)) &= \mathbb{P}(L(f^X, (x+k)) \cap L(g^X, y)) \\
&= \min\{\mathbb{P}(L(f^X, (x+k))), \mathbb{P}(L(g^X, y))\} \quad \text{by (28)} \\
&= \min\{\mathbb{P}(L(f, (x+k))), \mathbb{P}(L(g, y))\} \\
&\geq \mathbb{P}(L(f, (x+k)) \cap L(g, y)) \\
&= \mathbb{E}^{\mathbb{P}}(\mathbb{1}_L(f, x+k)(\omega)\mathbb{1}_L(g, y)(\omega)).
\end{aligned}
$$

$$(29)$$

Using the fact that f is bounded below and $\int_\Omega g\,\mathrm{d}\mathbb{P} < \infty$ we deduce that

$$\int_\Omega (fg)^-\,\mathrm{d}\mathbb{P} > -\infty.$$

By Lemma B.2, f^X is also bounded below and $g = g^X$ in distribution so $\int_\Omega g^X \mathrm{d}\mathbb{P} < \infty$. Hence

$$\int_\Omega (f^X g^X)^-\,\mathrm{d}\mathbb{P} > -\infty.$$

Therefore we may use the layer-cake representations of f, g, f^X and g^X together with Fubini's theorem and (29) to compute:

$$
\begin{aligned}
\int_\Omega fg \, \mathrm{d}\mathbb{P} &= k \int_\Omega g \, \mathrm{d}\mathbb{P} + \int_\Omega \int_\mathbb{R} \int_\mathbb{R} \mathbb{1}_L(f, x+k)(\omega)\mathbb{1}_L(g, y)(\omega)\mathrm{d}x \, \mathrm{d}y \, \mathrm{d}\mathbb{P} \\
&= k \int_\Omega g \, \mathrm{d}\mathbb{P} + \int_\mathbb{R} \int_\mathbb{R} \int_\Omega \mathbb{1}_L(f, x+k)(\omega)\mathbb{1}_L(g, y)(\omega) \, \mathrm{d}\mathbb{P} \, \mathrm{d}x \, \mathrm{d}y \\
&\leq k \int_\Omega g \, \mathrm{d}\mathbb{P} + \int_\mathbb{R} \int_\mathbb{R} \int_\Omega \mathbb{1}_L(f^X, x+k)(\omega)\mathbb{1}_L(g^X, y)(\omega) \, \mathrm{d}\mathbb{P} \, \mathrm{d}x \, \mathrm{d}y \\
&= \int_\Omega f^X g^X \, \mathrm{d}\mathbb{P}.
\end{aligned}
$$

$\square$

Lemma B.4. *If $f, g \in L^0(\Omega; \mathbb{R})$ and:*

(i) $\int fg \, \mathrm{d}\mathbb{P} > -\infty$;

(ii) $g \geq 0$;

(iii) $\int_\Omega g \, \mathrm{d}\mathbb{P}$ exists;

(iv) X has a continuous distribution;

then

$$
-\infty < \int_\Omega fg \, \mathrm{d}\mathbb{P} \leq \int_\Omega f^X g^X \, \mathrm{d}\mathbb{P} \leq \infty.
$$

Proof. In this proof, given a real k and random variable f, we will write f_k as an abbreviation for the random variable $\max\{f(\omega), k\}$. Lemma B.2 tells us $(f_k)^X = (f^X)_k$ so we may write f_k^X without ambiguity.

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We know $\int (fg)^- \mathrm{d}\mathbb{P} > -\infty$. Since $g \geq 0$, $(fg)^- = f^- g$, hence $\int f^- g \, \mathrm{d}\mathbb{P} > -\infty$. Also since $g \geq 0$ we have for any $k \in \mathbb{R}$

$$-\infty < \int_\Omega f^- g \, \mathrm{d}\mathbb{P} \leq \int_\Omega f_k^- g \, \mathrm{d}\mathbb{P}$$

By Lemma B.3 we then have

$$-\infty < \int_\Omega f^- g \, \mathrm{d}\mathbb{P} \leq \int_\Omega f_k^- g \, \mathrm{d}\mathbb{P} \leq \int_\Omega (f^-)_k^X g^X \, \mathrm{d}\mathbb{P} \quad \text{for all } k.$$

As $k \to -\infty$, $f_k^-(\omega) \downarrow f^-(\omega)$ and $(f^-)_k^X(\omega) \downarrow (f^-)^X(\omega)$ for all ω. So by the Montone Convergence Theorem

$$-\infty < \int_\Omega f^- g \, \mathrm{d}\mathbb{P} \leq \int_\Omega (f^-)^X g^X \, \mathrm{d}\mathbb{P}.$$

Lemma B.3 also tells us that

$$0 \leq \int_\Omega f^+ g \, \mathrm{d}\mathbb{P} \leq \int_\Omega (f^+)^X g^X \, \mathrm{d}\mathbb{P}.$$

Hence

$$\begin{aligned}
-\infty < \int_\Omega fg \, \mathrm{d}\mathbb{P} &= \int_\Omega (f^+ + f^-) g \, \mathrm{d}\mathbb{P} \\
&\leq \int_\Omega ((f^+)^X + (f^-)^X) g^X \, \mathrm{d}\mathbb{P} \\
&\leq \int_\Omega (f^+ + f^-)^X g^X \, \mathrm{d}\mathbb{P} \\
&= \int_\Omega f^X g^X \mathrm{d}P \leq \infty.
\end{aligned}$$

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□

Lemma B.5. *If $(\Omega, \mathcal{F}, \mathbb{P})$ is non-atomic and $Q \in L^0(\Omega; \mathbb{R})$ is a random variable on Ω, then there exists a uniform random variable $X \in L^0(\Omega; \mathbb{R})$ such that $Q(\omega) = F_Q^{-1}(X(\omega))$.*

Proof. Sierpiński's theorem on non-atomic measures tells us that for all $0 \leq \alpha \leq 1$ there is a measurable set $E \subseteq \Omega$ of measure α (Sierpiński, 1922).

One deduces that there is a uniformly distributed random variable U on Ω. To see this, first partition Ω into two subsets of measure $\frac{1}{2}$, we will call this partition Level 1. We define a random variable X_1 which is equal to $\frac{1}{2}$ on the first subset of Level 1 and equal to 1 on the second subset of Level 1. Inductively we partition each subset of Level n into two equally sized subsets and define X_n by $X_{n-1} - \frac{1}{2^n}$ on the first subsets and X_{n-1} on the second subsets. The distribution function of X_n will be a step function from 0 to 1 with 2^n uniform steps. By construction, $X^n(\omega)$ is decreasing for each $\omega \in \Omega$. Hence X^n converges pointwise, hence almost surely, hence in distribution to some random variable X. Thus X must be uniformly distributed.

At each point $x \in \mathbb{R}$ where there is a discontinuity of F_Q consider the set $\Omega_x = F_Q^{-1}(x)$. This set has non-zero measure $F_Q(x) - F_Q^-(x)$ where $F_Q^-(x)$ is the left limit of the distribution function at x. Note that $F_Q(x) - F_Q^-(x) = \mathbb{P}(Q = x)$. Hence by the above we can find a measurable function U_x on Ω_x

taking values uniformly between 0 and 1. Define a random variable X by

$$X(\omega) = \begin{cases} F_Q(Q(\omega)) & \text{if } F_Q \text{ is continuous at } Q(\omega) \\ F_Q(x) - U_x(\omega)\mathbb{P}(Q = x) & \text{if } F_Q \text{ is discontinuous at } x = Q(\omega). \end{cases}$$

Clearly $Q(\omega) = F_Q^{-1}(X(\omega))$. We must show that X is uniformly distributed, i.e. that $P(X \leq p) = p$ for all $p \in [0, 1]$.

Given $p \in [0, 1]$ define $p^- = \sup(\mathrm{Im}F_Q \cap (-\infty, p])$ and $p^+ = \inf(\mathrm{Im}F_Q \cap [p, \infty))$. We partition Ω into three sets A, B and C defined by

$$A = \{\omega \mid F_Q(Q(\omega)) \leq p^-\}$$
$$B = \{\omega \mid p^- < F_Q(Q(\omega)) \leq p^+\}$$
$$C = \{\omega \mid p^+ < F_Q(Q(\omega))\}.$$

For all ω, $X(\omega) \leq F_Q(Q(\omega))$. So if $\omega \in A$, $X(\omega) \leq F_Q(Q(\omega)) \leq p^- \leq p$. Hence

$$\mathbb{P}(X \leq p \mid A) = 1. \tag{30}$$

If $\omega \in B$ then $X(\omega) = p^+ - U_x(\omega)(p^+ - p^-)$. So

$$\mathbb{P}(X \leq p \mid B) = \frac{p - p^-}{p^+ - p^-}. \tag{31}$$

For all ω, $X(\omega) \geq F_Q^-(Q(\omega))$. Since F_Q^- is a left limit, $F_Q^-(Q(\omega)) \geq F_Q(x)$

if $x \leq Q(\omega)$. If $\omega \in C$ then $F_Q^{-1}(p^+) < Q(\omega)$. We deduce that if $\omega \in C$ then

$$X(\omega) \geq F_Q^-(Q(\omega)) \geq F_Q(F_Q^{-1}(p^+)) = p^+ \geq p.$$

We deduce that

$$\mathbb{P}(X \leq p \mid C) = \mathbb{P}(X = p \mid C).$$

and moreover this probability is equal to 0 unless $p^+ = p$. We may assume that each U_x takes values in $(0, 1)$ so that $X(\omega)$ never equals p^+ unless we have that $p^+ = p^-$ and F is continuous at $x = F^{-1}p^+$. But in this case we find that $\mathbb{P}(x = p^+) = \mathbb{P}(Q = x) = \mathbb{P}(Q \leq x) - \mathbb{P}(Q < x) = F_Q(x) - F_Q^-(x) = 0$. So $\mathbb{P}(x = p^+) = 0$ and hence

$$\mathbb{P}(X(\omega) \leq p \mid C) = 0. \tag{32}$$

Since $p^\pm \in \mathrm{Im}F_Q$ we compute that

$$\mathbb{P}(F_Q(Q(\omega)) \leq p^\pm) = \mathbb{P}(Q(\omega) \leq F_Q^{-1}(p^\pm)) = F_Q(F_Q^{-1}(p^\pm))) = p^\pm.$$

It follows that

$$\mathbb{P}(A) = p^-, \ P(A \cup B) = p^+, \ \text{hence } P(B) = p^+ - p^-. \tag{33}$$

Since A, B and C give a partition of Ω we may combine equations (30), (31),

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(32) and (33) to obtain

$$\mathbb{P}(X(\omega) \leq p) = p.$$

So X is uniformly distributed as claimed. $\qquad\square$

Proof of Theorem 3.1. Take Q to be $\frac{d\mathbb{Q}}{d\mathbb{P}}$ in Lemma B.5 to find X uniformly distributed with $\frac{d\mathbb{Q}}{d\mathbb{P}} = F^{-1}_{\frac{d\mathbb{Q}}{d\mathbb{P}}} V$ almost surely.

Suppose f satisfies the constraints of (7). We see that $-(-f)^X$ is equal to f in distribution, hence $-(-f)^X \in \mathcal{A}$ if $f \in \mathcal{A}$. Furthermore

$$
\begin{aligned}
-e^{rT}C &\leq \int_\Omega (-f)\frac{d\mathbb{Q}}{d\mathbb{P}}\, d\mathbb{P} &&\text{by (7),} \\
&\leq \int_\Omega (-f)^X \left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)^X d\mathbb{P} &&\text{by Lemma B.4,} \\
&= \int_\Omega (-f)^X \frac{d\mathbb{Q}}{d\mathbb{P}}\, d\mathbb{P} \leq \infty &&\text{by Lemma B.2.}
\end{aligned}
$$

So $-(-f)^X$ satisfies the constraints of (7).

Finally we note that

$$-(-f)^X(\omega) = -F^{-1}_{-f}F_X X(\omega) = -F^{-1}_{-f}X(\omega) = (1 - F_f)^{-1}X(\omega).$$

The result now follows by taking $U = 1 - X$. $\qquad\square$

Appendix C   Proof of Theorem 4.1

Proof. We consider functions φ of the form:

$$\varphi(x) = \begin{cases} k_1, & \text{if } x \geq \alpha \\ k_2, & \text{otherwise.} \end{cases} \tag{34}$$

We require that $0 < \alpha < p$ and $k_2 < k_1$. For functions of this form we can rewrite (8) as:

$$\mathcal{F}(\varphi) = \alpha u(k_2) + (1 - \alpha)u(k_1) \tag{35}$$

and equations (9) and (10) as:

$$\frac{pL - (p - \alpha)k_1}{\alpha} \leq k_2 \leq \frac{C - k_1 \int_\alpha^1 q(x)\,\mathrm{d}x}{\int_0^\alpha q(x)\,\mathrm{d}x.}$$

Let us restrict ourselves further to functions where:

$$k_2 = \frac{pL - (p - \alpha)k_1}{\alpha}.$$

So long as k_1 is sufficiently large, we will have $k_2 < k_1$. For such functions the ES constraint is automatically satisfied and the budget constraint becomes:

$$pL - (p - \alpha)k_1 \leq \frac{\alpha}{\int_0^\alpha q(x)\,\mathrm{d}x}\left(C - \int_\alpha^1 q(x)\,\mathrm{d}x\right). \tag{36}$$

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Taking the limit of the left hand side of (36) as $\alpha \to 0$ we obtain

$$pL - pk_1.$$

On the other hand the right hand side tends to zero as $\alpha \to 0$ because of our assumptions on the function $q(x)$. For all sufficiently large k_1 we can ensure that

$$pL - pk_1 < -1 < 0.$$

So for sufficiently large k_1 and sufficiently small α, the budget constraint will hold.

With the chosen value for k_2, our objective function can be written:

$$\mathcal{F}(\varphi) = \alpha u \left(\frac{pL - (p - \alpha)k_1}{\alpha} \right) + (1 - \alpha)u(k_1)$$

Our constraints are now simply that $0 < \alpha < \delta$ and $k_1 > M$ for some values $\delta > 0$ and $M > 0$.

We wish to show that for sufficiently large k_1, the limit of (C) as α tends to zero is $u(k_1)$. Let us choose constants c, N and η as given in (2). We may choose k_1 sufficiently large and α sufficiently small so that the following all hold:

$$k_1 > \frac{2M}{p},$$

$$\frac{p}{4}k_1 > pL,$$

$$\alpha < \frac{p}{2}.$$

It follows that

$$pL < \left(\frac{p}{2} - \alpha\right)k_1$$

and hence

$$pL < (p - \alpha)X_1 - \frac{p}{2}k_1 < (p - \alpha)k_1 - M\alpha.$$

Thus

$$\frac{pL - (p - \alpha)k_1}{\alpha} < -M$$

and we may conclude

$$u\left(\frac{pL - (p - \alpha)k_1}{\alpha}\right) > -c\left|\frac{L - (p - \alpha)k_1}{\alpha}\right|^{\eta}.$$

So for sufficiently large k_1 and sufficiently small $\alpha > 0$ the objective function is bounded below:

$$\mathcal{F}(\varphi) \geq -\alpha c\left|\frac{pL - (p - \alpha)k_1}{\alpha}\right|^{\eta} + (1 - \alpha)u(k_1).$$

$$= -c\alpha^{1-\eta}|pL - (p - \alpha)k_1|^{\eta} + (1 - \alpha)u(k_1)$$

$$\to u(k_1) \text{ as } \alpha \to 0$$

Hence the supremum of the objective function is bounded below by $(\sup_x u(x)) - \epsilon$ for any $\epsilon > 0$. On the other hand it is trivial that the objective function is bounded above by $\sup_x u(x)$. $\qquad \square$

Appendix D   The Black-Scholes-Merton case

We consider derivatives on the final stock price in a market where one can trade in either zero coupon bonds with (deterministic) risk free rate r or in a non-dividend paying stock whose price at time t, S_t, follows a geometric Brownian motion under the $\mathbb{P}$-measure

$$\mathrm{d}S_t = S_t(\mu \,\mathrm{d}t + \sigma \,\mathrm{d}W_t^{\mathbb{P}}), \quad S_0$$

with drift μ, volatility $\sigma > 0$ and initial condition $S_0 > 0$. The process $W_t^{\mathbb{P}}$ is a standard Brownian motion under the $\mathbb{P}$ probability measure.

The log stock price $s_T = \ln S_T$, under the $\mathbb{P}$-measure, is normally distributed with mean $s_0 + (\mu - \frac{1}{2}\sigma^2)T$ and standard deviation $\sigma\sqrt{T}$. Let us write the density function explicitly:

$$p_{s_T}^{\mathrm{BS}}(x) = \frac{1}{\sigma\sqrt{2\pi T}} \exp\left(-\frac{(x - (s_0 + (\mu - \frac{1}{2}\sigma^2)T))^2}{2\sigma^2 T} \right). \tag{37}$$

The standard pricing theory in the Black–Scholes–Merton market tells us that the price of a European derivative in this market can be computed using the discounted $\mathbb{Q}$-measure expectation of the payoff where the stock price process in the $\mathbb{Q}$-measure is

$$\mathrm{d}S_t = S_t(r \,\mathrm{d}t + \sigma \,\mathrm{d}W_t^{\mathbb{Q}}), \quad S_0,$$

where now $W_t^{\mathbb{Q}}$ is a standard Brownian motion under the measure $\mathbb{Q}$.

Hence we can write down the $\mathbb{Q}$-measure density function for s_T

$$q_{s_T}^{\mathrm{BS}}(x) = \frac{1}{\sigma\sqrt{2\pi T}}\exp\left(-\frac{(x-(s_0+(r-\frac{1}{2}\sigma^2)T))^2}{2\sigma^2 T}\right). \tag{38}$$

Now let $(\Omega^D, \mathcal{F}^D, \mathbb{P}^D)$ be the probability space for the final log stock price s_T. The superscript D stands for "derivatives market". This probability space is simply $\mathbb{R}$ with probability density given by p^{BS} and σ-field given by the Borel set. We can define a random variable $\frac{d\mathbb{Q}}{d\mathbb{P}}^D(s_T) := q^{\mathrm{BS}}(s_T)/p^{\mathrm{BS}}(s_T)$ on $(\Omega^D, \mathcal{F}^D, \mathbb{P}^D)$. Together, $\frac{d\mathbb{Q}}{d\mathbb{P}}^D$ and $(\Omega^D, \mathcal{F}^D, \mathbb{P}^D)$ define a market: the market of European derivatives on the final stock price s_T. For any payoff function f of s_T the price of the derivative is given by (5) so long as this integral exists and is less than ∞. This is a complete market.

Let U be the standard uniform random variable given by $F_{s_T}(s_T)$. We calculate

$$F_{s_T}^{-1}(U) = s_0 + T\left(\mu - \frac{\sigma^2}{2}\right) + \sigma\sqrt{T}\Phi^{-1}(U)$$

where as usual Φ us the cumulative distribution function of the standard normal. Using the explicit formulae for $q(s_T)$ and $p(s_T)$ we compute

$$\frac{d\mathbb{Q}}{d\mathbb{P}}^D(U) = \frac{q(F_{s_T}^{-1}(U))}{p(F_{s_T}^{-1}(U))} = \exp\left(\frac{\mu-r}{2\sigma^2}\left((\mu+r-\sigma^2)T+2(s_0-F_{s_T}^{-1}(U))\right)\right)$$

$$= \exp\left[\frac{\mu-r}{\sigma}\left(-\frac{\mu-r}{\sigma}\frac{T}{2}-\sqrt{T}\Phi^{-1}(U)\right)\right]$$

where we have highlighted the role of the market price of risk $\frac{\mu-r}{\sigma}$. If we

assume $\mu > r$, then this function is decreasing in U. Since U is uniform, we conclude that this expression is equal to $(1 - F_{\frac{d\mathbb{Q}}{d\mathbb{P}}}(U))^{-1}$. We see that if $\mu > r$ then $\frac{d\mathbb{Q}}{d\mathbb{P}}^D \to \infty$ as $U \to 0$. If $\mu < r$ then $\frac{d\mathbb{Q}}{d\mathbb{P}}^D \to \infty$ as $U \to 1$.

Thus European derivatives at time T in the Black–Scholes–Merton market satisfy the assumptions of Theorem 4.1.

Remark. *We have assumed in our analysis that the horizon of the investment, T_1, and the ES time horizon, T_2 coincide. Typically the ES time horizon is the time one estimates could be needed to liquidate the position in a hostile market so in practice one would have $T_2 < T_1$. However, an investor who wishes to maximize their utility at time T_1 could choose to restrict themselves to buying derivatives with maturity T_2 and then holding investing the payoff in zero coupon bonds until time T_1 in which case the payoff at time T_1 would be a function of the payoff at time T_2.*

Remark. *We have illustrated our results with the Black–Scholes–Merton model for simplicity. The key observation was that densities for p^{BS} and q^{BS} were normal but with different drifts, so the ratio q^{BS}/p^{BS} is unbounded. Over sufficiently short time horizons, the density of any stochastic process driven by Itô equations can be well approximated by multivariate normal distributions. This can be expressed rigorously using asymptotic formulae for the heat kernel of a stochastic process (see for example Hsu (2002)). Thus over short time horizons one expects to find that $\frac{d\mathbb{Q}}{d\mathbb{P}}$ will be unbounded for any market model defined using Itô calculus where the market price of risk is non-zero. One can easily devise examples of stochastic processes which*

converge to a fixed value at a future time T, so we cannot deduce that $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ will be unbounded at time T. Nevertheless, one expects that in a realistic market model $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ will indeed be unbounded at any time T.

D.1   Proof of Theorem 5.4

Proof. In the Black–Scholes–Merton market, the expectation $e(\gamma)$ is equal to

$$\int_{\mathbb{R}} \left(\frac{q_{s_T}^{BS}(x)}{p_{s_T}^{BS}(x)} \right)^{\frac{\gamma}{\gamma-1}} p_{s_T}^{BS}(x)\, \mathrm{d}x$$

where p^{BS} and q^{BS} are given by equations (37) and (38) respectively. On substituting in these formulae for p^{BS} and q^{BS} one obtains

$$e(\gamma) = \int_{\mathbb{R}} \frac{1}{\sigma\sqrt{2\pi T}} \exp\left(\frac{c_0 + c_1 x + c_2 x^2}{8(\gamma-1)\sigma^2 T} \right) \mathrm{d}x \qquad (39)$$

with

$$c_0 = T^2 \left(-\gamma\sigma^4 + 4\mu^2 - 4\mu\sigma^2 - 4\gamma r^2 + 4\gamma r\sigma^2 + \sigma^4 \right)$$
$$- 4s_0 T \left(-\gamma\sigma^2 - 2\mu + 2\gamma r + \sigma^2 \right) - 4(\gamma-1)s_0^2,$$
$$c_1 = 4 \left(T \left(-\gamma\sigma^2 - 2\mu + 2\gamma r + \sigma^2 \right) + 2(\gamma-1)s_0 \right),$$
$$c_2 = 4 - 4\gamma.$$

The overall coefficient of x^2 in the exponential in our expression (39) for $e(\gamma)$ is

$$\frac{4 - 4\gamma}{8(\gamma-1)\sigma^2 T} = -\frac{1}{2\sigma^2 T}.$$

This is always negative and so the expression (39) is a Gaussian integral and hence is finite.

It now follows from Theorem 5.3 that since u_I is risk-averse on the right and also unbounded on the right that the investor is difficult to satisfy. $\square$

Appendix E   Proof of Theorem 5.1

By Theorem 3.1, the optimization problem 7 is equivalent to solving

$$\sup_{\varphi:[0,1]\to\mathbb{R},\text{ with }\varphi\text{ increasing}} \int_0^1 u_I(\varphi(x))\,\mathrm{d}x$$

$$\text{subject to} \quad \int_0^1 u_R(\varphi(x))\,\mathrm{d}x \geq L \qquad (40)$$

$$\text{and} \quad -\infty \leq \int_0^1 \varphi(x)q(x)\,\mathrm{d}x \leq e^{rT}C.$$

where $q = (1 - F_{\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}})^{-1}$ and so is decreasing with integral 1.

Since in problem (40) we require that φ is increasing, there is some $p \in [0,1]$ such that $\varphi(x)$ is less than 0 for x less than p and $\varphi(x)$ is greater than 0 for x greater than p. Since the value of the integrals in the optimization problems is unaffected by the value of φ at the single point p, we may also assume that $\varphi(p) = 0$.

For a fixed p, we may define f_1 to be the restriction of φ to $[0, p]$ and f_2 to be the restriction of φ to $[p, 1]$. Let us write $\tilde{V}(p) \in \mathbb{R} \cup \{\pm\infty\}$ for the

value of the supremum in the problem.

$$\sup_{\substack{f_1:[0,p]\to[-\infty,0),\ \text{with } f_1 \text{ increasing} \\ f_2:[p,1]\to[0,\infty),\ \text{with } f_2 \text{ increasing}}} \int_p^1 u_I(f_2(x))\,\mathrm{d}x$$

$$\text{subject to} \quad \int_0^p u_R(f_1(x))\,\mathrm{d}x \geq L$$

$$\text{and} \quad -\infty \leq \int_0^p f_1(x)q(x) + \int_p^1 f_2(x)q(x)\,\mathrm{d}x \leq e^{rT}C.$$

We use the value $-\infty$ to indicate that the constraints cannot be satisfied as is conventional in convex analysis. The supremum of problem (40) and hence of (7) is given by

$$\sup_{p\in[0,1]} \tilde{V}(p).$$

It is obvious that $V(p) = \tilde{V}(p)$.

Appendix F   Proof of Theorem 5.3

We need first a Lemma.

Lemma F.1. *Let A and B be constants satisfying $-\infty \leq A < B \leq \infty$ and let a and b be finite constants satisfying $a < b$. We will write $\mathcal{I}$ for the set of increasing functions mapping $[a,b]$ to $[A,B]$.*

Suppose that $q : [a,b] \to \mathbb{R}$ is a positive decreasing function with finite integral. Suppose that u is a concave increasing function. Let $\partial u(x)$ be the set

$$\partial u(x) = \{y \in [0,\infty) \mid \forall x' \in [A,B], u(x') \leq u(x) + y(x' - x)\}.$$

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Apart from at the boundary points $\{A, B\}$, this is the subdifferential of the concave function u[4]. Let α be a constant and let $\varphi^\star \in F$ satisfy

$$\alpha q(x) \in \partial u(\varphi^*(x)) \tag{41}$$

for every x. Then φ^* is a solution to the maximization problem

$$\sup_{\varphi \in F} \int_a^b u(\varphi(x)) \, \mathrm{d}x$$
$$\text{subject to } \int_a^b \varphi(x) q(x) \, \mathrm{d}x \leq \int_a^b \varphi^*(x) q(x) \, \mathrm{d}x \tag{42}$$

and the minimization problem

$$\inf_{\varphi \in F} \int_a^b \varphi(x) q(x) \, \mathrm{d}x$$
$$\text{subject to } \int_a^b u(\varphi(x)) \, \mathrm{d}x \geq \int_a^b u(\varphi^*(x)) \, \mathrm{d}x. \tag{43}$$

Proof. Let $\varphi : [a, b] \to [A, B]$ be another function. By the assumption (41) we have

$$u(\varphi(x)) \leq u(\varphi^*(x)) + \alpha q(x)(\varphi(x) - \varphi^*(x)).$$

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[4]See Rockafellar (2015) for a discussion of subdifferentials. As remarked in Rockafellar (2015) the term subdifferential is used for both convex and concave functions even though superdifferential might be considered a more apt term for concave functions.

Integrating this

$$\int_a^b u(\varphi(x)) \, dx \leq \int_a^b u(\varphi^*(x)) \, dx + \alpha \int_a^b q(x)(\varphi(x) - \varphi^*(x)) \, dx.$$

So if φ satisfies the constraints of (42) we conclude

$$\int_a^b u(\varphi(x)) dx \leq \int_a^b u(\varphi^*(x)) \, dx.$$

Thus φ^* solves the problem (42). Similarly φ^* solves (43). □

Remark. *Lemma F.1 can also be used to solve portfolio optimization problems of the form (7) where the only constraints are bounds on the payoff function f. These problems are considered in more detail in Föllmer and Schied (2011), with a greater emphasis on the uniqueness of the solutions.*

With the Lemma in place, we can now move to proving Theorem 5.3.

Proof. u_R is smooth with derivative

$$u_R'(x) = \gamma_R(-x)^{\gamma_R - 1}.$$

We define $i_1(y) = ((u_R)')^{-1}(y) : [0, \infty) \to (-\infty, 0]$. So

$$i_1(y) = -\left(\frac{y}{\gamma_R}\right)^{\frac{1}{\gamma_R - 1}}.$$

Given $\alpha > 0$ we define $\varphi_{1,\alpha}^*(x) = i_1(\alpha q(x))$. By Lemma F.1, $\varphi_{1,\alpha}^*$ is a solution

of the problem:

$$\inf_{\varphi \in F} \int_a^b \varphi(x) q(x) \, \mathrm{d}x$$
$$\text{subject to } \int_a^b u_R(\varphi(x)) \, \mathrm{d}x \geq L(\alpha, a, b). \tag{44}$$

where $0 \leq a < b \leq 1$ and

$$L(\alpha, a, b) := \int_a^b u_R(\varphi_{1,\alpha}^*(x)) \mathrm{d}x$$
$$= -\int_a^b \left(\left(\frac{\alpha q(x)}{\gamma_R} \right)^{\frac{1}{\gamma_R - 1}} \right)^{\gamma_R} \mathrm{d}x$$
$$= -\left(\frac{\alpha}{\gamma_R} \right)^{\frac{\gamma_R}{\gamma_R - 1}} \int_a^b q(x)^{\frac{\gamma_R}{\gamma_R - 1}} \mathrm{d}x.$$

The optimum value of (44) is given by

$$C_1(\alpha, a, b) := \int_a^b q(x) i_1(\alpha q(x)) \, \mathrm{d}x$$
$$= \int_a^b -q(x) \left(\frac{\alpha q(x)}{\gamma_R} \right)^{\frac{1}{-1+\gamma_R}} \mathrm{d}x$$
$$= -\left(\frac{\alpha}{\gamma_R} \right)^{\frac{1}{\gamma_R - 1}} \int_a^b q(x)^{\frac{\gamma_R}{\gamma_R - 1}} \, \mathrm{d}x.$$

Let us write

$$I_1(a, b) := \int_a^b q(x)^{\frac{\gamma_R}{\gamma_R - 1}} e dx$$

To ensure $L(\alpha, a, b) = L$ we must take as α

$$\alpha^* = \gamma \left(\frac{-L}{I_1(a,b)} \right)^{\frac{\gamma_R - 1}{\gamma_R}}$$

which we note is finite and is greater than 0 whenever $I_1(a, b)$ is finite.

We compute that

$$C_1(\alpha^*, a, b) = - \left(\left(\frac{-L}{I_1(a,b)} \right)^{\frac{\gamma_R - 1}{\gamma_R}} \right)^{\frac{1}{\gamma_R - 1}} I_1(a, b)$$

$$= -(-L)^{\frac{1}{\gamma_R}} I_1(a, b)^{\frac{\gamma_R - 1}{\gamma_R}}.$$

If $I_1(0, p)$ is finite it follows from Lemma F.1 that the C_1 of Theorem 5.1 takes the value

$$C_1(p) = -(-L)^{\frac{1}{\gamma_R}} I_1(0, p)^{\frac{\gamma_R - 1}{\gamma_R}}. \tag{45}$$

If the investor is difficult to satisfy, it follows that the constraint is binding.

On the other hand if $I_1(0, p)$ is infinite, we may take $\varphi_1(x) = \varphi_{\alpha^*}^*(x) \mathbb{1}_{[a,b]}(x)$ to find a function satisfying the constraints of problem (11) with objective value $C(\alpha^*, a, b)$. Since this tends to $-\infty$ as $a \to 0$ we deduce that

$$C_1(p) = -\infty$$

if $I_1(0, p)$ is infinite. Since u_I is increasing we can achieve arbitrary large utilities below $\sup u_I$ given sufficient cash. Hence the constraint is not binding in this case.

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To determine when the investor is easily satisfied we solve the optimiza-
tion problem (12). We define

$$i_2(y) := ((u_I)')^{-1}(y) = \left(\frac{y}{\gamma_I}\right)^{\frac{1}{\gamma_I - 1}}.$$

We now define $\varphi_{2,\alpha}^* = i_2(\alpha(q(x)))$ for $\alpha > 0$.

By Lemma F.1, $\varphi_{2,\alpha}^*$ is a solution of the problem:

$$\sup_{\varphi \in F} \int_a^b u_I(\varphi(x))\, \mathrm{d}x$$

$$\text{subject to } \int_a^b \varphi(x)q(x)\, \mathrm{d}x \le C_2(\alpha, a, b). \tag{46}$$

where $0 \le a < b \le 1$ and

$$C_2(\alpha, a, b) := \int_a^b \varphi_{2,\alpha}^*(x)q(x)\mathrm{d}x$$

$$= \int_a^b \left(\frac{\alpha q(x)}{\gamma_I}\right)^{\frac{1}{\gamma_I - 1}} q(x)\mathrm{d}x$$

$$= \int_a^b \left(\frac{\alpha}{\gamma_I}\right)^{\frac{1}{\gamma_I - 1}} q(x)^{\frac{\gamma_I}{\gamma_I - 1}}\mathrm{d}x.$$

We define $I_2(a, b) = \int_a^b q(x)^{\frac{\gamma_I}{\gamma_I - 1}}\mathrm{d}x$ so we have

$$C_2(\alpha, a, b) = \left(\frac{\alpha}{\gamma_I}\right)^{\frac{1}{\gamma_I - 1}} I_2(a, b). \tag{47}$$

The corresponding supremum of (46) is then given by

$$
\begin{aligned}
u(\alpha, a, b) &:= \int_a^b \left(\frac{\alpha q(x)}{\gamma_I} \right)^{\frac{\gamma_I}{\gamma_I - 1}} \mathrm{d}x \\
&= \left(\frac{\alpha}{\gamma_I} \right)^{\frac{\gamma_I}{\gamma_I - 1}} I_2(a, b) \\
&= \left(\frac{C_2(\alpha, a, b)}{I_2(a, b)} \right)^{\gamma_I} I_2(a, b) \\
&= C_2(\alpha, a, b)^{\gamma_I} I_2(a, b)^{1 - \gamma_I}
\end{aligned}
\tag{48}
$$

We deduce that the investor is difficult to satisfy if and only if

$$
I_2(a, b)
$$

is finite. $\qquad \square$