Invariant measure for the stochastic Cauchy problem driven by a cylindrical Lévy process

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Abstract

In this work, we present sufficient conditions for the existence of a stationary solution of an abstract stochastic Cauchy problem driven by an arbitrary cylindrical Lévy process, and show that these conditions are also necessary if the semigroup is stable, in which case the invariant measure is unique. For typical situations such as the heat equation, we significantly simplify these conditions without assuming any further restrictions on the driving cylindrical Lévy process and demonstrate their application in some examples.

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1 Introduction

Cylindrical Lévy processes naturally extend the class of cylindrical Brownian motions and cover many examples of Lévy-type noise considered in the literature. A general framework of cylindrical Lévy processes in Banach spaces has been recently introduced by Applebaum and Riedle in [3]. Stochastic integration of deterministic operator-valued integrands with respect to cylindrical Lévy processes is developed in [19]. Based on this integration theory, the authors of the present article have developed a general theory of weak and mild solutions for the stochastic Cauchy problem driven by an arbitrary cylindrical Lévy process in [14].

More specifically, the stochastic Cauchy problem is a linear evolution equation driven by an additive noise of the form

$$dY(t) = AY(t)\,dt + B\,dL(t) \qquad \text{for all } t \in [0, T]. \tag{1.1}$$

Here, L is a cylindrical Lévy process on a separable Hilbert space U, the coefficient A is the generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on a separable Hilbert space V and B is a linear, bounded operator from U to V. In this general setting, we present sufficient conditions for the existence of a stationary solution of (1.1) and show that these conditions are also necessary if the semigroup is stable, in which case the invariant measure is unique. If the semigroup has a spectral decomposition, we significantly simplify these conditions without assuming any further restrictions on the driving cylindrical Lévy process.

For finite dimensional Lévy processes the existence of invariant measures and its relation to operator self-decomposibilty has been studied by Jurek [12], Jurek and Vervaat [13], Sato and Yamatado [21], [22], Wolfe [26], and Zabczyk [27]. The case of an infinite dimensional Lévy process in a Hilbert space was studied by Chojnowska-Michalik in [9] and [10]. The case of a cylindrical Lévy process was only considered for a specific example of a cylindrical Lévy process and under further assumptions on the semigroup in Priola and Zabczyk [17]. The assumptions in [17] enable the authors to reduce the problem of the existence of an invariant measure to the analogue problem in one dimension. The general setting in the present paper clearly excludes this approach. Our results in the general framework can easily be applied to the example considered in [17], and we are not only able to cover these results but even improve them; see Example 4.5.

In our general framework, having in hand the integration theory developed in [19] and the probabilistic description of cylindrical Lévy processes by their characteristics introduced in [18], we are able to generalise the conditions from the case of a genuine Lévy process in [9] to the cylindrical setting. The fact, that cylindrical processes are generalised processes not attaining values in the underlying Hilbert space, prevents us from directly adopting the methods from the classical case. Instead, we exploit some of the methods developed in [14] and [19] such as tightness of finite-dimensional approximations. As in the classical setting, the derived conditions are rather difficult to verify in the general case but can be significantly simplified in typical cases such as the heat equation. In the case of a genuine Lévy process, a modified version of the integrability condition on the logarithmic moments from Wolfe [26] for the finite-dimensional setting gives a necessary and sufficient condition for the existence of a stationary solution for the heat equation; see [10]. We succeed in establishing the exact analogue of the conditions on the logarithmic moments in case of a cylindrical driving noise, which requires some rather subtle estimates.

Our article begins with Section 2 where we fix most of our notations and introduce cylindrical Lévy processes and their stochastic integral. In section 3 we briefly demonstrate the equivalence of the existence of a stationary solution of (1.1) and of an invariant measure for the corresponding Mehler semigroup. Our first main result of this article is presented in this section, which provides sufficient conditions for the existence of a stationary solution in terms of the characteristics of the driving cylindrical Lévy process. Our second main result is in the final Section 4, where we significantly simplify the conditions from Section 3 if the semigroup has a spectral decomposition. We finish the article by demonstrating our results in some examples.

2 Preliminaries

Let U and V be real separable Hilbert spaces with norms $\|\cdot\|$ and inner products $\langle\cdot,\cdot\rangle$. Let $(e_k)_{k\in\mathbb{N}}$ and $(h_k)_{k\in\mathbb{N}}$ be the orthonormal bases of U and V, respectively. We identify the dual of a Hilbert space by the space itself. The space of all linear, bounded operators from U to V is denoted by $\mathcal{L}(U,V)$, equipped with the operator norm $\|\cdot\|_{\text{op}}$. By B_U, we denote the open unit ball in U, that is, $B_U := \{u \in U : \|u\| < 1\}$. The Borel σ-algebra of U is denoted by $\mathfrak{B}(U)$ and the space of Radon probability measures on $\mathfrak{B}(U)$ is denoted by $\mathcal{M}(U)$ and is equipped with the Prokhorov metric.

We fix a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, P)$, where the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfies the usual conditions of right continuity and completeness. By $L_P^0(\Omega;U)$, we denote the space of all equivalence classes of measurable functions $g\colon \Omega \to U$ and it is equipped with the topology of convergence in probability. The space of all regulated functions $g\colon [0,T] \to U$ is denoted by $R([0,T];U)$ and it is a Banach space when equipped with the supremum norm. Recall that a function $g\colon [0,T] \to U$ is called *regulated* if it can be uniformly approximated by step functions. In particular, a regulated function has only countable number of discontinuities; see [6, Ch.II.1.3] for this and other properties we will use.

Let Γ be a subset of U. The sets of the form

$$C(u_1, ..., u_n; B) := \{u \in U : (\langle u, u_1\rangle, ..., \langle u, u_n\rangle) \in B\},$$

for $u_1, ..., u_n \in \Gamma$ and $B \in \mathfrak{B}(\mathbb{R}^n)$ are called *cylindrical sets with respect to* Γ. The set of all these cylindrical sets is denoted by $\mathcal{Z}(U,\Gamma)$ and it is a σ-algebra if Γ is finite and otherwise an algebra. We write $\mathcal{Z}(U)$ for $\mathcal{Z}(U,U)$. A function $\mu\colon \mathcal{Z}(U) \to [0,1]$ is called a *cylindrical measure*, if for each finite subset $\Gamma \subseteq U$ the restriction of μ on the σ-algebra $\mathcal{Z}(U,\Gamma)$ is a measure. A cylindrical measure μ is only finitely additive and is said to extend to a measure ν on $\mathfrak{B}(U)$ if $\mu = \nu$ on $\mathcal{Z}(U)$. A cylindrical measure is called finite if $\mu(U) < \infty$ and a cylindrical probability measure if $\mu(U) = 1$. A *cylindrical random variable* Z in U is defined as a linear and continuous map $Z\colon U \to L^0_P(\Omega;\mathbb{R})$. Given a cylindrical random variable Z, we can define a cylindrical probability measure λ by

$$\lambda\colon \mathcal{Z}(U) \to [0,1], \qquad \lambda(Z) = P\big((Zu_1,\dots,Zu_n)\in B\big)$$

for cylindrical sets $Z = C(u_1,...,u_n;B)$. The cylindrical probability measure λ is called the *cylindrical distribution* of Z. The characteristic function of a cylindrical random variable Z is defined by

$$\varphi_Z\colon U \to \mathbb{C}, \qquad \varphi_Z(u) = E[\exp(iZu)],$$

and it uniquely determines the cylindrical distribution of Z.

A family $(Z(t) : t \geqslant 0)$ of cylindrical random variables is called a *cylindrical process*. By a *cylindrical Lévy process* we mean a cylindrical process $(L(t) : t \geqslant 0)$ such that for all $u_1,...,u_n \in U$ and $n\in\mathbb{N}$, the stochastic process $((L(t)u_1,...,L(t)u_n) : t \geqslant 0)$ is a Lévy process in $\mathbb{R}^n$ with respect to the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$. The characteristic function of $L(t)$ for all $t\geqslant 0$ is given by

$$\varphi_{L(t)}\colon U \to \mathbb{C}, \qquad \varphi_{L(t)}(u) = \exp\big(t\Psi(u)\big),$$

where $\Psi\colon U\to\mathbb{C}$ is called the (cylindrical) symbol of L, and is given by

$$\Psi(u) = ia(u) - \frac{1}{2}\langle Qu,u\rangle + \int_U \Big(e^{i\langle u,h\rangle} - 1 - i\langle u,h\rangle \mathbb{1}_{B_\mathbb{R}}(\langle u,h\rangle)\Big)\,\mu(\mathrm{d}h), \tag{2.1}$$

where $a\colon U\to\mathbb{R}$ is a continuous mapping with $a(0)=0$, the mapping $Q\colon U\to U$ is a positive, symmetric operator and μ is a cylindrical Lévy measure on $\mathcal{Z}(U)$, that is it is a cylindrical measure on $\mathcal{Z}(U)$ satisfying

$$\int_U \big(\langle u,h\rangle^2 \wedge 1\big)\,\mu(\mathrm{d}h) < \infty \qquad \text{for all } u\in U.$$

We call (a,Q,μ) the *(cylindrical) characteristics of* L. Cylindrical Lévy processes are introduced in [3] and its characteristics further studied in [18].

For a function $f\colon [0,T]\to\mathcal{L}(U,V)$ such that the map $f^*(\cdot)v\colon [0,T]\to U$ is a regulated function for each $v\in V$, one can define the stochastic integral

$$Z_A(v) := \int_0^T \mathbb{1}_A(s) f^*(s) v \,\mathrm{d}L(s)$$

for each set $A\in\mathfrak{B}([0,T])$. In this way, one obtains a cylindrical random variable $Z_A\colon V\to L^0_P(\Omega;\mathbb{R})$. The function f is called *stochastically integrable* with respect to L if for each Borel set $A \in \mathfrak{B}([0,T])$, the cylindrical random variable Z_A extends to a genuine V-valued random variable I_A, that is

$$\langle I_A, v\rangle = \int_0^T \mathbb{1}_A(s) f^*(s)v \, \mathrm{d}L(s) \qquad \text{for all } v \in V. \tag{2.2}$$

This stochastic integration theory is developed in [19] and applied in [14] to study the weak solution of abstract stochastic Cauchy problem driven by a cylindrical Lévy process.

3 Invariant measure

The main aim of this paper is to study the conditions for the existence of an invariant measure for the solution of the stochastic Cauchy problem

$$\begin{aligned} \mathrm{d}Y(t) &= AY(t)\,\mathrm{d}t + B\,\mathrm{d}L(t) \qquad \text{for all } t \geqslant 0, \\ Y(0) &= Y_0, \end{aligned} \tag{3.1}$$

where A is the generator of a C_0-semigroup $(T(t))_{t\geqslant 0}$ on a separable Hilbert space V, the driving noise L is a cylindrical Lévy process on a separable Hilbert space U and $B\colon U \to V$ is a bounded linear operator from U to V. The initial condition Y_0 is a V-valued $\mathcal{F}_0$-measurable random variable.

A V-valued process $(Y(t) : t \in [0,T])$ is called a weak solution of (3.1) on $[0,T]$ if it satisfies the following:

(1) Y is progressively measurable;

(2) the mapping $t \mapsto \langle Y(t), g(t)\rangle$ is integrable on $[0,T]$ for each $g \in C([0,T];V)$ and satisfies for each sequence $(g_n)_{n\in\mathbb{N}} \subseteq C([0,T];V)$ with $\|g_n\|_\infty \to 0$ that

$$\int_0^T \langle Y(s), g_n(s)\rangle \,\mathrm{d}s \to 0 \qquad \text{in probability as } n \to \infty;$$

(3) for every $v \in \mathcal{D}(A^*)$ and $t \in [0,T]$, P-almost surely, we have

$$\langle Y(t), v\rangle = \langle Y_0, v\rangle + \int_0^t \langle Y(s), A^*v\rangle \,\mathrm{d}s + L(t)(B^*v). \tag{3.2}$$

Theorem 4.3 in [14] shows that there exists a weak solution of the stochastic Cauchy problem (3.1) on an interval $[0,T]$ if and only if the map $s \to T(s)B$ is stochastically integrable with respect to L on $[0,T]$, in which case the solution is unique. Together with Lemma 3.1 below it follows that in this case the solution exists on each interval $[0, S]$ and is given by

$$Y(t) = T(t)Y_0 + \int_0^t T(t-s)B\,\mathrm{d}L(s) \qquad \text{for all } t \geqslant 0. \tag{3.3}$$

It remains to establish the following:

Lemma 3.1. If there exists a weak solution for the stochastic Cauchy problem (3.1) on $[0, T]$ for some $T > 0$, then there exists a weak solution on $[0, S]$ for any $S > 0$.

Proof. Choose $M \in \mathbb{N}$ such that $S/M \leqslant T$ and define the cylindrical random variable

$$Z \colon V \to L_P^0(\Omega; \mathbb{R}), \qquad Zv := \int_0^S B^*T^*(s)v\,\mathrm{d}L(s).$$

By [19, Lemma 5.4] and the semigroup property, we obtain for each $v \in V$ that

$$\begin{aligned}
\varphi_Z(v) &= \exp\left(\int_0^S \Psi(B^*T^*(s)v)\,\mathrm{d}s\right) \\
&= \prod_{i=0}^{M-1} \exp\left(\int_{\frac{iS}{M}}^{\frac{(i+1)S}{M}} \Psi(B^*T^*(s)v)\,\mathrm{d}s\right) \\
&= \prod_{i=0}^{M-1} \exp\left(\int_0^{\frac{S}{M}} \Psi\left(B^*T^*\left(s + \frac{iS}{M}\right)v\right)\mathrm{d}s\right) \\
&= \prod_{i=0}^{M-1} \exp\left(\int_0^{\frac{S}{M}} \Psi\left(B^*T^*(s)T^*\left(\frac{iS}{M}\right)v\right)\mathrm{d}s\right).
\end{aligned} \tag{3.4}$$

On the other hand, stochastic integrability of the map $s \mapsto T(s)B$ in $[0, T]$ implies that there exists a genuine probability distribution θ with characteristic function

$$\varphi_\theta(v) = \exp\left(\int_0^{\frac{S}{M}} \Psi\left(B^*T^*(s)v\right)\,\mathrm{d}s\right). \tag{3.5}$$

If for each $i \in \{0, \dots, M-1\}$, the image measure $\theta \circ T\left(\frac{iS}{M}\right)^{-1}$ is denoted by λ_i and $\lambda := \lambda_0 * \cdots * \lambda_{M-1}$, then it follows from (3.4) and (3.5) that

$$\varphi_\lambda(v) = \prod_{i=0}^{M-1} \varphi_\theta\left(T^*\left(\tfrac{iS}{M}\right)v\right) = \varphi_Z(v) \qquad \text{for all } v \in V.$$

Theorem IV.2.5 in [23] implies that Z is induced by a genuine V-valued random variable. Hence $s \mapsto T(s)B$ is stochastically integrable in $[0, S]$ which completes the proof by Theorem 4.3 in [14]. □

In the rest of this article we assume that the map $s \mapsto T(s)B$ is stochastically integrable with respect to L in $[0,T]$ for some (and hence each) $T>0$. In this case $\int_0^t T(s)B \,\mathrm{d}L(s)$ is an infinitely divisible, V-valued random variable and we define

$$\nu_t := \mathscr{L}\left(\int_0^t T(s)B \,\mathrm{d}L(s)\right) \qquad \text{for all } t \geqslant 0.$$

If (a,Q,μ) denotes the cylindrical characteristics of L, then according to [19, Le. 5.4] the usual semimartingale characteristics (c_t, S_t, ξ_t) of ν_t are given by

$$\langle c_t, v\rangle = \int_0^t a(B^*T^*(s)v)\,\mathrm{d}s + \int_V \langle h,v\rangle \big(\mathbb{1}_{B_V}(h) - \mathbb{1}_{B_{\mathbb{R}}}(\langle h,v\rangle)\big)\,\xi_t(\mathrm{d}h), \tag{3.6}$$

$$\langle v, S_t v\rangle = \int_0^t \langle B^*T^*(s)v, QB^*T^*(s)v\rangle \,\mathrm{d}s, \tag{3.7}$$

$$\xi_t = (\mathrm{leb}\otimes\mu)\circ\chi_t^{-1} \quad \text{on } \mathcal{Z}(V), \tag{3.8}$$

where $\chi_t \colon [0,\infty)\times U \to V$ is defined by $\chi_t(s,u) := \mathbb{1}_{[0,t]}(s)T(s)Bu$.

A probability measure ν on $\mathfrak{B}(V)$ is called a *stationary measure* for the process $(Y(t): t\geqslant 0)$ defined in (3.3) if it satisfies

$$\nu = T_t\nu * \nu_t \qquad \text{for all } t \geqslant 0, \tag{3.9}$$

where $T_t\nu$ denotes the forward measure $\nu\circ(T(t))^{-1}$. Equivalently, a measure satisfying (3.9) is also called an *operator self-decomposable measure.*

A stationary measure can also be defined as the invariant measure for the generalised Mehler semigroup of the process Y. The concept of a generalised Mehler semigroup has been studied in detail in [5] for the Gaussian case and [11] for the non-Gaussian case. First, we need to know that the family $(\nu_t : t \geqslant 0)$ defines a skew-convolution semi-group:

Lemma 3.2. The family $(\nu_t : t\geqslant 0)$ of probability measures on $\mathfrak{B}(V)$ satisfies

$$\nu_{t+s} = T_t\nu_s * \nu_t \qquad \text{for all } s,t \geqslant 0. \tag{3.10}$$

Proof. Let $\varphi_{T_t\nu_s*\nu_t} \colon V \to \mathbb{C}$ denotes the characteristic function of the probability measure $T_t\nu_s * \nu_t$. For each $v\in V$ and $s,t\geqslant 0$, we obtain,

$$\begin{aligned}\varphi_{T_t\nu_s*\nu_t}(v) &= \varphi_{\nu_s}(T^*(t)v)\varphi_{\nu_t}(v)\\ &= \exp\left(\int_0^s \Psi\left(B^*T^*(r+t)v\right)\,\mathrm{d}r + \int_0^t \Psi\left(B^*T^*(r)v\right)\,\mathrm{d}r\right)\end{aligned}$$

$$= \exp\left(\int_0^{t+s} \Psi\left(B^* T^*(r)v\right)\,\mathrm{d}r\right)$$
$$= \varphi_{\nu_{t+s}}(v),$$

which establishes (3.10). □

The generalised Mehler semigroup $(P_t : t \geqslant 0)$ for the family $(\nu_t : t \geqslant 0)$ is defined by

$$P_t \colon B_b(V) \to B_b(V), \qquad P_t f(v) = \int_V f\big(T(t)v + h\big)\,\nu_t(\mathrm{d}h),$$

where $B_b(V)$ denotes the space of all bounded and Borel measurable functions on V. The generalised Mehler semigroup is a semigroup by [5, Prop. 2.2] because $(\nu_t : t \geqslant 0)$ is a skew-convolution semigroup by Lemma 3.2. A measure ν is called an *invariant measure for the transition semigroup* $(P_t : t \geqslant 0)$ if for all $f \in B_b(V)$ and $t \geqslant 0$,

$$\int_V P_t f(v)\,\nu(\mathrm{d}v) = \int_V f(v)\,\nu(\mathrm{d}v). \tag{3.11}$$

The following equivalence result is from [1, Theorem 2.1], whose proof identically applies to the cylindrical case.

Theorem 3.3. *The following are equivalent for a measure ν on $\mathfrak{B}(V)$:*

(a) *ν is a stationary measure for the process* (3.3), *i.e. it satisfies* (3.9)*;*

(b) *ν is an invariant measure for the generalised Mehler semigroup* $(P_t : t \geqslant 0)$.

(c) *if Y_0 has probability distribution ν then the process* $(Y(t) : t \geqslant 0)$ *defined in* (3.3) *is strictly stationary.*

Proof. See Theorem 2.1 in [1]. □

A natural candidate for a stationary measure is the limit of ν_t in $\mathcal{M}(V)$ as $t \to \infty$. The following result relates the limit to the stochastic integral:

Lemma 3.4. The following conditions are equivalent:

(a) $(\nu_t : t \geqslant 0)$ converges in $\mathcal{M}(V)$ as $t \to \infty$;

(b) $\left(\int_0^t T(s)B\,\mathrm{d}L(s) : t \geqslant 0\right)$ converges in $L^0_P(\Omega; V)$ as $t \to \infty$.

In this case, the probability distribution of the limit in (b) coincides with the limit in (a).

Proof. Note that the process $\left(\int_0^t T(s)B\,\mathrm{d}L(s) : t \geqslant 0\right)$ has independent increments which follows from the definition of the stochastic integral as a limit of stochastic integrals of simple integrands. Consequently, Lemma A.2.1 in [13] guarantees that convergence in probability and weak convergence coincide. □

Lemma 3.5. If $(\nu_t : t \geqslant 0)$ converges to ν in $\mathcal{M}(V)$ as $t \to \infty$, then it follows that:

(a) the limit ν is a stationary measure for the process (3.3);

(b) any stationary measure λ for (3.3) has the form $\lambda = \beta * \nu$, where β is a probability measure satisfying $\beta = T_t\beta$ for all $t \geqslant 0$.

Proof. Lemma 3.2 guarantees $\nu_{t+s} = T_t\nu_s * \nu_t$ for any $s, t \geqslant 0$. By taking limit as $s \to \infty$, we obtain $\nu = T_t\nu * \nu_t$ for all $t \geqslant 0$, which proves (a). To establish (b), we follow the arguments in [9, Prop. 3.2]. Let λ be an invariant measure for (3.3) and $(t_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^+$ a sequence converging to ∞. By the definition of the invariant measure, we have

$$\lambda = T_{t_n}\lambda * \nu_{t_n} \qquad \text{for all } n \in \mathbb{N}\,. \tag{3.12}$$

Since $(\nu_{t_n} : n \in \mathbb{N})$ is relatively compact in $\mathcal{M}(V)$, and $\{\lambda\}$ is trivially relatively compact, Theorem III.2.1 in [16] guarantees that the sequence $(T_{t_n}\lambda : n \in \mathbb{N})$ is relatively compact in $\mathcal{M}(V)$. As a consequence of infinite divisibility of distributions ν and ν_t, we obtain $\varphi_\nu(v) \neq 0$ and $\varphi_{\nu_t}(v) \neq 0$ for all $v \in V$. It follows by (3.12) that,

$$\varphi_{T_{t_n}\lambda}(v) = \frac{\varphi_\lambda(v)}{\varphi_{\nu_{t_n}}(v)} \to \frac{\varphi_\lambda(v)}{\varphi_\nu(v)} \qquad \text{as } n \to \infty.$$

Hence, since $(t_n)_{n\in\mathbb{N}}$ is an arbitrary sequence, Lemma VI.2.1 in [16] implies that $(T_t\lambda : t \geqslant 0)$ converges weakly to some probability measure β, and thus we obtain $\lambda = \beta * \nu$ by (3.12). Using that both λ and ν are stationary measures for (3.3), we conclude

$$\beta * \nu = \lambda = T_t\lambda * \nu_t = T_t(\beta * \nu) * \nu_t = T_t\beta * (T_t\nu * \nu_t) = T_t\beta * \nu.$$

Consequently, $\varphi_\beta(v)\varphi_\nu(v) = \varphi_{T_t\beta}(v)\varphi_\nu(v)$ for all $v \in V$. Since $\varphi_\nu(v) \neq 0$ for all $v \in V$, we derive $\varphi_\beta(v) = \varphi_{T_t\beta}(v)$ implying $\beta = T_t\beta$. □

By Lemma 3.5, if the sequence $(\nu_t : t \geqslant 0)$ converges in $\mathcal{M}(V)$ then its limit is a stationary measure. Thus, conditions for the convergence of $(\nu_t : t \geqslant 0)$ provide conditions for the existence of a stationary measure. Later in the case of stable semigroups, we will see that the converse implication is also true, i.e. the existence of a stationary measure implies convergence of $(\nu_t : t \geqslant 0)$.

Theorem 3.6. *The sequence* $(\nu_t : t \geqslant 0)$ *converges in* $\mathcal{M}(V)$ *as* $t \to \infty$ *if and only if the characteristics of* ν_t *defined in* (3.6) - (3.8) *satisfy the following conditions:*

(a) *The limit of* c_t *exists in* V *as* $t \to \infty$ *where* c_t *is defined in* (3.6); (3.13)

(b) $$\int_0^\infty \operatorname{tr}\big[T(s)BQB^*T^*(s)\big]\,\mathrm{d}s < \infty; \tag{3.14}$$

(c) $$\sup_{n\geqslant 1} \int_0^\infty \int_U \left(\sum_{k=1}^n \langle u, B^*T^*(s)h_k\rangle^2 \wedge 1\right) \mu(\mathrm{d}u)\,\mathrm{d}s < \infty; \tag{3.15}$$

(d) $$\limsup_{m\to\infty} \sup_{n\geqslant m} \int_0^\infty \int_U \left(\sum_{k=m}^n \langle u, B^*T^*(s)h_k\rangle^2 \wedge 1\right) \mu(\mathrm{d}u)\,\mathrm{d}s = 0. \tag{3.16}$$

The limit of $(\nu_t : t \geqslant 0)$, if it exists, can be considered as the distribution of the improper stochastic integral $\int_0^\infty T(s)B\,\mathrm{d}L(s)$. Thus, naturally the proof of the theorem above resembles some arguments for establishing the existence of the integral $\int_0^T T(s)B\,\mathrm{d}L(s)$ for a finite time horizon from [19]. But since the underlying distribution of the driving noise L exists only in a generalised sense as a cylindrical measure, guaranteeing the existence and identifying the limit of the Levy measures of ν_t require some technical arguments, which we will provide in a few lemmas preceding the proof of Theorem 3.6. For this purpose we define a cylindrical measure ξ_∞ by

$$\xi_\infty := (\text{leb} \otimes \mu) \circ \chi_\infty^{-1} \colon \mathcal{Z}(V) \to [0,\infty], \tag{3.17}$$

where $\chi_\infty \colon [0,\infty) \times U \to V$ is defined by $\chi_\infty(s,u) := T(s)Bu$. The canonical projection onto the finite dimensional subspace is denoted by π_n, i.e.

$$\pi_n \colon V \to V, \qquad \pi_n(v) = \sum_{k=1}^n \langle v, h_k\rangle h_k,$$

where $(h_k)_{k\in\mathbb{N}}$ is an orthonormal basis of V.

Lemma 3.7. If (3.15) holds, then it follows that

$$\int_V \big(\langle h, v\rangle^2 \wedge 1\big)\, \xi_\infty(\mathrm{d}h) < \infty \qquad \text{for all } v \in V.$$

Proof. For any $v \in V$ and $n \in \mathbb{N}$ we obtain by the Cauchy-Schwarz inequality that

$$\int_0^\infty \int_U \big(\langle T(s)Bu, \pi_n(v)\rangle^2 \wedge 1\big)\, \mu(\mathrm{d}u)\,\mathrm{d}s$$

$$= \int_0^\infty \int_U \left(\left(\sum_{k=1}^n \langle T(s)Bu, h_k \rangle \langle v, h_k \rangle \right)^2 \wedge 1 \right) \mu(\mathrm{d}u)\,\mathrm{d}s$$

$$\leqslant \max\{1, \|v\|^2\} \int_0^\infty \int_U \left(\sum_{k=1}^n \langle u, B^*T^*(s)h_k \rangle^2 \wedge 1 \right) \mu(\mathrm{d}u)\,\mathrm{d}s.$$

Assumption (3.15) implies

$$\sup_{n \geqslant 1} \int_0^\infty \int_U \left(\langle T(s)Bu, \pi_n(v) \rangle^2 \wedge 1 \right) \mu(\mathrm{d}u)\,\mathrm{d}s < \infty. \tag{3.18}$$

Since for any sequence $(u_n)_{n\in\mathbb{N}} \subseteq U$ satisfying $u_n \to u$ in U, the finite measures $(|\beta|^2 \wedge 1)\,(\mu \circ \langle \cdot, u_n \rangle^{-1})$ converge weakly to $(|\beta|^2 \wedge 1)\,(\mu \circ \langle \cdot, u \rangle^{-1})$ according to [18, Lemma 4.4], it follows that

$$\begin{aligned}
&\lim_{n\to\infty} \int_U \left(\langle T(s)Bu, \pi_n(v) \rangle^2 \wedge 1 \right) \mu(\mathrm{d}u) \\
&\qquad = \lim_{n\to\infty} \int_{\mathbb{R}} \left(|\beta|^2 \wedge 1 \right) \left(\mu \circ \langle \cdot, B^*T^*(s)\pi_n(v) \rangle^{-1} \right) (\mathrm{d}\beta) \\
&\qquad = \int_{\mathbb{R}} \left(|\beta|^2 \wedge 1 \right) \left(\mu \circ \langle \cdot, B^*T^*(s)v \rangle^{-1} \right) (\mathrm{d}\beta) \\
&\qquad = \int_U \left(\langle T(s)Bu, v \rangle^2 \wedge 1 \right) \mu(\mathrm{d}u).
\end{aligned}$$

Consequently, Fatou's lemma guarantees for each $v \in V$ that

$$\begin{aligned}
\int_V \left(\langle h, v \rangle^2 \wedge 1 \right) \xi_\infty(\mathrm{d}h) &= \int_0^\infty \int_U \left(\langle T(s)Bu, v \rangle^2 \wedge 1 \right) \mu(\mathrm{d}u)\,\mathrm{d}s \\
&\leqslant \liminf_{n\to\infty} \int_0^\infty \int_U \left(\langle T(s)Bu, \pi_n(v) \rangle^2 \wedge 1 \right) \mu(\mathrm{d}u)\,\mathrm{d}s.
\end{aligned}$$

Applying (3.18) completes the proof. □

Lemma 3.8. If (3.15) holds, then the mapping

$$f\colon V \to \mathbb{C}, \qquad f(v) := \int_V \left(\cos(\langle h, v \rangle) - 1 \right) \xi_\infty(\mathrm{d}h)$$

satisfies $f(\pi_n v) \to f(v)$ as $n \to \infty$ for each $v \in V$.

Proof. We first note that by monotone convergence theorem and (3.15), it follows that

$$\int_0^\infty \sup_{n\geqslant 1} \int_U \left(\sum_{k=1}^n \langle u, B^*T^*(s)h_k \rangle^2 \wedge 1 \right) \mu(\mathrm{d}u)\,\mathrm{d}s$$

$$= \sup_{n\geqslant 1}\int_0^\infty \int_U \left(\sum_{k=1}^n \langle u, B^*T^*(s)h_k\rangle^2 \wedge 1\right) \mu(\mathrm{d}u)\,\mathrm{d}s < \infty. \tag{3.19}$$

Let $v \in V$ be fixed and define for each $n \in \mathbb{N}$ the function

$$r_n\colon [0,\infty) \to \mathbb{C}, \qquad r_n(s) := \int_U \left(\cos\left(\langle T(s)Bu, \pi_n v\rangle\right) - 1\right) \mu(\mathrm{d}u).$$

It follows that

$$f(\pi_n v) = \int_0^\infty \int_U \left(\cos(\langle T(s)Bu, \pi_n v\rangle) - 1\right) \mu(\mathrm{d}u)\,\mathrm{d}s = \int_0^\infty r_n(s)\,\mathrm{d}s. \tag{3.20}$$

Define the bounded and continuous function

$$g\colon \mathbb{R} \to \mathbb{C}, \qquad g(\beta) = \begin{cases} \frac{\cos(\beta)-1}{\beta^2\wedge 1}, & \text{if } \beta \neq 0, \\ -\frac{1}{2}, & \text{if } \beta = 0. \end{cases}$$

Lemma 4.4 in [18] implies that

$$\begin{aligned}
\lim_{n\to\infty} r_n(s) &= \lim_{n\to\infty} \int_U g\left(\langle u, B^*T^*(s)\pi_n v\rangle\right)\left(\langle u, B^*T^*(s)\pi_n v\rangle^2 \wedge 1\right) \mu(\mathrm{d}u) \\
&= \lim_{n\to\infty} \int_{\mathbb{R}} g(\beta)\left(|\beta|^2 \wedge 1\right) \left(\mu \circ \langle \cdot, B^*T^*(s)\pi_n v\rangle^{-1}\right)(\mathrm{d}\beta) \\
&= \int_{\mathbb{R}} g(\beta)\left(|\beta|^2 \wedge 1\right) \left(\mu \circ \langle \cdot, B^*T^*(s) v\rangle^{-1}\right)(\mathrm{d}\beta) \\
&= \int_U \left(\cos(\langle T(s)Bu, v\rangle) - 1\right) \mu(\mathrm{d}u).
\end{aligned} \tag{3.21}$$

For each $n \in \mathbb{N}$ and $s \geqslant 0$, we obtain by the Cauchy-Schwarz inequality that

$$\begin{aligned}
|r_n(s)| &= \left|\int_U g\big(\langle u, B^*T^*(s)\pi_n v\rangle\big)\left(\langle u, B^*T^*(s)\pi_n v\rangle^2 \wedge 1\right) \mu(\mathrm{d}u)\right| \\
&\leqslant \|g\|_\infty \int_U \left(\langle \pi_n T(s)Bu, v\rangle^2 \wedge 1\right) \mu(\mathrm{d}u) \\
&\leqslant \|g\|_\infty \max\{1, \|v\|^2\} \int_U \left(\sum_{k=1}^n \langle T(s)Bu, h_k\rangle^2 \wedge 1\right) \mu(\mathrm{d}u).
\end{aligned} \tag{3.22}$$

In view of (3.19), (3.21) and (3.22), applying Lebesgue's dominated convergence theorem to (3.20) completes the proof. □

Lemma 3.9. The following conditions are equivalent:

(a) the cylindrical measure ξ_∞ defined in (3.17) extends to a Lévy measure on $\mathfrak{B}(V)$.

(b) Conditions (3.15) and (3.16) are satisfied.

Proof. (a) $\Rightarrow$ (b). The result follows by making use of the definition of a Lévy measure, monotone convergence theorem and Lebesgue's theorem.
(b) $\Rightarrow$ (a). For any $N \in \mathbb{N}$ let $\rho_N := (\xi_\infty + \xi_\infty^-) \circ \pi_N^{-1}$, where $\xi_\infty^-(C) := \xi_\infty(-C)$ for all $C \in \mathcal{Z}(V)$. Then ρ_N extends to a measure as π_N is Hilbert-Schmidt, and satisfies

$$\int_V \left(\|v\|^2 \wedge 1\right) \, \rho_N(\mathrm{d}v) = 2 \int_V \left(\sum_{k=1}^N \langle v, h_k \rangle^2 \wedge 1 \right) \xi_\infty(\mathrm{d}v) < \infty.$$

Consequently, ρ_N is a genuine Lévy measure on $\mathfrak{B}(V)$. The Lévy-Khinchine Theorem implies that there exists an infinitely divisible probability measure θ_N on $\mathfrak{B}(V)$ with characteristic function

$$\varphi_{\theta_N} \colon V \to \mathbb{C}, \qquad \varphi_{\theta_N}(v) := \exp\left(\int_V \left(\cos\left(\langle v, h\rangle\right) - 1\right) \, \rho_N(\mathrm{d}h) \right).$$

By an application of the inequality $1 - \cos\beta \leqslant 2(\beta^2 \wedge 1)$ for all $\beta \in \mathbb{R}$, it follows that for every $v \in V$ we have

$$\begin{aligned} 1 - \varphi_{\theta_N}(v) &= 1 - \exp\left(\int_V \cos(\langle v, h \rangle - 1) \, \rho_N(\mathrm{d}h) \right) \\ &\leqslant \int_V (1 - \cos(\langle v, h\rangle) \, \rho_N(\mathrm{d}h) \leqslant 2 \int_V (\langle h, v\rangle^2 \wedge 1) \, \rho_N(\mathrm{d}h). \end{aligned}$$

By denoting the standard normal distribution on $\mathfrak{B}(\mathbb{R}^m)$ by γ_m, we obtain for every $m, n \in \mathbb{N}$ with $m \leqslant n$ and $N \in \mathbb{N}$ that

$$\begin{aligned} &\int_{\mathbb{R}^{n-m+1}} (1 - \mathrm{Re}\varphi_{\theta_N}(\beta_m h_m + \dots + \beta_n h_n)) \, \gamma_{n-m+1}(\,\mathrm{d}\beta_m, \dots, \,\mathrm{d}\beta_n) \\ &\leqslant 2 \int_{\mathbb{R}^{n-m+1}} \int_V \left(\left| \sum_{k=m}^n \beta_k \langle h, h_k \rangle \right|^2 \wedge 1 \right) \rho_N(\mathrm{d}h) \gamma_{n-m+1}(\,\mathrm{d}\beta_m, \dots, \,\mathrm{d}\beta_n) \\ &\leqslant 2 \int_V \left(\left(\int_{\mathbb{R}^{n-m+1}} \left| \sum_{k=m}^n \beta_k \langle h, h_k \rangle \right|^2 \gamma_{n-m+1}(\,\mathrm{d}\beta_m, \dots, \,\mathrm{d}\beta_n) \right) \wedge 1 \right) \rho_N(\mathrm{d}h) \\ &= 2 \int_V \left(\sum_{k=m}^n \langle h, h_k \rangle^2 \wedge 1 \right) \rho_N(\mathrm{d}h) \end{aligned}$$

$$
\begin{aligned}
&= 2\int_V \left(\sum_{k=m}^{n} \langle \pi_N h, h_k\rangle^2 \wedge 1\right) (\xi_\infty + \xi_\infty^-)(\mathrm{d}h) \\
&\leqslant 4\int_V \left(\sum_{k=m}^{n} \langle h, h_k\rangle^2 \wedge 1\right) \xi_\infty(\mathrm{d}h) \\
&= 4\int_0^\infty \int_U \left(\sum_{k=m}^{n} \langle u, B^*T^*(s)h_k\rangle^2 \wedge 1\right) \mu(\mathrm{d}u)\,\mathrm{d}s. \qquad (3.23)
\end{aligned}
$$

Furthermore, since $\theta_N \circ \pi_n^{-1} = \theta_n$ for all $N \geqslant n$ it follows that $\{\theta_N \circ \pi_n^{-1} : N \in \mathbb{N}\}$ is relatively compact for all $n \in \mathbb{N}$. This together with applying Condition (3.16) to (3.23) shows by Lemma VI.2.3 in [16] that the family $\{\theta_N : N \in \mathbb{N}\}$ is relatively compact in $\mathcal{M}(V)$. Furthermore, Lemma 3.8 implies for each $v \in V$ that

$$
\begin{aligned}
\lim_{N\to\infty} \varphi_{\theta_N}(v) &= \lim_{N\to\infty} \exp\left(\int_V (\cos(\langle v, h\rangle) - 1)\, \rho_N(\mathrm{d}h)\right) \\
&= \lim_{N\to\infty} \exp\left(\int_V (\cos(\langle \pi_N v, h\rangle) - 1)\, (\xi_\infty + \xi_\infty^-)(\mathrm{d}h)\right) \\
&= \exp\left(\int_V (\cos(\langle v, h\rangle) - 1)\, (\xi_\infty + \xi_\infty^-)(\mathrm{d}h)\right).
\end{aligned}
$$

It follows by [16, Lemma VI.2.1] that $(\theta_N)_{N\in\mathbb{N}}$ converges weakly to an infinitely divisible probability measure θ and the characteristic function φ_θ of θ is given by

$$
\varphi_\theta(v) = \exp\left(\int_V (\cos(\langle v, h\rangle) - 1)\, (\xi_\infty + \xi_\infty^-)(\mathrm{d}h)\right) \qquad \text{for all } v \in V.
$$

Consequently, $\xi_\infty + \xi_\infty^-$ extends to the Lévy measure of θ. Since

$$
\xi_\infty(C) \leqslant \xi_\infty(C) + \xi_\infty^-(C) \qquad \text{for all } C \in \mathcal{Z}(V),
$$

Theorem 3.4 in [19] implies that ξ_∞ extends to a Lévy measure on $\mathfrak{B}(V)$, which completes the proof. □

Proof of Theorem 3.6. Sufficiency: suppose that (3.13)–(3.16) hold. We first show that the family $(\nu_t : t \geqslant 0)$ of infinitely divisible probability measures with characteristics (c_t, S_t, ξ_t) is relatively compact in $\mathcal{M}(V)$, for which we use the compactness criterion for infinitely divisible probability measures as given in [16, Th. VI.5.3]. We only need to show that the set $(\xi_t : t \geqslant 0)$ restricted to the complement of any neighbourhood of the origin is relatively compact and the operators $R_t\colon V \to V$ defined by

$$
\langle R_t v, v\rangle := \langle S_t v, v\rangle + \int_{\|h\|\leqslant 1} \langle v, h\rangle^2\, \xi_t(\mathrm{d}h) \qquad (3.24)
$$

satisfy

$$\sup_{t\geqslant 0}\sum_{k=1}^{\infty}\langle R_t h_k, h_k\rangle < \infty \qquad \text{and} \qquad \lim_{m\to\infty}\sup_{t\geqslant 0}\sum_{k=m}^{\infty}\langle R_t h_k, h_k\rangle = 0.$$

This can be proved by following the same arguments as in the proof of part (ii) of Proposition 4.6 in [14] where we replace T by ∞ in the arguments and use (3.15) and (3.16).

Since $t \mapsto \operatorname{tr}(S_t)$ is increasing, Condition (3.14) implies that the operator $S_\infty := \int_0^\infty T(s)BQB^*T(s)\,\mathrm{d}s$ is well-defined and satisfies

$$\langle S_t v, v\rangle \to \langle S_\infty v, v\rangle \qquad \text{for all } v \in V. \tag{3.25}$$

Since $(\xi_t)_{t\geqslant 0}$ is a family of Lévy measures increasing to the Lévy measure ξ_∞ due to Lemma 3.9, we obtain by [11, Le. 3.3] for each $v \in V$ that

$$\int_V \left(e^{i\langle h,v\rangle} - 1 - i\langle h, v\rangle \mathbb{1}_{B_V}(v)\right)\xi_t(\mathrm{d}h) \\ \to \int_V \left(e^{i\langle h,v\rangle} - 1 - i\langle h, v\rangle \mathbb{1}_{B_V}(v)\right)\xi_\infty(\mathrm{d}h), \tag{3.26}$$

as $t \to \infty$. It follows from (3.13), (3.25) and (3.26) that the characteristic function φ_{ν_t} of ν_t converges to the characteristic function φ_ν of an infinitely divisible measure ν with characteristics $(c_\infty, S_\infty, \xi_\infty)$. Together with relative compactness of $(\nu_t : t \geqslant 0)$, Lemma VI.2.1 in [16] guarantees that $(\nu_t : t \geqslant 0)$ converges in $\mathcal{M}(V)$.

Necessity: if $(\nu_t : t \geqslant 0)$ converges weakly as $t \to \infty$ then (3.13)-(3.16) follow by the compactness criterion of infinitely divisible probability measures in Hilbert spaces as applied before. □

Example 3.10. In this example, we assume that $U = V$ and $B = \mathrm{Id}$ in equation (3.1). Let L be the canonical α-stable cylindrical Lévy process for $\alpha \in (0, 2)$, which is defined in [20] by requiring that its characteristic function is of the form

$$\varphi_{L(t)}(u) = \exp\left(-t\|u\|^\alpha\right) \qquad \text{for all } u \in U,\, t \geqslant 0.$$

Assume that there exists an orthonormal basis $(e_k)_{k\in\mathbb{N}}$ of U and an increasing sequence $(\lambda_k)_{k\in\mathbb{N}} \subseteq [0,\infty)$ with $T^*(t)e_k = e^{-\lambda_k t}e_k$ for all $t \geqslant 0$ and $k \in \mathbb{N}$. According to Theorem 4.1 in [20], the semigroup $(T(t))_{t\geqslant 0}$ is stochastically integrable on $[0, T]$ with respect to L if and only if

$$\int_0^T \|T(s)\|_{\mathrm{HS}}^\alpha\,\mathrm{d}s < \infty,$$

where $\|\cdot\|_{\mathrm{HS}}$ denotes the Hilbert-Schmidt norm. Theorem 3.6 guarantees that there exists a stationary solution if and only if

$$\int_0^\infty \|T(s)\|_{\mathrm{HS}}^\alpha \,\mathrm{d}s < \infty. \tag{3.27}$$

This can be seen by verifying Conditions (3.15) and (3.16) by similar arguments as exploited in the proof of Theorem 4.1 in [20].

For example, a sufficient assumption for the validity of (3.27) is $\sum_{k=1}^\infty \frac{1}{\lambda_k} < \infty$. This follows since Cauchy-Schwartz inequality implies

$$\begin{aligned}\int_0^\infty \left(\sum_{k=1}^\infty e^{-2\lambda_k s}\right)^{\alpha/2} \mathrm{d}s &\leqslant \left(\int_0^\infty e^{-\frac{\alpha}{2-\alpha}\lambda_1 s}\,\mathrm{d}s\right)^{\frac{2-\alpha}{2}} \left(\int_0^\infty \sum_{k=1}^\infty e^{-\lambda_k s}\,\mathrm{d}s\right)^{\alpha/2} \\ &= \left(\frac{2-\alpha}{\alpha\lambda_1}\right)^{\frac{2-\alpha}{2}} \left(\sum_{k=1}^\infty \frac{1}{\lambda_k}\right)^{\alpha/2}.\end{aligned}$$

Example 3.11. More specifically, we consider the heat equation on a bounded domain $\mathcal{O}$ in $\mathbb{R}^d$ with smooth boundary for some $d \in \mathbb{N}$ in the setting of the previous Example 3.10. In this case, the generator A is given by the Laplace operator Δ on $U = L^2(\mathcal{O})$ and L is the canonical α-stable cylindrical Lévy process for $\alpha \in (0,2)$. Weyl's law guarantees that the eigenvalues of A satisfy $\lambda_k = c_k k^{2/d}$ for all $k \in \mathbb{N}$, where $c_k \in [a,b]$ for some $a, b > 0$. Example 4.2 in [20] shows that there exists a solution if and only if $\alpha d < 4$. We claim that the same condition is sufficient and necessary for the existence of a stationary solution.

First, suppose $\alpha d < 4$. By the integral test for convergence of series we obtain for each $s > 0$ that

$$\begin{aligned}\|T(s)\|_{\mathrm{HS}}^2 = \sum_{k=1}^\infty e^{-2c_k s k^{2/d}} &\leqslant e^{-as} \sum_{k=1}^\infty e^{-ask^{2/d}} \\ &\leqslant e^{-as}\left(\int_0^\infty e^{-asx^{2/d}}\,\mathrm{d}x\right) = e^{-as}\frac{d\,\Gamma(\frac{d}{2})}{2a^{d/2}s^{d/2}}.\end{aligned}$$

Consequently, Condition (3.27) is satisfied since

$$\begin{aligned}\int_0^\infty \|T(s)\|_{\mathrm{HS}}^\alpha \,\mathrm{d}s &\leqslant c_a\left(\int_0^1 \frac{e^{-\frac{a\alpha s}{2}}}{s^{\frac{\alpha d}{4}}}\,\mathrm{d}s + \int_1^\infty \frac{e^{-\frac{a\alpha s}{2}}}{s^{\frac{\alpha d}{4}}}\,\mathrm{d}s\right) \\ &\leqslant c_a\left(\int_0^1 \frac{1}{s^{\frac{\alpha d}{4}}}\,\mathrm{d}s + \int_1^\infty e^{-\frac{a\alpha s}{2}}\,\mathrm{d}s\right) < \infty,\end{aligned}$$

where $c_a := \left(\frac{d\,\Gamma(\frac{d}{2})}{2a^{d/2}}\right)^{\alpha/2}$. On the other hand, the integral test for series implies

$$\|T(s)\|_{\mathrm{HS}}^2 = \sum_{k=1}^{\infty} e^{-2c_k s k^{d/2}} \geqslant -1 + \int_0^{\infty} e^{-2bsx^{d/2}}\,\mathrm{d}x = -1 + \frac{d\,\Gamma(\frac{d}{2})}{2(2b)^{d/2}s^{d/2}},$$

which results in $\int_0^1 \|T(s)\|_{\mathrm{HS}}^{\alpha}\,\mathrm{d}s = \infty$ for $\alpha d \geqslant 4$.

Remark 3.12. If L is the genuine Lévy process with (classical) characteristics (b, Q, μ), then the cylindrical characteristics of L are given by (a, Q, μ) where

$$a(u^*) = \langle b, u^*\rangle + \int_U \langle u, u^*\rangle \left(\mathbb{1}_{B_{\mathbb{R}}}(\langle u, u^*\rangle) - \mathbb{1}_{B_U}(u)\right)\mu(\mathrm{d}u). \tag{3.28}$$

Then for every $v \in V$, we have by (3.6) and (3.28),

$$\begin{aligned}\langle c_t, v\rangle &= \int_0^t a(B^*T^*(s)v)\,\mathrm{d}s \\ &\quad + \int_0^t \int_U \langle u, B^*T^*(s)v\rangle \big(\mathbb{1}_{B_V}(T(s)Bu) - \mathbb{1}_{B_{\mathbb{R}}}(\langle u, B^*T^*(s)v\rangle)\big)\mu(\mathrm{d}u)\,\mathrm{d}s \\ &= \int_0^t \langle T(s)Bb, v\rangle\,\mathrm{d}s + \int_0^t \int_U \langle T(s)Bu, v\rangle \big(\mathbb{1}_{B_V}(T(s)Bu) - \mathbb{1}_{B_U}(u)\big)\mu(\mathrm{d}u)\,\mathrm{d}s.\end{aligned}$$

As a consequence, we observe that in this case, Theorem 3.6 is equivalent to the well-known result from [9]: the sequence $(\nu_t : t \geqslant 0)$ converges weakly if and only if the following conditions are satisfied:

(i) There exists

$$\lim_{t\to\infty}\left(\int_0^t T(s)Bb\,\mathrm{d}s + \int_0^t \int_U T(s)Bu\big(\mathbb{1}_{B_V}(T(s)Bu) - 1_{B_U}(u)\big)\,\mu(\mathrm{d}u)\,\mathrm{d}s\right);$$

(ii) $$\int_0^{\infty} \mathrm{tr}\left(T(s)BQB^*T^*(s)\right)\,\mathrm{d}s < \infty; \tag{3.29}$$

(iii) $$\int_0^{\infty} \int_U \left(\|T(s)Bu\|^2 \wedge 1\right)\mu(\mathrm{d}u)\,\mathrm{d}s < \infty. \tag{3.30}$$

The equivalence of (3.30) and the Conditions (3.15) and (3.16) can be obtained by noting that in this case μ is a genuine Lévy measure and consequently $\xi_\infty = (\mathrm{leb} \otimes \mu) \circ \chi_{[0,\infty)}^{-1}$ is also a genuine measure. By Lemma 3.9, the Conditions (3.15) and (3.16) are equivalent to the Condition that ξ_∞ is a Lévy measure which is equivalent to (3.30).

It is well known that in general an invariant measure is not necessarily unique. As in the case of genuine Lévy processes, we obtain uniqueness if the semigroup $(T(t))_{t\geqslant 0}$ on V is stable, that is $T(t)v \to 0$ as $t \to \infty$ for each $v \in V$.

Theorem 3.13. *If the semigroup $(T(t))_{t\geqslant 0}$ is stable, then there exists a stationary measure ν for the process* (3.3) *if and only if $(\nu_t)_{t\geqslant 0}$ converges weakly; in this case the limit of $(\nu_t)_{t\geqslant 0}$ equals the stationary measure.*

Proof. Our claim in the cylindrical setting can be proved as in the classical situation; see [9, Prop. 6.1]. □

Combining Theorem 3.13 with Theorem 3.6, we obtain that, if the semigroup is stable, then Conditions (3.13)-(3.16) of Theorem 3.6 are necessary and sufficient for the existence of a stationary measure for the process (3.3), which in this case is unique.

Remark 3.14. In the case of a cylindrical Brownian motion as driving noise, a theory of existence of a solution of the Cauchy problem (3.1) in Banach spaces is developed in [7] and [24] and the existence of an invariant measure is derived in [25]. In the Gaussian setting, both the condition for the existence of a solution and for the existence of an invariant measure can be formulated in terms of the γ-radonifying norm. These results motivate the question whether our approach could be generalised to Banach spaces.

The existence result ([14, Th. 4.3]) in our setting of cylindrical Lévy processes heavily depends on the integration theory of deterministic operators and on a stochastic Fubini result ([14, Th. 3.1]). The integration theory developed in [19] covers the Banach space setting. However, due to the fact that Lévy measures in Banach spaces lack an explicit characterisation, see [15], an integrability condition, and thus a condition for the existence of a solution, can only be formulated in an abstract way. Only in certain Banach spaces in which a characterisation of Lévy measures are known, this condition can be simplified. The proof of the stochastic Fubini result in [14] utilises the radonifying property of Hilbert-Schmidt operators in Hilbert spaces. In a Banach space setting, this naturally leads to the class of radonifying operators, which however might result in an unnecessary restriction. This is due to the fact, that integrable operators need not to be radonifying, in contrast to the Gaussian setting in which integrable operators are necessarily γ-radonifying.

A similar comment applies for extending Theorem 3.6 on existence of a stationary measure to a Banach space setting. This is due to the fact, that Conditions (3.15) and (3.16) originate from the above mentioned characterisation of Lévy measures in Hilbert spaces.

4 The case of exponentially stable semigroups

In general the conditions of Theorem 3.6 may be difficult to verify in practice, in particular Condition (3.13) for the drift component c_t. If the semigroup is exponentially stable, i.e. there exists $C > 1$ and $\lambda > 0$ such that $\|T(t)\| \leqslant Ce^{-\lambda t}$ for all $t \geqslant 0$, and L is a genuine Lévy process, then a sufficient condition for the existence of stationary measure is that the Lévy measure μ of L satisfies the following simple condition

$$\int_U \log^+ \|u\| \, \mu(\mathrm{d}u) < \infty, \tag{4.1}$$

where $\log^+ x := \log x$ if $x \geqslant 1$ and 0 otherwise; see [9, Th. 6.7]. This condition is also necessary if V is finite dimensional (see [2, Th. 4.3.17] and references therein) or if the semigroup $(T(t))_{t\in\mathbb{R}}$ is a group (see [9, Prop. 6.8]) but in general is not necessary (see [8, Ex. 3.15]). In the case of a semigroup $(T(t))_{t\geqslant 0}$ with spectral decomposition $T(t)e_k = e^{-\lambda_k} e_k$ (e.g. the heat semigroup) where the eigenvalues (λ_k) satisfy some mild conditions (see (4.5)), the following weaker condition

$$\int_U \sup_{n\in\mathbb{N}} \left(\frac{\log^+ |\langle u, e_n\rangle|}{\lambda_n} \right) \mu(\mathrm{d}u) < \infty,$$

is shown in [10] to be both necessary and sufficient for the existence of a stationary measure when L is a genuine Lévy process. In the main result of this section, we generalise this condition for the case of cylindrical Lévy processes and give some examples.

Without loss of generality, we assume $U = V$ and $B = \mathrm{Id}$ in the rest of this section. We assume that A is a self-adjoint strictly negative operator with compact resolvent. Consequently, A has a purely point spectrum $(-\lambda_k)_{k\in\mathbb{N}}$, where

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \quad \text{and} \quad \lim_{k\to\infty} \lambda_k = \infty, \tag{4.2}$$

and there is an orthonormal basis $(e_k)_{k\in\mathbb{N}}$ in V consisting of eigenvectors e_k of A corresponding to the eigenvalues $-\lambda_k$. Then A is a generator of the C_0-semigroup $(T(t))_{t\geqslant 0}$ of bounded linear operators on V, given by the formula:

$$T(t)v = \sum_{k=1}^{\infty} e^{-\lambda_k t} \langle v, e_k\rangle e_k \quad \text{for } v \in V. \tag{4.3}$$

Clearly, the semigroup $(T(t))_{t\geqslant 0}$ satisfies

$$\|T(t)v\| \leqslant e^{-\lambda_1 t}\|v\| \quad \text{for all } v \in V \text{ and } t \geqslant 0, \tag{4.4}$$

and is therefore exponentially stable.

Theorem 4.1. *Assume that the semigroup* $(T(t))_{t\geqslant 0}$ *is given by* (4.3) *and satisfies*

$$\sum_{k=1}^{\infty}\frac{1}{\lambda_k}<\infty. \tag{4.5}$$

Then the following are equivalent:

(a) *There exists a stationary measure for the process* (3.3)*;*

(b) (i) $$\sup_{n\geqslant 1}\int_U \max_{1\leqslant k\leqslant n}\left(\frac{\log^+|\langle u,e_k\rangle|}{\lambda_k}\right)\mu(\mathrm{d}u)<\infty; \tag{4.6}$$

(ii) $$\limsup_{m\to\infty}\sup_{n\geqslant m}\int_U \max_{m\leqslant k\leqslant n}\left(\frac{\log^+|\langle u,e_k\rangle|}{\lambda_k}\right)\mu(\mathrm{d}u)=0. \tag{4.7}$$

Proof. (b) $\Rightarrow$ (a). We show that the conditions in Theorem 3.6 are satisfied. The spectral representation of the semigroup (4.3) implies

$$\int_0^\infty \mathrm{tr}(T(s)QT^*(s))\,\mathrm{d}s=\sum_{k=1}^{\infty}\langle Qe_k,e_k\rangle\int_0^\infty e^{-2\lambda_k s}\,\mathrm{d}s\leqslant\frac{1}{2}\|Q\|_{\mathrm{op}}\sum_{k=1}^{\infty}\frac{1}{\lambda_k},$$

which verifies Condition (3.14) due to our assumption(4.5). We next show that (3.15) and (3.16) are satisfied. It follows by Lemma 3.1 in [19] that for any $c>0$,

$$K_c:=\sup_{\|u^*\|\leqslant c}\int_U\left(\langle u,u^*\rangle^2\wedge 1\right)\mu(\mathrm{d}u)<\infty. \tag{4.8}$$

For $k,m,n\in\mathbb{N}$, $m\leqslant n$ and $s\geqslant 0$, we define the following sets

$$C_k(s):=\left\{u:|\langle u,e_k\rangle|<\exp\left(\frac{\lambda_k s}{2}\right)\right\} \tag{4.9}$$
$$B_{m,n}:=\left\{u:\sum_{k=m}^{n}\langle u,e_k\rangle^2\leqslant 1\right\}.$$

Using the spectral decomposition (4.3) of the semigroup, we have

$$\int_0^\infty\int_U\left(\sum_{k=m}^{n}\langle u,T^*(s)e_k\rangle^2\wedge 1\right)\mu(\mathrm{d}u)\,\mathrm{d}s$$
$$=\int_0^\infty\int_U\left(\sum_{k=m}^{n}e^{-2\lambda_k s}\langle u,e_k\rangle^2\wedge 1\right)\mu(\mathrm{d}u)\,\mathrm{d}s$$

$$
\begin{aligned}
&\leqslant \int_0^\infty \int_{B_{m,n}} \left(\sum_{k=m}^{n} e^{-2\lambda_k s} \langle u, e_k \rangle^2 \right) \mu(\mathrm{d}u)\, \mathrm{d}s \\
&\quad + \int_0^\infty \int_{\cap_{j=m}^n C_j(s) \cap B_{m,n}^c} \left(\sum_{k=m}^{n} e^{-2\lambda_k s} \langle u, e_k \rangle^2 \right) \mu(\mathrm{d}u)\, \mathrm{d}s \\
&\quad + \int_0^\infty \int_{\cup_{j=m}^n C_j^c(s) \cap B_{m,n}^c} \mu(\mathrm{d}u)\, \mathrm{d}s \\
&=: I^1_{m,n} + I^2_{m,n} + I^3_{m,n}.
\end{aligned} \tag{4.10}
$$

Since $B_{m,n} \subseteq \cap_{k=m}^n \{u : \langle u, e_k \rangle^2 \leqslant 1\}$, we have

$$
\begin{aligned}
I^1_{m,n} &\leqslant \sum_{k=m}^{n} \left(\int_0^\infty e^{-2\lambda_k s}\, \mathrm{d}s \right) \int_{\{u : \langle u, e_k \rangle^2 \leqslant 1\}} \langle u, e_k \rangle^2\, \mu(\mathrm{d}u) \\
&\leqslant \sum_{k=m}^{n} \left(\frac{1}{2\lambda_k} \right) \sup_{\|u^*\| \leqslant 1} \int_U \left(\langle u, u^* \rangle^2 \wedge 1 \right) \mu(\mathrm{d}u) = \frac{K_1}{2} \sum_{k=m}^{n} \frac{1}{\lambda_k}.
\end{aligned} \tag{4.11}
$$

Using the definition of the sets $C_k(s)$, we obtain

$$
\begin{aligned}
I^2_{m,n} &\leqslant \int_0^\infty \int_{\cap_{j=m}^n C_j(s)} \sum_{k=m}^{n} e^{-\lambda_k s} \left(e^{-\lambda_k s} \langle u, e_k \rangle^2 \wedge 1 \right) \mu(\mathrm{d}u)\, \mathrm{d}s \\
&\leqslant \sum_{k=m}^{n} \left(\int_0^\infty e^{-\lambda_k s}\, \mathrm{d}s \right) \int_U \left(\langle u, e_k \rangle^2 \wedge 1 \right) \mu(\mathrm{d}u) \\
&\leqslant \sum_{k=m}^{n} \frac{1}{\lambda_k} \sup_{\|u^*\| \leqslant 1} \int_U \left(\langle u, u^* \rangle^2 \wedge 1 \right) \mu(\mathrm{d}u) = K_1 \sum_{k=m}^{n} \frac{1}{\lambda_k}
\end{aligned} \tag{4.12}
$$

Noting that $\cup_{k=m}^n C_k^c(s) = \left\{ u : \max_{m \leqslant k \leqslant n} \left(\frac{2 \log^+ |\langle u, e_k \rangle|}{\lambda_k} \right) > s \right\}$, we have

$$
\begin{aligned}
I^3_{m,n} &\leqslant \int_0^\infty \int_{\cup_{k=m}^n C_k^c(s)} \mu(\mathrm{d}u)\, \mathrm{d}s \\
&= \int_0^\infty \mu \left(\left\{ u : \max_{m \leqslant k \leqslant n} \left(\frac{2 \log^+ |\langle u, e_k \rangle|}{\lambda_k} \right) > s \right\} \right) \mathrm{d}s \\
&= 2 \int_U \max_{m \leqslant k \leqslant n} \frac{\log^+ |\langle u, e_k \rangle|}{\lambda_k}\, \mu(\mathrm{d}u),
\end{aligned} \tag{4.13}
$$

where the last equality follows from a Fubini argument; see [4, App. 2] for details. Hence substituting (4.11), (4.12) and (4.13) in (4.10) and using (4.5) and (4.7) verifies

Condition (3.16). Similarly, by setting $m = 1$ in (4.11), (4.12) and (4.13) and using (4.5) and (4.6) verifies Condition (3.15).

It remains to prove that (3.13) is satisfied, that is there exists $\lim_{t\to\infty} c_t$ where for each $v \in V$,

$$\langle c_t, v\rangle := \int_0^t a(T^*(s)v)\,\mathrm{d}s + \int_V \langle h, v\rangle \big(\mathbb{1}_{B_V}(h) - \mathbb{1}_{B_\mathbb{R}}(\langle h, v\rangle)\big)\,\xi_t(\mathrm{d}h). \tag{4.14}$$

We first prove that $\int_0^\infty |a(T^*(s)v)|\,\mathrm{d}s < \infty$ for each $v \in V$. For this we will use the following equality which holds for all $u^* \in U$ and $\beta > 0$ as given by (3.9) in [19],

$$a(\beta u^*) = \beta a(u^*) + \beta \int_U \langle u, u^*\rangle \big(\mathbb{1}_{B_\mathbb{R}}(\beta\langle u, u^*\rangle) - \mathbb{1}_{B_\mathbb{R}}(\langle u, u^*\rangle)\big)\,\mu(\mathrm{d}u). \tag{4.15}$$

Let $\pi_n : U \to U$ be the projection operator defined by $\pi_n(v) := \sum_{k=1}^n \langle v, e_k\rangle e_k$. Then by (4.15) and assuming $\|T^*(s)\pi_n v\| \neq 0$, we obtain

$$\begin{aligned}
&|a(T^*(s)\pi_n v)| \\
&\leqslant \|T^*(s)\pi_n v\| \left| a\left(\frac{T^*(s)\pi_n v}{\|T^*(s)\pi_n v\|}\right)\right| \qquad (4.16)\\
&\quad + \int_U |\langle u, T^*(s)\pi_n v\rangle| \left| \mathbb{1}_{B_\mathbb{R}}(\langle u, T^*(s)\pi_n v\rangle) - \mathbb{1}_{B_\mathbb{R}}\left(\left\langle u, \frac{T^*(s)\pi_n v}{\|T^*(s)\pi_n v\|}\right\rangle\right)\right| \mu(\mathrm{d}u).
\end{aligned}$$

Since a maps bounded sets to bounded sets, it follows by (4.4) that

$$\begin{aligned}
\int_0^\infty \|T^*(s)\pi_n v\| \left| a\left(\frac{T^*(s)\pi_n v}{\|T^*(s)\pi_n v\|}\right)\right| \mathrm{d}s &\leqslant \|v\| \sup_{\|u^*\|\leqslant 1} |a(u^*)| \int_0^\infty e^{-\lambda_1 s}\,\mathrm{d}s \\
&= \frac{\|v\|}{\lambda_1} \sup_{\|u^*\|\leqslant 1} |a(u^*)|. \qquad (4.17)
\end{aligned}$$

Integrating the second term on the right side in (4.16) results in

$$\begin{aligned}
&\int_0^\infty \int_U |\langle u, T^*(s)\pi_n v\rangle| \left| \mathbb{1}_{B_\mathbb{R}}(\langle u, T^*(s)\pi_n v\rangle) - \mathbb{1}_{B_\mathbb{R}}\left(\left\langle u, \frac{T^*(s)\pi_n v}{\|T^*(s)\pi_n v\|}\right\rangle\right)\right| \mu(\mathrm{d}u)\,\mathrm{d}s \\
&\qquad = \int_0^\infty \int_{\{u:|\langle u, T^*(s)\pi_n v\rangle|>1\}\cap\left\{u:\left|\left\langle u, \frac{T^*(s)\pi_n v}{\|T^*(s)\pi_n v\|}\right\rangle\right|\leqslant 1\right\}} |\langle u, T^*(s)\pi_n v\rangle|\ \mu(\mathrm{d}u)\,\mathrm{d}s \\
&\qquad\quad + \int_0^\infty \int_{\{u:|\langle u, T^*(s)\pi_n v\rangle|\leqslant 1\}\cap\left\{u:\left|\left\langle u, \frac{T^*(s)\pi_n v}{\|T^*(s)\pi_n v\|}\right\rangle\right|>1\right\}} |\langle u, T^*(s)\pi_n v\rangle|\ \mu(\mathrm{d}u)\,\mathrm{d}s \\
&\qquad =: I_4 + I_5. \qquad (4.18)
\end{aligned}$$

Using (4.4) we estimate I_4 by

$$\begin{aligned}
I_4 &\leqslant \int_0^\infty \int_{\left\{u:\left|\left\langle u, \frac{T^*(s)\pi_n v}{\|T^*(s)\pi_n v\|}\right\rangle\right|\leqslant 1\right\}} \langle u, T^*(s)\pi_n v\rangle^2 \,\mu(\mathrm{d}u)\,\mathrm{d}s \\
&\leqslant \int_0^\infty \|T^*(s)\pi_n v\|^2 \int_U \left(\left\langle u, \frac{T^*(s)\pi_n v}{\|T^*(s)\pi_n v\|}\right\rangle^2 \wedge 1\right) \mu(\mathrm{d}u)\,\mathrm{d}s \\
&\leqslant \|v\|^2 \sup_{\|u^*\|\leqslant 1} \int_U \big(\langle u, u^*\rangle^2 \wedge 1\big)\,\mu(\mathrm{d}u) \int_0^\infty e^{-2\lambda_1 s}\,\mathrm{d}s = \frac{\|v\|^2}{2\lambda_1} K_1.
\end{aligned} \tag{4.19}$$

If $C_k(s)$ denotes the set defined in (4.9), we obtain

$$\begin{aligned}
I_5 \leqslant& \int_0^\infty \int_{\left\{u:\left|\left\langle u, \frac{T^*(s)\pi_n v}{\|T^*(s)\pi_n v\|}\right\rangle\right|>1\right\}\cap(C_1(s)\cap\dots\cap C_n(s))} |\langle u, T^*(s)\pi_n v\rangle| \,\mu(\mathrm{d}u)\,\mathrm{d}s \\
&+ \int_0^\infty \int_{\left\{u:\left|\left\langle u, \frac{T^*(s)\pi_n v}{\|T^*(s)\pi_n v\|}\right\rangle\right|>1\right\}\cap(C_1^c(s)\cup\dots\cup C_n^c(s))} \mu(\mathrm{d}u)\,\mathrm{d}s \\
\leqslant& \int_0^\infty \int_{\left\{u:\left|\left\langle u, \frac{T^*(s)\pi_n v}{\|T^*(s)\pi_n v\|}\right\rangle\right|>1\right\}\cap(C_1(s)\cap\dots\cap C_n(s))} \sum_{k=1}^n |\langle u, e_k\rangle||\langle v, e_k\rangle| e^{-\lambda_k s}\,\mu(\mathrm{d}u)\,\mathrm{d}s \\
&+ \int_0^\infty \mu\big(C_1^c(s)\cup\dots\cup C_n^c(s)\big)\,\mathrm{d}s \\
=:& I_6 + I_7.
\end{aligned} \tag{4.20}$$

For the integral I_6 we obtain

$$\begin{aligned}
I_6 &\leqslant \sum_{k=1}^n |\langle v, e_k\rangle| \int_0^\infty e^{-\frac{\lambda_k s}{2}} \int_{\left\{u:\left|\left\langle u, \frac{T^*(s)v}{\|T^*(s)v\|}\right\rangle\right|>1\right\}} \mu(\mathrm{d}u)\,\mathrm{d}s \\
&\leqslant 2\|v\| \sup_{\|u^*\|\leqslant 1} \int_U \big(\langle u, u^*\rangle^2 \wedge 1\big)\,\mu(\mathrm{d}u) \sum_{k=1}^n \frac{1}{\lambda_k} \\
&\leqslant 2\|v\| K_1 \sum_{k=1}^\infty \frac{1}{\lambda_k},
\end{aligned} \tag{4.21}$$

which is finite by using (4.5). Using the same equality from [4, App. 2] as in (4.13), we obtain

$$\begin{aligned}
I_7 &\leqslant \int_0^\infty \mu\left(\cup_{k=1}^n \left\{u : |\langle u, e_k\rangle| \geqslant e^{\frac{\lambda_k s}{2}}\right\}\right)\mathrm{d}s \\
&= \int_0^\infty \mu\left(\left\{u : \max_{1\leqslant k\leqslant n} \frac{2}{\lambda_k} \log^+ |\langle u, e_k\rangle| \geqslant s\right\}\right)\mathrm{d}s
\end{aligned}$$

$$= 2 \int_U \max_{1 \leqslant k \leqslant n} \frac{1}{\lambda_k} \log^+ |\langle u, e_k \rangle| \, \mu(\mathrm{d}u). \tag{4.22}$$

Applying (4.6) to (4.22) and using (4.17) – (4.22), it follows from (4.16) that there exists some $C_1 > 0$ such that

$$\sup_{n \in \mathbb{N}} \int_0^\infty |a(T^*(s)\pi_n v)| \, \mathrm{d}s \leqslant C_1(\|v\| + 1).$$

Fatou's Lemma implies that for any $\delta > 0$,

$$\begin{aligned} M_\delta &:= \sup_{\|v\| < \delta} \int_0^\infty |a(T^*(s)v)| \, \mathrm{d}s \\ &\leqslant \sup_{\|v\| < \delta} \liminf_{n \to \infty} \int_0^\infty |a(T^*(s)\pi_n v)| \, \mathrm{d}s \leqslant \sup_{\|v\| < \delta} C_1(\|v\| + 1) < \infty. \end{aligned} \tag{4.23}$$

This proves that $\int_0^\infty a(T^*(s)v) \, \mathrm{d}s$ exists, and, for each $v \in V$, we have

$$\lim_{t \to \infty} \int_0^t a(T^*(s)v) \, ds = \int_0^\infty a(T^*(s)v) \, ds. \tag{4.24}$$

For considering the second term in (4.14), define $f(h, v) := \langle h, v \rangle \left(\mathbb{1}_{B_V}(h) - \mathbb{1}_{B_\mathbb{R}}(\langle h, v \rangle) \right)$. Then for any $h, v \in V$,

$$\begin{aligned} |f(h, v)| &= |\langle h, v \rangle| \mathbb{1}_{B_V}(h) \mathbb{1}_{B_\mathbb{R}^c}(\langle h, v \rangle) + |\langle h, v \rangle| \mathbb{1}_{B_V^c}(h) \mathbb{1}_{B_\mathbb{R}}(\langle h, v \rangle) \\ &\leqslant |\langle h, v \rangle|^2 \mathbb{1}_{B_V}(h) + \mathbb{1}_{B_V^c}(h), \end{aligned}$$

from which the integrability of $f(\cdot, v)$ with respect to ξ_∞ follows by using the properties of Lévy measures. Consequently, since $\xi_t(C) \uparrow \xi_\infty(C)$ as $t \to \infty$ for each $C \in \mathcal{B}(V)$ and ξ_∞ is a Lévy measure due to Lemma 3.9, we obtain by the same arguments as in Lemma 3.3 in [11] that

$$\lim_{t \to \infty} \int_V f(h, v) \, \xi_t(\mathrm{d}h) = \int_V f(h, v) \, \xi_\infty(\mathrm{d}h).$$

Together with (4.24) it follows from (4.14) that $(\langle c_t, v \rangle)_{t \geqslant 0}$ converges for each $v \in V$ and

$$\lim_{t \to \infty} \langle c_t, v \rangle = \int_0^\infty a(T^*(s)v) \, \mathrm{d}s + \int_V f(h, v) \xi_\infty(\mathrm{d}h).$$

To prove that $(c_t)_{t \geqslant 0}$ converges in V, it is enough to show that $(c_t)_{t \geqslant 0}$ is relatively compact in V, which in this case, as $(c_t)_{t \geqslant 0}$ is bounded, reduces to establish

$$\lim_{m \to \infty} \sup_{t \geqslant 0} \sum_{k=m}^\infty \langle c_t, e_k \rangle^2 = 0. \tag{4.25}$$

Using (4.23), Cauchy-Schwarz inequality and the fact that $\xi_t \leqslant \xi_\infty$, we obtain

$$\begin{aligned}
&\langle c_t, e_k\rangle^2 \\
&= \left(\int_0^t a(T^*(s)e_k)\,\mathrm{d}s + \int_V \langle h, e_k\rangle \left(1_{B_V}(h) - 1_{B_{\mathbb{R}}}(\langle h, e_k\rangle)\right)\,\xi_t(\mathrm{d}h)\right)^2 \\
&\leqslant 2\left(\int_0^t |a(T^*(s)e_k)|\,\mathrm{d}s\right)^2 + 2\left(\int_{|\langle h,e_k\rangle|\leqslant 1<\|h\|} \langle h, e_k\rangle\,\xi_t(\mathrm{d}h)\right)^2 \\
&\leqslant 2\left(\sup_{\|v\|\leqslant 1}\int_0^\infty |a(T^*(s)v)|\,\mathrm{d}s\right)\int_0^\infty |a(T^*(s)e_k)|\,\mathrm{d}s \\
&\qquad + 2\xi_t(\|h\|>1)\int_{|\langle h,e_k\rangle|\leqslant 1<\|h\|} \langle h, e_k\rangle^2\,\xi_t(\mathrm{d}h) \\
&\leqslant 2M_1\int_0^\infty |a(T^*(s)e_k)|\,\mathrm{d}s + 2\xi_\infty(\{h : \|h\|>1\})\int_{|\langle h,e_k\rangle|\leqslant 1} \langle h, e_k\rangle^2\,\xi_\infty(\mathrm{d}h). \qquad (4.26)
\end{aligned}$$

It follows by (4.15) and Fubini's theorem that

$$\begin{aligned}
&\int_0^\infty |a(T^*(s)e_k)|\,\mathrm{d}s \\
&\qquad\leqslant |a(e_k)|\int_0^\infty e^{-\lambda_k s}\,\mathrm{d}s + \int_0^\infty e^{-\lambda_k s}\int_{1\leqslant|\langle u,e_k\rangle|\leqslant e^{\lambda_k s}} |\langle u, e_k\rangle|\,\mu(\mathrm{d}u)\,\mathrm{d}s \\
&\qquad= \frac{|a(e_k)|}{\lambda_k} + \int_0^\infty e^{-\lambda_k s}\int_{1\leqslant|\beta|\leqslant e^{\lambda_k s}} |\beta|\,\left(\mu\circ\langle\cdot, e_k\rangle^{-1}\right)(\mathrm{d}\beta)\,\mathrm{d}s \\
&\qquad= \frac{|a(e_k)|}{\lambda_k} + \int_{1\leqslant|\beta|} |\beta| \int_{\frac{1}{\lambda_k}\log|\beta|}^\infty e^{-\lambda_k s}\,\mathrm{d}s\,\left(\mu\circ\langle\cdot, e_k\rangle^{-1}\right)(\mathrm{d}\beta) \\
&\qquad= \frac{|a(e_k)|}{\lambda_k} + \frac{1}{\lambda_k}\int_{1\leqslant|\beta|} \left(\mu\circ\langle\cdot, e_k\rangle^{-1}\right)(\mathrm{d}\beta) \\
&\qquad= \frac{|a(e_k)|}{\lambda_k} + \frac{1}{\lambda_k}\mu\left(\{u : |\langle u, e_k\rangle| \geqslant 1\}\right) \\
&\qquad\leqslant \frac{1}{\lambda_k}\left(\sup_{\|u^*\|\leqslant 1}|a(u^*)| + K_1\right). \qquad (4.27)
\end{aligned}$$

Similar application of Fubini's theorem implies for the second term in (4.26) that

$$\begin{aligned}
&\int_{|\langle h,e_k\rangle|\leqslant 1} \langle h, e_k\rangle^2 \xi_\infty(\mathrm{d}h) \\
&= \int_0^\infty e^{-2\lambda_k s}\int_{|\langle u,e_k\rangle|\leqslant e^{\lambda_k s}} |\langle u, e_k\rangle|^2\,\mu(\mathrm{d}u)\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty e^{-2\lambda_k s} \int_{|\langle u,e_k\rangle|\leqslant 1} |\langle u, e_k\rangle|^2 \, \mu(\mathrm{d}u)\, \mathrm{d}s \\
&\qquad + \int_0^\infty e^{-2\lambda_k s} \int_{1<|\langle u,e_k\rangle|\leqslant e^{\lambda_k s}} |\langle u, e_k\rangle|^2 \, \mu(\mathrm{d}u)\, \mathrm{d}s \\
&\leqslant \frac{1}{2\lambda_k} \int_U \left(|\langle u, e_k\rangle|^2 \wedge 1\right) \mu(\mathrm{d}u) + \int_0^\infty e^{-2\lambda_k s} \int_{1<|\beta|\leqslant e^{\lambda_k s}} |\beta|^2 \, \left(\mu \circ \langle \cdot, e_k\rangle^{-1}\right)(\mathrm{d}\beta)\, \mathrm{d}s \\
&= \frac{1}{2\lambda_k} K_1 + \int_{1\leqslant|\beta|} |\beta|^2 \int_{\frac{1}{\lambda_k}\log|\beta|}^\infty e^{-2\lambda_k s}\, \mathrm{d}s \left(\mu \circ \langle \cdot, e_k\rangle^{-1}\right)(\mathrm{d}\beta) \\
&\leqslant \frac{1}{\lambda_k} K_1. \qquad (4.28)
\end{aligned}
$$

Using the estimates obtained in (4.27) and (4.28) in (4.26), it follows that there exists a constant $C_2 > 0$ such that

$$
\sup_{t\geqslant 0} \sum_{k=m}^\infty \langle c_t, e_k\rangle^2 \leqslant C_2 \sum_{k=m}^\infty \frac{1}{\lambda_k},
$$

which implies that $(c_t)_{t\geqslant 0}$ is relatively compact in V and hence (3.13) is satisfied.

(a) $\Rightarrow$ (b). Using the same equality from [4, App. 2] as in (4.13), we have for $m, n \in \mathbb{N}$ with $m \leqslant n$ that

$$
\begin{aligned}
\int_U \max_{m\leqslant k\leqslant n} \frac{\log^+ |\langle u, e_k\rangle|}{\lambda_k} \mu(\mathrm{d}u) &= \int_0^\infty \mu\left(\left\{u : \max_{m\leqslant k\leqslant n} \left(\frac{\log^+ |\langle u, e_k\rangle|}{\lambda_k}\right) > s\right\}\right) \mathrm{d}s \\
&= \int_0^\infty \mu\left(\cup_{k=m}^n \left\{u : e^{-\lambda_k s}|\langle u, e_k\rangle| > 1\right\}\right) \mathrm{d}s \\
&\leqslant \int_0^\infty \mu\left(\sum_{k=m}^n e^{-2\lambda_k s}|\langle u, e_k\rangle|^2 > 1\right) \mathrm{d}s \\
&\leqslant \int_0^\infty \int_U \left(\sum_{k=m}^n \langle u, T^*(s) e_k\rangle^2 \wedge 1\right) \mu(\mathrm{d}u)\, \mathrm{d}s.
\end{aligned}
$$

Since combining Theorem 3.13 with Theorem 3.6 implies Conditions (3.15) and (3.16), the above inequality verifies Conditions (4.6) and (4.7), which completes the proof. □

Remark 4.2. From the proof of Theorem 4.1 it also follows that without assuming (4.5), the following conditions:

(iii) $\displaystyle\sum_{k=1}^\infty \frac{|a(e_k)|}{\lambda_k} < \infty;$ (4.29)

(iv) $$\sum_{k=1}^{\infty} \frac{\langle Qe_k, e_k\rangle}{\lambda_k} < \infty; \tag{4.30}$$

(v) $$\sum_{k=1}^{\infty} \frac{1}{\lambda_k} \int_U \big(\langle u, e_k\rangle^2 \wedge 1\big)\, \mu(\mathrm{d}u) < \infty, \tag{4.31}$$

together with (4.6) and (4.7) are sufficient for the existence of an invariant measure.

Remark 4.3. Let L be a genuine Lévy process with classical characteristics (b, Q, μ). According to Theorem 1 in [10], if the semigroup satisfies (4.2), (4.3) and

$$\sum_{k=1}^{\infty} \frac{e^{-\lambda_k T}}{\lambda_k} < \infty, \tag{4.32}$$

then a necessary and sufficient condition for the existence of a stationary measure for the process (3.3) is given by

$$\int_U \sup_{n\in\mathbb{N}} \left(\frac{\log^+ |\langle u, e_n\rangle|}{\lambda_n} \right) \mu(\mathrm{d}u) < \infty. \tag{4.33}$$

In this case, our Conditions (4.6) and (4.7) are equivalent to (4.33). Since μ is a genuine Lévy measure, the monotone convergence theorem implies

$$\sup_{n\geqslant m} \int_U \max_{m\leqslant k\leqslant n} \left(\frac{\log^+ |\langle u, e_k\rangle|}{\lambda_k} \right) \mu(\mathrm{d}u) = \int_U \sup_{n\geqslant m} \left(\frac{\log^+ |\langle u, e_n\rangle|}{\lambda_n} \right) \mu(\mathrm{d}u),$$

which shows the equivalence of (4.6) and (4.33) by taking $m = 1$. Furthermore, Condition (4.2) implies for each $u \in U$ that

$$\sup_{n\geqslant m} \frac{\log^+ |\langle u, e_n\rangle|}{\lambda_n} \leqslant \sup_{n\geqslant m} \frac{\log^+ \|u\|}{\lambda_n} \leqslant \frac{\log^+ \|u\|}{\lambda_m} \to 0 \text{ as } m \to \infty.$$

An application of Lebesgue's theorem together with (4.33) implies (4.7).

It is well known from the Gaussian case, that a cylindrical driving noise requires stronger conditions for guaranteeing the existence of a solution of (3.1) than a genuine noise. In the same way, the weaker Condition (4.32) suffices to equivalently characterise the existence of a stationary solution by (4.33) in the case of a genuine noise, whereas in the cylindrical case we require Condition (4.5). However, in some special cases such as a symmetric distribution of the driving cylindrical noise L, the proof above can be easily modified by assuming (4.32) instead of (4.5).

Example 4.4. The following specific example of a cylindrical Lévy process is often considered in the literature: let $(\ell_k)_{k\in\mathbb{N}}$ be a sequence of symmetric, independent, real valued Lévy processes with characteristics $(0,0,\mu_k)$, and define

$$L(t)u := \sum_{k=1}^{\infty} \langle e_k, u\rangle \ell_k(t) \qquad \text{for all } u \in U,\ t \geqslant 0. \tag{4.34}$$

If the sum converges for each $u \in U$ and the family of characteristic functions of ℓ_k are equicontinuous in 0, then (4.34) defines a cylindrical Lévy process L; see [19, Le. 4.2].

Assume that the semigroup $(T(t))_{t\geqslant 0}$ satisfies the spectral representation (4.3). Since the independence of the real valued processes $(\ell_k)_{k\in\mathbb{N}}$ implies that the cylindrical Lévy measure μ is concentrated on the axes, Conditions (4.6) and (4.7) reduce to

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k} \int_{\mathbb{R}} \log^+ |\beta|\, \mu_k(\mathrm{d}\beta) < \infty. \tag{4.35}$$

Example 4.5. The authors of [17] consider a cylindrical Lévy process of the form (4.34) with $\ell_k = \sigma_k m_k$ for all $k \in \mathbb{N}$, where $(\sigma_k)_{k\in\mathbb{N}} \subseteq \ell^\infty(\mathbb{R})$ and $(m_k)_{k\in\mathbb{N}}$ is a sequence of identically distributed, independent, symmetric real valued Lévy processes without Gaussian part and with Lévy measure ρ. In this case, Condition (4.35) is equivalent to

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k} \int_{\mathbb{R}} \log^+ |\sigma_k\beta|\, \rho(\mathrm{d}\beta) < \infty. \tag{4.36}$$

It follows from Theorem 4.1 that, if the reciprocal eigenvalues $(1/\lambda_k)_{k\in\mathbb{N}}$ are summable, i.e. satisfy (4.5), then there exists a stationary solution if and only if

$$\int_1^\infty \log\beta\, \rho(\mathrm{d}\beta) < \infty. \tag{4.37}$$

This covers exactly the result in [17].

However, we can improve this result from [17] by directly applying Theorem 3.6. The latter guarantees without assuming that the reciprocal eigenvalues $(1/\lambda_k)_{k\in\mathbb{N}}$ are summable, that there exists a stationary measure if and only if Conditions (3.15) and (3.16) are satisfied which in this case is equivalent to

$$\sum_{k=1}^{\infty} \int_0^\infty \int_{\mathbb{R}} \left(e^{-2\lambda_k s} |\sigma_k\beta|^2 \wedge 1 \right) \rho(\mathrm{d}\beta)\, \mathrm{d}s < \infty. \tag{4.38}$$

Furthermore, Condition (4.38) is satisfied if and only if Condition (4.36) is true and

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k} \int_{\mathbb{R}} \left(|\sigma_k \beta|^2 \wedge 1\right) \rho(\mathrm{d}\beta) < \infty. \tag{4.39}$$

The last claim follows from the following calculation:

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}} \left(e^{-2\lambda_k s}|\sigma_k\beta|^2 \wedge 1\right) \rho(\mathrm{d}\beta)\,\mathrm{d}s \\
&\quad = \int_{\mathbb{R}} \int_0^\infty \left(e^{-2\lambda_k s}|\sigma_k\beta|^2 \wedge 1\right) \mathrm{d}s\,\rho(\mathrm{d}\beta) \\
&\quad = \int_{|\sigma_k\beta|\leqslant 1} \int_0^\infty e^{-2\lambda_k s}|\sigma_k\beta|^2 \,\mathrm{d}s\,\rho(\mathrm{d}\beta) + \int_{|\sigma_k\beta|>1} \int_0^\infty \left(e^{-2\lambda_k s}|\sigma_k\beta|^2 \wedge 1\right) \mathrm{d}s\,\rho(\mathrm{d}\beta) \\
&\quad = \frac{1}{2\lambda_k} \int_{|\sigma_k\beta|\leqslant 1} |\sigma_k\beta|^2 \rho(\mathrm{d}\beta) + \frac{1}{\lambda_k} \int_{|\sigma_k\beta|>1} \log|\sigma_k\beta|\,\rho(\mathrm{d}\beta) + \frac{1}{2\lambda_k} \int_{|\sigma_k\beta|>1} \rho(\mathrm{d}\beta) \\
&\quad = \frac{1}{2\lambda_k} \int_{\mathbb{R}} \left(|\sigma_k\beta|^2 \wedge 1\right) \rho(\mathrm{d}\beta) + \frac{1}{\lambda_k} \int_{\mathbb{R}} \log^+ |\sigma_k\beta|\,\rho(\mathrm{d}\beta).
\end{aligned}$$

For example, if each m_k is chosen as a symmetric, α-stable process with Lévy measure $\rho(\mathrm{d}\beta) = \frac{1}{2}|\beta|^{-1-\alpha}\mathrm{d}\beta$, then a simple calculation shows that (4.38) is satisfied if and only if $\sum_{k=1}^\infty \frac{|\sigma_k|^\alpha}{\lambda_k} < \infty$, which is a weaker condition than assuming that the reciprocal eigenvalues are summable.

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