Incentivising Participation in Liquid Democracy with Breadth-First Delegation

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ABSTRACT

Liquid democracy allows members of an electorate to either directly vote over alternatives, or delegate their voting rights to someone they trust. Most of the liquid democracy literature and implementations allow each voter to nominate only one delegate per election. However, if that delegate abstains, the voting rights assigned to her are left unused. To minimise the number of unused delegations, it has been suggested that each voter should declare a personal ranking over voters she trusts. In this paper, we show that even if personal rankings over voters are declared, the standard delegation method of liquid democracy remains problematic. More specifically, we show that when personal rankings over voters are declared, it could be undesirable to receive delegated voting rights, which is contrary to what liquid democracy fundamentally relies on. To solve this issue, we propose a new method to delegate voting rights in an election, called *breadth-first delegation*. Additionally, the proposed method prioritises assigning voting rights to individuals closely connected to the voters who delegate.

KEYWORDS

liquid democracy, social choice theory, economic mechanism design

1 INTRODUCTION

Liquid democracy is a middle ground between direct and representative democracy, as it allows each member of the electorate to directly vote on a topic, or temporarily choose a representative by delegating her voting rights to another voter. Therefore individuals who are either apathetic for an election, or trust the knowledge of another voter more than their own, can still have an impact on the election result (through delegating). An individual who casts a vote for themselves and for others is known as a *guru* (Christoff and Grossi [2017]). Liquid democracy has recently started gaining attention in a few domains which we discuss to show an overview of the general societal interest. In the political domain, local parties such as the Pirate Party in Germany, Demoex in Sweden, the Net Party in Argentina and Partido de Internet in Spain have been experimenting with liquid democracy implementations. Additionally, the local governments of the London Southwark borough and the Italian cities Turino and San Dona di Piave are working on integrating liquid democracy for community engagement processes (Boella et al. [2018]). In the technology domain, the online platform LiquidFeedback uses a liquid democracy system where a user selects a single guru for different topics (Behrens et al. [2014]; Kling et al. [2015]). Another prominent online example is GoogleVotes (Hardt and Lopes [2015]), where each user wishing to delegate can select a ranking over other voters.

Regardless of the increasing interest in liquid democracy, there exists outstanding theoretical issues. This work focuses on liquid democracy systems where each individual wishing to delegate can select a ranking over other voters. In such systems, given the common interpretation that delegations of voting rights are multi-step and transitive[1], we observe that: searching for a guru follows a depth-first search in a graph that illustrates all delegation preferences within an electorate, e.g. nodes represent the voters and directed edges the delegation choices for each voter. For this reason, we name this standard approach of delegating voting rights as *depth-first delegation*. Despite its common acceptance, we came across an important disadvantage even for the majority rule with binary issues. In particurlar, we show that when depth-first delegation is used, it could be undesirable to receive the voting rights of someone else. At this point, we emphasize that disincentivising voters to participate as gurus is in contrast to the ideology of liquid democracy due to the following. How can a liquid democracy system flourish if voters may not be incentivised to receive delegated votes? Motivated by this, we propose a simple idea solution to this issue: a new rule for delegating voting rights, called the *breadth-first delegation*, which guarantees that casting voters (those who do not delegate or abstain) weakly prefer to receive delegated voting rights, i.e. to participate as gurus. For these reasons, we consider this work of a high societal importance and immediate applications.

We outline this paper as follows: In the introduction we discuss the latest applied and theoretical developments in liquid democracy and give the preliminaries of our model. In the next two sections, we define delegation graphs, delegation rules and two types of participation. Afterwards, we formally introduce a new delegation rule and compare this rule with the standard one. Finally, we conclude this work with future research goals.

1.1 Related work

There currently exists a lack of theoretical analysis on liquid democracy. However, we summarise the main differences of our work to the main undertaking so far.

As outlined by Brill ([2018]), one of the main ongoing issues in liquid democracy is how to handle personal rankings over voters. His work discusses possible solutions around this issue without giving a formalised model, which this paper does. For two election alternatives where one is the ground truth, Kahng et al. ([2018]) find that: (a) there is no decentralised liquid democracy delegation rule that is guaranteed to outperform direct democracy and (b)

[1] We assume the following interpretation for the meaning of transitive delegations of voting rights: If a voter i delegates to a voter j, then i transfers to j the voting rights of herself and all the others that had been delegated to i.

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there is a centralised liquid democracy delegation rule that is guaranteed to outperform direct democracy as long as voters are not completely misinformed or perfectly informed about the ground truth. In comparison, our model can be used in a wider variety of elections, as it allows for multiple alternatives and no ground truth. Additionally our delegation rules can be used in a central or decentralised manner, thus the negative result (a) does not apply to our paper. The work of Christoff and Grossi ([2017]) focuses on the existence of delegation cycles and inconsistencies that can occur when there are several binary issues to be voted on with a different guru assigned for each issue. In comparison, we avoid delegation cycles by stating that a delegation chain (a path from a delegating voter to their assigned guru) cannot include the same voter more than once. Furthermore, individual rationality issues between multiple elections is out of scope for this work. Last, Brill and Talmon ([2018]) introduce a special case of Christoff and Grossi's model, which allows a single voter to be assigned several gurus. However, our model assigns one guru per voter. Therefore Brill and Talmon ([2018]) does allow more fine-grained delegations than our model, i.e. they allow for different pairs of alternatives to be ranked by different gurus. But the delegating voter cannot choose a preference relation over voters for any pair of alternatives, which is what this paper investigates how to do.

Similar to our work, GoogleVotes (Hardt and Lopes [2015]) allows a user to select a ranking over other voters and uses, what the authors describe as, a back-track breadth first search to assign a guru to a voter. We cannot complete a more comprehensive comparison to GoogleVotes as they have published only a general description of their system (without a formal model). However, we know that their delegation rule is different to our proposed breadth-first delegation rule as in the Tally/Coverage section of their video example (from minute 32 of Hardt [2014]), their rule assigns guru C to voter F, while our rule would assign guru A to voter F.

1.2 Preliminaries

Consider a set of voters $\mathcal{V}$ and a set of alternatives or outcomes $\mathcal{A}$. The set of possible electorates is given by $\mathcal{E}(\mathcal{V}) = 2^{\mathcal{V}} \backslash \{\emptyset\}$, i.e. non-empty subsets of $\mathcal{V}$. In our model, for every election there are three sets of electorates $V^a, V^c, V^d \in \mathcal{E}(\mathcal{V})$ such that $V^i \cap V^j = \emptyset$ for $i \neq j \in \{a, c, d\}$ and $V^a \cup V^c \cup V^d = \mathcal{V}$, where sets V^a, V^c, V^d consist of those who abstain, cast a vote and delegate their voting rights, respectively.

A *preference relation over alternatives* for a voter $i \in \mathcal{V}$ is denoted by $\succ_i^{\mathcal{A}}$ and is a binary relation on $\mathcal{A}$, i.e.: for $x, y \in A$ with $x \neq y$, the expression $x \succ_i^{\mathcal{A}} y$ indicates that voter i strictly prefers alternative x over alternative y. A *preference relation over voters* for voter $i \in \mathcal{V}$ is denoted by $\succ_i^{\mathcal{V}}$ and is a binary relation on $\mathcal{V}$, i.e.: for $i, x, y \in \mathcal{V}$ with[2] $i \neq x, y$ and $x \neq y$, the expression $x \succ_i^{\mathcal{V}} y$ indicates that voter i strictly prefers to delegate her voting rights to voter x instead of voter y. For both preference relations, we allow an index to identify ranking positions e.g. for any $i \in \mathcal{V}^d$, her m-th preferred voter is denoted by $\succ_{i,m}^{\mathcal{V}}$.

For a set W, consider a binary relation $\succ_i^W$. Then, $\succ_i^W$ is:

[2] A voter cannot include herself in her preference relation over voters.

(a) *complete* iff for every pair $x, y \in W$ either $x \succ_i^W y$ or $y \succ_i^W x$ holds,
(b) *antisymmetric* iff for every pair $x, y \in W$, if $x \succ_i^W y$ then $y \succ_i^W x$ does not hold, and
(c) *transitive* iff for all $x, y, z \in W$, if $x \succ_i^W y$ and $y \succ_i^W z$, then $x \succ_i^W z$.

Both preference relations over alternatives and preference relations over voters are antisymmetric and transitive but not complete (we do not enforce voters to rank every other member of the electorate as we consider this an unrealistic scenario for large electorates).

The set of all possible preference relations $\succ_i^{\mathcal{A}}$ and $\succ_i^{\mathcal{V}}$, for any $i \in \mathcal{V}$, are denoted by $R^{\mathcal{A}}$ and $R^{\mathcal{V}}$, respectively. A *preference profile over alternatives* is a function $P^{\mathcal{A}} : \mathcal{E}(\mathcal{V}) \to 2^{R^{\mathcal{A}}}$, where $P^A(N)$ returns a set of preference relations over alternatives (maximum one for each voter in N). For example, given an electorate $N = \{i, j, k\}$, a preference profile $P^A(N)$ could return $\{(i, \succ_i^{\mathcal{A}}), (j, \succ_j^{\mathcal{A}})\}$, meaning that agent i and j have been assigned a preference relation over alternatives but k has not (as k either delegates or abstains). Similarly, we define as a *preference profile over voters* a function $P^{\mathcal{V}} : \mathcal{E}(\mathcal{V}) \to 2^{R^{\mathcal{V}}}$, where $P^{\mathcal{V}}(N)$ returns a set of preference relations over voters (maximum one for each voter in N). Given profiles $P^{\mathcal{A}}$ and $P^{\mathcal{V}}$, voters are assigned to the V^a, V^c and V^d electorates as follows. If $(i, \succ_i^{\mathcal{A}}) \in P^{\mathcal{A}}(N)$, we infer that voter i casts a vote according to $\succ_i^{\mathcal{A}}$ and therefore becomes a member of the casting electorate V^c. If $(i, \succ_i^{\mathcal{A}}) \notin P^{\mathcal{A}}(N)$ and $(i, \succ_i^{\mathcal{V}}) \in P^{\mathcal{V}}(N)$, then i becomes a member of the delegating electorate V^d. If $(i, \succ_i^{\mathcal{A}}) \notin P^{\mathcal{A}}(N)$ and $(i, \succ_i^{\mathcal{V}}) \notin P^{\mathcal{V}}(N)$, i becomes a member of the abstaining electorate V^a.

Given an electorate N, adding or removing a preference relation over alternatives (or over voters) from a preference profile over alternatives $P^{\mathcal{A}}(N)$ (or over voters $P^{\mathcal{V}}(N)$), is denoted as follows. For a tuple $(i, \succ_i^{\mathcal{A}}) \in P^{\mathcal{A}}(N)$, a voter $j \in \mathcal{V} \setminus N$ and j's assigned preference relation over alternatives $\succ_j^{\mathcal{A}} \in R^{\mathcal{A}}$:

$$P^{\mathcal{A}}_{-i}(N) := P^{\mathcal{A}}(N) \setminus \{(i, \succ_i^{\mathcal{A}})\},$$
$$P^{\mathcal{A}}_{+(j,\succ_j^{\mathcal{A}})}(N) := P^{\mathcal{A}}(N) \cup \{(j, \succ_j^{\mathcal{A}})\}.$$

Similarly, for a tuple $(i, \succ_i^{\mathcal{V}}) \in P^{\mathcal{V}}(N)$, a voter $j \in \mathcal{V} \setminus N$ and j's assigned preference relation over voters $\succ_j^{\mathcal{V}} \in R^{\mathcal{V}}$:

$$P^{\mathcal{V}}_{-i}(N) := P^{\mathcal{V}}(N) \setminus \{(i, \succ_i^{\mathcal{V}})\},$$
$$P^{\mathcal{V}}_{+(j,\succ_j^{\mathcal{V}})}(N) := P^{\mathcal{V}}(N) \cup \{(j, \succ_j^{\mathcal{V}})\}.$$

To simplify the above cases notation, we will sometimes be using $P^{\mathcal{A}}, P^{\mathcal{V}}, P^{\mathcal{A}}_{-i}, P^{\mathcal{A}}_{+j}, P^{\mathcal{V}}_{-i}$ and $P^{\mathcal{V}}_{+j}$, accordingly.

2 DELEGATION GRAPH AND DELEGATION RULES

To model all possible delegation choices between voters, we use a weighted directed graph defined as follows.

Definition 1. *A* delegation graph *is a weighted directed graph* $G = (\mathcal{V}, E, w)$*, where*

- *$\mathcal{V}$ is the set of nodes representing the agents registered as voters;*
- *E is the set of directed edges representing delegations between voters; and*
- *w is the weight function $w : E \mapsto \mathbb{N}$ that assigns a value to an edge (i, j).*

To decide which values the weight function w has assigned to the outgoing edges of each vertex, we introduce the following function g that generates a delegation graph. Recall that we allow an index to identify ranking positions of trustees e.g. for any $i \in \mathcal{V}^d$, her m-th preferred voter is denoted by $>^{\mathcal{V}}_{i,m}$.

Definition 2. *Define as g the* delegation graph function *which takes as input a preference profile over voters $P^{\mathcal{V}}$ and returns the related delegation graph $G = (\mathcal{V}, E, w)$ for which the following property holds.*

Property 1. *For every $i, j \in \mathcal{V}$ and $i \neq j$, if there exists $(i, >^{\mathcal{V}}_i) \in P^{\mathcal{V}}$ with $>^{\mathcal{V}}_{i,x} = j$, then there exists $(i, j) \in E$ such that*

$$w((i, j)) = x.$$

We evaluate a delegation graph through the following.

Definition 3. *A* delegation rule function d *takes as input a preference profile over alternatives $P^{\mathcal{A}}$ together with a delegation graph G, and returns another preference profile over alternatives $\hat{P}^{\mathcal{A}}$ that resolves delegations as follows,*

- *if $(i, >^{\mathcal{A}}_i) \in \hat{P}^{\mathcal{A}}$, then i casts her vote,*
- *if $(i, >^{\mathcal{A}}_j) \in \hat{P}^{\mathcal{A}}$ for a voter $j \neq i$, then j becomes i's final delegate, i.e. her guru,*
- *if $(i, >^{\mathcal{A}}_k) \notin \hat{P}^{\mathcal{A}}$ for every $k \in \mathcal{V}$, then i abstains.*

For each voter $i \in \mathcal{V}$, a delegation rule analyses the subtree of the delegation graph rooted at node i and decides whether i casts, delegates or abstains. If voter i is found to delegate, the chosen delegation rule function will traverse i's subtree to find i's guru.

To get the outcome of an election, we use a voting rule function f. In our model, f takes as input the modified preference profile over alternatives $\hat{P}^{\mathcal{A}}$ (which incorporates delegations) and returns a single winner or a ranking over alternatives (depending on the voting rule used). In Section 5, we show that the output of the voting rule depends on the chosen delegation rule, meaning that we could get different election results when only the delegation rule function is different, i.e.

$$f(d(P^{\mathcal{A}}, g(P^{\mathcal{V}}))) \neq f(d'(P^{\mathcal{A}}, g(P^{\mathcal{V}}))),$$

when $d \neq d'$.

3 CAST AND GURU PARTICIPATION

The key property that we investigate is participation. The participation property holds if a voter, by joining an electorate, is at least as satisfied as before joining. This property has been defined only in the context of vote casting (Fishburn and Brams [1983]; Moulin [1988]). Due to the addition of delegations in our model, we establish two separate definitions of participation to reflect this new functionality[3].

For both of the following definitions, note that for an electorate $N \in \mathcal{E}(\mathcal{V})$, the set of all preference profiles over alternatives is given by $\mathcal{P}^{\mathcal{A},N}$, while the set of all preference profiles over delegates is given by $\mathcal{P}^{\mathcal{V},N}$.

A voting rule f satisfies the *cast participation* property when every voter $i \in \mathcal{V}$ weakly prefers joining any possible voting electorate V^c compared to abstaining and regardless of who is in the delegating electorate V^d. For the next two definitions, we add the notation $\succeq_i$ of weakly preferring as in these two cases the resulting outcomes might be identical. For non-identical outcomes, a voter i has a strict preference (as indicated in the model preliminaries).

Definition 4. *The* Cast Participation *property holds for a voting rule f iff:*

$$f(d(P^{\mathcal{A}}, g(P^{\mathcal{V}}))) \;\succeq^{\mathcal{A}}_i\; f(d(P^{\mathcal{A}}_{-i}, g(P^{\mathcal{V}}))),$$

for every possible disjoint casting and delegating electorates $V^c, V^d \in \mathcal{E}(\mathcal{V})$, where $i \in V^c$, and every possible preference profile for these electorates $P^A \in \mathcal{P}^{\mathcal{A},V^c}$ and $P^{\mathcal{V}} \in \mathcal{P}^{\mathcal{V},V^d}$.

For any casting and delegating electorates V^c and V^d, a voting rule f satisfies the *guru participation* property when any voter $i \in V^c$ weakly benefits from receiving additional voting rights of any voter $j \in \mathcal{V}$.

Definition 5. *The* Guru Participation *property holds for a voting rule f iff:*

$$f(d(P^{\mathcal{A}}, g(P^{\mathcal{V}}))) \;\succeq^{\mathcal{A}}_i\; f(d(P^{\mathcal{A}}, g(P^{\mathcal{V}}_{-j}))), \tag{1}$$

for every possible disjoint casting and delegating electorates $V^c, V^d \in \mathcal{E}(\mathcal{V})$, where $i \in V^c$, $j \in V^d$, and every possible profile $P^A \in \mathcal{P}^{\mathcal{A},V^c}$ and $P^{\mathcal{V}} \in \mathcal{P}^{\mathcal{V},V^d}$ that assign j's vote to guru i, i.e.

$$(j, >^{\mathcal{A}}_i) \;\in\; d(P^{\mathcal{A}}, g(P^{\mathcal{V}})).$$

Let F be the set of all voting rules. It is known that only a subset $\bar{F} \subset F$ satisfy (cast) participation: for example, Fishburn and Brams ([1983]) show that single transferable vote does not satisfy (cast) participation, while Moulin ([1988]) shows there is no Condorcet-consistent voting rule satisfying this property given 25 or more voters.

We are interested in exploring guru participation for voting rules in the subset $\bar{F}$ and this paper focuses on the majority rule with binary issues. Our results build on Observation 1, which we intuitively describe as follows. First, recall that if a voting rule satisfies cast participation, then any voter weakly prefers casting a vote instead of abstaining. When delegations of votes are allowed in the election and there is only one guru who gains an additional vote, then this guru weakly prefers receiving this additional vote (and therefore casting it, since cast participation is satisfied). Therefore guru participation is satisfied. Formally, we write the above as follows (see preliminaries for full form notation and definitions of the profiles used).

[3]There could be other interesting participation properties for liquid democracy, such as incentivising deviation from delegating to casting. But this is out of the paper's scope, as we focus on finding delegation rules that weakly benefit casting voters who become gurus.

Observation 1. *Let $f \in \bar{F}$ be the majority rule. Consider voters $i, j \in \mathcal{V}$, a profile $\hat{P}^{\mathcal{A}}$ returned by $d(P^{\mathcal{A}}, g(P^{\mathcal{V}}))$ and a profile $\hat{P}'^{\mathcal{A}}$ returned by $d(P^{\mathcal{A}}, g(P^{\mathcal{V}}_{+j}))$, where i has been assigned as j's guru, i.e. $(j, \succ^{\mathcal{V}}_i) \in \hat{P}'^{\mathcal{A}}$. Then guru i weakly benefits after j delegates if for every $k \in \mathcal{V}$:*

(a) *k's vote is assigned to guru $l \in \mathcal{V}$ by both returned preference profiles, i.e. $(k, \succ^{\mathcal{V}}_l) \in \hat{P}^{\mathcal{A}} \cap \hat{P}'^{\mathcal{A}}$, or*

(b) *k's vote is assigned to guru i after j joins the delegating electorate, i.e. $(k, \succ^{\mathcal{A}}_i) \in \hat{P}'^{\mathcal{A}}$.*

4 INTRODUCING BREADTH-FIRST DELEGATION

Recall that liquid democracy allows for multi-step delegations. Therefore, the guru of any $i \in V^d$ could be any voter $j \in V^c$ who is in the sub tree of the delegation graph with root i. Furthermore, the assigned guru j may not be included in i's preference relation $\succ^{\mathcal{V}}_i$, i.e. it could be that $\nexists\, x$ such that $\succ^{\mathcal{V}}_{i,x} = j$. In this case, there is at least one intermediate delegator between voter i and the assigned guru j. To find who exactly are intermediate delegators, we introduce the delegation chains as follows.

A *delegation chain* for a voter $i \in V^d$ starts with i, then lists the intermediate voters in V^d who have further delegated i's voting rights and ends with i's assigned guru $j \in V^c$. These chains (see Definition 6) must satisfy the following conditions: (a) no voter occurs more than once in the chain (to avoid infinite delegation cycles that could otherwise occur) and (b) each member of the chain must be linked to the next member through an edge in the delegation graph, which is generated from the given preference profile over voters.

Definition 6. *Given profiles $P^{\mathcal{A}}$ and $P^{\mathcal{V}}$, a voter $i \in V^d$ and her guru $j \in V^c$, we define a* delegation chain *for i to be an ordered tuple $C_i = \langle i, \ldots, j \rangle$ such that:*

(a) *for an integer $x \in [1, |C_i|]$, notation $C_{i,x}$ indicates the voter at the x-th position in C_i,*

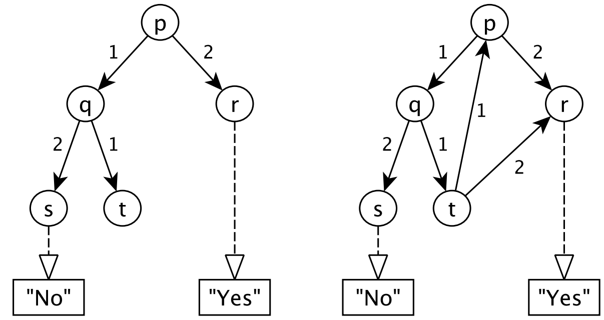
(b) *for every pair of integers $x, y \in [1, |C_i|]$ with $x \neq y$,*
$$C_{i,x} \neq C_{i,y},$$

(c) *for every integer $z \in [1, |C_i| - 1]$, there exists an edge*
$$(C_{i,z}, C_{i,z+1}) \in E \in g(P^{\mathcal{V}}),$$

(d) *for every integer $z \in [1, |C_i| - 1]$, the voter positioned at $C_{i,z}$ belongs in the V^d electorate , and*

(e) *the voter positioned at $C_{i,|C_i|}$ belongs in the V^c electorate.*

Observe that the variable x in the expression $C_{i,x}$ also indicates how deep the voter $C_{i,x}$ is in the delegation graph subtree rooted with vertex i. Thus sometimes we refer to x as the depth of $C_{i,x}$ in C_i. The function w takes as input a delegation chain and returns a list of the weights assigned to each edge among voters in C_i, that is,
$$w(C_i) = [w(C_i)_1, \ldots, w(C_i)_x, \ldots, w(C_i)_{n-1}],$$



(a) (b)

Figure 1: (a) A delegation graph with electorates $V^a = \{t\}$, $V^c = \{s, r\}$ and $V^d = \{p, q\}$, meaning that t abstains, s, r cast, and p, q delegate. The preference relations over alternatives are: "No" $\succ^{\mathcal{A}}_s$ "Yes" and "Yes" $\succ^{\mathcal{A}}_r$ "No". (b) A delegation graph with electorates $V^a = \{\}$, $V^c = \{s, r\}$ and $V^d = \{p, q, t\}$. The preference relations over alternatives remain the same. The only difference to (a) is that voter t decides to delegate with a preference relation over voters $p \succeq^{\mathcal{V}}_t r$.

where $w(C_i)_x$ is the weight of edge $(C_{i,x}, C_{i,x+1})$ and $n = |C_i|$.

Delegation chains can be used as a tool to find a guru for a voter $i \in V^d$. The standard interpretation of liquid democracy delegations prioritises all possible delegation chains involving i and i's most preferred voter $\succ^{\mathcal{V}}_{i,1}$ before all possible delegation chains involving i and i's second preferred voter $\succ^{\mathcal{V}}_{i,2}$ and so on. Note that this priority rule hold for the deeper levels of the delegation graph subtree rooted at i. In other words, we observe that the standard way to select i' guru is to choose the first casting voter found through a depth first search in i's subtree, which motivates the next definition.

A *depth-first delegation* rule d^D assigns guru j to i iff: (a) there is a delegation chain C_i that can be formed from i to j, and (b) there is no other delegation chain C'_i leading to a different guru k that has a smaller weight at the earliest depth after the root, compared to C_i.

Definition 7. *For $i, j, k \in \mathcal{V}$, a* depth-first delegation rule d^D *returns a profile $\hat{P}^{\mathcal{A}}$ with $(i, \succ^{\mathcal{A}}_j) \in \hat{P}^{\mathcal{A}}$ iff (a) and (b) hold:*

(a) *$\exists$ delegation chain C_i with $C_{i,|C_i|} = j$,*

(b) *$\nexists$ any delegation chain C'_i such that:*
  - *b1. $C'_{i,|C'_i|} = k$ for $k \neq j$,*
  - *b2.* • *$\exists y$: $w(C'_i)_y < w(C_i)_y$, and*
    • *$w(C'_i)_x \leq w(C_i)_x$ for all $0 < x < y$.*

Example 4.1. Consider the delegation graph in Figure 1 (a). There are two delegation chains[4] available for voter $p \in \mathcal{V}$: $C_p = \langle p, r \rangle$ and $C'_p = \langle p, q, s \rangle$ with weights $w(C_p) = [2]$ and $w(C'_p) = [1, 2]$, respectively. The d^D rule returns profile $\hat{P}^{\mathcal{A}}$ that assigns s as the guru of p, i.e. $(p, \succ^{\mathcal{A}}_s) \in \hat{P}^{\mathcal{A}}$, due to inequality $w(C'_p)_1 < w(C_p)_1$. Note that C'_p satisfies Definition 7.

[4] Recall that $\langle p, q, t \rangle$ does not satisfy the delegation chain definition as $t \notin V^c$.

DELEGATION GRAPH	DELEGATION RULE	Yes	No
Figure 1 (a)	d^D	1	3
Figure 1 (b)	d^D	3	2
Figure 1 (a)	d^B	2	2
Figure 1 (b)	d^B	3	2

Table 1: Election results for Figure 1 when using either the depth-first or the breadth-first delegation rule.

In Example 4.1, p's voting right is assigned to guru s, but why should s (who is the second preference of q) outrank agent p's explicit second preference r? This question gains even more importance the longer the depth-first delegation chain is. Given this issue, we define a novel delegation rule that prioritises a voter's explicit preferences as follows. A *breadth-first delegation* rule d^B assigns guru j to i iff: (a) there is a delegation chain C_i that can be formed from i to j; and (b) there is no other delegation chain C'_i leading to a different guru k with: either a shorter length or, an equal length and a smaller weight at the earliest depth after the root, compared to C_i.

Definition 8. *For $i, j, k \in \mathcal{V}$, a* breadth-first delegation rule d^B *returns a profile $\hat{P}^{\mathcal{A}}$ with $(i, \succ_j^{\mathcal{A}}) \in \hat{P}^{\mathcal{A}}$ iff (a) and (b) hold:*

(a) *$\exists$ delegation chain C_i with $C_{i,|C_i|} = j$,*
(b) *$\nexists$ any delegation chain C'_i such that $C'_{i,|C_i|} = k$, for $k \neq j$, and*

- *b1.* $|C'_i| < |C_i|$, *or*
- *b2.*
  - $|C'_i| = |C_i|$ *and*
  - $\exists y$: $w(C'_i)_y < w(C_i)_y$ *and*
  - $w(C'_i)_x \leq w(C_i)_x$ *for all* $0 < x < y$.

Example 4.2. Consider the delegation graph in Figure 1 (a). There are two delegation chains available for voter $p \in \mathcal{V}$: $C_p = \langle p, r\rangle$ and $C'_p = \langle p, q, s\rangle$ with weights $w(C_p) = [2]$ and $w(C'_p) = [1, 2]$, respectively. The d^B rule returns profile $\hat{P}^{\mathcal{A}}$ that assigns r as the guru of p, i.e. $(p, \succ_r^{\mathcal{A}}) \in \hat{P}^{\mathcal{A}}$, due to inequality $|C_p| < |C'_p|$. Note that C_p satisfies Definition 8.

5 DEPTH-FIRST VERSUS BREADTH-FIRST DELEGATION

Through the next two examples, we show that different delegation rules can have different properties. More specifically, we present an instance where the depth-first delegation rule cannot guarantee guru participation, while the breadth-first delegation rule does.

Example 5.1. Consider the delegation graph in Figure 1(a) and all possible delegation chains available to each voter in V^d: $C_p = \langle p, r\rangle$, $C'_p = \langle p, q, s\rangle$ and $C_q = \langle q, s\rangle$.

Using rule d^D, voter p is assigned guru s through chain C'_p (see Example 4.1), while voter q is also assigned guru s through chain C_q. Therefore d^D returns the preference profile over alternatives

$$\{(p, \succ_s^{\mathcal{A}}), (q, \succ_s^{\mathcal{A}}), (s, \succ_s^{\mathcal{A}}), (r, \succ_r^{\mathcal{A}})\}.$$

Using rule d^B instead, voter p is assigned guru r through C_p (see Example 4.2), while q's guru remains the same. Therefore d^B returns another preference profile over alternatives:

$$\{(p, \succ_r^{\mathcal{A}}), (q, \succ_s^{\mathcal{A}}), (s, \succ_s^{\mathcal{A}}), (r, \succ_r^{\mathcal{A}})\}.$$

In the next example we focus on the case where the previously abstaining voter t decides to delegate and show that the election result is inversed only when d^D is used (see Table 1).

Example 5.2. Consider the delegation graph in Figure 1(b) and all possible delegation chains available to each voter in V^d with their respective edge weights:

DELEGATION CHAIN	EDGE WEIGHTS
$C_p = \langle p, r\rangle$	$w(C_p) = [2]$,
$C'_p = \langle p, q, s\rangle$	$w(C'_p) = [1, 2]$,
$C''_p = \langle p, q, t, r\rangle$	$w(C''_p) = [1, 1, 2]$,
$C_q = \langle q, s\rangle$	$w(C_q) = [2]$,
$C'_q = \langle q, t, r\rangle$	$w(C'_q) = [1, 2]$,
$C''_q = \langle q, t, p, r\rangle$	$w(C''_q) = [1, 1, 2]$,
$C_t = \langle t, r\rangle$	$w(C_t) = [2]$,
$C'_t = \langle t, p, r\rangle$	$w(C'_t) = [1, 2]$,
$C''_t = \langle t, p, q, s\rangle$	$w(C''_t) = [1, 1, 2]$.

Using rule d^D, observe that voter p is assigned guru r through the chain C''_p due to $w(C''_p)_1 < w(C_p)_1$, $w(C''_p)_1 = w(C'_p)_1$ and $w(C''_p)_2 < w(C'_p)_2$. Voter q is also assigned guru r through chain C''_q since $w(C''_q)_1 < w(C_q)_1$, $w(C''_q)_1 = w(C'_q)_1$ and $w(C''_q)_2 < w(C'_q)_2$. Last, voter t is assigned guru s through chain C''_t because $w(C''_t)_1 < w(C_t)_1$, $w(C''_t)_1 = w(C'_t)_1$ and $w(C''_t)_2 < w(C'_t)_2$. Therefore rule d^D returns the preference profile over alternatives

$$\{(p, \succ_r^{\mathcal{A}}), (q, \succ_r^{\mathcal{A}}), (s, \succ_s^{\mathcal{A}}), (r, \succ_r^{\mathcal{A}}), (t, \succ_s^{\mathcal{A}})\}.$$

Using rule d^B instead, voter p is assigned guru r through the chain C_p due to inequalities $|C_p| < |C'_p| < |C''_p|$. Voter q is assigned guru s through C_q due to $|C_q| < |C'_q| < |C''_q|$ and voter t is assigned guru r through C_t because of $|C_t| < |C'_t| < |C''_t|$. Therefore, rule d^B returns the profile over alternatives

$$\{(p, \succ_r^{\mathcal{A}}), (q, \succ_s^{\mathcal{A}}), (s, \succ_s^{\mathcal{A}}), (r, \succ_r^{\mathcal{A}}), (t, \succ_r^{\mathcal{A}})\}.$$

Examples 5.1 and 5.2 show that guru participation may not hold for depth-first delegation when a cycle exists in the delegation graph. Due to this cycle, when t joins the election, both r and s receive new delegated voting rights, thus Observation 1 does not occur[5]. We summarise the above for the majority rule $f \in \bar{F}$.

Theorem 5.3. *Given the majority rule $f \in \bar{F}$, guru participation is not guaranteed to hold when using the depth-first delegation rule d^D.*

[5] Observation 1 states how guru participation can be satisfied when a voting rule satisfying cast participation is used: when a voter joins the delegating electorate, if only one voter increase the number of times assigned as a guru, then this voter is weakly better off.

Proof. Consider the preference profile over alternatives and the preference profile over voters of Figure 1(b),

$$P^{\mathcal{A}} = \{(r, \succ_r^{\mathcal{A}}), (s, \succ_s^{\mathcal{A}})\}$$
$$P^{\mathcal{V}} = \{(p, \succ_p^{\mathcal{V}}), (q, \succ_q^{\mathcal{V}}), (t, \succ_t^{\mathcal{V}})\},$$

where the preferences over alternatives for r and s are: "Yes" $\succ_r^{\mathcal{A}}$ "No", "No"$\succ_s^{\mathcal{A}}$ "Yes" and the preferences over voters for p, q, t are: $q \succ_p^{\mathcal{V}} r$, $t \succ_q^{\mathcal{V}} s$ and $p \succ_t^{\mathcal{V}} r$. We prove this theorem using Examples 5.1 and 5.2. In Example 5.1, where voter t abstains, rule d^D returns profile

$$d^D(P^{\mathcal{A}}, g(P^{\mathcal{V}}_{-t})) =$$
$$\{(p, \succ_s^{\mathcal{A}}), (q, \succ_s^{\mathcal{A}}), (s, \succ_s^{\mathcal{A}}), (r, \succ_r^{\mathcal{A}})\},$$

which gives three votes (via s) for alternative "No" and one vote (via r) for alternative "Yes" (see also Table 1). From Example 5.2 where voter t delegates, rule d^D returns profile

$$d^D(P^{\mathcal{A}}, g(P^{\mathcal{V}})) =$$
$$\{(p, \succ_r^{\mathcal{A}}), (q, \succ_r^{\mathcal{A}}), (s, \succ_s^{\mathcal{A}}), (r, \succ_r^{\mathcal{A}}), (t, \succ_s^{\mathcal{A}})\},$$

which gives three votes for "Yes" and two votes for "No". Observe that the election result changes from "No" to "Yes" despite the fact that t votes for "No" through her guru s, i.e. $(t, \succ_s^{\mathcal{A}}) \in d^D(P^{\mathcal{A}}, g(P^{\mathcal{V}}))$. Note that due to the preference "No"$\succ_s^{\mathcal{A}}$ "Yes", we get

$$f(d^D(P^{\mathcal{A}}, g(P^{\mathcal{V}}))) \prec_s^{\mathcal{A}} f(d^D(P^{\mathcal{A}}, g(P^{\mathcal{V}}_{-t}))), \quad (2)$$

where f is a voting rule satisfying cast participation. However, the preference expressed by (2) implies that guru s becomes worst off after t delegates to her, which violates the definition of guru participation (1), proving this theorem. □

We highlight that if a delegation graph has no cycle then guru participation is guaranteed to hold for the depth-first delegation rule, which show through Lemma 5.4 and Theorem 5.5.

Lemma 5.4. *When using depth-first delegation rule d^D, if there is no cycle in the delegation graph then Observation 1 holds.*

Proof. Assume there exists a delegation graph with no cycles where Observation 1 does not hold. We show that the only case where Observation 1 does not hold is when a cycle exists.

Recall that (by Observation 1) guru participation is guaranteed to hold if whenever a voter j joins the delegating electorate, there exists only one voter, say i, in the casting electorate who increases the number of times she becomes a guru. Consider a voter k who changes her assigned guru to another voter l after j joins the delegating electorate, where $l \neq i$ and $k \neq j$. This means that, apart from i, voter l also increases the times she becomes a guru. Next we describe that, when d^D is used, this case can only arise through the following circumstance. Let guru i be assigned to voter j through delegation chain $C_j = \langle j, \ldots, i \rangle$ and guru l be assigned to voter k through delegation chain $C_k = \langle k, ..., j, ..., l \rangle$. Chain C_k must pass through j because all chains without j are available before j delegates. Note that even if both chains pass through voter j, they end at different gurus. For d^D, this only occurs if there exists a voter h with $h \neq i, j, l$, such that

$$C_j = \langle j, \ldots, h, \ldots, i \rangle \quad \text{and} \quad (3)$$
$$C_k = \langle k, \ldots, h, \ldots, j, \ldots, l \rangle. \quad (4)$$

The reason for the above is that k's delegation goes through h to reach j, but then the preferred delegation from j passes through h (see chain C_j). As k's delegation already includes h before j, and an intermediator voter cannot be repeated (definition 3), k uses another route to guru l (through a less preferred option of j). From (3) and (4), observe that there exists a cycle in the graph, i.e. the cycle $\langle h, ..., j, ..., h \rangle$, which contradicts our assumption and proves the lemma. □

Theorem 5.5. *Given the majority rule $f \in \bar{F}$ and a delegation graph with no cycles, guru participation is guaranteed to hold when using the depth-first delegation rule d^D.*

Proof. We prove this using Lemma 5.4 and Observation 1. □

We have previously shown that depth-first delegation does not guarantee guru participation when the delegation graph contains cycles. The next theorem states that breadth-first delegation always guarantees guru participation. To show this, we first introduce the following observation and lemma.

Observation 2. *Consider two voters j and k in a delegating electorate. Using the breadth-first delegation rule d^B, if k is assigned guru l through a delegation chain C_k with $j \notin C_k$, then k is assigned guru l even when j abstains.*

The above observation occurs because rule d^B has used C_k ahead of any possible delegation chain that includes j. Therefore chain C_k will still be used by d^B when j is in the abstaining electorate and no possible delegation chain that includes j can be formed.

Lemma 5.6. *Consider two voters j and k in a delegating electorate. Using the breadth-first delegation rule d^B, if voter k is assigned her guru through a delegation chain C_k with $j \in C_k$, then k is assigned the same guru as j.*

Proof. Assume that, using d^B, voter j is assigned guru i through delegation chain $C_j = \langle j \ldots, i \rangle$ and k is assigned a different guru l through a delegation chain that includes j, i.e.

$$C_k = \langle k, \ldots, j, \ldots, l \rangle.$$

Then either (a) or (b) occurs:

(a) rule d^B should use chain $C'_j = \langle j, \ldots, l \rangle$, which contradicts the assumption that C_j is used,

(b) there exists a shared intermediate voter e such that

$$C_j = \langle j, \ldots, e, g, \ldots, i \rangle \quad \text{and}$$
$$C_k = \langle k, \ldots, f, j, \ldots, l \rangle,$$

where $e \in \langle k, \ldots, f \rangle$. Recall that d^B prioritises shorter length delegation chains (see definition 8). We show that voter k has a shorter delegation chain available that does not include j, i.e. there exists a C'_k such that $|C'_k| < |C_k|$ and $j \notin C'_k$. Let $C'_k = \langle k, \ldots, e, g, \ldots, i \rangle$. According to d^B, the delegation chain used to assign j's guru, $\langle j \ldots, e, g, \ldots, i \rangle$, is shorter or equal in length to any other alternative, thus $|\langle j, \ldots, e, g, \ldots, i\rangle| \leq |\langle j, \ldots, l\rangle|$. Observe that

$$|\langle g, \ldots, i\rangle| < |\langle j, \ldots, e, g, \ldots, i\rangle| \leq |\langle j, \ldots, l\rangle| \Rightarrow$$
$$|\langle k, \ldots, e\rangle| + |\langle g, \ldots, i\rangle| < |\langle k, \ldots, e\rangle| + |\langle j, \ldots, l\rangle|.$$

Since $e \in \langle k, \ldots, f\rangle$, we can rewrite the previous as

$$|\langle k, ..., e\rangle| + |\langle g, ..., i\rangle| < |\langle k, ..., f\rangle| + |\langle j, ..., l\rangle|.$$

Therefore, rule d^B should use C'_k to assign k's guru. However, since $j \notin C'_k$, the assumption is contradicted.

The contradictions of both (a) and (b) prove this lemma. □

THEOREM 5.7. *Given the majority rule $f \in \bar{F}$, guru participation is guaranteed to hold when using the breadth-first delegation rule d^B.*

PROOF. By Observation 2, given voters j and k in the delegating electorate, if a voter k does not delegate through j, then k's assigned guru (if any) is the same even if j abstained. By Lemma 5.6, if a voter k delegates through j, then the guru of k is the same as the guru of j. Combining the above cases, we show that (regardless of k delegating through j or not), whenever a voter j joins the delegating electorate and is assigned to a guru i, then i is the only casting voter who increases the number of times she becomes a guru. Since also $f \in \bar{F}$, then Observation 1 holds, meaning that the breadth-first delegation rule d^B is guaranteed to satisfy guru participation. □

6 CONCLUSION AND FUTURE WORK

In this paper, we discuss the depth-first and the breadth-first delegation rule proving that only the latter has the desirable property that every guru weakly prefers receiving delegated voting rights under the majority rule with two alternatives. The immediate future questions that arise are to investigate what holds for more than two alternatives in this model or for other voting rules that satisfy cast participation such as approval voting (Felsenthal [2010]). However, there could be delegation rules that satisfy other interesting properties which improve the concept of liquid democracy. Towards this path, we note that one of the main issues that current liquid democracy implementations suffer from, is that large parts of an electorate might end up being represented by only a small subset of gurus (Kling et al. [2015]). Since the breadth-first delegation rule favours keeping delegated voting rights close to their origin, could this issue be resolved by using this rule? Other interesting future directions are investigating guru participation with voting rules that do not satisfy cast participation, relaxing the assumption of strict personal rankings over voters, and analysing other types of participation.

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