King's Research Portal

DOI:

[10.1016/j.ins.2020.09.043](https://doi.org/10.1016/j.ins.2020.09.043)

Document Version

Peer reviewed version

[Link to publication record in King's Research Portal](#)

Citation for published version (APA):

Zheng, H., Xie, W. B., Lam, H. K., & Wang, L. (2021). Membership-function-dependent stability analysis and local controller design for T–S fuzzy systems: A space-enveloping approach. *INFORMATION SCIENCES*, 548, 233-253. <https://doi.org/10.1016/j.ins.2020.09.043>

Citing this paper

Please note that where the full-text provided on King's Research Portal is the Author Accepted Manuscript or Post-Print version this may differ from the final Published version. If citing, it is advised that you check and use the publisher's definitive version for pagination, volume/issue, and date of publication details. And where the final published version is provided on the Research Portal, if citing you are again advised to check the publisher's website for any subsequent corrections.

General rights

Copyright and moral rights for the publications made accessible in the Research Portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognize and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the Research Portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the Research Portal

Take down policy

If you believe that this document breaches copyright please contact librarypure@kcl.ac.uk providing details, and we will remove access to the work immediately and investigate your claim.

Membership-Function-Dependent Stability Analysis and Local Controller Design for T–S Fuzzy Systems: A Space-Enveloping Approach

Hua Zheng^a, Wen-Bo Xie^{*a,b}, *Member, IEEE*, Hak-Keung Lam^c, *Fellow, IEEE*, Likui Wang^d

^aCollege of Automation, Harbin University of Science and Technology, Harbin, 150080, P. R. China

^bShanghai Key Laboratory of Power Station Automation Technology, and the school of Mechatronic Engineering and Automation, Shanghai University, Shanghai, 200072, P. R. China

^cDepartment of Informatics, King's College London, London, WC2R 2LS, United Kingdom

^dSchool of Mathematics and Information Science, Nanchang Hangkong University, Nanchang 330063, China

Abstract

For Takagi–Sugeno (T–S) fuzzy systems, relaxed stability conditions are obtained by more effectively enveloping the trajectory of the membership functions (MFs) in a unified space of MF. Considering an open-loop T–S fuzzy system, the system premise variables operation domain can be divided into a series of subdomains. Based on an MF unified space extremum calculation technique, the MFs extremum values in each premise variable corresponding to the unified space subdomain are calculated. With these extremes, a tight local convex polyhedron enveloping the MF trajectory is constructed in each subdomain. Thus, linear matrix inequality (LMI) stability conditions are derived via a piecewise Lyapunov function. Then, a state-feedback local controller is designed to close the system loop. From a geometric viewpoint, MF extremum enveloping and piecewise linear-approximation methods are both utilized to achieve relaxed stability and robustness conditions. Finally, several examples are adopted to illustrate the metrics of the proposed approaches.

Keywords: Takagi–Sugeno fuzzy system, membership function, piecewise Lyapunov function, local controller, conservatism

*corresponding author, e-mail: xiewenbo@hrbust.edu.cn

1. Introduction

Takagi–Sugeno (T–S) fuzzy systems can approximate nonlinear systems with arbitrary accuracy. In other word, with a fuzzy system, a nonlinear system can be described as a convex combination of sublinear systems blended by membership functions (MFs) [1]–[2]. Thus, abundant linear control technologies can be applied to nonlinear control problems [3]–[5].

For research on T–S fuzzy control systems, conservatism in stability analysis can be seen as a primary obstacle. Many methods have been proposed to reduce this constraint. Classic methods, such as quadratic Lyapunov function (QLF) methods, can be found in [6, 7], and some improved results are provided in [8, 9]. To reduce the conservatism caused by the QLF, piecewise Lyapunov function (PLF) was investigated in [10]. Meanwhile, fuzzy Lyapunov function (FLF) approaches are illustrated in [11, 12]. Further, the line-integral FLF is discussed in [13]. According to a structural description in [14], the above results can be called MF-independent approaches, which do not involve MF information in stability analysis.

To further reduce conservatism, MF-dependent methodologies have been investigated widely [14]. To begin, the MF boundary information was artfully brought into stability analysis, as in [15]. Further, by dividing the premise variable operating domain into connected subdomains, the local boundary information of the MF was employed, as in [16]. Moreover, a parameterized LF-based systematic method with a slack matrix technique can be found in [17]. Recently, some results have focused on the MF-approximation approach, which can more effectively introduce MF information into stability analysis. In [18, 19], staircase functions are used to approximate the original MFs through a series of sample points. The piecewise linear-approximation approach can be found in [20, 21], which can provide better fitting precision. The polynomial MF-approximation method with even smaller approximation errors was found in [22]. The sum-of-square (SOS) convex optimization technique could be used to solve the derived polynomial form stability conditions. However, when the SOS technique is used, the stability conditions must be solved on the entire region of the premise variable, through which, some conservatism is involved. Some research results focus

on improving LF forms to tackle the conservatism issue, such as PLF approaches in [23]–[25], FLF methods in [26, 27], switching Lyapunov function (SLF) techniques in [28, 29], and the switching polynomial Lyapunov function in [30]. Recently, a unified MF space extremum-based method was proposed [31]. It provides a new geometrical perspective to investigate process involving MFs. In this approach, although the MF is utilized in the synthesis process with high approximation precision, the controller form still restricts the conservatism-reduction effects.

For the fuzzy controller form, parallel distribution compensation (PDC) methods are very popular in current research [32]–[34]. Thus, Pólya’s theorem can be applied to greatly reduce conservatism. In addition, robust H_∞ control with PDC control was investigated in [35]–[37]. Some imperfect premise-matching methods (imperfect matched cases, as in [38], are induced by different premise variables, MFs, or the rule number in the system plant and controller) have been investigated in [39]–[41]. However, many PDC methods and some imperfect premise-matching controllers are globally applicable throughout the entire operating domain of the system state. To improve such methods, switching fuzzy control approaches, where different sets of feedback gains are employed for the control process according to the operating subdomains, can be found in [42]–[43]. For the imperfect premise matching case, a type of MF-dependent local observer-controller/state feedback method is investigated in [44], where the design freedom is further expanded. However, one can find that the controller MFs still cause conservatism in stability analysis, even in MF-dependent methods.

Motivated by the above issues, this study investigates system stability from a geometrical perspective. A new form of imperfect premise-matching controller design method is proposed. For an open-loop T–S fuzzy system, the MF trajectory is divided into specific segments, and each region is independently enclosed in a local convex polyhedron formed by the MF’s extremum values. The division method is carefully designed such that the equilibrium of the premise variable is enclosed by only one subregion. Thus, the switching actions are guaranteed to be finite. The corresponding vertices of the local polyhedron and PLF are utilized to acquire numerical solvable stability conditions. The global stability property is ensured. From a geometric viewpoint, under the same MF-approximation precision, the conservatism in stability conditions

is relaxed effectively. For a closed-loop T–S fuzzy system, a novel imperfect premise-matching local controller method is designed, and stability analysis is carried out both by the MF extremum and piecewise linear-approximation approaches. Compared with existing results, reduced conservatism is achieved.

The remainder of this paper is organized as follows. Section 2 describes the extreme-based MF-dependent method and analyzes the stability of an open-loop T–S fuzzy system. Section 3 describes the piecewise linear approximation method, designs the local controller, and analyzes the stability of a closed-loop T–S fuzzy system. Section 4 presents the simulation results. Some conclusions are presented in Section 5.

Notations: In this paper, $\mathbb{R}^n$ represents the n -dimensional Euclidean space. A^T and A^{-1} represent the transpose and inverse of a matrix A , respectively. $\star$ denotes the transpose of the corresponding block matrix. I denotes an identity matrix of appropriate dimension. $\| \star \| = \sqrt{\star^T \star}$ is the Euclidean norm of a vector, and $\|X\|_2 = \{X : \int_0^\infty \|X(t)\|^2 dt < \infty\}$ is an L_2 norm.

2. Stability Analysis for Open-Loop T–S Fuzzy Systems

2.1. Open-loop T–S fuzzy system

Consider the following T–S fuzzy system:

Rule i : IF $z_1(t)$ is M_{i1} ... and $z_p(t)$ is M_{ip} , THEN

$$\dot{x}(t) = A_i x(t), \quad (1)$$

where M_{ij} is a fuzzy set, where $i = 1, 2, \dots, l$ with l representing the number of model fuzzy rules and $j = 1, 2, \dots, p$ with p representing the number of premise variables; $z(t) = [z_1(t) \dots z_p(t)]^T \in \mathbb{R}^p$ is the premise variable; $x(t) \in \mathbb{R}^n$ is the system state vector; and $A_i \in \mathbb{R}^{n \times n}$ is the system matrix in the i th rule. The fuzzy system is referred to as

$$\dot{x}(t) = \sum_{i=1}^l h_i(z(t)) A_i x(t), \quad (2)$$

where

$$\sum_{i=1}^l h_i(z(t)) = 1, h_i(z(t)) = \frac{\varpi_i(z(t))}{\sum_{i=1}^l \varpi_i(z(t))} \geq 0, \varpi_i(z(t)) = \prod_{j=1}^p \psi_{ij}(z_j(t)).$$

$h_i(z(t))$ denotes the normalized grade of membership, $\psi_{ij}(z_j(t))$ is the grade of membership of $z_j(t)$ in M_{ij} , and $\varpi_i(z(t))$ is the weight of the i th rule. For simplicity, t will be

omitted for variables without confusion.

Assumption 1. In this *study*, we assume that the premise variable z is a function of x : $z = \gamma(x)$. $\gamma(x) \rightarrow z_0$ (monotonically or *non-monotonically*) when $x \rightarrow 0$. An energy function $V_z = (z - z_0)^T Q (z - z_0)$ is defined, where $Q = Q^T > 0$. The converging property
85 of z can be indicated by V_z as $V_z \rightarrow 0$ monotonically or *non-monotonically*.

2.2. Extreme-based convex polyhedron envelope method

Lemma 1. ^[31] $\forall i, j = 1, 2, \dots, l$, *system* (2) is asymptotically stable *if* there exists a matrix $P > 0$ *such that*

$$\sum_{j=1}^l \eta_{ij} (A_j^T P + P A_j) < 0, \quad (3)$$

where η_{ij} *represents the* vertices of a convex polyhedron *that* contains points $h(z) = (h_1(z), h_2(z), \dots, h_l(z))$, *and* constraints $0 \leq \eta_{ij} \leq 1$ and $\sum_{j=1}^l \eta_{ij} = 1$ are satisfied.

From Lemma 1, one can see that stability conditions can be derived via the vertices
90 of the convex polyhedron. By defining the vertices as *checkpoints and comparing* the convex polyhedrons described by *the checkpoints*, it is easy to find that smaller convex polyhedrons can lead to more relaxed conservatism.

2.3. Stability *analysis*

The operation domain of *the* premise variable z can be divided into q adjacent and
95 non-overlapping *subregions* as $z = \bigcup_{k=1}^q \Lambda_k$, where Λ_k denotes *the k th subregion*. In *this section*, we *provide an equilibrium-based* division method. It is worth *noting* that there should be one *subregion* containing a *neighborhood* of the equilibrium of z as $z_0 = \gamma(0)$. Define the *neighborhood is defined* as $E_1 = \{z | (z - z_0)^T Q (z - z_0) \leq \delta, \delta > 0\}$. Suppose *that* $E_1 \subseteq \Lambda_1$; a case of *a* two-dimensional premise variable by setting $q = 9$
100 is shown in Fig. 1.

Lemma 2. The premise variable z is divided into q *subregions* by the *equilibrium-based* division method *above*. *System* (2) is locally asymptotically stable in each *z subregion*, *and* *system* (2) is globally asymptotically stable.

Proof: With the **equilibrium-based** division method and Assumption 1, if the system is locally stable, z will **go along with** the trajectory $\gamma(x)$. When z crosses the boundary between two adjacent **subdomains**, x **dose** not jump. Considering x **that** is locally stable, x will attenuate **perpetually**. According to Assumption 1, $z \rightarrow z_0$. If $V_z = (z - z_0)^T Q(z - z_0)$ monotonically attenuates, for any given small $\delta > 0$, $V_z \leq \delta$ will be satisfied in finite time. If V_z **attenuates non-monotonically** as in Fig. 10, $V_z(T) \leq \delta$ for $t \geq T$ will also be guaranteed in finite time. If $V_z \leq \delta$, z will always **remain as** $E_1 = \{z | (z - z_0)^T Q(z - z_0) \leq \delta, \delta > 0\}$. Therefore, one can say that the switching actions are finite. For **system** state x , the premise variable z **does** not change at the switching moment. Then, **system** (2) can be guaranteed to be globally asymptotically stable. This completes the proof. $\square$

Fig 1. The switching trajectory of z with $q = 9$

Remark 1. If the system equilibrium is located at **the** boundaries of the **subregions**, infinite switching may **occur**. **System (2) can be guaranteed to be stable only under the average dwell time (ADT) structure [45]. However, with the equilibrium-based division method, such an issue can be avoided due to its finite switching actions. One can also notice that a larger value of δ reduces the switching times. Correspondingly, a smaller δ can lead to more switching actions.**

Remark 2. With a **subregion-dependent PLF** $V_{\Lambda_k}(x) = x^T P(\Lambda_k)x$, the **variation of**

each V_k during the switch process is described in Fig. 2. If the premise variable z enters the specified neighborhood, the final PLF (usually denoted as V_{Λ_1}) converges to zero without switching.

Fig 2. Schematic diagram of PLF

Remark 3. One can also divide the MFs into q sub-regions, and there exists a mapping of $F : \mathbb{R}^p \rightarrow \mathbb{R}^l$, which makes $\Xi_k = F(h_1(\Lambda_k), \dots, h_l(\Lambda_k))$, where Ξ_k is the k th MF subdomain corresponding to the k th premise variable subdomain.

Theorem 1. $\forall s, j = 1, 2, \dots, l, v = 1, 2, \dots, l-1, k = 1, 2, \dots, q$, and q is the number of subregions divided by z . System (2) is asymptotically stable if there exist matrices $P(\Lambda_k) = P^T(\Lambda_k) \in \mathbb{R}^{n \times n}$, and thus the following inequalities hold:

$$\sum_{j=1}^l \lambda_{ksvj}(\Lambda_k)(P(\Lambda_k)A_j + A_j^T P(\Lambda_k)) < 0, \quad (4)$$

where Λ_k denotes the k th subregion divided over the operation region of premise variables and $\lambda_{ksv} = (\lambda_{ksv1}, \lambda_{ksv2}, \dots, \lambda_{ksvl})$ are the checkpoints obtained by Algorithm 1 in [31] satisfying $0 \leq \lambda_{ksvj} \leq 1$ and $\sum_{j=1}^l \lambda_{ksvj} = 1$.

Proof: Construct a PLF as $V_{\Lambda_k}(x) = x^T P(\Lambda_k)x$. Based on Lemmas 1 and 2, asymptotic stability can be achieved with condition (4). This completes the proof. $\square$

Remark 4. Compared with some switch Lyapunov function methods as in [23] and [30], the continuity of the Lyapunov function is not required for the proposed method. Moreover, unlike the noncontinuous switching approach in [29], the starting/ending points of the PLF of the current subregion are not required to be smaller than those of the adjacent region, which yields more design flexibility.

Remark 5. One can clearly see that the Lyapunov function in Theorem 1 is differentiable except at the boundaries. Considering the division boundaries, it can be concluded that the left and right derivatives exist for vertices $z_{r_i=1k}$ and $z_{r_i=2k}$, respectively.

$$\frac{\partial V_{\Lambda_k}}{\partial z_r}(z_r = z_{r_i=1k}) = \lim_{z_r \rightarrow z_{r_i=1k}^+} \frac{\Psi_{r_i=1k}}{z_r - z_{r_i=1k}}, \quad (5)$$

$$\frac{\partial V_{\Lambda_k}}{\partial z_r}(z_r = z_{r_i=2k}) = \lim_{z_r \rightarrow z_{r_i=2k}^-} \frac{\Psi_{r_i=2k}}{z_{r_i=2k} - z_r}, \quad (6)$$

where $\Psi_{r_i=1k} = V_{\Lambda_k}(z_1, \dots, z_r, \dots, z_p) - V_{\Lambda_k}(z_1, \dots, z_{r_i=1k}, \dots, z_p)$, and $\Psi_{r_i=2k} = V_{\Lambda_k}(z_1, \dots, z_{r_i=2k}, \dots, z_p) - V_{\Lambda_k}(z_1, \dots, z_r, \dots, z_p)$.

Remark 6. The analysis was carried out in a unified MF space. Thus, compared with traditional approximation approaches for MF, such as the staircase method in [19] and linear piecewise method in [21], one does not need to consider the number of divisions in the premise variables. However, the number of fuzzy rules is necessary for the subdivision process. Thus, one can decide which type of method is appropriate. For example, in a fuzzy system with few premise variables but a large number of fuzzy rules, a premise variable division method should be adopted. However, for a system with large dimensions of premise variables and few fuzzy rules, the unified MF space division method would be more suitable. The computational complexity of the different methods is summarized in Table 1. It should be noted that all the results described are revised for the open-loop system (2).

Table 1. Computational Complexity

Methods	Number of LMIs	Number of decision variables
Theorem 1 in [19]	$q + 2l + 1$	$n(n + 1)$
Theorem 1 in [21]	$2^p(q_1 + q_2) + 4l$	$n(n + 1)(l + 1/2)$
Theorem 2 in [31]	$ql(l - 1)$	$\frac{1}{2}n(n + 1)$
Theorem 1 in this paper	$ql(l - 1)$	$\frac{1}{2}n(n + 1)q$

Remark 7. A three-rule fuzzy system example with $q = 3$ for Theorem 1 is shown in Fig. 3, which is obviously different from previous works. Theorem 2 in [31] investigates

Fig 3. Local convex polyhedron in **three-dimensional** space of MF with $q = 3$

the local convex polyhedron in **an** MF space. However, the stability conditions were obtained **using a QLF**. It also leads to an **imperfect** involvement of MFs. One can refer to the region of $(\bigcup_{k=1}^q \Xi_k) \cup \Pi$ in Fig. 3. With Theorem 1 in this paper, only **the** region $\bigcup_{k=1}^q \Xi_k$ is included in the stability analysis. Π in Fig. 3 is excluded, which **greatly reduces** the conservatism. Thus, Theorem 2 in [31] is a special case of Theorem 1 in this study.

3. Stability of **Closed-Loop** T-S Fuzzy Systems

Based on system (2), given a closed-loop T-S fuzzy system,

$$\dot{x} = \sum_{i=1}^l h_i(A_i x + B_i u), \quad (7)$$

where $B_i \in \mathbb{R}^{n \times m}$ are system control parameter matrices **and** u is the control input.

3.1. Piecewise **approximation of membership functions**

Proposition 1. ^[46] Functions $h_i(z)$ can be written as follows:

$$h_i(z) = \sum_{k=1}^q \sum_{i_1=1}^2 \cdots \sum_{i_p=1}^2 \prod_{r=1}^p v_{ri_r k}(z_r) (\hat{h}_i(z_{i_1 \dots i_p k}) + \Delta h_i(z_{i_1 \dots i_p k})), \quad (8)$$

where $v_{r_i,k}(z_r)$ are normalized linear MFs with properties $0 \leq v_{r_i,k}(z_r) \leq 1$ and $\sum_{k=1}^q$
 $\sum_{i_1=1}^2 \dots \sum_{i_p=1}^2 \prod_{r=1}^p v_{r_i,k}(z_r) = 1$, $v_{r_i,1k} = (z_r - z_{r_i=2k}) / (z_{r_i=1k} - z_{r_i=2k})$ and $v_{r_i,2k} =$
 $1 - v_{r_i,1k}$. z_r is the r th premise variable, i_r represents the vertices of piecewise MFs for
 z_r , and k denotes the k th subdomain Λ_k on z , and q is the number of premise-variable
 165 division domains. It can be described as $\Lambda_k = \{z_1^k, z_2^k, \dots, z_p^k\}$, where $z_{r1k} \leq z_r \leq z_{r2k}$.
 $\Delta h_i(z_{i_1 \dots i_p k})$ is defined as the approximation errors in the domain constructed by vertices
 $z_{(i_r=2)k}$ and $z_{(i_r=1)k}$ satisfying $\Delta \underline{h}_i(z_{i_1 \dots i_p k}) \leq \Delta h_i(z_{i_1 \dots i_p k}) \leq \Delta \bar{h}_i(z_{i_1 \dots i_p k})$, where $\Delta \bar{h}_i(z_{i_1 \dots i_p k})$
 and $\Delta \underline{h}_i(z_{i_1 \dots i_p k})$ are the upper and lower bounds of $\Delta h_i(z_{i_1 \dots i_p k})$, respectively.

170 Motivated by [20], an example of a piecewise linear function is shown in Fig. 4,
 which illustrates a two-dimensional premise-variables case. One can find that the nor-
 malized MFs $v_{r_i,k}(z_r)$ are linear. $\Lambda_{(k=1)}$ represents the square shaded region in Fig. 4.

Fig 4. k th subdomain and normalized membership functions

Lemma 3. ^[21] With the piecewise linear approximation method, the related function
 has the following property:

$$\sum_{i=1}^l h_i = \sum_{i=1}^l \hat{h}_i, \sum_{i=1}^l \Delta h_i = 0. \quad (9)$$

3.2. Local controller design

Here, a piecewise-linear local controller in the k th subdomain is constructed as

$$u = K(\Lambda_k)x, \quad (10)$$

where $K(\Lambda_k) \in \mathbb{R}^{m \times n}$ are local control gain matrices. After substituting u into system (7), the closed-loop system can be written as

$$\dot{x} = \sum_{i=1}^l h_i(\Lambda_k)(A_i + B_i K(\Lambda_k))x. \quad (11)$$

3.3. Extreme-based stability

Theorem 2. $\forall s, i, j = 1, \dots, l, v = 1, 2, \dots, l-1, k = 1, \dots, q$, q is the number of subregions divided by z . System (11) is asymptotically stable if there exist matrices $N(\Lambda_k) = N^T(\Lambda_k) > 0 \in \mathbb{R}^{n \times n}$, and $Y(\Lambda_k) \in \mathbb{R}^{m \times n}$; thus, making the following inequalities hold:

$$\sum_{i=1}^l \lambda_{k,svi}(\Lambda_k) \Phi_i(\Lambda_k) < 0, \quad (12)$$

where $\Phi_i(\Lambda_k) = N(\Lambda_k)A_i^T + A_i N(\Lambda_k) + Y^T(\Lambda_k)B_i^T + B_i Y(\Lambda_k)$, and $\lambda_{k,sv} = (\lambda_{k,sv1}, \lambda_{k,sv2}, \dots, \lambda_{k,svl})$ is the checkpoint obtained from Algorithm 1 in [31].

Proof: Construct a local Lyapunov function as $V_{\Lambda_k}(x) = x^T P(\Lambda_k)x$. Then, $\dot{V}$ can be given in each subregion MF Λ_k as

$$\dot{V}_{\Lambda_k} = \sum_{i=1}^l h_i(\Lambda_k)x^T (P(\Lambda_k)A_i + A_i^T P(\Lambda_k) + P(\Lambda_k)B_i K(\Lambda_k) + K(\Lambda_k)^T B_i^T P(\Lambda_k))x. \quad (13)$$

To guarantee

$$\sum_{i=1}^l h_i(\Lambda_k)(P(\Lambda_k)A_i + A_i^T P(\Lambda_k) + P(\Lambda_k)B_i K(\Lambda_k) + K(\Lambda_k)^T B_i^T P(\Lambda_k)) < 0, \quad (14)$$

pre- and post-multiply matrices $N(\Lambda_k) = P^{-1}(\Lambda_k)$ and their transposes with the above conditions, respectively. Define $Y(\Lambda_k) = K(\Lambda_k)N(\Lambda_k)$. Then, we have

$$\sum_{i=1}^l h_i(\Lambda_k) \Phi_i(\Lambda_k) < 0. \quad (15)$$

Based on Lemma 1 and Lemma 2, the proof is complete. $\square$

Remark 8. The local controller can be treated as a state-dependent switch controller.

Once the premise variables enter the k th subregion, the k th controller starts working. It can be seen that the controller gain matrices of the adjacent subdomains are all

suitable for the boundary vertices. In other words, the vertices can be stabilized by any *neighboring subdomain* controller. The number of applicable controllers for each vertex is dependent on the divided *subdomain* and position of the vertex. For example, when $q = 2$, any boundary *vertex* can be stabilized by two adjacent controllers (apart from the starting $z_{i_1=1,k=1}$ or the ending $z_{i_1=2,k=q}$).

To apply the results of Theorem 2 to a practical system, a H_∞ robust case is investigated here. Consider the fuzzy system (7) with an external disturbance:

$$\begin{aligned}\dot{x} &= \sum_{i=1}^l h_i(A_i x + B_i u + E_i \omega), \\ y &= \sum_{i=1}^l h_i C_{2i} x,\end{aligned}\tag{16}$$

where $E_i \in \mathbb{R}^{n \times \tau}$ and $C_{2i} \in \mathbb{R}^{o \times n}$ are system parameter matrices, $\omega \in \mathbb{R}^\tau$ is the unknown external disturbance with $\omega \in \mathbf{L}_2(0, \infty)$, and $y \in \mathbb{R}^o$ is the interested output;

Assumption 2. Similar to Assumption 1, the premise variable z is assumed to be a function of y as $z = \gamma(y)$. $\gamma(y) \rightarrow z_0$ (monotonically or non-monotonically) when $y \rightarrow 0$. An energy function $V_z = (z - z_0)^T Q (z - z_0)$ is defined, where $Q = Q^T > 0$. The convergence of z can be indicated by V_z as: $V_z \rightarrow 0$ monotonically or non-monotonically converges to zero.

Corollary 1. $\forall s, i, j = 1, \dots, l, v = 1, 2, \dots, l-1, k = 1, \dots, q$, q is the number of *subregions* divided on z ; if there exist matrices $N(\Lambda_k) = N^T(\Lambda_k) > 0 \in \mathbb{R}^{n \times n}$, $Y(\Lambda_k) \in \mathbb{R}^{m \times n}$ and scalar γ making the following inequalities hold, system (16) is robust and stable, and the index $\|y\|_2^2 \leq \gamma^2 \|\omega\|_2^2$ is satisfied:

$$\sum_{i=1}^l \lambda_{ksvi} \Gamma_i(\Lambda_k) < 0,\tag{17}$$

where

$$\Gamma_i(\Lambda_k) = \begin{bmatrix} \Phi_i(\Lambda_k) & E_i & N(\Lambda_k) C_{2i}^T \\ \star & -\gamma^2 I & 0 \\ \star & \star & -I \end{bmatrix}$$

and $\Phi_i(\Lambda_k) = N(\Lambda_k) A_i^T + A_i N(\Lambda_k) + Y^T(\Lambda_k) B_i^T + B_i Y(\Lambda_k)$; $\lambda_{ksv} = (\lambda_{ksv1}, \lambda_{ksv2}, \dots, \lambda_{ksvl})$ are the checkpoints obtained from Algorithm 1 in [31].

Proof: Construct a local Lyapunov function as $V_{\Lambda_k}(x) = x^T P(\Lambda_k) x$. Then, $\dot{V}$ can

be given in each subregion Λ_k as

$$\dot{V}_{\Lambda_k} = \sum_{i=1}^l h_i(\Lambda_k) (x^T (P(\Lambda_k)A_i + A_i^T P(\Lambda_k) + P(\Lambda_k)B_i K(\Lambda_k) + K(\Lambda_k)^T B_i^T P(\Lambda_k))x + x^T P(\Lambda_k)E_i \omega + \omega^T E_i^T P(\Lambda_k)x). \quad (18)$$

Considering the zero-initial state situation and defining J for each Λ_k as

$$\begin{aligned} J &= \int_0^T (y^T y - \gamma^2 \omega^T \omega + \frac{d}{dt} V) dt - V_T \\ &= \int_0^T \begin{bmatrix} x \\ \omega \end{bmatrix}^T \begin{bmatrix} \Psi_h(\Lambda_k) + C_{2h}^T(\Lambda_k)C_{2h}(\Lambda_k) & P(\Lambda_k)E_h(\Lambda_k) \\ \star & -\gamma^2 I \end{bmatrix} \begin{bmatrix} x \\ \omega \end{bmatrix} dt - V_T, \end{aligned} \quad (19)$$

where $\Psi_h(\Lambda_k) = \sum_{i=1}^l h_i(\Lambda_k)\Psi_i(\Lambda_k)$, $\Psi_i(\Lambda_k) = P(\Lambda_k)A_i + A_i^T P(\Lambda_k) + P(\Lambda_k)B_i K(\Lambda_k) + K(\Lambda_k)^T B_i^T P(\Lambda_k)$, $C_{2h}(\Lambda_k) = \sum_{i=1}^l h_i(\Lambda_k)C_{2i}$, $E_h(\Lambda_k) = \sum_{i=1}^l h_i(\Lambda_k)E_i$, and $V_T = x_T^T P(\Lambda_k)x_T$ is the value of the Lyapunov function at time T .

Because $V_{\Lambda_k} > 0$, $J < 0$ will be satisfied if the matrices in (19) are negative. Pre- and post-multiply matrices $\text{diag}\{N(\Lambda_k), I\}$ and their transposes with the matrices in (19), respectively. Define $N(\Lambda_k) = P^{-1}(\Lambda_k)$ and $\Phi_h(\Lambda_k) = N(\Lambda_k)\Psi_h(\Lambda_k)N(\Lambda_k)$. Then, the equivalent conditions can be obtained:

$$\begin{bmatrix} \Phi_h(\Lambda_k) + N(\Lambda_k)C_{2h}^T(\Lambda_k)C_{2h}(\Lambda_k)N(\Lambda_k) & E_h(\Lambda_k) \\ \star & -\gamma^2 I \end{bmatrix} < 0. \quad (20)$$

With the Schur complement, we have

$$\sum_{i=1}^l h_i(\Lambda_k)\Gamma_i(\Lambda_k) < 0. \quad (21)$$

Based on Lemmas 1-2, the proof is completed. $\square$

To achieve optimal robust control effects, (17) should be solved with minimized γ .

3.4. Piecewise approximation stability

Theorem 3. $\forall i = 1, \dots, l, i_1, \dots, i_p = 1, 2$, and $k = 1, \dots, q$, q is the number of subregions divided by z . System (11) is asymptotically stable if there exist matrices $N(\Lambda_k) = N^T(\Lambda_k) > 0 \in \mathbb{R}^{n \times n}$, $W(z_{i_1 \dots i_p k}) = W(z_{i_1 \dots i_p k})^T \in \mathbb{R}^{n \times n}$, $T_i(z_{i_1 \dots i_p k}) = T_i^T(z_{i_1 \dots i_p k}) \in \mathbb{R}^{n \times n}$, and $Y(z_{i_1 \dots i_p k}) \in \mathbb{R}^{m \times n}$, making the following inequalities hold:

$$T_i(z_{i_1 \dots i_p k}) > 0, \quad (22)$$

$$\Phi_i(z_{i_1 \dots i_{pk}}) + W(z_{i_1 \dots i_{pk}}) + T_i(z_{i_1 \dots i_{pk}}) > 0, \quad (23)$$

$$\begin{aligned} & \sum_{i=1}^l \hat{h}_i(z_{i_1 \dots i_{pk}}) \Phi_i(z_{i_1 \dots i_{pk}}) + \Delta \bar{h}_i(z_{i_1 \dots i_{pk}}) (\Phi_i(z_{i_1 \dots i_{pk}}) + W(z_{i_1 \dots i_{pk}}) + T_i(z_{i_1 \dots i_{pk}})) \\ & - \Delta \underline{h}_i(z_{i_1 \dots i_{pk}}) T_i(z_{i_1 \dots i_{pk}}) < 0, \end{aligned} \quad (24)$$

where $\Phi_i(z_{i_1 \dots i_{pk}}) = N(\Lambda_k)A_i^T + A_iN(\Lambda_k) + B_iY(z_{i_1 \dots i_{pk}}) + Y^T(z_{i_1 \dots i_{pk}})B_i^T$; $Y(z_{i_1 \dots i_{pk}}) = K(z_{i_1 \dots i_{pk}})N(\Lambda_k)$; $\Delta \underline{h}_i(z_{i_1 \dots i_{pk}})$ and $\Delta \bar{h}_i(z_{i_1 \dots i_{pk}})$, are predefined scalars satisfying $\Delta \underline{h}_i(z_{i_1 \dots i_{pk}}) \leq \Delta h_i(z_{i_1 \dots i_{pk}}) \leq \Delta \bar{h}_i(z_{i_1 \dots i_{pk}})$.

Proof: Considering the piecewise linear approximation method with the local controller (10), one can construct a PLF $V_{\Lambda_k}(x) = x^T P(\Lambda_k)x$. Then, $\dot{V}$ can be given in each subregion Λ_k as

$$\dot{V}_{\Lambda_k} = \sum_{i=1}^l h_i(\Lambda_k) (P(\Lambda_k)A_j + A_j^T P(\Lambda_k) + P(\Lambda_k)B_iK(\Lambda_k) + K(\Lambda_k)^T B_i^T P(\Lambda_k)). \quad (25)$$

Based on Proposition 1, a series of sufficient conditions for guaranteeing the negativity of (25) is

$$\begin{aligned} & \zeta_{qp} \sum_{i=1}^l (\hat{h}_i(z_{i_1 \dots i_{pk}}) + \Delta h_i(z_{i_1 \dots i_{pk}})) (P(\Lambda_k)A_j + A_j^T P(\Lambda_k) + P(\Lambda_k)B_iK(\Lambda_k) \\ & + K^T(\Lambda_k) B_i^T P(\Lambda_k)) < 0. \end{aligned} \quad (26)$$

Pre- and post-multiply matrices $N(\Lambda_k) = P^{-1}(\Lambda_k)$ and their transposes with the above conditions, respectively. Based on Lemma 3 and conditions (22)-(23), one can determine that condition (24) can ensure the negativity of $\dot{V}_{\Lambda_k}$. Then, based on Lemma 2, the proof is complete. $\square$

Remark 9. To illustrate the results of the piecewise linear approximation, with $p = 1$ and $q = 3$, a three-rule example is shown in Fig. 5. One can see that the original MFs are contained in a parallelogram surrounded by black dashed lines in each subregion. Regarding the vertices of the parallelogram area as the vertices of polyhedron in MF space, one can obtain the checkpoints $h(z_{i_1 \dots i_{pk}}) = (\hat{h}_1(z_{i_1 \dots i_{pk}}) + \Delta \bar{h}_1(z_{i_1 \dots i_{pk}}) / \Delta \underline{h}_1(z_{i_1 \dots i_{pk}}), \dots, h_l(z_{i_1 \dots i_{pk}}) + \Delta \bar{h}_l(z_{i_1 \dots i_{pk}}) / \Delta \underline{h}_l(z_{i_1 \dots i_{pk}}))$ in MF space (red circles). The local convex polyhedrons and vertices are shown in Fig. 6. The MF trajectory in a three-dimensional space is enveloped by a convex polyhedron in each region. It is worth noting that some vertices in the space of premise variables are involved in the convex

polyhedron (red dots). Compared with the plane envelope approach in Theorem 2, it is easy to find that more useless information is introduced in the piecewise approach. However, more slack matrices and decision variables can be adopted in Theorem 3. Thus, it is difficult to state which method is more beneficial for reducing conservatism.

Fig 5. Piecewise approximation method with $q = 3$ in premise variable space

Fig 6. Piecewise approximation envelope with $q = 3$ in membership function space

Remark 10. The computational complexity of Theorems 2 and Theorem 3 is shown

in Table 2. The number of LMIs and decision variables in Theorem 3 exponentially increases as the dimension of the premise variables becomes large, whereas Theorem 2 does not.

Table 2. Computational Complexity

Methods	Number of LMIs	Number of decision variables
Theorem 2	$ql(l-1)$	$\frac{1}{2}n(n+1)q + mnq$
Theorem 3	$2^p q + 2ql$	$n(n+1)q + mnq + \frac{1}{2}n(n+1)ql$

Corollary 2. $\forall i = 1, \dots, l, i_1, \dots, i_p = 1, 2$, and $k = 1, \dots, q$, q is the number of subregions divided by z . If there exist matrices $N(\Lambda_k) = N^T(\Lambda_k) > 0 \in \mathbb{R}^{n \times n}$, $W(z_{i_1 \dots i_p k}) = W(z_{i_1 \dots i_p k})^T \in \mathbb{R}^{(n+\tau+o) \times (n+\tau+o)}$, $T_i(z_{i_1 \dots i_p k}) = T_i^T(z_{i_1 \dots i_p k}) \in \mathbb{R}^{(n+\tau+o) \times (n+\tau+o)}$, $Y(z_{i_1 \dots i_p k}) \in \mathbb{R}^{m \times n}$ and a scalar γ that make the following inequalities hold, system (16) is robust stable and the index $\|y\|_2^2 \leq \gamma^2 \|\omega\|_2^2$ is satisfied:

$$T_i(z_{i_1 \dots i_p k}) > 0, \quad (27)$$

$$\Upsilon_i(z_{i_1 \dots i_p k}) + W(z_{i_1 \dots i_p k}) + T_i(z_{i_1 \dots i_p k}) > 0, \quad (28)$$

$$\begin{aligned} & \sum_{i=1}^l \hat{h}_i(z_{i_1 \dots i_p k}) \Upsilon_i(z_{i_1 \dots i_p k}) + \Delta \bar{h}_i(z_{i_1 \dots i_p k}) (\Upsilon_i(z_{i_1 \dots i_p k}) + W(z_{i_1 \dots i_p k}) + T_i(z_{i_1 \dots i_p k})) \\ & - \Delta \underline{h}_i(z_{i_1 \dots i_p k}) T_i(z_{i_1 \dots i_p k}) < 0, \end{aligned} \quad (29)$$

where

$$\Upsilon_i(z_{i_1 \dots i_p k}) = \begin{bmatrix} \Phi_i(z_{i_1 \dots i_p k}) & E_i & N(\Lambda_k) C_{2i}^T \\ \star & -\gamma^2 I & 0 \\ \star & \star & -I \end{bmatrix}$$

and $\Phi_i(z_{i_1 \dots i_p k}) = N(\Lambda_k) A_i^T + A_i N(\Lambda_k) + B_i Y(z_{i_1 \dots i_p k}) + Y^T(z_{i_1 \dots i_p k}) B_i^T$; $Y(z_{i_1 \dots i_p k}) = K(z_{i_1 \dots i_p k}) N(\Lambda_k)$, $\Delta \underline{h}_i(z_{i_1 \dots i_p k})$ and $\Delta \bar{h}_i(z_{i_1 \dots i_p k})$ are predefined scalars satisfying $\Delta \underline{h}_i(z_{i_1 \dots i_p k}) \leq \Delta \bar{h}_i(z_{i_1 \dots i_p k})$.

Proof: With a similar proof process to those of Corollary 1 and Theorem 3, the proof can be completed. $\square$

Remark 11. For Corollaries 1-2, y will converge to y'_0 instead of equilibrium y_0 due to the disturbance w . Considering Assumption 2, the system premise variable z will converge to z'_0 instead of equilibrium z_0 . Thus, $E_1 = \{z|(z - z_0)^T Q(z - z_0) \leq \delta, \delta > 0\}$ should be suitable for z'_0 inside. Moreover, the distance between z'_0 and the boundary of E_1 should be smaller than δ .

Remark 12. In this section, we mainly focus on the control problem of dynamic systems described by (7) or (16). If the parameters in these closed-loop systems are unknown, one can refer to the piecewise affine identification method mentioned in [47] to finish the parameter estimation task in practical engineering applications.

Remark 13. For real engineering systems, tracking/following control tasks or uncertainties are commonly observed. By using the sliding mode control (SMC) technique in Theorems 2-3 and Corollaries 1-2, one can tackle the above issues effectively and even improve the stability converging speed by adopting various reaching laws of SMC, as in [3, 48].

4. Simulation Examples

Example 1. A three-rule open-loop T-S fuzzy model described in [31] is given as

$$A_1 = \begin{bmatrix} 0 & 1 \\ -0.06 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & a \\ -1.94 & -1 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & b \\ -0.5 & -1.5 \end{bmatrix},$$

where a and b are constant parameters to be determined, and the MFs are described as

$$\begin{cases} h_1(x_1) = \frac{1}{3} + \frac{0.7}{\sqrt{36}} \cos x_1 - \frac{0.7}{\sqrt{12}} \sin x_1 \\ h_2(x_1) = \frac{1}{3} + \frac{0.7}{\sqrt{36}} \cos x_1 + \frac{0.7}{\sqrt{12}} \sin x_1. \\ h_3(x_1) = \frac{1}{3} - \frac{0.7}{\sqrt{9}} \cos x_1 \end{cases}$$

The operation region is divided into $\Lambda_1 = \{x_1 | 2\alpha\pi - \frac{\pi}{3} \leq x_1 \leq 2\alpha\pi + \frac{\pi}{3}\}$, $\Lambda_2 = \{x_1 | 2\alpha\pi + \frac{\pi}{3} \leq x_1 \leq 2\alpha\pi + \pi\}$, and $\Lambda_3 = \{x_1 | 2\alpha\pi + \pi \leq x_1 \leq \frac{5\pi}{3}\}$, where α is an integer. We carry out a conservatism-comparison experiment between Theorem 1 in this paper and Theorem 2 in [31]. The number of MF divisions is both set to $q = 3$.

With Algorithm 1 in [31], the checkpoints are calculated as

$$\begin{aligned}
\lambda_{111} &= (0.5667, 0.3333, 0.1000), \lambda_{112} = (0.5667, 0.2167, 0.2167), \\
\lambda_{121} &= (0.4060, 0.4890, 0.1050), \lambda_{122} = (0.4110, 0.4890, 0.1000), \\
\lambda_{131} &= (0.5667, 0.2167, 0.2167), \lambda_{132} = (0.4060, 0.3773, 0.2167), \\
\lambda_{211} &= (0.4053, 0.4896, 0.1051), \lambda_{212} = (0.4053, 0.4896, 0.1051), \\
\lambda_{221} &= (0.1448, 0.5667, 0.2885), \lambda_{222} = (0.3282, 0.5667, 0.1051), \\
\lambda_{231} &= (0.2019, 0.4896, 0.3085), \lambda_{232} = (0.1448, 0.5467, 0.3085), \\
\lambda_{311} &= (0.1444, 0.5464, 0.3092), \lambda_{312} = (0.1444, 0.3092, 0.5464), \\
\lambda_{321} &= (0.1000, 0.5464, 0.3536), \lambda_{322} = (0.1444, 0.5464, 0.3092), \\
\lambda_{331} &= (0.1444, 0.3092, 0.5464), \lambda_{332} = (0.1000, 0.3536, 0.5464).
\end{aligned}$$

Fig 7. The feasible region of Theorem 1 (x) and Theorem 2 in [31] (o) for $q = 3$

The comparison results are shown in Fig. 7, which clearly demonstrates the conservatism reduction. For Theorem 1 in this paper, the cases of $q = 3$ and $q = 5$ are also compared in Fig. 8, which illustrates that with increasing MF division number, conservativeness is further relaxed. For $q = 3$, with $a = 10, b = 16.5$, the phase trajectories in Theorem 1, are shown in Fig. 9. One can see that the system state converges to zeros. Given $V_z = x_1^T x_1$, choose $\delta = 1 < \frac{\pi}{3}$. It can be easily found that the chosen δ can be

seen as a **neighborhood** of the equilibrium $z_0 = 0$. **When the** initial state is $x_0 = [\frac{\pi}{3}, 8]^T$, from Fig. 10, after the peak of V_z converging into δ , V in Fig. 11 will **no longer switch**.

Fig 8. The feasible region of Theorem 1 (.) with $q=3$ and (o) for $q=5$

Fig 9. Phase trajectory for Theorem 1 with $q=3$

Fig 10. The attenuation effects of V_z

Fig 11. Piecewise Lyapunov function for Theorem 1

Example 2. A three-rule T-S fuzzy control model, as described in [34], is as follows:

$$A_1 = \begin{bmatrix} -1 & -4 \\ 8 & -9 \end{bmatrix}, A_2 = \begin{bmatrix} -2+a & -3 \\ -7+b & -9 \end{bmatrix}, A_3 = \begin{bmatrix} -4 & 0 \\ 0 & -1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} -0.45 \\ -3 \end{bmatrix}, B_2 = \begin{bmatrix} -1 \\ -3 \end{bmatrix}, B_3 = \begin{bmatrix} -1 \\ -2 \end{bmatrix},$$

where a and b are adjustable parameters that can test the system's conservatism. The MFs of the fuzzy system are given as

$$\begin{cases} h_1(x_1) = \frac{1}{2(1+e^{x_1})} \\ h_2(x_1) = \frac{1}{2(1+e^{x_1+1})} \\ h_3(x_1) = 1 - h_1(x_1) - h_2(x_1) \end{cases}$$

The operation region of the premise variable is $-10 \leq x_1 \leq 10$. If q is odd, the subregion is divided into $\Lambda_k = \{x_1 | 20(k - q/2 - 1)/q \leq x_1 \leq 20(k - q/2)/q, k = 1, 2, \dots, q\}$. If q is even and $q = 2$, $\Lambda_1 = \{x_1 | -10 \leq x_1 \leq \delta, \delta > 0\}$ and $\Lambda_2 = \{x_1 | \delta \leq x_1 \leq 10, \delta > 0\}$. If q is even and $q \geq 4$, $\Lambda_{k1} = \{x_1 | -10 + 20(k_1 - 1)(q - 1)/(q(q - 2)) \leq x_1 \leq -10 + 20k_1(q - 1)/(q(q - 2)), k_1 = 1, 2, \dots, q/2 - 1\}$, $\Lambda_{\frac{q}{2}} = \{x_1 | -10/q \leq x_1 \leq 10/q\}$, $\Lambda_{q/2+k_2} = \{x_1 | 10/q + 20(k_2 - 1)(q - 1)/q^2 \leq x_1 \leq 10/q + 20k_2(q - 1)/q^2, k_2 = 1, 2, \dots, q/2\}$. The feasible regions of Theorem 1 in [34], and of Theorems 2 and 3 ($q = 15$ for these results) in this paper are compared in Fig. 12. One can see that conservatism was reduced.

Setting the division number $q = 20$, the local stable design parameters in [21] are set as $|\dot{x}_1|_{max} = 0.4$ and $|h^{x_1}| = 0.24$. The comparisons of feasible regions for Theorem 1 in [21], Theorems 2 and 3 in this paper are shown in Fig. 13. And Theorem 3 in this paper achieves a larger feasible region. From Remark 8, the local controller gain $K(\Lambda_k)$ and $K(\Lambda_{k+1})$ can be used to stabilize the closed-loop system at vertices, $20(k - q/2)/q$ except $x_1 = -10$ or $x_1 = 10$. The control scheme at vertices can be randomly selected within two adjacent controllers. The phase portraits for $a = 17, b = 19$ of Theorem 2 and $a = 18, b = 19$ of Theorem 3 are shown in Fig. 14 and 15, respectively, which demonstrate satisfactory stabilization effects. With the initial state $x_0 = [6, -6]^T$ for Theorem 2 and $x_0 = [-10, -20]^T$ for Theorem 3, the PLF V converges to zero and ends

inside the $\Lambda_{10} = [-0.5, 0.5]$ subregion, as shown in Fig. 16 and 17. Given $V_z = x_1^T x_1$ and choosing $\delta = 0.5$, we can see from Fig. 17 (d) that there is no switching after the system enters the Λ_{10} domain.

Fig 12. The feasible region of Theorem 3 in [34] (o), Theorem 2 (x) and Theorem 3 (.) for $q = 15$

Fig 13. The feasible region of Theorem 1 in [21] (x), Theorem 2 (o) and Theorem 3 (.) with $q = 20$

Considering the property that $h_{i=1,2,3}$ remains almost unchanged when $x_1 \geq 10$ or $x_1 \leq -10$, $u = K(\Lambda_1)x_1$ can be utilized for the area $x_1 \leq -10$. In addition, $u = K(\Lambda_{20})x_1$ can be employed for the area $x_1 \geq 10$. This will provide a locally stable result for

$x_1 \geq 10$ and $x_1 \leq -10$. To improve the local stable effects, more vertices in the region of $x_1 \geq 10$ and $x_1 \leq -10$ can be involved in stability conditions, which will certainly reduce the conservatism.

Fig 14. Phase trajectory for Theorem 2 with $q = 20$

Fig 15. Phase trajectory for Theorem 3 with $q = 20$

Fig 16. Piecewise Lyapunov function for Theorem 2

Fig 17. Piecewise Lyapunov function for Theorem 3

Example 3. Based on Corollaries 1-2, a floating production storage and offloading (FPSO) vessel system in [49] is adopted here for carrying out dynamic positioning control. Vessel length $L_s = 200.6\text{m}$ and water displacement $\Delta = 73097.15\text{t}$. With the dimensionless bio-system in [21], the parameter matrices are given as

$$A_i = \begin{bmatrix} 0_{3 \times 3} & \mathfrak{R}_i \\ 0_{3 \times 3} & G \end{bmatrix}, B_i = \begin{bmatrix} 0_{3 \times 3} & M \end{bmatrix}^T, E_i = B_i, C_{2i} = \begin{bmatrix} I_{3 \times 3} & 0_{3 \times 3} \end{bmatrix}, i = 1, 2, 3,$$

where $\alpha = 0.0349$ rad is given, as in [21]

$$\mathfrak{R}_1 = \begin{bmatrix} \alpha & 1 & 0 \\ -1 & \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathfrak{R}_2 = \begin{bmatrix} 1 & \alpha & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathfrak{R}_3 = \begin{bmatrix} \alpha & -1 & 0 \\ 1 & \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$G = \begin{bmatrix} -0.0797 & 0 & 0 \\ 0 & -0.0818 & -0.1224 \\ 0 & -0.2254 & -0.2468 \end{bmatrix}, M = \begin{bmatrix} 0.9215 & 0 & 0 \\ 0 & 0.7802 & 1.4811 \\ 0 & 1.4811 & 7.4562 \end{bmatrix}.$$

The premise variable is $x_3 \in [-\pi/2, \pi/2]$, and the MFs are

$$h_1(x_3) = \begin{cases} 1 & x_3 < -\pi/2 \\ 0.5 - 0.5 \sin(2x_3 + \pi/2) & -\pi/2 \leq x_3 \leq 0, \\ 0 & x_3 > 0 \end{cases}$$

$$h_3(x_3) = \begin{cases} 0 & x_3 < 0 \\ 0.5 + 0.5 \sin(2x_3 - \pi/2) & 0 \leq x_3 \leq \pi/2, \\ 1 & x_3 > \pi/2 \end{cases}$$

$$h_2(x_3) = 1 - h_1(x_3) - h_3(x_3).$$

The external disturbance of the system is induced by sea wind, waves, and current. Considering the typical wind, wave, and current coefficients for an FPSO vessel, one can obtain the following simplified low-frequency dimensionless force and moment of ocean disturbance for a bio-system:

$$w(t) = 0.0025e^{-0.00001t}[-0.01 \cos(x_3 - \pi/6), 0.1 \sin(x_3 - \pi/6), \sin(2x_3 - \pi/6)]^T,$$

where the initial phase $-\pi/6$ denotes the disturbance coming from north-by-east 30° in

a north-east earth coordinate described in [49]. For this practical system, **overly** large control gain matrices are unacceptable. Thus, it is necessary to **impose** the following **constraints** to avoid **great** control gain matrices:

$$\begin{bmatrix} -\varepsilon_1 & Y^T(\Lambda_k) \\ \star & -I \end{bmatrix} < 0, \begin{bmatrix} -\varepsilon_2 & Y^T(z_{i_1, \dots, i_{p_k}}) \\ \star & -I \end{bmatrix} < 0, \begin{bmatrix} -\sigma & I \\ \star & N(\Lambda_k) \end{bmatrix} < 0, \quad (30)$$

where **the** predefined scalars $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, and $\sigma > 0$. The constraint parameters are set as $\varepsilon_1 = \varepsilon_2 = 0.1$ and $\sigma = 1$.

For $q = 3$, the operation domain formed by **the** premise variable is divided as $\Lambda_k = \{x_1 | \pi(k - q/2 - 1)/q \leq x_1 \leq \pi(k - q/2)/q, k = 1, 2, 3\}$. For $q = 1$ and $q = 3$, **the** mincx function is used to optimize γ^2 in Corollaries 1-2 via **MATLAB** R2019a. The comparisons among **the** indices γ_{min} are shown in Table 3. When $q = 3$, with the initial state $x_0 = [2m \ 3m \ -70^\circ \ 0 \ 0 \ 0]$, **the trajectory of the system state is depicted** in Fig. 18 and 19. One can see **that** the system state converges to zero rapidly. From Fig. 20, the PLF V converges to zero and enters the subregion $\{\Lambda_2[-30^\circ, 30^\circ]\}$. Given $V_z = x_3^T x_3$ and **choosing** $\delta = 2^\circ < 30^\circ$, it is easy to find that no switching actions **occur** after x_3 entering Λ_2 . The external disturbance is shown in Fig. 21. The robust performance with **the** zero-initial condition is shown in Fig. 22.

Table 3. γ_{min} Comparison

Method	γ_{min}
Theorem 1 in [36]	no solution
Theorem 2 in [35]	11.8717
Theorem 1 in [37]	8.7557
Theorem 1 in [21]	23.7980 ($q = 1$), 11.0206 ($q = 3$)
Corollary 1	3.4456 ($q = 1$), 3.4246 ($q = 3$)
Corollary 2	3.4449 ($q = 1$), 3.4122 ($q = 3$)

Fig 18. Vessel state for Corollary 1

Fig 19. Vessel state for Corollary 2

Fig 20. Piecewise Lyapunov function for Corollaries 1–2

Fig 21. External disturbance for Corollaries 1–2

Fig 22. Robust performance for Corollaries 1–2

5. Conclusions

In this study, a geometrical perspective is adopted to investigate the extremum-based and piecewise approximation stability analysis methods. Conservatism is reduced by independently considering the local polyhedron with the PLF approach. From the comparisons in the simulation section, one can find that the extremum-based MF envelope method can more effectively involve the MF information. However, the piecewise approximation method can introduce more slack matrices and decision variables to further reduce the conservatism.

References

- [1] T. Takagi, M. Sugeno, Fuzzy identification of systems and its applications to modeling and control, IEEE Transactions on Systems, Man, and Cybernetics 15 (1) (1985) 116–132.
- [2] C. Liu, H. K. Lam, X. J. Ban, X. D. Zhao, Design of polynomial fuzzy observer-controller with membership functions using unmeasurable premise variables for nonlinear systems, Information Sciences 355-356 (2016) 186–207.
- [3] A. Ferrara, G. Ferrari-Trecate, C. Vecchio, Sliding mode control for coordination

in multi-agent systems with directed communication graphs, in: 2007 European Control Conference (ECC), Kos, Greece, 2007, pp. 1477–1484.

- 315 [4] H. B. Chen, P. Shi, C. C. Lim, A new unified input-to-state stability criterion for impulsive stochastic delay systems with markovian switching, *International Journal of Robust and Nonlinear Control* 30 (1) (2020) 159–181.
- [5] Y. Yang, C. Xu, D. Yue, X. P. Xie, Output feedback tracking control of a class of continuous-time nonlinear systems via adaptive dynamic programming approach, 320 *Information Sciences* 469 (2018) 1–13.
- [6] H. O. Wang, K. Tanaka, M. F. Griffin, An approach to fuzzy control of nonlinear systems: Stability and design issues, *IEEE Transactions on Fuzzy Systems* 4 (1996) 14–23.
- [7] J. Yoneyama, M. Nishikawa, H. Katayama, A. Ichikawa, Output stabilization of 325 Takagi-Sugeno fuzzy systems, *Fuzzy Sets and Systems* 111 (2) (2000) 253–266.
- [8] T. Zhao, J. Xiao, H. M. Sheng, T. Wang, H_∞ control of continuous-time interval type-2 T-S fuzzy systems via dynamic output feedback controllers, *Neurocomputing* 165 (2015) 133–143.
- [9] F. S. Yang, S. P. Guan, D. H. Wang, Quadratically convex combination approach 330 to stability of T-S fuzzy systems with time-varying delay, *Journal of the Franklin Institute* 351 (7) (2014) 3752–3765.
- [10] J. B. Qiu, G. Feng, H. J. Gao, Static-output-feedback H_∞ control of continuous-time T-S fuzzy affine systems via piecewise Lyapunov functions, *IEEE Transactions on Fuzzy Systems* 21 (2) (2013) 245–261.
- 335 [11] T. M. Guerra, L. Vermeiren, LMI-based relaxed nonquadratic stabilization conditions for nonlinear systems in the Takagi-Sugeno’s form, *Automatica* 40 (2004) 823–829.
- [12] H. J. Yang, X. Li, Z. X. Liu, L. Zhao, Robust fuzzy-scheduling control for nonlinear systems subject to actuator saturation via delta operator approach, 340 *Information Sciences* 272 (2014) 158–172.

- [13] B. J. Rhee, S. C. Won, A new fuzzy Lyapunov function approach for a Takagi-Sugeno fuzzy control system design, *Fuzzy Sets and Systems* 157 (9) (2006) 1211–1228.
- [14] H. K. Lam, A review on stability analysis of continuous-time fuzzy-model-based control systems: From membership-function-independent to membership-function-dependent analysis, *Engineering Applications of Artificial Intelligence* 67 (2018) 390–408.
- [15] H. K. Lam, F. H. F. Leung, Stability analysis of fuzzy control systems subject to uncertain grades of membership, *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)* 35 (6) (2005) 1322–1325.
- [16] M. Narimani, H. K. Lam, Relaxed LMI-based stability conditions for Takagi-Sugeno fuzzy control systems using regional-membership-function-shape-dependent analysis approach, *IEEE Transactions on Fuzzy Systems* 17 (5) (2009) 1221–1228.
- [17] L. A. Mozelli, R. M. Palhares, F. O. Souza, E. M. A. M. Mendes, Reducing conservativeness in recent stability conditions of T-S fuzzy systems, *Automatica* 45 (6) (2009) 1580–1583.
- [18] J. T. Pan, S. M. Fei, Y. D. Ni, M. X. Xue, New approaches to relaxed stabilization conditions and H_∞ control designs for T-S fuzzy systems, *Journal of Control Theory and Applications* 10 (1) (2012) 82–91.
- [19] H. K. Lam, N. Mohammad, Quadratic-stability analysis of fuzzy-model-based control systems using staircase membership functions, *IEEE Transactions on Fuzzy Systems* 18 (1) (2010) 125–137.
- [20] H. K. Lam, Y. B. Zhao, L. G. Wu, Linear matrix inequalities-based membership-function-dependent stability analysis for non-parallel distributed compensation fuzzy-model-based control systems, *IET Control Theory and Applications* 8 (8) (2014) 614–625.

- [21] W. B. Xie, Y. L. Wang, J. Zhang, M. Y. Fu, Novel separation principle based H_∞ Observer-Controller design for a class of T-S fuzzy systems, IEEE Transactions on Fuzzy Systems 26 (6) 3206–3221.
- [22] M. Narimani, H. K. Lam, SOS-based stability analysis of polynomial fuzzy-model-based control systems via polynomial membership functions, IEEE Transactions on Fuzzy Systems 18 (5) (2010) 862–871.
- [23] G. Feng, H_∞ controller design of fuzzy dynamic systems based on piecewise lyapunov functions, IEEE transactions on systems, man, and cybernetics 34 (1) (2004) 283–92.
- [24] Y. J. Chen, H. Ohtake, K. Tanaka, W. Wang, H. O. Wang, Relaxed stabilization criterion for T-S fuzzy systems by minimum-type piecewise-lyapunov-function-based switching fuzzy controller, IEEE Transactions on Fuzzy Systems 20 (6) (2012) 1166–1173.
- [25] C. Z. Zhang, H. K. Lam, J. B. Qiu, C. J. Liu, Q. J. Chen, A new design of membership-function-dependent controller for T-S fuzzy systems under imperfect premise matching, IEEE Transactions on Fuzzy Systems 27 (7) (2019) 1428–1440.
- [26] H. K. Lam, J. Lauber, Membership-function-dependent stability analysis of fuzzy-model-based control systems using fuzzy Lyapunov functions, Information Sciences 232 (5) 253–266.
- [27] X. P. Xie, Z. W. Liu, X. L. Zhu, An efficient approach for reducing the conservatism of LMI-based stability conditions for continuous-time T-S fuzzy systems, Fuzzy Sets and Systems 263 (2015) 71–81.
- [28] D. Zhai, A. Y. Lu, J. X. Dong, Q. L. Zhang, Stability analysis and state feedback control of continuous-time T-S fuzzy systems via a new switched fuzzy Lyapunov function approach, Applied Mathematics and Computation 293 (2017) 586–599.

- [29] H. K. Lam, Polynomial Fuzzy Model-Based Control Systems: Stability Analysis and Control Synthesis Using Membership Function Dependent Techniques, 395
Spring, 2016.
- [30] H. K. Lam, M. Narimani, H. Y. Li, H. H. Liu, Stability analysis of polynomial-fuzzy-model-based control systems using switching polynomial Lyapunov functions, IEEE Transactions on Fuzzy Systems 21 (5) (2013) 800–813.
- [31] X. Z. Yang, H. K. Lam, L. G. Wu, Membership-dependent stability conditions for 400
type-1 and interval type-2 T-S fuzzy systems, Fuzzy Sets and Systems 356 (2018) 44–62.
- [32] A. Sala, C. Ariño, Local stability of open- and closed-loop fuzzy systems, in: 2006 IEEE International Symposium on Intelligent Control, Munich, Germany, 405
2006, pp. 2384–2389.
- [33] S. K. Hong, R. Langari, An LMI-based H_∞ fuzzy control system design with TS framework, Information sciences 123 (3-4) (2000) 163–179.
- [34] J. Chen, S. Y. Xu, B. Y. Zhang, Y. M. Chu, Y. Zou, New relaxed stability and stabilization conditions for continuous-time T-S fuzzy models, Information Sciences 410
329 (2016) 447–460.
- [35] B. Y. Zhang, S. S. Zhou, T. Li, A new approach to robust and non-fragile H_∞ control for uncertain fuzzy systems, Information Sciences 177 (22) (2007) 5118–5133.
- [36] S. H. Kim, Nonquadratic H_∞ stabilization conditions for observer-based T-S fuzzy 415
control systems, IEEE Transactions on Fuzzy Systems 22 (3) (2014) 699–706.
- [37] W. B. Xie, H. Li, Z. H. Wang, J. Zhang, Observer-based controller design for a T-S fuzzy system with unknown premise variables, International Journal of Control, Automation and Systems 17 (4) (2018) 907–915.
- [38] H. K. Lam, Stability analysis and performance design for fuzzy-model-based control system under imperfect premise matching, IEEE Transactions on Fuzzy Systems 420
17 (4) (2008) 949–961.

- [39] X. H. Chang, G. H. Yang, Relaxed stabilization conditions for continuous-time Takagi-Sugeno fuzzy control systems, *Information Sciences* 180 (17) (2010) 3273–3287.
- 425 [40] D. H. Lee, J. B. Park, Y. H. Joo, A fuzzy Lyapunov function approach to estimating the domain of attraction for continuous-time Takagi-Sugeno fuzzy systems, *Information Sciences* 185 (1) (2012) 230–248.
- [41] C. Peng, S. D. Ma, X. P. Xie, Observer-based non-PDC control for networked T-S fuzzy systems with an event-triggered communication, *IEEE Transactions on Cybernetics* 47 (8) (2017) 2279–2287.
- 430 [42] L. K. Wang, H. K. Lam, Further study on observer design for continuous-time Takagi-Sugeno fuzzy model with unknown premise variables via average dwell time, *IEEE Transactions on Cybernetics*, doi : 10.1109/TCYB.2019.2933696.
- [43] L. K. Wang, H. K. Lam, New stability criterion for continuous-time Takagi-Sugeno fuzzy systems with time-varying delay, *IEEE Transactions on Cybernetics* 49 (4) (2019) 1551–1556.
- 435 [44] C. Z. Zhang, H. K. Lam, J. B. Qiu, C. J. Liu, Q. J. Chen, A new design of membership-function-dependent controller for T-S fuzzy systems under imperfect premise matching, *IEEE Transactions on Fuzzy Systems* 27 (7) (2019) 1428–1440.
- 440 [45] S. Shi, Z. Y. Fei, P. Shi, C. K. Ahn, Asynchronous filtering for discrete-time switched T-S fuzzy systems, *IEEE Transactions on Fuzzy Systems* (2019). doi : 10.1109/TFUZZ.2019.2917667.
- [46] H. K. Lam, Polynomial fuzzy-model-based control systems: stability analysis via piecewise-linear membership functions, *IEEE Transactions on Fuzzy Systems* 19 (3) (2011) 588–593.
- 445 [47] G. Ferrari-Trecate, M. Muselli, D. Liberati, M. Morari, A clustering technique for the identification of piecewise affine systems, *Automatica* 39 (2) (2003) 205–217.

- 450
- [48] H. Y. Li, J. H. Wang, H. P. Du, H. R. Karimi, Adaptive sliding mode control for Takagi-Sugeno fuzzy systems and its applications, *IEEE Transactions on Fuzzy Systems* 26 (2) (2018) 531–542.
- [49] T. I. Fossen, *Handbook of marine craft hydrodynamics and motion control*, John Wiley & Sons, 2011.