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# Surface operators in the 6d $\mathcal{N} = (2, 0)$ theory

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## Abstract

The 6d  $\mathcal{N} = (2, 0)$  theory has natural surface operator observables, which are akin in many ways to Wilson loops in gauge theories. We propose a definition of a “locally BPS” surface operator and study its conformal anomalies, the analog of the conformal dimension of local operators. We study the abelian theory and the holographic dual of the large  $N$  theory refining previously used techniques. Introducing non-constant couplings to the scalar fields allows for an extra anomaly coefficient, which we find in both cases to be related to one of the geometrical anomaly coefficients, suggesting a general relation due to supersymmetry. We also comment on surfaces with conical singularities.

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# 1 Introduction

Understanding the six dimensional  $\mathcal{N} = (2, 0)$  superconformal field theory is one of the most intriguing problems in theoretical physics. In this paper we revisit the most natural observables in this theory, surface operators [1]. If we define the theory as arising from  $N$  coincident M5-branes, the simplest surface operators correspond to the endpoints of M2-branes [2].

In some ways the surface operators in six dimensions are analogous to Wilson loops in lower dimensional gauge theories. Wilson loops are the boundaries of fundamental strings, which are the dimensional reduction of M2-branes, and indeed one obtains Wilson loops in compactifications of the 6d theory with surface operators. Wilson loops are not only interesting due to their physical importance, they are also accessible to many perturbative and non-perturbative calculational tools in supersymmetric field theories: Feynman diagrams, holographic descriptions [3–5], localization [6], the defect CFT framework and associated OPE techniques [7, 8], integrability [9, 10], duality to scattering amplitudes [11] and more. See for instance a recent survey of these techniques, as applied to supersymmetric Wilson loops in ABJM theory [12].

We do not expect all these techniques to extend to surface operators in six dimensions, but it is worthwhile to examine which of them may work, and we hope that some calculations may lead to exact results applicable for all  $N$ . Here we take the first step in such an examination, defining the notion of a “locally BPS surface operator” and studying basic properties of their anomalies. This is mainly based on previous work [13–17], which we modify and refine in several ways.

As reviewed in the next section, the evaluation of generic surface operators leads to logarithmic divergences. The anomaly depends on the geometry of the surface, as well as intrinsic properties of the operator which are captured by three numbers, known as *anomaly coefficients* [18].

The “locally BPS” operator couples to the scalar fields via a unit 5-vector  $n^i$ . This can be viewed as a coupling to an R-symmetry background, and for non-constant  $n^i$  we find a new anomaly, proportional to  $(\partial n)^2$ , with its own anomaly coefficient.

We perform explicit calculations of the three geometrical and one background coefficients in both the free theory at  $N = 1$  and the holographic description valid at large  $N$ . An examination of our results reveals that the new anomaly coefficient matches (up to a sign) one of the geometric ones in both regimes. We present here a simple argument, relying on supersymmetry, why we expect this relation to hold for all  $N$ . A more rigorous proof of this relation based on the application of defect CFT techniques to surface operators will be presented in [19].

Beyond the study of  $\mathcal{N} = (2, 0)$  superconformal symmetry, surface operators in conformal field theories have drawn interest within a number of different contexts. Recent work on

entangling surfaces in 4d [20–23] and theories with boundaries [24–26] uses some techniques which apply in our case as well. In particular, the classification of local conformal invariants of surfaces is independent of the codimension and translates to the 6d case [27].

Surface operators in the  $\mathcal{N} = (2, 0)$  theory have been studied both from a field theory perspective [14–17] and using holography [28, 13]. Corresponding soliton solutions of the M5-brane equations of motion have been discussed in the literature under the moniker of self-dual strings [1].

The resemblance to Wilson loops is evident in both the field theoretic and the holographic approach. In the former, for  $N = 1$  as is studied in Section 3, we define the surface operator in analogy to the Maldacena-Wilson loops [4] as

$$V_\Sigma = \exp \int_\Sigma (iB^+ - n^i \Phi_i \text{vol}_\Sigma), \quad (1.1)$$

where  $B^+$  is the pullback of the chiral 2-form to the surface  $\Sigma$  and  $\Phi^i$  are the scalar fields.

Since for  $N > 1$  there is no realisation of the theory in terms of fundamental fields, we cannot give an analogous definition of the surface operator. However, by analogy with Wilson loops [3–5], in the large  $N$  limit, these operators in the fundamental representation have a nice holographic dual as M2-branes ending on the surface and extending into the  $AdS_7 \times S^4$  bulk, as discussed in Section 4. In the absence of a scalar coupling breaking the  $\mathfrak{so}(5)$  R-symmetry, these would be delocalised on the  $S^4$  [29, 30]. At leading order, we need only consider minimal 3-volumes [4, 13] (similar to the minimal surfaces of interest in the Wilson loop case [3–5]), and to find the anomaly, which is a local quantity, it is enough to understand the volume close to the  $AdS$  boundary. High-rank (anti-)symmetric representations are dual to configurations involving M5 branes shrinking to the surface on the boundary of  $AdS_7$  and have been considered in [31–34].

The definition in (1.1) includes BPS operators. Simple examples are the plane or sphere with constant unit  $n^i$ . Other examples are briefly discussed in Section 3 and will be explored in more detail elsewhere [35]. We call operators with generic  $\Sigma$  and unit length  $n^i$  “locally BPS”, and show that they possess some nice properties, in particular that all power law divergences cancel.

In the next section we recall the structure of surface operator anomalies and introduce the anomaly coefficients. We evaluate these anomaly coefficients for the two known realisations of the  $\mathcal{N} = (2, 0)$  theory; first as the theory of a single M5-brane ( $N = 1$ ) [36], for which the equations of motion are known [37], and second, using holography (for the large  $N$  limit) from M-theory on the  $AdS_7 \times S^4$  background [38] found in [39]. The resulting anomaly coefficients are presented in equations (3.24) and (4.18). After performing the free field and holographic calculations, we address in Section 5 surfaces with singularities. We discuss our results in Section 6 and offer a simple argument for the relation between two of the anomaly coefficients. We collect some technical tools in appendices. Our conventions can be

found in Appendix A. Details of the geometry of submanifolds are compiled in Appendix B. Appendix C contains an alternative, more geometric derivation of the field theory results in Section 3.

## 2 Surface anomalies

The most natural quantities associated to surface operators in conformal field theories are their anomaly coefficients. To understand their origin, note that, unlike line operators, the expectation values of surface operators typically suffer from ultraviolet divergences, which cannot be removed by the addition of local counterterms. The regularised expectation value satisfies

$$\log \langle V_\Sigma \rangle \sim \log \epsilon \int_\Sigma \text{vol}_\Sigma \mathcal{A}_\Sigma + \text{finite}, \quad (2.1)$$

where  $\epsilon$  is a regulator,  $\mathcal{A}_\Sigma$  is known as the *anomaly density*, and we suppressed possible power-law divergences.

$\mathcal{A}_\Sigma$  is scheme independent and indicates an anomalous Weyl symmetry, since for a constant rescaling  $g \rightarrow e^{2\omega} g$ , the expectation value varies as

$$\log \langle V_\Sigma \rangle_{e^{2\omega} g} - \log \langle V_\Sigma \rangle_g = \omega \int_\Sigma \text{vol}_\Sigma \mathcal{A}_\Sigma, \quad (2.2)$$

where the subscript  $\langle \bullet \rangle_g$  denotes the background metric.

The anomaly is constrained by the Wess-Zumino consistency condition [18, 40] to be conformally invariant. In dimensions  $d \geq 3$ , the local geometric conformal invariants for a 2d submanifold, which have been classified in [27], are

$R^\Sigma$ : The Ricci scalar of the induced metric  $h_{ab}$  on  $\Sigma$ .

$H^2 + 4 \text{tr} P$ :  $H^\mu$  is the mean curvature,  $P_{ab}$  the pullback of the Schouten tensor (B.2).

$\text{tr} W$ :  $W_{abcd}$  is the pullback of the Weyl tensor.

Under conformal transformations, the first two change by a total derivative (type A anomalies) and the last is itself conformally invariant (type B).

As we allow for variable couplings to the scalars, parametrised by a unit 5-vector  $n^i$ , we find an extra potential type B Weyl anomaly associated to it:

$$(\partial n)^2 \equiv \partial^a n^i \partial_a n_i.$$

This is (up to total derivatives) the only quantity of the correct dimension that can be constructed using only  $n$ .

The anomaly of a surface operator in any 6d  $\mathcal{N} = (2, 0)$  theory then takes the form

$$\mathcal{A}_\Sigma = \frac{1}{4\pi} [a_1 \mathcal{R}^\Sigma + a_2 (H^2 + 4 \text{tr} P) + b \text{tr} W + c (\partial n)^2]. \quad (2.3)$$

The anomaly coefficients  $a_1$ ,  $a_2$ ,  $b$  and  $c$  depend on the theory (that is on  $N$ ) and the type of surface operator (which, at least at large  $N$ , is specified by the representation of the  $A_{N-1}$  algebra [41, 42]), but not on its geometry or  $n$ . They are the focus of this paper.

Let us mention that there exists another commonly used basis where

$$\mathcal{A}_\Sigma = \frac{1}{4\pi} \left[ aR^\Sigma + b_1 \text{tr} \tilde{\Pi}^2 + b_2 \text{tr} W + c(\partial n)^2 \right], \quad (2.4)$$

where  $\tilde{\Pi}_{ab}^\mu$  is the traceless part of the second fundamental form (see (B.8)). These bases are related through the Gauss-Codazzi equation (B.7). The relation between the coefficients is then

$$\begin{aligned} a_1 &= -b_1 + a, & 2a_2 &= b_1, & b &= b_2 + b_1, \\ a &= a_1 + 2a_2, & b_1 &= 2a_2, & b_2 &= b - 2a_2. \end{aligned} \quad (2.5)$$

Some results about these anomaly coefficients are known for surface defects in generic CFTs. The bound  $b_1 < 0$  was derived in [22] by showing that  $b_1$  captures the 2-point function of the displacement operator, which is positive by unitarity. Similarly, it was shown in [43, 22] that  $b_2$  is calculated by the one-point function of the stress tensor in the presence of the surface defect (this was also conjectured in [44]). Assuming that the average null energy condition holds in the presence of defects also leads to a bound  $b_2 > 0$  [23].

For the surface operators at hand, these anomaly coefficients were also calculated previously. At large  $N$ , the first such result was a calculation of the 1/2-BPS sphere [28], with total anomaly  $-4N$ , implying  $a_1^{(N)} + 2a_2^{(N)} = -2N$ , at leading order at large  $N$ . This was soon followed by the more detailed result  $a_2^{(N)} = -N$  and  $a_1^{(N)} = b^{(N)} = 0$  [13].

More recently, it was conjectured that  $\mathcal{N} = (2, 0)$  supersymmetry imposes  $b = 0$  (or  $b_1 = -b_2$ ) for any  $N$  [45].  $a$  and  $b_2$  were calculated at any  $N > 1$  (and for any representation) by studying the holographic entanglement entropy in the presence of surface operators [46, 34, 23, 47]. This result is also supported by a recent calculation based on the superconformal index [48], which suggests that it is exact.

The anomaly coefficient  $c$  has previously not been discussed, to our knowledge.

### 3 Abelian theory with $N = 1$

In this section we study the anomaly coefficients of the surface operator in the abelian  $(2, 0)$  theory. This is the theory of a single M5-brane and the degrees of freedom form the tensor supermultiplet of the  $\mathfrak{osp}(8^*|4)$  symmetry algebra. It consists of three fields [49] (see also [50, 51])

- A real closed self-dual 3-form  $H = dB^+$ .
- A chiral spinor  $\psi_{\alpha\dot{\alpha}}$  subject to the symplectic Majorana condition  $\bar{\psi} = -c\Omega\psi$  (A.14) where  $c$  and  $\Omega$  are charge conjugation matrices, see (A.12).

- Five real scalar fields  $\Phi^i$ .

These fields transform into each other under superconformal transformations as [36]

$$\begin{aligned}\delta_\varepsilon B_{\mu\nu}^+ &= \varepsilon(x)\gamma_{\mu\nu}\psi, \\ \delta_\varepsilon\psi &= -\gamma^\mu\partial_\mu\Phi^i\check{\gamma}_i\bar{\varepsilon}(x) + \frac{1}{12}\gamma^{\mu\nu\rho}H_{\mu\nu\rho}\bar{\varepsilon}(x) + 4\Phi^i\check{\gamma}_i\varepsilon^1, \\ \delta_\varepsilon\Phi^i &= -\varepsilon(x)\check{\gamma}^i\psi.\end{aligned}\tag{3.1}$$

The parameter  $\bar{\varepsilon}(x)$  is an antichiral spinor of the form  $\bar{\varepsilon}_{\dot{\alpha}\alpha}(x) = \bar{\varepsilon}_{\dot{\alpha}\alpha}^0 + (\bar{\gamma}_\mu)_{\dot{\alpha}}{}^\beta x^\mu \varepsilon_{\beta\dot{\alpha}}^1$ , where  $\bar{\varepsilon}^0$  and  $\varepsilon^1$  are constant spinors parametrising, respectively, the supersymmetry and special supersymmetry transformations. Our spinor conventions are summarised in Appendix A.

### 3.1 Surface operators and BPS condition

We define the surface operators  $V_\Sigma$  of the abelian theory as in (1.1). To avoid complications arising for null surfaces (which could be interesting, but lie beyond the scope of this work), we restrict to space-like surfaces in flat 6d Minkowski space (with mostly positive signature).

A surface operator is BPS provided that its variation under the supersymmetry transformations (3.1) vanishes

$$\delta_\varepsilon V_\Sigma = -\int \varepsilon(x) \left[ \frac{i}{2}\gamma_{\mu\nu}\partial_a x^\mu\partial_b x^\nu\epsilon^{ab} - n^i\check{\gamma}_i \right] \psi \text{vol}_\Sigma V_\Sigma = 0.\tag{3.2}$$

Since this is an integral over the insertion of an operator  $\psi$  along the surface, this is satisfied only when the integrand vanishes at every point along the surface, leading to the projector equation

$$\varepsilon\Pi_- = 0, \quad \Pi_- = \frac{1}{2} - \frac{i}{4}\partial_a x^\mu\partial_b x^\nu\epsilon^{ab}\frac{n^i}{n^2}\gamma_{\mu\nu}\check{\gamma}_i.\tag{3.3}$$

If we impose that  $n^2 \equiv n^i n^i = 1$ , then  $\Pi_-$  is a half rank projector and otherwise it is a full rank matrix. In the case of a planar surface with constant unit  $n^i$ , this is a single condition, so the surface preserves 16 supercharges, i.e. is 1/2-BPS.<sup>1</sup>

In analogy to Wilson loops in 4d theories, it is natural to discuss “locally BPS operators” [5], where the equations (3.3) are satisfied at every point along the surface, but without a global solution. This amounts to the requirement  $n^2 = 1$ , and as shown below, leads to the cancellation of all power-like divergences in the evaluation of the surface operator.

One can also look for surfaces, other than planes, that preserve some smaller fraction of the supersymmetry by relating  $n^i(\sigma)$  to  $x^\mu(\sigma)$  and its derivatives. One simple way to realise

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<sup>1</sup>The BPS condition for a surface operator extended in the time-like direction can be obtained by Wick-rotation to

$$V_\Sigma^{\text{timelike}} = \exp \left[ i \int_\Sigma B^+ - \Phi \text{vol}_\Sigma \right].$$

this is for surfaces with the geometry  $\mathbb{R} \times S$ , for some curve  $S \subset \mathbb{R}^{1,4}$ . Upon dimensional reduction this becomes a Wilson loop in 5d maximally supersymmetric Yang-Mills (or 4d upon further dimension reduction). Then one can choose  $n^i$  to follow the construction of globally BPS Wilson loops of [52] or [53] to find globally BPS surface operators. Indeed this was realised recently in [54] (see also [55]).

There are further examples of globally BPS surface operators, which do not follow this construction. The simplest is the spherical surface, but there are several other classes of such operators, which will be explored elsewhere [35].

## 3.2 Propagators

Since the abelian theory is non-interacting, the expectation value of  $V_\Sigma$  reduces to

$$\log \langle V_\Sigma \rangle = \frac{1}{2} \int \left[ -\langle B^+(\sigma)B^+(\tau) \rangle + \langle \Phi_i(\sigma)\Phi_j(\tau) \rangle n^i(\sigma)n^j(\tau) \sqrt{h(\sigma)h(\tau)} d^2\sigma d^2\tau \right], \quad (3.4)$$

where  $h$  is the determinant of the induced metric on  $\Sigma$ . Evaluating this requires expressions for the propagators of the tensor and scalar fields.

While one would preferably derive the propagators from an action, none is readily available. Many actions for the abelian  $\mathcal{N} = (2, 0)$  theory have been proposed over the years, but they all suffer from some pathologies regarding the self-dual 2-form (see [56, 57, 36, 58, 59] for examples of available actions, and [60–62] and references therein for recent accounts of the various approaches in the abelian theory). In any case, gauge fixing and inverting the kinetic operator is not straightforward.

### 3.2.1 Tensor structure

We sidestep these obstacles by determining the propagators in other ways. The scalar propagator in flat 6d is fixed by conformal symmetry to be

$$\langle \Phi_i(x)\Phi_j(y) \rangle = \frac{C_\Phi \delta_{ij}}{|x-y|^4}. \quad (3.5)$$

The proportionality constant depends on the normalisation of the fields. It could be determined from an action, but in its absence it is fixed by supersymmetry below.

The more complicated question is the self-dual 2-form propagator. Let us start by considering an unconstrained 2-form field  $B$  with a free Maxwell type action

$$S_{\text{tot}} \propto \int d^6x B^{\mu\nu} \left( -(\delta_\mu^\rho \delta_\nu^\sigma - \delta_\nu^\rho \delta_\mu^\sigma) \partial^2 + 4(1-\alpha) \partial_\mu \partial^\rho \delta_\nu^\sigma \right) B_{\rho\sigma}, \quad (3.6)$$

where  $\alpha$  is a gauge fixing parameter. In Feynman gauge  $\alpha = 1$ , this gives the propagator

$$\langle B^{\mu\nu}(x)B_{\rho\sigma}(y) \rangle = \frac{C_B(\delta_\mu^\rho \delta_\nu^\sigma - \delta_\nu^\rho \delta_\mu^\sigma)}{|x-y|^4}. \quad (3.7)$$

Now we decompose the field into its self-dual and anti-self-dual parts  $B_{\mu\nu} = B_{\mu\nu}^+ + B_{\mu\nu}^-$  and try to deduce the propagators for each component.

Since there is no covariant 4-tensor satisfying the self-duality properties of a mixed correlator  $\langle B^+ B^- \rangle$ , we can decompose

$$\langle BB \rangle = \langle B^+ B^+ \rangle + \langle B^- B^- \rangle. \quad (3.8)$$

The two terms on the right hand side need not be identical, but the difference between them should be parity-odd.<sup>2</sup> The only such term of the right scaling dimension which we can write down is

$$\langle B^+ B^+ \rangle - \langle B^- B^- \rangle \propto \epsilon^{\mu\nu\rho\sigma\kappa\lambda} \frac{x^\kappa y^\lambda}{|x - y|^6}. \quad (3.9)$$

However, terms of this type do not contribute to (3.4), since the integration is symmetric in  $x$  and  $y$ . Therefore, for the purpose of our calculation we can take  $\langle B^+ B^+ \rangle = \langle BB \rangle/2$ . Note that in curved space we can add to the right hand side a term proportional to the Weyl tensor with all the required symmetries.

### 3.2.2 Normalisation

The normalisation of the tensor field propagator is fixed by the assumption that the surface operator defined in (1.1) corresponds to a single unit of quantised charge. First, for any closed surface  $\Sigma$ , we can rewrite the surface operator (without scalars) in terms of the field strength as

$$\exp \int_{\Sigma} iB^+ = \exp \int_V iH, \quad (3.10)$$

where  $\partial V = \Sigma$ . In order for this to be well-defined, any two such  $V$  with the same boundary must yield the same result. Equivalently, for every closed 3-manifold  $V$

$$\int_V H \in 2\pi\mathbb{Z}, \quad (3.11)$$

and similarly for  $*H$ .

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<sup>2</sup>The two-dimensional analogue is instructive. The propagator of a free boson in complex coordinate  $z$  is given by

$$\langle \phi(z)\phi(0) \rangle = \log |z|^2,$$

while for a (anti-)chiral boson one finds

$$\langle \phi_+(z)\phi_+(0) \rangle = \log z, \quad \langle \phi_-(z)\phi_-(0) \rangle = \log \bar{z}.$$

Indeed the sum reproduces the free boson propagator, but the two differ by a parity-violating imaginary part.

Now consider a flat surface operator in the  $(x^1, x^2)$  plane, which we view as a source for the unconstrained  $B$  field. The solution to the equations of motion would be given by convoluting the propagator with this source. Using the expression in (3.7), we get

$$B_{\mu\nu}(x) = \int_{\mathbb{R}^2} \frac{C_B(\delta_\mu^1 \delta_\nu^2 - \delta_\nu^1 \delta_\mu^2)}{|x - y|^4} dy^1 dy^2. \quad (3.12)$$

Again, because we don't know the self-dual propagator, the field strength we obtain is not self-dual, but the quantisation condition should still be satisfied. Imposing that the charge enclosed in a transverse sphere is quantised leads to

$$\int_{S^3} *H = 4\pi^3 C_B = 2\pi \quad \Rightarrow \quad C_B = \frac{1}{2\pi^2}. \quad (3.13)$$

The normalisation of the scalar propagator is then fixed by supersymmetry. A simple way to implement that is to compare with the classical BPS solution of the self-dual string [1] which gives<sup>3</sup>  $C_\Phi = 2C_B$ . Overall, we are left with

$$\langle \Phi_i(x) \Phi_j(y) \rangle = \frac{\delta_{ij}}{\pi^2 |x - y|^4}, \quad (3.14a)$$

$$\langle B_{\mu\nu}^+(x) B_{\rho\sigma}^+(y) \rangle = \frac{\delta_{\mu\rho} \delta_{\nu\sigma} - \delta_{\mu\sigma} \delta_{\nu\rho}}{4\pi^2 |x - y|^4}. \quad (3.14b)$$

We emphasise that this normalisation is obtained by imposing a quantisation condition on the unconstrained  $B$ -field, where we treat  $B^+$  as the self-dual subsector of a general 2-form. This follows the discussion in [56], however some caution is warranted. The interplay between the quantisation and self-duality conditions could lead to obstructions, resulting in halving the self-dual source on the right hand side of (3.12). In that case, the overall normalisation of both propagators would increase by a factor of two.

With the flat space propagators we are able to determine the anomaly coefficients  $a_1$ ,  $a_2$ , and  $c$ . The calculation of  $b$ , however, requires a the curved space propagator, where the right-hand side of (3.9) could pick up contributions whose integral does not vanish. Since we do not know how to fix these terms, we cannot determine  $b$ .

Note though that we can calculate the contribution of the scalars to the anomaly coefficient  $b$ . The propagator of a conformal scalar in a curved background can be expanded in powers of the geodesic distance [17], and the contribution to the anomaly coefficient  $b$  is read off as  $-2/3$ .

If we give up the requirement of self-duality, we can use the short-distance expansion of an unconstrained 2-form propagator on curved space, which has been computed in [14, 16], and again, the Weyl tensor of the background explicitly contributes to the curvature corrections.

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<sup>3</sup>The absence of power-law divergences in the calculation in the next section is also a hint that this is indeed the correct proportionality.

Halving that to try to account for self-duality and adding to it the contribution from the scalars, one obtains  $b = -4/3$  [17]. This is in disagreement with the conjecture  $b = 0$  [45] and therefore one may not trust it.

### 3.3 Evaluation of the anomaly

With the propagators at hand, we can compute the expectation value of the surface operator by evaluating the integrals in (3.4). Generically, these integrals are divergent and must be regularised.

In this section we take a rather naive approach of placing a hard UV cutoff on the double integral (3.4), so as to restrict  $|\sigma - \tau| > \epsilon$  (where the distance is measured with the induced metric), the same regularisation that is used in [14]. A different regularisation is employed in [17], where the surface is assumed to be contained within a 5d linear subspace of  $\mathbb{R}^6$  and the two copies of the surface are displaced by a distance  $\epsilon$  in the 6th direction. This restriction to  $\mathbb{R}^5$  must still yield the correct answer, since even for surfaces in 4d the geometric invariants in the anomaly (2.3) are independent of each other. Still, in Appendix C we redo the calculation removing this assumption by displacing the two copies of the surface along geodesics in the direction of an arbitrary normal vector field. That approach could be important for the calculation of surface operators in four dimensions, where the restriction to a 3d linear subspace does not allow to resolve all the anomaly coefficients.

To find the anomalies we only need the short-distance behaviour of the propagators, so we use normal coordinates  $\eta^a$  about a point  $\sigma$  on  $\Sigma$ . The notations and required geometry are presented in Appendix B.

Starting from the scalar contribution to (3.4), the integrand is

$$\frac{1}{2\pi^2} \frac{n^i(\sigma)n^i(\tau)}{|x(\sigma) - x(\tau)|^4} \sqrt{h(\sigma)}\sqrt{h(\tau)}. \quad (3.15)$$

Using  $n^i n^i = 1$  and (B.14), (B.12) we have

$$\begin{aligned} n^i(\sigma)n^i(\tau) &= 1 - \frac{1}{2} (\partial_a n^i \partial_b n^i) \eta^a \eta^b + \mathcal{O}(\eta^3), \\ \sqrt{h(\tau)} &= 1 - \frac{1}{6} R_{ab}^\Sigma \eta^a \eta^b + \mathcal{O}(\eta^3), \\ |x(\sigma) - x(\tau)|^2 &= \eta^a \eta_a - \frac{1}{12} \Pi_{ab} \cdot \Pi_{cd} \eta^a \eta^b \eta^c \eta^d + \mathcal{O}(\eta^5). \end{aligned} \quad (3.16)$$

The integral computing the density of the scalar contribution to  $\log \langle V_\Sigma \rangle$  is then

$$\frac{1}{2\pi^2} \int \frac{d^2 \eta}{|\eta|^4} \left[ 1 - \left( \frac{1}{6} R_{ab}^\Sigma + \frac{1}{2} \partial_a n^i \partial_b n^i \right) \eta^a \eta^b + \frac{1}{6|\eta|^2} \Pi_{ab} \cdot \Pi_{cd} \eta^a \eta^b \eta^c \eta^d + \mathcal{O}(\eta^3) \right]. \quad (3.17)$$

Using polar coordinates  $\eta^a = \eta e^a(\varphi)$ , where  $e$  is a 2d unit vector, and the identities

$$\int_0^{2\pi} d\varphi e^a e^b = \pi \delta^{ab}, \quad \int_0^{2\pi} d\varphi e^a e^b e^c e^d = \frac{\pi}{4} (\delta^{ab} \delta^{cd} + \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc}), \quad (3.18)$$

we are left with the radial integral, for which we introduce the cutoff  $\epsilon$

$$\begin{aligned} & \frac{1}{\pi} \int_{\epsilon} \frac{d\eta}{\eta^3} \left( 1 - \frac{\eta^2}{48} (4R^{\Sigma} + 12(\partial n)^2 - H^2 - 2\Pi^{ab} \cdot \Pi_{ab}) + \mathcal{O}(\eta^3) \right) \\ & = \frac{1}{2\pi\epsilon^2} + \frac{1}{16\pi} (2R^{\Sigma} - H^2 + 4(\partial n)^2) \log \epsilon + \text{finite}. \end{aligned} \quad (3.19)$$

To get the expression in the second line we also used the Gauss-Codazzi equation (B.7).

The calculation of the contribution of the 2-form field is very similar. Expanding the tensor structure, we have

$$\frac{1}{2} \langle B^+(\sigma) B^+(\tau) \rangle = \frac{1}{4\pi^2} \frac{\delta_{\mu\rho} \delta_{\nu\sigma}}{|x(\sigma) - x(\tau)|^4} dx^{\mu}(\sigma) \wedge dx^{\nu}(\sigma) \otimes dx^{\rho}(\tau) \wedge dx^{\sigma}(\tau). \quad (3.20)$$

In terms of  $\eta^a$ , the differential forms read (see (B.9))

$$\begin{aligned} dx^{\mu} \wedge dx^{\nu} \Big|_{\sigma} &= \varepsilon^{ab} v_a^{\mu} v_b^{\nu} d^2\eta, \\ dx^{\rho} \wedge dx^{\sigma} \Big|_{\tau} &= \varepsilon^{cd} \left( v_c^{[\rho} v_d^{\sigma]} + 2v_c^{[\rho} v_{de}^{\sigma]} \eta^e + \left( v_{ce}^{[\rho} v_{df}^{\sigma]} + v_c^{[\rho} v_{def}^{\sigma]} \right) \eta^e \eta^f + \mathcal{O}(\eta^3) \right) d^2\eta. \end{aligned} \quad (3.21)$$

Collecting terms and introducing a radial cutoff as above, we find the contribution

$$- \frac{1}{2\pi\epsilon^2} - \frac{1}{16\pi} (-2R^{\Sigma} + 3H^2) \log \epsilon + \text{finite}. \quad (3.22)$$

Finally, combining (3.19) and (3.22) we find that the quadratic divergences cancel and we are left with

$$\log \langle V_{\Sigma} \rangle = \frac{1}{4\pi} \log \epsilon \int_{\Sigma} \text{vol}_{\Sigma} [R^{\Sigma} - H^2 + (\partial n)^2] + \text{finite}. \quad (3.23)$$

Comparing to (2.3), we can read off the anomaly coefficients

$$a_1^{(1)} = +1, \quad a_2^{(1)} = -1, \quad c^{(1)} = +1. \quad (3.24)$$

As discussed above, since we do not know the contribution of the Weyl tensor to the  $B$ -field propagator, we cannot determine  $b^{(1)}$ . According to the conjecture of [45] however, it should vanish. This relation is the subject of work in progress [19].

Equation (3.23) differs from (2.3) by the absence of the  $\text{tr } P$  term, which also vanishes in flat space. Since  $H^2$  doesn't vanish in flat space, it determines  $a_2$  unambiguously and in curved space  $H^2$  is necessarily accompanied by  $4 \text{tr } P$ , based on the general argument for the form of the anomaly reviewed in Section 2.

Finally, we reiterate that, depending on the form of the quantisation condition, the result for the anomaly coefficients may be multiplied by 2, see the discussion following (3.14). In any case, the abelian theory should have surface operators with an integer multiple of  $iB^+ - n^i \Phi_i$  in (1.1), and for all of them it is still true that  $a_1^{(1)} = -a_2^{(1)} = c^{(1)}$ .

### 3.4 Generalising the scalar coupling

Note that the preceding calculation is applicable regardless of whether the operator is locally BPS or not, so we may relax the condition  $n^2 = 1$ . In that case the result for the anomaly coefficients is

$$a_1^{(1)} = \frac{n^2 + 1}{2}, \quad a_2^{(1)} = -\frac{n^2 + 3}{4}, \quad c^{(1)} = 1. \quad (3.25)$$

If we replace  $n^i \rightarrow in^i$ , we recover the expressions for the surface operator studied in [17]. An operator with  $n^2 = 0$  was studied in [14], but assuming a non-self-dual 2-form. The anomaly coefficients computed in [14] are half of the ones we obtain by setting  $n^2 = 0$  in (3.25), due to a difference in the overall normalisation of the propagator.

It would be interesting to study this system in the large  $n^2$  limit. This is similar to the “ladder” limit of the cusped Wilson loop in  $\mathcal{N} = 4$  SYM in 4d first suggested in [63] which is related to a special scaling limit of that theory, dubbed the “fishnet” model, which also has a 6d version [64].

## 4 Holographic description at large $N$

The holographic calculation of the Weyl anomaly for surface operators was pioneered by Graham and Witten in [13]. Here we present a rewriting of their argument, which we also generalise slightly to include operators extended on the  $S^4$ .

### 4.1 Surface operators

The  $\mathcal{N} = (2, 0)$  theory is described at large  $N$  by 11d supergravity on an asymptotically  $AdS_7 \times S^4$  geometry [38]

$$ds^2 = \frac{L^2}{y^2} (dy^2 + g^{(0)} + g^{(1)}y^2) + \frac{L^2}{4} g_{S^4}^{(0)} + \mathcal{O}(y^2), \quad L = (8\pi N)^{1/3} l_P, \quad (4.1)$$

such that  $g^{(0)}$  is the metric of the dual field theory<sup>4</sup> and  $g_{S^4}^{(0)}$  is the metric of  $S^4$ .

The background also includes  $N$  units of  $F_4$  flux

$$\frac{1}{(2\pi)^2 l_P^3} \int_{S^4} F_4 = 2\pi N. \quad (4.2)$$

The full form of the metric is determined by the supergravity equations of motion in the presence of fluxes and by requiring the geometry to close smoothly in the interior. While the latter requires nonlocal information, the near-boundary expansion is fixed to the required

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<sup>4</sup>Or in the same conformal class.

order by local information about the boundary. Following [65, 66], the first term in this expansion was found in [13] as

$$g_{\mu\nu}^{(1)} = -P_{\mu\nu}^{(0)} \equiv -P_{\mu\nu}|_{g=g^{(0)}}. \quad (4.3)$$

At this order the  $S^4$  is round, so to leading order the solution to (4.2) is simply

$$F_4 = \frac{3}{8}L^3 \text{vol}_{S^4}. \quad (4.4)$$

The holographic description of the surface operators (1.1) is by M2-branes anchored along  $\Sigma$  on the boundary of  $AdS$  [4]. Using  $\hat{\Sigma}$  for the world-volume of the M2-brane, it has a boundary at  $y = 0$  with  $\partial\hat{\Sigma} = \Sigma$ . The expectation value of the surface operators is then given by the minimum of the M2-brane action, reading (in Euclidean signature and with all fermionic terms suppressed) [67]

$$\log \langle V_\Sigma \rangle \simeq -S_{M2} = -T_{M2} \int_{\hat{\Sigma}} (\text{vol}_{\hat{\Sigma}} + iA_3), \quad T_{M2} = \frac{1}{4\pi^2 l_p^3} = \frac{2N}{\pi L^3}, \quad (4.5)$$

where  $T_{M2}$  is the tension of the brane, proportional to  $N$ .  $\text{vol}_{\hat{\Sigma}}$  is the volume form calculated from the induced metric and  $A_3$  is the pullback of the 3-form potential.

## 4.2 Local supersymmetry

Before studying the M2-brane embeddings, let us note that the M2-brane minimizing (4.5) is also locally supersymmetric. The supergravity fields appearing there sit in the supergravity multiplet, which transform as

$$\begin{aligned} \delta A_{MNP} &= -3\bar{\varepsilon}\Gamma_{[MN}\Psi_{P]}, \\ \delta\Psi_M &= D_M\varepsilon + \frac{1}{288} \left( \Gamma^{PQRS}{}_M - 8\Gamma^{QRS}\delta_M^P \right) F_{PQRS}\varepsilon, \\ \delta E_M^{\bar{M}} &= \bar{\varepsilon}\Gamma^{\bar{M}}\Psi_M, \end{aligned} \quad (4.6)$$

where  $E_M^{\bar{M}}$ ,  $\Psi_M$  and  $A_3$  are respectively the vielbein, gravitino and 3-form potential of  $F_4$  ( $\bar{M} = 1, \dots, 11$  is the frame index). Using these transformations, the variation of (4.5) is

$$\delta_\varepsilon S = T_{M2} \int_{\hat{\Sigma}} \bar{\varepsilon} \left( \Gamma^{\hat{a}} - \frac{i}{2} \varepsilon^{\hat{a}\hat{b}\hat{c}} \Gamma_{\hat{b}\hat{c}} \right) \Psi_{\hat{a}} \text{vol}_{\hat{\Sigma}} = 0. \quad (4.7)$$

We here denote the coordinates on the world-volume by  $\hat{\sigma}^{\hat{a}}$ . The projector equation is then

$$\bar{\varepsilon}\Pi_- = 0, \quad \Pi_- = \frac{1}{2} \left[ 1 - \frac{i}{6} \varepsilon^{\hat{a}\hat{b}\hat{c}} \Gamma_{\hat{a}\hat{b}\hat{c}} \right]. \quad (4.8)$$

The projector is again half-rank, so that the M2-brane locally preserves half of the supersymmetries (16 supercharges). These supercharges can be shown to agree with the field theory BPS condition (3.3) on  $\Sigma$  once we decompose  $x^M$  into coordinates on the boundary of  $AdS$ ,  $x^\mu$ , and the  $S^4$  coordinates  $n^i$ .

### 4.3 Holographic calculation

To find the saddle points of the action (4.5), we parametrise the M2-brane by  $y, \sigma^a$  where  $\sigma^a$  are coordinates for  $\Sigma$ . We then use the static gauge to describe the embedding by  $\{u^{a'}(y, \sigma), n^i(y, \sigma)\}$ , where  $u^{a'}$  are the normal directions to the surface  $\Sigma$  at  $y = 0$ . In this setup, the boundary conditions are  $u^{a'}(y = 0, \sigma) = 0$  and  $n^i(y = 0, \sigma) = n^i(\sigma)$  (where the right hand side has the  $n^i$  from (1.1)).

Because the metric (4.1) diverges at the boundary of  $AdS$ , the volume element on the M2-brane diverges as  $y^{-3}$ , which leads to divergences in the action. Finding the shape of the embedding requires knowledge of the full surface and is generally a hard problem. But since we are only interested in the logarithmically divergent part of the action, it is sufficient to solve the equations of motion for small  $y$ . We do this perturbatively following [13], mirroring the solution of the background supergravity equations above.

Using (4.3), the lowest order terms in the metric for our coordinates normal and tangent to the surface, are

$$\begin{aligned} g_{ab}(y, \sigma, u) &= h_{ab} - P_{ab}^{(0)} y^2 + \partial_{a'} g_{ab}^{(0)} \Big|_{u=0} u^{a'} + \mathcal{O}(y^4, u^2), \\ g_{aa'}(y, \sigma, u) &= \mathcal{O}(y^2, u), \\ g_{a'b'}(y, \sigma, u) &= g_{a'b'}^{(0)} \Big|_{u=0} + \mathcal{O}(y^2, u). \end{aligned} \quad (4.9)$$

Here  $h_{ab} = g_{ab}^{(0)} \Big|_{u=0}$  is the metric on  $\Sigma$ . Note that away from  $y = 0$ , this metric depends on  $u^{a'}$  (for  $y \neq 0$ , generically  $u^{a'} \neq 0$ ), as in the first line.

To write down the M2-brane action we need the induced metric  $\hat{h}_{ab} = \partial_a X^M \partial_b X^N g_{MN}$  (including also the  $S^4$  directions). We expand the embedding coordinates as

$$\begin{aligned} u^{a'}(y, \sigma) &= \mathcal{O}(y^2), \\ n^i(y, \sigma) &= n^i(\sigma) + \mathcal{O}(y^2). \end{aligned} \quad (4.10)$$

It is easy to check that higher order terms are not required. Then the  $S^4$  metric can be replaced with  $g_{S^4}^{(0)} = \delta_{ij} dn^i dn^j$  and the second fundamental form is  $\Pi_{ab}^{a'} = -\frac{1}{2} g^{a'b'} \partial_{b'} g_{ab}$ .

Dropping the explicit  $\mathcal{O}(y^*)$  as well as the subscript  $|_{u=0}$  along with the superscript  $^{(0)}$ , since all the quantities are evaluated on the surface, we find

$$\begin{aligned} \hat{h}_{yy} &\simeq \frac{L^2}{y^2} \left[ 1 + \partial_y u^{a'} \partial_y u^{b'} g_{a'b'} \right], \\ \hat{h}_{ay} &\simeq 0, \\ \hat{h}_{ab} &\simeq \frac{L^2}{y^2} \left[ h_{ab} + \left( -P_{ab} + \frac{1}{4} \partial_a n^i \partial_b n^j \delta_{ij} \right) y^2 - 2\Pi_{ab}^{a'} u^{b'} g_{a'b'} \right]. \end{aligned} \quad (4.11)$$

The determinant of the metric is then

$$\det \hat{h} \simeq \frac{L^6}{y^6} \left( 1 + \partial_y u^{a'} \partial_y u^{b'} g_{a'b'} - 2H^{a'} u^{b'} g_{a'b'} + \left( -\text{tr} P + \frac{1}{4} (\partial n)^2 \right) y^2 \right) \det h, \quad (4.12)$$

while the pullback of the 3-form

$$A_3 = \frac{1}{3!} A_{ijk} dn^i \wedge dn^j \wedge dn^k \sim \mathcal{O}(y), \quad (4.13)$$

does not contribute to the divergences. We thus find the action

$$S_{M2} \simeq \frac{L^3}{(2\pi)^2 l_P^3} \int_{\Sigma} \text{vol}_{\Sigma} \int_{y \geq \epsilon} \frac{dy}{y^3} \left[ 1 + \frac{1}{2} \left( \partial_y u^{a'} \right)^2 - H \cdot u + (-4 \text{tr} P + (\partial n)^2) \frac{y^2}{8} \right]. \quad (4.14)$$

At order  $\mathcal{O}(y^2)$ , we need only solve for  $u^{a'}(y)$ , which has the equation of motion

$$y^3 \partial_y \left( y^{-3} \partial_y u^{a'} \right) + H_{a'} \simeq 0 \quad \Rightarrow \quad u^{a'} \simeq \frac{1}{4} H^{a'} y^2. \quad (4.15)$$

The action evaluated at the classical solution is then

$$S_{M2} \simeq \frac{L^3}{(2\pi)^2 l_P^3} \int_{\Sigma} \text{vol}_{\Sigma} \int_{y \geq \epsilon} \frac{dy}{y^3} \left[ 1 - \frac{y^2}{8} (H^2 + 4 \text{tr} P) + \frac{y^2}{8} (\partial n)^2 \right] \quad (4.16)$$

where we see that the anomaly indeed takes the form (2.3). The result is

$$\log \langle V_{\Sigma} \rangle = \frac{N}{4\pi} \log \epsilon \int_{\Sigma} \text{vol}_{\Sigma} \left[ - (H^2 + 4 \text{tr} P) + (\partial n)^2 \right] \log \epsilon + \text{finite}, \quad (4.17)$$

where we discarded an irrelevant term proportional to  $\epsilon^{-2}$  (see the discussion below).

This result agrees with the original calculation of [13] and adds to it the coupling to  $(\partial n)^2$ . It is also consistent with the explicit calculation of the 1/2-BPS sphere [28], for which the anomaly is  $-4N$ . The anomaly coefficients at leading order in  $N$  are then

$$\begin{aligned} a_1^{(N)} &= \mathcal{O}(N^0), & b^{(N)} &= \mathcal{O}(N^0), \\ a_2^{(N)} &= -N + \mathcal{O}(N^0), & c^{(N)} &= +N + \mathcal{O}(N^0). \end{aligned} \quad (4.18)$$

As in the case of Wilson loops in  $\mathcal{N} = 4$  SYM in 4d, we expect this holographic description to be correct in the locally BPS case when the scalar couplings satisfy  $n^2 = 1$ . Following [29, 30], the case of  $n^2 = 0$  should be described by the same surface inside  $AdS_7$ , but completely smeared over the  $S^4$ . In this case we find the same result for the geometric anomaly coefficients as above, and, since the corresponding anomaly term vanishes,  $c^{(N)}$  does not apply.

### 4.3.1 Power-law divergence

Note that in addition to the log divergence in (4.17), (4.16) produces also a power-law divergence

$$\frac{L^3}{(2\pi)^2 l_P^3} \frac{\text{Area}(\Sigma)}{2\epsilon^2}. \quad (4.19)$$

While such divergences can be removed by the addition of a local counter-terms, in the field theory result (3.23), they cancelled without extra counter-terms (for the locally BPS operator).

A more elegant way of eliminating the power law divergences also in this holographic calculation follows the example of the locally BPS Wilson loops [5]. A careful treatment of the boundary conditions suggests that the natural action is a Legendre transform of (4.5), which differs from the action we used by a total derivative. This modification does not change the equations of motion, but gives a contribution on the boundary, where it precisely cancels the divergence above.

By looking at the M5-brane metric before the decoupling limit, we can identify the coordinate to use in the transform as  $r^i = L^3 n^i / 2y^2$ . Defining its conjugate momentum by differentiating with respect to the boundary value of the coordinate (where  $y = \epsilon$ )

$$p_i(\sigma) = \frac{\delta S[x^\mu, r^i]}{\delta r^i} = -\frac{\epsilon^3 n^i \delta S[x^\mu, n^i, \epsilon]}{L^3 \delta \epsilon} = \frac{\epsilon^3 n^i}{L^3} \frac{L^3}{(2\pi)^2 l_p^3} \left( \frac{1}{\epsilon^3} + \mathcal{O}\left(\frac{1}{\epsilon}\right) \right). \quad (4.20)$$

In the last equality we used the value of the classical action (4.16), undoing the integration, so the classical Lagrangian density.

The Legendre transformed action is then

$$\tilde{S}[x^\mu, p^i] = S[x^\mu, r^i] - \int_\Sigma p_i r^i \text{vol}_\Sigma = S[x^\mu, n^i, \epsilon] - \frac{L^3}{2(2\pi)^2 l_p^3 \epsilon^2} \int_\Sigma \text{vol}_\Sigma. \quad (4.21)$$

The last term exactly cancels the power law divergence in (4.19).

## 5 Surfaces with singularities

An interesting class of surface operators that has received some attention recently is surfaces with conical singularities. For these surfaces, it was found that the regularised expectation value typically diverges as [68, 21, 69, 70]

$$\log \langle V_{\Sigma_c} \rangle \sim A \log^2 \epsilon + \mathcal{O}(\log \epsilon). \quad (5.1)$$

Let us consider a conical defect (on flat space) of the form

$$x^\mu(r, s) = r \gamma^\mu(s), \quad \gamma^2 = 1, \quad n^i(r, s) = \nu^i(s). \quad (5.2)$$

We allow here also a ‘‘conical singularity’’ in the scalar couplings, which has  $s$  dependence even as  $r \rightarrow 0$ . It is possible to also allow  $x^\mu$  and  $n^i$  to have higher order terms in  $r$ , but since those lead to subleading divergences, they are unimportant.

We can try to use the usual formula for the anomaly (2.3) by plugging in the geometric invariants

$$R^\Sigma = \Omega \delta(r), \quad H^2 = \frac{\kappa^2 - 1}{r^2}, \quad (\partial n)^2 = \frac{(\partial_s \nu)^2}{r^2}, \quad (5.3)$$

where  $\Omega$  is the deficit angle,  $\kappa = \ddot{\gamma}^2/|\dot{\gamma}|^2$  is the curvature of  $\gamma$ . Plugging into (2.3), the Ricci scalar gives a finite contribution, but  $H^2$  and  $(\partial n)^2$  diverge as  $r \rightarrow 0$ . Introducing a cutoff  $\hat{\epsilon}$  on the  $r$  integration, this gives

$$\frac{1}{4\pi} \log \epsilon \log \hat{\epsilon} \int_{\gamma} a_2 (1 - \kappa^2(s)) - c(\partial_s \nu)^2 ds + \mathcal{O}(\log \epsilon). \quad (5.4)$$

This expression is a bit naive, as we should treat all divergences on the same footing and should identify  $\hat{\epsilon} = \epsilon$ . But then we should not use (2.3), rather go back one step and regularise the divergences that gave rise to the original  $\log \epsilon$  divergence while also applying it to the  $r$  integration. As we show below, this leads to the expression in (5.4) with  $\log \epsilon \log \hat{\epsilon} \rightarrow \frac{1}{2} \log^2 \epsilon$ . In both the free field case and the holographic realisation this factor of  $1/2$  is a simple consequence of the usual coefficient of the quadratic term in the Taylor expansion, or in other words of an integral of the form  $\int \log r d \log r$ .

This factor of  $1/2$  was noticed already in the calculations of [68, 21] and justified in [70] by a careful treatment of the holographic calculation, which is repeated below. We think that the comparison of this to the free-field calculation and the universal nature of our result further elucidates this mismatch from the naive expectation. Our calculation is also more generic, for allowing arbitrary conical singularities and incorporating the scalar singularities too.

We should note, as already noticed in [21], that surfaces with “creases”, i.e. co-dimension one singularities, do not lead to additional  $\log^2 \epsilon$  divergences and the expression (2.3) can be immediately applied to them.

## 5.1 Field theory

Here we do not rely on (3.23), but go further back to where the  $\log \epsilon$  arises from an integral of the form (3.19)

$$\int_{\epsilon}^{\rho} \frac{d\eta}{\eta} = -\log \epsilon + \text{finite}, \quad (5.5)$$

where  $\eta$  is a radial coordinate around the point  $x$ , and  $\rho$  is an IR cutoff related to the overall size of the surface, or at least a large smooth patch where we defined our local coordinate. Near the cone the smooth patch is bounded by the distance from  $x$  to the apex, which we denote by  $r$ . The integral instead gives

$$\int_{\epsilon}^r \frac{d\eta}{\eta} = -\log \frac{\epsilon}{r}. \quad (5.6)$$

With this careful treatment of the log, we can go back to (2.3), plug in the expressions from (5.3) and integrate over  $r$  and with the same UV cutoff to find

$$\begin{aligned} \log \langle V_\Sigma \rangle &= -\frac{1}{4\pi} \int_\gamma ds \int_\epsilon^{\frac{\epsilon}{r}} \frac{dr}{r} [a_2 (1 - \kappa^2) - c(\partial_s n)^2] \log \frac{\epsilon}{r} + \text{finite} \\ &= \frac{1}{8\pi} \log^2 \epsilon \int_\gamma [a_2(1 - \kappa^2(s)) - c(\partial_s \nu)^2] ds + \mathcal{O}(\log \epsilon). \end{aligned} \quad (5.7)$$

## 5.2 Holography

The derivation in holography is similar. We first note that conformal symmetry fixes the form of the solution as

$$y(r, s) = ru(s) \quad (5.8)$$

To get to (4.17), we integrate over  $y$ , but the conformal ansatz suggests to impose the range  $\epsilon \leq y \leq ru_{\max}$ . Plugging the curvatures from (5.3) into equation (4.17) we arrive at

$$\log \langle V_\Sigma \rangle = -\frac{1}{4\pi} \int_\gamma ds \int_\epsilon^{\frac{\epsilon}{ru_{\max}(s)}} \frac{dr}{r} [a_2(1 - \kappa^2) - c(\partial_s n)^2] \log \frac{\epsilon}{ru_{\max}(s)} + \text{finite}. \quad (5.9)$$

which again gives the  $\log^2 \epsilon$  divergence with the same  $1/2$  prefactor, as in the field theory (5.7).

## 5.3 Example: circular cone

As a simple example of a singular surface we compute explicitly the anomaly of a cone. Denoting the deficit angle by  $\phi$  (see figure 1) and including an internal angle  $\theta$  for the scalar coupling  $n^i$ , we parametrise the cone as follows

$$\gamma^\mu(s) = \begin{pmatrix} \cos \phi \sin s \\ \cos \phi \cos s \\ \sin \phi \end{pmatrix}, \quad n^i(s) = \begin{pmatrix} \sin \theta \sin s \\ \sin \theta \cos s \\ \cos \theta \end{pmatrix}, \quad 0 \leq r, \quad 0 \leq s < 2\pi. \quad (5.10)$$

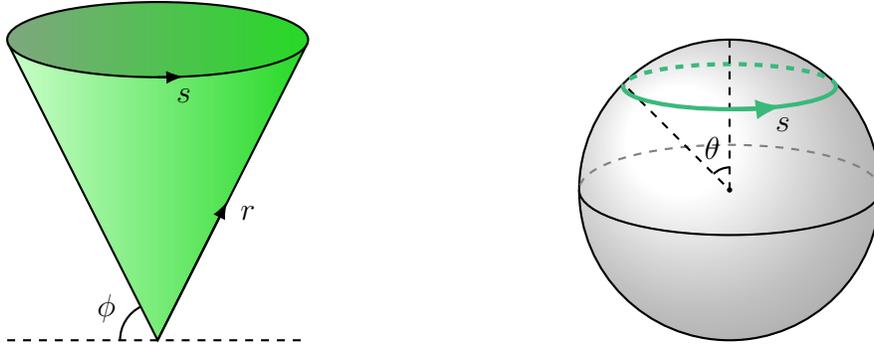
The conformal invariants are explicitly

$$\kappa^2 = \frac{1}{\cos^2 \phi}, \quad (\partial_s n)^2 = \frac{\sin^2 \theta}{\cos^2 \phi}. \quad (5.11)$$

The divergence is then

$$\log \langle V_\Sigma \rangle = -\frac{a_2 \sin^2 \phi + c \sin^2 \theta}{4 \cos \phi} \log^2 \epsilon + \mathcal{O}(\log \epsilon). \quad (5.12)$$

Notice that as long as the anomaly coefficients satisfy the relation  $a_2 = -c$ , which we have shown to hold in the abelian and large  $N$  case, the anomaly vanishes for configurations  $\theta = \pm\phi$ , which correspond generically to  $1/8$ -BPS configurations.



**Figure 1:** On the left, the surface wraps a (circular) cone with a deficit angle  $\phi$ . On the right, the scalar coupling follows a circle at angle  $\theta$  on  $S^2$ . For a fixed  $r$ , we have a curve that simultaneously traces the circles  $\gamma(s)$  and  $n^i(s)$ .

## 6 Conclusion

In this paper we calculated the anomaly coefficients of locally supersymmetric surface operators in the  $\mathcal{N} = (2, 0)$  theory in 6d, refining and generalising the calculations of [13, 17]. We first introduced a new anomaly coefficient  $c$  (2.3) arising from non-constant dependence on the internal R-symmetry directions. These are explicit scalar couplings in the abelian theory and motion on  $S^4$  in the holographic realisation.

We then presented an explicit calculation for the abelian theory and for the large  $N$  limit (using holography). The results are in equations (3.24) and (4.18). Although we are not able to compute the anomaly coefficient  $b$  at  $N = 1$  because we do not know the general curved space propagator for the self-dual 2-form, we found the others in both cases.

Making all  $N$  conjectures based on the asymptotics is a fool's errand, which we carefully tread. This is especially true given that the abelian theory is not the same as the  $A_{N-1}$  theory at  $N = 1$ , since the latter is the empty theory. Nevertheless, in both cases we see that  $a_2 = -c$ , and we expect this to hold generally. The argument is based on the BPS Wilson loops of [52], where  $n^i$  is parallel to  $\dot{x}^\mu$  and which have trivial expectation values. If we uplift them to the 6d theory we expect to find surface operators with no anomaly (and no finite part as well). These operators satisfy  $H^2 = (\partial n)^2$  and indeed they do not contribute to the anomaly<sup>5</sup> for  $a_2 = -c$ . A proof of this relation as well as properties of  $b$ , based on defect CFT techniques, will be presented elsewhere [19].

Two more results are the formalism for regularising surface operators presented in Appendix C and the expression for the divergences due to conical singularities over arbitrary curves in Section 5.

All our calculations are for a surface operator in the fundamental representation. It is expected that 1/2-BPS surface operators are classified by representations of the  $A_{N-1}$  algebra

<sup>5</sup>In the uplift we find only surfaces with trivial topology, so the anomaly vanishes regardless of  $a_1$ .

of the theory. At large  $N$  this is proven, since the asymptotically  $AdS_7 \times S^4$  solutions of 11d supergravity preserving the symmetry algebra of 1/2-BPS surface operators can be classified in terms of Young diagrams [41, 42].

A calculation of anomalies of surface operators in arbitrary representation, based on the bubbling geometries and holographic entanglement entropy was undertaken in [47]. If we assume  $b = 0$ , then for a the fundamental representation, their result reads

$$a_1^{(N)} = \frac{1}{2} - \frac{1}{2N}, \quad a_2^{(N)} = -N + \frac{1}{2} + \frac{1}{2N}. \quad (6.1)$$

This is supported by an independent calculation using the superconformal index [48]. In the large  $N$  limit, our result [13] indeed agrees with theirs. These calculations do not determine the remaining anomaly coefficients in generic representations. But if we believe the  $b = 0$  conjecture of [45] and our argumentation above for  $c = -a_2$ , this fixes the remaining ones.

It would be interesting to reproduce these finite  $N$  corrections using other methods as well as do direct holographic calculations for higher-dimensional surface operators.

The anomalies studied here are the most basic properties of surface operators, but finding them is only a first step in understanding these observables and the mysterious theory they belong to. Planar/spherical surface operators preserve part of the conformal group (and with the scalar coupling also half the supersymmetries) and their deformations behave like operators in a defect CFT. A natural next step is to study the defect CFT data: spectrum and structure constants.

Another natural question is the classification of globally BPS surface operators (and local operators within the surface operators) beyond the case of the plane/sphere.

We hope to report progress on these questions in the near future.

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# A Conventions and notation

In this paper we work in Minkowski space with mostly positive signature. We make use of the following indices:

Index	Usage
$M = 1, \dots, 11$	11d spacetime vector $X^M$
$A = 1, \dots, 32$	11d spinors
$\mu = 1, \dots, 6$	6d spacetime vectors $x^\mu$
$\alpha (\hat{\alpha}) = 1, \dots, 4$	6d chiral (antichiral) spinors
$i = 1, \dots, 5$	R-symmetry vectors
$\check{\alpha} = 1, \dots, 4$	R-symmetry spinors
$a' = 1, \dots, 4$	spacetime vectors orthogonal to the surface
$a = 1, 2$	worldsheet coordinates $\sigma^a$
$\hat{a} = 1, 2, 3$	worldvolume coordinates $\hat{\sigma}^{\hat{a}}$

Our usage of spinors is restricted to the supersymmetry transformations (3.1) and (4.6) but we include our conventions for completeness. In general we follow the NW-SE convention for indices summation

$$\bar{\Phi}\Psi \equiv \bar{\Phi}^A\Psi_A, \quad (\text{A.1})$$

The conjugate and transpose act as

$$(\Psi_A)^* = (\Psi^*)^A, \quad (\mathcal{C}^{AB})^T = \mathcal{C}^{BA}. \quad (\text{A.2})$$

Below we detail the properties of gamma matrices in  $d = 11$  and  $d = 6$ , and we state the reality condition on spinors. More details can be found in [36] and references therein.

## A.1 $d = 11$ Clifford algebra

The 11d Clifford algebra is generated by the set of matrices  $(\Gamma_M)_A^B$  satisfying

$$\{\Gamma_M, \Gamma_N\} = 2\eta_{MN}. \quad (\text{A.3})$$

Here for readability  $M$  is used for flat spacetime, unlike (4.6) where it denotes curved spacetime.

The matrices may be chosen such that  $\Gamma_0^\dagger = -\Gamma_0$  is antihermitian while the others are hermitian  $\Gamma_M^\dagger = \Gamma_M$  ( $M \neq 0$ ). In addition, there is an orthogonal, real anti-symmetric matrix  $\mathcal{C}_{AB}$  such that  $\Gamma_M\mathcal{C} = -(\Gamma_M\mathcal{C})^T$ .  $\mathcal{C}$  naturally defines a real structure by relating  $\Psi$  and  $\Psi^\dagger$  as

$$\bar{\Psi} \equiv -i\Gamma_0\Psi^\dagger = \mathcal{C}^\dagger\Psi. \quad (\text{A.4})$$

This is the Majorana condition.

## A.2 $d = 6$ Clifford algebra

An easy way to construct the 6d Clifford algebra is to decompose  $\Gamma_M = \{\Gamma_\mu, \Gamma_i\}$  by introducing a chirality matrix  $\Gamma_* = \Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5$ . The matrices are then (in the chiral basis)

$$\Gamma_\mu = \begin{pmatrix} 0 & \bar{\gamma}_\mu \\ \gamma_\mu & 0 \end{pmatrix} \otimes I_4, \quad \Gamma_i = \begin{pmatrix} -I_4 & 0 \\ 0 & I_4 \end{pmatrix} \otimes \check{\gamma}_i, \quad \Gamma_* = \begin{pmatrix} -I_4 & 0 \\ 0 & I_4 \end{pmatrix} \otimes I_4, \quad (\text{A.5})$$

where the algebra is

$$\bar{\gamma}_\mu\gamma_\nu + \bar{\gamma}_\nu\gamma_\mu = 2\eta_{\mu\nu}, \quad \gamma_\mu\bar{\gamma}_\nu + \gamma_\nu\bar{\gamma}_\mu = 2\eta_{\mu\nu}, \quad \{\check{\gamma}_i, \check{\gamma}_j\} = 2\delta_{ij}. \quad (\text{A.6})$$

Since  $\gamma_\mu$  and  $\check{\gamma}_i$  commute, they define independent spinor representations. Explicitly, we decompose  $A = (\dot{\alpha} \oplus \alpha) \otimes \check{\alpha}$ , so that the indices are  $(\gamma_\mu)_\alpha^{\dot{\beta}}$ ,  $(\bar{\gamma}_\mu)_{\dot{\alpha}}^\beta$  and  $(\check{\gamma}_i)_{\check{\alpha}}^{\check{\beta}}$ . The chiral and antichiral representations are related through

$$\bar{\gamma}_\mu^\dagger = \gamma_0\bar{\gamma}_\mu\gamma_0 \Rightarrow \begin{cases} \bar{\gamma}_0^\dagger = -\gamma_0, \\ \bar{\gamma}_\mu^\dagger = \gamma_\mu, \end{cases} \quad \mu \neq 0. \quad (\text{A.7})$$

The chirality operator gives 2 additional constraints

$$\gamma_{012345} = I, \quad \bar{\gamma}_{012345} = -I, \quad (\text{A.8})$$

with  $\gamma_{\mu\nu\dots\rho} \equiv \gamma_{[\mu}\bar{\gamma}_\nu\dots\gamma_{\rho]}$  the antisymmetrised product of  $\gamma$ -matrices.<sup>6</sup> The charge conjugation matrix takes the form

$$C_{AB} = \begin{pmatrix} 0 & c_{\dot{\alpha}\beta} \\ c_{\alpha\dot{\beta}} & 0 \end{pmatrix} \otimes \Omega_{\check{\alpha}\check{\beta}}, \quad c \equiv c_{\dot{\alpha}\beta}, \quad (\text{A.9})$$

and is used to lower (or raise) spinor indices. The matrix  $\Omega_{\check{\alpha}\check{\beta}}$  is the real, antisymmetric symplectic metric of  $\mathfrak{sp}(4)$  and  $c$  is unitary:

$$c^\dagger c = c^{\alpha\dot{\alpha}}c_{\dot{\alpha}\beta} = \delta_\alpha^\beta, \quad c^* c^T = c^{\dot{\alpha}\alpha}c_{\alpha\dot{\beta}} = \delta_{\dot{\alpha}}^{\dot{\beta}}, \quad \Omega^\dagger\Omega = \Omega^{\check{\alpha}\check{\beta}}\Omega_{\check{\beta}\check{\gamma}} = \delta_{\check{\gamma}}^{\check{\alpha}}. \quad (\text{A.10})$$

They satisfy

$$(\gamma_\mu c) = -(\gamma_\mu c)^T, \quad (\bar{\gamma}_\mu c^T) = -(\bar{\gamma}_\mu c^T)^T, \quad (\check{\gamma}_i \Omega) = -(\check{\gamma}_i \Omega)^T. \quad (\text{A.11})$$

A representation of this algebra is given by

$$\begin{aligned} \gamma_0 &= \bar{\gamma}_0 = iI_2 \otimes I_2, & \gamma_1 &= -\bar{\gamma}_1 = -i\sigma_1 \otimes I_2, & \gamma_2 &= -\bar{\gamma}_2 = -i\sigma_2 \otimes I_2, \\ \gamma_3 &= -\bar{\gamma}_3 = i\sigma_3 \otimes \sigma_1, & \gamma_4 &= -\bar{\gamma}_4 = i\sigma_3 \otimes \sigma_2, & \gamma_5 &= -\bar{\gamma}_5 = -i\sigma_3 \otimes \sigma_3, \\ \check{\gamma}_1 &= \sigma_1 \otimes \sigma_2, \quad \check{\gamma}_2 = \sigma_2 \otimes \sigma_2, \quad \check{\gamma}_3 = \sigma_3 \otimes \sigma_2, \quad \check{\gamma}_4 = I_2 \otimes \sigma_1, \quad \check{\gamma}_5 = I_2 \otimes \sigma_3, \\ c &= -c^T = \sigma_1 \otimes i\sigma_2, & \Omega &= i\sigma_2 \otimes I_2. \end{aligned} \quad (\text{A.12})$$

<sup>6</sup>(Anti-)symmetrisation is understood with the appropriate combinatorial factors, i.e.  $A_{[ab]} = \frac{1}{2}A_{ab} - A_{ba}$ .

### A.3 Symplectic Majorana condition

In 6d the spinor  $\Psi$  decomposes into a chiral and an antichiral 6d spinor as

$$\Psi_A = \begin{pmatrix} \bar{\chi}^{\dot{\alpha}\alpha} \\ \psi_{\alpha\dot{\alpha}} \end{pmatrix}, \quad \bar{\Psi}^A \equiv (-i(\psi^\dagger)^{\alpha\dot{\alpha}}(\gamma_0)_{\dot{\alpha}\alpha} \quad -i(\bar{\chi}^\dagger)^{\dot{\alpha}\alpha}(\bar{\gamma}_0)_{\dot{\alpha}\alpha}) \equiv (\bar{\psi}^{\dot{\alpha}\alpha} \quad \chi^{\alpha\dot{\alpha}}). \quad (\text{A.13})$$

The Majorana condition on  $\Psi$  then translates to

$$\chi^{\alpha\dot{\alpha}} = (c^\dagger \Omega^\dagger \bar{\chi})^{\alpha\dot{\alpha}} = (c \Omega \bar{\chi})^{\alpha\dot{\alpha}}, \quad \bar{\psi}^{\dot{\alpha}\alpha} = (c^* \Omega^\dagger \psi)^{\dot{\alpha}\alpha} = -(c \Omega \psi)^{\dot{\alpha}\alpha}, \quad (\text{A.14})$$

where in the second equality we use the properties of our representation. The inclusion of the symplectic form  $\Omega$  in (A.14) is the reason these equations are known as the *symplectic Majorana condition*. The spinors  $\varepsilon^1$ ,  $\bar{\varepsilon}^2$ , and  $\psi$  in (3.1) are of this type.

## B Geometry of submanifolds

In this appendix we assemble the geometry results used throughout the main text and in Appendix C. Sections B.1 and B.2 contain our conventions for Riemann curvature and the definition of the second fundamental form of an embedded submanifold as well as some standard results relating the two. In Section B.3 the second fundamental form is related to the coefficients of the normal coordinate expansion of the embedding.

### B.1 Riemann curvature

We adopt the convention where the Riemann tensor is defined as

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\rho\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\mu_{\sigma\lambda} \Gamma^\lambda_{\nu\rho}. \quad (\text{B.1})$$

It is convenient to split it into a conformally invariant Weyl tensor  $W_{\mu\nu\rho\sigma}$  and the Schouten tensor  $P_{\mu\nu}$ ,

$$P_{\mu\nu} = \frac{1}{d-2} \left( R_{\mu\nu} - \frac{R}{2(d-1)} g_{\mu\nu} \right), \quad (\text{B.2})$$

$$W_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - g_{\mu\rho} P_{\nu\sigma} + g_{\mu\sigma} P_{\nu\rho} + g_{\nu\rho} P_{\mu\sigma} - g_{\nu\sigma} P_{\mu\rho}. \quad (\text{B.3})$$

### B.2 Extrinsic curvature

We define the second fundamental form to be

$$\Pi_{ab}^\mu = (\partial_a \partial_b x^\lambda + \partial_a x^\rho \partial_b x^\sigma \Gamma_{\rho\sigma}^\lambda) (\delta_\lambda^\mu - g_{\kappa\lambda} \partial^c x^\kappa \partial_c x^\mu). \quad (\text{B.4})$$

The second part is the projector to the components orthogonal to the surface (defined by its embedding  $x^\mu(\sigma)$ ), while the first part is the action of the covariant derivative on the (pullback) of  $x^\lambda(\sigma)$ . The mean curvature vector is then

$$H^\mu = h^{ab}\Pi_{ab}^\mu. \quad (\text{B.5})$$

These invariants are related to the intrinsic curvature of  $\Sigma$  and  $M$  by the Gauss-Codazzi equation

$$R_{abcd}^\Sigma = R_{abcd}^M + 2\Pi_{a[b}^\mu\Pi_{c]d}^\nu g_{\mu\nu}. \quad (\text{B.6})$$

Contracting twice with  $h^{-1}$  and expanding the Riemann tensor in terms of the Weyl and Schouten tensors, we obtain

$$(H^2 + 4 \operatorname{tr} P) = 2R^\Sigma + 2 \operatorname{tr} \tilde{\Pi}^2 - 2 \operatorname{tr} W, \quad (\text{B.7})$$

where  $\tilde{\Pi}_{ab}^\mu$  is the traceless part of the second fundamental form

$$\tilde{\Pi}_{ab}^\mu = \Pi_{ab}^\mu - \frac{H^\mu}{2} h_{ab}. \quad (\text{B.8})$$

### B.3 Embedding in normal coordinates

Using these standard geometry results, we now derive the expressions needed for (3.16) and (3.21). Unlike in Section 3, we state here the result for a generic curved spacetime  $M$ . This allows us to perform the calculation in Appendix C on curved space.

Let  $x^\mu$  and  $\eta^a$  be Riemann normal coordinates on  $M$  and  $\Sigma$  about the same point. In terms of these, the embedding  $\Sigma \hookrightarrow M$  may be expanded as

$$x^\mu(\eta) = x^\mu(0) + \eta^a v_a^\mu + \frac{1}{2} \eta^a \eta^b v_{ab}^\mu + \frac{1}{6} \eta^a \eta^b \eta^c v_{abc}^\mu + \mathcal{O}(\eta^4). \quad (\text{B.9})$$

These coefficients are constrained by the condition that straight lines in normal coordinates correspond to geodesics. In particular, a curve on  $\Sigma$  given by a straight line in  $\eta$  has constant speed and its curvature in  $M$  is normal to  $\Sigma$  at every point, which gives the constraints

$$\begin{aligned} \delta_{ab} &= v_a \cdot v_b, \\ 0 &= v_{ab} \cdot v_c, \\ 0 &= 3 v_d \cdot v_{abc} + v_{ab} \cdot v_{cd} + v_{ac} \cdot v_{bd} + v_{ad} \cdot v_{bc}. \end{aligned} \quad (\text{B.10})$$

Using (B.4) one easily checks that the second order coefficient equals the second fundamental form

$$\Pi_{ab}^\mu|_{\eta=0} = v_{ab}^\mu. \quad (\text{B.11})$$

The geodesic distance between  $\xi(\eta)$  and the origin of the normal frame is found from (B.9)

$$|x(\eta) - x(0)|^2 = \eta^a \eta_a - \frac{1}{12} \Pi_{ab} \cdot \Pi_{cd} \eta^a \eta^b \eta^c \eta^d + \mathcal{O}(\eta^5). \quad (\text{B.12})$$

Furthermore, in normal coordinates, the metrics take the form

$$\begin{aligned} g_{\mu\nu} &= \delta_{\mu\nu} - \frac{1}{3}R_{\mu\rho\nu\sigma}^M \xi^\rho \xi^\sigma + \mathcal{O}(\xi^3), \\ h_{ab} &= \delta_{ab} - \frac{1}{3}R_{acbd}^\Sigma \eta^c \eta^d + \mathcal{O}(\eta^3), \end{aligned} \tag{B.13}$$

which yields an expansion for the volume factor

$$\sqrt{h(\eta)} = 1 - \frac{1}{6}R_{ab}^\Sigma \eta^a \eta^b + \mathcal{O}(\eta^3). \tag{B.14}$$

## C Geodesic point-splitting

In this appendix we present an alternative regularisation of (3.4), essentially point splitting, displacing one copy of the surface operator by a distance  $\epsilon$  in an arbitrary normal direction  $\nu$ . This regularisation is used in [16, 17], but there the vector  $\nu$  is taken to be a constant, and therefore the method is only applicable if the operators are restricted to a codimension-one subspace.

The technology used to define this regularisation scheme applies for generic smooth embedded surfaces in a Riemannian manifold, and we present here a curved space calculation, as opposed to Section 3.3, where for brevity we restricted ourselves to flat space. However, we still have to restrict to conformally flat backgrounds, since otherwise we do not have a short-distance expansion for the propagator and therefore still cannot infer the anomaly coefficient  $b$ .

As expected, we recover the result (3.23) exactly, and thus verify scheme-independence.

### C.1 Displacement map

We can regularise the integral (3.4) by displacing a copy of the surface a distance  $\epsilon$  along a unit normal vector field  $\nu$ . Under that map, which we denote by  $\mathcal{T}$ , the geodesic distance admits an expansion of the form

$$|\mathcal{T}(x^\mu(\sigma)) - x^\mu(\sigma + \eta)|^2 = \epsilon^2 + \eta^2 + \sum_{k=3}^{\infty} \sum_{l=0}^k f_l^{(k)} \eta^l \epsilon^{k-l}. \tag{C.1}$$

We can combine the terms of fixed  $k$  in terms of degree  $k$  polynomials  $f^{(k)}$

$$\sum_{l=0}^k f_l^{(k)} \eta^l \epsilon^{k-l} = \epsilon^k f^{(k)}(\eta/\epsilon). \tag{C.2}$$

We calculate the higher order terms in (C.1) explicitly in (C.7), but first we note that the only terms contributing to the divergent part are  $f^{(3)}$  and  $f^{(4)}$ . To see that, the integrals

computing the expectation value take the form

$$\int_0^\rho \frac{\eta^{m+1} d\eta}{|\mathcal{T}(x^\mu(\sigma)) - x^\mu(\sigma + \eta)|^4}, \quad (\text{C.3})$$

where  $\rho$  is an arbitrary but fixed IR cutoff. We can evaluate (C.3) by expanding the integrand in  $\epsilon$ . Writing  $s \equiv \eta/\epsilon$ , we obtain

$$\epsilon^{m-2} \int_0^{\rho/\epsilon} \frac{s^{m+1}}{(1+s^2)^2} \left[ 1 - \frac{2f^{(3)}(s)}{1+s^2} \epsilon + \left( \frac{3(f^{(3)}(s))^2}{(1+s^2)^2} - \frac{2f^{(4)}(s)}{1+s^2} \right) \epsilon^2 + \mathcal{O}(\epsilon^3) \right] ds. \quad (\text{C.4})$$

By application of Faà di Bruno's formula one checks that the terms in brackets of order  $\epsilon^n$  contribute to the divergence only if  $m+n \leq 2$ . We can therefore safely ignore higher orders in  $\epsilon$ . Only a finite number of terms remains to be computed and we find that the only divergent integrals (C.3) are:

$$m=0: \quad \frac{1}{2\epsilon^2} - \frac{1}{8\epsilon} \left( 4f_0^{(3)} + \pi f_1^{(3)} + 4f_2^{(3)} + 3\pi f_3^{(3)} \right) + \left( -3(f_3^{(3)})^2 + 2f_4^{(4)} \right) \log \epsilon, \quad (\text{C.5a})$$

$$m=1: \quad \frac{\pi}{4\epsilon} + 2f_3^{(3)} \log \epsilon, \quad (\text{C.5b})$$

$$m=2: \quad -\log \epsilon. \quad (\text{C.5c})$$

The relevant coefficients can be read off of the expansion of the geodesic distance up to combined order of 4 in  $\eta$  and  $\epsilon$ . The second term on the left hand side of (C.1) can be expanded simply using the embedding (B.9). For the first term, we solve the geodesic equation order by order in the displacement  $\epsilon$  to obtain

$$\mathcal{T}(x^\mu) = x^\mu + \epsilon \nu^\mu - \frac{\epsilon^2}{2} \Gamma_{\kappa\lambda}^\mu \nu^\kappa \nu^\lambda + \frac{\epsilon^3}{6} \left( -\partial_\nu \Gamma_{\rho\sigma}^\mu + 2\Gamma_{\nu\lambda}^\mu \Gamma_{\rho\sigma}^\lambda \right) \nu^\nu \nu^\rho \nu^\sigma + \mathcal{O}(\epsilon^4). \quad (\text{C.6})$$

Combining these expressions, and writing  $\eta^a = \eta e^a(\varphi)$  as in (3.18) and onwards, the only two non-vanishing relevant coefficients read

$$\begin{aligned} f_2^{(3)} &= -e^a e^b \Pi_{ab} \cdot \nu, \\ f_4^{(4)} &= -\frac{1}{12} e^a e^b e^c e^d \Pi_{ab} \cdot \Pi_{cd}. \end{aligned} \quad (\text{C.7})$$

The first contributes to a scheme-dependent divergence  $\epsilon^{-1}$ , while the second contributes to the anomaly.

## C.2 Evaluation of the anomaly

With the displacement map (C.6) in hand, we can evaluate (3.4). The propagators on a conformally flat background can be obtained by considering curved space actions for a

conformal scalar and a Maxwell-type 2-form and inverting the kinetic operators order by order, following [17] and [14]. We find:

$$\langle \Phi_i(x) \Phi_j(x + \xi) \rangle = \frac{\delta_{ij}}{\pi^2 |\xi|^4} \left[ 1 + \frac{1}{3} P_{\mu\nu} \xi^\mu \xi^\nu + \mathcal{O}(\xi^3) \right], \quad (\text{C.8})$$

$$\begin{aligned} \langle B^{+\mu\nu}(x) B_{\rho\sigma}^+(x + \xi) \rangle &= \frac{1}{4\pi^2 |\xi|^4} \left[ \delta_\mu^\rho \delta_\nu^\sigma - \delta_\nu^\rho \delta_\mu^\sigma \right. \\ &\quad \left. - \frac{4}{3} \left( 4P_{[\rho}^{[\mu} \delta_{\sigma]}^{\nu]} \delta_{\lambda\tau} + P_{\lambda[\rho} \delta_{\sigma]}^{[\mu} \delta_{\tau]}^{\nu]} + \delta_{\lambda[\rho} P_{\sigma]}^{[\mu} \delta_{\tau]}^{\nu]} \right) \xi^\lambda \xi^\tau + \mathcal{O}(\xi^3) \right]. \end{aligned} \quad (\text{C.9})$$

To apply our regularisation, we should replace  $\xi$  by (C.1) in the denominator of the propagators before performing the integral over  $\eta$ . A priori, we should also perform the displacement in the numerator, since a term of order  $\mathcal{O}(\epsilon)$  can contribute to the  $\epsilon^{-1}$  divergence by multiplying (C.5a). However, one easily checks that the only terms of that order are accompanied by nonzero powers of  $\eta$ , and therefore do not contribute to the divergence of (3.4). We therefore drop the  $\epsilon$  in the numerators of the propagators.

The expansion of the numerators is then assembled, as before, from (3.16) and (3.21), but in addition, since we are working on curved space, we obtain an additional term at  $\mathcal{O}(\eta^2)$  explicitly involving  $\text{tr } P$  from the propagators (C.8). Collecting terms in analogy to Section 3.3, and integrating out the angular coordinate using (3.18), we obtain the scalar contribution

$$\frac{1}{2\pi\epsilon^2} + \frac{H \cdot \nu}{4\pi\epsilon} + \frac{1}{16\pi} (2R^\Sigma - (H^2 + 4 \text{tr } P) + 4(\partial n)^2) \log \epsilon + \text{finite}, \quad (\text{C.10})$$

while the tensor field yields

$$-\frac{1}{2\pi\epsilon^2} - \frac{H \cdot \nu}{4\pi\epsilon} - \frac{1}{16\pi} (-2R^\Sigma + 3(H^2 + 4 \text{tr } P)) \log \epsilon + \text{finite}. \quad (\text{C.11})$$

Combining these terms, we find

$$\log \langle V_\Sigma \rangle = \frac{1}{4\pi} \log \epsilon \int_\Sigma \text{vol}_\Sigma [R^\Sigma - (H^2 + 4 \text{tr } P) + (\partial n)^2] + \text{finite}, \quad (\text{C.12})$$

which agrees exactly with (3.23). Note that the scheme dependence, which is present in the simple pole of both (C.10) and (C.11), cancels in the final result, and the terms  $H^2$  and  $\text{tr } P$  combine to an anomaly term as in (2.3), as required.

## References

- [1] P. S. Howe, N. D. Lambert, and P. C. West, “The selfdual string soliton,” *Nucl. Phys.* **B515** (1998) 203–216, [hep-th/9709014](#).
- [2] A. Strominger, “Open  $p$ -branes,” *Phys. Lett.* **B383** (1996) 44–47, [hep-th/9512059](#).

- [3] S.-J. Rey and J.-T. Yee, “Macroscopic strings as heavy quarks in large  $N$  gauge theory and anti-de Sitter supergravity,” *Eur. Phys. J.* **C22** (2001) 379–394, [hep-th/9803001](#).
- [4] J. M. Maldacena, “Wilson loops in large  $N$  field theories,” *Phys. Rev. Lett.* **80** (1998) 4859–4862, [hep-th/9803002](#).
- [5] N. Drukker, D. J. Gross, and H. Ooguri, “Wilson loops and minimal surfaces,” *Phys. Rev.* **D60** (1999) 125006, [hep-th/9904191](#).
- [6] V. Pestun, “Localization of gauge theory on a four-sphere and supersymmetric Wilson loops,” *Commun. Math. Phys.* **313** (2012) 71–129, [arXiv:0712.2824](#).
- [7] M. Cooke, A. Dekel, and N. Drukker, “The Wilson loop CFT: Insertion dimensions and structure constants from wavy lines,” *J. Phys.* **A50** no. 33, (2017) 335401, [arXiv:1703.03812](#).
- [8] S. Giombi, R. Roiban, and A. A. Tseytlin, “Half-BPS Wilson loop and  $AdS_2/CFT_1$ ,” *Nucl. Phys.* **B922** (2017) 499–527, [arXiv:1706.00756](#).
- [9] N. Drukker, “Integrable Wilson loops,” *JHEP* **10** (2013) 135, [arXiv:1203.1617](#).
- [10] D. Correa, J. Maldacena, and A. Sever, “The quark anti-quark potential and the cusp anomalous dimension from a TBA equation,” *JHEP* **08** (2012) 134, [arXiv:1203.1913](#).
- [11] L. F. Alday and J. M. Maldacena, “Gluon scattering amplitudes at strong coupling,” *JHEP* **06** (2007) 064, [arXiv:0705.0303](#).
- [12] N. Drukker *et al.*, “Roadmap on Wilson loops in 3d Chern-Simons-matter theories,” [arXiv:1910.00588](#).
- [13] C. R. Graham and E. Witten, “Conformal anomaly of submanifold observables in AdS/CFT correspondence,” *Nucl. Phys.* **B546** (1999) 52–64, [hep-th/9901021](#).
- [14] M. Henningson and K. Skenderis, “Weyl anomaly for Wilson surfaces,” *JHEP* **06** (1999) 12, [hep-th/9905163](#).
- [15] M. Henningson, “Surface observables and the Weyl anomaly,” in *High energy physics and quantum field theory. Proceedings, 14th International Workshop, QFTHEP’99, Moscow, Russia, May 27-June 2, 1999*, pp. 384–386. 1999. [hep-th/9908183](#).
- [16] A. Gustavsson, “On the Weyl anomaly of Wilson surfaces,” *JHEP* **12** (2003) 59, [hep-th/0310037](#).
- [17] A. Gustavsson, “Conformal anomaly of Wilson surface observables: A field theoretical computation,” *JHEP* **07** (2004) 74, [hep-th/0404150](#).
- [18] S. Deser and A. Schwimmer, “Geometric classification of conformal anomalies in arbitrary dimensions,” *Phys. Lett.* **B309** (1993) 279–284, [hep-th/9302047](#).
- [19] N. Drukker, M. Probst, and M. Trépanier to appear.
- [20] S. N. Solodukhin, “Entanglement entropy, conformal invariance and extrinsic geometry,” *Phys. Lett.* **B665** (2008) 305–309, [arXiv:0802.3117](#).
- [21] R. C. Myers and A. Singh, “Entanglement entropy for singular surfaces,” *JHEP* **09** (2012) 13, [arXiv:1206.5225](#).

- [22] L. Bianchi, M. Meineri, R. C. Myers, and M. Smolkin, “Rényi entropy and conformal defects,” *JHEP* **07** (2016) 76, [arXiv:1511.06713](#).
- [23] K. Jensen, A. O’Bannon, B. Robinson, and R. Rodgers, “From the Weyl anomaly to entropy of two-dimensional boundaries and defects,” [arXiv:1812.08745](#).
- [24] D. Fursaev, “Conformal anomalies of CFTs with boundaries,” *JHEP* **12** (2015) 112, [arXiv:1510.01427](#).
- [25] S. N. Solodukhin, “Boundary terms of conformal anomaly,” *Phys. Lett.* **B752** (2016) 131–134, [arXiv:1510.04566](#).
- [26] C. P. Herzog and K.-W. Huang, “Boundary conformal field theory and a boundary central charge,” *JHEP* **10** (2017) 189, [arXiv:1707.06224](#).
- [27] A. Schwimmer and S. Theisen, “Entanglement entropy, trace anomalies and holography,” *Nucl. Phys.* **B801** (2008) 1–24, [arXiv:0802.1017](#).
- [28] D. E. Berenstein, R. Corrado, W. Fischler, and J. M. Maldacena, “The operator product expansion for Wilson loops and surfaces in the large  $N$  limit,” *Phys. Rev.* **D59** (1999) 105023, [hep-th/9809188](#).
- [29] L. F. Alday and J. Maldacena, “Comments on gluon scattering amplitudes via AdS/CFT,” *JHEP* **11** (2007) 068, [arXiv:0710.1060](#).
- [30] J. Polchinski and J. Sully, “Wilson loop renormalization group flows,” *JHEP* **10** (2011) 059, [arXiv:1104.5077](#).
- [31] B. Chen, W. He, J.-B. Wu, and L. Zhang, “M5-branes and Wilson Surfaces,” *JHEP* **08** (2007) 067, [arXiv:0707.3978](#).
- [32] B. Chen and J.-B. Wu, “Wilson-Polyakov surfaces and M-theory branes,” *JHEP* **05** (2008) 46, [arXiv:0802.2173](#).
- [33] H. Mori and S. Yamaguchi, “M5-branes and Wilson surfaces in  $AdS_7/CFT_6$  correspondence,” *Phys. Rev.* **D90** no. 2, (2014) 26005, [arXiv:1404.0930](#).
- [34] R. Rodgers, “Holographic entanglement entropy from probe M-theory branes,” *JHEP* **03** (2019) 092, [arXiv:1811.12375](#).
- [35] N. Drukker and M. Trépanier to appear.
- [36] P. Claus, R. Kallosh, and A. Van Proeyen, “M5-brane and superconformal (0,2) tensor multiplet in 6 dimensions,” *Nucl. Phys.* **B518** (1998) 117–150, [hep-th/9711161](#).
- [37] P. S. Howe, E. Sezgin, and P. C. West, “Covariant field equations of the M-theory five-brane,” *Phys. Lett.* **B399** (1997) 49–59, [hep-th/9702008](#).
- [38] J. M. Maldacena, “The Large  $N$  limit of superconformal field theories and supergravity,” *Int. J. Theor. Phys.* **38** (1999) 1113–1133, [hep-th/9711200](#).
- [39] P. G. O. Freund and M. A. Rubin, “Dynamics of dimensional reduction,” *Phys. Lett.* **97B** (1980) 233–235.
- [40] N. Boulanger, “General solutions of the Wess-Zumino consistency condition for the Weyl anomalies,” *JHEP* **07** (2007) 069, [arXiv:0704.2472](#).

- [41] E. D’Hoker, J. Estes, M. Gutperle, and D. Krym, “Exact half-BPS flux solutions in M-theory II: Global solutions asymptotic to  $AdS_7 \times S^4$ ,” *JHEP* **12** (2008) 044, [arXiv:0810.4647](#).
- [42] C. Bachas, E. D’Hoker, J. Estes, and D. Krym, “M-theory solutions invariant under  $D(2, 1; \gamma) \oplus D(2, 1; \gamma)$ ,” *Fortsch. Phys.* **62** (2014) 207–254, [arXiv:1312.5477](#).
- [43] A. Lewkowycz and E. Perlmutter, “Universality in the geometric dependence of Renyi entropy,” *JHEP* **01** (2015) 080, [arXiv:1407.8171](#).
- [44] N. Drukker, J. Gomis, and S. Matsuura, “Probing  $\mathcal{N} = 4$  SYM With Surface Operators,” *JHEP* **10** (2008) 048, [arXiv:0805.4199](#).
- [45] L. Bianchi and M. Lemos, “Superconformal surfaces in four dimensions,” [arXiv:1911.05082](#).
- [46] S. A. Gentle, M. Gutperle, and C. Marasinou, “Entanglement entropy of Wilson surfaces from bubbling geometries in M-theory,” *JHEP* **08** (2015) 019, [arXiv:1506.00052](#).
- [47] J. Estes, D. Krym, A. O’Bannon, B. Robinson, and R. Rodgers, “Wilson surface central charge from holographic entanglement entropy,” *JHEP* **05** (2019) 032, [arXiv:1812.00923](#).
- [48] A. Chalabi, A. O’Bannon, B. Robinson, and J. Sisti, “Central charges of 2d superconformal defects,” [arXiv:2003.02857](#).
- [49] M. Gunaydin, P. van Nieuwenhuizen, and N. P. Warner, “General construction of the unitary representations of Anti-de Sitter superalgebras and the spectrum of the  $S^4$  compactification of eleven-dimensional supergravity,” *Nucl. Phys.* **B255** (1985) 63–92.
- [50] M. Gunaydin and S. Takemae, “Unitary supermultiplets of  $OSp(8^*|4)$  and the  $AdS_7/CFT_6$  duality,” *Nucl. Phys.* **B578** (2000) 405–448, [hep-th/9910110](#).
- [51] C. Cordova, T. T. Dumitrescu, and K. Intriligator, “Multiplets of superconformal symmetry in diverse dimensions,” *JHEP* **03** (2019) 163, [arXiv:1612.00809](#).
- [52] K. Zarembo, “Supersymmetric Wilson loops,” *Nucl. Phys.* **B643** (2002) 157–171, [hep-th/0205160](#).
- [53] N. Drukker, S. Giombi, R. Ricci, and D. Trancanelli, “Supersymmetric Wilson loops on  $S^3$ ,” *JHEP* **05** (2008) 017, [arXiv:0711.3226](#).
- [54] M. Mezei, S. S. Pufu, and Y. Wang, “Chern-Simons theory from M5-branes and calibrated M2-branes,” *JHEP* **08** (2019) 165, [arXiv:1812.07572](#).
- [55] K.-M. Lee and H.-U. Yee, “BPS String Webs in the 6-dim (2,0) Theories,” *JHEP* **03** (2007) 057, [hep-th/0606150](#).
- [56] E. Witten, “Five-brane effective action in M-theory,” *J. Geom. Phys.* **22** (1997) 103–133, [hep-th/9610234](#).
- [57] P. Pasti, D. P. Sorokin, and M. Tonin, “On Lorentz invariant actions for chiral  $p$  forms,” *Phys. Rev.* **D55** (1997) 6292–6298, [hep-th/9611100](#).

- [58] J. Bagger, N. Lambert, S. Mukhi, and C. Papageorgakis, “Multiple membranes in M-theory,” *Phys. Rept.* **527** (2013) 1–100, [arXiv:1203.3546](#).
- [59] N. Lambert, “MBranes: Lessons from M2’s and hopes for M5’s,” *Fortsch. Phys.* **67** no. 8-9, (2019) 1910011, [arXiv:1903.02825](#).
- [60] A. Sen, “Self-dual forms: action, hamiltonian and compactification,” *J. Phys.* **A53** no. 8, (2020) 084002, [arXiv:1903.12196](#).
- [61] N. Lambert, “(2,0) lagrangian structures,” *Phys. Lett.* **B798** (2019) 134948, [arXiv:1908.10752](#).
- [62] E. Andriolo, N. Lambert, and C. Papageorgakis, “Geometrical aspects of an abelian (2,0) action,” [arXiv:2003.10567](#).
- [63] D. Correa, J. Henn, J. Maldacena, and A. Sever, “The cusp anomalous dimension at three loops and beyond,” *JHEP* **05** (2012) 098, [arXiv:1203.1019](#).
- [64] N. Gromov, V. Kazakov, G. Korchemsky, S. Negro, and G. Sizov, “Integrability of conformal fishnet theory,” *JHEP* **01** (2018) 095, [arXiv:1706.04167](#).
- [65] C. Fefferman, “Conformal invariants,” “*Elie Cartan et les mathematiques d’aujourd’hui*,” *Asterisque, hors serie* (1985) 95–116.
- [66] C. Fefferman and C. R. Graham, “The ambient metric,” *Ann. Math. Stud.* **178** (2011) 1–128, [arXiv:0710.0919](#).
- [67] E. Bergshoeff, E. Sezgin, and P. K. Townsend, “Supermembranes and eleven-dimensional supergravity,” *Phys. Lett.* **B189** (1987) 75–78.
- [68] I. R. Klebanov, T. Nishioka, S. S. Pufu, and B. R. Safdi, “On shape dependence and RG flow of entanglement entropy,” *JHEP* **07** (2012) 001, [arXiv:1204.4160](#).
- [69] P. Bueno and R. C. Myers, “Universal entanglement for higher dimensional cones,” *JHEP* **12** (2015) 168, [arXiv:1508.00587](#).
- [70] H. Dorn, “On the logarithmic divergent part of entanglement entropy, smooth versus singular regions,” *Phys. Lett.* **B763** (2016) 134–138, [arXiv:1608.04900](#).