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Proportional resource allocation in dynamic n-player Blotto games*

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Abstract

We introduce a novel and general model of dynamic n-player Blotto contests. The players have asymmetric resources, the battlefields are heterogenous, and contest success functions are general as well. We obtain one possibility and one impossibility result. When players maximize the expected value of the battles, the strategy profile in which players allocate their resources proportional to the sizes of the battles at every history—whether their resources are fixed from the beginning or can be subject to shocks in time—is a subgame perfect equilibrium. However, when players maximize the probability of winning, there is always a distribution of values over the battles such that proportional resource allocation cannot be supported as an equilibrium. *JEL: C73, D72*

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1 Introduction

Social, economic, and political interactions that resemble contests are ubiquitous. Examples include rent-seeking, political campaigns, sports competitions, litigation, lobbying, wars and so on. These wide-ranging applications led economists theoretically model contests to understand the strategic incentives behind them. Traditionally, the first contest model of "Colonel Blotto" game was introduced by Borel (1921). A Colonel Blotto game is a two-person static game in which each player allocates a limited resource over a number of battlefields, each of which is of the same "size." Fast forward, the literature on contests is now enormous, and a Blotto contest denotes any contest in which two or more players allocate a limited resource over some "battlefield."

In this paper, we introduce, to the best of our knowledge, the first model of dynamic multi-battle n-player Blotto games in a very general framework as follows. The players each have possibly asymmetric resources, the battlefields are heterogenous, and the contest success functions are quite general as well. In our benchmark model, each player starts the dynamic contest with a limited budget and distributes this budget over a finite number of battlefields, which could be of varying sizes. For example, one battlefield may be the double or triple the area of another battle. Since the battles take place in a sequential order, players can condition their strategies on the outcomes of previous battles. At time t, players choose simultaneously their allocation on battle t to win $x^t > 0$, which could be interpreted as the area of the battlefield. The winner in a battlefield is determined by a contest success function satisfying the axioms (A1-A6) of Skaperdas (1996). The winner and the resulting resource allocations are revealed to every player before the next battle takes place. The overall winner of this dynamic game is the player who wins the most area. We also introduce an extension of our model to the case in which the initial resources of the players might be subject to an exogeneous shock and can change throughout the game.

The proportional allocation of resources is generally considered to be a benchmark in resource distribution games, and especially in Blotto contests it is arguably one of the prominent strategies. In experimental static Blotto games, proportional distribution is usually considered to be a level-0 behavior.[1] Studying a static, simultaneous-move

[1]For an experimental Blotto game see Arad and Rubinstein (2012); for level-k reasoining, see Stahl

model of resource allocation in U.S. presidential campaigns in a prominent paper, Brams and Davis (1974) highlighted the concepts of 'population of states' as well as 'population proportionality' in campaign resource allocation.[2] Brams and Davis (1974, p. 113) showed that populous states receive disproportionately more investments with regards to their population. More specifically, the winner-take-all feature of the Electoral College—i.e., that the popular-vote winner in each battle wins all the electoral votes of that battle—induces candidates to allocate campaign resources roughly in proportion to the 3/2's power of the electoral votes of each state. The question of why some small states "punch above their weight"–i.e., attract attention and resources more than proportional to their weight—in political campaigns has been puzzling researchers (for an analysis in a non-Blotto setting, paying attention to "momentum", see, e.g., Klumpp and Polborn, 2006).

Here we obtain one possibility and one impossibility result. Our solution concept is subgame-perfect equilibrium. First, we find that, when players maximize the expected value of the battles, the strategy profile in which players allocate their resources proportional to the sizes of the battles at every history is a subgame perfect equilibrium (see Theorem 2 and Theorem 4). This overal result is very robust: it does not depend on the number of players, asymmetry in the resources, exogenous shocks to resources, number or the sizes of battlefields, or the type of contest success functions satisfying the aforementioned axioms. Second, it turns out that, when players maximize the probability of winning the dynamic contest, there is always a distribution of player values over the battles such that resource proportionality cannot be supported as an equilibrium (see Proposition 1 and Proposition 3), showing that a recent possibility result obtained in a two-player and same-size battles setting of Klumpp, Konrad and Solomon (2019) does not generalize to a setup with more than two players and uneven battle sizes.

(1993).

[2]As noted by Brams and Davis (1974), the population of a state need not exactly reflect the proportion of the voting-age population who are registered and actually vote in a presidential election.

2 Relevant Literature

Our paper contributes to the literature on dynamic contests and campaign resource allocation in sequential elections. This brief section summarizes related work apart from the ones mentioned in the Introduction.

In a recent closely related paper, Klumpp, Konrad, and Solomon (2019) consider dynamic Blotto games where two players fight in odd number of battle fields, which are identical.[3] The player who wins the majority of battles wins the game. Accordingly, they show that under general contest success functions players allocate their resources evenly across battles in all subgame perfect equilibria, one of which is in pure strategies.[4] Their result generalizes our two-player example with three symmetric battlefields presented in section 4.

Note that their very interesting finding does not contradict our preceding impossibility result because they study two-player symmetric dynamic contests where battlefields are of the same size.

Sela and Erez (2013) studied a two-player dynamic Tullock contest, in which each player maximizes the sum of the expected payoffs (similar to expected value maximization in our setting) for all districts. Suppose that the value of the battles is equal across the stages and that for each resource unit that a player allocates, he loses $0 \leq \alpha \leq 1$ units of resource from his budget. Then they identified a subgame prefect equilibrium such that the players' resource allocations are weakly decreasing over the stages. Duffy and Matros (2015) studied static contests (stochastic asymmetric Blotto games) in up to four battles with two players having asymmetric yet similar budgets and generalizing Lake's (1979) paper.[5] In a similar setting, Deck, Sarangi, and Wiser (2017) have recently studied symmetric static contests with two players who do not have budget constraints. They identified the Nash equilibrium

[3]Colonel Blotto games were first introduced by Borel (1921). Among others, recent contributions to Blotto games include Roberson (2006), Kvasov (2007), and Rinott, Scarsini, and Yu (2012). There is also a huge literature on non-Blotto contests initiated by Tullock (1967; 1974), and see also, e.g., Krueger (1974). The early literature on non-Blotto contests are motivated by rent-seeking.

[4]For a discussion of dynamics in contests, see Konrad (2009).

[5]For experimental results on Blotto games see, e.g., Deck and Sheremeta (2012) and Duffy and Matros (2017) and the references therein.

of the symmetric game (Electoral College).

In another static presidential campaign model, Lake (1979) argued that one would need to assume that the candidates maximize only their probability of winning the election, i.e., one would simply try to receive a majority of electoral votes, instead of complying with Brams and Davis' (1973, 1974) assumption that they maximize their expected electoral vote. Nevertheless, Lake's (1979) main result echoes Brams and Davis' (1974) impossibility of population proportionality result in that in Lake's model too it turns out that presidential candidates find it optimal to spend a disproportionately large amount of their funds in the larger states.

Harris and Vickers (1985) construed a patent race as a multi-battle contest, in which two players alternate in expending resources in a sequence of single battles. These battles or sub-contests serve as the components of the overall R&D contest. Just like in a singles tennis match, the player who is first to win a given number of battles wins the contest, by obtaining the patent.[6]

Additional work on dynamic resource allocation contests is as follows. Dziubinski, Goyal and Minarsch (2017) have recently studied multi-battle dynamic contests on networks in which neighboring 'kingdoms' battle in a sequential order. Hinnosaar (2018) chracterizes equilibria of sequential contests in which efforts are exerted sequentially to win a (single-battle) contest. In a two-player and two-stage campaign resource allocation game, Kovenock and Roberson (2009) characterized the unique subgame perfect equilibrium. In a two-player best-of-three multi-battle dynamic contest, Konrad (2018) anaylzes resource carryover effects between the battles. Brams and Davis (1982) examined a model of resource allocation in the U.S. presidential primaries to study the effects of momentum transfer from one primary to another. Klumpp and Polborn (2006) also focused on momentum issues; they considered a two-player model in which an early primary victory increases the likelihood of victory for one player and creates an asymmetry in campaign spending, which in turn magnifies the player's advantage. This asymmetry of campaign

[6]In the PGA Tour, which brings professional male golfers together to play in a number of tournaments each year (LPGA does so for female golfers), each tournament consists of multiple battles in that golfers attempt to minimize the total number of shots they take across 72 holes.

spending generates a momentum which can propel an early winner to the overall victory. Strumpf (2002), on the other hand, discussed a countervailing force to momentum, which favors later winners.

3 Model

We consider dynamic Blotto contests where there are m heterogeneous battle fields, indexed by $t = 1, 2, ..., m$, and n players, indexed by $i = 1, 2, ..., n$. The battles take place in a predetermined *sequential* order. Players have possibly asymmetric budgets: Each player i has a budget $W_i \geq 0$ that he or she can allocate over the battles. In each battle t, the value of the battle is denoted by $x^t > 0$, which could be interpreted as the "area" of the battlefield t. Each time period t, the battle at t takes place, and each player i *simultaneously* chooses a pure action (allocation) denoted by w_i^t which is smaller than or equal to the budget, W_i, minus the already spent allocation by player i until battle t. Given the chosen actions in battle t, $w^t := (w_1^t, \dots, w_n^t)$, the probability of player i winning battle t is defined by a contest success function, which has the following form.

$$p_i^t(w^t) = \begin{cases} \frac{f(w_i^t)}{\sum_j f(w_j^t)} & \text{if } \sum_j w_j^t > 0 \\ \frac{1}{n} & \text{if } \sum_j w_j^t = 0, \end{cases} \tag{3.1}$$

where $f(.)$ satisfies Skaperdas's (1996) axioms (A1-A6), which characterizes a wide range of contest success functions used in the literature. More specfically, it is of the following form: $f(w_j^t) = \beta(w_j^t)^\alpha$ for some $\alpha > 0$ and $\beta > 0$.

To avoid trivial cases, we assume that for any t, $x^t < \sum_{t' \neq t} x^{t'}$, that is, there is no "dictatorial" battle. Let x_i^t be the value player i wins at battle t, which is x^t with probability $p_i^t(w^t)$ or 0 with probability $1 - p_i^t(w^t)$.

The set of histories of length t is denoted by H^t. A history of length $t \geq 1$ is a sequence

$$h^t := (((w_1^1, x_1^1), \dots, (w_n^1, x_n^1)), \dots, ((w_1^t, x_1^t), \dots, (w_n^t, x_n^t))) \tag{3.2}$$

satisfying the following conditions

(i) For each $1 \leq i \leq n$ and for each $1 \leq t' \leq t$, $w_i^{t'} \in [0, W_i - \sum_{j<t'} w_i^j]$.

(ii) For each battle $t' \leq t$, there exists a unique player i such that $x_i^{t'} = x^{t'}$ and for all $j \neq i$, $x_j^{t'} = 0$.

The first property states that each action at any given battle t is bounded by the budget set which diminishes after each action taken in previous battles. The second property states that the battles each have winner-take-all structure.

The history H^0 consists of only the empty sequence $\emptyset$. Let $H = H^0 \cup H^1 \cup \ldots \cup H^m$. Note that, the history h^{t-1} is presented to all players at time t. There is a subset $\overline{H^t} \subset H$ consisting of histories of length t where the game comes to an end at battle t. We call $\overline{H^t}$ the set of terminal histories of length t. If the game has not ended before battle m then the game ends at battle m. We will specify terminal histories in detail later on.

The remaining budget of player i after history $h^t \in H$ is defined as $B_i(h^t) = W_i - \sum_{j \leq t} w_i^j$ where for every $j \leq t$, w_i^j is a realized spending of history h^t. The realized winning schedule of a given history $h^t \in H$, denoted by $V(h^t)$, is the sequence of players that won the battles at battles $1, \ldots, t$. Thus $V(h^t) \in \{1, \ldots, n\}^t$. For example, if $h^3 = (((w_1^1, x_1^1), (w_2^1, 0)), ((w_1^2, 0), (w_2^2, x_2^2)), ((w_1^3, 0), (w_2^3, x_2^3)))$ in a two-player dynamic contest with $m > 3$ battles, then $V(h^3) = (1, 2, 2)$.

For player i, a pure strategy σ_i is a sequence of σ_i^t's such that for each t, σ_i^t assigns, to every $h^{t-1} \in H^{t-1}$, allocation $\sigma_i^t(h^{t-1}) \in [0, B_i(h^{t-1})]$. A pure strategy profile is denoted by $\sigma = (\sigma_i)_{i \leq n}$. The set of pure strategies of player $i \leq n$ is denoted by Σ_i and the set of pure strategy profiles by $\Sigma = \times_{i \leq n} \Sigma_i$. For any $\sigma \in \Sigma$, let $(\sigma|h) = ((\sigma_1|h), \ldots, (\sigma_n|h))$ denote the strategy profile induced by σ in the subgame starting from history h.

In our setup, we analyze two *different dynamic contests* in which the players either (i) maximize the expected value that players get from the battles, or (ii) maximize the probability of winning the contest.

Maximizing the expected value: Now suppose that players maximize the expected value they get from the battles (i.e., the sum of battlefield areas), so the terminal histories are exactly the histories with length m. The set of terminal histories is denoted by $\overline{H^m}$, which is equal to H^m.

For any $\bar{h}^m \in \overline{H^m}$, player i receives a payoff equal to

$$\bar{u}_i(\bar{h}^m) = \sum_{t \leq m} x_i^t, \tag{3.3}$$

where $\bar{h}^m := (((w_1^1, x_1^1), \ldots, (w_n^1, x_n^1)), \ldots, ((w_1^m, x_1^m), \ldots, (w_n^m, x_n^m)))$.

Analogous to the maximizing-probability-of-winning case, the set of terminal histories induced by a strategy profile σ conditional on reaching history h is denoted by $\rho(\sigma|h)$, which is a subset of $\overline{H^m}$. The payoff for player $i \leq n$ induced by a pure strategy profile $\sigma \in \Sigma$ at any $h^t \in H^t$ is

$$v_i(\sigma|h^t) = \sum_{\bar{h}^m \in \rho(\sigma|h^t)} q(\sigma, \bar{h}^m|h^t)\bar{u}_i(\bar{h}^m). \tag{3.4}$$

Maximizing probability of winning: A player wins the multi-battle dynamic contest if he or she wins the most battlefield areas. In the context of sequential elections, this winning rule is equivalent to the *plurality* rule in which the candidate who receives the plurality of votes wins the election. In this part, we assume that players maximize the probability of winning.

An element $\bar{h}^t \in H$ is called terminal if the contest ends with battle at battle t—if either $t = m$ or there exists a player i such that

$$\sum_{j \leq t} x_i^j > \max\{\sum_{j \leq t} x_{i'}^j |\ i' \neq i\} + x^{t+1} + \ldots + x^m. \tag{3.5}$$

If at some history a player is guaranteed to lose, then the player's remaining budget after that history is 0. Thus players who guaranteed to lose stay in the game and spend 0 at the remaining battles. Furthermore, if at some history a player is already guaranteed to win, then the contest ends at this history.

Let $\overline{H} = \bigcup_{t \leq m} \overline{H^t}$ be the set of all terminal histories. For any given $\bar{h}^t \in \overline{H}$, let $C(\bar{h}^t)$ be the set of players that have won the highest number of battlefield areas up to and including battle t, which is defined by $C(\bar{h}^t) = \arg\max_{i \leq n} \sum_{j \leq t} x_i^j$. For $\bar{h}^t \in \overline{H}$, player i receives a payoff equal to

$$u_i(\bar{h}^t) = \begin{cases} \frac{1}{|C(\bar{h}^t)|} & \text{if } i \in C(\bar{h}^t) \\ 0 & \text{otherwise.} \end{cases} \tag{3.6}$$

For every $t \leq m$, we define $\rho : \Sigma \times H^t \to \mathcal{P}(\overline{H})$ where $\mathcal{P}(\overline{H})$ is the power set of $\overline{H}$ such that $\rho(\sigma|h^t)$ denotes the set of terminal histories that are reached with positive probability with respect to σ conditional on reaching history $h^t \in H^t$. The probability of a terminal history $\bar{h}$ being reached with respect to σ conditional on reaching h^t is denoted as $q(\sigma, \bar{h}|h^t)$. The payoff for player $i \leq n$ induced by a pure strategy profile $\sigma \in \Sigma$ at any history $h^t \in H^t$ is defined as

$$v_i(\sigma|h^t) = \sum_{\bar{h} \in \rho(\sigma|h^t)} q(\sigma, \bar{h}|h^t) u_i(\bar{h}), \tag{3.7}$$

We denote $v_i(\sigma|\emptyset)$ by $v_i(\sigma)$. Equation (3.7) and $q(\sigma, \bar{h}|h^t)$ define the payoff of each player i as follows. For any given terminal history $\bar{h}$ in $\rho(\sigma|h^t)$, we multiply the probability of reaching the given history $\bar{h}$ with the utility player i gets at terminal history $\bar{h}$ and then we sum over all terminal histories in $\rho(\sigma|h^t)$.

Our solution concept is subgame perfect equilibrium in pure strategies.

Subgame perfect equilibrium: A pure strategy profile $\sigma \in \Sigma$ is a *subgame perfect equilibrium* if for every battle $t \leq m$, for every history $h \in H^t$, for every player $i \leq n$, and for every strategy $\sigma'_i \in \Sigma_i$

$$v_i(\sigma|h) \geq v_i(\sigma_{-i}, \sigma'_i|h).$$

A strategy profile $\sigma \in \Sigma$ is a subgame perfect equilibrium if and only if for every $h \in H$, σ induces an equilibrium in the subgame starting with history h.

We will also need the following definition:

Proportional strategy profile: Here we define a very specific pure strategy profile σ. For any t, for any nonterminal history $h^{t-1} \in H - \overline{H}$, and for any player i, let

$$\sigma_i^t(h^{t-1}) = B_i(h^{t-1}) \frac{x^t}{x^t + \ldots + x^m}. \tag{3.8}$$

We call σ the proportional pure strategy profile. Note that under σ, no matter what the other players do, every player proportionally allocates his or her available budget over the remaining battles. A dynamic contest is said to admit *proportionality* if the proportional strategy profile is a subgame perfect equilibrium.

4 Dynamic Blotto games where players maximize the probability of winning

In this section, the players maximize their probability of winning the dynamic contest. Here we analyze three examples of dynamic contests, which will help establishing our main result in the section while also shedding some light on the specifics of the dynamics of our contest: I) 3-identical-battle dynamic contest. II) 4-battle dynamic contest where the first battle is larger (i.e., has a higher value) than the 3 other identical battles. III) 4-battle dynamic contest where the last battle is larger than the first 3 identical battles. We show that the dynamic contest in Example I satisfies proportionality. Furthermore, we provide nontrivial dynamic contests in Examples II and III that do not admit proportionality.

Finally, we show that for any n-player dynamic contest with at least four battles and at least two players, there is always a distribution of values over the battles such that proportionality does not satisfy.

4.1 Example I: The dynamic contest with 3 identical battles

Suppose that there are two players A and B with equal budget $W = 100$ and three identical battles. To solve this dynamic contest, we use backward induction. If for given h^2, $V(h^2)$ is equal to (A, A) or (B, B), which means player A has already won or lost the first two battles, then the contest comes to an end and player A wins or loses, respectively. If for given h^2, $V(h^2)$ is equal to (A, B) or (B, A) then the dynamic contest continues to the last battle and in the subgame after h^2, the unique best response is to allocate all of the remaining budget to the last battle. Therefore, in any subgame after any nonterminal history h^2 the strategy profile (σ_A^3, σ_B^3), which is to spend the remaining budget to the last battle, is the unique Nash equilibrium.

Now, suppose that for h^1, $V(h^1) = (A)$. Then, player A's best response to B is to maximize his payoff

$$v_A(\sigma|h^1) = v_A(((\sigma_A^1, \sigma_B^1), (\sigma_A^2, \sigma_B^2), (\sigma_A^3, \sigma_B^3))|h^1) \tag{4.1}$$

with respect to σ_A^2. Given a fixed σ_B^2, player A solves the following maximization problem

$$\begin{aligned}
\max_{\sigma_A^2} v_A(\sigma|h^1) &= \max_{\sigma_A^2(h^1)} (p_A^2(\sigma_A^2(h^1), \sigma_B^2(h^1)) \\
&\qquad + (1 - p_A^2(\sigma_A^2(h^1), \sigma_B^2(h^1))) \\
&\qquad \times p_A^3(B_A(h^1) - \sigma_A^2(h^1), B_B(h^1) - \sigma_B^2(h^1))) \qquad (4.2) \\
&= \max_{\sigma_A^2(h^1)} (\frac{\sigma_A^2(h^1)}{\sigma_A^2(h^1) + \sigma_B^2(h^1)} + \frac{\sigma_B^2(h^1)}{\sigma_A^2(h^1) + \sigma_A^2(h^1)} \\
&\qquad \times \frac{100 - w_A^1 - \sigma_A^2(h^1)}{200 - w_A^1 - \sigma_A^2(h^1) - w_B^1 - \sigma_B^2(h^1)}) \qquad (4.3) \\
&= \max_{w_A^2} (\frac{w_A^2}{w_A^2 + w_B^2} + \frac{w_B^2}{w_A^2 + w_B^2} \\
&\qquad \times \frac{100 - w_A^1 - w_A^2}{200 - w_A^1 - w_A^2 - w_B^1 - w_B^2}). \qquad (4.4)
\end{aligned}$$

Equality (4.2) holds, since player A already won the first battle, in order to win the contest player A either needs to win the second battle—given $\bar{h}^2$ with $V(\bar{h}^2) = (A, A)$—, or if he loses the second then to win the third battle—given h^2 with $V(h^2) = (A, B)$. Moreover, if the game continues to the last battle, then spending the remaining budget is the unique best reponse in the subgame after given history. Equality (4.3) follows from the contest success function. Equality (4.4) follows from the definition of a strategy which maps histories to actions in the following battle.

As player A wants to maximize $v_A(\sigma|h^1)$, player B wants to minimize it. We derive best response functions from the first-order condition of $v_A(\sigma|h^1)$ with respect to $\sigma_A^2(h^1)$ and $\sigma_B^2(h^1)$, respectively. The intersection of best responses provides us the following two conditions[7]

$$w_A^2 = \frac{1}{2}(100 - w_A^1), \qquad (4.5)$$

$$w_B^2 = \frac{1}{2}(100 - w_B^1). \qquad (4.6)$$

So far, we have showed that the pair (σ_A^3, σ_B^3)—in which players each allocate the remaining budget to the last battle—, and the pair ((σ_A^2, σ_B^2),(σ_A^3, σ_B^3))—in which strategies satisfy (4.5) and (4.6)—are Nash equilibria in their respective subgames. Now, we suppose that

[7] Detailed calculations are available in the Appendix.

$h = \emptyset$. Then, player A best responds to B by maximizing his payoff $v_A(\sigma|\emptyset) = v_A(\sigma)$ with respect to $\sigma_A^1(\emptyset)$

$$\max_{\sigma_A^1(\emptyset)} v_A(\sigma) = \max_{\sigma_A^1(\emptyset)} (p_A^1(w^1)(p_A^2(w^2) + (1 - p_A^2(w^2))p_A^3(w^3)) + (1 - p_A^1(w^1))p_A^2(w^2)p_A^3(w^3)). \tag{4.7}$$

where $w^1 = (\sigma_A(\emptyset), \sigma_B(\emptyset))$, and w^2, w^3 are elements of $[0, 100 - w_A^1] \times [0, 100 - w_B^1]$ and $[0, 100 - w_A^1 - w_A^2] \times [0, 100 - w_B^1 - w_B^2]$, respectively. But since we have already concluded that whoever wins the first battle, both players should invest equally all their remaining budget to the last two battles by equations (4.5) and (4.6), we can write $w_A^2 = w_A^3 = (1/2)(100 - w_A^1)$ and $w_B^2 = w_B^3 = (1/2)(100 - w_B^1)$. Thus, we can rewrite equation (4.7) as

$$\max_{\sigma_A(\emptyset) = w_A^1} \frac{(w_A^1 - 100)\left((w_A^1)^2 + 3(w_A^1)(w_B^1 - 100) - 100 w_B^1\right)}{(w_A^1 + w_B^1 - 200)^2 (w_A^1 + w_B^1)}. \tag{4.8}$$

Equations (4.7) and (4.8) follows from the analogous reasoning for equations (4.2),(4.3) and (4.4). By the same method as for $v_A(\sigma|h^1)$, the first order condition of $v_A(\sigma)$ with respect to $\sigma_A(\emptyset)$ and $\sigma_B(\emptyset)$ yields the best response functions of players A and B. The intersection of the best responses leads to the condition $w_A^1 = w_B^1 = 100/3$. From equations (4.5), (4.6) and $w_A^1 = w_B^1$=100/3, we deduce that $w_A^1 = w_A^2 = w_A^3 = w_B^1 = w_B^2 = w_B^3 = 100/3$, which concludes that every subgame perfect equilibrium strategy profile of Example I satisfies proportionality.[8]

[8] This example is similar to the benchmark example of Fu, Lu, and Pan (2015, p. 9); though, our settings differ when the battles are not identical.

4.2 Examples II and III: Dynamic contests with 4 battles where one is larger than the others

To illustrate our model with a nontrivial example, suppose that there are two players A and B with equal budget $W = 100$ and 4 battles, one of which is a 2-value battle and the other battles each have 1 value as presented below.

Battle #	1	2	3	4
Area of the battle	2	1	1	1

Thus, if a player wins the 2-value battle and one other battle, or wins all the 1-value battles, then the player ends up winning the dynamic contest. We next provide two examples for 4-battle case in which the 2-value battle is the first and the last battle, respectively. We show that neither contest admits a proportional subgame perfect equilibrium strategy profile.[9]

Example II[10]: Let the 2-value battle be the first battle. If for given h^2, $V(h^2)$ is equal to (A, A) or (B, B), which means player A has already won or lost the first two battles, then the contest comes to an end and player A wins or loses, respectively. If for given h^3, $V(h^3)$ is equal to (A, B, B) or (B, A, A), then the dynamic contest continues to the last battle. And, for the subgame after the given history the unique best response for each player (σ_A^4 and σ_B^4) is to allocate all the remaining budget to the last battle.

For given h^2, $V(h^2) = (A, B)$, player A best responds to B by maximizing his payoff $v_A(\sigma|h^2)$ with respect to σ_A^3

$$\max_{\sigma_A^3} v_A(\sigma|h^2) = \max_{\sigma_A^3(h^2)} v_A(\sigma|h^2) = \max_{w_A^3}(p_A^3 + (1 - p_A^3)p_A^4),$$

which is analogous to the dynamic contest with 3 identical battles. Hence, each player allocating equally her remaining budget to battles 3 and 4 is the unique best response, (σ_A^3, σ_A^4) and (σ_B^3, σ_B^4), for subgames starting with history h^2. Now suppose that for given h^1, if $V(h^1) = (A)$, then player A best responds to B by maximizing his payoff

$$\max_{\sigma_A^2(h^1)} v_A(\sigma|h^1) = \max_{w_A^2}(p_A^2 + (1 - p_A^2)p_A^3 + (1 - p_A^2)(1 - p_A^3)p_A^4). \tag{4.9}$$

[9] Calculations are available upon request.

[10] From this point on, we take p_i^t as $p_i^t(w^t)$.

We derive Equation (4.9) by an analogous method from Example I. To obtain the best response functions, we take the first order conditions of the payoff functions with respect to $\sigma_A^2(h^1)$ and $\sigma_B^2(h^1)$, respectively. The intersection of the best responses yields the following two conditions

$$w_A^2 = \frac{1}{3}(100 - w_A^1), \tag{4.10}$$

$$w_B^2 = \frac{1}{3}(100 - w_B^1). \tag{4.11}$$

Given that each player wins one battle from the first two battles, we already concluded that players allocate their remaining budget to battles 3 and 4 equally. Thus, by equations (4.10) and (4.11), we conclude that whoever is the winner of the first battle, players allocating equally their remaining budget to three identical battles is the best response for any subgame after the first battle.

Now, we suppose that $h^1 = \emptyset$. Then, player A best responds to B by maximizing his payoff $v_A(\sigma|\emptyset) = v_A(\sigma)$ with respect to $\sigma_A^1(\emptyset)$ which is $\max_{\sigma_A^1(\emptyset)} v_A(\sigma)$, i.e.,

$$\begin{aligned}
\max_{\sigma_A^1(\emptyset)} v_A(\sigma) &= \max_{w_A^1}(p_A^1(p_A^2 + (1-p_A^2)p_A^3 \\
&\qquad + (1-p_A^2)(1-p_A^3)p_A^4) + (1-p_A^1)p_A^2p_A^3p_A^4) \\
&= \frac{(w_A^1 - 100)\left(4(w_A^1)^2\left(w_B^1 - 125\right) + (w_A^1)^3 + 10000w_B^1\right)}{(w_A^1 + w_B^1 - 200)^3\left(w_A^1 + w_B^1\right)} \\
&\quad + \frac{(w_A^1 - 100)(w_A^1\left(3(w_B^1)^2 - 1100w_B^1 + 70000\right))}{(w_A^1 + w_B^1 - 200)^3\left(w_A^1 + w_B^1\right)}
\end{aligned} \tag{4.12}$$

Equation (4.12) is derived from a similar method applied to equation (4.9). First order condition of equation (4.12) with respect to w_A^1 and w_B^1 yields the best response functions of players A and B. We deduce that $w_A^1 = w_B^1 = 50$ is at the intersection of the best responses. Hence, equations (4.10) and (4.11) imply that $w_A^2 = w_A^3 = w_A^4 = w_B^2 = w_B^3 = w_B^4 = 50/3$. In conclusion, there exists a subgame perfect equilibrium strategy profile σ such that on an equilibrium path of σ, players spend half of their budget to the 2-value battle and then equally split the remaining budget among 1-value battles. Thus, Example II does not admit proportionality.

Example III: Let the 2-value battle be the last battle as presented below.

Battle #	1	2	3	4
Value of the battle	1	1	1	2

If for given h^2, $V(h^2)$ is equal to (A, B) or (B, A), then players allocating all their remaining budget to the 2-value battle is the unique best response for the subgame starting from history h^2. If for given h^2, $V(h^2)$ is equal to (A, A), then in the following subgame of the related history, player A needs only one of the remaining battles and player B needs both battles to win the contest which is the same case as in Example I after the first battle. Thus, the unique best responses are players distributing equally their remaining budget on the last two battles. Therefore, in any subgame after any history $h^2 \in H$, there is no equilibrium that is proportional. Hence, Example III does not admit proportionality.

The following proposition provides a class of dynamic contests that does not admit proportionality whenever players maximize probability of winning.

Proposition 1 *For any contest success function à la Skaperdas (A1–A6), any $m \geq 4$ number of battles and any $n \geq 2$ players, there exists a dynamic contest, where players maximize their probability of winning, for which proportionality fails.*

Proof: Consider a dynamic contest such that $x^1 = \ldots = x^{m-1} = 1$ and $x^m = 3$. Consider the history h^{m-2} where player 1 wins battle 1, player 2 wins battle 2, player 1 wins battle 3 and so on up to and including battle $m-2$. Thus, for history h^{m-2}, the maximum difference between the total values of player 1 and player 2 is 1. In the subgame after history h^{m-2}, player 1 and player 2 spending all their budget to the last battle is the unique Nash equilibrium. Since we reach history h^{m-2} with positive probability under the proportional strategy profile, the dynamic contest does not admit proportionality. □

This simple impossibility result shows that Klumpp, Konrad, and Solomon's (2019) result on two-player symmetric dynamic Blotto games with same-size battlefields does not extend to more general dynamic contests with heterogenous battlefields. To gain some further insight, let's consider the following example.

Battle #	1	2	3	4	5
Value of the battle	7	7	8	7	7

If this dynamic contest is played by two players, then it is clear that the player who wins any of the three battles wins the contest. Thus, the larger battle, which gives 8 values to the winner, should be treated the same as the other battles in equilibrium. Put differently, the equilibrium behaviour of this dynamic contest would be no different than the equilibrium behaviour in the two player dynamic contest in which the value of every battle is equal to 7. This simple intuition reveals that if the value of a battlefield is not "too" different from the others, then we cannot expect propotionality to hold in such dynamic contests. Note that if this dynamic contest is played by more than two players, then the larger battle can be pivotal in some histories reached with positive probability, and hence may receive more than proportional allocation in equilibrium. This is an example of a dynamic game in which a small difference in battle values can make a huge difference in equilibrium behavior.

5 Dynamic Blotto games where players maximize the expected value of the battles

In this section, we assume that players maximize the expected value that they receive from the battlefield in the n-player sequential multi-battle contest. We first define the well-known Tullock contest success function in this context.

$$p_i^t(w^t) = \begin{cases} \frac{w_i^t}{\sum_j w_j^t} & \text{if } \sum_j w_j^t > 0 \\ \frac{1}{n} & \text{if } \sum_j w_j^t = 0. \end{cases} \tag{5.1}$$

The following theorem provides our second main result. Note that in the following section we generalize our model and extend this result to more general contest success functions.

Theorem 2 *For any dynamic contest with Tullock contest success function where players maximize winning the expected value of the battles, proportionality is satisfied.*

Proof: We show that the proportional strategy profile $\sigma = (\sigma_i)_{i \leq n}$ is robust to one-shot deviations, which implies that σ is a subgame perfect equilibirum. That is, any player i

at any nonterminal history h^t can not improve his payoff by changing σ_i^t, given that all other players, $j \neq i$, follow the proportional strategy. If player i switches to a strategy $\bar{\sigma}_i = (\bar{\sigma}_i^{t+1}, \sigma_i^{t+2}, \ldots, \sigma_i^m)$ after history h^t such that $\bar{\sigma}_i^{t+1}(h^t) \neq \sigma_i^{t+1}(h^t)$, then the expected value player i wins after battle t given the history h^t is denoted as $v_{i,t+1}(\bar{\sigma}_i, \sigma_{-i}|h^t)$, which satisfies

$$v_{i,t+1}(\bar{\sigma}_i, \sigma_{-i}|h^t) = \frac{x^{t+1}\bar{\sigma}_i^{t+1}(h^t)}{\bar{\sigma}_i^{t+1}(h^t) + \sum_{j\neq i}\sigma_j^{t+1}(h^t)} + v_{i,t+2}(\sigma|h_{dev}^{t+1}), \tag{5.2}$$

where h_{dev}^{t+1} is a successor of h^t with the property that at battle $t+1$ player i spent $\bar{\sigma}_i^{t+1}(h^t)$, and each player $j \neq i$ spent proportionally. And the expected value of player i after history h^t if she follows σ,

$$v_{i,t+1}(\sigma|h^t) = \frac{x^{t+1}\sigma_i^{t+1}(h^t)}{\sum_{1\leq j\leq n}\sigma_j^{t+1}(h^t)} + v_{i,t+2}(\sigma|h^{t+1}), \tag{5.3}$$

where h^{t+1} is a successor of h^t with the property that at battle $t+1$, each player spent proportionally. For simplicity we take

$$\frac{x^{t+1} + \ldots + x^m}{x^{t+1}} = k,$$

$$B_i(h^t) = a,$$

$$\sum_{j\neq i} B_j(h^t) = b,$$

$$\bar{\sigma}_i^{t+1}(h^t) = \sigma_i^{t+1}(h^t) + \Delta = \frac{a}{k} + \Delta.$$

where Δ is a real number. We can rewrite player i's probability of winning battle $t+1$ if he plays $\bar{\sigma}_i^{t+1}(h^t)$ as

$$\frac{\bar{\sigma}_i^{t+1}(h^t)}{\bar{\sigma}_i^{t+1}(h^t) + \sum_{j\neq i}\sigma_j^{t+1}(h^t)} = \frac{\frac{a}{k} + \Delta}{\frac{a}{k} + \Delta + \frac{b}{k}} = \frac{a + \Delta k}{a + \Delta k + b},$$

and player i's probability of winning battle $t+1$ if he plays $\sigma_i^{t+1}(h^t)$ as

$$\frac{\sigma_i^{t+1}(h^t)}{\sum_{1\leq j\leq n}\sigma_j^{t+1}(h^t)} = \frac{\frac{a}{k}}{\frac{a}{k} + \frac{b}{k}} = \frac{a}{a+b}.$$

Since σ is a proportional strategy profile, for any t, for any h^t, and for successor of histories where h^{t+1} is a successor of h^t, h^{t+2} is a successor of h^{t+1}, and so on up to and including h^m is a successor of h^{m-1}, given that players follow proportional strategy profile, we have

$$\frac{\sigma_i^{t+1}(h^t)}{\sum_{1\leq j\leq n}\sigma_j^{t+1}(h^t)} = \frac{\sigma_i^{t+2}(h^{t+1})}{\sum_{1\leq j\leq n}\sigma_j^{t+2}(h^{t+1})} = \ldots = \frac{\sigma_i^m(h^{m-1})}{\sum_{1\leq j\leq n}\sigma_j^m(h^{m-1})} = \frac{a}{a+b},$$

which means that player i wins each battle after h^t with equal probability if he/she follows σ_i. That is, if players follow the proportional strategy profile, the proportions of the remaining budgets stay constant throughout the battles. The same property satisfies for the strategy profile $(\bar{\sigma}_i, \sigma_{-i})$ after history h^{t+1}_{dev}. Hence for successor of histories where h^{t+2}_{dev} is a successor of h^{t+1}_{dev}, h^{t+3}_{dev} is a successor of h^{t+2}_{dev}, and so on up to and including h^{m}_{dev} is a successor of h^{m-1}_{dev}, given that players follow proportional strategy profile after history h^{t+1}_{dev}, we have

$$\frac{\sigma_i^{t+2}(h^{t+1}_{dev})}{\sum_{1\le j\le n}\sigma_j^{t+2}(h^{t+1}_{dev})} = \frac{\sigma_i^{t+3}(h^{t+2}_{dev})}{\sum_{1\le j\le n}\sigma_j^{t+3}(h^{t+2}_{dev})} = \ldots = \frac{\sigma_i^{m}(h^{m-1}_{dev})}{\sum_{1\le j\le n}\sigma_j^{m}(h^{m-1}_{dev})}.$$

Now we can simply calculate player i's probability of winning any battle after history h^{t+1}_{dev}, if player i follows the strategy $\bar{\sigma}_i$

$$\frac{\sigma_i^{t+2}(h^{t+1}_{dev})}{\sum_{1\le j\le n}\sigma_j^{t+2}(h^{t+1}_{dev})} = \frac{a-\frac{a}{k}-\Delta}{a-\frac{a}{k}-\Delta+b-\frac{b}{k}}.$$

Therefore we can rewrite equation (5.2) as

$$v_{i,t+1}(\bar{\sigma}_i, \sigma_{-i}|h^t) = x^{t+1}\frac{a+\Delta k}{a+\Delta k+b} + \frac{a-\frac{a}{k}-\Delta}{a-\frac{a}{k}-\Delta+b-\frac{b}{k}}(x^{t+2}+\ldots+x^m),$$

And we can rewrite equation (5.3) as

$$v_{i,t+1}(\sigma|h^t) = \frac{a}{a+b}(x^{t+1}+\ldots+x^m).$$

We show that $v_i(\sigma|h^t) - v_i(\bar{\sigma}_i, \sigma_{-i}|h^t) \ge 0$, in other words show that

$$x^{t+1}\left(\frac{a}{a+b} - \frac{a+\Delta k}{a+\Delta k+b}\right) + (x^{t+2}+\ldots+x^m)\left(\frac{a}{a+b} - \frac{a-\frac{a}{k}-\Delta}{a-\frac{a}{k}-\Delta+b-\frac{b}{k}}\right) \ge 0. \quad (5.4)$$

Since $k-1 = (x^{t+2}+\ldots+x^m)/(x^{t+1})$, we can rewrite inequality (5.4) as

$$\left(\frac{a}{a+b} - \frac{a+\Delta k}{a+\Delta k+b}\right) + (k-1)\left(\frac{a}{a+b} - \frac{a-\frac{a}{k}-\Delta}{a-\frac{a}{k}-\Delta+b-\frac{b}{k}}\right) \ge 0. \quad (5.5)$$

We can simplify inequality (5.5) as

$$\frac{b\Delta^2k^3}{(a+b)(a(k-1)+b(k-1)-\Delta k)(a+b+\Delta k)} \ge 0. \quad (5.6)$$

Inequality (5.6) satisfies because we have the following conditions

$$a \ge \frac{a}{k} + \Delta,$$

$$b(k-1) \geq 0,$$

$$\frac{a}{k} + \Delta \geq 0.$$

Thus, for any Δ we have $v_i(\sigma|h^t) - v_i(\bar{\sigma}_i, \sigma_{-i}|h^t) \geq 0$. $\square$

We next present extensions of this theorem to a more general setting with variable budgets and with any contest success function satisfying Skaperdas's (1996) axioms A1–A6.

6 Dynamic Blotto games with exogenous budget shocks and general contest success functions

In this section, we show that our results are robust to exogenous budget shocks during the dynamic contest. We call this extension dynamic contests with variable budgets.

Changes in the model: A history of length $t \geq 1$ is a sequence

$$h^t := (((w_1^1, x_1^1), \ldots, (w_n^1, x_n^1)), \ldots, ((w_1^t, x_1^t), \ldots, (w_n^t, x_n^t))) \tag{6.1}$$

satisfying the following conditions

(i) For each $1 \leq i \leq n$ and for each $1 \leq t' \leq t$, $w_i^{t'} \in [0, W_i + z_i^{t'} - \sum_{j<t'} w_i^j]$, where $z_i^{t'} \in \mathbb{R}$ represents the exogenous budget shock player i receives, if any, before battle t' takes place. The amount of the exogenous shock $z_i^{t'}$ is announced publicly to all players before battle t'.

(ii) For each battle $t' \leq t$, there exists a unique player i such that $x_i^{t'} = x^{t'}$ and for all $j \neq i$, $x_j^{t'} = 0$.

Part (i) introduces a new variable $z_i^{t'}$ which needs more eloboration. Note that everyone receives the news about the exogenous budget shock at the same time and before the next battle takes place, so the dynamic game is still a game under complete information. In the context of sequential elections, an example would be that a private donor decides to contribute resources to the campaign of a (presidential) candidate after several mini-elections have taken place.

Accordingly, the remaining budget of player i after history $h^t \in H$ is defined as $\hat{B}_i(h^t) = \max\{W_i + \sum_{j \leq t+1} z_i^j - \sum_{j \leq t} w_i^j, 0\}$ where for every $j \leq t$, w_i^j is a realized spending of history h^t. For player i, a pure strategy σ_i is a sequence of σ_i^t's such that for each t, σ_i^t assigns, to every $h^{t-1} \in H^{t-1}$, allocation $\sigma_i^t(h^{t-1}) \in [0, \hat{B}_i(h^{t-1})]$.

Proportional strategy profile: For any t, for any nonterminal history $h^{t-1} \in H - \overline{H}$, and for any player i, let

$$\sigma_i^t(h^{t-1}) = \hat{B}_i(h^{t-1}) \frac{x^t}{x^t + \ldots + x^m}. \tag{6.2}$$

Note that for any subgame starting at history h^{t-1} and for any player i, the knowledge of player i on the variable budget set is $\{z_1^t, \ldots, z_n^t\}$. Therefore the strategy σ is defined on $\hat{B}_i(h^t)$ but not on future variable budgets, which is exogenously given each time before a battle takes place. We call σ the proportional (pure) strategy profile. Note that under σ, no matter what the other players do, every player proportionally allocates her available budget over the remaining battles.

A dynamic contest is said to satisfy *proportionality* if the proportional strategy profile is a subgame perfect equilibrium. Mutatis mutandis, the rest of the model remains the same as the model presented in section 3.

The following proposition extends Proposition 1 to the case with variable budgets. Its proof echoes the one of Proposition 1, and hence it is not reproduced.

Proposition 3 *For any contest success function à la Skaperdas (A1–A6), for any $m \geq 4$ number of battles and any $n \geq 2$ players, there exists a dynamic contest with variable budgets, where players maximize their probability of winning, for which proportionality fails.*

We now state our last main result, which extends Theorem 2 to the case with variable budgets and general contest success functions.

Theorem 4 *For any dynamic contest with variable budgets where players maximize the expected value of the battles, the dynamic contest satisfies proportionality.*

Proof: First, we show that a strategy profile $\sigma^* \in \Sigma$ that satisfies proportionality in the dynamic contest is a Nash equilibrium by proving that for any player i a strategy $\sigma_i \in \Sigma_i$ is a best response to σ^*_{-i} whenever $\sigma_i = \sigma^*_i$ with fixed budgets, and then we extend this result to contests with variable budgets. Let w be the spending sequence associated with $(\sigma_i, \sigma^*_{-i})$ and $w_{-i} = (w^t_{-i})_{t \leq m}$ denote the spending sequence excluding player i, where for all t, $w^t_{-i} = (w^t_1, \ldots, w^t_{i-1}, w^t_{i+1}, \ldots, w^t_n)$.[11] We show that

$$\sigma_i \in \arg\max_{\sigma'_i} v_i(\sigma^*_{-i}, \sigma'_i | \varnothing), \tag{6.3}$$

that is, player i's best response to σ^*_{-i} associated with w_{-i} is σ_i associated with w_i. Given that all players but i follow the spending sequence w_{-i}, player i's expected value from a 1-value battle t_1 for any $w^{t_1}_i$, which we treat as a variable, is given by

$$\frac{\beta(w^{t_1}_i)^\alpha}{(\beta(w^{t_1}_i)^\alpha + \beta \sum_{j \neq i}(w^{t_1}_j)^\alpha)}. \tag{6.4}$$

Differentiating (6.4) with respect to $w^{t_1}_i$ gives

$$\frac{\alpha(w^{t_1}_i)^{\alpha-1} \sum_{j \neq i}(w^{t_1}_j)^\alpha}{((w^{t_1}_i)^\alpha + \sum_{j \neq i}(w^{t_1}_j)^\alpha)^2}, \tag{6.5}$$

which is player i's marginal gain from the battle t_1. We next consider a k-value battle t_k for some $k \in \{1, ..., m\}$. By our supposition each player except player i spends in proportion to the value of the battle, i.e., for each $j \neq i$, $w^{t_k}_j = k w^{t_1}_j$. We next show that player i's best response to proportional allocation is also to spend in proportion to the value at battle t_k, i.e., $w^{t_k}_i = k w^{t_1}_i$. In this case, player i's expected value for any $w^{t_k}_i$ from k-value battle t_k is

$$\frac{k\beta(w^{t_k}_i)^\alpha}{(\beta(w^{t_k}_i)^\alpha + \beta \sum_{j \neq i} w^{t_k}_j)} = \frac{k(w^{t_k}_i)^\alpha}{((w^{t_k}_i)^\alpha + k^\alpha \sum_{j \neq i}(w^{t_1}_j)^\alpha)}. \tag{6.6}$$

Differentiating (6.6) with respect to $w^{t_k}_i$ gives

$$\frac{\alpha k^{\alpha+1}(w^{t_k}_i)^{\alpha-1} \sum_{j \neq i}(w^{t_1}_j)^\alpha}{((w^{t_k}_i)^\alpha + k^\alpha \sum_{j \neq i}(w^{t_1}_j)^\alpha)^2}, \tag{6.7}$$

[11] To be sure, one may condition his strategy on the winners of the previous battles and also on the previous battle spendings. However, without loss of generality, we can confine attention to the spending sequence, w, that is associated with the given strategy profile, because the utility received from the previous battles does not affect the utility that can be received from the remaining ones as the utility function is additive. This is, of course, not true under winning probability maximization because winning a battle alters the ways in which the dynamic contest can be won.

which is player i's marginal gain from the k-value battle t_k. Next, we show that for $w_i^{t_k} = kw_i^{t_1}$, Expression 6.7 equals Expression 6.5. First, Expression 6.7 equals

$$\frac{\alpha k^{\alpha+1}(w_i^{t_k})^{\alpha-1}\sum_{j\neq i}(w_j^{t_1})^{\alpha}}{((w_i^{t_k})^{\alpha}+k^{\alpha}\sum_{j\neq i}(w_j^{t_1})^{\alpha})^2} = \frac{\alpha k^{\alpha+1}(kw_i^{t_1})^{\alpha-1}\sum_{j\neq i}(w_j^{t_1})^{\alpha}}{((kw_i^{t_1})^{\alpha}+k^{\alpha}\sum_{j\neq i}(w_j^{t_1})^{\alpha})^2}. \tag{6.8}$$

Cancelling out k's leads to

$$\frac{\alpha(w_i^{t_1})^{\alpha-1}\sum_{j\neq i}(w_j^{t_1})^{\alpha}}{((w_i^{t_1})^{\alpha}+\sum_{j\neq i}(w_j^{t_1})^{\alpha})^2}, \tag{6.9}$$

which is Expression 6.5. We showed that if player i allocates proportionally to k-value battle, then his marginal gain from that battle is equal to his marginal gain from 1-value battle provided that others allocate proportionally. Thus, there is no incentive for player i to deviate from proportional allocation when others allocate proportionally. Hence, the proportional strategy profile σ is a Nash equilibrium of the dynamic contest.

Now we show that σ^* is a subgame perfect equilibrium. In other words, for every $h \in H$, σ^* induces an equilibrium in the subgame starting with history h. By definition, the subgame starting with history h is a game (i.e., dynamic contest) and $(\sigma^*|h)$ is a proportional strategy profile. Thus, by an analogous argument used in the first part of the proof, $(\sigma^*|h)$ is a Nash equilibrium in the subgame starting with history h.[12] That is, we obtain for every i and every h

$$(\sigma_i^*|h) \in \arg\max_{\sigma_i'} v_i(\sigma_{-i}^*, \sigma_i'|h), \tag{6.10}$$

Therefore, σ^* is a subgame perfect equilibrium, so the dynamic contest satisfies proportionality.

So far, we have shown that in the beginning of the dynamic contest with fixed budgets players allocating their budgets proportionally is a subgame perfect Nash equilibirum of the dynamic contest. It implies that in the dynamic contest with variable budgets, players allocating their budgets proportionally is also a Nash equilibirum in the beginning of the game, because the initial budgets and payoffs are the same in the begining. After the competition in the first battle, players' budgets are updated, which defines a new dynamic

[12]Note that in the first part we showed that a proportional strategy profile is a Nash equilibrium in any dynamic contest.

contest. Applying the same argument again on this game shows that proportional allocation is a Nash equilibrium in this subgame as well. By repeating this process, we conclude that there is a subgame perfet equilibrium in which players allocate their budgets proportionally to each battle however the budgets might evolve throughout the contest. □

7 Concluding Remarks

In this paper, we have introduced a novel model of dynamic n-player Blotto games. This is a very general framework as the budgets and battlefields are possibly asymmetric, and there are no restrictions on the number of players or the number of battlefields (e.g., odd or even). In this framework, we studied the proportional allocation of resources and the equilibrium behavior. We showed that the strategy profile in which players allocate their resources proportionally at every history is a subgame perfect equilibrium whenever players maximize the expected value of the battles. By contrast, when players maximize the probability of winning the dynamic contest, there is always a distribution of values over the battles such that proportionality cannot be supported as an equilibrium. We have shown that our results are robust to exogeneous shocks on budgets. Moreover, the results do not depend on the specfic contest success function used in the competition as long as it satisfies Skaperdas's (1996) axioms (A1–A6). While we have focused on proportional equilibrium behavior in this paper, we believe that characterizing non-proportional equilibrium behavior—especially when players maximize their probability of winning—in n-player dynamic Blotto contests is a promising future research direction.

As we have mentioned, Blotto games can be applied to many economic and political situations as Borel (1921) himself envisioned. As an example, consider sequential elections as an n-player dyanmic multi-battle contest where political candidates choose how they distribute their limited resources over multiple "battlefields" or states as in the U.S. presidential primaries. In this context, our results imply that proportionality is immediately rectified once one has candidates who maximize their electoral vote instead of simply maximizing their probability of winning, despite the presence of the winner-take-all feature.

To achieve proportionality in at least the U.S. presidential primaries which have the winner-take-all feature, a viable policy suggestion could be to provide additional incentives for players to induce them to win as many delegates as possible in the entire presidential primaries overall. For instance, the electoral system may provide them additional funding in the ensuing presidential race where these incentives are positively linked to the number of delegates won by the presidential player in the primaries. Such incentives can be very effective at the margin. As a matter of fact, even in the absence of any such additional pecuniary incentives, for one reason or another, players seem to already have the behavioral trait of maximizing their expected delegates themselves and do not want to stop pumping campaign funding to their remaining primaries even though they have already guaranteed winning the majority of the delegates in the primaries. The main reason that the players may try win more delegates beyond what they would need to guarantee their presidential candidacy (i.e., the main reason that they still might keep investing in the remaining primaries even though they know that it will not affect their chances of winning any further delegates) could be that they care about entering the U.S. presidential race with an impressive momentum gained in the presidential primaries, as Hillary Clinton tried to do against the late surge of Bernie Sanders in the U.S. Democratic primaries in 2016 even though she had already accumulated more than sufficiently many delegates to win her party's presidential candidacy up to that point. Nevertheless, to ensure proportionality, the parties or the electoral system may consider boosting players' tendency to maximize their expected delegates via some additional pecuniary incentives, which may help at the margin at least for the players who may simply try to maximize their winning probability in their U.S. presidential primaries.

Appendix

Example I

The first order conditions of $v_A(\sigma|h^1)$ with respect to $\sigma_A^2(h^1)$ and $\sigma_A^2(h^1)$ are

$$\begin{aligned}\frac{\partial v_A(\sigma|h^1)}{\partial \sigma_A^2(h^1)} &= \frac{\partial v_A(\sigma|h^1)}{\partial w_A^2} \\ &= \frac{w_B^2\,(w_B^1 + w_B^2 - 100)\,(w_A^1 + 2w_A^2 + w_B^1 + 2w_B^2 - 200)}{(w_A^2 + w_B^2)^{\,2}\,(w_A^1 + w_A^2 + w_B^1 + w_B^2 - 200)^{\,2}} = 0. \end{aligned} \tag{7.1}$$

$$\begin{aligned}\frac{\partial v_A(\sigma|h^1)}{\partial \sigma_B^2(h^1)} &= \frac{\partial v_A(\sigma|h^1)}{\partial w_B^2} \\ &= -\frac{(w_A^2)^2\,(w_B^1 + 2\,(w_B^2 - 50))}{(w_A^2 + w_B^2)^2(w_A^1 + w_A^2 + w_B^1 + w_B^2 - 200)^2} \\ &\quad - \frac{w_A^2\,(w_A^1\,(w_B^1 + 2\,(w_B^2 - 50)) + (w_B^1)^2)}{(w_A^2 + w_B^2)^2(w_A^1 + w_A^2 + w_B^1 + w_B^2 - 200)^2} \\ &\quad - \frac{w_A^2\,(2\,(w_B^2 - 150)\,w_B^1 + 2\,(w_B^2 - 100)^{\,2})}{(w_A^2 + w_B^2)^2(w_A^1 + w_A^2 + w_B^1 + w_B^2 - 200)^2} \\ &\quad - \frac{(w_A^1 - 100)\,(w_B^2)^2}{(w_A^2 + w_B^2)^2(w_A^1 + w_A^2 + w_B^1 + w_B^2 - 200)^2} = 0. \end{aligned} \tag{7.2}$$

From equations (7.1) and (7.2) we conclude best response functions BR_A and BR_B such that

$$BR_A(\sigma_B^2(h^1)) = \frac{1}{2}\left(-w_A^1 - w_B^1 - 2w_B^2 + 200\right),$$

$$\begin{aligned}BR_B(\sigma_A^2(h^1)) &= \frac{-w_A^2\,(w_A^1 + w_A^2 + w_B^1 - 200)}{w_A^1 + 2\,(w_A^2 - 50)} \\ &\quad + \frac{\sqrt{w_A^2\,(w_A^1 + w_A^2 - 100)}}{w_A^1 + 2\,(w_A^2 - 50)} \\ &\quad \times \frac{\sqrt{(w_A^2 - w_B^1 + 100)\,(w_A^1 + w_A^2 + w_B^1 - 200)}}{w_A^1 + 2\,(w_A^2 - 50)}. \end{aligned}$$

The intersection of best responses provides us the following two conditions

$$\begin{aligned} w_A^2 &= \frac{1}{2}(100 - w_A^1), \\ w_B^2 &= \frac{1}{2}(100 - w_B^1). \end{aligned}$$

The first order conditions of $v_A(\sigma)$ with respect to $\sigma_A^1(\varnothing)$ and $\sigma_A^1(\varnothing)$ are

$$\begin{aligned}\frac{\partial v_A(\sigma)}{\partial w_A^1} &= \frac{2(w_B^1 - 100)(w_A^1)^2(3w_B^1 - 100)}{(w_A^1 + w_B^1 - 200)^3(w_A^1 + w_B^1)^2} \\ &\quad + \frac{2(w_B^1 - 100)\,(w_A^1 w_B^1(3w_B^1 - 500) - 200(w_B^1 - 100)w_B^1)}{(w_A^1 + w_B^1 - 200)^3(w_A^1 + w_B^1)^2} = 0, \end{aligned}$$

$$\frac{\partial v_A(\sigma)}{\partial w_B^1} = -\frac{2(w_A^1-100)\left((w_A^1)^2(3w_B^1-200)\right)}{(w_A^1+w_B^1-200)^3(w_A^1+w_B^1)^2} - \frac{2(w_A^1-100)w_A^1\left(3(w_B^1)^2-500w_B^1+20000\right)}{(w_A^1+w_B^1-200)^3(w_A^1+w_B^1)^2} + \frac{2(w_A^1-100)\left(100(w_B^1)^2\right)}{(w_A^1+w_B^1-200)^3(w_A^1+w_B^1)^2} = 0.$$

From the above equations $\partial v_A(\sigma)/\partial w_A^1 = 0$ and $\partial v_A(\sigma)/\partial w_B^1 = 0$, we find the best response functions of player A and B.

$$BR_A(\sigma_B^1(\emptyset)) = \frac{-3(w_B^1)^2+500w_B^1}{6w_B^1-200} - \frac{\sqrt{9(w_B^1)^3-600(w_B^1)^2-70000w_B^1+8000000}\sqrt{w_B^1}}{6w_B^1-200}.$$

$$BR_B(\sigma_A^1(\emptyset)) = \frac{-3(w_A^1)^2+500w_A^1}{6w_A^1-200}. - \frac{\sqrt{9(w_A^1)^3-600(w_A^1)^2-70000w_A^1+8000000}\sqrt{w_A^1}}{6w_A^1-200}.$$

The intersection of the best responses leads to the condition $w_A^1 = w_B^1 = 100/3$.

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