Further study on observer design for
continuous-time Takagi-Sugeno fuzzy model with
unknown premise variables via average dwell time

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Abstract—This paper further studies the problem of observer design for continuous-time T-S (Takagi-Sugeno) fuzzy system with unmeasurable premise variables. A membership function dependent Lyapunov function is designed to get the observer-based controller. Different from the existing results, a switching method is proposed to deal with the time derivative of membership functions. Several problems such as, too many parameters, small local stabilization region in the existing papers are solved by applying the switching method. In addition, two algorithms are designed to get the controller gains and observer gains. In the end, two examples are provided to demonstrate the effectiveness of the proposed approach.

Index Terms—Takagi-Sugeno’s fuzzy model, Observer design, Non-parallel distributed compensation law, Unknown premise variables, Linear matrix inequality

I. INTRODUCTION

In recent years, as the development of artificial intelligence, there are lots of results about nonlinear system\cite{1}-\cite{2} and fuzzy systems\cite{3}-\cite{16} especially for T-S fuzzy model\cite{17}-\cite{31} (see\cite{32} and the references therein) among which, observer design is a hot topic. As shown in\cite{33}-\cite{35}, before designing the observer, one should know whether the premise variables are dependent on the states estimated by the fuzzy observer. There are some papers about the case that the premise variables are not dependent on the estimated states\cite{36}-\cite{38}. In practice, the premise variables generally depend on the unmeasurable states, so there are also some papers about the case that the premise variables are unknown or partly unknown\cite{39},\cite{40},\cite{41} and\cite{28} where a novel integral sliding surface function is proposed on the observer space and the sliding mode dynamics and error dynamics are obtained in accordance with estimated premise variables. In\cite{29}, an observer was designed to deal with the case that the inputs are unknown and the disturbance affect both states and outputs of the system. New robust $H_{\infty}$ observer with unknown premise variables is designed by applying the Finsler’s lemma in\cite{34} where better $H_{\infty}$ performances can be obtained compared to previous results, but the used Lyapunov function is quadratic. Generally speaking, the results based on quadratic Lyapunov function is more conservative than the ones based on the membership function dependent Lyapunov function, however, for continuous-time system, the time derivatives of the membership functions are hard to be dealt with because, in most of the cases, the membership functions contain system states. Recently, the time derivatives of the membership function are explored in\cite{42} where some restrictive assumptions are made on the system states and the floor function, mod function are used to get local stabilization region, in addition, the local stabilization region can be enlarged by searching some parameters which are used to bound the time derivatives of the membership functions. It is naturally that the method of dealing with the time derivative of membership functions in\cite{42} is used in\cite{35} to design observer and some less conservative results have been obtained, but there are still some problems: (1) In order to deal with the time derivative of membership functions, the floor function and mod function are used and the system states must be restricted in a known area which means the results in\cite{35},\cite{42},\cite{43} are only local; (2) In order to get the results expressed as linear matrix inequalities (LMIs), too many parameters $\delta_i$ are generated and searching these parameters is a too time-consuming work; (3) The $H_{\infty}$ performance is in inverse proportion to the stabilization region; (4), The method can not deal with the case that the observed state is out of the local stability region.

Based on the above discussion, the problem of $H_{\infty}$ observer design for continuous-time T-S fuzzy systems with unknown premise variables is further investigated in this paper. The contributions of this paper are summarized as follows: (I) A membership function dependent Lyapunov function is designed and a switching control method is applied to deal with the time derivatives of membership functions, in addition, the average dwell time (ADT) technique is used to guarantee that the switching times are finite; (II) Two algorithms are designed to get the switching controller gains and observer gains such that the fuzzy system is globally asymptotically stable with better $H_{\infty}$ performance.

II. PROBLEM STATEMENT AND PRELIMINARIES

Consider the following T-S fuzzy model\cite{34}-\cite{36}:

\begin{align*}
\dot{x}(t) &= A_{\theta}x(t) + B_{\theta}u(t) + N_{\theta}\omega(t), \\
z(t) &= C_{\theta}x(t) + D_{\theta}u(t), \\
y(t) &= E_{\theta}x(t),
\end{align*}

(1)
where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, $\omega(t) \in \mathbb{R}^d$ is the exogenous disturbances input, $z(t) \in \mathbb{R}^v$ is the controlled output, $y(t) \in \mathbb{R}^w$ is the measured output. The fuzzy system (1) is obtained by applying the sector nonlinear method in [33], where $\theta(t) \in \mathbb{R}^w$ with $\theta_1(t), \theta_2(t) \cdots \theta_w(t)$ which are premise variables and $h_i(\theta(t))$ are the membership functions, $A_\theta = \sum_{i=1}^r h_i(\theta(t)) A_i$, $B_\theta = \sum_{i=1}^r h_i(\theta(t)) B_i$, $N_\theta = \sum_{i=1}^r h_i(\theta(t)) C_i$, $\sum_{i=1}^r h_i(\theta(t)) D_i$, $E_\theta = \sum_{i=1}^r h_i(\theta(t)) E_i$. Note, for simplicity, we only consider a special case that $\sum_{i=1}^r h_i(\theta(t)) = 1$. For simplicity, single and double sums are written similarly as [42] that

$$\Upsilon_{\theta} = \sum_{i=1}^r h_i(\theta) \Upsilon_{\theta i}, \quad \Upsilon_{\theta}^{-1} = \left( \sum_{i=1}^r h_i(\theta) \Upsilon_{i} \right)^{-1}.$$  

In this paper, we consider the following observer-based controller

$$\dot{x} = A_\theta \hat{x} + B_\theta u + N_\theta \omega + T(\hat{\theta})(y - \hat{y}),$$
$$\dot{\hat{y}} = E \hat{x}, u = K(\hat{\theta}) \hat{x},$$

(2)

where $K(\hat{\theta})$ and $T(\hat{\theta})$ are the controller gains and observer gains to be designed.

Combining (1) with (2), the closed-loop fuzzy system is written as

$$\dot{x} = G(\theta, \hat{\theta}) \dot{x} + W(\theta, \hat{\theta}) \omega,$$
$$z = H(\theta, \hat{\theta}) \dot{x},$$

(3)

where

$$G(\theta, \hat{\theta}) = \begin{bmatrix} A_\theta + B_\theta K(\hat{\theta}) & T(\hat{\theta}) E \\ (B_\theta - B_\theta K(\hat{\theta})) & A_\theta - T(\hat{\theta}) E \end{bmatrix},$$
$$W(\theta, \hat{\theta}) = \begin{bmatrix} N_\theta \\ N_\theta - N_\theta \end{bmatrix},$$
$$H(\theta, \hat{\theta}) = \begin{bmatrix} C_\theta + D_\theta K(\hat{\theta}) & C_\theta \end{bmatrix}.$$  

In [44], [45] where $\dot{X}_h \leq 0$, $\dot{Y}_h \leq 0$, while in this paper, we need $\dot{X}_h \geq 0$, $\dot{Y}_h \geq 0$ which are defined below where $X_i > 0$ and $Y_i > 0$ are variables to be designed.

$$\dot{X}_h = \sum_{i=1}^r h_i X_i = \sum_{k=1}^{r-1} \dot{h}_k (X_k - X_r),$$
$$\dot{Y}_h = \sum_{i=1}^r h_i Y_i = \sum_{k=1}^{r-1} \dot{h}_k (Y_k - Y_r).$$

(4)  

(5)

A switching idea is used as follows:

$$\begin{cases}
\text{if } \dot{h}_k \leq 0, & \text{then } X_k - X_r \leq 0, Y_k - Y_r \leq 0, \\
\text{if } \dot{h}_k > 0, & \text{then } X_k - X_r > 0, Y_k - Y_r > 0.
\end{cases}$$

(6)

There are $2^{r-1}$ possible cases in (6). Let $H_p$ be the set that contains the possible permutations of $\dot{h}_k$ where $p \in S$, $S = \{1, 2, \cdots, 2^{r-1}\}$ and $C_p$ be the set that contains the constraints of $X_i$ and $Y_i$. (6) can be presented as

$$\text{if } H_p, \text{ then } C_p.$$ (7)

Based on the above discussions and Lemma 2 in [44], the following observer-based switching controller is designed to stabilize the fuzzy system [35]:

$$u_p = K_p(\hat{\theta}) \hat{x},$$
$$\dot{x} = A_\theta \hat{x} + B_\theta u_p + N_\theta \omega + T_p(\hat{\theta})(y - \hat{y}).$$

(8)

**Definition 1:** (see [46]) For a switching signal $\sigma(t)$ and $T \geq t \geq 0$, let $N_\sigma(T, t)$ denote the number of discontinuities of $\sigma(t)$ in the interval $(t, T)$. We say that $\sigma(t)$ has an ADT $\sigma_\alpha$ if there exist two positive numbers $N_0$ (we call $N_0$ the chatter bound here) and $\alpha$ such that

$$N_\sigma(T, t) \leq N_0 + \frac{T - t}{\sigma_\alpha}.$$  

Without loss of generality, in this paper we let $N_0 = 0$.

### III. MAIN RESULTS

**Theorem 1:** Given $\varepsilon > 0$, $\alpha > 0$, $\mu > 1$, the closed-loop T-S fuzzy system (2) is globally exponentially asymptotically stable with $H_\infty$ performance $\gamma_p$ for any switching signal $\sigma(t)$ generated by the time derivative of membership functions and the ADT satisfying $\sigma_\alpha > \frac{\ln \mu}{\alpha}$, if there exist matrices $P_i^p > 0$, $R_i^q > 0$, $G_i^p$, $F_i^q$ such that $C_p$ and the following LMIs hold for $i, j, l = 1, \cdots, r$, and $p, q \in S$

$$P_i^p \geq \frac{1}{\mu} P_i^p,$$
$$R_i^q \geq \frac{1}{\mu} R_i^q,$$

(9)  

(10)

$$\Upsilon_{	ext{ij}} < 0,$$

(11)

$$\frac{2}{r-1} \Upsilon_{	ext{ii}} + \Upsilon_{	ext{ij}} + \Upsilon_{	ext{ji}} \leq 0, i \neq j,$$

(12)

where

$$A_{11} = A_i P_i^p + B_i F_i^q + P_i^p A_i^T + (B_i F_i^q)^T + \alpha P_i^p,$$
$$A_{21} = (A_i - A_j) P_i^p + (B_i - B_j) F_i^q + (M_{3i})^T,$$
$$A_{22} = A_i R_i^q - M_{3i}^T + (A_i R_i^q)^T - (M_{3i})^T + \alpha R_i^q.$$
\[
\Lambda_{41} = C_l P'_i + D_l F'_p, \\
\Lambda_{52} = R'_i - Q M'_{1ij} - \varepsilon (M_{2ij})^T, \\
\Lambda_{55} = -\varepsilon Q M'_{2ij} - \varepsilon (Q M'_{2ij})^T,
\]
\[
\Upsilon_{ij} = \begin{bmatrix}
\Lambda_{11} & \ast & \ast & \ast & \ast \\
\Lambda_{21} & \Lambda_{22} & \ast & \ast & \ast \\
N^T_i & N^T_i - N^T_j & -\gamma^T_2 I & \ast & \ast \\
\varepsilon M'_{3i} & \Lambda_{52} & 0 & 0 & \Lambda_{55}
\end{bmatrix},
\]
\[
M_{1ij} = \begin{bmatrix}
G^p_i & 0 & \ast & \ast & \ast \\
G^p_{2ij} & G^p_{3ij}
\end{bmatrix}, M_{2ij} = \begin{bmatrix}
G^p_1 & 0 & \ast & \ast & \ast \\
G^p_{4ij} & G^p_{5ij}
\end{bmatrix},
\]
\[
M^p_{3i} = \begin{bmatrix}
G^p_{6i} & 0
\end{bmatrix}, E_Q = \begin{bmatrix}
I & 0
\end{bmatrix},
\]
then, the \( H_\infty \) performance is ensured by the observer-based controller (8). The controller gains, observer gains are designed as follows
\[
K^p (\hat{\theta}) = \left( \sum_{i=1}^{r} h_i (\hat{\theta}) F^p_i \right) \left( \sum_{i=1}^{r} h_i (\hat{\theta}) P^p_i \right)^{-1},
\]
\[
T^p (\hat{\theta}) = \sum_{i=1}^{r} h_i (\hat{\theta}) L^p_i, L^p_i = G^p_{6i} (G^p_i)^{-1}.
\]

**Proof:** Design the non-quadratic Lyapunov candidate function as
\[
V (\hat{x}) = \hat{x}^T \hat{\sigma}^\ast(t) (\hat{\theta}) \hat{x},
\]
where
\[
\hat{\sigma}^\ast(t) (\hat{\theta}) = \begin{bmatrix}
(P^\ast(t))^{-1} & 0 \\
0 & (R^\ast(t))^{-1}
\end{bmatrix}.
\]
Considering time \( t \in [t_k, t_{k+1}) \), the \( p \)th observer-based controller is activated (that is to say the switching signal \( \sigma(t) = p \)). Let \( J^p = \frac{dV (\hat{x})}{dt} + \alpha V (\hat{x}) + \gamma_p^2 z^T \tau - \omega^T \omega \), we have
\[
J^p = \hat{x}^T \left( \Delta_1 + \gamma_p^2 H^T (\hat{\theta}, \hat{\theta}) H^p (\hat{\theta}, \hat{\theta}) \right) \hat{x} + 2 \hat{x}^T \hat{\sigma}^\ast(t) (\hat{\theta}) W (\hat{\theta}, \hat{\theta}) \omega - \omega^T \omega,
\]
\[
\Delta_1 = G^p (\theta, \hat{\theta})^T \hat{\sigma}^\ast(t) (\hat{\theta}) + \hat{\sigma}^\ast(t) (\hat{\theta}) G^p (\theta, \hat{\theta})
+ \frac{\alpha P^p (\hat{\theta})}{dt} + \alpha P^p (\hat{\theta}),
\]
\[
J^p = \begin{bmatrix}
\hat{x} \\
\omega
\end{bmatrix}^T \Theta (\hat{\theta}, \hat{\theta}) \begin{bmatrix}
\hat{x} \\
\omega
\end{bmatrix},
\]
\[
\Theta (\theta, \hat{\theta}) = \begin{bmatrix}
\Delta_1 & \ast & \ast \\
W (\theta, \hat{\theta})^T \hat{\sigma}^\ast(t) (\hat{\theta}) - \gamma_p^2 I & \ast & \ast \\
H^p (\theta, \hat{\theta}) & 0 & -I
\end{bmatrix}.
\]
Then, \( J^p < 0 \) is ensured by \( \Theta (\theta, \hat{\theta}) < 0 \). Substituting \( \hat{\sigma}^\ast(t) (\hat{\theta}), G^p (\theta, \theta), H^p (\theta, \hat{\theta}), W (\theta, \hat{\theta}), K^p (\hat{\theta}) = F^p (\hat{\theta})^{-1}, T^p (\hat{\theta}) = L^p \theta, \) in \( \Theta (\theta, \hat{\theta}) < 0 \), we have
\[
\begin{bmatrix}
\Delta_2 & \ast & \ast & \ast \\
\Delta_3 & \Delta_4 & \ast & \ast \\
N^T_{\theta} & \Delta_5 & -\gamma^2_2 I & \ast \\
\Delta_6 & C_\theta R_{\theta} & 0 & -I
\end{bmatrix} < 0,
\]
\[
\Delta_2 = Re \left( A_{\theta} P^p_{\theta} + B_{\theta} F^p \right) - \frac{dP^p_{\theta}}{dt} + \alpha P^p_{\theta},
\]
\[
\Delta_3 = (A_{\theta} - A_{\hat{\theta}}) P^p_{\theta} + (B_{\theta} - B_{\hat{\theta}}) F^p_{\hat{\theta}} + \left( L^p_{\theta} E_{\theta} \right)^T,
\]
\[
\Delta_4 = Re \left( A_{\hat{\theta}} R^p_{\theta} - L^p_{\theta} E_{\theta} \right) - \frac{dR^p_{\theta}}{dt} + \alpha R^p_{\theta},
\]
\[
\Delta_5 = N^T_{\theta} - N^T_{\hat{\theta}}, \Delta_6 = C_\theta P^p_{\theta} + D_{\theta} F^p_{\hat{\theta}}.
\]
Introducing auxiliary vector variables
\[
X^T = \begin{bmatrix}
X^T_1 & X^T_2 & X^T_3 & X^T_4 & X^T_5
\end{bmatrix},
\]
because of \( C_p \), (16) holds if
\[
\begin{bmatrix}
\Delta_2 & \ast & \ast & \ast & \ast \\
\Delta_3 & \Delta_4 & \ast & \ast & \ast \\
N^T_{\theta} & \Delta_5 & -\gamma^2_2 I & \ast & \ast \\
\Delta_6 & C_\theta R_{\theta} & 0 & -I & \ast \\
0 & R_{\theta} & 0 & 0 & 0
\end{bmatrix} X^T < 0
\]
holds subject to \( E^T L_{\theta}^T - E^T L_{\hat{\theta}}^T 0 0 -I \) \( X = 0 \)
where
\[
\Delta_2 = Re \left( A_{\theta} P^p_{\theta} + B_{\theta} F^p \right) + \alpha P^p_{\theta},
\]
\[
\Delta_3 = (A_{\theta} - A_{\hat{\theta}}) P^p_{\theta} + (B_{\theta} - B_{\hat{\theta}}) F^p_{\hat{\theta}}.
\]
Applying Finsler’s Lemma in [34], (17) holds if
\[
\begin{bmatrix}
\Delta_2 & \ast & \ast & \ast & \ast \\
\Delta_3 & \Delta_4 & \ast & \ast & \ast \\
N^T_{\theta} & \Delta_5 & -\gamma^2_2 I & \ast & \ast \\
\Delta_6 & C_\theta R_{\theta} & 0 & -I & \ast \\
0 & R_{\theta} & 0 & 0 & 0
\end{bmatrix} = \text{Re} \left( M^T E \right) < 0
\]
holds where
\[
M = \begin{bmatrix}
0 & Q M^p_{166\hat{\theta}} & 0 & 0 & Q M^p_{266\hat{\theta}}
\end{bmatrix},
\]
\[
E = \begin{bmatrix}
E^T (L^p_{\theta})^T - E^T (L^p_{\hat{\theta}})^T & 0 & 0 & -I
\end{bmatrix}.
\]
Using the condition \( EQ = [I 0] \) where \( Q = \begin{bmatrix}
E^T \left( E E^T \right)^{-1} E^\perp \end{bmatrix} \) (\( E^\perp \) is an orthogonal basis of the null(E) which can be obtained by matlab [47]) and letting \( M^p_{166\hat{\theta}} = \begin{bmatrix}
G^p_{16\theta} & 0 & G^p_{266\hat{\theta}} & G^p_{366\hat{\theta}}
\end{bmatrix} \),
\[ M_{2\theta\theta} = \begin{bmatrix} \varepsilon G_1^P & 0 \\ G_4^{\theta \theta} & G_5^{\theta \theta} \end{bmatrix}, \]

we have
\[ L_\theta^P Q M_{1\theta \theta}^P = L_\theta^P \begin{bmatrix} I & 0 \end{bmatrix} M_{1\theta \theta}^P = \begin{bmatrix} L_\theta^P G_1^P & 0 \end{bmatrix}, \]
\[ L_\theta^P Q M_{2\theta \theta}^P = \begin{bmatrix} \varepsilon G_2^P & 0 \end{bmatrix}. \]

Defining \( L_\theta^P G_1^P = M_{1\theta \theta}^P \) and applying Schur complement to (18), we know \( J^P < 0 \) is ensured by (11)-(12), in addition, \( J^P < 0 \) implies \( \frac{dV(x)}{dt} < -\alpha V(x) \) which shows the system (3) with \( \omega = 0 \) is exponentially asymptotically stable.

From the above analysis, if \( C_p \) and (11)-(12) hold, the fuzzy system (3) is exponentially stabilized by the \( p \) controller with \( H_\infty \) performance \( \gamma_p \). Since \( P_i^P \geq \frac{1}{\mu} P_i^P, R_i^P \geq \frac{1}{\mu} R_i^P \), we have \( (P_i^P)^{-1} \leq \mu (P_i^P)^{-1}, R_i^P \leq \mu R_i^P \) which means \( P^P \leq \mu P^P \).

So, for time \( t \in [t_k, t_k+1) \) we have
\[ V(\tilde{x}(t)) \leq e^{-\alpha(t-t_k)} V(\tilde{x}(t_k)). \]  
(19)

Suppose the membership functions are continuous such that at the switching moment \( h_i(t_k^-) = h_i(t_k^+), \) then, using the constraints (9)-(10) we have  
\[ V(\tilde{x}(t)) \leq \mu V(\tilde{x}(t_k^-)). \]  
(20)

and at the switching moment  
\[ V(\tilde{x}(t_k-)) \leq \mu V(\tilde{x}(t_k^+)). \]  
(21)

Working on (22) recursively with (23), we have
\[ V(\tilde{x}(t)) \leq e^{-\alpha(t-t_k)} V(\tilde{x}(t_k^-)) \leq e^{-\alpha(t-t_k)} \mu V(\tilde{x}(t_k^-)) \leq e^{-\alpha(t-t_k)} e^{-\alpha(t_k-t_k-1)} V(\tilde{x}(t_k-1)) \leq e^{-\alpha(t-t_k)} e^{-\alpha(t_k-t_k-1)} e^{\alpha(t_k-t_k-2)} V(\tilde{x}(t_k-2)) \leq \ldots \]

Since \( k \leq \frac{t_k-t_k}{\sigma_a} \), we have
\[ V(\tilde{x}(t)) \leq e^{-(t-t_k)}(\alpha-\frac{\ln\mu}{\alpha}) V(\tilde{x}(t_k^-)), \]  
(24)

\begin{algorithm}
\begin{enumerate}
\item Step 1: For different \( p \in \mathcal{S} \), solving the LMIs in \( C_p \) and (11)-(12) to find the corresponding minimal performance \( \gamma \) denoted as \( \gamma_p \) and reordering the performance from big to small as \( \gamma_{p_1} \geq \gamma_{p_2} \geq \cdots \geq \gamma_{p_{r-1}} \) where \( p_t \in \mathcal{S} \).
\item Step 2: Given \( \alpha \) denoted as \( \alpha_0 \), solving the LMIs in \( C_{p_1} \) and (11)-(12) with \( p = p_1 \), we get \( P_{i_1}^P, F_{i_1}^P, R_{i_1}^P, L_{i_1}^P \) and \( \gamma \) denoted as \( \gamma_0 \); then using the obtained \( P_{i_1}^P, R_{i_1}^P \) to solve the LMIs in \( C_{p_2}, (11)-(12) \) with \( p = p_2 \) and (9)-(10) with \( p = \{p_1, p_2\}, q = \{p_1, p_2\} \) by searching \( \mu \) and \( \alpha \) (denoted as \( \mu_{p_1}, \gamma_{p_1} \)) we get \( P_{i_2}^P, F_{i_2}^P, R_{i_2}^P, L_{i_2}^P \) and \( \gamma_{p_1} \); then using the obtained \( P_{i_2}^P, R_{i_2}^P \) to solve the LMIs in \( C_{p_3}, (11)-(12) \) with \( p = p_3 \) and (9)-(10) with \( p = \{p_2, p_3\}, q = \{p_2, p_3\} \) by searching \( \mu \) and \( \alpha \) (denoted as \( \mu_{p_2}, \gamma_{p_2} \)) we get \( P_{i_3}^P, R_{i_3}^P, F_{i_2}^P, R_{i_2}^P \) and \( \gamma_{p_2} \).
\item Step 3: Repeat step 2 till we find all the feedback gains. Let \( \mu = \mu_{p_1}, \mu_{p_2}, \ldots, \mu_{p_{r-1}}, \alpha = \max \{\alpha_{p_1}, \alpha_{p_2}, \ldots, \alpha_{p_{r-1}}\} \), \( \gamma = \max \{\gamma_{p_1}, \gamma_{p_2}, \ldots, \gamma_{p_{r-1}}\} \). The fuzzy system is stabilized by the switching controller if the ADT of the time derivative of the membership function satisfies \( \sigma_a > \frac{\ln\mu}{\alpha} \).
\end{enumerate}
\end{algorithm}

If the switching times are finite, we need not consider the dwell time and can use a more simple algorithm to get the feedback gains and observer gains as follows.

\begin{algorithm}
\begin{enumerate}
\item For different \( p \in \mathcal{S} \), solving the LMIs in \( C_p \) and (11)-(12) to find the corresponding minimal performance \( \gamma_p \), the final performance is \( \gamma = \max \{\gamma_{1, \gamma_2, \ldots, \gamma_{r-1}}\} \) and the corresponding feedback gains and observer gains are \( \{P_{i_1}^P, F_{i_1}^P, R_{i_1}^P, L_{i_1}^P\}, \{P_{i_2}^P, F_{i_2}^P, R_{i_2}^P, L_{i_2}^P\}, \ldots, \{P_{i_{r-1}}^P, F_{i_{r-1}}^P, R_{i_{r-1}}^P, L_{i_{r-1}}^P\} \).
\end{enumerate}
\end{algorithm}

Remark 1: The introduced variable \( M \) in (18) is in a special structure. Actually, we can introduce a more general variable as
\[ M = \begin{bmatrix} Q M_{1\theta \theta}^P & Q M_{2\theta \theta}^P & Q M_{3\theta \theta}^P & Q M_{4\theta \theta}^P & Q M_{5\theta \theta}^P \end{bmatrix}, \]  
(25)

where
\[ M_{1\theta \theta}^P = \begin{bmatrix} G_1^P & 0 \\ 0 & G_3^{\theta \theta} \end{bmatrix}, \]
\[ M_{2\theta \theta}^P = \begin{bmatrix} \varepsilon_1 G_1^P & 0 \\ \varepsilon_2 G_1^P & 0 \end{bmatrix}, \]
\[ M_{3\theta \theta}^P = \begin{bmatrix} \varepsilon_3 G_1^P & 0 \\ \varepsilon_4 G_1^P & 0 \end{bmatrix}, \]
\[ M_{4\theta \theta}^P = \begin{bmatrix} \varepsilon_5 G_1^P & 0 \\ \varepsilon_6 G_1^P & 0 \end{bmatrix}, \]
\[ M_{5\theta \theta}^P = \begin{bmatrix} \varepsilon_7 G_1^P & 0 \\ \varepsilon_8 G_1^P & 0 \end{bmatrix}. \]

More relaxed results can be obtained by using (25), but the parameters \( \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \) increase at the same time.

Remark 2: Note that the switching signal in this paper is generated by the time derivative of the membership function which is dependent on the system state, so the real ADT is hard to get but fortunately, the parameter \( \mu \) and \( \alpha \) can be searched to get satisfied performance \( \gamma \) and ADT condition. With respect to number of rules \( r \), number of state variables \( n \), number of control inputs \( m \), number of system outputs \( w \) for Theorem 1, the decision variables in Theorem 1 is \( n (2n + m + 1) r + 2 (n^2 - n w + w^2) r^2 \). The computation burden increases, as the number of rules increase.
IV. SIMULATION EXAMPLES

Example 1: Consider the following two-rule fuzzy system

\[
A_1 = \begin{bmatrix} 1.59 & -7.29 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2.6 & -2 \\ 1 & -3.5 + a \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad N_1 = N_2 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}.
\]

In this example, there is a perturbation parameter \( a \) in the system matrix \( A_2 \). As \( a \) varies, the \( H_{\infty} \) performance index \( \gamma \) obtained by different methods are shown in Table I.

<table>
<thead>
<tr>
<th>Methods</th>
<th>( \gamma(a = 0.1) )</th>
<th>( \gamma(a = 0.2) )</th>
<th>( \gamma(a = 0.3) )</th>
<th>( \gamma(a = 0.35) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>[34]</td>
<td>2.4217</td>
<td>13.5695</td>
<td>Infeasible</td>
<td>Infeasible</td>
</tr>
<tr>
<td>Theorem 1*</td>
<td>2.2029</td>
<td>7.8734</td>
<td>Infeasible</td>
<td>Infeasible</td>
</tr>
<tr>
<td>[35]</td>
<td>1.9615</td>
<td>7.6642</td>
<td>5.0361</td>
<td>Infeasible</td>
</tr>
<tr>
<td>Algorithm 1</td>
<td>0.5624</td>
<td>0.8065</td>
<td>1.5467</td>
<td>3.7583</td>
</tr>
<tr>
<td>Algorithm 2</td>
<td>0.4366</td>
<td>0.6214</td>
<td>1.1377</td>
<td>2.0569</td>
</tr>
</tbody>
</table>

TABLE II

The data of Algorithm 1 as \( \varepsilon = 0.01 \).

<table>
<thead>
<tr>
<th>( a = 0.1 )</th>
<th>( a = 0.2 )</th>
<th>( a = 0.3 )</th>
<th>( a = 0.35 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step 1</td>
<td>( \gamma_1 = 0.4366 )</td>
<td>( \gamma_1 = 0.6214 )</td>
<td>( \gamma_1 = 1.1377 )</td>
</tr>
<tr>
<td>Step 2</td>
<td>( \mu_1 = 0.145 )</td>
<td>( \mu_1 = 0.350 )</td>
<td>( \mu_1 = 0.225 )</td>
</tr>
</tbody>
</table>

Applying Corollary 3 in [34] with \( \alpha_1 = 0.01 \), \( T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \), we get the second line in Table 1 which shows that this method is infeasible as \( a \geq 0.3 \). Theorem 1\* means the Lyapunov function used in Theorem 1 of this paper is quadratic (\( P_1 = P_2, R_1 = R_2 \)). Theorem 1\* is also infeasible as \( a \geq 0.3 \). The Lyapunov function used in [34] and Theorem 1\* are both quadratic, but for \( a \leq 0.2 \), Theorem 1\* is less conservative than Corollary 3 in [34], for example, applying Corollary 1 in [35] with \( \delta = 10, \sigma_1 = 0.01, \rho = 1, \varepsilon = 0.001, \mu_1 = 0.3, l_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, l_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}, l_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}, l_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}, \epsilon_1 = \epsilon_2 = \frac{\pi}{2}, \epsilon_3 = \epsilon_4 = \pi, V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \), we get the fourth line which shows the method in [35] is less conservative than [34] but Theorem 1\* is still infeasible as \( a \geq 0.4 \). As shown in the Introduction, although the method in [35] is less conservative than [34] and Theorem 1\*, it has many shortcomings, for this example, 1. The stability region must belong to the intersection of \( l_1 x, l_2 x, l_3 x \) and \( l_4 x \), so the result is only local; 2. There too many parameters \( \delta, \rho, \varepsilon, \mu_1 \), and searching these parameters is a too time-consuming work; 3. The better the \( H_{\infty} \) performance index \( \gamma \) is, the smaller the stability region is; 4. Since the state is observed randomly, the method can’t deal with the case that the observed state is out of the local stability region.

Algorithm 2 is less conservative than Algorithm 1 because Algorithm 1 requires exponential stability in the average dwell time and there are more constraints such as (9), (10) than Algorithm 2. In Algorithm 1, small decay rate \( \alpha \) will lead to good performance \( \gamma \). As \( \alpha \to 0 \), we will almost get the same result as by Algorithm 2, but the parameter \( \mu \) will become bigger which leads to longer average dwell time. In practice, long average dwell time would not be satisfied for some of the cases, so the average dwell time corresponding to different performance \( \gamma \) obtained by Algorithm 1 is less than 5 (As shown in Table 2, all the results satisfy this requirement). In the following, we will show how to get the switching observer-based controller by using Algorithm 1.

Since \( r = 2 \), we have \( S = \{1, 2\} \). In Step 1, let \( \alpha = 0 \) applying (11), (12) and \( C_1 \left( P_1 \geq P_2, Q_1 \geq Q_2, \bar{Z}_1 \geq \bar{Z}_2 \right) \) we get \( \gamma = 2.0569 \); applying (11), (12) \( C_2 \left( P_1 < P_2, Q_1 < Q_2, \bar{Z}_1 < \bar{Z}_2 \right) \) we get \( \gamma = 0.9461 \), so \( \gamma_1 = 2.0569 \) and \( \gamma_2 = 0.9461 \). Since \( \gamma_1 > \gamma_2, C_1 \) can be searched first. In step 2, let \( \alpha = 0.01 \) (Denoted as \( \alpha_0 \)), applying (11), (12) and \( C_1 \), we get \( \gamma_0 = 2.9902 \) and the following matrices

\[
R_1 = \begin{bmatrix} 0.1509 & -0.0004 \\ -0.0004 & 0.0057 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0.1451 & -0.0022 \\ -0.0022 & 0.0051 \end{bmatrix},
\]

\[
P_1 = \begin{bmatrix} 0.6373 & 0.1591 \\ 0.1591 & 0.0797 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.5200 & 0.1324 \\ 0.1324 & 0.0736 \end{bmatrix},
\]

\[
F_1 = \begin{bmatrix} -0.1542 & -0.0830 \\ -0.0830 & -0.0770 \end{bmatrix}, \quad F_2 = \begin{bmatrix} -0.1309 & -0.0770 \\ -0.0770 & -0.0707 \end{bmatrix},
\]

\[
L_1 = \begin{bmatrix} 5.4066 \\ 0.7109 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 11.9481 \\ 0.9040 \end{bmatrix}.
\]

Then, let \( \alpha = 0.025, \mu = 1.13 \) (Denoted as \( \alpha_1, \mu_1 \)), applying (11), (12), \( C_2 \) and (9), (10) with the above known matrices, we get \( \gamma_2 = 3.7583 \) and the following matrices

\[
R_1 = \begin{bmatrix} 0.1345 & -0.0008 \\ -0.0008 & 0.0052 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0.1366 & -0.0018 \\ -0.0018 & 0.0057 \end{bmatrix},
\]

\[
P_1 = \begin{bmatrix} 0.5722 & 0.1431 \\ 0.1431 & 0.0712 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.5723 & 0.1425 \\ 0.1425 & 0.0799 \end{bmatrix},
\]

\[
F_1 = \begin{bmatrix} -0.1404 & -0.0749 \\ -0.0749 & -0.0836 \end{bmatrix}, \quad F_2 = \begin{bmatrix} -0.1398 & -0.0836 \end{bmatrix}.
\]
\[
L_1^2 = \begin{bmatrix}
5.3888 \\
0.6967
\end{bmatrix}, L_2^2 = \begin{bmatrix}
13.1148 \\
0.8880
\end{bmatrix}.
\]

So, the final \(H_\infty\) performance is \(\gamma = 3.7583\) and the fuzzy system is stable with \(\gamma\) if ADT satisfies \(\sigma_\alpha > \frac{\ln(\mu)}{\alpha} = \frac{\ln(1.13)}{0.025} = 4.8887\) (Of course, small decay rate \(\alpha\) will lead to better \(H_\infty\) performance, for example, if \(\alpha_1 = 0.01, \mu_1 = 1.13\) in Step 2, we will get \(\gamma = 2.1834\) which is better than the result when \(\alpha_2 = 0.025, \mu_1 = 1.13\), however, small decay rate will also lead to larger ADT which means the controller should not switch for a long time).

As \(\dot{\bar{x}}(0) = [1, -3, 2, 3]^T\), choosing \(u = u_2 = (h_1F_1^2 + h_2F_2^2) (h_1P_1^1 + h_2P_2^1)^{-1} \dot{x}\), we get \(\bar{h}_1(0) = 1.4057\) which satisfies \(H_1: h_1 > 0\). Figure 1 shows the responses of the error system \(e = x - \dot{x}\) and the time derivative of the membership function \(h_1\). The time of point A is \(t = 0.22\) and \(h_1(0.22) = -0.1438 < 0\), so at this point the controller is switched to \(u_1 = (h_1F_1^1 + h_2F_2^1) (h_1P_1^1 + h_2P_2^1)^{-1} \dot{x}\) and will not change in the following time. The ADT = \(10/2 = 5 > 4.8887\) satisfies the requirement and the error system is stabilized by the observer-based switching controller. If the initial condition is \(\bar{x}(0) = [1, -1, 2, 1]^T\), choosing \(u = u_1 = (h_1F_1^1 + h_2F_2^1) (h_1P_1^1 + h_2P_2^1)^{-1} \dot{x}\), we get \(\bar{h}_1(0) = -0.7149\) which satisfies \(H_2: h_1 \leq 0\). Figure 2 shows the evolution of the error system \(e = x - \dot{x}\) and \(h_1\). As shown in Figure 2, \(h_1\) is always negative, so the controller need not switch and the error system is stabilized by the controller \(u_1\).

**Example 2:** Consider the following continuous-time T-S fuzzy model proposed in [34] where
\[
A_1 = \begin{bmatrix}
1 & 0 \\
-1 & -1
\end{bmatrix}, A_2 = \begin{bmatrix}
2.5 & 0 \\
-2.3 & -1
\end{bmatrix},
\]
\[
N_1 = N_2 = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}, B_1 = B_2 = \begin{bmatrix}
1 & \\
0
\end{bmatrix}.
\]

![Fig. 1. Responses of dh1/dt, u and the error system e = x - \dot{x}](image)

![Fig. 2. Responses of dh1/dt, u and the error system e = x - \dot{x}](image)

V. Conclusions

In this paper, we have studied the problem of observer design for continuous-time T-S fuzzy systems with unknown premise variables. In order to deal with the time derivative of membership functions coming from the membership function dependent Lyapunov function, a switching method has been applied and two algorithms are designed to get the controller gains and observer gains such that the system is asymptotically stable if the average dwell time is larger than \(\frac{\ln(\mu)}{\alpha}\). In the end, two examples have been given to show the effectiveness of the proposed method. The method in this paper can be extended to deal with other problems such as \(H_\infty\) filtering design, estimating the domain of attraction of fuzzy system with input saturation, robust reliable control.

REFERENCES

with immeasurable premise variables via sliding mode observer”,
IEEE Trans. on Cybernetics., Available online, articles in Press,
DOI:10.1109/TCYB.2018.2874166.
[31] P. Selvaraj, B. Kaviarasan, R. Sakthivel and H. R. Karimi, “Fault-
[32] H. K. Lam, “A review on stability analysis of continuous-time fuzzy-