Symplectic domination

Joel Fine* and Dmitri Panov†

Abstract

Let M be a compact oriented even-dimensional manifold. This note constructs a compact symplectic manifold S of the same dimension and a map $f\colon S \to M$ of strictly positive degree. The construction relies on two deep results: the first is a theorem of Ontaneda that gives a Riemannian manifold N of tightly pinched negative curvature which admits a map to M of degree equal to one; the second is a result of Donaldson on the existence of symplectic divisors. Given Ontaneda's negatively curved manifold N, the twistor space Z is symplectic. The manifold S is then a suitable multisection of the twistor space, found via Donaldson's theorem.

The aim of this short note is to prove the following theorem.

Theorem 1. *Let M be a compact oriented manifold of even dimension. There exists a map of positive degree $f\colon S \to M$ from a compact symplectic manifold S of the same dimension.*

This result says, in some sense, that there are "a lot" of symplectic manifolds. This fits with the philosophy behind a folklore conjecture in symplectic topology, stated as Conjecture 6.1 in the article [3] of Eliashberg. The conjecture asserts that if X is a compact manifold of dimension $2n \geqslant 6$, which admits an almost complex structure and a cohomology class $\kappa \in H^2(X, \mathbb{R})$ with $\kappa^n \neq 0$, then M carries a symplectic structure.

Theorem 1 follows rather quickly from two deep results (stated as Theorems 2 and 4 below). The first is a spectacular construction by Ontaneda of Riemannian manifolds with tightly pinched negative curvatures.

Theorem 2 (Ontaneda)**.** *Let M be a compact oriented manifold and $\epsilon > 0$. There exists a degree one map $f\colon N \to M$ from a compact oriented Riemannian manifold N of the same dimension, with sectional curvatures in the interval $[-1-\epsilon, -1]$.*

This is the main result of a lengthy preprint [8], which was subsequently broken up into a series of articles for publication [9, 10, 11, 12, 13, 14, 15]. The pinched manifolds constructed by Ontaneda are smoothings of singular negatively curved manifolds constructed by Charney and Davis using a procedure called strict hyperbolisation [1]. This in turn builds on the hyperbolisation of polyhedra by Gromov [6].

For our purposes, the important consequence of the curvature pinching is that the twistor space of N carries a natural symplectic form. We recall that the twistor space $Z \to N$ of an oriented Riemannian manifold N is the bundle of compatible almost complex structures on the tangent spaces. I.e. the fibre of Z over $x \in N$ is the set of all linear orthogonal complex structures on T_xN which induce the given orientation. The fibres are homogeneous spaces, identified with $F = \mathrm{SO}(2n)/\mathrm{U}(n)$. The symplectic form on Z is provided by a construction due to Reznikov (which is, in fact, a special case of Weinstein's "fat bundles" [17]).

*Département de mathématiques, Université libre de Bruxelles, Belgium. joel.fine@ulb.ac.be

†Department of Mathematics, King's College London, United Kingdom. dmitri.panov@kcl.ac.uk

Theorem 3 (Reznikov [16]). *Let N be an oriented even-dimensional Riemannian manifold with twistor space Z. There is a natural closed 2-form ω on Z with integral cohomology class $[\omega] \in H^2(Z, \mathbb{Z})$ which is symplectic when restricted to each fibre of $Z \to N$. Moreover, there is a positive number $\epsilon > 0$, depending only on the dimension of N, such that if the sectional curvatures of N lie in the interval $[-1-\epsilon, -1]$ then ω is symplectic.*

Since this is central to the proof of Theorem 1, we explain briefly how the construction goes. The key to the existence of an integral closed 2-form is that the model fibre $F = \mathrm{SO}(2n)/\mathrm{U}(n)$ of twistor space is a homogeneous integral symplectic manifold. In other words, there is a principal S^1-bundle $P_F \to F$ with a connection A_F whose curvature is a symplectic form on F; moreover P_F carries an action of $\mathrm{SO}(2n)$ covering the action on F and leaving A_F invariant.

This can be seen via the theory of integral coadjoint orbits (see, for example, [7]), but it is also simple to describe it explicitly. Let $P_F = \mathrm{SO}(2n)/\mathrm{SU}(n)$. This is the total space of a principal circle bundle $P_F \to F$ and the $\mathrm{SO}(2n)$-action on P_F covers that on F. Fix a point $x \in F$ and a point $p \in P_F$ over x, with stabilisers $\mathrm{SU}(n) \subset \mathrm{U}(n) \subset \mathrm{SO}(2n)$. Denote by $\mathfrak{p}$ the orthogonal complement of $\mathfrak{u}(n) \subset \mathfrak{so}(2n)$ via the Killing form. The orthogonal complement of $\mathfrak{su}(n)$ is $\mathfrak{p} \oplus \mathbb{R}$, where the second summand corresponds to the subspace of $\mathfrak{u}(n)$ spanned by multiples of the matrix $i\mathrm{Id} \in \mathfrak{u}(n)$. It follows that $T_xF \cong \mathfrak{p}$ whilst $T_pP_F \cong \mathfrak{p} \oplus \mathbb{R}$ and the second summand is tangent to the fibres of $P_F \to F$. So the splitting of $T_pP_F \cong T_xF \oplus \mathbb{R}$ determines a horizontal distribution in P_F, defining the $\mathrm{SO}(2n)$-invariant connection A_F.

The curvature $2\pi i \omega_F$ of A_F is a closed $\mathrm{SO}(2n)$-invariant 2-form on F. Unwinding the above definition we can describe it as follows. We have $\mathfrak{so}(2n) = \mathfrak{u}(n) \oplus \mathfrak{p}$ and one can check that $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{u}(n)$. This gives a map $\Lambda^2\mathfrak{p} \to \mathfrak{u}(n)$ and composing with $-\frac{i}{2\pi}\mathrm{Tr}\colon \mathfrak{u}(n) \to \mathbb{R}$ gives the element $\omega_F \in \Lambda^2\mathfrak{p}^*$. To check ω_F is non-degenerate, note that its kernel at x is a $\mathrm{U}(n)$-invariant subspace of $\mathfrak{p}$ and $\mathfrak{p}$ is an irreducible $\mathrm{U}(n)$-representation. One way to see this is to write $\mathfrak{so}(2n) \cong \Lambda^2(\mathbb{R}^{2n})$, then $\mathfrak{u}(n)$ is identified with the real $(1,1)$-forms, and $\mathfrak{p}$ with the real parts of $(2,0)$-forms. So, as a $\mathrm{U}(n)$-representation, $\mathfrak{p}$ is isomorphic to $\Lambda^2(\mathbb{C}^n)$ which is indeed irreducible. It follows that ω_F is either non-degenerate or identically zero. Suppose for a contradiction that it were zero, or equivalently that $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{su}(n)$. We then define a map $\rho\colon \mathfrak{so}(2n) \to \mathbb{R}$ by setting it equal to $-i\,\mathrm{Tr}$ on $\mathfrak{u}(n)$ and zero on $\mathfrak{p}$. The condition $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{su}(n)$ together with $[\mathfrak{u}(n), \mathfrak{p}] \subset \mathfrak{p}$ implies ρ vanishes on $[\mathfrak{so}(2n), \mathfrak{so}(2n)]$. But by simplicity, $[\mathfrak{so}(2n), \mathfrak{so}(2n)] = \mathfrak{so}(2n)$ and so ρ vanishes identically, giving a contradiction.

We now return to the twistor space $Z \to N$ and carry out this construction on every fibre. The result is a principal S^1-bundle $P \to Z$ fitting together the fibrewise bundles $P_F \to F$. Moreover, the connection A_F gives a fibrewise connection in Z. To promote this to a genuine connection in all of $P \to Z$ we must specify the horizontal distribution transverse to the fibres of $Z \to N$; but this is precisely what the Levi-Civita connection does. This gives a connection A in $P \to Z$ whose curvature determines a closed integral 2-form ω which is symplectic on each fibre.

One can now ask for ω to be symplectic, which becomes a curvature inequality for the Riemannian metric on N. Reznikov observed that this inequality is satisfied by hyperbolic space and so, by openness, it is also satisfied by all negatively curved metrics which are sufficiently pinched. In the case $\dim N = 4$, the article [4] gives the full curvature inequality explicitly.

The next step in the proof is to invoke another deep theorem, namely Donaldson's result on symplectic hypersurfaces.

Theorem 4 (Donaldson [2])**.** *Let (Z,ω) be a compact symplectic manifold with $[\omega]$ an integral cohomology class. There exists a symplectic submanifold S of codimension 2, with $[S]$ Poincaré dual to a positive multiple $k[\omega]$ of the symplectic class.*

Proof of Theorem 1. By Ontaneda's Theorem it suffices to prove the result for all compact oriented even-dimensional Riemannian manifolds N with sectional curvatures pinched arbitrarily close to -1.

Suppose first that $\dim N = 4$. In this case, the twistor space $Z \to N$ has fibres S^2. By Reznikov's result we know that there is an integral symplectic form on Z for which the twistor fibres are symplectic. Now let $S \subset Z$ be a Donaldson hypersurface, with $[S] = k[\omega]$ for $k > 0$. The twistor projection restricts to a smooth map $f\colon S \to N$ and we claim the degree of this map is positive. To prove this write $[F]$ for the homology class of a fibre of $Z \to N$. The intersection number $[S]\cdot[F] = k\int_F \omega$ is positive since it is a positive multiple of the symplectic area of F. It follows that f is surjective. Now Sard's theorem implies the existence of a point $x \in N$ which is not a critical value of f. This means that S meets the fibre F_x over x transversely. The local degree of f at each point of $F_x \cap S$ is equal to the local intersection of F_x and S at that point, hence the degree of f equals $[S]\cdot[F]$ which we have just seen is positive.

In higher dimensions the argument is similar. When $\dim N = 2n$ the twistor space has dimension $n(n+1)$ and the fibre has dimension $n(n-1)$. We start as before with a Donaldson hypersurface $S_1 \subset Z$, with $[S_1]$ Poincaré dual to $k_1[\omega]$. We apply Donaldson's theorem again, this time to (S_1, ω_{S_1}), to obtain a symplectic submanifold $S_2 \subset S_1 \subset Z$, where S_2 has codimension 4 in Z with $[S_2]$ Poincaré dual to $k_2[\omega]^2$. We continue in this way, producing a chain $S_d \subset S_{d-1} \subset \cdots \subset S_1 \subset Z$ of symplectic submanifolds where $d = n(n-1)/2$. Each S_j is a symplectic submanifold of Z of codimension $2j$ and so S_d has complimentary dimension to a fibre of $Z \to N$. Moreover, $[S_d]$ is Poincaré dual to $k[\omega]^d$ for some $k > 0$. It follows that $[S_d]\cdot[F] = k\int_F \omega^d$ which is positive since it is a positive multiple of the symplectic volume of the fibre. From here the same argument as before shows that the twistor projection $f\colon S_d \to N$ has positive degree. □

We close with a remark, that the symplectic manifolds (S,ω) produced in the proof of Theorem 1 are of "general type" in the sense that $c_1(S) = -p[\omega]$ where $p > 0$. This follows from adjunction and the fact, proved in [5], that when $dimN = 2n$, the symplectic structures on the twistor space satisfy $c_1(Z) = (n-2)[\omega]$.

Acknowledgements

We would like to thank Igor Belegradek and Anton Petrunin for discussions. JF was supported by ERC consolidator grant 646649 "SymplecticEinstein".

References

[1] Charney, R. M. and Davis, M. W. *Strict hyperbolization,* Topology 34 329–350 (1995)

[2] Donaldson, S. K. *Symplectic submanifolds and almost-complex geometry,* J. Differential Geom. 44:4, 666–705 (1996)

[3] Eliashberg, Y. *Recent advances in symplectic flexibility,* Bull. Am. Math. Soc. 52:1 1–26 (2015)

[4] Fine, J. and Panov, D. *Symplectic Calabi–Yau manifolds, minimal surfaces and the hyperbolic geometry of the conifold,* J. Differential Geom. 82:1, 155–205, (2009)

[5] Fine, J. and Panov, D. *Hyperbolic geometry and non-Kähler manifolds with trivial canonical bundle,* Geom. Topol. 14:3, 1723–1763 (2010)

[6] Gromov, M. *Hyperbolic groups* in Essays in Group Theory, ed. S. M. Gersten, MSRI Publ. Springer, New York, 75–284 (1987)

[7] Kirillov, A. A. *Lectures on the Orbit Method,* Graduate Studies in Mathematics 64, Am. Math. Soc. (2004)

[8] Ontaneda, P. *Pinched smooth hyperbolization,* preprint, `arXiv:1110.6374`

[9] Ontaneda, P. *Riemannian hyperbolization,* preprint, `arXiv:1406.1730`

[10] Ontaneda, P. *Normal smoothings for smooth cube manifolds,* preprint, `arXiv:14060.1735`

[11] Ontaneda, P. *Normal smoothings for Charney–Davis strict hyperbolizations,* J. Topol. Anal. 9:1, 127–165 (2017)

[12] Ontaneda, P. *On the Farrell and Jones warping deformation,* J. Lond. Math. Soc. (2) 92:3, 566–582 (2015)

[13] Ontaneda, P. *Deforming an ϵ-close to hyperbolic medtric to a warp metric,* preprint, Michigan Math. J. 65:4, 693–701 (2016)

[14] Ontaneda, P. *Deforming an ϵ-close to hyperbolic metric to a hyperbolic metric,* Proc. Roy. Soc. Edinburgh 148:3, 629–641 (2018)

[15] Ontaneda, P. *Cut limits on hyperbolic extensions,* Michigan Math. J. 65:4, 703–714 (2016)

[16] Reznikov, A. G. *Symplectic twistor spaces,* Ann. Global Anal. Geom. 11:2, 109–118 (1993)

[17] Weinstein, A. *Fat bundles and symplectic manifolds,* Adv. in Math. 37:3, 239–250 (1980)