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# ASYMPTOTIC PERIOD RELATIONS FOR JACOBIAN ELLIPTIC SURFACES

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**Abstract**

We describe the image of the locus of hyperelliptic curves of genus $g$ under the period mapping in a neighbourhood of the diagonal locus $\mathfrak{Diag}_g$. There is just one branch for each of the alkanes $C_gH_{2g+2}$ of elementary organic chemistry, and each branch has a simple linear description in terms of the entries of the period matrix.

This picture is replicated for simply connected Jacobian elliptic surfaces, which form the next simplest class of algebraic surfaces after K3 and abelian surfaces. In the period domain for such surfaces of geometric genus $g$ there is a locus $\mathcal{W}_{1^g}$ that is analogous to $\mathfrak{Diag}_g$, and the image of the moduli space under the period map has just one branch through $\mathcal{W}_{1^g}$ for each alkane. Each branch is smooth and has an explicit description as a vector bundle of rank $g-1$ over a domain that contains $\mathcal{W}_{1^g}$.

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# 1 Introduction

The classical Schottky problem is that of describing the period locus $\mathfrak{J}_g$ of period matrices of complex algebraic curves of genus $g$ as a subvariety of Siegel space $\mathfrak{H}_g$.

This problem naturally extends to higher dimensions; for example, an algebraic surface of positive geometric genus $g$ has a period matrix that arises from integrating 2-forms around 2-cycles, and then the Schottky problem becomes that of describing the image of the moduli space under the multi-valued period map. This image we refer to as the period locus. As described below, we are only concerned here with local aspects of the geometry of the situation, for which the fact that the period map is multi-valued is irrelevant.

We consider this problem for simply connected Jacobian elliptic surfaces of geometric genus $g$. In the classification of surfaces these are the simplest beyond K3 and abelian surfaces (or those which are quotients of such surfaces); for these latter varieties the Schottky problem scarcely arises, since the period map is an isomorphism. Of course, beyond those lie the surfaces of general type, the well known complexity of whose moduli spaces suggests that it is reasonable to focus on some particular classes of surfaces such as the ones considered here.

The main result of this paper is that the picture for these elliptic surfaces is analogous to that for hyperelliptic curves and that both pictures are described by the *alkanes* of elementary organic chemistry. These are the acyclic saturated hydrocarbons and their molecular formula is $C_gH_{2g+2}$. The connexion with algebraic geometry is that on a hyperelliptic curve of genus $g$ the hyperelliptic involution has $2g+2$ fixed points and that when such curves degenerate to trees of elliptic curves then each elliptic curve $E$ plays the rôle of a quadrivalent carbon atom C where the 2-torsion points of $E$ appear as the bonds (either C–C or C–H) of C.

In fact, there is a subdomain $\mathcal{W}_{1^g}$ of the period domain $\mathcal{V}_g$ for simply connected Jacobian elliptic surfaces which corresponds to trees of $g$ *special Kummer surfaces*, whose definition is recalled below, and which is isomorphic to $\mathfrak{H}_1^{g+1}$. We regard this as the analogue of the locus $\mathfrak{Diag}_g$ of diagonal matrices in $\mathfrak{H}_g$, which corresponds to trees of elliptic curves. In both cases there is an action of the symmetric group $\mathrm{Sym}_g$ on the germs $(\mathcal{V}_g, \mathcal{W}_{1^g})$ and $(\mathfrak{H}_g, \mathfrak{Diag}_g)$ that stems from the fact that the stabilizer of $\mathcal{W}_{1^g}$ (respectively, $\mathfrak{Diag}_g$) in the relevant discrete group is $(SL_2(\mathbb{Z})\wr \mathrm{Sym}_g)\times SL_2(\mathbb{Z})$ (respectively, $SL_2(\mathbb{Z})\wr \mathrm{Sym}_g$), where $\wr$ denotes the wreath product.

In slightly more detail, the main result can be summarized like this. Let $\mathcal{JE}_g$ denote the stack of simply connected Jacobian elliptic surfaces and $PL_g$, the period locus, its image under the period map. Then

(1) the branches of $PL_g$ through $\mathcal{W}_{1^g}$ and the branches of the period locus $\mathfrak{Hyp}_g$ of hyperelliptic curves through $\mathfrak{Diag}_g$, taken modulo the action of $\mathrm{Sym}_g$, both correspond to the alkanes $C_gH_{2g+2}$ and

(2) each of these branches has, to first order, a straightforward and explicit linear description in terms of matrices.

In outline, the proof goes as follows: an elaboration of a plumbing construction introduced by Fay [F1], as corrected in [F2], leads to the construction and description of one branch for each alkane, and then a stable reduction theorem based on the Minimal Model Program (MMP) establishes that there are no further branches.

In fact, our approach also provides a slight variant of Chakiris' proof [C1] [C2] of the generic Torelli theorem for these surfaces.

**Notation**: If $\mathfrak{X}$ is a separated Deligne–Mumford stack, then we denote its geometric quotient by $[\mathfrak{X}]$. if $G$ is a finite group acting on a space $X$, then $X/G$ will denote the quotient stack and $[X/G]$ the geometric quotient.

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# 2 Some further details

We begin by recalling some constructions that involve algebraic curves. Later we shall extend these constructions to include algebraic surfaces.

Fay [F1] constructed certain degenerating families of complex algebraic curves (that is, compact Riemann surfaces) via means of explicit plumbing constructions that are recalled below and then derived formulae for the derivative of the period matrix of each of these families.

Akira Yamada [Y] then pointed out that Fay's formulae are wrong, and gave correct formulae for Fay's constructions.

In [F2] (bottom of page 123), Fay corrects his error by pointing out that his plumbing constructions should have been done differently, and that for these different plumbing constructions his formulae are correct.

In short, the resulting confusion can be resolved as follows. Fay in [F1] in fact made two different plumbing constructions of 1-parameter degenerating families of curves without monodromy (so the curve degenerates but its Jacobian does not) for which there are explicit formulae for the derivative of the period matrix. For one construction the correct formula is that given in [F1] and for the other the correct formula is given in [Y]. (There are also two different plumbing constructions of families of curves with monodromy but we shall not use such constructions in this paper.) We shall refer to the plumbing constructions for which Yamada's formula [Y] is correct as Yamada plumbings, and those for which Fay's formula [F1] is correct as Fay plumbings.

We shall recall the detailed construction of Fay plumbings in Section 4. In Section 12 we will give his formula for the derivative of the resulting period matrix; it is convenient to point out here that in our version of his formula there is a minus sign that does not appear in [F1]. This is because we have chosen a different normalization which slightly increases the flexibility of the construction.

In fact, and this is the crux of this paper, we do this in higher dimensions; one advantage of Fay plumbings is that they can be generalized to plumb not only curves but also, at least in certain circumstances, morphisms from curves to stacks.

In the body of the paper we discuss surfaces first, and then, in Sections 12 to 14, proceed to consider curves. We do this because the formulae for curves are, in essence, special cases of those for surfaces. However, in this introduction we shall take curves first.

After recovering Fay's formulae we also recover his version [F1] of Poincaré's "asymptotic period relations". These were discovered by Poincaré [P] when $g = 4$ and generalized by Fay to all values of $g$. According to Igusa's account (see p. 167 of [I]) Poincaré exploited the geometry of the theta divisor of a Jacobian (specifically, that it is a hypersurface of translation type), so his argument cannot extend to the case of surfaces, while Fay uses his plumbing construction. We point out how these relations describe, both intrinsically and in terms of co-ordinates, the tangent cone to the closure $\mathfrak{J}_g^c$ of the Jacobian locus along the locus $\mathfrak{Diag}_g$ of diagonal matrices in Siegel space $\mathfrak{H}_g$ in terms of the Grassmannian $Grass(2, g)$ that classifies lines in $\mathbb{P}^{g-1}$.

These Poincaré–Fay asymptotic period relations have also been recovered by Farkas, Grushevsky and Salvati Manni [FGSM] in the course of proving their global weak solution to the Schottky problem. More precisely, they show that differentiating the identities obtained by substituting the Schottky–Jung proportionalities into Riemann's quartic theta identities leads to the Poincaré–Fay relations; they deduce their global weak solution as an immediate consequence.

Then we describe, to first order along $\mathfrak{Diag}_g$, the closure $\mathfrak{Hyp}_g^c$ of the hyperelliptic locus in $\mathfrak{H}_g$. This description is given in terms of the *alkanes* of elementary organic chemistry, which were first enumerated by Cayley; see sequence A000602 in [OEIS] for corrections. These are the hydrocarbons whose molecular formula is of the type $C_gH_{2g+2}$ (and so are exactly the saturated acyclic hydrocarbons) and have uses ranging from fuel to furniture polish, depending on their molecular weight. The $g$-alkane is the one whose carbon skeleton is a chain of length $g$. It is distinguished from the others of the same molecular formula by having a higher boiling point. We shall refer to $g$ as the genus of the alkane. The number $2g+2$ is the number of fixed points of the hyperelliptic involution on a hyperelliptic curve of genus $g$. It turns out that, modulo the action of the symmetric group $\mathrm{Sym}_g$ on $\mathfrak{H}_g$ given by $s(\tau_{ij}) = \tau_{s(i),s(j)}$, there is one branch of $\mathfrak{Hyp}_g^c$ along $\mathfrak{Diag}_g$ for each alkane, each branch is smooth and there are explicit first-order equations for each branch in terms of the entries $\tau_{ij}$ of the period matrix.

Here are more precise statements. Recall that a square matrix $(a_{ij})$ is *tridiagonal* if $a_{ij} = 0$ whenever $|i-j| \geq 2$ and *quadridiagonal* if $a_{ij} = 0$ whenever $|i-j| \geq 3$. The locus of tridiagonal symmetric $g \times g$ matrices has dimension $2g-1$.

**Theorem 2.1** *(a special case of Theorem 14.6) To first order (that is, modulo the square of the defining ideal) the branch of $\mathfrak{Hyp}_g^c$ through $\mathfrak{Diag}_g$ in $\mathfrak{H}_g$ that corresponds to the $g$-alkane is the locus of symmetric tridiagonal matrices.*

There is a similar, although slightly more complicated, description of the branches corresponding to the other alkanes. Poincaré's asymptotic period relations suggest that, on the other hand, the fact that the locus of symmetric quadridiagonal matrices has dimension $3g-3$ has no parallel significance.

Now turn to surfaces. We use Fay's plumbing to make similar constructions and calculations for degenerating families of simply connected Jacobian elliptic surfaces, which can be regarded as the simplest surfaces of strictly positive Kodaira dimension and also as the simplest surfaces for which, thanks mainly to our understanding of the period map for K3 surfaces, the Schottky problem is not vacuous.

Fix an integer $h \geq 2$ and consider Jacobian elliptic surfaces $X$ over $\mathbb{P}^1$ of geometric genus $h$. These have $10h+8$ moduli, and the coarse moduli space is rational. The primitive (co)homology $H = H_{prim}$ of $X$ is the orthogonal complement in $H^2(X)$ of the section and a fibre; $h^2(X) = 12h+10$ and rank $H = 12h+8$. We can describe a chart of the period domain

$$\mathcal{V}_h = \{\xi \in Grass(h, H_{\mathbb{C}}) : (u,v) = 0 \text{ and } (u,\overline{u}) > 0 \ \forall u,v \in \xi\}$$

as follows.

Pick a totally isotropic sublattice $L$ of $H$ whose rank is $h$.

**Definition 2.2** *The surface $X$ is in $L$-*general position *if the pairing $L \otimes \mathbb{C} \times H^0(X, \Omega^2_X) \to \mathbb{C}$ given by integration is non-degenerate.*

Assume that $X$ is in $L$-general position. Then a basis $(A_1, ..., A_h)$ of $L$ defines a basis $(\omega_1, ..., \omega_h)$ of $H^0(X, \Omega^2_X)$ which is normalized by the requirement $\int_{A_i} \omega_j = \delta_{ij}$. Extend $(A_1, ..., A_h)$ first to a basis of $L^\perp$ and then to a basis of $H$ such that the induced basis of $H/L^\perp$ is dual to $(A_1, ..., A_h)$ and $H$ is decomposed as

$$H = L \oplus (L^\perp/L) \oplus (H/L^\perp).$$

Then the normalized period matrix of $X$ has, when we ignore the $h \times h$ identity matrix that arises from integrating around the cycles $A_i$, two blocks: the first is an $h \times (10h+8)$ block that arises from integrating the forms $\omega_i$ around cycles in $L^\perp/L$ and the second is an $h \times h$ block that is skew-symmetric. So, if $h = 1$, this last block is zero and can be ignored to yield a vector of length 18. Different isotropic lattices $L$ will give different charts.

Let $PL_h$, the *period locus*, denote the image of the moduli stack in the period domain $\mathcal{V}_h$ under the period map. We shall recall the definition of the domain of the period map later; the presence of RDPs creates a slight subtlety.

**Definition 2.3** *A Jacobian elliptic surface is* special *if it is birational to a geometric quotient $[C \times E/\iota]$ where $C$ is an elliptic or hyperelliptic curve, $E$ is an elliptic curve and $\iota$ acts on $C$ as $(-1_C)$ if $C$ is elliptic and as its hyperelliptic involution otherwise, and on $E$ as $(-1_E)$.*

Special surfaces $X$ are characterized as the simply connected Jacobian elliptic surfaces whose global monodromy on the cohomology of the generic fibre of the elliptic fibration is of order two [C1]. Their geometric genus is given by $h = p_g(X) = g(C)$. Their bad fibres are all of type $\widetilde{D}_4$ (or $I_0^*$ in Kodaira's notation) and there are exactly $2h+2$ such fibres; they correspond to the fixed points of $\iota$.

Here are some trivial remarks, some definitions and some notation.

**Definition 2.4**

*(1)* $\dim \mathcal{V}_h = h(10h+8) + h(h-1)/2$.

*(2) If $h = \sum h_i$ then the period domain $\mathcal{V}_h$ contains a copy of $\mathcal{V}_{h_1} \times \cdots \times \mathcal{V}_{h_r}$.*

(3) *Each* $\mathcal{V}_{h_i}$ *contains a copy* $PL_{h_i,special}$ *of the period locus of special elliptic surfaces of genus* $h_i$*.* $PL_{h_i,special}$ *is isomorphic to* $\mathfrak{Hyp}_{h_i} \times \mathfrak{H}_1$ *and so has dimension* $2h_i$*.*

(4) *If* $j \in [0,4]$ *then* $\mathcal{K}_j$ *denotes the period domain for Jacobian elliptic K3 surfaces with at least* $j$ *fibres of type* $\widetilde{D}_4$*. So* $\mathcal{K}_j$ *is a subdomain of* $\mathcal{V}_1$*,* $\dim \mathcal{K}_j = 18 - 4j$ *and* $\mathcal{K}_4 = PL_{1,special}$*.*

(5) *Fix an alkane* $\Gamma$ *of genus* $h$ *and let* $\gamma_j$*, for* $j \in [1,4]$*, denote the number of vertices (carbon atoms) in* $\Gamma$ *that are joined to* $j$ *other carbon atoms in* $\Gamma$*. So, in particular,* $h = \sum \gamma_j$ *and* $h - 1 = \sum j\gamma_j$*. There is then a closed subvariety* $\mathcal{V}_\Gamma$ *of the product* $\mathcal{K}_1^{\gamma_1} \times \cdots \times \mathcal{K}_4^{\gamma_4}$ *(which is, in turn, a subdomain of* $\mathcal{V}_h$*) that is defined by the requirement that the* $\widetilde{D}_4$*-fibres on K3s that are adjacent in* $\Gamma$ *should be isomorphic. It is easy to see that* $\dim \mathcal{V}_\Gamma = 9h + 9$*.*

(6) *In* $PL_{h_1,\ special} \times \cdots \times PL_{h_r,\ special}$*, which is isomorphic to* $\prod \mathfrak{Hyp}_{h_i} \times \mathfrak{H}_1^r$ *and so has dimension* $2h$*, there is a subvariety* $\mathcal{W}_{h_1,\dots,h_r}$ *defined by the property that the factors in each copy of* $\mathfrak{H}_1$ *are equal. This is the period locus for unions of special elliptic surfaces that are birational to geometric quotients* $[C_i \times E_i/\iota]$ *of genera* $h_1, \dots, h_r$ *where the elliptic curves* $E_i$ *are isomorphic, so that* $\dim \mathcal{W}_{h_1,\dots,h_r} = 2h - (r-1)$*. In particular,* $\dim \mathcal{W}_{1^h} = h + 1$*.*

(7) *For each* $\Gamma$*,* $\mathcal{W}_{1^h}$ *is isomorphic to the subvariety of* $\mathcal{V}_\Gamma$ *defined by the condition that each K3 surface should be special.*

We regard $\mathcal{W}_{1^h}$, which is isomorphic to $(\mathfrak{H}_1)^{h+1}$, as the analogue in the period domain $\mathcal{V}_h$ of the diagonal locus $\mathfrak{Diag}_g$ in $\mathfrak{H}_g$. There is an action of $(SL_2(\mathbb{Z}) \wr \mathrm{Sym}_h) \times SL_2(\mathbb{Z})$ on the pair $(\mathcal{V}_h, \mathcal{W}_{1^h})$, where the action on $\mathcal{W}_{1^h}$ arise from the permutation action of $\mathrm{Sym}_h$ on the first term in the isomorphism $\mathcal{W}_{1^h} \cong (\mathfrak{H}_1)^h \times \mathfrak{H}_1$.

The next result is stated in terms of a certain vector bundle $E_\Gamma \to \mathcal{V}_\Gamma$ of rank $h-1$. The fibre of $E_\Gamma$ over a point $(Y_1, \dots, Y_h)$ of $\mathcal{V}_\Gamma$ is the vector space spanned by certain matrices $\Pi_e$ of rank 1, where $e = (i,j)$ runs over the edges (the carbon-carbon bonds) of $\Gamma$. Each $\Pi_e$ is a tensor product $\Pi_e = \underline{\omega}_e \otimes \underline{I}_e$ of two vectors, where each vector is computed from the surfaces $Y_i$ and $Y_j$. The vector $\underline{\omega}_e$ is comprised of projective data while $\underline{I}_e$ is a vector of integrals, so consists of transcendental data. These vectors are described explicitly in Proposition 8.16.

**Theorem 2.5** *(= Theorem 11.11) Fix an alkane* $\Gamma$*.*

*(1) There is a branch* $B_\Gamma$ *of* $PL_h$ *through* $\mathcal{W}_{1^h}$ *that contains* $\mathcal{V}_\Gamma$*.*

*(2) To first order,* $B_\Gamma$ *is, in a neighbourhood of* $\mathcal{W}_{1^h}$*, the vector bundle* $E_\Gamma$ *over* $\mathcal{V}_\Gamma$*.*

*(3) The zero section of* $E_\Gamma$ *is the image of* $\mathcal{V}_\Gamma$ *embedded in* $B_\Gamma$*.*

This leads to the main result. It is an analogue of Theorem 14.6.

**Theorem 2.6** *(= Theorem 11.12) (1) The branch $B_\Gamma$ described above is the unique branch of the period locus $PL_h$ that contains $\mathcal{V}_\Gamma$.*

*(2) In a neighbourhood of $\mathcal{W}_{1^h}/\operatorname{Sym}_h$, the period locus $PL_h$ is, to first order, the union of the vector bundles $E_\Gamma$.*

# 3 The domain of the period map for Jacobian elliptic surfaces

In this section we elaborate the point that, while for moduli spaces of polarized surfaces it is natural to allow the surfaces to have RDPs, for the period map it is better to resolve the RDPs and to allow quasi–polarizations.

We shall refer to a given Jacobian elliptic surface $f : X \to C$ with section $C_0$ as smooth if it is smooth and also relatively minimal and as an RDP surface if it has only RDPs, $C_0$ lies in the smooth locus of $f$ and $C_0$ is $f$-ample. That is, if every geometric fibre of $f$ is a reduced and irreducible curve of arithmetic genus 1. They are objects of two of the stacks that we shall consider:

(1) $\mathcal{JE}^{sm}$ (resp., $\mathcal{JE}_h^{sm}$) is the stack of smooth Jacobian elliptic surfaces (resp., simply connected such surfaces of geometric genus $h$);

(2) $\mathcal{JE}^{RDP}$ (resp., $\mathcal{JE}_h^{RDP}$) is the stack of RDP Jacobian elliptic surfaces (resp., simply connected such surfaces of geometric genus $h$).

So $\mathcal{JE}^{RDP}$ is separated and there is a natural morphism $\mathcal{JE}^{sm} \to \mathcal{JE}^{RDP}$ given by passing to the relative canonical model. According to Artin's results [Ar], which we now recall, this morphism is representable, 1-to-1 on field-valued points but not separated.

Suppose that $(X \to C \to S, C_0)$ is an object of $\mathcal{JE}^{RDP}$ over $S$. Then we have Artin's functor $Res_{X/S}$, whose $T$-points, for an $S$-scheme $T$, are isomorphism classes of diagrams

$$\begin{array}{ccccc} \widetilde{X_T} & \xrightarrow{\pi} & X_T & \longrightarrow & X \\ & {\scriptstyle F}\searrow & \downarrow & & \downarrow \\ & & C_T & \longrightarrow & C \\ & & \downarrow & & \downarrow \\ & & T & \longrightarrow & S \end{array}$$

where $F$ is smooth, projective and relatively minimal (this is equivalent to saying that $\widetilde{X_T} \to C_T$ is an object of $\mathcal{JE}^{sm}$), and $\pi$ is projective and birational, in the sense that $\pi_*\mathcal{O} = \mathcal{O}$. Then $Res_{X/S}$ is represented by a locally quasi-separated algebraic space $R$ over $S$, such that $R \to S$ is a bijection on all field-valued points. The morphism $\mathcal{JE}^{sm} \to \mathcal{JE}^{RDP}$ described above can be localized to yield an isomorphism $\mathcal{JE}^{sm} \times_{\mathcal{JE}^{RDP}} S \xrightarrow{\cong} R$.

Now restrict attention to simply connected Jacobian elliptic surfaces $X$ of geometric genus $p_g = h$. Fix a unimodular lattice $\Lambda = \Lambda_h$ of rank $12h + 10$ and signature $(2h+1, 10h+9)$ and elements $\sigma, \phi \in \Lambda$ such that $\sigma^2 = -(h+1), \sigma.\phi = 1$ and $\phi^2 = 0$. So $\Lambda = \mathbb{Z}\{\sigma, \phi\} \perp H$ where $H = H_h$ is unimodular, its rank is $12h+8$ and its signature is $(2h, 10h + 8)$.

Assume also that $H$ is even; these requirements specify $\Lambda_h$ and $H_h$ uniquely. (For example, $H_{prim}(X)$ is even, from the fact that $c_1(X)$ is equivalent to a multiple of a fibre $\phi$ and the Wu formula, which says that $x^2 + x.c_1(X)$ is even for all classes $x \in H^2(X, \mathbb{Z})$.)

Consider the subgroup $\mathfrak{G}$ of the orthogonal group $O_\Lambda(\mathbb{Z})$ given by

$$\mathfrak{G} = \{\gamma \in O_\Lambda(\mathbb{Z}) | \gamma(\sigma) = \sigma \text{ and } \gamma(\phi) = \phi\};$$

then $\mathfrak{G}$ is naturally identified with $O_H(\mathbb{Z})$. There is a $\mathfrak{G}$-torsor $\mathcal{JE}_\Lambda \to \mathcal{JE}_h^{sm}$ defined by the fact that a $T$-point of $\mathcal{JE}_\Lambda$ consists of a $T$-point of $\mathcal{JE}_h^{sm}$ and an isometry $\Psi : \Lambda_T \to R^2F_*\mathbb{Z}$ such that $\Psi$ maps $\sigma$ to the class of the section $(C_0)_T$ of $\widetilde{X_T} \to C_T$ and $\phi$ to the class of a fibre.

The discussion in the third paragraph on p. 228 of [C1] can be translated into the language of stacks to say that $\mathcal{JE}_\Lambda$ is the domain of the period map. That is, the period map is a $\mathfrak{G}$-equivariant holomorphic morphism

$$\widetilde{per} : \mathcal{JE}_\Lambda \to \mathcal{V}_h,$$

where $\mathcal{V}_h$ is the period domain. Equivalently, the period map is the quotient morphism

$$per : \mathcal{JE}_h^{sm} = \mathcal{JE}_\Lambda/\mathfrak{G} \to \mathcal{V}_h/\mathfrak{G}$$

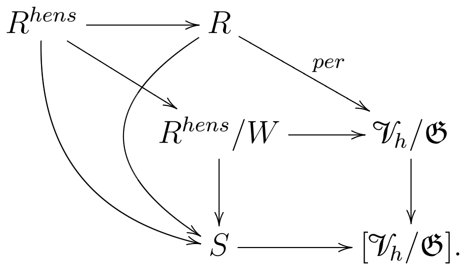
of quotient stacks. This fits into a commutative diagram

$$\begin{array}{ccc} \mathcal{JE}_h^{sm} & \xrightarrow{per} & \mathcal{V}_h/\mathfrak{G} \\ \downarrow & & \downarrow \\ \mathcal{JE}_h^{RDP} & \longrightarrow & [\mathcal{V}_h/\mathfrak{G}]. \end{array}$$

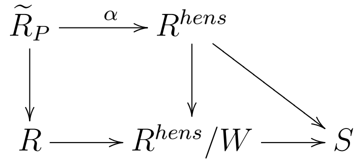
If $S$ is the henselization of $\mathcal{JE}_h^{RDP}$ at a closed point, so that $S$ is the base of a miniversal deformation of a simply connected RDP Jacobian elliptic surface $X_s$, then [Ar] the henselization $R^{hens}$ of $R = \mathcal{JE}_h^{sm} \times_{\mathcal{JE}_h^{RDP}} S$ at its unique closed point is the base of a miniversal deformation of the minimal resolution $\widetilde{X}_s$ of $X_s$. (In fact Artin shows that $R^{hens}$ is the base of a versal deformation, but in characteristic zero his argument proves the miniversality of $R^{hens}$.) So, for local purposes such as those of this paper, we can take the domain of the period map to be a miniversal deformation space of a smooth Jacobian elliptic surface.

In fact, slightly more is true. The finite Weyl group $W$ associated to the configuration of singularities on $X_s$ acts on $R^{hens}$ in such a way that $[R^{hens}/W] =$

$S$ and there is a commutative diagram




This is because there is a non-separated union $\widetilde{R}_P = \cup_{\mathcal{C}} R^{hens}_{\mathcal{C}}$ of copies of $R^{hens}$ on which $W$ acts freely, where there is one copy $R^{hens}_{\mathcal{C}}$ of $R^{hens}$ for each chamber $\mathcal{C}$ that is defined in the usual way and two charts $R^{hens}_{\mathcal{C}}, R^{hens}_{\mathcal{E}}$ in $\widetilde{R}_P$ are glued in a way that depends upon the relative position of the chambers $\mathcal{C}, \mathcal{E}$ with respect to $W$. (This glueing always induces an isomorphism over the complement of the discriminant locus in $S$.) Moreover $W$ permutes the charts $R^{hens}_{\mathcal{C}}$ simply transitively, and $\widetilde{R}_P/W = \widetilde{\mathcal{J}\mathcal{E}}_h \times_{\mathcal{J}\mathcal{E}_h} S = R$. The isomorphisms $R^{hens}_{\mathcal{C}} \to R^{hens}$ glue to a morphism $\alpha : \widetilde{R}_P \to R^{hens}$ that exhibits $R^{hens}$ as the maximal separated quotient of $\widetilde{R}_P$ in the category of schemes. So $W$ also acts on $R^{hens}$, $\alpha$ is $W$-equivariant and $S = [R^{hens}/W]$. All this is documented in [SB], to which we add one comment: since, in the commutative diagram




the vertical maps are quotients by $W$, so that the square is 2-Cartesian, it follows that the scheme $S$ is the maximal separated quotient of the algebraic space $R$ in the category of algebraic spaces and that the stack $R^{hens}/W$ is the maximal separated quotient of $R$ in the 2-category of algebraic stacks.

Finally, consider the forgetful morphism $\alpha : S \to \prod_x Def_{X_s,x}$ from $S$ to the product of the miniversal deformation spaces of the germs $(X_s, x)$ of $X_s$ at its singularities $x$ and the morphisms $Def_{\widetilde{X}_s,x} \to Def_{X_s,x}$. These latter are geometric quotients by the relevant finite Weyl group and $R^{hens}$ is isomorphic to the fibre product

$$S \times_{\prod_x Def_{X_s,x}} \prod_x Def_{\widetilde{X}_s,x},$$

so that if $\alpha$ is smooth then so is $R^h$.

## 4 Fay's plumbing for curves and stacky curves

In this section we give a detailed description of Fay plumbings. All of this is taken from [F1] and [F2] but we repeat it here to avoid confusion.

Fix a real number $\delta$ with $0 < \delta \ll 1$. Let $\Delta = \Delta_t$ be the open disc in the complex plane $\mathbb{C}$ with co-ordinate $t$ defined by $|t| < \delta^2$ and let $F$ be the open submanifold of $\mathbb{C}^2$ with co-ordinates $q, v$ defined by $|q| < \delta^{1/2}$, $|v| < \delta$ and $|v \pm q| < \delta$ (so that, in particular, $|v^2 - q^2| < \delta^2$). Set $t = v^2 - q^2$. Note that the morphism $t : F \to \Delta$ is smooth outside the origin in $\Delta$ and the fibre $t^{-1}(0)$ consists of two discs $U'_a$ and $U'_b$, where $U'_a$ is given by $q - v = 0$ and $U'_b$ by $q + v = 0$. These discs cross normally at the point 0 given by $q = v = 0$.

We shall use $F$, with these co-ordinates, as the basic plumbing fixture.

Now suppose that $C_a, C_b$ are curves of genus $g_a, g_b$, that $a \in C_a, b \in C_b$ and that if $i = a, b$ then we have chosen a local co-ordinate $z_i$ on some simply connected neighbourhood $U_i$ of $i$ in $C_i$ such that $U_i$ is defined by the inequality $|z_i| < \delta^{1/2}$. In particular, $U_i$ is an open disc that is embedded in $C_i$.

The suffix $i$ will stand for either $a$ or $b$ in what follows.

Let $U_i^c$ denote the closure of $U_i$ in $C_i$. We shall assume that each $U_i^c$ is a (closed) disc; this is slightly stronger than assuming $U_i$ is an open disc. We shall assume also *either* that $C_a$ and $C_b$ are distinct *or* that $C_a = C_b$ and the closed discs $U_a^c$ and $U_b^c$ are disjoint.

Of course, all of these assumptions can be fulfilled after decreasing $\delta$ if necessary.

In $F$ we have open subsets $V_a$ and $V_b$ defined, respectively, by the inequalities

$$|v - q| < |q|, \ |t| < \delta|q|^2 \text{ and} \tag{4.1}$$
$$|v + q| < |q|, \ |t| < \delta|q|^2. \tag{4.2}$$

Note that $V_a$ and $V_b$ are disjoint and that $V_i$ contains $U'_i - \{0\}$.

Define $\rho_i : V_i \to U_i \times \Delta$ by $z_i = q,\ t = q^2 - v^2$. It is easy to check that $\rho_i$ is unramified (since the ramification locus is defined by $v = 0$) and is injective. Therefore $\rho_i$ is an isomorphism from $V_i$ to its image $Y_i$, which is open in $U_i \times \Delta$ and so open in $C_i \times \Delta$.

Let $Y_i$ denote the image of $V_i$. Then $Y_i$ is contained in the region $|t| < \delta|z_i|^2$ and contains $(U_i - \{i\}) \times \{0\}$. Note that $\rho_i$ maps $U'_i - \{0\}$ isomorphically onto $U_i - \{i\}$.

In $C_i \times \Delta$ consider the closed subset $\{(P, t) \ : \ P \in U_i^c \text{ and } (P, t) \notin Y_i\}$. Define $W_i$ to be the open subset of $C_i \times \Delta$ obtained by deleting this closed set.

Note that $W_i \cap (C_i \times \{0\}) = C_i - \{i\}$.

**Proposition 4.3** *We can glue the three charts $W_a$, $W_b$ and $F$ via the isomorphisms $\rho_i : V_i \overset{\cong}{\to} Y_i$ to get a separated 2-dimensional complex manifold $\mathscr{C}$ with a holomorphic map $\pi : \mathscr{C} \to \Delta_t$ such that $\pi^{-1}(0) = C_a \cup_{a \sim b} C_b$, where $C_a$ and $C_b$ cross normally at a single point. After shrinking $\Delta_t$ if necessary, the map is proper.*

PROOF: This is routine. □

We call this a *Fay plumbing* because it is, implicitly, constructed and considered by Fay in the final paragraph of p. 37 of [F1].

If $C_a$ and $C_b$ are distinct then for all $t \neq 0$ the fibre $\mathscr{C}_t$ is a curve of genus $g = g_a + g_b$ and there is no monodromy on its cohomology. This is what Fay calls "pinching a cycle homologous to zero" [F1], p. 37 *et seq.* We shall call it a *homologically trivial Fay plumbing of $C_a$ to $C_b$ that identifies $a$ with $b$* or just a *Fay plumbing of $C_a$ to $C_b$* without explicit mention of the points or co-ordinates that are chosen.

If $C_a = C_b$, of genus $g$, and also $U_a^c \cap U_b^c = \emptyset$ (so that the plumbing is possible), then for all $t \neq 0$ the fibre $\mathscr{C}_t$ is of genus $g+1$, $\mathscr{C}_0$ is the nodal curve $C/a \sim b$ and there is non-trivial monodromy on the cohomology of $\mathscr{C}_t$. Fay calls this "pinching a non-zero homology cycle" [F1], p. 50.

Note that on $F \cap W_i$ we have $z_i = q$.

In a homologically trivial Fay plumbing the derivative at $t = 0$ of the period matrix of $\mathscr{C}_t$ is given as Corollary 3.2 on p. 41 of [F1]. We shall prove this, or, rather, a version of it in a higher-dimensional context, later on, in Proposition 8.12.

When a non-zero homology cycle is pinched in a Fay plumbing the period matrix is described by by Corollary 3.8 on p. 53 of [F1].

However, the families that are written down on pp. 37 and 50 of *loc. cit.*, which we call the *Yamada plumbing* because they are considered explicitly by Yamada, are given by the glueing $t = z_a z_b$ and (provided that $g_a + g_b \geq 1$) are different families because, for example, their period matrices are different. Their expansions at $t = 0$ are given as Corollary 2 on p. 129 and Corollary 6 on pp. 137-138 of [Y].

In this paper we shan't have any need to pinch cycles that are not homologous to zero. For one application of such pinchings, however, see [CSB]. Note that there Yamada plumbings are used, although an argument could also be based on the use of Fay plumbings.

It will also be useful to be able to plumb certain stacky curves.

**Definition 4.4** *A* $\mathbb{Z}/2$-curve *is a reduced connected proper Deligne–Mumford stack $\widetilde{C}$ of dimension* 1 *such that at each generic point the stabilizer (the isotropy group) is trivial and there are finitely many points where the stabilizer is $\mathbb{Z}/2$.*

For example, if $i \in C_i$ then there is a unique smooth $\mathbb{Z}/2$-curve $\widetilde{C}_i$ such that the geometric quotient $[\widetilde{C}_i]$ of $\widetilde{C}_i$ is given by $[\widetilde{C}_i] = C_i$, the quotient map $\widetilde{C}_i \to C_i$ is an isomorphism over $C_i - \{i\}$ and there is a unique geometric point on $\widetilde{C}_i$ lying over $i$. There are local co-ordinates $\tilde{z}_i$ and $z_i$ on $\widetilde{C}_i$ and on $C_i$, respectively, such that the non-trivial element $\iota$ of $\mathbb{Z}/2$ acts via $\iota^* \tilde{z}_i = -\tilde{z}_i$ and $z_i = \tilde{z}_i^2$.

Take the plumbing fixture $F$, with co-ordinates $q, v$ as before, and let $\iota$ act on $F$ by $\iota^* q = -q$, $\iota^* v = -v$. So the quotient stack $\widetilde{F} = F/\langle \iota \rangle$ is a smooth separated 2-dimensional Deligne–Mumford stack.

Suppose $\widetilde{C}_i$, for $i = a, b$, are smooth $\mathbb{Z}/2$-curves and that each of $a, b$ the stabilizer is $\langle\iota\rangle \cong \mathbb{Z}/2$. Fix a local co-ordinate $\tilde{z}_i$ on $\widetilde{C}_i$ at $i$ such that $\iota^*\tilde{z}_i = -\tilde{z}_i$.

**Proposition 4.5** *We can glue together the charts $\widetilde{C}_a$, $\widetilde{C}_b$ and $\widetilde{F}$ via the formulae $\tilde{z}_i = q$ and $t = q^2 - v^2$ to get a smooth 2-dimensional Deligne–Mumford stack $\widetilde{\mathscr{C}}$ with a proper separated morphism $\widetilde{\mathscr{C}} \to \Delta_t$.*

PROOF: As before. □

Taking geometric quotients gives the following result. Define $G = [F/\langle\iota\rangle]$.

**Proposition 4.6** *The curves $C_i$ can be plumbed via the plumbing fixture $G$ to $\mathscr{B} \to \Delta_t$. On the surface $\mathscr{B}$ there is an $A_1$-singularity at 0 and $\mathscr{B}_0 = C_a \cup C_b/a \sim b$.*

PROOF: This is clear. □

The minimal resolution of $\mathscr{B}$ gives a semi-stable family of curves over $\Delta_t$.

# 5 Plumbing families of curves

The plumbing construction of the previous section can be extended so as to plumb families of curves.

So suppose that $C_a \to S_a$, $C_b \to S_b$ are analytic families of semi-stable curves of genera $g_a$ and $g_b$, respectively, with sections $\sigma_i : S_i \to C_i$ each of which lies in the relevant smooth locus. Assume also that $U_i$ is a tubular neighbourhood of the section $i = \sigma_i(S_i)$ and that $z_i$ is a fibre co-ordinate on $U_i$ such that $U_i$ is isomorphic to $S_i \times \Delta_{z_i}$, where $\Delta_{z_i}$ is the disc with co-ordinate $z_i$ such that $|z_i| < \delta$. Such a neighbourhood and such a co-ordinate will exist if each $S_i$ is sufficiently small, for example a small polydisc.

Take the same 2-dimensional plumbing fixture $F$ and disjoint open subsets $V_i$ as before and consider the morphisms $\rho_i : V_i \times S_i \to U_i \times \Delta_t$ defined by

$$\rho_i(q, v, s_i) = (\sigma_i(s_i), z_i, t)$$

where $z_i = q$ and $t = q^2 - v^2$.

As before, $\rho_i$ is an isomorphism to an open analytic subvariety $Y_i$ of $U_i \times \Delta_t$.

In $C_i \times \Delta_t$ consider the closed subset that is the intersection of $U_i^c \times \Delta_t$ and the complement of $Y_i$, and define $W_i$ to be the open subvariety of $C_i \times \Delta_t$ obtained by deleting this closed subset.

Now glue the chart $W_a \times S_b$ to $F \times S_a \times S_b$ via the isomorphism $V_a \times S_a \times S_b \to Y_a \times S_b$, and glue $W_b \times S_a$ to $F \times S_a \times S_b$ via the isomorphism $V_b \times S_a \times S_b \to Y_b \times S_a$.

The result of this glueing is an analytic space $\mathscr{C}$ with a proper morphism $\mathscr{C} \to S_a \times S_b \times \Delta_t$ that is a family of semi-stable curves of genus $g_a + g_b$. If $D_i \subset S_i$ is the discriminant locus of $C_i \to S_i$ then the discriminant locus of $\mathscr{C} \to S_a \times S_b \times \Delta_t$ is $D_a \times S_b \times \Delta_t \cup D_b \times S_a \times \Delta_t \cup S_a \times S_b \times \{0\}$.

We can concatenate plumbings as follows: suppose that $C_i \to S_i$ are families of stable curves for $i = a, b, c$ and that $\sigma_a, \sigma_{b_1}, \sigma_{b_2}, \sigma_c$ are sections of $C_a, C_b, C_b, C_c$ respectively and that $\sigma_{b_1}$ and $\sigma_{b_2}$ are disjoint.

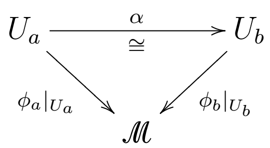
Then there are two choices: we can first plumb $C_a$ to $C_b$, obtaining $\mathscr{C}' \to S_a \times S_b \times \Delta$, in a way that identifies the sections $\sigma_a$ and $\sigma_{b_1}$, and then plumb $\mathscr{C}'$ to $C_c$ to get a family over $S_a \times S_b \times \Delta \times S_c \times \Delta$, or we can first plumb $C_b$ to $C_c$, obtaining $\mathscr{C}''$, by identifying $\sigma_{b_2}$ with $\sigma_c$, and then plumb $\mathscr{C}''$ to $C_a$ to get another family over $S_a \times S_b \times \Delta \times S_c \times \Delta$. It will be important for us to notice that these two families are the same; that is, the final result is independent of the order of the plumbings.

Similarly we can construct stacks $\widetilde{C}_i \to S_i$ by introducing isotropy groups $\mathbb{Z}/2$ along each section $\sigma(S_i)$ and then plumb together the stacks $\widetilde{C}_i$.

# 6 Plumbing morphisms

Keep the notation of Section 4 and suppose that $\mathscr{C} = W_a \cup W_b \cup F \to \Delta$ is a Fay plumbing of curves $C_a$ to $C_b$ that identifies $a$ with $b$ and that $z_i$ is the local co-ordinate on $C_i$ at $i$ that is used in the plumbing.

Suppose that $\mathscr{M}$ is some holomorphic stack and that $\phi_i : C_i \to \mathscr{M}$ are morphisms such that for the choice of co-ordinates $z_a$ and $z_b$ the morphisms $\phi_a$ and $\phi_b$ are isomorphic on $U_a$ and on $U_b$. That is, there is a 2-commutative diagram




where $\alpha$ is an isomorphism such that $\alpha^* z_b = z_a$.

**Lemma 6.1** *There is a morphism $\Phi : \mathscr{C} \to \mathscr{M}$ such that the restriction of $\Phi$ to the closed fibre $\mathscr{C}_0 = C_a \cup C_b$ coincides with $\phi_i$ on each $C_i$ and the restriction $\Phi_i$ of $\Phi$ to each of the two charts $W_i$ of $\mathscr{C}$ factors through the projection $pr_i : W_i \to C_i$ as $\Phi_i = \phi_i \circ pr_i$.*

PROOF: Suppose that $w_1, ..., w_n$ are local co-ordinates on a smooth chart $X \to \mathscr{M}$ and that a local lift $\widetilde{\phi}_i : U_i \to X$ of $\phi_i|_{U_i}$ is given, in local co-ordinates, by a formula

$$\widetilde{\phi}_i(z_i) = (\phi_{i,1}(z_i), ..., \phi_{i,n}(z_i)).$$

Then the hypothesis is that $\widetilde{\phi}_a(z_a)$ is equivalent to $\widetilde{\phi}_b(z_b)$, so that we can define $\Psi : F \to \mathscr{M}$ by $\Psi(q, v) = (\phi_{a,1}(q), ..., \phi_{a,n}(q)) \sim (\phi_{b,1}(q), ..., \phi_{b,n}(q))$.

Then define $\Phi_i : W_i \to \mathscr{M}$ as $\Phi_i = \phi_i \circ pr_i$ where $pr_i : W_i \to C_i$ is the projection. Since the morphisms $\Psi$ and $\Phi_i$ agree on the overlaps the lemma is proved. □

**Remark**: If instead we consider the Yamada plumbing defined by the glueing $t = z_a z_b$ then it is not clear how to construct such a morphism $\Phi$. This is why we focus here on Fay plumbings.

**Definition 6.2**

*(1)* $\mathcal{E}\ell\ell^0$ *denotes the stack of (smooth) elliptic curves.*

*(2)* $\mathcal{E}\ell\ell$ *is the stack of reduced and irreducible curves of genus* 1 *with planar singularities, provided with a marked smooth point.*

*(3)* $\mathcal{E}\ell\ell^{Kod}$ *is the stack of Kodaira fibres.*

For example, suppose that $\mathcal{M}$ is one of the stacks $\mathcal{E}\ell\ell$ or $\mathcal{E}\ell\ell^{Kod}$. Then, for either of these choices of $\mathcal{M}$, the assumptions hold if the points $\phi_i(i)$ are both isomorphic to the same smooth elliptic curve and each $\phi_i$ has the same finite order of ramification at the point $i \in C_i$.

In general, the hypotheses imply that the curves $C_a, C_b$ have the same image in the geometric quotient of $\mathcal{M}$ if that quotient exists.

By definition, the datum of a *Jacobian elliptic fibration* $f : X \to Y$ over an irreducible base is equivalent to the datum of a morphism $\phi : Y \to \mathcal{E}\ell\ell$ such that $Y$ meets $\mathcal{E}\ell\ell^0$. Then the next corollary, which is an immediate consequence of the lemma, shows that Jacobian elliptic surfaces can be plumbed.

**Corollary 6.3** *Suppose that, for each* $i = a, b$, $f_i : X_i \to C_i$ *is a smooth (resp., RDP) Jacobian elliptic surface defined by morphisms* $\phi_i : C_i \to \mathcal{E}\ell\ell^{Kod}$ *(resp.,* $\phi_i : C_i \to \mathcal{E}\ell\ell$*), that the curves* $\phi_a(a)$ *and* $\phi_b(b)$ *are isomorphic to the same smooth elliptic curve* $E$ *and that the orders of ramification of* $\phi_a$ *at* $a$ *and of* $\phi_b$ *at* $b$ *are finite and equal.*

*Then there is a Jacobian elliptic fibration* $f : \mathfrak{X} \to \mathcal{C}$ *and a proper morphism* $g : \mathcal{C} \to \Delta$ *with composite* $h = g \circ f : \mathfrak{X} \to \Delta$ *where*

*(1)* $\mathfrak{X}$ *is a smooth (resp., c-DV) threefold,*

*(2)* $\mathcal{C}$ *is a smooth surface,*

*(3) if each* $X_i$ *is simply connected and* $C_a \neq C_b$ *then* $g$ *is a homologically trivial Fay plumbing of two copies of* $\mathbb{P}^1$ *and so is a conic bundle whose singular fibre* $\mathcal{C}_0$ *has two components,*

*(4) if* $C_a \neq C_b$ *then the fibre* $\mathfrak{X}_0 = X_a \cup X_b = h^{-1}(0)$ *is a reduced divisor in* $\mathfrak{X}$ *with normal crossings along the common curve* $E$*,*

*(5) if* $\mathcal{W}_i$ *is the inverse image in* $\mathfrak{X}$ *of the open subset* $W_i$ *of* $\mathcal{C}$*, then there is a holomorphic projection* $\mathcal{W}_i \to X_i - E$ *such that the induced morphism* $\mathcal{W}_i \to (X_i - E) \times \Delta$ *is an isomorphism to an open submanifold that contains* $(X_i - E) \times \{0\}$ *and*

*(6)* $\mathfrak{X}_t \to \mathcal{C}_t$ *is a smooth (resp., RDP) Jacobian elliptic surface for* $t \neq 0$ *and*

*(7) the configuration of* $(-2)$*-curves on* $\mathfrak{X}_t$ *is constant and specializes to the union of the* $(-2)$*-configurations on* $X_a$ *and on* $X_b$ *if* $X_a$ *and* $X_b$ *are smooth, while if they are RDP surfaces then the analogous statement for configurations of RDPs is true.*

PROOF: The hypotheses concerning ramification imply that $\phi_a$ and $\phi_b$ are locally isomorphic. Then everything except the last statement, which is easy, is automatic. □

In particular, suppose that each $X_i$ is simply connected. Then $C_i$ is isomorphic to $\mathbb{P}^1$ and $\mathcal{C}_t$ is also isomorphic to $\mathbb{P}^1$. It follows that if $\mathfrak{X} \to \Delta$ is the composite morphism then for $t \neq 0$ the fibre $\mathfrak{X}_t$ is a simply connected Jacobian elliptic surface and that, if $p_g(X_i) = g_i$, $\mathfrak{X}_0 = X_a \cup X_b$, $\mathfrak{X}_0$ is reduced with normal crossings and $X_a \cap X_b = E$. Moreover, $p_g(\mathfrak{X}_t) = g_a + g_b + 1 = g$, say.

**Definition 6.4** $\mathcal{JE}^{RDP,sst}$ *denotes the stack of* semi-stable RDP Jacobian elliptic surfaces. *Its objects over a scheme* $B$ *are pairs* $(\mathfrak{X} \xrightarrow{f} S \to B, \widetilde{S})$ *such that*

*(1)* $f : \mathfrak{X} \to S$ *and* $S \to B$ *are projective, flat, Cohen–Macaulay and of relative dimensions* 2 *and* 1, *respectively,*

*(2)* $\widetilde{S}$ *is a section of* $f$ *lying in the relatively smooth locus of* $f$ *(so is an effective Cartier divisor on* $\mathfrak{X}$*),*

*(3)* $\widetilde{S}$ *is* $f$*-ample,*

*(4)* $\mathfrak{X} \to B$ *and* $S \to B$ *are semi-stable, in the sense that all geometric fibres are reduced with normal crossings, and the family of curves* $S \to B$ *is of compact type,*

*(5) for every geometric point* $\delta$ *of* $B$ *and every irreducible component* $C$ *of* $S_\delta$, $f^{-1}(C) \to C$ *is an RDP Jacobian elliptic surface whose origin is* $\widetilde{S} \cap f^{-1}(C)$ *and*

*(6)* $f : \mathfrak{X} \to S$ *is smooth over the nodes of* $S_\delta$.

**Definition 6.5** $\mathcal{JE}^{sm,sst}$ *is the stack obtained from* $\mathcal{JE}^{RDP,sst}$ *by dropping the requirement 3 above and imposing instead the requirement that each* $f^{-1}(C) \to C$ *should be a smooth and relatively minimal Jacobian elliptic surface.*

Note that $\mathcal{JE}^{sm,sst}$ (resp., $\mathcal{JE}^{RDP,sst}$) contains $\mathcal{JE}^{sm}$ (resp., $\mathcal{JE}^{RDP}$) as an open substack.

In terms of this notation, the objects $\mathfrak{X} \to \mathcal{C} \to \Delta$ of Corollary 6.3 are objects over $\Delta$ of either $\mathcal{JE}^{sm,sst}$ or $\mathcal{JE}^{RDP,sst}$, as appropriate.

There is an obvious extension of this to plumbings of several curves, as follows.

Suppose that $C_1, ..., C_r$ are curves, that $a_1 \in C_1,\ b_1, a_2 \in C_2, \dots, b_{r-2}, a_{r-1} \in C_{r-1},\ b_{r-1} \in C_r$ and that $z_x$ is a local co-ordinate at $x$ on the relevant curve. Then there is a Fay plumbing $\mathscr{C} \to S_{r-1}$ where $S_{r-1}$ is an $(r-1)$-dimensional polydisc and the curves $C_1, ..., C_r$ are attached in a chain that identifies $a_i$ with $b_i$. As a manifold, $\mathscr{C}$ is the union of charts $W_i$, each of which is open in $C_i \times S_{r-1}$ and contains $C_i - \{a_i, b_{i-1}\}$, together with plumbing fixtures.

Suppose also that there are given morphisms $\phi_i : C_i \to \mathscr{M}$ such that $\phi_i$ and $\phi_{i+1}$ are locally isomorphic in terms of the given co-ordinates $z_{a_i}$ and $z_{b_i}$. Then these morphisms can be plumbed to give a morphism $\Phi : \mathscr{C} \to \mathscr{M}$ such that the restriction $\Phi_i$ of $\Phi$ to each chart $W_i$ factors through the projection $pr_i : W_i \to C_i$ as $\Phi_i = \phi_i \circ pr_i$.

Moreover, these plumbings can be constructed in any order.

# 7 Plumbings modulo $t^2$

Suppose that $C_i$, for $i = a, b$, are curves, that $i \in C_i$ and that $z_i$ is a local co-ordinate on $C_i$ at $i$, as before, and that $\mathscr{C} \to \Delta$ is the corresponding Fay plumbing of $C_a$ to $C_b$. Suppose also that $\phi_i : C_i \to \mathscr{M}$ are morphisms that are merely *isomorphic to first order.* That is, $\phi'_a : \mathbf{Spec}\, \mathbb{C}[[z_a]]/(z_a^2) \to \mathscr{E}\ell\ell$ is isomorphic to $\phi'_b : \mathbf{Spec}\, \mathbb{C}[[z_b]]/(z_b^2) \to \mathscr{E}\ell\ell$ in the sense that there is a 2-commutative diagram

$$\begin{array}{ccc} \mathbf{Spec}\, \mathbb{C}[[z_a]]/(z_a^2) & \xrightarrow[\cong]{\alpha} & \mathbf{Spec}\, \mathbb{C}[[z_b]]/(z_b^2) \\ & {\scriptstyle \phi'_a} \searrow \quad \swarrow {\scriptstyle \phi'_b} & \\ & \mathscr{M} & \end{array}$$

where $\alpha$ is an isomorphism such that $\alpha^* z_b = z_a$. Put $\Delta' = \mathbf{Spec}\, \mathbb{C}[[t]]/(t^2)$ and $\mathscr{C}' = \mathscr{C} \times_\Delta \Delta'$.

**Proposition 7.1** *$\phi'_a$ and $\phi'_b$ can be plumbed to give a morphism $\Phi' : \mathscr{C}' \to \mathscr{M}$.*

PROOF: Exactly as before. □

**Corollary 7.2** *If $f_i : X_i \to C_i$ are Jacobian elliptic surfaces such that the classifying morphisms $\phi_i : C_i \to \mathscr{E}\ell\ell$ are isomorphic to first order at $a$ and $b$, then $f_a$ and $f_b$ can be plumbed to $\mathfrak{X}' \xrightarrow{f} \mathscr{C}' \to \Delta'$.*

If each $f_i : X_i \to C_i$ is a smooth (resp, RDP) Jacobian elliptic surface then $\mathfrak{X}' \xrightarrow{f} \mathscr{C}' \to \Delta'$ is an object over $\Delta'$ of $\mathscr{J}\mathscr{E}^{sm,sst}$ (resp., $\mathscr{J}\mathscr{E}^{RDP,sst}$).

**Lemma 7.3** *Suppose that each $f_i : X_i \to C_i$ is an RDP Jacobian elliptic surface with minimal resolution $\widetilde{X}_i \to X_i$. Then $\mathfrak{X}' \xrightarrow{f} \mathscr{C}' \to \Delta'$ can be lifted to an object of $\mathscr{J}\mathscr{E}^{sm,sst}$ over $\Delta'$.*

PROOF: By construction, $\mathfrak{X}' \to \Delta'$ induces a trivial deformation of each RDP. □

**Lemma 7.4** *The forgetful morphism* $\mathcal{JE}^{RDP,sst} \to \mathcal{M}^c = \coprod \mathcal{M}_g^c$ *to the stack of stable curves of compact type is smooth and has irreducible fibres.*

PROOF: Given an object $C$ of $\mathcal{M}^c$ and an object $p : Y \to C$ of $\mathcal{JE}^{RDP,sst}$ over $C$, define $\mathcal{L} = p_*\mathcal{O}_Y(C_0)$. This is an $\iota$-linearized $p$-ample invertible sheaf on $C$ and $\mathbb{G}_m$ acts $\iota$-equivariantly on the vector bundle $\mathcal{L} \oplus \mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3}$ with weight $n$ on $\mathcal{L}^{\otimes n}$. Define $\mathbb{P} = [(\mathcal{L} \oplus \mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3}) - \{0\}]/\mathbb{G}_m$; this is a $\mathbb{P}(1,2,3)$-bundle over $C$ and $Y$ is a sextic Cartier divisor in $\mathbb{P}$.

Therefore an object of $\mathcal{JE}^{RDP,sst}$ lying over an object $C$ of $\mathcal{M}^c$ is defined by the data of a line bundle $\mathcal{L}$ on $C$ and a sextic divisor in the $\mathbb{P}(1,2,3)$ bundle associated to $\mathcal{L}$; it follows that $\mathcal{JE}^{RDP,sst} \to \mathcal{M}^c$ is smooth, with irreducible fibres. □

**Corollary 7.5** *The object* $\mathcal{X}' \xrightarrow{f'} \mathcal{C}' \to \Delta'$ *of* $\mathcal{JE}^{RDP,sst}$ *over* $\Delta'$ *can be extended to an object* $\mathcal{X} \xrightarrow{f} \mathcal{C} \to \Delta$ *of* $\mathcal{JE}^{RDP,sst}$ *over* $\Delta$ *where the* 3*-fold* $\mathcal{X}$ *is smooth.*

PROOF: The existence of $\mathcal{X}$ is clear and the only thing to check is that $\mathcal{X}$ is smooth. For this, notice that in suitable local co-ordinates $\mathcal{X}' \to \Delta'$ is defined by $t = xy$; then the same holds for $\mathcal{X}$ and the smoothness follows. □

In Lemma 11.7 we shall prove a version of this result for another relevant partial compactification of $\mathcal{JE}^{sm}$, which will be denoted by $\mathcal{JE}^{sm,n.m.}$. The distinction between these compactifications is that a typical boundary point of $\mathcal{JE}^{sm,sst}$ corresponds to a singular surface that consists of two smooth surfaces that meet transversely in a smooth elliptic curve, while in $\mathcal{JE}^{sm,n.m.}$ the two surfaces are singular and meet in a model of a $\widetilde{D}_4$-fibre.

# 8 Homologically trivial plumbings of surfaces

Here we construct families of Jacobian elliptic surfaces that degenerate in a homologically trivial fashion, so that the period matrix specializes to a matrix lying in the interior of the period domain and there is an explicit formula for the derivative of the period matrix that involves no derivatives but is instead a matrix of rank one, just as for curves.

This leads to a result analogous to the asymptotic relations that will be derived later, in Section 14, for the locus of hyperelliptic curves.

We begin with a modification of Kodaira's degenerate fibres and his notation for them.

(1) $I_{n-4}^* = \widetilde{D}_n$ for $n \geq 4$: contract the four $(-2)$ curves at the extremities and call the result $\overline{D}_n$. This has four $A_1$ singularities.

(2) $II = Cu$: blow up the cusp three times, then contract the resulting $(-2), (-3)$ and $(-6)$ curves, and call the result $\overline{II}$. This has three singularities, one of each type $\frac{1}{2}(1,1) = A_1$, $\frac{1}{3}(1,1)$ and $\frac{1}{6}(1,1)$.

(3) $III = Ta$: blow up the tacnode twice, then contract the resulting $(-2)$ curve and two $(-4)$ curves and call the result $\overline{III}$. This has one $A_1$ singularity and two singularities of type $\frac{1}{4}(1,1)$.

(4) $IV = Tr$: blow up the triple point, then contract the three resulting $(-3)$ curves and call the result $\overline{IV}$. This has three singularities of type $\frac{1}{3}(1,1)$.

(5) $II^* = \widetilde{E}_8, III^* = \widetilde{E}_7$ and $IV^* = \widetilde{E}_6$: contract all curves except the central one and call the result $\overline{R}^*$ for $R = II, III, IV$. This has three singularities, all of type $A$.

Now suppose that $Y_i \to C_i$ are smooth simply connected Jacobian elliptic surfaces, each of which has a fibre $\phi_i$ (with its structure as a scheme) of type $\widetilde{D}_4$ over $i$. Let $\pi_i : Y_i \to \overline{Y}_i$ denote the contraction of the four $(-2)$-curves at the extremities, so that $\overline{Y}_i \to C_i$ has a fibre $\overline{\phi}_i$, with its structure as a scheme, of type $\overline{D}_4$. So $\overline{\phi}_i$ is a copy of $\mathbb{P}^1$ but with multiplicity 2. Say $p_g(Y_i) = h_i$.

**Lemma 8.1** *For each $i$ there is a smooth $\mathbb{Z}/2$-curve $\widetilde{C}_i$ and a smooth proper Deligne–Mumford surface $\widetilde{Y}_i$ with a projective (in particular, representable) morphism $\widetilde{Y}_i \to \widetilde{C}_i$ such that*

*(1) the geometric quotients $[\widetilde{C}_i]$ and $[\widetilde{Y}_i]$ are $[\widetilde{C}_i] = C_i$ and $[\widetilde{Y}_i] = \overline{Y}_i$,*

*(2) the quotient morphism $\widetilde{C}_i \to C_i$ is an isomorphism outside $i$ and the stabilizer group at the unique point $\tilde{i}$ of $\widetilde{C}_i$ that lies over $i$ is $\mathbb{Z}/2$,*

*(3) $\widetilde{Y}_i \to \widetilde{C}_i$ is smooth over $\tilde{i}$ and the fibre $\widetilde{\phi}_{\tilde{i}}$ over $\tilde{i}$ is $E/(-1)$ for some elliptic curve $E$,*

*(4) the quotient morphism $\rho_i : \widetilde{Y}_i \to \overline{Y}_i$ is an isomorphism outside the four points that map to the four nodes on $\overline{Y}_i$, where the stabilizer group is $\mathbb{Z}/2$,*

*(5) the morphism $\widetilde{Y}_i \to \overline{Y}_i \times_{C_i} \widetilde{C}_i$ is finite and birational and*

*(6) the induced morphisms $\widetilde{C}_i \to \mathcal{E}\ell\ell$ are ramified at the points $\tilde{i}$ and so are isomorphic to first order at $\tilde{a}, \tilde{b}$.*

PROOF: Locally, each $A_1$ singularity on $\overline{Y}_i$ is defined by an equation $z^2 = ty$, where $t$ is a local co-ordinate on $C_i$. So, if $B_i \to C_i$ is a double cover that is branched at $i$, then $B_i$ has a local co-ordinate $s$ such that $t = s^2$ and then $\overline{Y}_i \times_{C_i} B_i$ is defined by the equation $z^2 = s^2 y$ and the normalization of this is smooth. The group $\mathbb{Z}/2$ acts on $B_i$, and then we construct $\widetilde{C}_i$ locally in such a way that near $i$ it is $B_i$ while away from $i$ it is $C_i$. Then $\widetilde{C}_i = B_i/(\mathbb{Z}/2)$ and $\widetilde{Y}_i$ is the normalization of $\overline{Y}_i \times_{C_i} \widetilde{C}_i$. □

**Lemma 8.2** *(1) There are natural isomorphisms*

$$\begin{aligned}\pi_i^*\omega_{\overline{Y}_i} &\overset{\cong}{\to} \Omega^2_{Y_i},\\ \pi_i^*\omega_{\overline{Y}_i}(\overline{\phi}_i) &\overset{\cong}{\to} \Omega^2_{Y_i}(\phi_i),\\ \rho_i^*\omega_{\overline{Y}_i} &\overset{\cong}{\to} \Omega^2_{\widetilde{Y}_i} \text{ and}\\ \rho_i^*\omega_{\overline{Y}_i}(\phi_i) &\overset{\cong}{\to} \Omega^2_{\widetilde{Y}_i}(2\widetilde{\phi}_{\tilde{i}}).\end{aligned}$$

*(2) These isomorphisms induce isomorphisms*

$$\begin{aligned}\pi_i^* : H^0(\overline{Y}_i, \omega_{\overline{Y}_i}) &\overset{\cong}{\to} H^0(Y_i, \Omega^2_{Y_i}),\\ \pi_i^* : H^0(\overline{Y}_i, \omega_{\overline{Y}_i}(\overline{\phi}_i)) &\overset{\cong}{\to} H^0(Y_i, \Omega^2_{Y_i}(\phi_i)),\\ \rho_i^* : H^0(\overline{Y}_i, \omega_{\overline{Y}_i}) &\overset{\cong}{\to} H^0(\widetilde{Y}_i, \Omega^2_{\widetilde{Y}_i}) \text{ and}\\ \rho_i^* : H^0(\overline{Y}_i, \omega_{\overline{Y}_i}(\overline{\phi}_i)) &\overset{\cong}{\to} H^0(\widetilde{Y}_i, \Omega^2_{\widetilde{Y}_i}(2\widetilde{\phi}_{\tilde{i}})).\end{aligned}$$

PROOF: This is a straightforward (and well known) local calculation for $A_1$ singularities that depends on two facts: one is that the non-trivial element $\iota$ of a stabilizer group at a point on $\widetilde{Y}_i$ that lies over a singularity on $\overline{Y}_i$ acts trivially on a local generator $d\tilde{z}_i \wedge dw_i$ of $\Omega^2_{\widetilde{Y}_i}$ where $\tilde{z}_i$ is a local co-ordinate on $\widetilde{C}_i$ such that $\iota^*\tilde{z}_i = -\tilde{z}_i$ and $w_i$ is a fibre co-ordinate, and the other is that $Y_i \to \overline{Y}_i$ is a crepant resolution. □

Now assume that the two $\overline{D}_4$ fibres are isomorphic. That is, two cross-ratios are equal.

**Corollary 8.3** *Choose local co-ordinates $\tilde{z}_i$ on $\widetilde{C}_i$ at $\tilde{i}$ such that $\iota^*\tilde{z}_i = -\tilde{z}_i$ where $\iota$ is a generator of the stabilizer of the point $\tilde{i}$.*

*(1) In terms of these local co-ordinates there is a Fay plumbing of the stacks $\widetilde{Y}_i$ modulo $t^2$ via the morphisms $\widetilde{C}_i \to \mathscr{E}\ell\ell$ to give $\widetilde{\mathscr{Y}}' \to \Delta'$.*

*(2) If $\widetilde{\mathscr{Y}} \to \Delta$ is any lifting of $\widetilde{\mathscr{Y}}' \to \Delta'$ then $\widetilde{\mathscr{Y}}$ is a smooth 3-dimensional Deligne–Mumford stack, whose only non-trivial stabilizers are copies of $\mathbb{Z}/2$ at each of the four 2-torsion points of $\widetilde{Y}_a \cap \widetilde{Y}_b$ (which is isomorphic to the quotient $E/\iota$ of an elliptic curve $E$ by $(-1)$).*

*(3) Let $\overline{\mathscr{Y}} = [\widetilde{\mathscr{Y}}]$. Then $\overline{\mathscr{Y}}_0 = \overline{Y}_a \cup \overline{Y}_b$ and $\overline{Y}_a \cap \overline{Y}_b$ is a fibre of type $\overline{D}_4$ on each of them.*

*(4) $p_g(\widetilde{\mathscr{Y}}_t) = p_g(Y_a) + p_g(Y_b)$ and there is no monodromy on $H^2(\widetilde{\mathscr{Y}}_t, \mathbb{Z})$.*

PROOF: (1): note that, since the $\overline{D}_4$ fibres are isomorphic, the classifying morphisms $\widetilde{C}_i \to \mathscr{E}\ell\ell$ are then isomorphic to first order, and the plumbing is constructed just as in Corollary 7.2.

(2): the proof of smoothness is the same as in that of Corollary 7.5 and the rest is immediate.

(3): this is clear.

(4): A holomorphic 2-form $\omega_t$ on $\widetilde{\mathcal{Y}}_t$ will specialize to a pair $(\omega_a, \omega_b)$ where $\omega_i$ lies in $H^0(\widetilde{Y}_i, \Omega^2_{\widetilde{Y}_i}(E/\iota))$ Since $\iota$ acts as $(-1)$ on $E$ the residue of $\omega_i$ is zero, so $\omega_i$ is holomorphic. (In contrast, we had $p_g(\mathcal{X}_t) = g_a + g_b + 1$ because there was no constraint imposed by anti-invariance to force the vanishing of the residue of the relevant 2-form.)

The absence of monodromy is a well known consequence. $\square$

Now suppose that there is a lifting $\widetilde{\mathcal{Y}} \to \Delta$, and fix one. Then there is a $C^0$ collapsing map $\gamma : \widetilde{\mathcal{Y}}_t \to \widetilde{\mathcal{Y}}_0$, in the context of orbifolds, unique up to homotopy. There is a totally isotropic sublattice $L$ of $H_2(\widetilde{\mathcal{Y}}_t, \mathbb{Z})$ with a basis $(A_1, ..., A_{h_a}, A_{h_a+1}, ..., A_{h_a+h_b})$ such that the image of $(A_1, ..., A_{h_a})$ under $\gamma_*$ is a basis of an isotropic lattice $L_a$ in $H_2(\widetilde{Y}_a, \mathbb{Z})$ and the image of $(A_{h_a+1}, ..., A_{h_a+h_b})$ is a basis of an isotropic lattice $L_b$ in $H_2(\widetilde{Y}_b, \mathbb{Z})$. We shall assume that each $\widetilde{Y}_i$ is in $L_i$-general position (see Definition 2.2).

**Lemma 8.4** *$\widetilde{\mathcal{Y}}_t$ is in $L$-general position for $t \neq 0$.*

PROOF: If not, then there is a holomorphic 2-form $\omega_t$ on $\widetilde{\mathcal{Y}}_t$ such that $\int_A \omega_t = 0$ for every $A \in L$. We can assume that $\omega_t$ is not divisible by $t$. By the argument used in the proof of Lemma 8.3 4 $\omega_t$ specializes to a pair $(\omega_a, \omega_b)$, where $\omega_i$ is holomorphic on $\widetilde{Y}_i$ and $\int_A \omega_i = 0$ for every $A \in L_i$. Then $\omega_a$ and $\omega_b$ both vanish; this means that $\omega_t$ is divisible by $t$, contradiction. $\square$

Poincaré duality on $\widetilde{Y}_i$ identifies $H_2(\widetilde{Y}_i, \mathbb{Z})$ with a subgroup of $H^2(\widetilde{Y}_i, \mathbb{Z})$ whose quotient is 2-elementary. Correspondingly, by Lemma 8.4, there is a unique normalized basis $(\omega^{(1)}(t), ..., \omega^{(h)}(t))$ of $H^0(\widetilde{\mathcal{Y}}_t, \Omega^2_{\widetilde{\mathcal{Y}}_t})$ such that $\int_{A_l} \omega^{(j)}(t) = \delta^j_l$.

There are holomorphic 3-forms $\Omega^{(j)}$ on the smooth 3-fold stack $\widetilde{\mathcal{Y}}$ such that

$$\omega^{(j)}(\lambda) = \mathrm{Res}_{\widetilde{\mathcal{Y}}_\lambda} \Omega^{(j)}/(t - \lambda).$$

We can expand $\Omega^{(j)}$ locally, on the inverse image in $\widetilde{\mathcal{Y}}$ of the plumbing fixture $\widetilde{F}$, as

$$\Omega^{(j)} = \sum_{m,n \geq 0} c^{(j)}_{m,n} q^m v^n dq \wedge dv \wedge dw,$$

so that, since $t \equiv q^2 - v^2 \pmod{t^2}$,

$$\omega^{(j)}(t) \equiv -\tfrac{1}{2} \sum_{m,n \geq 0} c^{(j)}_{m,n} q^m v^{n-1} dq \wedge dw.$$

(Here and from now on all congruences are taken modulo $t^2$.)

In a neighbourhood of $\widetilde{Y}_a$ we have $v = q(1 - tq^{-2})^{1/2}$, so that

$$v^{n-1} = q^{n-1}(1 - tq^{-2})^{(n-1)/2} \equiv q^{n-1}(1 - (n-1)tq^{-2}/2).$$

Therefore $\omega^{(j)}(t)|_{\widetilde{Y}_a} \equiv \omega^{(j)}_{\widetilde{Y}_a} + t\eta^{(j)}_{\widetilde{Y}_a}$ where

$$\begin{aligned} \omega^{(j)}_{\widetilde{Y}_a} &= -\tfrac{1}{2}\sum c^{(j)}_{m,n} q^{m+n-1} dq \wedge dw \text{ and} \\ \eta^{(j)}_{\widetilde{Y}_a} &= \tfrac{1}{4}\sum c^{(j)}_{m,n}(n-1) q^{m+n-3} dq \wedge dw \end{aligned}$$

are 2-forms (the first holomorphic, the second meromorphic) on $\widetilde{Y}_a$. On $\widetilde{Y}_a$ we have $q = \tilde{z}_a$ and $w = w_a$, so that

$$\begin{aligned} \omega^{(j)}_{\widetilde{Y}_a} &= -\tfrac{1}{2}\sum_{p\geq 0}\left(\sum_{m+n=p} c^{(j)}_{m,n}\right) \tilde{z}_a^{p-1} d\tilde{z}_a \wedge dw_a \text{ and} \\ \eta^{(j)}_{\widetilde{Y}_a} &= \tfrac{1}{4}\sum_{p\geq 0}\left(\sum_{m+n=p} (n-1)c^{(j)}_{m,n}\right) \tilde{z}_a^{p-3} d\tilde{z}_a \wedge dw_a. \end{aligned}$$

In a neighbourhood of $\widetilde{Y}_b$ we have $v = -q(1-tq^{-2})^{1/2}$ and on $\widetilde{Y}_b$ we have $q = \tilde{z}_b$ and $w = w_b$, so that $\omega^{(j)}(t)|_{\widetilde{Y}_b} \equiv \omega^{(j)}_{\widetilde{Y}_b} + t\eta^{(j)}_{\widetilde{Y}_b}$ where

$$\begin{aligned} \omega^{(j)}_{\widetilde{Y}_b} &= -\tfrac{1}{2}\sum_{p\geq 0}\left(\sum_{m+n=p} (-1)^{n-1}c^{(j)}_{m,n}\right) \tilde{z}_b^{p-1} d\tilde{z}_b \wedge dw_b \text{ and} \\ \eta^{(j)}_{\widetilde{Y}_b} &= \tfrac{1}{4}\sum_{p\geq 0}\left(\sum_{m+n=p} (-1)^{n-1}(n-1)c^{(j)}_{m,n}\right) \tilde{z}_b^{p-3} d\tilde{z}_b \wedge dw_b. \end{aligned}$$

The involution $\iota$ acts via $\iota^*\tilde{z}_i = -\tilde{z}_i$ and $\iota^* w_i = -w_i$, so that, since each $\omega^{(j)}_{\widetilde{Y}_i}$ and $\eta^{(j)}_{\widetilde{Y}_i}$ is $\iota$-invariant, we need only consider odd values of the index $p$ in the expansions above. That is,

$$\omega^{(j)}_{\widetilde{Y}_a} = -\tfrac{1}{2}\sum_{r\geq 0}\left(\sum_{m+n=2r+1} c^{(j)}_{m,n}\right) \tilde{z}_a^{2r} d\tilde{z}_a \wedge dw_a, \tag{8.5}$$

$$\eta^{(j)}_{\widetilde{Y}_a} = \tfrac{1}{4}\sum_{r\geq 0}\left(\sum_{m+n=2r+1} (n-1)c^{(j)}_{m,n}\right) \tilde{z}_a^{2r-2} d\tilde{z}_a \wedge dw_a, \tag{8.6}$$

$$\omega^{(j)}_{\widetilde{Y}_b} = -\tfrac{1}{2}\sum_{r\geq 0}\left(\sum_{m+n=2r+1} (-1)^{n-1}c^{(j)}_{m,n}\right) \tilde{z}_b^{2r} d\tilde{z}_b \wedge dw_b \text{ and} \tag{8.7}$$

$$\eta^{(j)}_{\widetilde{Y}_b} = \tfrac{1}{4}\sum_{r\geq 0}\left(\sum_{m+n=2r+1} (-1)^{n-1}(n-1)c^{(j)}_{m,n}\right) \tilde{z}_b^{2r-2} d\tilde{z}_b \wedge dw_b \tag{8.8}$$

and $c^{(j)}_{m,n} = 0$ if $m+n$ is even. In particular, these formulae show that each $\eta^{(j)}_{\widetilde{Y}_i}$ has only a double pole along $\widetilde{\phi}_{\tilde{i}}$, and is otherwise holomorphic.

The next lemma is well known. It holds for any one-parameter degeneration of surfaces, not only for the families that we have constructed via Fay plumbings. However, the results such as Lemma 8.10 below that follow for the derivatives $\eta_{X_i}^{(j)}$ do not hold in such generality, because for more general glueing data a formula such as $t \equiv q^2 - v^2 \pmod{t^2}$ will not hold. It follows that for a general one-parameter degeneration it is not possible to control the orders of the poles of the forms $\eta_{\widetilde{Y}_i}^{(j)}$ (which are the restrictions to $\widetilde{Y}_i$ of the derivatives at $t=0$ of the forms $\omega^{(j)}(t)$) along $\widetilde{\phi}_{\bar{i}}$.

**Lemma 8.9** *(1) $\omega_{\widetilde{Y}_i}^{(j)}$ is holomorphic on $\widetilde{Y}_i$ for every $j \in [1, h_a + h_b]$.*

*(2) $\omega_{\widetilde{Y}_b}^{(j)} = 0$ for every $j \in [1, h_a]$ and $\omega_{\widetilde{Y}_a}^{(j)} = 0$ for every $j \in [h_a+1, h_a+h_b]$.*

*(3) $(\omega_{\widetilde{Y}_a}^{(1)}, ..., \omega_{\widetilde{Y}_a}^{(h_a)})$ and $(\omega_{\widetilde{Y}_b}^{(h_a+1)}, ..., \omega_{\widetilde{Y}_b}^{(h_a+h_b)})$ are bases of the vector spaces $H^0(\widetilde{Y}_a, \omega_{\widetilde{Y}_a})$ and $H^0(\widetilde{Y}_b, \omega_{\widetilde{Y}_b})$, respectively.*

*(4) These bases are normalized with respect to the given A-cycles on $\widetilde{Y}_a$ and $\widetilde{Y}_b$.*

PROOF: This is a restatement of Lemmas 8.3 4 and 8.4. □

**Lemma 8.10** *(1) Up to scalars there is a unique meromorphic 2-form $\widetilde{\eta}_{\widetilde{Y}_i} \in H^0(\widetilde{Y}_i, \Omega^2_{\widetilde{Y}_i}(2\widetilde{\phi}_{\bar{i}}))$ such that every $\int_{A_l} \widetilde{\eta}_{\widetilde{Y}_i} = 0$.*

*(2) Every $\eta_{\widetilde{Y}_i}^{(j)}$ is a multiple of $\widetilde{\eta}_{\widetilde{Y}_i}$.*

PROOF: (1) follows from the facts that $\dim H^0(\widetilde{Y}_i, \Omega^2_{\widetilde{Y}_i}(2\widetilde{\phi}_{\bar{i}})) = p_g(\widetilde{Y}_i) + 1$ and that $\widetilde{Y}_i$ is in $L_i$-general position. (2) is then an immediate consequence of (1) and the observation above that each $\eta_{\widetilde{Y}_i}^{(j)}$ lies in $H^0(\widetilde{Y}_i, \Omega^2_{\widetilde{Y}_i}(2\widetilde{\phi}_{\bar{i}}))$. □

We shall refer to the forms $\widetilde{\eta}_{\widetilde{Y}_i}$ as being *normalized* by the requirements that $\widetilde{\eta}_{\widetilde{Y}_i} \in H^0(\widetilde{Y}_i, \Omega^2_{\widetilde{Y}_i}(2\widetilde{\phi}_{\bar{i}}))$ and every $\int_{A_l} \widetilde{\eta}_{\widetilde{Y}_i} = 0$.

Construct four row vectors

$$
\begin{aligned}
\underline{\eta}_{\widetilde{Y}_i} &= [\eta_{\widetilde{Y}_i}^{(1)}, ..., \eta_{\widetilde{Y}_i}^{(h_a+h_b)}], \\
\underline{\omega}_{\widetilde{Y}_a} &= [\omega_{\widetilde{Y}_a}^{(1)}, ..., \omega_{\widetilde{Y}_a}^{(h_a)}] \text{ and} \\
\underline{\omega}_{\widetilde{Y}_b} &= [\omega_{\widetilde{Y}_b}^{(h_a+1)}, ..., \omega_{\widetilde{Y}_b}^{(h_a+h_b)}],
\end{aligned}
$$

each consisting of meromorphic 2-forms on the surface indicated. As an abbreviation, define two further row vectors, each consisting of numbers, by

$$\underline{\omega}_{\widetilde{Y}_i}(i) = \frac{\underline{\omega}_{\widetilde{Y}_i}}{d\tilde{z}_i \wedge dw_i}(i)\,.$$

**Proposition 8.11** *We can scale the forms $\widetilde{\eta}_{\widetilde{Y}_i}$ such that in terms of the local co-ordinates $\tilde{z}_i$ they are given by*

$$\begin{aligned}\widetilde{\eta}_{\widetilde{Y}_a} &= \tfrac{1}{4}(\tilde{z}_a^{-2} + \text{ h.o.t.})d\tilde{z}_a \wedge dw_a \text{ and}\\ \widetilde{\eta}_{\widetilde{Y}_b} &= -\tfrac{1}{4}(\tilde{z}_b^{-2} + \text{ h.o.t.})d\tilde{z}_b \wedge dw_b\end{aligned}$$

*and there is an equality of row vectors*

$$\underline{\eta}_{\widetilde{Y}_i} = \widetilde{\eta}_{\widetilde{Y}_i}[\underline{\omega}_{\widetilde{Y}_a}(a), -\underline{\omega}_{\widetilde{Y}_b}(b)].$$

PROOF: Examining the first few terms in the expansion provided by the formulae (8.5) *et seq.* gives

$$\begin{aligned}\omega^{(j)}_{\widetilde{Y}_a}(a) &= -\tfrac{1}{2}(c^{(j)}_{1,0} + c^{(j)}_{0,1}),\\ \omega^{(j)}_{\widetilde{Y}_b}(b) &= -\tfrac{1}{2}(-c^{(j)}_{1,0} + c^{(j)}_{0,1}),\\ \eta^{(j)}_{\widetilde{Y}_a} &= \tfrac{1}{4}(-c^{(j)}_{1,0}\tilde{z}_a^{-2} + \text{h.o.t.})d\tilde{z}_a \wedge dw_a \text{ and}\\ \eta^{(j)}_{\widetilde{Y}_b} &= \tfrac{1}{4}(c^{(j)}_{1,0}\tilde{z}_b^{-2} + \text{h.o.t.})d\tilde{z}_b \wedge dw_b\end{aligned}$$

for all $j = 1, ..., h_a + h_b$.

If $j \leq h_a$ then $\omega^{(j)}_{\widetilde{Y}_b} = 0$, so that $c^{(j)}_{1,0} = c^{(j)}_{0,1} = -\omega^{(j)}_{\widetilde{Y}_a}(a)$.

If $j \geq h_a + 1$ then $\omega^{(j)}_{\widetilde{Y}_a} = 0$, so that $c^{(j)}_{1,0} + c^{(j)}_{0,1} = 0$ and $\omega^{(j)}_{\widetilde{Y}_b}(b) = c^{(j)}_{1,0}$. The result is proved. □

Via the identifications of Lemma 8.2 we also regard these as vectors of forms on $Y_i$ and on $\overline{Y}_i$ and also write them as $\underline{\eta}_{Y_i}$, etc.

Let $\Psi_{Y_i}$ denote the period matrix of $Y_i$, normalized with respect to the 2-cycles $A_1, ..., A_{h_i}$ above. Recall that normalizing means that $\int_{A_j} \omega^{(k)}_{Y_i} = \delta^k_j$ for each $i$ and for the appropriate values of $j, k$ and that $\Psi_{Y_i}$ is an $h_i \times (11h_i + 8)$ matrix while $\Psi_t := \Psi_{\widetilde{\mathcal{Y}}_t}$ is an $h \times (11h + 8)$ matrix. Note, however, that each $Y_i$ contains four disjoint $(-2)$-curves (components of the $\widetilde{D}_4$-fibre that is designated on each), so that each $\Psi_{Y_i}$ contains an $h_i \times 4$ block of zeroes, which can be discarded to give matrices that we shall continue to denote by $\Psi_{Y_i}$, each of whose shape is $h_i \times (11h_i \times 4)$.

**Proposition 8.12** *(1) The normalized period matrix $\Psi_t$ of $\widetilde{\mathcal{Y}}_t$ is given, after a suitable re-arrangement of the blocks that comprise it, by*

$$\Psi_t = \begin{bmatrix} \Psi_{Y_a} & 0 \\ 0 & \Psi_{Y_b} \end{bmatrix} + t[\underline{\omega}_{Y_a}(a), -\underline{\omega}_{Y_b}(b)] \otimes [\underline{I}_a, \underline{I}_b] + h.o.t.$$

*where $\underline{I}_i$ is a vector of integrals of $\widetilde{\eta}_{Y_i}$ over cycles on $Y_i$.*

*(2) The derivative $(d\Psi_t/dt)|_{t=0}$ of the period matrix of $\mathcal{Y}_t$ at $t=0$ is the rank 1 matrix $[\underline{\omega}_{Y_a}(a), -\underline{\omega}_{Y_b}(b)] \otimes [\underline{I}_a, \underline{I}_b]$.*

PROOF: This is an immediate consequence of Proposition 8.11 and the fact that

$$\omega^{(j)}(t)|_{\widetilde{Y}_i} \equiv \omega^{(j)}_{\widetilde{Y}_i} + t\eta^{(j)}_{\widetilde{Y}_i},$$

modulo $t^2$, for each $j = 1, ..., h_a + h_b$ and for each $i = a, b$.

Note that the vector $[\underline{I}_a, \underline{I}_b]$ can be seen to have the correct length (so that the matrix that describes $(d\Psi_t/dt)|_{t=0}$ has the correct shape) by omitting the zeroes, four in each $[\underline{I}_i]$, that arise from integrating $\widetilde{\eta}_i$ around the four exceptional $(-2)$ curves on $Y_i$ that are contracted in $\overline{Y}_i$. □

**Lemma 8.13** *Suppose that $h_i \geq 1$ and that $Y_i$ is* either *a special elliptic surface* or *generic. Then the class $[\widetilde{\eta}_{Y_i}]$ in $H^2(Y_i, \mathbb{C})$ is non-zero.*

PROOF: Suppose first that $Y_i$ is special. Then $Y_i$ is birational to the geometric quotient $C \times E/\iota$, where $C$ is hyperelliptic of genus $h_i$. Suppose that $P \in C$ is a fixed point of $\iota|_C$, so that the image of $\{P\} \times E$ in $Y_i$ is a $\widetilde{D}_4$ fibre $\phi_i$.

There is an $\iota$–anti-invariant section $\sigma$ of $H^0(C, \Omega^1_C(2P))$ that is not holomorphic; modulo the subspace $H^0(C, \Omega^1_C)$ this section is unique up to scalars. Moroever, $\sigma$ has no residues and defines a non-trivial class in $H^{0,1}(C)$. Choose a non-zero section $\tau$ of $H^0(E, \Omega^1_E)$. Then $\tau$ is $\iota$–anti-invariant, and $\sigma \wedge \tau$ defines a non-zero section of $H^0(Y_i, \omega_{Y_i}(\phi_i))/H^0(Y_i, \omega_{Y_i})$. Inspection of the leading term of $\widetilde{\eta}_{Y_i}$ shows that $\widetilde{\eta}_{Y_i}$ defines a non-zero section of the same 1-dimensional vector space, so that $\sigma \wedge \tau$ is equivalent to $\lambda\widetilde{\eta}_{Y_i}$ modulo $H^0(Y_i, \omega_{Y_i})$, for some $\lambda \in \mathbb{C}$. But $\sigma \wedge \tau$ defines a non-zero class in $H^{1,1}(Y_i)$, and the lemma is proved when $Y_i$ is special.

The result for generic $Y_i$ follows at once. □

We can now re-state Proposition 8.12 in intrinsic terms. We shall make use of Remark 8.14 1 below, to the effect that $[\widetilde{\eta}_{Y_i}]$ in fact lies in $Fil^1(H^2(Y_i, \mathbb{C}))$.

**Corollary 8.14** *$(d\Psi_t/dt)|_{t=0}$ is a linear map*

$$H^0(Y_a, \omega_{Y_a}) \oplus H^0(Y_b, \omega_{Y_b}) \to H^{1,1}(Y_a) \oplus H^{1,1}(Y_b)$$

*whose rank is at most 1.*

*If at least one of the surfaces $Y_a$ or $Y_b$ is special or generic, then the rank of $(d\Psi_t/dt)|_{t=0}$ equals 1, its kernel is the hyperplane $\{(\sigma_a, \sigma_b) | \sigma_a(a) + \sigma_b(b) = 0\}$ in $H^0(Y_a, \omega_{Y_a}) \oplus H^0(Y_b, \omega_{Y_b})$ and its image is the line spanned by the element $([\widetilde{\eta}_{Y_a}], [\widetilde{\eta}_{Y_b}])$.*

**Remark**: (1) Recall also that the final $h \times h$ block (with respect to an appropriate choice of basis) of the period matrix, and so of the derivative $(d\Psi/dt)|_{t=0}$, is skew-symmetric. But a skew-symmetric matrix of rank 1 vanishes identically; for $(d\Psi/dt)|_{t=0}$ this is predicted by Griffiths transversality. In terms of the bases that we have used, this means that the final piece of length $h_i$ in each vector $\underline{I}_i$ vanishes. Stated in intrinsic terms, this means that each class $[\widetilde{\eta}_{Y_i}]$ lies in $H^{1,1}(Y_i)$.

(2) Suppose that $Y_b$ is rational. Then $(d\Psi_t/dt)|_{t=0}$ is given by

$$(d\Psi_t/dt)|_{t=0} = [\underline{\omega}_{Y_a}(a)] \otimes [\underline{I}_a, \underline{I}_b].$$

Note that $Y_b$ has 4 moduli if the constraint that its $\widetilde{D}_4$ fibre $\phi_b$ should be isomorphic to $\phi_a$ is ignored. This constraint reduces the number of moduli of $Y_b$ to 3.

There is a desingularization $\mathcal{Y} \to [\widetilde{\mathcal{Y}}]$ of the geometric quotient such that the closed fibre $\mathcal{Y}_0$ of $\mathcal{Y} \to \Delta$ is semi-stable and has six components: two are the surfaces $Y_i$ for $i = a, b$ and four are copies of $\mathbb{P}^2$ each of whose normal bundle is $\mathcal{O}(-2)$. So $\mathcal{Y} \to \Delta$ has a birational model $\mathcal{Y}' \to \Delta$ with good reduction, which is obtained from $\mathcal{Y}$ after flopping four times (each time in a curve in $Y_b$ that is a section of the elliptic fibration and meets one of the $V_i$), then contracting the strict transforms of the $V_i$ to curves and finally contracting the strict transform of $Y_b$ to a curve. The closed fibre $\mathcal{Y}_0'$ is isomorphic to $Y_a$. That is, a Fay plumbing gives a one-parameter deformation of a surface $Y_a$ where the derivative of the period matrix is given explicitly, provided that $Y_a$ contains a $\widetilde{D}_4$-fibre. Since $Y_b$ has three moduli the vector $\underline{I}_b$ has three moduli, so $(d\Psi_t/dt)|_{t=0}$ has three moduli.

This is an analogue of Example 3.5 on p. 45 of [F1].

The next lemma is needed for the proof of Proposition 8.16.

**Lemma 8.15** *If $\delta$ is a vertical $(-2)$-curve on $Y_i$ that does not lie in the designated $\widetilde{D}_4$-fibre $\phi_i$, then $\int_\delta \widetilde{\eta}_{Y_i} = 0$.*

PROOF: We can take $i = a$.

By construction, the first order plumbing $\mathcal{Y}' \to \Delta'$ is trivial along $Y_a - T_a$, where $T_a$ is a suitable neighbourhood of $\phi_a$. So $\int_\delta \omega^{(j)}(t) \equiv 0 \pmod{t^2}$ for $j = 1, ..., h_a$. But $\omega^{(j)}(t)|_{Y_a} = \omega^{(j)}_{Y_a} + t\eta^{(j)}_{Y_a} + h.o.t.$ and the lemma follows. □

Now suppose that $f_1 : Y_1 \to C_1, ..., f_r : Y_r \to C_r$ are smooth simply connected Jacobian elliptic surfaces such that $Y_i \to C_i$ has $\widetilde{D}_4$-fibres over a finite non-empty set $\{P_{ij}\}$ of points in $C_i$. Let $Y_i \to \overline{Y}_i$ denote the contraction of each of these designated $\widetilde{D}_4$-fibres to a $\overline{D}_4$-fibre.

Suppose also that $\Gamma$ is a connected tree with $r$ vertices and that we can associate the surfaces $\overline{Y}_i$ to the vertices of $\Gamma$ such that two vertices $i, j$ are joined in $\Gamma$ if and only if the $\overline{D}_4$-fibre on $\overline{Y}_i$ that lies over $P_{ij}$ is isomorphic to the $\overline{D}_4$-fibre on $\overline{Y}_j$ that lies over $P_{ji}$.

Then the plumbings modulo $t^2$ of the stacks $\widetilde{Y}_i$ that has just been described can be iterated to give

$$\mathcal{Y}' = \mathcal{Y}'_\Gamma \to \Delta'_{r-1} = \mathbf{Spec}\,\mathbb{C}[t_1, ..., t_{r-1}]/(t_1^2, ..., t_{r-1}^2)$$

where the closed fibre is $\sum \widetilde{Y}_i$ arranged according to the tree $\Gamma$. Let $\underline{\omega}_{Y_i}$ denote a normalized basis of $H^0(Y_i, \Omega^2_{Y_i})$ and $\widetilde{\eta}_{ij}$ a normalized element of $H^0(Y_i, \Omega^2_{Y_i}(\phi_{ij}))$.

**Proposition 8.16** *Assume that $\mathcal{Y}' \to \Delta'_{r-1}$ can be lifted to a family $\mathcal{Y} \to S_{t_e}$ over an $(r-1)$-dimensional polydisc.*

*(1) Then there is no monodromy on $H^2$ of the geometric generic fibre $\mathcal{Y}_t$ and the period matrix $\Psi(\mathcal{Y}_t)$ is given by*

$$\Psi(\mathcal{Y}_t) = [\Psi(Y_1), ..., \Psi(Y_r)] + \sum_e t_e \Pi_e + h.o.t.,$$

*where $\Pi_e$ is the rank 1 matrix*

$$[\underline{\omega}_{Y_i}(P_{ij}), -\underline{\omega}_{Y_j}(P_{ji})] \otimes [\underline{I}_{ij}, \underline{I}_{ji}]$$

*and $\underline{I}_{ij}$ is a vector of integrals of $\widetilde{\eta}_{ij}$ around 2-cycles on $Y_i$.*

*(2) If each $Y_i$ is either special or generic, then the image of the tangent space $T_0 S_{t_e}$ under the derivative of the period map is of dimension $r-1$ and is the vector space spanned by the matrices $\Pi_e$.*

PROOF: (1) is a straightforward consequence of Proposition 8.12. Lemma 8.15 tells us that, when considering integrals of $\widetilde{\eta}_{ij}$ around cycles on $Y_i$, we can omit the $(-2)$ curves that lie in the $\widetilde{D}_4$-fibres on $Y_i$ and are disjoint from $\phi_{ij}$. This makes it possible to write the vectors $\underline{I}_{ij}$ in such a way that the matrices $\Pi_e$, when suitably enlarged by adding various blocks of zero matrices, lie in the same vector space of matrices.

(2) then follows from Lemma 8.13. □

We shall see, in Lemma 11.7, that the liftings $\mathcal{Y} \to S_{t_e}$ exist when each surface $Y_i$ is a special elliptic surface. Then, in Theorem 11.11, which will lead to the main result, Theorem 11.12, we shall take all the surfaces $Y_i$ to be K3 surfaces and $\Gamma$ will be an alkane.

# 9 Stable reduction of surfaces

Chakiris proved [C1], [C2] a generic Torelli theorem for Jacobian elliptic surfaces over $\mathbb{P}^1$ by reducing the problem to special elliptic surfaces.

To achieve this reduction he extended the domain of the period map to make the map proper. In turn, he did this by first showing that a one-parameter degeneration of Jacobian elliptic surfaces over $\mathbb{P}^1$ without monodromy can be put into a certain standard form. See, e.g., the statement $(**)$ in [C2], top of p. 174 or the "stable reduction" theorem on p. 231 of [C1]. Note, however, that this version of a stable reduction theorem gives a closed fibre that contains curves of cusps, and so is not semi log canonical (slc) in the sense of the MMP.

In this section we shall refine his result, so that the period map becomes proper over each of the loci $\mathcal{W}_{h_1,...,h_r}$ defined in Definition 2.4. For this, we use his ideas strengthened by the MMP, which was not available to him. The main result is Theorem 9.1.

We say that a special elliptic surface $X$ is *sesqui-special* if it is associated to a product $E \times C$ where the Jacobians $E = \mathrm{Jac}(E)$, $\mathrm{Jac}(C)$ of $E$ and $C$ have no multiplication and $\mathrm{Hom}(E, \mathrm{Jac}(C)) = 0$. If $C$ and $E$ are generic then $X$ is sesqui-special [Z]. A very general point in $\mathcal{W}_{h_1,...,h_r}$ is defined by a configuration of sesqui-special surfaces $X_1, ..., X_r$ where $X_i$ is associated to $E \times C_i$ and also $\mathrm{Hom}(\mathrm{Jac}(C_i), \mathrm{Jac}(C_j)) = \mathbb{Z}\delta_{ij}$.

**Theorem 9.1** *Suppose that $\mathcal{X} \to \Delta$ is a 1-parameter degeneration of simply connected Jacobian elliptic surfaces of geometric genus $h \geq 1$ which is semi-stable (in the usual sense that the closed fibre $\mathcal{X}_0$ is reduced with normal crossings) and that there is no monodromy on the cohomology of the geometric generic fibre $\mathcal{X}_{\bar{\eta}}$. Assume also that, under the period map, the image of $0 \in \Delta$ is the direct sum of the period matrices of sesqui-special elliptic surfaces $\widetilde{V}_1, ..., \widetilde{V}_r$ that define a point in $\mathcal{W}_{h_1,...,h_r}$.*

*Then there is a birationally equivalent model $\mathcal{Y} \to \Delta$ with the following properties:*

*(1) $\mathcal{Y}$ has $\mathbb{Q}$-factorial canonical singularities;*

*(2) the closed fibre $\mathcal{Y}_0$ has slc singularities;*

*(3) the irreducible components of $\mathcal{Y}_0$ are the singular models $V_i$ of the $\widetilde{V}_i$ with $\overline{D}_4$-fibres;*

*(4) if $V_i \cap V_j$ is not empty then it is a copy of $\mathbb{P}^1$ and contains 4 points at all of which $V_i$ and $V_j$ each has a node;*

*(5) each triple intersection $V_i \cap V_j \cap V_k$ is empty;*

*(6) $\mathcal{Y}_0$ is formed by arranging the surfaces $V_i$ in a tree.*

PROOF: To begin, suppose that $\mathcal{X} \to \Delta$ is an arbitrary semi-stable family of smooth minimal elliptic surfaces of Kodaira dimension 1.

Then run the MMP in two steps, as follows.

(1) Run a $K_{\mathcal{X}/\Delta}$ MMP on $\mathcal{X} \to \Delta$ and let $\mathcal{X}_1 \to \Delta$ be the output. Then $\mathcal{X}_1$ has $\mathbb{Q}$-factorial terminal singularities and $K_{\mathcal{X}_1/\Delta}$ is semi-ample So some relative pluricanonical system $|mK_{\mathcal{X}_1/\Delta}|$ defines an algebraic fibre space $f : \mathcal{X}_1 \to S$ where $S \to \Delta$ is a semi-stable family of curves (so that $S$ has singularities of type $A$) and $K_{\mathcal{X}_1/\Delta}$ pulls back from an ample $\mathbb{Q}$-line bundle on $S$. Moreover, the closed fibre $\mathcal{X}_{1,0}$ has slc singularities. Replace $\mathcal{X}$ by $\mathcal{X}_1$.

(2) If there are surfaces $E_i$ in $\mathcal{X}$ such that $f(E_i)$ is a point then there are only finitely many such. For suitable $\alpha_i \in \mathbb{Q}$ with $0 < \alpha_i \ll 1$ run a $(K_{\mathcal{X}/S}, \sum \alpha_i E_i)$ MMP on $\mathcal{X} \to S$. The output is a birational map $\mathcal{X} - \to \mathcal{X}_1$ under which the strict transform of each $E_i$ is of dimension at most 1. Since $K_{\mathcal{X}}$ is trivial in a neighbourhood of $\sum E_i$, the rational map $\mathcal{X} - \to \mathcal{X}_1$ is regular outside $\sum E_i$, so that the rational map $\mathcal{X}_1 - \to S$ is a morphism, all its fibres are 1-dimensional, $\mathcal{X}_1$ has canonical singularities and $\mathcal{X}_{1,0}$ has

slc singularities. Moreover, $\mathfrak{X}_1$ is $\mathbb{Q}$-factorial, by general properties of the MMP. Replace $\mathfrak{X}$ by $\mathfrak{X}_1$.

At this stage of the argument $\mathfrak{X}$ has $\mathbb{Q}$-factorial canonical singularities and $\mathfrak{X}_0$ has slc singularities. Moreover, there is a surface $S$, a semi-stable morphism $g : S \to \Delta$ (so that $S$ has singularities of type $A$) and a morphism $f : \mathfrak{X} \to S$ with only one-dimensional fibres such that $K_{\mathfrak{X}/\Delta}$ is the pullback under $f$ of a $g$-ample $\mathbb{Q}$-line bundle on $S$.

Now assume that there is no monodromy on the cohomology $H^2(\mathfrak{X}_{\bar\eta})$ of the geometric generic fibre. This is equivalent to $p_g(\mathfrak{X}_{\bar\eta}) = \sum p_g(\widetilde{V})$, where $\widetilde{V}$ runs over the minimal resolutions of the components of $\mathfrak{X}_0$. Assume also that $\mathfrak{X}_{\bar\eta}$ is simply connected and that the generic fibre $\mathfrak{X}_\eta$ is Jacobian. However, we shall make no assumption about the image of 0 under the period map until after Lemma 9.9.

Say $S_0 = \sum C_i$, $X_i = f^{-1}(C_i)$, $f_i : X_i \to C_i$ the restriction of $f : \mathfrak{X} \to S$, $\nu_i : X_i^\nu \to X_i$ the normalization, $\widetilde{X}_i \to X_i^\nu$ the minimal resolution. We have $K_{\mathfrak{X}/\Delta}|_{X_i} \sim f_i^*(\alpha_i)$ for some $\alpha_i \in \mathbb{Q}$ with $\alpha_i > 0$.

**Lemma 9.2** *(1) Each $C_i$ is isomorphic to $\mathbb{P}^1$.*

*(2) $f_i : X_i \to C_i$ has a section.*

PROOF: $S_0$ is a specialization of $\mathbb{P}^1$, and (1) follows.

(2): Suppose that $D \subset \mathfrak{X}$ is the Zariski closure of the given generic section. Then $D \cap X_i$ consists of a section with also some vertical components. □

**Lemma 9.3** *$X_i$ and $X_j$ are Cohen–Macaulay and are smooth at each generic point of $f^{-1}(P_{ij})$.*

PROOF: This is a consequence of the classification of slc singularities. In particular, smoothness at each generic point of $f^{-1}(P_{ij})$ follows from the fact that slc singularities have normal crossings in codimension 1. □

By the classification of slc singularities, the generic fibre of each $f_i$ is elliptic or a rational cycle. For each $i$, put $C_i^0 = C_i - \mathrm{Sing}(S_0)$.

**Lemma 9.4** *The maps $H^2(\mathfrak{X}_0, \mathcal{O}) \to \oplus H^2(\widetilde{X}_i, \mathcal{O})$ and $H^2(X_i^\nu, \mathcal{O}) \to H^2(\widetilde{X}_i, \mathcal{O})$ are all isomorphisms.*

PROOF: $H^2(\mathfrak{X}_0, \mathcal{O}) \to \oplus H^2(\widetilde{X}_i, \mathcal{O})$ is surjective. Since the formation of the groups $H^i(\mathfrak{X}_t, \mathcal{O})$ commutes with specialization the lemma follows from the assumption that $p_g(\mathfrak{X}_{\bar\eta}) = \sum p_g(\widetilde{X}_i)$. □

**Lemma 9.5** *Suppose that $X_i = f^{-1}(C_i) \to C_i$ is generically smooth and that $h^2(X_i, \mathcal{O}) > 0$. Then $X_i$ is normal with rational singularities, and over $C_i^0$ it is Gorenstein.*

PROOF: Let $Q \in C_i^0$ and let superscript $h$ denote henselization at $Q$. Note that $S_0^h = C_i^h$. Then $\mathfrak{X}^h - X_i^h \to S^h - S_0^h$ is a Jacobian elliptic fibration, so that

$K_{\mathfrak{X}^h - X_i^h}$ pulls back from a line bundle on $S^h - S_0^h$ and so is trivial. Since $X_i^h$ is an irreducible and principal Weil divisor in $\mathfrak{X}^h$, the restriction homomorphism $\mathrm{Cl}(\mathfrak{X}^h) \to \mathrm{Cl}(\mathfrak{X}^h - X_i^h)$ of Weil divisor class groups is an isomorphism. Therefore $K_{\mathfrak{X}^h}$ is trivial. So $\mathfrak{X}$ is Gorenstein along $f^{-1}(Q)$ and $K_{\mathfrak{X}^h}$ pulls back from a line bundle on $S^h$. Therefore $X_i^h$ is also Gorenstein and $\omega_{X_i^h}$ also pulls back from a line bundle on $C_i^h$.

There is a commutative diagram

$$\begin{array}{ccccc} & & \widetilde{X}_i & & \\ & \swarrow^{k_i} & & \searrow^{g_i} & \\ X_i^{min} & & & & X_i \\ & \searrow_{f_i^{min}} & & \swarrow_{f_i} & \\ & & C_i & & \end{array}$$

where $g_i : \widetilde{X}_i \to X_i$ is the minimal resolution, $k_i : \widetilde{X}_i \to \widetilde{X}_i^{min}$ is the minimal model relative to $C_i$ and $f_i^{min} : \widetilde{X}_i^{min} \to C_i$ is the resulting morphism. Then

$$f_i^{min*} K_{\widetilde{X}_i^{min}} + E \sim K_{\widetilde{X}_i} \sim g_i^* K_{X_i} - Z$$

for some non-negative integral divisors $Z, E$. Moreover, $Z > 0$ if $X_i$ is not normal along $f_i^{-1}(Q)$, since there is a non-zero contribution to $Z$ made by the conductor ideal. But $k_i^* K_{X_i^{min}} \sim g_i^* K_{X_i} - (Z + E)$ then shows that $h^2(X_i^{min}, \mathcal{O}_{X_i^{min}}) < h^2(X_i, \mathcal{O}_{X_i})$, contradiction. So $X_i$ is normal along $f_i^{-1}(Q)$, so that over $C_i^0$ it is normal and Gorenstein.

That $X_i$ is normal everywhere follows from Lemma 9.3; it remains to prove that $X_i$ has rational singularities.

So suppose that there is at least one irrational singularity. We know that $H^2(X_i, \mathcal{O}_{X_i}) \to H^2(\widetilde{X}_i, \mathcal{O}_{\widetilde{X}_i})$ is an isomorphism; then $\widetilde{X}_i$ is elliptic over $\mathbb{P}^1$ with a section and $p_g(\widetilde{X}_i) > 0$, so that, by considering the Albanese variety of $\widetilde{X}_i$, we see that $H^1(\widetilde{X}_i, \mathcal{O}_{\widetilde{X}_i}) = 0$. Via the Leray spectral sequence

$$E_2^{pq} = H^p(X_i, R^q g_{i*} \mathcal{O}_{\widetilde{X}_i}) \Rightarrow H^{p+q}(\widetilde{X}_i, \mathcal{O}_{\widetilde{X}_i})$$

we then see that $R^q g_{i*} \mathcal{O}_{\widetilde{X}_i} = 0$, and the singularities are rational. □

Suppose that $C, D$ are irreducible components of $S_0$, $C \cap D = P$, $Y = f^{-1}(C)$, $Z = f^{-1}(D)$. Since $S$ has an isolated singularity of type $A$ at $P$, we can write the henselization $S^h$ of $S$ at $P$ as a geometric quotient $S^h = [S'/(\mathbb{Z}/N)]$ where $S'$ is smooth and local, $S' \to \Delta$ is semi-stable and the closed fibre $S'_0$ has two smooth branches $C', D'$. Moreover, $\mathbb{Z}/N$ acts with opposite weights on these two branches, since $S^h$ is of type $A$. In particular, it acts effectively on each of them.

Let $\mathfrak{X}'$ denote the normalization of $\mathfrak{X} \times_S S'$, with induced morphism $f' : \mathfrak{X}' \to S'$; this morphism is proper. Let $P' \in S'$ be the point lying over $P$.

Set $\mathfrak{X}^h = \mathfrak{X} \times_S S^h$ and let $f^h : \mathfrak{X}^h \to S^h$ denote the induced morphism.

Since $\mathbb{Z}/N$ acts freely in codimension 1 on $S'$, it also does so on $\mathfrak{X}'$. So the quotient map

$$\mathfrak{X}' \to [\mathfrak{X}'/(\mathbb{Z}/N)] = \mathfrak{X}^h$$

is étale outside the 1-dimensional locus $f^{-1}(P)$. Since $\mathfrak{X}^h$ has canonical singularities, so does $\mathfrak{X}'$. In particular, $\mathfrak{X}'$ is Cohen–Macaulay. Since $f' : \mathfrak{X}' \to S'$ is dominant with equi-dimensional fibres and $S'$ is regular, it follows that $f'$ is flat.

Suppose that $\Delta \to \Delta$ is a finite base change. Then the same argument shows that $\mathfrak{X}' \times_\Delta \Delta$ also has canonical singularities, and so $\mathfrak{X}'_0$ has slc singularities.

The group $\mathbb{Z}/N$ acts effectively on the branches $(C', P')$ and $(D', P')$, and $[C'/(\mathbb{Z}/N)] = C^h$, $[D'/(\mathbb{Z}/N)] = D^h$. So $\mathfrak{X}'_0 = Y' \cup Z'$, where $Y' = f'^{-1}(C')$, $Z' = f'^{-1}(D')$.

Let $p : Y' \to C'$ and $q : Z' \to D'$ denote the induced morphisms; they are proper, and $Y'$, $Z'$ are partial normalizations of $Y \times_C C'$, $Z \times_D D'$, respectively.

Since $Y, Z$ are Cohen–Macaulay, we can identify $Y^h := Y \times_C C^h = [Y'/(\mathbb{Z}/N)]$ and $Z^h := Z \times_D D^h = [Z'/(\mathbb{Z}/N)]$.

Since $C', D'$ are principal divisors on $S'$, the divisors $Y'$ and $Z'$ are principal on $\mathfrak{X}'$.

Extend the $ADE$ notation in the usual way, to include $A_0 = \mathbb{A}^2$, $A_\infty = (xy = 0)$ and $D_\infty = (x^2 = y^2 z)$.

Suppose that $\xi$ is a closed point of $f'^{-1}(P')$. Henselize $\mathfrak{X}', Y', Z'$ at $\xi$ to get a 3-fold germ $\mathfrak{X}''$ and principal divisors $Y'', Z''$ on it. The stabilizer of $\xi$ is a subgroup $H \cong \mathbb{Z}/M$ of $\mathbb{Z}/N$ that acts freely in codimension 1 on $\mathfrak{X}''$. Put $\mathfrak{X}^{loc} = [\mathfrak{X}''/H]$, $Y^{loc} = [Y''/H]$, $Z^{loc} = [Z''/H]$; these are localizations of $\mathfrak{X}, Y$ and $Z$, respectively.

**Lemma 9.6**

*(1) There are just two possibilities:*

*(a) either $\mathfrak{X}''$ is smooth, $\mathfrak{X}''_0 = A_\infty$, $Y'' \cap Z''$ is smooth and $\mathfrak{X}^{loc}_0 = [A_\infty/\frac{1}{M}(1, -1, 1)]$ where $M \in \{1, 2, 3, 4, 6\}$,*

*(b) or $\mathfrak{X}''_0$ is a degenerate cusp of multiplicity at most 4, $Y'', Z''$ are of type A, $Y'' \cap Z''$ is a nodal curve, $M \in \{1, 2\}$ and $\mathfrak{X}^{loc}_0$ is either (A) a degenerate cusp of multiplicity $\leq 4$ or (B) the geometric quotient of such a degenerate cusp by $\mathbb{Z}/2$.*

*(2) In case (1b), $Y^{loc}$ and $Z^{loc}$ are both of type $A_{\neq\infty}$ and $Y^{loc} \cap Z^{loc}$ is a smooth curve.*

*(3) $\mathfrak{X}''$ and $\mathfrak{X}''_0$ are Gorenstein.*

PROOF: We know that $\mathfrak{X}''$ is canonical, $\mathfrak{X}''_0 = Y'' \cup Z''$ is slc and that $Y'', Z''$ are principal divisors on $\mathfrak{X}''$.

Next, the classification of slc singularities with at least two branches shows that the curve $Y'' \cap Z''$ is either smooth or a plane node. Then in the first case $\mathfrak{X}''$ is smooth, $\mathfrak{X}''_0 = A_\infty$ and $\mathfrak{X}_0^{loc} = A_\infty / \frac{1}{M}(a, -a, 1)$, where $a$ is prime to $M$, while in the second case $\mathfrak{X}''_0$ is a degenerate cusp and $Y'', Z''$ are of type $A$.

Because $p : Y' \to C'$ is a Jacobian semi-stable family of elliptic curves and $H$ acts effectively on $Y'$, the classification of automorphism groups of elliptic curves shows that $M \in \{1, 2, 3, 4, 6\}$. If the fibre $Y' \cap Z'$ over $P'$ of $Y' \to C'$ is singular, then $M \in \{1, 2\}$. In either case we can take $a = 1$.

For (2) we can suppose that $M = 2$ and $H = \langle \iota \rangle$.

Suppose that $\iota$ switches the two branches of $Y'' \cap Z''$. Then each of $Y^{loc} = [Y''/\iota]$ and $Z^{loc} = [Z''/\iota]$ is of type $A$ or $D$, and $Y^{loc} \cap Z^{loc}$ is smooth. The classification of non-isolated slc singularities then shows that $Y^{loc}$ and $Z^{loc}$ are both of type $A$.

If $\iota$ fixes the two branches then $\mathfrak{X}_0^{loc}$ is a degenerate cusp, $Y^{loc}$ and $Z^{loc}$ are both of type $A$ and $Y^{loc} \cap Z^{loc}$ is nodal.

For (3) it is enough to show that $\mathfrak{X}''_0$ is Gorenstein. This follows from (1). □

**Lemma 9.7** *(1) $\mathfrak{X}'$ is Gorenstein, $Y'$ and $Z'$ have only singularities of type $A$ and their canonical classes are linearly equivalent to zero.*

*(2) The fibre $f'^{-1}(P')$ is either an elliptic curve or a cycle of rational curves.*

PROOF: (1): By Lemma 9.6 $\mathfrak{X}''_0$ has Gorenstein singularities so $\mathfrak{X}'_0$ is everywhere locally Gorenstein. Since $Y'$ and $Z'$ have singularities of type $A$ so do $Y'$ and $Z'$.

Now $f' : \mathfrak{X}' \to S'$ is generically Jacobian, and so Jacobian over $S' - \{P'\}$. Therefore $K_{\mathfrak{X}' - f'^{-1}(P')}$ is linearly equivalent to the pull back of a line bundle on $S' - \{P'\}$, and so $K_{\mathfrak{X}' - f'^{-1}(P')} \sim 0$. Since $\mathfrak{X}'$ is Gorenstein, $K_{\mathfrak{X}'} \sim 0$; the triviality of $K_{Y'}$ and $K_{Z'}$ is then immediate.

(2): The fact that $f'^{-1}(P')$ is reduced and nodal follows from the description given in Lemma 9.6. The rest follows from the triviality of $K_{\mathfrak{X}'}$. □

**Lemma 9.8** *(1) $f' : \mathfrak{X}' \to S'$ is a semi-stable family of curves of genus 1.*

*(2) $f^h : \mathfrak{X}^h \to S^h$ is the geometric quotient of $f' : \mathfrak{X}' \to S'$ by an equivariant action of a cyclic group of order $1, 2, 3, 4$ or $6$.*

*(3) Suppose that $Y \to C$ is generically smooth and has finite local monodromy around $P$. Then $f'^{-1}(P')$ is smooth and $Z \to D$ is generically smooth with finite local monodromy around $P$.*

*(4) If $f'^{-1}(P')$ is smooth, then $Y \to C$ and $Z \to D$ are both generically smooth with finite local monodromy around $P$ and the fibres of the pair $(Y \to C,\ Z \to D)$ over $P$ are of types either $(I_0, I_0)$ or $(\overline{D}_4, \overline{D}_4)$ or $(\overline{R}, \overline{R}^*)$, where $R = II, III, IV$.*

PROOF: This is an immediate consequence of Lemma 9.7 (2). □

**Lemma 9.9** *If $Y$ is birational to a special elliptic surface, then $f'^{-1}(P')$ is smooth and the fibres of the pair $(Y,Z)$ over $P$ are of type either $(\overline{D}_4,\overline{D}_4)$ or $(I_0,I_0)$.*

PROOF: This follows at once from Lemma 9.8. □

Now suppose that, under the period map, the image of $0\in\Delta$ under the period map is the direct sum of matrices of special elliptic surfaces $\widetilde{V}_1,...,\widetilde{V}_r$.

**Lemma 9.10** *If $f_i:X_i\to C_i$ is generically smooth and $h^2(X_i,\mathcal{O}_{X_i})>0$ then $X_i$ is a special elliptic surface.*

PROOF: The Hodge structure on $H^2(X_i)$ embeds into the direct sum $\oplus H^2(\widetilde{V}_i)$. Since each $\widetilde{V}_i$ is special, it follows that the global monodromy on $H^1$ of the generic fibre of $f_i:X_i\to C_i$ is a group of order 2, so that $X_i$ is a special elliptic surface. (Recall that $f_i:X_i\to C_i$ has a section, so that if $X_i$ is not simply connected then it is ruled, which would contradict $h^2(X_i,\mathcal{O}_{X_i})>0$. So $X_i$ is simply connected.) □

**Lemma 9.11** *If $X_i$ is birational to a special elliptic surface $U$ then it is isomorphic to the $\overline{D}_4$-model of $U$.*

PROOF: This follows from Lemma 9.5 and Lemma 9.9. □

We now prove Theorem 9.1.

Say $\mathcal{S}=\{i:\ f_i$ is generically smooth and $h^2(X_i,\mathcal{O}_{X_i})>0\}$. So $\mathcal{S}$ is the set of indices $i$ such that $X_i$ is not a union of ruled surfaces. Say $|\mathcal{S}|=s$.

Recall that $K_{\mathfrak{X}/\Delta}|_{X_i}\sim f_i^*(\alpha_i)$, where $\alpha_i\in\mathbb{Q}_{>0}$. Suppose $i\in\mathcal{S}$; then the fibres of $f_i:X_i\to C_i$ are all of type either $\overline{D}_4$ or $I_0$. Suppose that $X_i$ has $b_i$ double curves of type $\overline{D}_4$ and $c_i$ of type $I_0$. Then, by the adjunction formula,

$$K_{X_i}\sim f_i^*(\alpha_i-\tfrac{1}{2}b_i-c_i).$$

So $h_i-1=\alpha_i-\frac{1}{2}b_i-c_i$.

The objects $f_j^{-1}(C_j)$ form a connected tree, so, for every pair $i,j\in\mathcal{S}$, there is a unique path that links them. So there is a total of $2\times(s-1)$ double curves on $\sqcup_{i\in\mathcal{S}}X_i$ each of which is linked to a double curve on some other surface $X_j$ with $j\in\mathcal{S}$. Therefore

$$\begin{aligned}\sum_k\alpha_k &= h-1=\sum_{i\in\mathcal{S}}h_i-1=\sum_{i\in\mathcal{S}}(h_i-1)+(s-1)\\ &= \sum_{i\in\mathcal{S}}(\alpha_i-\tfrac{1}{2}b_i-c_i)+(s-1).\end{aligned}$$

It follows that

$$\tfrac{1}{2}\sum_{i\in\mathcal{S}}b_i+\sum_{i\in\mathcal{S}}c_i+\sum_{k\notin\mathcal{S}}\alpha_k=s-1.$$

Since $\sum b_i + \sum c_i$ is the total number of double curves on $\sqcup_{i \in \mathcal{S}} X_i$, we have

$$\sum b_i + \sum c_i \geq 2(s-1).$$

Comparison of these shows that $c_i = 0$ for all $i \in \mathcal{S}$ and, since every $\alpha_k > 0$, that $\mathcal{S}$ is the full set of indices. That is, $\mathfrak{X}_0$ is the union of the special elliptic surfaces $X_1, ..., X_s$.

Therefore the direct sums $\oplus_1^s H^2_{prim}(X_i)$ and $\oplus_1^r H^2_{prim}(\widetilde{V}_i)$ of integral Hodge structures are isomorphic. Moreover, there are elliptic curves $E, F$ and hyperelliptic curves $A_1, ..., A_s, B_1, ..., B_r$ such that $X_i \cong [(E \times A_i)/\langle\iota\rangle]$ and $V_j \cong [(F \times B_j)/\langle\iota\rangle]$. The product $E \times A_i$ can then be recovered from $X_i$ as the unique double cover of $X_i$ ramified in its nodes, and the same holds for $F \times B_j$. Therefore there is an isomorphism

$$H^1(E) \otimes \big(\oplus H^1(A_i)\big) \to H^1(F) \otimes \big(\oplus H^1(B_j)\big)$$

of integral Hodge structures of weight 2.

From Theorem 10.1, to be proved in Section 10 below, and the Torelli theorem for curves, it follows that $E \cong F$, that $r = s$ and that, after re-ordering if necessary, $A_i \cong B_i$ for every $i$. So $X_i$ is birational to $V_i$, and Theorem 9.1 is proved. □

We also prove some further results about the object $\mathcal{Y}$ of Theorem 9.1 that will be useful later.

Say that $V_i$ meets $s_i$ other surfaces $V_j$ in $\mathcal{Y}_0$. That is, $V_i$ contains $s_i$ double curves $\delta_{ij}$, each of which is of type $\overline{D}_4$. Since each $V_i$ has only nodes, we can assume that, after some finite base change $\Delta \to \Delta$ if necessary, each $V_i$ is smooth outside the double curves $\delta_{ij}$.

**Lemma 9.12** *If $B_i \subset V_i$ is a section of $f_i : V_i \to C_i$ then $B_i$ meets each $\delta_{ij}$ in a node and $B_i^2 = -(h_i+1) + s_i/2$.*

PROOF: The strict transform $\widetilde{B}_i$ of $B_i$ on $\widetilde{V}_i$ is a section, so meets each $\widetilde{D}_4$ fibre in an end curve of the $\widetilde{D}_4$ configuration. So $B_i$ meets $\delta_{ij}$ is a node. Since $\widetilde{B}_i^2 = -(h_i+1)$ the lemma follows. □

**Proposition 9.13** *$\mathcal{Y} \to T$ has a section.*

PROOF: There is a section of $\mathfrak{X}_\eta$, and so an irreducible Weil divisor $D \subset \mathcal{Y}$ that restricts to a section of $\mathcal{Y} \to T$ over the generic point of $\Delta$. We can write $D \cap V_i = D_i + \psi_i$ where $D_i$ is a section of $f_i : V_i \to C_i$ and $\psi_i$ is an $f_i$-vertical curve that meets $D_i$. To show that $D$ is a section it is enough to show that each $\psi_i = 0$.

Recall that $p_g(V_i) = h_i$ and $p_g(\mathcal{Y}_t) = h$ for $t \neq 0$. Then $D^2.\mathcal{Y}_t = -(h+1)$ and $(D_i)^2_{V_i} = -(h_i+1) + s_i/2$, since $D_i$ passes through $s_i$ nodes on $V_i$. So

$$\begin{aligned} -(h+1) &= D^2.\mathcal{Y}_t = \sum D^2.V_i = \sum (D_i + \psi_i)^2_{V_i} \\ &= -\sum (h_i+1) + \sum s_i/2 + 2\sum (D_i.\psi_i)_{V_i}. \end{aligned}$$

Now $\sum s_i = 2(r-1)$ and $\sum h_i = h$, so that $\sum (D_i.\psi_i)_{V_i} = 0$. It follows that each $\psi_i = 0$. □

**Lemma 9.14** *The singularities of $\mathcal{Y}$ are of index at most* 2.

PROOF: Inspection. □

# 10 Unscrewing tensor products of weight 1 Hodge structures

Here we prove a result that yields a Torelli theorem for sesqui-special elliptic surfaces. The phrases "Principally polarized abelian variety" and "irreducible representation" are abbreviated to "ppav" and "irrep".

**Theorem 10.1** *Suppose that $A_0, B_0$ are elliptic curves, that $A_1, ..., A_r, B_1, ..., B_s$ are ppav's and that $\mathrm{Hom}(A_i, A_j) = \mathbb{Z}\delta_{ij} = \mathrm{Hom}(B_i, B_j)$ for all $i, j \geq 0$. Say $A = \sum_{i>0} A_i$ and $B = \sum_{j>0} B_j$ and give them the product principal polarizations. Assume that $\dim A = h \geq 2$ and that there is an isomorphism $H^1(A_0) \otimes H^1(A) \to H^1(B_0) \otimes H^1(B)$ of $\mathbb{Z}$-Hodge structures.*

*Then $A_0$ is isomorphic to $B_0$ and $A$ is isomorphic to $B$ as ppav's.*

PROOF: We first prove a result (Proposition 10.2) about polarizable $\mathbb{Q}$-Hodge structures, and then at the end clear denominators.

We shall use some elementary facts about Mumford–Tate groups of polarizable Hodge structures, as described in [Mo].

The category $\mathbb{Q}\mathrm{HS}^{pol}$ of polarizable $\mathbb{Q}$-Hodge structures is Tannakian and semi-simple; the corresponding group $MT$ (the universal Mumford–Tate group) is a pro-reductive group over $\mathbb{Q}$. There is a cocharacter $\nu : \mathbb{G}_{m,\mathbb{Q}} \to MT$ such that if $\rho_V : MT \to GL(V)$ is the representation corresponding to the polarizable $\mathbb{Q}$-Hodge structure $V$ of pure weight $n$ then $\rho_V(\nu(r))(v) = r^{-n}v$ for all $v \in V$. The image of $\rho_V$ is $MT(V)$.

Moreover the Deligne torus $\mathbb{S} = R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ embeds into $MT_{\mathbb{R}}$ so as to extend the embedding $\nu_{\mathbb{R}}$ of $\mathbb{G}_{m,\mathbb{R}}$. The action of $\mathbb{S}_{\mathbb{C}}$ on the subspace $V^{p,q}$ of $V_{\mathbb{C}}$ is given by $(z_1, z_2)(v) = z_1^{-p} z_2^{-q} v$ and $\mathbb{G}_{m,\mathbb{C}}$ appears as the diagonal subgroup of $\mathbb{S}_{\mathbb{C}}$.

We have polarized $\mathbb{Z}$-Hodge structures $A_i' = H^1(A_i, \mathbb{Z})$, $B_j' = H^1(B_j, \mathbb{Z})$ of weight 1 and dimensions $2, 2, 2h, 2h$. Tensor these Hodge structures with $\mathbb{Q}$; write $A_i = A_i' \otimes \mathbb{Q}$, etc., so that there are decompositions

$$A = \sum_1^r A_i \text{ and } B = \sum_1^s B_j$$

into irreducible components. Assume that $A_0 \otimes A$ is isomorphic to $B_0 \otimes B$ as $\mathbb{Q}$-Hodge structures, that $h \geq 2$ and that $MT(A_0)$ and $MT(B_0)$ are both isomorphic to $GL_2$.

**Proposition 10.2** *$A_0$ is isomorphic to $B_0$ and $A$ is isomorphic to $B$ as $\mathbb{Q}$-Hodge structures.*

PROOF: There are representations $\rho = \rho_{A_0} : MT \to GL(A_0)$ and $\sigma_i = \rho_{A_i} : MT \to GL(A_i)$. By assumption, $\rho$ is surjective. Say $G_i = \mathrm{im}(\sigma_i) = MT(A_i)$ and $H_i = \mathrm{im}(\rho \oplus \sigma_i) = MT/(\ker(\rho) \cap \ker(\sigma_i) = MT(A_0 + A_i)$.

Consider $\rho \otimes \sigma_i : H_i \to GL(A_0 \otimes A_i)$ and let $Z_i \subset H_i$ denote its kernel.

**Lemma 10.3** *$Z_i$ is central in $H_i$.*

PROOF: It is enough to show that, if $x$ and $y$ are two matrices such that $x \otimes y = 1$, then $x$ and $y$ are scalars.

Every eigenvalue of $y$ must be the inverse of every eigenvalue of $x$, so $x$ and $y$ each have a unique eigenvalue, which we can assume is 1. If $x \neq 1$ consider a vector $u$ with $(x-1)u \neq 0$ and a non-zero vector $v$ with $yv = v$. Then $(x \otimes y)(u \otimes v) \neq u \otimes v$, a contradiction. □

**Proposition 10.4** *(1) The derived subgroup $G^{der}$ of a reductive group $G$ defined over a field has a simply connected universal cover $G^{sc}$.*

*(2) Any simply connected (or adjoint) reductive group over a field is a product of simple simply connected (or adjoint) reductive groups.*

*(3) If $G_1, G_2$ are groups over a field then every irrep $W$ of $G_1 \times G_2$ is of the form $W = V_1 \otimes V_2$ where $V_i$ is an irrep of $G_i$.*

PROOF: (1): For split groups this is pointed out in 7.3.4 of [SGA3] XXIV. In general see Corollary A.4.11 of [CGP].

(2): For adjoint groups this is Prop. 5.5 of [SGA3] XXIV. As stated in 5.3 of *op. cit.* the same holds for simply connected groups.

(3): This is well known. □

The construction of $G^{sc}$ is functorial in $G$, so $MT^{sc}$ exists, and the homomorphisms

$$MT \to H_i \to GL(A_0) \times MT(A_i)$$

give homomorphisms

$$MT^{sc} \to H_i^{sc} \xrightarrow{\pi} MT(A_0)^{sc} \times MT(A_i)^{sc} = SL_2 \times MT(A_i)^{sc}$$

of simply connected reductive groups over $\mathbb{Q}$, and $H_i^{sc}$ maps surjectively to both factors.

**Lemma 10.5** *If there is an $SL_2$-factor in $MT(A_i)^{sc}$ then $MT(A_i)^{sc} = SL_2$ and* $\dim(A_i) = 2$.

PROOF: Suppose $MT(A_i)^{sc} = SL_2 \times K_i$, where $K_i$ is simply connected. Since $\mathrm{End}(A_i) = \mathbb{Q}$, it follows that the connected part $Z_i$ of the centre of $MT(A_i)$ is $Z_i \cong \mathbb{G}_m$. Therefore $MT(A_i)/MT(A_i)^{der} \cong \mathbb{G}_m$, so that $A_i$ is also an irrep of $MT(A_i)^{sc}$.

There are isogenies

$$Z_i \times SL_2 \times K_i \to Z_i \times MT(A_i)^{der} \to MT(A_i),$$

so that $A_i = X_i \otimes Y_i$ where $X_i$ is an irrep of $Z_i \times SL_2$ and $Y_i$ is an irrep of $K_i$.

Consider also the isogeny

$$\widetilde{\mathbb{S}} = \mathbb{S} \times_{MT(A_i)_{\mathbb{R}}} (Z_i \times SL_2)_{\mathbb{R}} \times K_{i,\mathbb{R}} \to \mathbb{S}.$$

Say that $\widetilde{\mathbb{S}}_{\mathbb{C}}$ acts on $X_{i,\mathbb{C}}$ with distinct characters $\chi_1, ..., \chi_a$ and on $Y_{i,\mathbb{C}}$ with distinct characters $\psi_1, ..., \psi_b$. Since $\mathbb{S}_{\mathbb{C}}$ acts on $A_{i,\mathbb{C}}$ via just 2 characters, $\alpha_1 : (z_1, z_2) \mapsto z_1^{-1}$ and $\alpha_2 : (z_1, z_2) \mapsto z_2^{-1}$, it follows that $\{\chi_i + \psi_j\} = \{\alpha_1, \alpha_2\}$, so that $a, b \leq 2$.

It is easy to see that $a = b = 2$ is impossible, so that there are two possibilities.

(1) $a = 1$. Then there is a non-trivial torus in $(\mathbb{G}_m \times SL_2)_{\mathbb{C}}$ that acts on $X_{i,\mathbb{C}}$ with only 1 weight. But $X_i$ is an irrep of $\mathbb{G}_m \times SL_2$, so it is trivial. But this is a contradiction since $X_i \otimes Y_i$ is a faithful rep of $MT(A_i)$.

(2) $b = 1$. The same argument shows that $Y_i \otimes \mathbb{C} = Y_i^{p,p}$ for some $p$, so that $Y_i(p)$ is trivial, so 1-dimensional. Then $K_i = 1$, so $MT(A_i)^{sc} = SL_2$ and $X_i = A_i$ is an irrep of $SL_2$ for which some non-trivial torus in $SL_2$ has only two weights. So $\dim(A_i) = 2$ and Lemma 10.5 is proved.

□

**Lemma 10.6** *The homomorphism $\pi : H_i^{sc} \to MT(A_0)^{der} \times MT(A_i)^{sc}$ is an isomorphism and $A_0 \otimes A_i$ is irreducible.*

PROOF: Since both projections $H_i^{sc} \to MT(A_0)^{der}$ and $H_i^{sc} \to MT(A_i)^{sc}$ is surjective, it follows from the claim that we have just established that *either* $\pi : H_i^{sc} \to MT(A_0)^{der} \times MT(A_i)^{sc}$ is an isomorphism *or* $G_i^{sc} = SL_2$ and the image of $\pi$ is the diagonal subgroup of $SL_2 \times SL_2$. In this latter case, however, we have $A_0 \cong A_i$, contradiction. □

So the equations

$$A_0 \otimes A = \sum_1^r A_0 \otimes A_i \text{ and } B_0 \otimes B = \sum_1^s B_0 \otimes B_j$$

are decompositions into irreducibles. So $r = s$ and $A_0 \otimes A_i \cong B_0 \otimes B_i$ (after re-ordering if necessary).

Since $A_0 \otimes A_i$ is irreducible, we can recover the factors $A_0$ and $A_i$ as representations of $MT$ up to tensoring by mutually inverse characters. Considering the action of $\mathbb{S}$ determines these characters. Since also $\dim(A) = \dim(B) > 2$, it follows that that $A_i \cong B_i$ for all $i \geq 0$.

This completes the proof of Proposition 10.2. □

**Lemma 10.7** *The terms $A_0 \otimes A_i$ are pairwise non-isomorphic.*

PROOF: The image of $MT(A_0 + A_1 + A_2)^{der}$ in $MT(A_0) \times MT(A_1) \times MT(A_2)$ equals $MT(A_0)^{der} \times MT(A_1)^{der} \times MT(A_2)^{der}$, since $\mathrm{Hom}(A_i, A_j) = 0$ for $i \neq j$. The result follows. □

**Lemma 10.8** *(1)* $\mathrm{End}(A_0 \otimes A_i) = \mathbb{Q}$ *for* $i > 0$ *and* $\mathrm{End}(\sum_{i>0} A_0 \otimes A_i) = \mathrm{End}(A) = \mathbb{Q}^r$.

*(2) Every homomorphism $\phi : A_0 \otimes A \to B_0 \otimes B$ of $\mathbb{Q}$-Hodge structures can be written as $\phi = \phi_1 \otimes \phi_2$ where $\phi_1 : A_0 \to B_0$ and $\phi_2 : A \to B$ are homomorphisms of $\mathbb{Q}$-Hodge structures.*

PROOF: (1) follows at one from the equality $MT(A_0 \otimes A_i)^{der} = MT(A_0)^{der} \times MT(A_i)^{der}$ and the fact that $\mathrm{Hom}(A_0 \otimes A_i, A_0 \otimes A_j) = \mathrm{Hom}(A_i, A_j) = \mathbb{Q}\delta_{ij}$. (2) is an immediate consequence. □

**Proposition 10.9** *Suppose that $\phi' : A_0' \otimes A' \to B_0' \otimes B'$ is an isomorphism of $\mathbb{Z}$-Hodge structures. Then there are isomorphisms $\phi_1' : A_0' \to B_0'$ and $\phi_2' : A' \to B'$ of $\mathbb{Z}$-Hodge structures such that $\phi' = \phi_1' \otimes \phi_2'$.*

PROOF: Write $\phi = \phi' \otimes \mathbb{Q}$ and $\phi = \phi_1 \otimes \phi_2$. The structure theorem for modules over a PID shows that there are $\mathbb{Z}$-bases $\{e_i\}$, $\{f_i\}$, $\{a_j\}$ and $\{b_j\}$ of $A_0', B_0', A'$ and $B'$ and scalars $\lambda_i, \mu_j \in \mathbb{Q}^*$ such that $\phi_1(e_i) = \lambda_i f_i$ and $\phi_2(a_j) = \mu_j b_j$ for all $i, j$. Then $\phi'(e_i \otimes a_j) = \lambda_i \mu_j f_i \otimes b_j$, so that $\lambda_i \mu_j = \pm 1$ for all $i, j$. It follows that $\lambda_i = \pm\lambda_1$ and $\mu_j = \pm\mu_1$ for all $i, j$. After replacing $f_i$ by $\pm f_i$ and $b_j$ by $\pm b_j$ if necessary, we see that there are scalars $\lambda, \mu$ such that $\phi_1(e_i) = \lambda f_i$, $\phi_2(a_j) = \mu b_j$ for all $i, j$ and $\lambda\mu = 1$.

Then $\phi' = \phi_1' \otimes \phi_2'$ where $\phi_1' = \lambda^{-1}\phi_1$ and $\phi_2' = \lambda\phi_2$, and each $\phi_i'$ is a $\mathbb{Z}$-isomorphism. □

Now we can complete the proof of the theorem. We have ppav's $A_0, A_1, ..., A_r$ and $B_0, B_1, ..., B_s$ such that $\mathrm{Hom}(A_i, A_j) = \mathbb{Z}\delta_{ij} = \mathrm{Hom}(B_i, B_j)$. There is an isomorphism $H^1(A_0) \otimes \sum H^1(A_i) \to H^1(B_0) \otimes \sum H^1(B_j)$ of $\mathbb{Z}$-Hodge structures and $\dim \sum_{i>0} A_i \geq 2$. We want to prove that $A_0 \cong B_0$ and $\sum_{i>0} A_i$ is isomorphic to $\sum_{j>0} B_j$ as ppav's.

For this, it only remains to observe that, if $A = \sum A_i$ where each $A_i$ is a ppav and $\mathrm{Hom}(A_i, A_j) = \mathbb{Z}\delta_{ij}$ for all $i, j$, then the only principal polarization on $A$ is the standard product principal polarization. □

# 11 The main results for surfaces

Our goal is to give a first order description of the period locus $PL_h$ in a neighbourhood of $\mathcal{W}_{1^h}$, or of $\mathcal{W}_{1^h}/(SL_2(\mathbb{Z}) \wr \mathrm{Sym}_h) \times SL_2(\mathbb{Z})$.

**Definition 11.1** *Denote by* $\mathcal{J}\mathcal{E}^{RDP,n.m.}$ *(resp.,* $\mathcal{J}\mathcal{E}^{sm,n.m.}$*) the stack of* generalized *RDP (resp., smooth) semi-stable Jacobian elliptic surfaces with no monodromy. Its objects over a scheme* $\Delta$ *are pairs* $(\mathfrak{X}\xrightarrow{f}S \to \Delta, S')$ *with the following properties:*

*(1)* $\mathfrak{X} \to S$ *and* $S \to \Delta$ *are projective;*

*(2)* $\mathfrak{X} \to \Delta$ *and* $S \to \Delta$ *are flat, Cohen–Macaulay and of relative dimensions* 2 *and* 1*, respectively;*

*(3)* $S'$ *is a section of* $\mathfrak{X} \to S$*;*

*(4)* $S \to \Delta$ *is a semi-stable family of curves of compact type;*

*(5) for every geometric point* $\delta$ *of* $\Delta$ *the fibre* $\mathfrak{X}_\delta$ *of* $\mathfrak{X} \to \Delta$ *is a reduced sum* $\mathfrak{X}_\delta = \sum V_i$ *of irreducible components* $V_i$ *each of which is of the form* $V_i = f^{-1}(C_i)$ *for some irreducible component* $C_i$ *of* $S_\delta$*;*

*(6) over the complement of the nodes of* $S_\delta$ *each* $V_i$ *has only RDPs (resp., is smooth) and* $V_i \to C_i$ *is relatively minimal;*

*(7)* $2S'$ *is a Cartier divisor on* $\mathfrak{X}$*;*

*(8)* $2S'$ *is* $f$*-ample (resp., is* $f$*-nef);*

*(9) the restriction* $f_i = f|_{V_i} : V_i \to C_i$ *gives the minimal resolution* $\widetilde{V}_i$ *of* $V_i$ *the structure of an elliptic surface of genus* $h_i$ *over* $C_i$ *on which the strict transform of* $S' \cap V_i$ *is the identity section;*

*(10)* $\mathfrak{X}_\delta$ *has slc singularities;*

*(11) if* $P = C_i \cap C_j$ *is a node of* $S_\delta$ *then* $f^{-1}(P)$ *is a fibre of type* $\overline{D}_4$ *on each of* $V_i$ *and* $V_j$ *and the section* $S'$ *meets* $f^{-1}(P)$ *in one of the four points on it that is an* $A_1$ *singularity on both* $V_i$ *and* $V_j$*;*

*(12) the rank one sheaf* $\omega^{[2]}_{\mathfrak{X}/\Delta} = \mathcal{O}(2K_{\mathfrak{X}/\Delta})$ *is the pullback of an ample invertible sheaf on* $S$*;*

*(13)* $\sum h_i$ *is independent of the point* $\delta \in \Delta$*. (This last is the "no monodromy" condition.)*

Observe that there is a commutative diagram

$$\begin{array}{ccc}
\mathcal{J}\mathcal{E}^{sm} & \xhookrightarrow{open} & \mathcal{J}\mathcal{E}^{sm,n.m.} \\
\downarrow & & \downarrow \\
\mathcal{J}\mathcal{E}^{RDP} & \xhookrightarrow{open} & \mathcal{J}\mathcal{E}^{RDP,n.m.}
\end{array}$$

where each vertical arrow is given by passing to the relative canonical model. It that describes the fact that $\mathcal{JE}^{RDP,n.m.}$ (resp., $\mathcal{JE}^{sm,n.m.}$) is a partial compactification of the stack $\mathcal{JE}^{RDP}$ (resp., $\mathcal{JE}^{sm}$).

**Proposition 11.2** *(1) The relative canonical model $\mathcal{Y} \to T \to \Delta$ of the object $\mathcal{Y} \to \Delta$ of Theorem 9.1 (an output of the MMP) is an object over $\Delta$ of $\mathcal{JE}^{RDP,n.m.}$.*

*(2) The relative canonical model $\overline{\mathcal{Y}} \to \mathcal{B} \to \Delta$ of the object $\overline{\mathcal{Y}} \to \Delta$ of Corollary 8.3 (an output of a plumbing construction) is an object over $\Delta$ of $\mathcal{JE}^{RDP,n.m.}$ (resp., $\mathcal{JE}^{sm,n.m.}$) if, away from the curve $\overline{Y}_a \cap \overline{Y}_b = \overline{\phi}$, each surface $\overline{Y}_i$ is an RDP (resp., smooth) elliptic surface.*

PROOF: For $\overline{\mathcal{Y}}$ of Corollary 8.3 this is clear. For $\mathcal{Y}$ this follows from Theorem 9.1, Proposition 9.13 and Lemma 9.14. □

**Proposition 11.3** *$\mathcal{JE}^{RDP,n.m.}$ is smooth.*

PROOF: It is enough to find a smooth stack $\mathcal{S}$ and a surjective smooth morphism $\mathcal{S} \to \mathcal{JE}^{RDP,n.m.}$.

Recall that a stable hyperelliptic curve of compact type over a base $\Delta$ is a stable curve $q : C \to \Delta$ of compact type with a $q$-equivariant involution $\iota$ such that $\iota$ covers the identity map $id_\Delta$, the fixed point locus of $\iota$ is finite over $\Delta$, the geometric quotient $[C/\iota] \to \Delta$ is a pre-stable curve of genus zero and, for every geometric point $\delta \in \Delta$, $\iota$ preserves each component of the geometric fibre $C_\delta$. In particular, every node of $C_\delta$ is, therefore, $\iota$-invariant. These are the objects of the stack $\mathcal{H}yp^c = \coprod_g \mathcal{H}yp^c_g$, which is smooth and is the closure of the hyperelliptic locus in the stack of stable curves of compact type.

Consider the stack $\mathcal{S}$ whose objects over $\Delta$ are triples $(Y \xrightarrow{p} C \xrightarrow{q} \Delta, \iota, \iota_Y)$ with the properties that

(1) $(q : C \to \Delta, \iota)$ is a stable hyperelliptic curve of compact type,

(2) $p : Y \to C$ is flat, projective and of relative dimension 1,

(3) there is a section $C_0$ of $p$ that lies in the relatively smooth locus of $p$ and is a $p$-ample Cartier divisor on $Y$,

(4) $p$ is smooth over the fixed points of $\iota$ (this includes all nodes on each $C_\delta$),

(5) for each geometric point $\delta$ of $\Delta$ and each irreducible component $D$ of $C_\delta$, the inverse image $p^{-1}(D)$ has only RDPs,

(6) for each $D$ as just described, $p^{-1}(D) \to D$ is a Jacobian elliptic surface on which $C_0$ is the identity,

(7) $\iota_Y$ is an involution of $Y$ that lifts $\iota$ and preserves $C_0$ and

(8) for every geometric point $\delta \in \Delta$ and for each node $x$ in $C_\delta$, $\iota_Y$ acts on the fibre $p^{-1}(x)$ as $(-1)$.

It follows that $\iota_Y$ acts as $(-1)$ on every smooth fibre of $p$ and that the locus of fixed points of $\iota_Y$ consists of the 2-torsion points on the fibres $p^{-1}(x)$ over the fixed points $x$ of $\iota$.

**Lemma 11.4** *The forgetful morphism $\mathcal{S} \to \mathcal{H}yp^c$ is smooth.*

PROOF: $\mathcal{S} \to \mathcal{H}yp^c$ is the base change under $\mathcal{H}yp^c \to \mathcal{M}^c$ of the morphism $\mathcal{J}\mathcal{E}^{RDP,sst} \to \mathcal{M}^c$ from Lemma 7.4. The result now follows from that lemma. □

Since $\mathcal{H}yp^c$ is smooth the same is true of $\mathcal{S}$.

Taking geometric quotients by $\iota$ gives a morphism $\pi : \mathcal{S} \to \mathcal{J}\mathcal{E}^{RDP,n.m.}$.

**Lemma 11.5** $\pi : \mathcal{S} \to \mathcal{J}\mathcal{E}^{RDP,n.m.}$ *is smooth.*

PROOF: Given an object $\mathcal{X} \xrightarrow{f} S \to \Delta$ in $\mathcal{J}\mathcal{E}^{RDP,n.m.}$ the objects in its pre-image are obtained by taking a double cover of $\mathcal{X}$ whose branch locus is an $f$-vertical divisor $B$ and a finite scheme that is disjoint from $B$ and supported on the $A_1$ singularities in the $\overline{D}_4$-fibres of $f$.

Suppose that $\Delta \hookrightarrow \Delta_1$ is a thickening of Artin schemes and that we are given $\mathcal{X}_1 \xrightarrow{f_1} S_1 \to \Delta_1$ that restricts to $\mathcal{X} \xrightarrow{f} S \to \Delta$. Suppose also that we have an object $Y$ in the pre-image of $\mathcal{X} \xrightarrow{f} S \to \Delta$ as above. Certainly $B$ can be lifted to a vertical divisor $B_1$, so that $Y$ can be extended to a double cover $Y_1^0$ over $\Delta_1$ outside the singular points on the $\overline{D}_4$-fibres. Moreover, the local triviality of $\omega^{[2]}_{\mathcal{X}_1/S_1}$ implies that $Y_1^0$ can be extended across these singular points. □

This completes the proof of Proposition 11.3. □

Now restrict to the closed substacks $\mathcal{J}\mathcal{E}^{*,n.m.}_{s.c.}$ of $\mathcal{J}\mathcal{E}^{*,n.m.}$ consisting of simply connected Jacobian elliptic surfaces, where $*$ denotes *sm* or *RDP*, and write

$$\mathcal{J}\mathcal{E}^{*,n.m.}_{s.c.} = \coprod_h \mathcal{J}\mathcal{E}^{*,n.m.}_h$$

where $h$ denotes the geometric genus. Then the 2-Cartesian diagram above restricts to give a 2-Cartesian diagram

$$\begin{array}{ccc} \mathcal{J}\mathcal{E}^{sm}_h & \xhookrightarrow{open} & \mathcal{J}\mathcal{E}^{sm,n.m.}_h \\ \downarrow & & \downarrow \\ \mathcal{J}\mathcal{E}^{RDP}_h & \xhookrightarrow{open} & \mathcal{J}\mathcal{E}^{RDP,n.m.}_h \end{array}$$

Recall the loci $\mathcal{W}_{h_1,\ldots,h_r}$ described in Definition 2.4. Recall also that $h = \sum h_i$.

**Proposition 11.6** *The period map* $per : \mathcal{JE}_h^{sm} \to \mathcal{V}_h/\mathfrak{G}$ *extends to a morphism* $per^+ : \mathcal{JE}_h^{sm,n.m.} \to \mathcal{V}_h/\mathfrak{G}$ *that is proper over a neighbourhood of the generic point of each* $\mathcal{W}_{h_1,...,h_r}$ *and that fits into a 2-commutative diagram*

$$\begin{array}{ccc} \mathcal{JE}_h^{sm,n.m.} & \xrightarrow{per^+} & \mathcal{V}_h/\mathfrak{G} \\ \downarrow & & \downarrow \\ \mathcal{JE}_h^{RDP,n.m.} & \longrightarrow & [\mathcal{V}_h/\mathfrak{G}]. \end{array}$$

PROOF: The properness follows from Theorem 9.1 and Proposition 11.2. □

**Lemma 11.7** *For each* $(h_1, ..., h_r)$ *and each point* $x$ *in* $\mathcal{W}_{h_1,...,h_r}$, *the stack* $\mathcal{JE}_h^{sm,n.m.}$ *is smooth at each point lying over* $x$.

PROOF: Suppose that $(X \to S, S')$ is a geometric point of $\mathcal{JE}_h^{sm,n.m.}$ that maps to $x$. Let $r : X \to Y$ be the contraction of all vertical $(-2)$-curves that lie in the smooth locus of $X$ and are disjoint from $S'$. Recall the combinatorial description of $Y$.

(1) $X = \sum_1^r X_i$ and $Y = \sum_1^r Y_i$ where, by assumption, each $Y_i$ is birational to a geometric quotient $[(E_i \times C_i)/\iota]$.

(2) The elliptic curves $E_i$ are all isomorphic since $Y$ is connected.

(3) The configurations $X = \sum X_i$, $Y = \sum Y_i$ and $S = \sum C_i$ are isomorphic trees.

(4) $S'' = r(S')$ is a section of $Y \to S$. $2S''$ is Cartier and is ample relative to $S$.

(5) Say $\sigma_{ij} = X_i \cap X_j$ if this is non-empty and $\phi_{ij} = r(\sigma_{ij})$. Then $\sigma_{ij}$ and $\phi_{ij}$ are fibres of type $\overline{D}_4$ on each of $X_i, X_j, Y_i, Y_j$, as appropriate.

(6) $Y_i$ has 4 singularities of type $A_1$ on each $\phi_{ij}$ and has $D_4$-singularities disjoint from $S''$ and from the $\phi_{ij}$.

The divisor $2S''$ defines a finite morphism $\rho : Y \to Z = \sum_1^r Z_i$ of degree 2. Say $\psi_{ij} = \rho(\phi_{ij})$. Then, because of the nature of special Jacobian elliptic surfaces,

(1) $Z \to S$ is a $\mathbb{P}^1$-bundle,

(2) the branch locus $B \subset Z$ is $B = B_0 + B_1 + \sum_{ij} \psi_{ij}$ where

(3) $B_0 = \rho(S'')$ is a section of $Z \to S$ and a Cartier divisor on $Z$,

(4) $B_1$ is disjoint from $B_0$,

(5) $B_1$ is a sum $B_1 = \sum_1^3 D_i$ of three sections $D_i$ of $Z \to S$,

(6) each $D_i$ is a Cartier divisor on $Z$,

(7) the $D_i$ are linearly equivalent and

(8) $B_1$ has ordinary triple points over the complement of the nodes in $S$ and, as does $B_0$, meets each $\psi_{ij}$ transversely.

Conversely, given $(Z, B)$, we recover $Y$ as the double cover of $Z$ branched along $B + \sum_{ij} \psi_{ij}$.

We next prove that we can deform the triple points of $B_1$ independently, while fixing $Z$. For this, let $\Sigma$ denote the set of triple points of $B_1$ and $\mathcal{I}_\Sigma$ its sheaf of ideals, and consider the short exact sequence

$$0 \to \mathcal{I}_\Sigma^3(B_1) \to \mathcal{O}_Z(B_1) \to \mathcal{O}_Z/\mathcal{I}_\Sigma^3 \to 0$$

of coherent sheaves on $Z$. It is straightforward to verify that $H^0(Z, \mathcal{I}_\Sigma^3(B_1)) = \operatorname{Sym}^3 H^0(Z, \mathcal{I}_\Sigma(D_1))$ and that then a count of dimensions shows that the map $H^0(Z, \mathcal{O}_Z(B_1)) \to H^0(Z, \mathcal{O}_Z/\mathcal{I}_\Sigma^3)$ is surjective.

It follows that the morphism $\operatorname{Def}_Y \to \prod \operatorname{Def}_{Y,P}$ of deformation spaces, where $P$ runs over the $D_4$ singularities of $Y$, is formally smooth. □

Now fix an alkane $\Gamma$ of genus $h$. Recall the period space $\mathcal{V}_h$ and its $9h+9$-dimensional closed subvariety $\mathcal{V}_\Gamma$ whose points correspond to configurations of special elliptic Kummer surfaces that are arranged in a way defined by $\Gamma$, and the arithmetic group $\mathfrak{G}$ that acts on $\mathcal{V}_h$. Let $R_\Gamma \to \mathcal{V}_\Gamma \times \mathcal{V}_\Gamma$ denote the groupoid induced from the action of $\mathfrak{G}$ on $\mathcal{V}_h$ and the closed embedding $\mathcal{V}_\Gamma \hookrightarrow \mathcal{V}_h$. Via the Torelli theorem and the surjectivity of the period map for K3 surfaces, the points of the stack $\mathcal{V}_\Gamma/R_\Gamma$ correspond to $h$-tuples $(Y_1, ..., Y_h)$ of Jacobian elliptic K3 surfaces, one surface for each vertex of $\Gamma$, where adjacent surfaces (that is, surfaces that correspond to adjacent vertices of $\Gamma$) have isomorphic $\widetilde{D}_4$-fibres.

Let $\mathcal{JE}_\Gamma$ denote the reduced closed substack of $\mathcal{JE}_h^{sm,n.m.}$ whose geometric points are configurations $x = (V_i | i \in \Gamma)$. So each $V_i$ is K3 and is smooth outside the double locus $D_i = V_i \cap (\cap_{j \neq i} V_j)$. Each component of $D_i$ is a fibre of type $\overline{D}_4$. So $\mathcal{JE}_\Gamma$ equals $(per^+)^{-1}(\mathcal{V}_\Gamma/R_\Gamma)$. Let $\mathcal{JE}_{1^h}$ denote the substack defined by the condition that each $V_i$ is a special Kummer surface. So $\mathcal{JE}_{1^h}$ equals $(per^+)^{-1}(\mathcal{W}_{1^h}/R_{1^h})$, where $R_{1^h}$ is the groupoid over $\mathcal{W}_{1^h}$ induced from the the groupoid $R_\Gamma$ (or, equivalently, from the action of $\mathfrak{G}$ on $\mathcal{V}_h$). So there is a diagram with 2-Cartesian squares

$$\begin{array}{ccc} \mathcal{JE}_{1^h} & \xrightarrow{per^+_{1^h}} & \mathcal{W}_{1^h}/R_{1^h} \\ \big\downarrow & & \big\downarrow \\ \mathcal{JE}_\Gamma & \xrightarrow{per^+_\Gamma} & \mathcal{V}_\Gamma/R_\Gamma \\ \big\downarrow & & \big\downarrow \\ \mathcal{JE}_h^{sm,n.m.} & \xrightarrow{per^+} & \mathcal{V}_h/\mathfrak{G} \end{array}$$

whose vertical arrows are closed embeddings.

**Remark**: It is clear that the groupoid $R_{1^h} \to \mathcal{W}_{1^h} \times \mathcal{W}_{1^h}$ is isomorphic to an action of the group $\mathfrak{G}_{1^h} := (SL_2(\mathbb{Z}) \wr \mathrm{Sym}_g) \times SL_2(\mathbb{Z})$.

**Lemma 11.8** *$\mathcal{JE}_\Gamma$ is smooth along $\mathcal{JE}_{1^h}$.*

PROOF: Recall first the easy fact that $\dim \mathcal{JE}_\Gamma = 9h + 9$.

Let $x = (V_i | i \in \Gamma)$ be a point in the subvariety $\mathcal{JE}_{1^h}$ of $\mathcal{JE}_\Gamma$. Let $\widetilde{V}_i \to V_i$ be the minimal resolution and $\widetilde{D}_i \subset \widetilde{V}_i$ denote the total transform of $D_i$. So $\widetilde{D}_i$ is a sum of $\widetilde{D}_4$ fibres. Let $\widetilde{\sigma}_i \subset \widetilde{V}_i$ be the given section. Then the Zariski tangent space $T_x\mathcal{JE}_\Gamma$ is given by

$$T_x\mathcal{JE}_\Gamma = \left\{ (\xi_i) \in H^1(\widetilde{V}_i, T_{\widetilde{V}_i}(-\log(\widetilde{D}_i + \widetilde{\sigma}_i))) : \xi_i|_{V_i \cap V_j} = \xi_j|_{V_i \cap V_j} \right\};$$

this is because each $H^1$ classifies first order deformations of $\widetilde{V}_i$ that preserve the combinatorial structure of the configuration $\widetilde{D}_i + \widetilde{\sigma}_i$, and then we must impose the condition that when the cross-ratio of the four marked points on each $\overline{D}_4$ fibre varies, it does so in a way that is compatible with the fact that it lies on $V_i$ and $V_j$.

Then, if $V_i$ is a vertex of $\Gamma$ whose valency is $r$,

$$\dim H^1(\widetilde{V}_i, T_{\widetilde{V}_i}(-\log(\widetilde{D}_i + \widetilde{\sigma}_i))) = 20 - (1 + 5r) + r - 1 = 18 - 4r;$$

this is because the first Chern classes of the curves in the configuration $\widetilde{D}_i + \widetilde{\sigma}_i$ on $\widetilde{V}_i$ are not linearly independent in $H^1(\widetilde{V}_i, \Omega^1_{\widetilde{V}_i})$, but rather satisfy $r - 1$ linear conditions. So, if $\Gamma$ has $\gamma_j$ vertices of valency $j$, then

$$\dim T_x\mathcal{JE}_\Gamma = \sum_1^4 \gamma_j(18 - 4j) - (h - 1) = 9h + 9,$$

since $\sum j\gamma_j = 2e$, where $e = h - 1$ is the number of edges in $\Gamma$. Therefore $\dim T_x\mathcal{JE}_\Gamma = \dim \mathcal{JE}_\Gamma$ and the smoothness is established. □

**Lemma 11.9** *(1) $per^+_\Gamma : \mathcal{JE}_\Gamma \to \mathcal{V}_\Gamma / R_\Gamma$ is an isomorphism over a neighbourhood of the generic point of $\mathcal{W}_{1^h}/\mathfrak{G}_{1^h}$.*

*(2) $\mathcal{V}_\Gamma$ is smooth along $\mathcal{W}_{1^h}$.*

PROOF: We show first that the derivative $per^+_*$ of $per^+$ is injective at all points $x$ of $\mathcal{JE}_\Gamma$.

As above, $x = (V_i | i \in \Gamma)$ and

$$T_x\mathcal{JE}_\Gamma = \left\{ (\xi_i) \in H^1(\widetilde{V}_i, T_{\widetilde{V}_i}(-\log(\widetilde{D}_i + \widetilde{\sigma}_i))) : \xi_i|_{V_i \cap V_j} = \xi_j|_{V_i \cap V_j} \right\}.$$

Choose a generator $\omega_i$ of $H^0(\widetilde{V}_i, \Omega^2_{\widetilde{V}_i}) = H^0(V_i, \omega_{V_i})$; then $per^+_*$ is the linear map defined by contraction of $(\xi_i)$ against the various vectors $(0, ..., 0, \omega_i, 0, ..., 0)$, so is clearly injective.

It follows that $per_\Gamma^+$ is étale over a neighbourhood of $\mathscr{W}_{1^h}/\mathfrak{G}_{1^h}$ and then that $\mathscr{V}_\Gamma$ is smooth along $\mathscr{W}_{1^h}$. Finally, the surjectivity of the period map for K3 surfaces completes the proof of the lemma. □

**Corollary 11.10** *(Chakiris [C1], [C2]) The generic Torelli theorem holds for simply connected Jacobian elliptic surfaces.*

PROOF: This is an immediate consequence of the fact that $per^+$ is an isomorphism over a neighbourhood of $\mathscr{W}_{1^h}$. □

This proof of generic Torelli does not rely on Chakiris' version of a stable reduction theorem [C1] but rather on Theorem 9.1, which derives from the MMP. However, beyond that, the proof relies on his ideas.

Define the vector bundle $E_\Gamma \to \mathscr{V}_\Gamma$ by the property that its fibre over the point $(Y_1, ..., Y_h)$ of $\mathscr{V}_\Gamma$ is the vector space spanned by the $h-1$ matrices $\Pi_e$, each of rank one, where $e = (i, j)$ runs over the edges of $\Gamma$ and, in the notation of Proposition 8.16,

$$\Pi_e = [\omega_{Y_i}(P_{ij}), -\omega_{Y_j}(P_{ji})] \otimes [\underline{I}_i, \underline{I}_j].$$

This is a vector bundle of rank $h-1$.

**Theorem 11.11** *(1) There is a branch $B_\Gamma$ of $PL_h$ that contains $\mathscr{V}_\Gamma$.*

*(2) To first order $B_\Gamma$ is, in a neighbourhood of $\mathscr{W}_{1^h}$, the vector bundle $E_\Gamma$.*

PROOF: Choose a smooth neighbourhood $\mathscr{V}_\Gamma^0$ of $\mathscr{W}_{1^h}$ in $\mathscr{V}_\Gamma$ and let $E_\Gamma^0$ denote the restriction of $E_\Gamma$ to $\mathscr{V}_\Gamma^0$. Then Proposition 8.16 gives a family of surfaces of genus $h$ parametrized by $\mathscr{V}_\Gamma^0 \times S_{h-1}$, where $S_{h-1}$ is an $h-1$-dimensional polydisc, and the image of $\mathscr{V}_\Gamma^0 \times S_{h-1}$ under the period map equals, to first order, precisely $E_\Gamma^0$. Since $\dim E_\Gamma = 10h + 8$, which is the number of moduli, and, by Corollary 11.10, the period map is generically injective, $E_\Gamma^0$ is, to first order, the image of some open piece of the moduli space. □

**Theorem 11.12** *(1) The branch $B_\Gamma$ of $PL_h$ is the unique branch of $PL_h$ that contains $\mathscr{V}_\Gamma$.*

*(2) To first order, the period locus $PL_h$ equals the union $\cup_\Gamma E_\Gamma$ of the vector bundles $E_\Gamma \to \mathscr{V}_\Gamma$ in a neighbourhood of $\mathscr{W}_{1^h}$.*

PROOF: Proposition 11.6 shows that, in particular, $per^+$ is proper over some neighbourhood $\mathscr{N}_h$ of $\mathscr{W}_{1^h}$ in $PL_h$. Set $\mathscr{J}\mathscr{E}_h^0 = (per^+)^{-1}(\mathscr{N}_h)$. Note that $\mathscr{V}_\Gamma \subset PL_h$ and put $\mathscr{V}_\Gamma^0 = \mathscr{N}_h \cap \mathscr{V}_\Gamma$. (We could have used this choice of $\mathscr{V}_\Gamma^0$ in the proof of Theorem 11.11.)

Let $E_\Gamma^0 \to \mathscr{V}_\Gamma^0$ be the restriction of $E_\Gamma$ to $\mathscr{V}_\Gamma^0$. The plumbing construction of Section 8 and the formula of Proposition 8.12 for the derivative of the period matrix show that $E_\Gamma^0$ is, to first order, a closed subvariety of $\mathscr{N}_h$. That is, there is, for each $\Gamma$, a closed substack $\mathscr{F}_\Gamma$ of $\mathscr{J}\mathscr{E}_h^0$ such that $per^+$ induces an isomorphism $\mathscr{F}_\Gamma \to E_\Gamma^0$ to first order.

Now on one hand $\mathscr{J}\mathscr{E}_h^{sm,n.m.}$, and so $\mathscr{J}\mathscr{E}_h^0$, is smooth and, on the other hand, for each $x = (V_1, ..., V_h) \in \mathscr{W}_{1^h}$, Theorem 9.1 shows that $(per^+)^{-1}(x)$ is a finite set which consists of exactly one point for each alkane $\Gamma$ of genus $h$. Therefore $(per^+)^{-1}(\mathscr{W}_{1^h}) \cong \coprod_\Gamma \mathscr{W}_{1^h}$ and, in a neighbourhood of $\mathscr{W}_{1^h}$, $\mathscr{J}\mathscr{E}_h^0 = \coprod_\Gamma \mathscr{F}_\Gamma$.

So $PL_h = \cup_\Gamma E_\Gamma^0$ in a neighbourhood of $\mathscr{W}_{1^h}$. □

## 12 Fay's formulae for homologically trivial plumbings of curves

For a homologically trivial Fay plumbing of curves, the arguments of Section 8 go through to recover Fay's Corollary 3.2, as follows.

Suppose that $\mathscr{C} \to \Delta_t$ is a homologically trivial Fay plumbing of $C_a$ to $C_b$ that identifies $a$ with $b$ and $(\omega^{(1)}(t), ..., \omega^{(g)}(t))$ is a normalized basis of $H^0(\mathscr{C}_t, \omega_{\mathscr{C}_t})$.

**Proposition 12.1**

$$\omega^{(j)}(t) \equiv \omega_{C_i}^{(j)} + t\eta_{C_i}^{(j)} \pmod{t^2}$$

*where*

(1) $(\omega_{C_a}^{(j)})_{1\le j\le g_a}$ *is a normalized basis of* $H^0(C_a, \omega_{C_a})$, $(\omega_{C_b}^{(j)})_{g_a+1\le j\le g_a+g_b}$ *is a normalized basis of* $H^0(C_b, \omega_{C_b})$ *and* $\omega_{C_i}^{(j)} = 0$ *otherwise, and*

(2) *there is a unique element* $\tilde\eta_i \in H^0(C_i, \omega_{C_i}(2i))$, *normalized by the requirements that, firstly,* $\int_{A_k} \tilde\eta_i = 0$ *for every A-cycle* $A_k$ *on* $C_i$ *and that, secondly,* $\tilde\eta_a = \frac{1}{4}(z_a^{-2} + \text{ h.o.t.})dz_a$ *while* $\tilde\eta_b = -\frac{1}{4}(z_b^{-2} + \text{ h.o.t.})dz_b$, *such that there is an equality*

$$[\underline{\eta}_{C_i}] = \tilde\eta_i \left[\underline{\omega}_{C_a}(a), -\underline{\omega}_{C_b}(b)\right]$$

*of row vectors of length* $g$. *The definition of these row vectors is analogous to that of the vectors which are defined immediately after Lemma 8.10.*

It follows that

$$\tau(\mathscr{C}_t) \equiv \begin{bmatrix} \tau(C_a) & 0 \\ 0 & \tau(C_b) \end{bmatrix} + t\left[\underline{\omega}_{C_a}(a), -\underline{\omega}_{C_b}(b)\right] \otimes \underline{v} \pmod{t^2}$$

where $\underline{v} = [\underline{v}_a, \underline{v}_b]$ and $\underline{v}_i$ is the vector of integrals of the form $\tilde\eta_i$ around the $B$-cycles on $C_i$. Since, by the bilinear relations for integrals of the first kind, $\tau(\mathscr{C}_t)$ is symmetric, it follows that $v = \lambda\left[\underline{\omega}_{C_a}(a), -\underline{\omega}_{C_b}(b)\right]$ for some scalar $\lambda$ and

$$\tau(\mathscr{C}_t) \equiv \begin{bmatrix} \tau(C_a) & 0 \\ 0 & \tau(C_b) \end{bmatrix} + \lambda t\left[\underline{\omega}_{C_a}(a), -\underline{\omega}_{C_b}(b)\right]^{\otimes 2} \pmod{t^2}.$$

We can calculate $\lambda$ from the bilinear relations for integrals of the second kind on $C_a$ with only poles at $a$:

$$\sum_{l=1}^{g_a}\left(\int_{A_l}\phi\int_{B_l}\psi - \int_{A_l}\psi\int_{B_l}\phi\right) = \int_\gamma f\psi$$

where $\gamma$ is a loop in $C_a$ around $a$ and $\phi = df$. Taking $\psi = \omega_{C_a}^{(j)}$ and $\phi = \tilde{\eta}_a$ gives, via the expansions above in terms of power series of $\omega_{C_a}^j$ and $\eta_{C_a}^j$,

$$\int_{B_j} \tilde{\eta}_a = \frac{\pi\sqrt{-1}}{2}\omega_{C_a}^{(j)}(a),$$

so that $\lambda = \frac{2\pi\sqrt{-1}}{4}$. This differs from the formula given in Fay's Corollary 3.2, in which $\lambda = \frac{1}{4}$, because our 1-forms are normalized by the requirement that $\int_{A_l} \omega^{(j)} = \delta_{jl}$ and not $2\pi\sqrt{-1}\delta_{jl}$.

**Remark**: As already mentioned, on p. 41 of [F1] the minus sign in front of $\underline{\omega}_{C_b}(b)$ is missing. The reason is a different choice of normalization in the plumbing construction, which replaces $z_b$ by $-z_b$. Logically, however, there is no difference.

# 13 Poincaré's asymptotic period relations

Suppose that $E_1, ..., E_g$ are disjoint curves of genus 1 and that $D$ is a copy of $\mathbb{P}^1$. Fix points $a_i \in E_i$ and a local co-ordinate $z_i$ on $E_i$ at $a_i$. On $D$ fix a point $\infty$ and a global co-ordinate $u$ on $D - \{\infty\}$. Fix distinct points $b_1, ..., b_g \in D - \{\infty\}$ given by $u - b_j = 0$.

Then successively making Fay plumbings of the $E_i$ to $D$ using these data in a way that identifies $a_j \in E_j$ to $b_j \in D - \{\infty\}$ leads to a family $\mathscr{C} \to S$ of genus $g$ curves over a $g$-dimensional polydisc $S = S_g = \Delta_{t_1,...,t_g}$ with co-ordinates $t_1, ..., t_g$. We fix a symplectic basis $(A_j, B_j)$ of each $H_1(E_j, \mathbb{Z})$ and let $v_j$ be the corresponding normalized 1-form on $E_j$. We write $v_i(a_i) = \frac{v_i}{dz_i}(a_i)$; this notation will be used many times.

The next result is due to Poincaré [P] when $g = 4$ and to Fay [F1] in general. According to Igusa ([I], p. 167) Poincaré's proof exploited the fact that the theta divisor on a Jacobian is of translation type, and so can not in any obvious way be extended to an analysis of period matrices of varieties of higher dimensions. However, Fay's approach, of which we include some details that he omitted, goes via his plumbing construction and so can be extended. It turns out that hyperelliptic curves are particularly interesting from this point of view.

As already mentioned, this result is derived in [FGSM] from their global results.

**Theorem 13.1** *("Poincaré's asymptotic period relations", [F1], p.45) For $i \neq j$ the entries $\tau_{ij}$ of the period matrix $\tau$ of $\mathscr{C}_t$ can be written as $\tau_{ij} = \overline{\tau}_{ij}u_{ij}$ where $u_{ij} \equiv 1 \pmod{(t)}$ and*

$$\overline{\tau}_{ij} = \frac{2\pi\sqrt{-1}}{16}t_i t_j v_i(a_i)v_j(a_j)/(b_i - b_j)^2.$$

PROOF: Induction on $g$.

Construct a family $\mathscr{C}' \to S_1$ of curves of genus 1 by plumbing $a_1 \in E_1$ to $b_1 \in D$. This has a normalized 1-form $\omega_1(\mathscr{C}'_{t_1})$. Then near a point of $D - \{b_1\}$ we have

$$\omega_1(\mathscr{C}'_{t_1}) = \tfrac{1}{4} t_1 v_1(a_1) \frac{du}{(u-b_1)^2} + O(t_1^2).$$

Now construct a genus 2 family $\mathscr{C} \to \Delta_{t_1,t_2}$ by plumbing $\mathscr{C}'$ to $E_2$ in a way that identifies $b_2 \in D$ with $a_2 \in E_2$. There are normalized 1-forms $\Omega_j(\mathscr{C}_{t_1,t_2})$ on $\mathscr{C}_{t_1,t_2}$, where $j = 1, 2$. Then, near a point in $E_2$, we have

$$\Omega_1(\mathscr{C}_{t_1,t_2}) = t_2 \omega_1(\mathscr{C}')(b_2)\eta_2 + O(t_2^2),$$

where $\eta_2$ is a meromorphic 1-form on $E_2$ with a double pole at $a_2$ and $\eta_2 = \frac{1}{4} v_2(a_2)(z_2^{-2} + \text{ h.o.t.}) dz_2$. Moreover, $\int_{A_2} \eta_2 = 0$. So

$$\begin{aligned} \tau_{12} = \int_{B_2} \Omega_1(\mathscr{C}) &\equiv t_2 \tfrac{1}{4} t_1 v_1(a_1) \frac{1}{(b_2-b_1)^2} \int_{B_2} \eta_2 \pmod{(t_1^2, t_2^2)} \\ &\equiv \frac{2\pi\sqrt{-1}}{16} t_1 t_2 v_1(a_1) v_2(a_2)/(b_2-b_1)^2. \end{aligned}$$

Now assume that $g \geq 3$ and that the result is true for plumbings of genus $\leq g-1$. Write

$$\frac{2\pi\sqrt{-1}}{16} v_i(a_i) v_j(a_j)/(b_i - b_j)^2 = c_{ij}.$$

Note that $c_{ij} \in \mathbb{C}^*$ since $v_i(a_i) \neq 0$.

Suppose that $k \in [3, g]$ and that $j \in [1, 2]$. Then, by induction,

$$\tau_{12} = c_{12} t_1 t_2 u_{12k} + t_k X_{12k}$$

where $u_{12k} \equiv 1 \pmod{(t)}$ and $X_{12k}$ is some function.

Moreover, $\tau_{12} \equiv 0 \pmod{t_j}$ since setting $t_j = 0$ gives a plumbing where the curve $\mathscr{C}_t$ remains singular and one of its irreducible components is the non-varying curve $E_j$ of genus 1.

Therefore $X_{12k}$ is divisible by $t_j$ and then we can write

$$\tau_{12} = t_1 t_2 (c_{12} u_{12k} + t_k Y_{12k}).$$

Since $c_{12} \neq 0$ the induction is complete. □

It follows that the off-diagonal quantities $y_{ij} = \overline{\tau}_{ij}^{-1/2}$ satisfy the $\binom{g}{4}$ Plücker relations

$$y_{ij} y_{kl} - y_{ik} y_{jl} + y_{il} y_{jk} = 0.$$

Clearing denominators gives an octic polynomial $f_{ijkl}(\tau)$ for each Plücker relation, defined by

$$\begin{aligned} f_{ijkl}(\tau) &= 2\tau_{ij}\tau_{kl}\tau_{il}\tau_{jk}\tau_{ik}\tau_{jl}(\tau_{ik}\tau_{jl} + \tau_{il}\tau_{jk} + \tau_{ij}\tau_{kl}) \\ &- (\tau_{ij}^2\tau_{il}^2\tau_{jk}^2\tau_{jl}^2 + \tau_{ik}^2\tau_{il}^2\tau_{jk}^2\tau_{jl}^2 + \tau_{ij}^2\tau_{ik}^2\tau_{jl}^2\tau_{kl}^2). \end{aligned}$$

Denote by $T$ the ideal generated by $\{\tau_{ij} | i \neq j\}$ and denote by $\mathcal{J}_g^{plumb}$ the locus in $\mathfrak{H}_g$ that consists of the period matrices of the curves $\mathcal{C}_t$ that arise from the plumbing construction. Then $f_{ijkl}(\tau) \equiv 0 \pmod{T^9}$ on $\mathcal{J}_g^{plumb}$. We say that "the equations $f_{ijkl}(\tau) = 0$ are asymptotic period relations". That is, there are holomorphic functions $F_{ijkl}$ on $\mathfrak{H}_g$ such that each $F_{ijkl}$ vanishes on $\mathcal{J}_g^{plumb}$ and $F_{ijkl}$ is congruent to $f_{ijkl}$ modulo $T^9$.

We shall see later (Corollary 13.4) that, in a neighbourhood of the diagonal locus, $\mathcal{J}_g^{plumb}$ coincides with the Jacobian locus.

Let $\mathcal{J}_g^c$ denote the closure of the Jacobian locus $\mathcal{J}_g$ inside the stack $\mathcal{A}_g$ along the closed substack $\mathcal{D}iag_g$ of $\mathcal{A}_g$ that parametrizes products of elliptic curves. Note that $\mathcal{D}iag_g$ is isomorphic to the quotient $(\mathcal{A}_1)^g / \mathrm{Sym}_g$ of $\mathcal{A}_1^g$ by the symmetric group. In transcendental terms, $\mathcal{A}_g = \mathfrak{H}_g / Sp_{2g}(\mathbb{Z})$ and $\mathcal{D}iag_g = \mathfrak{Diag}_g/(SL_2(\mathbb{Z}) \wr \mathrm{Sym}_g)$. Let $\overline{\mathcal{M}}_g^c$ denote the open substack of the stack $\overline{\mathcal{M}}_g$ of stable curves of genus $g$ that parametrizes curves of compact type. Then the Jacobian morphism $\mathcal{M}_g \to \mathcal{A}_g$ extends to a proper morphism $\overline{\mathcal{M}}_g^c \to \mathcal{A}_g$ whose image is $\mathcal{J}_g^c$. We let $\mathfrak{J}_g$ and $\mathfrak{J}_g^c$ denote the inverse images in $\mathfrak{H}_g$ of $\mathcal{J}_g$ and $\mathcal{J}_g^c$.

We show next, in Proposition 13.2, that these "asymptotic relations" are exactly the defining equations, in terms of the entries $\tau_{ij}$ of the period matrix, of the associated graded ring belonging to the closed subvariety $\mathfrak{Diag}_g$ of $\mathfrak{J}_g^c$.

We shall use the term *multi-elliptic* to refer to a stable curve of genus $g$ that contains $g$ elliptic components (and maybe some smooth rational components). Such a curve is necessarily of compact type. Then $\mathcal{D}iag_g$ is the image of the stack of multi-elliptic stable curves.

Let $\mathcal{G}r_{\mathcal{J}_g^c}$ denote the sheaf of graded $\mathcal{O}_{\mathcal{D}iag_g}$-algebras that is associated to the closed embedding $\mathcal{D}iag_g \hookrightarrow \mathcal{J}_g^c$ and $\mathcal{G}r_{\mathfrak{J}_g^c}$ its pullback to $\mathcal{O}_{\mathfrak{Diag}_g}$.

The Grassmannian $Grass(2, g)$ is embedded in $\mathrm{Proj}\, \mathbb{C}[\{y_{ij}\}] = \mathbb{P}_y^{\binom{g}{2}-1}$ via the Plücker co-ordinates $y_{ij}$. Let $X$ denote the closure of the image of $Grass(2, g)$ under the generically finite rational map

$$\mathbb{P}_y^{\binom{g}{2}-1} - \to \mathbb{P}_{\overline{\tau}}^{\binom{g}{2}-1}$$

given by $y_{ij} \mapsto y_{ij}^{-2} = \overline{\tau}_{ij}$ and let $\widehat{X} \subseteq \mathbb{A}^{\binom{g}{2}}$ be the affine cone over $X$ with affine co-ordinate ring $\mathbb{C}[\widehat{X}]$. The quadratic Plücker identities relating the $y_{ij}$ give rise to octic relations between the $\overline{\tau}_{ij}$ that are the defining equations of $\widehat{X}$ (or of $X$).

**Proposition 13.2** *$\mathcal{G}r_{\mathfrak{J}_g^c}$ is isomorphic, as a sheaf of graded $\mathcal{O}_{\mathfrak{Diag}_g}$-algebras, to the constant sheaf $\mathcal{O}_{\mathfrak{Diag}_g} \otimes_{\mathbb{C}} \mathbb{C}[\widehat{X}]$.*

PROOF: Let $\mathcal{E} \to \mathfrak{Diag}_g$ be the $g$-fold universal elliptic curve over $\mathfrak{Diag}_g$, so that $\dim \mathcal{E} = 2g$. Set $U = ((\mathbb{P}^1)^g - \Delta)/PGL_2$ where $\Delta$ is the union of all the diagonals. Let $S$ be a $g$-dimensional polydisc with co-ordinates $t_1, ..., t_g$ and put $L = \mathcal{E} \times U \times S$. Then a suitable Fay plumbing is a family of curves over $L$ which then gives a period map

$$h : L \to \mathfrak{H}_g$$

such that $\tau_{ij} = \overline{\tau}_{ij} u_{ij}$ as above. Moreover, the sublocus of $L$ defined by $t_1 = \cdots = t_g = 0$ is everywhere locally the base of a miniversal deformation of a singular stable curve $C_0$ which is of the form $C_0 = \mathbb{P}^1 + \sum_1^g E_i'$ for varying curves $E_i'$ of genus 1. Therefore, by the local Torelli theorem for smooth curves, there is a dense open subspace of $L$ along which the derivative of $h$ is injective. So the image of $h$ is open inside some branch of $\mathfrak{J}_g^c$ along $\mathfrak{Diag}_g$. In particular, the dimension of $h(L)$ is $3g - 3$.

**Lemma 13.3** *$\mathcal{J}_g^c$ is unibranched along $\mathcal{D}iag_g$.*

PROOF: Suppose that $E_1, ..., E_g$ are curves of genus 1 and that $\overline{\mathcal{M}}_{\sum E_i} \subset \overline{\mathcal{M}}_g$ is the locus of multi-elliptic curves $C$ that contain each $E_i$. That is, $C$ is a tree whose components are the curves $E_i$ together with some copies of $\mathbb{P}^1$.

To prove the lemma it is enough to show that $\overline{\mathcal{M}}_{\sum E_i}$ is connected. We do this by induction on $g$.

This is clearly true when $g = 1$. Say $\overline{\mathcal{M}}' = \overline{\mathcal{M}}_{E_g}$ and $\overline{\mathcal{M}}'' = \overline{\mathcal{M}}_{\sum_1^{g-1} E_i}$. These are connected, by the induction hypothesis. Let $\overline{\mathcal{M}}_1'$ and $\overline{\mathcal{M}}_1''$ be their inverse images in the stack $\overline{\mathcal{M}}_{g,1}$ of 1-pointed stable curves of genus $g$. There are forgetful morphisms $\overline{\mathcal{M}}_1' \to \overline{\mathcal{M}}'$ and $\overline{\mathcal{M}}_1'' \to \overline{\mathcal{M}}''$. Since their fibres are connected, both $\overline{\mathcal{M}}_1'$ and $\overline{\mathcal{M}}_1''$ are connected. There is a clutching morphism $\overline{\mathcal{M}}_1' \times_{\mathbb{C}} \overline{\mathcal{M}}_1'' \to \overline{\mathcal{M}}_{\sum_1^g E_i}$; since this is surjective the lemma is proved. □

**Corollary 13.4** *$\mathcal{J}_g^{plumb}$ coincides with $\mathcal{J}_g^c$ in a neighbourhood of the diagonal locus.*

PROOF: This follows from Lemma 13.3 and the fact that $\mathcal{J}_g^{plumb}$ is exactly $h(L)$. □

Set $R = \mathcal{O}_{\mathfrak{H}_g}$ and $S = \mathcal{G}r_T R$, the sheaf of graded $\mathcal{O}_{\mathfrak{Diag}_g}$-algebras that is associated to the ideal $T$. Since the spaces concerned are Stein we shall not emphasize the distinction between a ring and a sheaf of rings. Given a graded $\mathcal{O}_{\mathfrak{Diag}_g}$-algebra such as $S$, $\operatorname{Proj} S$ will denote the projectivization relative to $\mathfrak{Diag}_g$ in the analytic category.

Let $I$ be the ideal of $R$ that defines $\mathfrak{J}_g^c$, $J$ the ideal of $R$ generated by the functions $F_{ijkl}$ and $K$ the ideal of $R$ that defines $(\mathfrak{Diag}_g \times \widehat{X}) \cap \mathfrak{H}_g$. Note that $K$ is defined by the functions $f_{ijkl}$. Let $\overline{I}$, $\overline{J}$ $\overline{K}$ be the associated graded ideals of $S$.

Note that $\overline{K}$ is the defining ideal of $\mathfrak{Diag}_g \times X$ inside $\operatorname{Proj} S$ and so is prime.

By Corollary 13.4, $J \subset I$, so that $\overline{J} \subset \overline{I}$. Modulo $T^9$, the functions $F_{ijkl}$ and $f_{ijkl}$ are equal, and so $\overline{J} = \overline{K}$. Therefore $\overline{K} \subseteq \overline{I}$.

Suppose that $\overline{K} \neq \overline{I}$. Now $\operatorname{Proj}(S/\overline{K}) = \mathfrak{Diag}_g \times X$, which is reduced and irreducible of dimension $3g - 4$. So $\dim \operatorname{Proj}(S/\overline{I}) \leq 3g - 5$. On the other hand, $\mathrm{Proj}(S/\overline{I})$ is the exceptional divisor in the blow-up $\mathrm{Bl}_{\mathfrak{Diag}_g}\mathfrak{J}_g^c$ and so has dimension $3g-4$. Therefore $\overline{K}=\overline{I}$ and the proposition follows. □

**Corollary 13.5** *Up to graded equivalence the singularity of $\mathcal{J}_g^c$ at a given point $x$ of $\mathcal{D}iag_g$ is isomorphic to the cone $\widehat{X}$ and is independent of $x$.*

When $g=4$ (the case considered in detail by Poincaré) $\widehat{X}$ is then an octic hypersurface in $\mathbb{A}^6$ so that $\mathcal{J}_4^c$ does not have rational singularities along $\mathcal{D}iag_4$.

# 14 Hyperelliptic curves and alkanes

We continue with the notation of the previous section, *except* that "elliptic curve" will mean "curve $E$ of genus 1 provided with an involution $\iota$ with 4 fixed points". By abuse of notation, we denote $\iota$ by $[-1]_E$ and the fixed locus of $\iota$ by $E[2]$.

Suppose instead that we choose points $a_1$ on $E_1$, $b_{i-1}$ and $a_i$ on $E_i$ for $i=2,...,g-1$ and $b_{g-1}$ on $E_g$ and a local co-ordinate $z_x$ at each point $x$, and then plumb $E_i$ to $E_{i+1}$ by identifying $a_i$ to $b_i$ in a chain. This leads to a family $\mathcal{C}\to S$ of genus $g$ curves where $S=\Delta_{t_1,...,t_{g-1}}$ is a $(g-1)$-dimensional polydisc with co-ordinates $t_1,...,t_{g-1}$.

Let $\tau_i$ be the period of $E_i$, defined, as before, after the choice of a symplectic basis $(A_j,B_j)$ of $H_1(E_j,\mathbb{Z})$ and normalized 1-form $\omega_j$ on $E_j$. Construct a symplectic basis of $H_1(\mathcal{C}_t,\mathbb{Z})$ as a union of the symplectic bases $(A_j,B_j)$.

**Lemma 14.1** *If each point $b_{i-1},a_i$ lies in $E_i[2]$ and at each point $b_{i-1},a_i$ on $E_i$ a local co-ordinate is chosen that is anti-invariant under the involution $[-1_{E_i}]$ then the family $\mathcal{C}\to S$ is hyperelliptic.*

PROOF: The constraint on the points $a_1,...,a_{g-1}$ implies that the chain $\mathcal{C}_0=E_1\cup\cdots\cup E_g$ is a double cover of a chain $\mathcal{B}_0=\Gamma_1\cup\cdots\cup\Gamma_g$ of copies of $\mathbb{P}^1$ and that the map $\mathcal{C}_0\to\mathcal{B}_0$ is not étale at the nodes. So, via Proposition 4.6, Fay plumbings can be constructed simultaneously to give a double cover $\mathcal{C}\to\mathcal{B}$ over $S$ where $\mathcal{B}\to S$ is a family of genus zero curves. □

We next give a description of the closure $\mathcal{H}yp_g^c$ of the hyperelliptic locus in $\overline{\mathcal{M}}_g^c$ near $\mathcal{D}iag_g$ in combinatorial terms.

**Lemma 14.2** *Suppose that $C$ is multi-elliptic and that $E_1,...,E_g$ are its components of genus* 1.

*(1) $C$ is hyperelliptic if and only if $C$ contains no rational curves, each component $E_i$ of $C$ is elliptic and contains at most* 4 *nodes of $C$, and each node of $C$ that lies on $E_i$ also lies in $E_i[2]$.*

*(2) If $C$ is hyperelliptic then the hyperelliptic involution $\iota_C$ preserves each component $E_i$ of $C$ and restricts to $[-1_{E_i}]$, and the fixed point locus of $\iota_C$ consists of the nodes of $C$ and each $E_i[2]$.*

PROOF: This is an elementary exercise. □

So we can describe stable multi-elliptic hyperelliptic curves $C$ in terms of alkanes of the same genus, where the genus of an alkane is the number of carbon atoms in it.

**Corollary 14.3** *Each fibre in $\mathcal{Hyp}_g^c$ over a point in $\mathcal{Diag}_g$ is a finite set that corresponds naturally to the set of alkanes of genus $g$.*

PROOF: This is a translation of the preceding lemma. The carbon atoms correspond to the elliptic curves $E_i$, the bonds on the atom correspond to the points in $E_i[2]$ and the hydrogen atoms to the points in $E_i[2]$ that are not nodes. That is, the the hydrogen atoms correspond to the fixed points of the hyperelliptic involution that are not nodes. □

**Corollary 14.4** *The branches of $\mathcal{Hyp}_g^c$ through $\mathcal{Diag}_g$ in $\mathcal{A}_g$ correspond to the alkanes of genus $g$.*

We shall also use the term alkane to refer to a hyperelliptic multi-elliptic stable curve.

Suppose that $C = \sum_1^g E_i$ is an alkane. Suppose that $K$ is the set of edges, so that $\{i, j\} \in K$ if and only if $i \neq j$ and $E_i$ meets $E_j$. Choose a local co-ordinate on each $E_i$ at each node of $C$ (not necessarily anti-invariant under the involution $[-1_{E_i}]$) and let $\mathcal{C} \to S_{g-1} = \Delta_{\{t_k|k\in K\}}$ be the result of plumbing together the curves $E_i$ according to these data.

**Theorem 14.5** *Modulo $(\{t_l|l \in K\})^2$, the off-diagonal entry $\tau(\mathcal{C}_t)_k$ of the period matrix of $\mathcal{C}_t$ is a multiple of $t_k$ if $k \in K$. The other off-diagonal entries vanish.*

PROOF: We can suppose the curves $E_i$ ordered so that $E_g$ meets only $E_{g-1}$ and then argue by induction on $g$.

Suppose that $a \in E_{g-1}$ is identified with $b \in E_g$. Suppose that $\mathcal{C}' \to S' = \Delta_{t_1,...,t_{g-2}}$ is the result of plumbing $E_1, ..., E_{g-1}$ according to the data, so that $\mathcal{C} \to S$ is the plumbing of $\mathcal{C}$ to $E_g \times S'$ that identifies $\{a\} \times S'$ with $\{b\} \times S'$. Write $(t') = (t_1, ..., t_{g-2})$ and $(t) = (t', t_{g-1})$ and suppose that $(\Omega'_1, ..., \Omega'_{g-1})$ is a basis of normalized 1-forms on $\mathcal{C}'$. By Fay's formula, $\tau(\mathcal{C}_t)$ is congruent to

$$\begin{bmatrix} \tau(\mathcal{C}'_{t'}) & 0 \\ 0 & \tau_g \end{bmatrix} + \frac{\pi\sqrt{-1}}{2} t_{g-1}[\Omega'_1(\{a\} \times S'), ..., \Omega'_{g-1}(\{a\} \times S'), -\omega_g(b)]^{\otimes 2}$$

modulo $(t)^2$. The restriction $\Omega'_i|_{E_{g-1}}$ is identically zero for all $i \leq g-2$, and so $\Omega'_i(\{a\} \times S') \equiv 0 \pmod{t'}$ for $i \leq g-2$. Moreover, according to the induction hypothesis, $\tau(\mathcal{C}'_{t'})$ has the required form, and the result follows. □

Let $\mathfrak{Hyp}_g^c$ denote the closure of the hyperelliptic locus in $\mathfrak{H}_g$.

**Theorem 14.6** *(Asymptotic period relations for the hyperelliptic locus) To first order in a neighbourhood of $\mathfrak{Diag}_g/\operatorname{Sym}_g$ each branch of $\mathfrak{Hyp}_g^c$ is described as a subvariety of $\mathfrak{H}_g$ by the vanishing of the entries $\tau_k$ in the period matrix $\tau$ where $k$ runs over the set of pairs that are not edges of the corresponding alkane.*

*In particular, the branch corresponding to the linear alkane equals, to first order, the locus $\mathfrak{T}_g$ of tridiagonal matrices.*

PROOF: We prove this only for the linear alkane. The proof in general is the same but with more complicated notation.

Let $E_\tau = \mathbb{C}/(\mathbb{Z}\tau \oplus \mathbb{Z})$ be the elliptic curve corresponding to $\tau \in \mathfrak{H}_1$. Pick a co-ordinate $z$ on $\mathbb{C}$ such that the involution $[-1_{E_\tau}]$ acts via $z \mapsto -z$. Let $\mathscr{F} \to \mathfrak{Diag}_g$ be the family whose fibre over $(\tau_1, ..., \tau_g)$ is $E_{\tau_2} \times \cdots \times E_{\tau_{g-1}}$. Then the Fay plumbing just described, using the co-ordinates provided, gives a family $\mathscr{C} \to \mathscr{F} \times S_{g-1}$ of stable curves which is minimally versal at each point where $t_1 = \cdots = t_{g-1} = 0$ and so is minimally versal everywhere.

Fixing a non-zero 2-torsion point on each of $E_2, ..., E_{g-1}$ defines a section of $\mathscr{F} \to \mathfrak{Diag}_g$. Then the restriction of $\mathscr{C}$ to the corresponding subvariety $\mathfrak{Diag}_g \times S_{g-1}$ of $\mathscr{F} \times S_{g-1}$ is then, over the complement of the discriminant, a family of hyperelliptic curves.

Since $\dim(\mathfrak{Diag}_g \times S_{g-1}) = 2g-1$ this is everywhere a minimally versal family of hyperelliptic curves. Then, by the local Torelli theorem for hyperelliptic curves, the image $T$ of $\mathfrak{Diag}_g \times S_{g-1}$ in $\mathfrak{H}_g$ is of dimension $2g-1$. To first order $T$ lies in $\mathfrak{T}_g$, and the theorem is proved. □

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