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QUANTUM UNIQUE ERGODICITY FOR HALF-INTEGRAL WEIGHT AUTOMORPHIC FORMS

STEPHEN LESTER AND MAKSYM RADZIWIŁŁ

ABSTRACT. We investigate the analogue of the Quantum Unique Ergodicity (QUE) conjecture for half-integral weight automorphic forms. Assuming the Generalized Riemann Hypothesis (GRH) we establish both QUE for half-integral weight Hecke Maaß cusp forms for $\Gamma_0(4)$ lying in Kohnen's plus subspace and mass equidistribution for half-integral weight holomorphic Hecke cusp forms for $\Gamma_0(4)$ lying in Kohnen's plus subspace. By combining the former result along with an argument of Rudnick, it follows that under GRH the zeros of these holomorphic Hecke cusp forms equidistribute with respect to hyperbolic measure on $\Gamma_0(4)\backslash\mathbb{H}$ as the weight tends to infinity.

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1. INTRODUCTION

1.1. Quantum Unique Ergodicity. Let (M, g) be a compact, smooth Riemannian manifold without boundary and denote by $-\Delta_g$ the Laplace-Beltrami operator for M . Also, let vol_g denote the normalized Riemannian volume form for M . Given an orthonormal basis $\{\phi_\ell\}$ of $L^2(M, \text{dvol}_g)$ -normalized eigenfunctions of Δ_g with eigenvalues λ_ℓ the Quantum Ergodicity Theorem of Shnirel'man [46], Colin de Verdière [4] and Zelditch [55] implies that if the geodesic flow is ergodic on the unit cotangent bundle S^*M with respect to the normalized Liouville measure then there exists a density one subsequence of eigenfunctions $\{\phi_{\ell_j}\}$ along which the measures $|\phi_{\ell_j}|^2 \text{dvol}_g$ weakly converge to dvol_g as $j \rightarrow \infty$.¹

If in addition M has negative curvature then more is known about the geodesic flow beyond ergodicity, e.g., exponential decay of correlations, central limit theorem. In this setting Rudnick and Sarnak [40] have conjectured that the quantum limit is unique, that is, $|\phi_\ell|^2 \text{dvol}_g$ weakly converges to dvol_g along the entire sequence of eigenfunctions $\{\phi_\ell\}$. This is known as the Quantum Unique Ergodicity (QUE) conjecture. The general form of this conjecture appears to be far from being solved, however, in the special arithmetic setting where $M = \text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ Lindenstrauss [25] and Soundararajan [49] have shown QUE holds for a Hecke basis of eigenfunctions.

¹The Quantum Ergodicity Theorem gives the stronger result that ϕ_{ℓ_j} equidistributes within S^*M as $j \rightarrow \infty$. The weak convergence $|\phi_{\ell_j}|^2 \text{dvol}_g \Rightarrow \text{dvol}_g$ follows from projecting to configuration space.

Prior to these breakthroughs, this result was known to follow conditionally under the assumption of the Generalized Riemann Hypothesis (GRH) from Watson’s formula [54]. Additionally, other cases of arithmetic QUE have been established by Lindenstrauss [25] and Silberman-Venkatesh [45].

The hyperbolic surface $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ also has the structure of a complex analytic Riemannian surface and it is natural to wonder if the holomorphic analogue of QUE holds. The analogue of the eigenfunctions of $\Delta_{\mathbb{H}}$ lying in L^2 are the holomorphic modular forms, which are intrinsic to number theory. In this context Lindenstrauss’s method does not seem to apply. However, by combining two different approaches, one of which uses Watson’s formula [54], Holowinsky and Soundararajan [12, 50, 13] established mass equidistribution for Hecke cusp forms in the limit as the weight tends to infinity. Their result has the beautiful corollary proved by Rudnick [39] that the zeros of all such Hecke cusp forms equidistribute with respect to hyperbolic measure within compact subsets of $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ with boundary measure zero as the weight tends to infinity. Holomorphic QUE has been established in other cases as well by Marshall [29], Nelson [33, 34], and Nelson-Pitale-Saha [35].

1.2. Half-integral weight automorphic forms. Let

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}.$$

Also, let

$$\theta(z) = \sum_{n=1}^{\infty} e(n^2 z) \quad (e(z) = e^{2\pi iz})$$

denote the classical theta function. For $\gamma \in \Gamma_0(N)$ and $z \in \mathbb{H}$ set $j(\gamma, z) := \theta(\gamma z)/\theta(z)$. Let N be divisible by 4 and χ a character modulo N . For $k \in \mathbb{N}$, a holomorphic function $f : \mathbb{H} \rightarrow \mathbb{C}$ is a *modular form of weight $k + \frac{1}{2}$* , level N and character χ provided that f transforms in the following way

$$f(\gamma z) = \chi(d)j(\gamma, z)^{2k+1}f(z) \quad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N), z \in \mathbb{H}$$

and that f grows at most polynomially at the cusps of $\Gamma_0(N)$. We say that f is a cusp form if f vanishes at all the cusps of $\Gamma_0(N)$.

Let $\theta_1(z) = \mathrm{Im}(z)^{1/4} \theta(z)$ and $J(\gamma, z) := \theta_1(\gamma z)/\theta_1(z)$. A *half-integral weight Maaß cusp form* for $\Gamma_0(4)$ is a smooth function $f : \mathbb{H} \rightarrow \mathbb{C}$, which transforms as follows

$$f(\gamma z) = J(\gamma, z)f(z) \quad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N), z \in \mathbb{H}$$

and is also an eigenfunction of

$$\Delta_{1/2} = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - \frac{1}{2}iy \cdot \frac{\partial}{\partial x}.$$

Additionally, f has at most polynomial growth at the cusps of $\Gamma_0(4)$ and satisfies

$$\iint_{\Gamma_0(4)\backslash\mathbb{H}} |f(z)|^2 d\text{vol}(z) < \infty, \quad d\text{vol}(z) := \frac{dx dy}{y^2}, \quad z = x + iy.$$

If f vanishes at the cusps of $\Gamma_0(4)$ it is called a cusp form.

Modular forms of half-integral weights emerge naturally in number theory. An example of such a modular form of weight $\ell + \frac{3}{2}$ for $\Gamma_0(4)$ is

$$\theta_P(z) = \sum_{m \in \mathbb{Z}^3} P(m) e(|m|^2 z), \quad (e(z) = e^{2\pi i z})$$

where P is a homogeneous harmonic polynomial on \mathbb{R}^3 of degree ℓ . Other examples of half-integral weight holomorphic forms are known to encode information on periods of integral weight forms (see [44]), and in the level aspect, assuming the Birch and Swinnerton-Dyer conjecture, their coefficients reflect the order of the Tate-Shafarevich group of quadratic twists of elliptic curves with rank 0.

Both the holomorphic and real analytic half-integral weight modular forms play special roles in number theory. For instance, Duke [6] established the equidistribution of CM points on $\text{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ by combining a formula of Maaß (which needed to be extended to the non-compact setting) along with proving a subconvexity bound for the Fourier coefficients of half-integral weight Maaß forms. This built upon the pioneering work of Iwaniec [15] who had established similar estimates for half-integral weight holomorphic modular forms, which has the application to the equidistribution of lattice points on a sphere.

In this paper we set out to investigate the analogues of QUE in the setting of holomorphic half-integral weight forms for $\Gamma_0(4)\backslash\mathbb{H}$, as well as for half-integral weight Maaß forms. The two cases are rather similar. For this reason, despite stating the holomorphic case first, we will give a detailed proof only in the case of Maaß forms. The rationale for our choice is that the literature dealing with half-integral weight Maaß forms is less developed and certain additional difficulties arise. We sketch the analogous argument for holomorphic half-integral weight forms in Section 9.

1.3. Holomorphic half-integral weight forms.

1.3.1. *Setting.* Write $S_{k+1/2}(\Gamma_0(4))$ for the space of weight $k + \frac{1}{2}$ holomorphic cusp forms for $\Gamma_0(4)$. Every $g \in S_{k+1/2}(\Gamma_0(4))$ has a Fourier expansion of the form

$$g(z) = \sum_{n \geq 1} c(n) e(nz)$$

and for odd p the Hecke operator T_{p^2} is defined on $S_{k+1/2}(\Gamma_0(4))$ as

$$(1.1) \quad T_{p^2}g(z) = \sum_{n \geq 1} \left(c(p^2n) + \left(\frac{(-1)^k n}{p} \right) p^{k-1} c(n) + p^{2k-1} c\left(\frac{n}{p^2}\right) \right) e(nz).$$

Here we have used the convention that $c(x) = 0$ if $x \notin \mathbb{Z}$. We call a half-integral weight cusp form g a *Hecke cusp form* if $T_{p^2}g = \lambda_p g$ for all $p > 2$. We recall that the half-integral weight Hecke operators vanish on the primes, thus $T_p = 0$ for all p (see Shimura [43], p. 450). We will restrict our attention to Hecke forms in what follows.

One of the main tools in understanding half-integral weight forms is the Shimura lift [43], which to a half-integral weight cusp form associates a cusp form of integer weight. Following Kohnen-Zagier [20], we focus on forms belonging to *Kohnen's plus subspace* on which the behavior of the Shimura lift is very well understood. Precisely the Kohnen plus space $S_{k+1/2}^+(\Gamma_0(4))$ denotes the subspace of $S_{k+1/2}(\Gamma_0(4))$ of cusp forms whose coefficients satisfy $c(n) = 0$ for $(-1)^k n \equiv 2, 3 \pmod{4}$ and has a basis consisting of simultaneous eigenfunctions of the operators T_{p^2} for all odd p . We note that as $k \rightarrow \infty$ asymptotically one-third of half-integral weight Hecke cusp forms belong to the Kohnen space (since $\dim S_{k+1/2}^+(\Gamma_0(4)) \sim k/6$ and $\dim S_{k+1/2}(\Gamma_0(4)) \sim k/2$ by [21, 36] and dimension formulas).

1.3.2. Results. Compared to the case of integral weight forms, the study of the distribution of mass for half-integral weight forms presents several new difficulties. First of all, an analogue of Watson's formula [54] is not currently known to exist. Consequently, the simple conditional (on GRH) proof of mass equidistribution for the integer weight case does not carry over to the half-integral weight setting. Secondly, the unconditional techniques of Holowinsky-Soundararajan [13] are not directly applicable since they rely crucially on the multiplicativity of the coefficients. The coefficients of half-integral weight forms are not multiplicative (except at squares) and in general correspond to central values of L -functions, following Waldspurger [53]. In particular, one does not expect the analogue of the Riemann Hypothesis to hold for Dirichlet series built from Fourier coefficients of half-integral weight Hecke cusp forms, due to the absence of an Euler product. Our first main result establishes mass equidistribution for holomorphic half-integral weight Hecke cusp forms lying in Kohnen's plus space, assuming GRH.

Theorem 1. *Assume the Generalized Riemann Hypothesis. Let g_k be a holomorphic half-integral weight cusp form for $\Gamma_0(4)$ of weight $k + \frac{1}{2}$, with k a positive integer. Suppose that g_k*

- (1) *is normalized so that $\iint_{\Gamma_0(4) \backslash \mathbb{H}} y^{k+1/2} |g_k(z)|^2 \, d\text{vol}(z) = 1$,*
- (2) *lies in the Kohnen subspace,*
- (3) *is a simultaneous eigenfunction of the Hecke operators T_{p^2} , $p \neq 2$.*

Let \mathcal{D} be a compact subset of $\Gamma_0(4)\backslash\mathbb{H}$ with boundary measure zero. Then as $k \rightarrow \infty$

$$\iint_{\mathcal{D}} y^{k+\frac{1}{2}} |g_k(z)|^2 d\text{vol}(z) \rightarrow \frac{\text{vol}(\mathcal{D})}{\text{vol}(\Gamma_0(4)\backslash\mathbb{H})}$$

where $d\text{vol}(z) = dx dy / y^2$ is the hyperbolic area measure.

In the proof of Theorem 1 we specifically require the Riemann Hypothesis for $L(s, f)$ with $f \in S_{2k}(\text{SL}_2(\mathbb{Z}))$, all of the quadratic twists $L(s, f \otimes \chi_d)$, with d fundamental discriminants with $(-1)^k d > 0$ and for the symmetric square L -function $L(s, \text{Sym}^2 f)$.

Following the method of Rudnick [39], which is closely related to ideas of Shiffman-Zelditch [42], Theorem 1 gives an immediate consequence for the distribution of zeros of holomorphic half-integral weight forms (see also [23] for a recent refinement of Rudnick's result).

Corollary 1.1. *Assume the Generalized Riemann Hypothesis. Let g_k be as in Theorem 1. Then as $k \rightarrow \infty$ the zeros of g_k equidistribute with respect to hyperbolic measure within compact subsets of $\Gamma_0(4)\backslash\mathbb{H}$ with boundary measure zero.*

This means that for g_k as in Theorem 1 we have for a compact subset $\mathcal{D} \subset \Gamma_0(4)\backslash\mathbb{H}$ with boundary measure zero that

$$\sum_{\substack{g_k(\varrho)=0 \\ \varrho \in \mathcal{D}}} 1 = \frac{k}{2} \cdot \frac{\text{vol}(\mathcal{D})}{\text{vol}(\Gamma_0(4)\backslash\mathbb{H})} + o(k)$$

as $k \rightarrow \infty$, with the zeros counted with multiplicity.

1.4. Half-integral weight Maaß forms.

1.4.1. *Setting.* Let V denote the space spanned by the half-integral weight Maaß cusp forms for $\Gamma_0(4)$. Every half-integral weight Maaß cusp form, g , with eigenvalue $\lambda = -(\frac{1}{4} + t^2)$ has a Fourier expansion at the cusp at ∞ of the following form

$$g(z) = \sum_{n \neq 0} b_{g,\infty}(n) W_{\frac{1}{4} \text{sgn}(n), it}(4\pi|n|y) e(nx), \quad (z = x + iy)$$

where W is the Whittaker function. For $p > 2$ the action of a Hecke operator $T_{p^2} : V \rightarrow V$ on a Maaß cusp form g is given explicitly by

$$T_{p^2} g(z) = \sum_{n \neq 0} c(n) W_{\frac{1}{4} \text{sgn}(n), it}(4\pi|n|y) e(nx),$$

where

$$c(n) = p b_{g,\infty}(np^2) + p^{-1/2} \left(\frac{n}{p}\right) b_{g,\infty}(n) + p^{-1} b_{g,\infty}\left(\frac{n}{p^2}\right)$$

(see Theorem 1.7 of Shimura [43]). The operators T_{p^2} are self-adjoint and commute with each other as well as with $\Delta_{1/2}$ (see Theorem 1 of [21] for the holomorphic setting and Proposition 1.4 of [18] for the real analytic one). As before, we will only consider *Hecke Maaß cusp forms*, which are half-integral weight Maaß forms which are eigenfunctions of T_p^2 , for p odd.

Similarly to the holomorphic case, we have at our disposal the Shimura lift, which is well understood on the Kohnen subspace. In particular, this associates a half-integral weight form, g , with $b_{g,\infty}(n) = 0$ for $n \equiv 2, 3 \pmod{4}$ with a weight 0 Hecke Maaß cusp form for $\mathrm{SL}_2(\mathbb{Z})$ (see Katok-Sarnak [18]). The Kohnen space V^+ is the subspace spanned by such forms.

1.4.2. *Results.* Before stating our results for Maaß forms we highlight that it might be possible to apply the ergodic techniques of Lindenstrauss in the setting of half-integral weight Maaß forms. Additionally, Lindenstrauss' method would establish the full version of QUE which shows equidistribution on the unit cotangent bundle of $\Gamma_0(4)\backslash\mathbb{H}$. Lindenstrauss's method would not however rule out the possibility of escape of mass into the cusps. In our next theorem we establish, conditionally on GRH, QUE for half-integral weight Hecke Maaß cusp forms and also eliminate the possibility of escape of mass. It is also worth pointing out that our result can be made effective, and yields a slow rate of convergence to equidistribution. On the other hand making Lindenstrauss's result effective remains a challenging open problem.

Theorem 2. *Assume the Generalized Riemann Hypothesis. Let $\{g_j\}$ be a basis of the Kohnen space V^+ , such that $\iint_{\Gamma_0(4)\backslash\mathbb{H}} |g_j(z)|^2 d\mathrm{vol}(z) = 1$ for all j and each g_j is a simultaneous eigenfunction of the Hecke operators T_{p^2} for all $p > 2$ as well as of $\Delta_{1/2}$ with $\Delta_{1/2}g_j = -(\frac{1}{4} + t_j^2)g_j$. Let \mathcal{D} be a compact subset of $\Gamma_0(4)\backslash\mathbb{H}$ with boundary measure zero. Then as $|t_j| \rightarrow \infty$*

$$\iint_{\mathcal{D}} |g_j(z)|^2 d\mathrm{vol}(z) \rightarrow \frac{\mathrm{vol}(\mathcal{D})}{\mathrm{vol}(\Gamma_0(4)\backslash\mathbb{H})}$$

where $d\mathrm{vol}(z) = dx dy/y^2$ is the hyperbolic area measure.

In the proof of Theorem 2 we specifically require the Riemann Hypothesis for L -functions of weight 0 Hecke Maaß eigencuspforms for the full modular group, all of their quadratic twists and the symmetric square L -function $L(s, \mathrm{Sym}^2 f)$ with f a weight 0 Hecke Maaß eigencuspform for $\mathrm{SL}_2(\mathbb{Z})$.

The problem of equidistribution of half-integral weight Eisenstein series has been recently addressed by Petridis-Raulf-Risager [37], where they prove QUE for half-integral weight Eisenstein series under the assumption of a subconvexity bound for a multiple Dirichlet series. Establishing such a subconvexity bound is still open,

apparently even under GRH. In the weight 0 case Luo-Sarnak [27] and Jakobson [17] have unconditionally established QUE for the Eisenstein series.

1.5. Comments on the proof and further work. Our proof takes elements from Holowinsky's [12] unconditional argument for holomorphic integer weight forms and Soundararajan's conditional upper bounds for moments of L -functions [48]. Similarly to Holowinsky we have to estimate a certain main term and an off-diagonal term consisting of coefficients of half-integral weight forms. Holowinsky's treatment of the off-diagonal is based on sieve estimates, while ours is based on Soundararajan's work on moments. It is interesting to notice that Soundararajan's proof could have been carried out also in the multiplicative setting where it would deliver the corresponding sieve bound. Similarly to Holowinsky's treatment [12] we also experience considerable difficulties with estimating the main term. In our case the estimation of the main term corresponds to averaging a Dirichlet polynomial of length X^2 over quadratic characters of conductor $\asymp X$. This is a notoriously difficult scenario because an application of the Poisson summation gives back a sum of the same length. We explain how this significant obstruction is resolved in our work in the next section.

In the case of weight zero level 1 Hecke Maaß forms Watson's formula and the Generalized Lindelöf Hypothesis imply an upper bound on the rate of convergence in QUE of size $\ll \lambda^{-1/4+\varepsilon}$. Currently, no effective rate is unconditionally known in this setting, but has been worked out for the holomorphic case, i.e. Hecke cusp forms for $\mathrm{SL}_2(\mathbb{Z})$, in [23], where the rate obtained is a small negative power of the logarithm of the weight. It also emerges from an inspection of our proof that in the half-integral case we only get a bound of $\ll (\log \lambda)^{-\delta}$, for some $\delta > 0$, on the rate of convergence to uniform distribution. It would be very interesting to obtain a bound of size $\ll \lambda^{-\delta}$ for some $\delta > 0$.

Finally in a future paper we plan to address the problem of Quantum Ergodicity for half-integral weight forms in the holomorphic case, that is, to show that the mass of almost all holomorphic forms in the Kohnen plus space tends to equidistribution as the weight increases. This means that at least as far holomorphic forms go, the situation in the half-integral case is qualitatively the same as in the integer weight case before the breakthroughs of Holowinsky-Soundararajan. It will be therefore very interesting to see if unconditional results are also possible to obtain.

2. OVERVIEW OF THE ARGUMENT

2.1. Reduction to sums of Fourier coefficients. For simplicity write $g = g_j$ and $t = t_j$ so that $\Delta_{1/2}g_j = -(\frac{1}{4} + t^2)g_j$. Note that $t \in \mathbb{R}$ since there are no half-integral

Maaß forms in V^+ with exceptional eigenvalues.² Since we will take $|t| \rightarrow \infty$ we assume outright that $|t| > 100$. By an approximation argument it suffices to show that for a test function $\Psi \in C_c^\infty(\Gamma_0(4)\backslash\mathbb{H})$

$$\iint_{\Gamma_0(4)\backslash\mathbb{H}} \Psi(z)|g(z)|^2 \frac{dx dy}{y^2} \rightarrow \frac{1}{\text{vol}(\Gamma_0(4)\backslash\mathbb{H})} \iint_{\Gamma_0(4)\backslash\mathbb{H}} \Psi(z) \frac{dx dy}{y^2} \quad (|t| \rightarrow \infty).$$

For $\Psi \in C_c^\infty(\Gamma_0(4)\backslash\mathbb{H})$ one has $\Psi(z+1) = \Psi(z)$ and we can define $\tilde{\Psi}(z)$ to be the extension of Ψ to \mathbb{H} by Γ_∞ -periodicity. Additionally, we see that $\tilde{\Psi}_y(x) := \tilde{\Psi}(x+iy)$ has a Fourier expansion. Using this expansion and adapting an argument of Luo and Sarnak, (see Section 4 of [27], but some care must be taken at the elliptic points of $\Gamma_0(4)\backslash\mathbb{H}$), one can expand Ψ into a sum of Poincaré series

$$P_{h,\ell}(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4)} h(\text{Im}(\gamma z)) e(\ell \text{Re}(\gamma z))$$

where $h(y)$ is the ℓ th Fourier coefficient of $\tilde{\Psi}_y(x)$. Thus, we see that Theorem 2 follows from estimates for the diagonal term $\ell = 0$ (here $P_{h,0}(z) = E(z|h)$ is the incomplete Eisenstein series)

$$\begin{aligned} \iint_{\Gamma_0(4)\backslash\mathbb{H}} E(z|h)|g(z)|^2 \frac{dx dy}{y^2} &= \frac{1}{\text{vol}(\Gamma_0(4)\backslash\mathbb{H})} \iint_{\Gamma_0(4)\backslash\mathbb{H}} E(z|h) \frac{dx dy}{y^2} + o(1) \quad (|t| \rightarrow \infty) \\ &= \frac{1}{\text{vol}(\Gamma_0(4)\backslash\mathbb{H})} \int_0^\infty h(y) \frac{dy}{y^2} + o(1) \end{aligned}$$

along with the bound for the off-diagonal terms $\ell \neq 0$

$$\iint_{\Gamma_0(4)\backslash\mathbb{H}} P_{h,\ell}(z)|g(z)|^2 \frac{dx dy}{y^2} = \frac{1}{\text{vol}(\Gamma_0(4)\backslash\mathbb{H})} \iint_{\Gamma_0(4)\backslash\mathbb{H}} P_{h,\ell}(z) \frac{dx dy}{y^2} + o(1) = o(1) \quad (|t| \rightarrow \infty).$$

The left-hand sides of both equations above are estimated in terms of the sums of Fourier coefficients of $g(z)$. By the unfolding technique it is easy to see that

$$(2.1) \quad \iint_{\Gamma_0(4)\backslash\mathbb{H}} E(z|h)|g(z)|^2 \frac{dx dy}{y^2} = \sum_{n \neq 0} |b_{g,\infty}(n)|^2 \int_0^\infty W_{\frac{1}{4} \text{sgn}(n), it}(4\pi y|n|)^2 h(y) \frac{dy}{y^2}$$

²This follows from the fact that the Shimura lift of a Maaß form $g \in V^+$ with eigenvalue $-(\frac{1}{4}+t^2)$ is a Maaß cusp form for $\text{SL}_2(\mathbb{Z})$ with eigenvalue $-(\frac{1}{4}+(2t)^2)$. Since Δ on $\text{SL}_2(\mathbb{Z})$ has no exceptional eigenvalues it follows that neither does V^+ .

and for $\ell \neq 0$

$$(2.2) \quad \iint_{\Gamma_0(4) \backslash \mathbb{H}} P_{h,\ell}(z) |g(z)|^2 \frac{dx dy}{y^2} \\ = \sum_{n \neq 0, -\ell} b_{g,\infty}(n) b_{g,\infty}(n + \ell) \int_0^\infty W_{\frac{1}{4} \operatorname{sgn}(n), it}(4\pi|n|y) W_{\frac{1}{4} \operatorname{sgn}(n+\ell), it}(4\pi|n + \ell|y) h(y) \frac{dy}{y^2}.$$

Thus the problem is reduced to estimating (2.1) and (2.2) as $|t| \rightarrow \infty$.

2.2. Estimating the sums of Fourier coefficients. At this point we appeal to a consequence of a Waldspurger type formula, obtained in the setting of half-integral weight Maaß forms by Baruch and Mao [1] (with related works by Biró [2] and Katok-Sarnak [18]). It follows from this formula that if $g \in V^+$ is a half-integral weight Hecke Maaß cusp form then for d a fundamental discriminant and $\delta \geq 1$ any integer

$$|b_{g,\infty}(d\delta^2)| \ll_\varepsilon \frac{1}{\sqrt{|n|}} \left(\frac{L(\frac{1}{2}, f \otimes \chi_d)}{L(1, \operatorname{Sym}^2 f)} \right)^{1/2} \cdot \delta^{7/64+\varepsilon} \cdot |t|^{-\operatorname{sgn}(n)/4} \cdot e^{\pi|t|/2}, \quad n = d\delta^2$$

where f is the Shimura lift of g and in particular f is a Hecke Maaß cusp form for $\operatorname{SL}_2(\mathbb{Z})$ (see Section 10). The exponent $7/64$ is the exponent towards the Ramanujan conjecture obtained by Kim and Sarnak [19]. In fact any exponent $< \frac{1}{2}$ would do (with some additional work). Using the triangle inequality, this reduces the problem of estimating (2.2) to showing that for $\ell \neq 0$

$$\frac{1}{L(1, \operatorname{Sym}^2 f)} \sum_{\substack{d_1 \delta_1^2 - d_2 \delta_2^2 = \ell \\ |t| \leq d_1 \delta_1^2 \leq 2|t|}} L(\frac{1}{2}, f \otimes \chi_{d_1})^{1/2} L(\frac{1}{2}, f \otimes \chi_{d_2})^{1/2} = o(|t|)$$

where the summation is over fundamental discriminants d_1, d_2 and integers δ_1, δ_2 . On the other hand the estimation of the main term (2.1) is, through the use of an exact Waldspurger formula, closely related to showing that

$$(2.3) \quad \frac{1}{L(1, \operatorname{Sym}^2 f)} \sum_{|t| \leq d \leq 2|t|} L(\frac{1}{2}, f \otimes \chi_d) \sim c|t|$$

for some constant $c > 0$, where the summation is over fundamental discriminants d (although we note that while (2.3) is useful to explain the technical issues that arise, we never deal with (2.3) directly but rather prefer to phrase the problem of estimating (2.1) in terms of the coefficients $|b_{g,\infty}(d)|^2$).

We obtain the first bound as a consequence of Soundararajan's work on moments of L -functions (in fact gaining a small logarithmic saving). That such a saving

is possible is suggested by the following ‘‘Sato-Tate’’ property of the coefficients $L(\frac{1}{2}, f \otimes \chi_d)$

$$(2.4) \quad \sum_{X \leq |d| \leq 2X} L(\frac{1}{2}, f \otimes \chi_d)^{1/2} = o(X)$$

which is analogous to

$$(2.5) \quad \sum_{X \leq n \leq 2X} |\lambda_f(n)| = o(X),$$

where $\lambda_f(n)$ are the Hecke eigenvalues of f . We remark that for holomorphic forms the first bound (2.4) is now known unconditionally [38], while the second (2.5) has been known for quite some time [8], and played a critical role in Holowinsky’s work [12].

Let us now turn our attention to the second estimate (2.3) that we need to establish QUE. The main difficulty here is the following: since $d \asymp |t|$ and f has also spectral parameter $\asymp |t|$ the length of the Dirichlet polynomial approximating $L(\frac{1}{2}, f \otimes \chi_d)$ is $\asymp |t|^2$. Therefore applying Poisson summation for quadratic characters with conductor $\asymp |t|$ will return a sum of the same length. This means that we are confronted with the notorious ‘‘deadlock situation’’ discussed for example by Munshi in [32], but in any case well recognized by experts.

We overcome the deadlock situation by using once again another idea of Holowinsky. The starting point is to notice that $L(\frac{1}{2}, f \otimes \chi_d)$ is proportional to $|b_{g,\infty}(d)|^2$ so instead of directly evaluating the moment (2.3) we head back to the expression for this moment in terms of the Rankin-Selberg integral (2.1). In order to evaluate the Rankin-Selberg convolution we consider the following expression

$$\iint_{\Gamma_0(4) \backslash \mathbb{H}} |g(z)|^2 E(z|h) E^Y(z|k) \, d\text{vol}(z)$$

where

$$E^Y(z|k) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4)} k(Y \cdot \text{Im}(\gamma z))$$

and k a smooth function. Evaluating the above in two different ways shows that the summation in (2.3) can be extended by a factor of Y provided that we have a saving of say, Y^{100} , in the shifted convolution problem (2.2). This allows us to extend the length of summation in (2.3) by a small power of $\log |t|$, and consequently after applying Poisson summation (which now returns a sum slightly shorter than $|t|$) and using once again the work of Soundararajan [48] on moments, we obtain an asymptotic estimate for (2.3). In the actual proof we phrase these details slightly differently by establishing a ‘‘convexity bound’’ for the Dirichlet series with coefficients $|b_{g,\infty}(n)|^2$

which then gives an asymptotic evaluation of the average of $|b_{g,\infty}(n)|^2$ when slightly more than $|t|$ terms are added up.

It is worthwhile to point out that in similar problems involving the second or fourth moment of L -functions (as considered by Soundararajan-Young [47]) one benefits from the presence of a power greater than the first. However, this feature is not present in our moment problem. Finally, we notice that if we could obtain unconditionally a power saving in the shifted convolution problem (2.2), then we could extend the length of summation in the first moment by a small power of $|t|$ and then evaluate this first moment unconditionally. This would lead to an unconditional proof of QUE for half-integral weight forms, with a power-saving in the rate of convergence. A power-saving in the shifted convolution problem (2.2) seems to however remain far out of reach, in particular due to the sum being averaged in a range which corresponds to $|t|$ where t is the spectral parameter.

3. ESTIMATES FOR CHARACTER SUMS

The results on moments of L -functions which appear in the next section depend on a character sum estimate which we obtain below.

Proposition 3.1. *Let $A > 0$ and $X \geq A$ be given. Let F be a Schwartz function with \widehat{F} compactly supported in $(-A, A)$. Let ℓ be an integer, a, b be positive integers and r, s be positive odd integers such that $(ab, rs) = 1$ and $1 \leq abrs \leq X/A$. Write $r = r_0 q^2$ and $s = s_0 w^2$ where r_0, s_0 are square-free and let*

$$\mathcal{M}(r_0, q, w, a, b, \ell) = \sum_{\substack{d_1 | q, d_2 | w \\ (d_1 d_2, r_0) = 1 \\ (ad_1, bd_2) | \ell}} \frac{\mu(d_1) \mu(d_2) c_{r_0}(\ell / (ad_1, bd_2))}{[d_1, d_2] r_0}$$

where $c_r(\ell) := \sum_{(a,r)=1} e(a\ell/r)$ is a Ramanujan sum. If $s_0 = r_0$ then

$$\sum_{\substack{m, n \\ am = bn + \ell}} \binom{m}{r} \binom{n}{s} F\left(\frac{am}{X}\right) = \left(\frac{ab/(a, b)^2}{r_0}\right) \widehat{F}(0) \frac{X}{[a, b]} \cdot \mathcal{M}(r_0, q, w, a, b, \ell)$$

and if $r_0 \neq s_0$

$$\sum_{\substack{m, n \\ am = bn + \ell}} \binom{m}{r} \binom{n}{s} F\left(\frac{am}{X}\right) = 0.$$

Remark 1. *In the simple case where $a = b = 1$, $\ell = 0$ and $s = 1$ this result reduces to the formula*

$$\sum_m \binom{m}{r} F\left(\frac{m}{X}\right) = \begin{cases} \widehat{F}(0) \varphi(r) \cdot \frac{X}{r} & \text{if } r = \square, \\ 0 & \text{otherwise,} \end{cases}$$

uniformly for odd $1 \leq r \leq X/A$.

Additionally, it is easily seen from the proof that if F is only a Schwartz function then for any fixed $B > 0$ the results of Proposition 3.1 remain true up to an additional error term of $O_B(X^{-B})$ uniformly in $|\ell| \leq X^B$.

The proof of Proposition 3.1 depends on the following version of Poisson summation, which we recall here.

Lemma 3.1. *Let F be a Schwartz function. Suppose that d, r are positive co-prime integers. Then, for any integer t*

$$\sum_{n \equiv t \pmod{d}} \left(\frac{n}{r}\right) F(n) = \frac{1}{dr} \left(\frac{d}{r}\right) \sum_k \widehat{F}\left(\frac{k}{dr}\right) e\left(\frac{tk\bar{r}}{d}\right) \tau_k(r)$$

where \bar{r} is the multiplicative inverse of $r \pmod{d}$,

$$\tau_k(n) := \sum_b \left(\frac{b}{n}\right) e\left(\frac{kb}{n}\right),$$

and

$$\widehat{F}(x) := \int_{\mathbb{R}} F(u) e(-xu) du$$

is the Fourier transform of F .

Proof. We have

$$\sum_{n \equiv t \pmod{d}} \left(\frac{n}{r}\right) F(n) = \sum_b \left(\frac{b}{r}\right) \sum_{\substack{n \equiv b \pmod{r} \\ n \equiv t \pmod{d}}} F(n).$$

Since $(r, d) = 1$, the congruence condition can be re-written as $n \equiv b\bar{d} + tr\bar{r} \pmod{rd}$. This way we re-write the inner sum over n as

$$\sum_{n \equiv b\bar{d} + tr\bar{r} \pmod{rd}} F(n).$$

By Poisson summation

$$\sum_{n \equiv b\bar{d} + tr\bar{r} \pmod{rd}} F(n) = \frac{1}{rd} \sum_k \widehat{F}\left(\frac{k}{rd}\right) e\left(\frac{kb\bar{d}}{r} + \frac{kt\bar{r}}{d}\right)$$

and the claim follows. \square

We are now ready to prove the proposition.

Proof of Proposition 3.1. Dividing the linear condition $am = bn + \ell$ by (a, b) we see that we need to have $(a, b) | \ell$ and that upon substituting $a/(a, b)$ for a , $b/(a, b)$ for b , $\ell/(a, b)$ for ℓ and finally $X/(a, b)$ for X , we see that we can as well assume that a and b are co-prime from the outset, which we will do now.

Consider first the case where both r and s are square-free. Write $r = dr^*$ and $s = ds^*$ with $d = (r, s)$. Since r was assumed to be square-free, $(d, r^*) = 1$. Given $t \pmod{d}$ if we require that $m \equiv t \pmod{d}$ then the condition $ma = nb + \ell$ implies that $\bar{b}(ta - \ell) \equiv n \pmod{d}$. Therefore,

$$\sum_{\substack{m, n \\ am=bn+\ell}} \binom{m}{r} \binom{n}{s} F\left(\frac{am}{X}\right) = \sum_t \binom{\bar{b}(ta - \ell)}{d} \sum_{\substack{m \equiv t \pmod{d} \\ ma=nb+\ell}} \binom{m}{r^*} \binom{n}{s^*} F\left(\frac{am}{X}\right).$$

Moreover, given $u \pmod{s^*}$ we see that summing over m, n such that $n \equiv u \pmod{s^*}$ and $ma - nb = \ell$ is the same as summing over m, n such that $n \equiv u \pmod{s^*}$ and $ma - \ell \equiv ub \pmod{bs^*}$. With this in mind we re-write the inner sum over m as

$$\begin{aligned} S &:= \sum_u \binom{u}{s^*} \sum_{\substack{m \equiv t \pmod{d} \\ ma-\ell \equiv ub \pmod{bs^*}}} \binom{m}{r^*} F\left(\frac{ma}{X}\right) \\ &= \sum_u \binom{u}{s^*} \sum_{m \equiv t \pmod{d}} \binom{m}{r^*} \cdot \frac{1}{bs^*} \sum_{0 \leq v < bs^*} e\left(\frac{v(ma - ub - \ell)}{bs^*}\right) F\left(\frac{ma}{X}\right) \\ &= \frac{1}{bs^*} \sum_{0 \leq v < bs^*} e\left(\frac{-v\ell}{bs^*}\right) \sum_u \binom{u}{s^*} e\left(-\frac{vu}{s^*}\right) \sum_{m \equiv t \pmod{d}} \binom{m}{r^*} e\left(\frac{vma}{bs^*}\right) F\left(\frac{ma}{X}\right). \end{aligned}$$

Applying Lemma 3.1 we get

$$\sum_{m \equiv t \pmod{d}} \binom{m}{r^*} e\left(\frac{avm}{bs^*}\right) F\left(\frac{ma}{X}\right) = \frac{X}{ar^*d} \binom{d}{r^*} \sum_k \widehat{F}\left(\frac{X}{a} \left(\frac{k}{dr^*} - \frac{va}{bs^*}\right)\right) e\left(\frac{ktr^*}{d}\right) \tau_k(r^*)$$

where we have used the assumption $(d, r^*) = 1$. Since $abdr^*s^* \leq abrs \leq X/A$ and \widehat{F} is compactly supported in $(-A, A)$ the above term is equal to 0 unless $kbs^* = vadr^*$. Since s is square-free $(s^*, d) = 1$ so $(bs^*, adr^*) = 1$. Therefore $bs^* | v$, but since $0 \leq v < bs^*$ this implies that $v = 0$, and hence that also $k = 0$. Therefore

$$S = \frac{X}{abr^*s^*d} \widehat{F}(0) \binom{d}{r^*} \sum_u \binom{u}{s^*} \sum_{u' \pmod{r^*}} \binom{u'}{r^*}.$$

The main term is clearly zero unless $s^* = r^* = 1$, which implies that $r = s$. Hence,

$$(3.1) \quad \sum_{\substack{m, n \\ am=bn+\ell}} \binom{m}{r} \binom{n}{s} F\left(\frac{am}{X}\right) = \mathbf{1}_{r=s} \frac{X}{abr} \widehat{F}(0) \sum_t \binom{\bar{b}(ta - \ell)}{r},$$

where $\mathbf{1}_{r=s}$ equals one if $r = s$ and is zero otherwise.

To complete the proof in the square-free case we now compute the sum on the RHS of (3.1). First observe that

$$(3.2) \quad \frac{1}{r} \sum_{0 \leq t < r} e\left(\frac{-t\ell}{r}\right) \tau_{at}(r) \overline{\tau_{bt}(r)} = \sum_{t \pmod{r}} \left(\frac{\bar{b}t(at - \ell)}{r}\right).$$

Next, for odd square-free r a little calculation using Gauss sums and quadratic reciprocity gives

$$\tau_v(r) = \left(\frac{1+i}{2} + \left(\frac{-1}{r}\right) \frac{1-i}{2}\right) \left(\frac{v}{r}\right) \sqrt{r}.$$

In particular $\tau_v(r) = 0$ unless $(v, r) = 1$ and if $(v, r) = 1$ then

$$\tau_{av}(r) \overline{\tau_{bv}(r)} = \left(\frac{av}{r}\right) \sqrt{r} \left(\frac{bv}{r}\right) \sqrt{r} = \left(\frac{ab}{r}\right) r.$$

Thus,

$$\frac{1}{r} \sum_{0 \leq t < r} e\left(\frac{-t\ell}{r}\right) \tau_{at}(r) \overline{\tau_{bt}(r)} = \left(\frac{ab}{r}\right) c_r(\ell)$$

where $c_r(\ell)$ is a Ramanujan sum. Combining this with (3.1) and (3.2) completes the proof for square-free r, s .

Write now $r = r_0 q^2$ and $s = s_0 w^2$ with r_0 and s_0 square-free. Then we can re-write the sum as

$$\sum_{\substack{m, n \\ am=bn+\ell \\ (m, q)=(n, w)=1}} \left(\frac{m}{r_0}\right) \left(\frac{n}{s_0}\right) F\left(\frac{am}{X}\right).$$

Introducing Möbius inversion this is equal to

$$\sum_{\substack{d_1 | q, d_2 | w \\ (d_1, r_0) = (d_2, s_0) = 1}} \mu(d_1) \mu(d_2) \left(\frac{d_1}{r_0}\right) \left(\frac{d_2}{s_0}\right) \sum_{\substack{m, n \\ ad_1 m = bd_2 n + \ell}} \left(\frac{m}{r_0}\right) \left(\frac{n}{s_0}\right) F\left(\frac{ad_1 m}{X}\right).$$

By the result for square-free r and s , this equals zero unless $r_0 = s_0$ and in this case this equals

$$\begin{aligned} & \hat{F}(0) \frac{X}{[a, b]} \sum_{\substack{d_1 | q, d_2 | w \\ (d_1 d_2, r_0) = 1 \\ (ad_1, bd_2) | \ell}} \frac{\mu(d_1) \mu(d_2)}{[d_1, d_2]} \left(\frac{d_1 d_2}{r_0}\right) \cdot \left(\frac{ad_1 b d_2 / (ad_1, bd_2)^2}{r_0}\right) \frac{c_{r_0}(\ell / (ad_1, bd_2))}{r_0} \\ & = \left(\frac{ab / (a, b)^2}{r_0}\right) \hat{F}(0) \frac{X}{[a, b]} \cdot \sum_{\substack{d_1 | q, d_2 | w \\ (d_1 d_2, r_0) = 1 \\ (ad_1, bd_2) | \ell}} \frac{\mu(d_1) \mu(d_2)}{[d_1, d_2]} \frac{c_{r_0}(\ell / (ad_1, bd_2))}{r_0}, \end{aligned}$$

since $(ad_1, bd_2) = (a, b)(d_1, d_2)$ because $d_1|r$ and $d_2|s$ and $(rs, ab) = 1$ by assumption. \square

4. A MOMENT CALCULATION

The main result of this section is an estimate for moments of a short Dirichlet polynomial, presented in Lemma 4.1 below. This will be the crucial ingredient in our bound for

$$\sum_{\substack{d_1, d_2 \\ a|d_1 \leq X \\ ad_1 = bd_2 + \ell}} L(\tfrac{1}{2}, f \otimes \chi_{d_1})^{1/2} L(\tfrac{1}{2}, f \otimes \chi_{d_2})^{1/2}$$

obtained in the next section, where the summation is over fundamental discriminants d_1, d_2 and f is an even, arithmetically normalized, Hecke Maaß cusp form for $\mathrm{SL}_2(\mathbb{Z})$.

Throughout this section we use the following notation. Let $\delta \in (0, \frac{1}{2}]$. Also, let $\{a(p)\}_{p \geq 2}$ be a sequence of real numbers such that $|a(p)| \leq 2p^{1/2-\delta}$ for all odd p and $a(2) = 0$. For $x \geq 1$ and an integer $\ell \neq 0$ define

$$(4.1) \quad B(x) := \frac{1}{2} \sum_{p \leq x} \frac{a(p)^2}{p} \quad \text{and} \quad A(\ell) := 1 + \sum_{p|\ell} \frac{a(p)^2}{p}.$$

Lemma 4.1. *Let $X \geq 10$ and an integer $k > 0$ be given. Then, uniformly in integers $1 \leq a, b \leq (\log X)^{100}$, $\ell \neq 0$ and real numbers $x \in [2, (X/10)^{1/(2k)}]$ such that $2B(x) \geq A(\ell)$, we have*

$$\sum_{\substack{d_1, d_2 \\ a|d_1 \leq X \\ ad_1 = bd_2 + \ell}} \left(\sum_{\substack{p \leq x \\ p|ab}} \frac{a(p)(\chi_{d_1}(p) + \chi_{d_2}(p))}{\sqrt{p}} \right)^{2k} \ll_{\delta} \frac{X}{[a, b]} \cdot \frac{(2k)!}{2^k k!} \left(2 \sum_{p \leq x} \frac{a(p)^2}{p} + O(A(\ell)) \right)^k$$

where d_1, d_2 are fundamental discriminants.

Remark 2. *Here and in Lemma 4.3 we exclude $p = 2$ and $p|ab$ from the sum over primes so that we can apply Proposition 3.1, which was proved for odd integers r, s that are co-prime to ab .*

The proof of Lemma 4.1 depends on the following lemma, which uses the character sum estimate obtained in the previous section.

Lemma 4.2. *Let F be a Schwartz function with \widehat{F} compactly supported in $(-10, 10)$. Let $X \geq 10$ and integers $k, j > 0$ be given. Then uniformly in integers $\ell \neq 0$, $1 \leq a, b \leq (\log X)^{100}$ and real numbers x such that $2 \leq x^{k+j} \leq (X/10)^{1/2}$ and*

$$B(x) \geq A(\ell)$$

$$\begin{aligned} & \sum_{\substack{m,n \\ am=bn+\ell}} \left(\sum_{\substack{p \leq x \\ p \nmid ab}} \frac{a(p) \left(\frac{m}{p}\right)}{\sqrt{p}} \right)^k \left(\sum_{\substack{p \leq x \\ p \nmid ab}} \frac{a(p) \left(\frac{n}{p}\right)}{\sqrt{p}} \right)^j F\left(\frac{am}{X}\right) \\ & \ll_{\delta} \mathbf{1}_{2|k-j} \cdot \left| \widehat{F}(0) \right| \frac{X}{[a, b]} \cdot C(k)C(j) \cdot \left(2B(x) + C_{\delta}A(\ell) \right)^{(k+j)/2-\eta(k,j)} \cdot A(\ell)^{\eta(k,j)} \end{aligned}$$

with $C_{\delta} > 0$ depending only on δ , and where $C(k) = k!/(\lfloor k/2 \rfloor! 2^{k/2})$, $\eta = \eta(k, j)$ equals 0 if $k \equiv j \equiv 0 \pmod{2}$ and $\eta = 1$ if $k \equiv j \equiv 1 \pmod{2}$, and $\mathbf{1}_{2|k-j}$ equals 1 if $2|k-j$ and is 0 otherwise.

We will prove Lemma 4.2 and Lemma 4.1 shortly, but beforehand we state and prove the following fairly standard lemma (see for example [48, Lemma 3]) that will also be required later on.

Lemma 4.3. *Let F be a Schwartz function such that \widehat{F} has compact support in $(-10, 10)$. Let $J > 0$, $X \geq 10$ and integers $k_1, k_2, \dots, k_J > 0$ be given. Let $\{a(p)\}_{p \geq 2}$ be a sequence of real-numbers with $a(2) = 0$. Then for real numbers x such that $1 \leq x^{k_1 + \dots + k_J} \leq X/10$ and disjoint intervals $I_1, I_2, \dots, I_J \subset [3, x]$*

$$\left| \sum_m \prod_{j \leq J} \left(\sum_{p \in I_j} \frac{a(p) \left(\frac{m}{p}\right)}{\sqrt{p}} \right)^{k_j} F\left(\frac{m}{X}\right) \right| \leq \left| \widehat{F}(0) \right| X \prod_{j \leq J} \left(\widetilde{C}(k_j) \cdot \left(\sum_{p \in I_j} \frac{a(p)^2}{p} \right)^{k_j/2} \right),$$

where $\widetilde{C}(k)$ is the k th moment of a normal random variable with mean 0 and variance 1, that is $\widetilde{C}(2\ell) = (2\ell)!/(\ell! 2^{\ell})$ for ℓ a non-negative integer, and $\widetilde{C}(2\ell + 1) = 0$.

Proof. First, we extend $a(p)$ to a completely multiplicative function $a(n)$ by setting $a(p^{\alpha}) = a(p)^{\alpha}$. Also, let $p_j(n)$ equal one if $\Omega(n) = j$ and zero otherwise. We also define the multiplicative function $\nu(p^{\alpha}) = \alpha!$. In this notation, we expand the moment as

$$k_1! \cdots k_J! \sum_{\substack{r_1, \dots, r_J \\ p_j | r_j \Rightarrow p_j \in I_j, j=1, \dots, J}} \frac{a(r_1) \cdots a(r_J) p_{k_1}(r_1) \cdots p_{k_J}(r_J)}{\sqrt{r_1 \cdots r_J} \nu(r_1) \cdots \nu(r_J)} \sum_m \left(\frac{m}{r_1 \cdots r_J} \right) F\left(\frac{m}{X}\right).$$

Since $r_1 \cdots r_J \leq x^{k_1 + \dots + k_J} \leq X/10$ we can apply Proposition 3.1 (see Remark 1) to see that the inner sum over m equals 0 unless $r_1 \cdots r_J = \square$, and this implies that $r_j = \square$ for each $j = 1, \dots, J$, since $(r_i, r_j) = 1$ for $i \neq j$. Since $\Omega(r_j) = k_j$ this completes the proof in the case where k_j is odd for some j .

To handle the remaining case write $r_j = n_j^2$ and $k_j = 2h_j$ so by Proposition 3.1 the above equation equals

$$X\widehat{F}(0)(2h_1)!\cdots(2h_J)!\sum_{\substack{n_1,\dots,n_J \\ p_j|n_j \Rightarrow p_j \in I_j, j=1,\dots,J}} \frac{a(n_1)^2 \cdots a(n_J)^2 p_{h_1}(n_1) \cdots p_{h_J}(n_J)}{n_1 \cdots n_J \nu(n_1^2) \cdots \nu(n_J^2)} \frac{\varphi(n_1^2 \cdots n_J^2)}{n_1^2 \cdots n_J^2}.$$

Now apply the inequalities $\nu(n^2) \geq 2^{\Omega(n)}\nu(n)$ and $\varphi(n)/n \leq 1$ then rearrange the sum to see that the absolute value of the above is

$$\begin{aligned} &\leq X \left| \widehat{F}(0) \right| (2h_1)!\cdots(2h_J)! \sum_{\substack{n_1,\dots,n_J \\ p_j|n_j \Rightarrow p_j \in I_j, j=1,\dots,J}} \frac{a(n_1)^2 \cdots a(n_J)^2 p_{h_1}(n_1) \cdots p_{h_J}(n_J)}{n_1 \cdots n_J 2^{\Omega(n_1)+\cdots+\Omega(n_J)} \nu(n_1) \cdots \nu(n_J)} \\ &= X \left| \widehat{F}(0) \right| \prod_{j \leq J} \frac{(2h_j)!}{2^{h_j} h_j!} \left(\sum_{p \in I_j} \frac{a(p)^2}{p} \right)^{h_j}. \end{aligned}$$

□

We now prove Lemma 4.2.

Proof of Lemma 4.2. We can assume at the outset that $(a, b) | \ell$ otherwise the result is vacuously true. In the notation of the proof of the previous lemma we have

$$(4.2) \quad \begin{aligned} &\sum_{\substack{m,n \\ am=bn+\ell}} \left(\sum_{\substack{2 < p \leq x \\ p|ab}} \frac{a(p) \binom{m}{p}}{\sqrt{p}} \right)^k \left(\sum_{\substack{2 < p \leq x \\ p|ab}} \frac{a(p) \binom{n}{p}}{\sqrt{p}} \right)^j F\left(\frac{am}{X}\right) \\ &= k!j! \sum_{\substack{r,s \\ p|rs \Rightarrow 2 < p \leq x \\ (rs, ab)=1}} \frac{a(r)a(s)p_k(r)p_j(s)}{\sqrt{rs}\nu(r)\nu(s)} \sum_{\substack{m,n \\ am=bn+\ell}} \binom{m}{r} \binom{n}{s} F\left(\frac{am}{X}\right). \end{aligned}$$

Let $\rho_\ell(n)$ be the multiplicative function given by $\rho_\ell(p^\alpha) = (p^\alpha, \ell)/p^\alpha$ and $\zeta(n)$ be the multiplicative function given by $\zeta(p^\alpha) = (1 + 3/p)^\alpha$. Notice that the main term $\mathcal{M}(r_0, q, w, a, b, \ell)$ in Proposition 3.1 is bounded by

$$\left| \sum_{\substack{d_1|q, d_2|w \\ (d_1 d_2, r_0)=1 \\ (ad_1, bd_2)|\ell}} \frac{\mu(d_1)\mu(d_2)}{[d_1, d_2]} \frac{c_{r_0}(\ell/(ad_1, bd_2))}{r_0} \right| \leq \rho_\ell(r_0)\zeta(qw).$$

Write $r = r_0 q^2$ and $s = s_0 w^2$ where r_0 and s_0 are square-free and apply Proposition 3.1 to see that only the terms with $r_0 = s_0 = t$ survive and the absolute value of the

RHS of (4.2) is

$$\leq \left| \widehat{F}(0) \right| \cdot \frac{X}{[a, b]} k! j! \sum_{\substack{t, q, w \\ p|tqw \Rightarrow p \leq x}} \frac{a^2(tqw) \rho_\ell(t) \varsigma(qw) p_k(tq^2) p_j(tw^2) \mu^2(t)}{tqw \nu(tq^2) \nu(tw^2)}.$$

Now notice that if k and j are of different parities, then $p_k(tq^2) p_j(tw^2) = 0$. Set $\eta = 1$ if k and j are both odd and set $\eta = 0$ if k and j are both even. Without loss of generality assume $k \leq j$ and write $k_0 = (k - \eta)/2$ and $j_0 = (j - \eta)/2$. Thus, using the inequalities $1 \leq \nu(m) \nu(n) \leq \nu(mn)$ and $\nu(n^2) \geq 2^{\Omega(n)} \nu(n)$ we bound the above sum by

$$(4.3) \quad \leq \left| \widehat{F}(0) \right| \cdot \frac{X}{[a, b]} k! j! \sum_{h=0}^{k_0} \sum_{\substack{t, q, w \\ p|tqw \Rightarrow p \leq x}} \frac{a^2(tqw) \rho_\ell(t) \varsigma(q) \varsigma(w) p_{2h+\eta}(t) p_{k_0-h}(q) p_{j_0-h}(w)}{tqw 2^{\Omega(q)+\Omega(w)} \nu(t) \nu(q) \nu(w)}.$$

Let

$$\alpha_\ell(x) = \sum_{p \leq x} \frac{a(p)^2 \rho_\ell(p)}{p}.$$

Rearranging (4.3) we see that it is

$$(4.4) \quad \leq \left| \widehat{F}(0) \right| \cdot \frac{X}{[a, b]} k! j! \sum_{h=0}^{k_0} \frac{\alpha_\ell(x)^{2h+\eta} (B(x) + C_\delta)^{k_0+j_0-2h}}{(k_0-h)! (j_0-h)! (2h+\eta)!}$$

for some $C_\delta > 0$ depending only on δ . To bound the above sum use the inequality $m^n (m-n)! \geq m!$ to get

$$\begin{aligned} & \sum_{h=0}^{k_0} \frac{\alpha_\ell(x)^{2h+\eta} \cdot (B(x) + C_\delta)^{k_0+j_0-2h}}{(k_0-h)! (j_0-h)! (2h+\eta)!} \\ & \leq \frac{\alpha_\ell(x)^\eta (B(x) + C_\delta)^{k_0+j_0}}{k_0! j_0!} \sum_{h=0}^{k_0} \frac{1}{(2h+\eta)!} \cdot \left(\frac{\sqrt{k_0 j_0} \alpha_\ell(x)}{B(x)} \right)^{2h}. \end{aligned}$$

Note that the inner sum over h is

$$\leq \exp \left(\frac{\sqrt{k_0 j_0} \alpha_\ell(x)}{B(x)} \right) \leq \exp \left(\frac{k_0 \alpha_\ell(x)}{2B(x)} \right) \cdot \exp \left(\frac{j_0 \alpha_\ell(x)}{2B(x)} \right).$$

Also, $\alpha_\ell(x) \leq \widetilde{C}_\delta A(\ell)$, for some $\widetilde{C}_\delta > 1$ which depends only on δ . It follows that $\alpha_\ell(x)/(2B(x)) \leq \widetilde{C}_\delta A(\ell)/(2B(x)) \leq \widetilde{C}_\delta/2$, by the assumption $B(x) \geq A(\ell)$. Hence,

$e^{\alpha_\ell(x)/(2B(x))} = 1 + O_\delta(A(\ell)/B(x))$. Combining the two previous observations it follows that (4.4) is

$$\begin{aligned} &\ll \left| \widehat{F}(0) \right| \cdot \frac{X}{[a, b]} \cdot \frac{k!j!}{k_0!j_0!} \cdot (e^{\alpha_\ell(x)/(2B(x))} \cdot (B(x) + C_\delta))^{k_0+j_0} \cdot \alpha_\ell(x)^\eta \\ &\ll_\delta \left| \widehat{F}(0) \right| \cdot \frac{X}{[a, b]} \cdot \frac{k!j!}{k_0!j_0!} \cdot (B(x) + O_\delta(A(\ell)))^{k_0+j_0} \cdot A(\ell)^\eta. \end{aligned}$$

□

We are finally ready to prove the main result of this section.

Proof of Lemma 4.1. For fundamental discriminants d_1, d_2 we have

$$\chi_{d_1}(p) = \left(\frac{d_1}{p} \right) \text{ and } \chi_{d_2}(p) = \left(\frac{d_2}{p} \right).$$

Thus after replacing χ_d by the corresponding Jacobi symbol, we extend the summation over d_1, d_2 to all integers, not just fundamental discriminants. Also, let F be a Schwartz function such that \widehat{F} has compact support in $(-10, 10)$ and $F(x) \geq \mathbf{1}_{[-1,1]}(x)$ for all $x \in \mathbb{R}$. Using the binomial theorem we see that the $2k$ -th moment over fundamental discriminants is bounded by

$$\begin{aligned} &\sum_{\substack{m,n \\ am=bn+\ell}} \left(\sum_{\substack{2 < p \leq x \\ p \nmid ab}} \frac{a(p) \left(\binom{m}{p} + \binom{n}{p} \right)}{\sqrt{p}} \right)^{2k} F\left(\frac{am}{X}\right) \\ &= \sum_{j \leq 2k} \binom{2k}{j} \sum_{\substack{m,n \\ am=bn+\ell}} \left(\sum_{\substack{2 < p \leq x \\ p \nmid ab}} \frac{a(p) \binom{m}{p}}{\sqrt{p}} \right)^j \left(\sum_{\substack{2 < p \leq x \\ p \nmid ab}} \frac{a(p) \binom{n}{p}}{\sqrt{p}} \right)^{2k-j} F\left(\frac{am}{X}\right). \end{aligned}$$

Applying Lemma 4.2 we bound this as

$$(4.5) \quad \ll \frac{X}{[a, b]} (2B(x) + C_\delta A(\ell))^k \sum_{j \leq 2k} \binom{2k}{j} C(2k-j) C(j) \left(\frac{A(\ell)}{B(x)} \right)^{\eta(2k-j, j)}$$

for some $C_\delta > 0$ depending only on δ . Since $\eta(2k-j, j)$ equals zero if j is even and is one if j is odd, the contribution of the even j to the above sum equals

$$= \frac{(2k)!}{2^k} \sum_{j=0}^k \frac{1}{(k-j)!j!} = \frac{(2k)!}{k!}.$$

The contribution of the odd j to the sum in (4.5) equals

$$= \frac{A(\ell)}{B(x)} \frac{(2k)!}{2^k} \sum_{j=0}^{k-1} \frac{1}{(k-1-j)!j!} = \frac{kA(\ell)}{B(x)} \cdot \frac{(2k)!}{2 \cdot k!} \leq \frac{(2k)!}{2 \cdot k!} \left(1 + \frac{A(\ell)}{B(x)}\right)^k,$$

where in the last inequality we used the elementary estimate $kw < (1+w)^k$ for $k \geq 1, w > 0$. Using the above two estimates in (4.5) completes the proof. \square

5. LEMMAS ON L -FUNCTIONS

Let f be an even, arithmetically normalized, weight 0 Hecke Maaß cusp form for $\mathrm{SL}_2(\mathbb{Z})$ with Laplace eigenvalue $-(\frac{1}{4} + t^2)$. We will need rather sharp estimates for the following two moments:

$$\sum_{|d| \leq X} L(\tfrac{1}{2}, f \otimes \chi_d) \quad \text{and} \quad \sum_{\substack{d_1, d_2 \\ a|d_1| \leq X \\ ad_1 = bd_2 + \ell}} L(\tfrac{1}{2}, f \otimes \chi_{d_1})^{1/2} L(\tfrac{1}{2}, f \otimes \chi_{d_2})^{1/2}$$

where the summation is over fundamental discriminants. Before stating our lemmas let us explain heuristically what to expect.

5.1. Heuristics. Heuristically for $|d| \gg |t|^\varepsilon$ we expect that if $L(\frac{1}{2}, f \otimes \chi_d) \neq 0$ then $\log L(\frac{1}{2}, f \otimes \chi_d)$ is approximated by

$$\sum_{p^k \leq |d|^\varepsilon} \frac{(\alpha_p^k + \beta_p^k) \chi_d(p)^k}{kp^{k/2}}$$

where α_p, β_p are the Satake parameters of f . We also expect that as d varies over fundamental discriminants $\chi_d(p)$ behaves on average as an independent random variable $X(p)$ taking values ± 1 with equal probability $\frac{p}{2(p+1)}$ and the value 0 with probability $\frac{1}{p+1}$. We note that in the sum the terms with $k \geq 3$ contribute $O(1)$, while for $k = 2$ we have $\alpha_p^2 + \beta_p^2 = \lambda_f(p^2) - 1$. Therefore the above sum heuristically behaves like

$$\sum_{p \leq |d|^\varepsilon} \frac{\lambda_f(p) X(p)}{\sqrt{p}} + \frac{1}{2} \sum_{p \leq |d|^\varepsilon/2} \frac{\lambda_f(p^2)}{p} - \frac{1}{2} \log \log |d| + O(1),$$

which is the same as

$$\sum_{p \leq |d|^\varepsilon} \frac{\lambda_f(p) X(p)}{\sqrt{p}} + \frac{1}{2} \log L(1, \mathrm{Sym}^2 f) - \frac{1}{2} \log \log |d| + O(1).$$

In addition, the sum over $p \leq |d|^\varepsilon$ behaves approximately like a Gaussian random variable with mean 0 and variance $\sum_{p \leq |d|^\varepsilon} \lambda_f(p)^2/p = \log \log |d| + \log L(1, \mathrm{Sym}^2 f) + O(1)$, since $\lambda_f(p)^2 = \lambda_f(p^2) + 1$. Note that we have to include $L(1, \mathrm{Sym}^2 f)$ since a

priori its contribution could easily overwhelm $\log \log |d|$ when $|t|^\varepsilon \leq |d| \leq t^2$ (this is due to our lack of knowledge of the Ramanujan-Petersson conjecture in the case of weight 0 Maaß forms). It follows from these considerations (or from a formal application of [5]) that we expect heuristically, in the range $|t|^\varepsilon \leq X$, that

$$\begin{aligned} \sum_{X \leq |d| \leq 2X} L\left(\frac{1}{2}, f \otimes \chi_d\right) &\asymp X \cdot \mathbb{E} \left[\exp \left(\sum_{2 < p \leq X} \frac{\lambda_f(p) X(p)}{\sqrt{p}} \right) \right] (\log X)^{-1/2} \cdot L(1, \text{Sym}^2 f)^{1/2} \\ &\asymp L(1, \text{Sym}^2 f) X \end{aligned}$$

using the fact that $\mathbb{E}[e^{\lambda X}] = \exp(\frac{1}{2}\sigma^2\lambda^2)$ for a Gaussian random variable with variance σ^2 and mean 0. In the shifted moment the same heuristic applies, but in addition we expect $L(\frac{1}{2}, f \otimes \chi_{d_1})$ to behave approximately independently from $L(\frac{1}{2}, f \otimes \chi_{d_2})$. This leads to the following conjecture, for $1 \leq a, b, |\ell| < X^{1-\varepsilon}$

$$\begin{aligned} \sum_{\substack{d_1, d_2 \\ X \leq a|d_1| \leq 2X \\ ad_1 = bd_2 + \ell}} L\left(\frac{1}{2}, f \otimes \chi_{d_1}\right)^{1/2} L\left(\frac{1}{2}, f \otimes \chi_{d_2}\right)^{1/2} \\ &\asymp \frac{X}{[a, b]} \left(\mathbb{E} \left[\exp \left(\frac{1}{2} \sum_{p \leq X} \frac{\lambda_f(p) X(p)}{\sqrt{p}} \right) \right] \right)^2 \cdot (\log X)^{-1/2} L(1, \text{Sym}^2 f)^{1/2} \\ &\asymp \frac{X}{[a, b]} \cdot L(1, \text{Sym}^2 f)^{3/4} \cdot (\log X)^{-1/4}. \end{aligned}$$

5.2. Statements of results under GRH and some preliminary lemmas. In this section we establish two main lemmas, which combine Soundararajan's method for moments of L -functions along with our character sum estimate. The first of these results is the following lemma.

Lemma 5.1. *Let $\varepsilon > 0$. Assume the Generalized Riemann Hypothesis. Let f be an even, arithmetically normalized, Hecke Maaß eigencuspform with eigenvalue $-(\frac{1}{4} + t^2)$. Then uniformly for $|t|^\varepsilon \leq X$, $0 \neq |\ell| \leq \log X$ and $1 \leq a, b \leq (\log X)^{100}$*

$$\sum_{\substack{d_1, d_2 \\ a|d_1| \leq X \\ ad_1 = bd_2 + \ell}} L\left(\frac{1}{2}, f \otimes \chi_{d_1}\right)^{1/2} L\left(\frac{1}{2}, f \otimes \chi_{d_2}\right)^{1/2} \ll_\varepsilon L(1, \text{Sym}^2 f)^{3/4+\varepsilon} \frac{X}{[a, b]} (\log X)^{-1/4+\varepsilon},$$

where the summation is over fundamental discriminants d_1, d_2 .

We also need an upper bound for the first moment, however Soundararajan's method only gives

$$\sum_{|d| \leq X} L\left(\frac{1}{2}, f \otimes \chi_d\right) \ll L(1, \text{Sym}^2 f)^{1+\varepsilon} \cdot X(\log X)^\varepsilon$$

for any $\varepsilon > 0$, where the summation is over fundamental discriminants. This bound differs from the one predicted in the heuristic of the previous section by an extra factor of $L(1, \text{Sym}^2 f)^\varepsilon (\log X)^\varepsilon$ and the term $L(1, \text{Sym}^2 f)^\varepsilon$ is especially problematic in our range of interest $|t|^\varepsilon \leq X \leq t^2$ since even on GRH we only know that

$$(5.1) \quad L(1, \text{Sym}^2 f) \ll \exp\left((\log |t|)^{1/8+\varepsilon}\right),$$

for any $\varepsilon > 0$, which is proved by Li [24]. However the loss of ε on the exponent $L(1, \text{Sym}^2 f)$ is due to a non-optimal treatment of the small primes $p \ll (\log |t|)^{2+\varepsilon}$ in Soundararajan's method [48]. We are in luck since small primes are analytically easy to deal with, and we remedy this loss by a small refinement of Soundararajan's method which is more efficient on small primes. We also note that in principle the loss of $(\log X)^\varepsilon$ could have been also averted by the use of Harper's [10] refinement of Soundararajan's method. In this way, we will establish the second main result of this section, which is the following lemma.

Lemma 5.2. *Let $\varepsilon > 0$. Assume the Generalized Riemann Hypothesis. Let f be an even, arithmetically normalized Hecke Maaß eigencuspform with eigenvalue $-(\frac{1}{4} + t^2)$. Then uniformly in $|t|^\varepsilon \leq X$*

$$\sum_{|d| \leq X} L\left(\frac{1}{2}, f \otimes \chi_d\right) \ll_\varepsilon L(1, \text{Sym}^2 f) X (\log X)^\varepsilon$$

where the summation is over fundamental discriminants.

Lemma 5.1 and Lemma 5.2 are proved in Sections 5.3 and 5.4, respectively. In the remainder of this section we prove several preliminary lemmas.

When using Lemma 5.1 in combination with a Waldspurger type formula we will be dividing by $L(1, \text{Sym}^2 f)$. We therefore record the following simple (but useful!) observation.

Lemma 5.3. *Let f be an even, arithmetically normalized, Hecke Maaß eigencuspform with eigenvalue $-(\frac{1}{4} + t^2)$. Assume the Generalized Riemann Hypothesis for $L(s, \text{Sym}^2 f)$. Then for $x \geq (\log |t|)^{2+\varepsilon}$*

$$(5.2) \quad \sum_{p \leq x} \frac{\lambda_f(p)^2}{p} = \log \log x + \log L(1, \text{Sym}^2 f) + O(1).$$

Additionally,

$$(5.3) \quad L(1, \text{Sym}^2 f) \gg \frac{1}{\log \log |t|}.$$

Proof. By Perron's formula, for $x \geq 2$

$$(5.4) \quad \frac{1}{2\pi i} \int_{1-ix \log(|t|+x)}^{1+ix \log(|t|+x)} \log \left(L(s+1, \text{Sym}^2 f) \right) \frac{x^s}{s} ds = \sum_{p \leq x} \frac{\lambda_f(p^2)}{p} + O(1).$$

Under GRH, $\log \left(L(s+1, \text{Sym}^2 f) \right)$ is analytic in the half-plane $\text{Re}(s) > -\frac{1}{2}$. Additionally, repeating a classical argument of Littlewood (see Titchmarsh [52] (14.2.2)) it follows that under GRH, for $\text{Re}(s) \geq -\frac{1}{2} + \frac{1}{\log x}$

$$(5.5) \quad |\log L(s+1, \text{Sym}^2 f)| \ll \log x \cdot \log(|t| + |s|).$$

Now, in (5.4) shift contours to $\text{Re}(s) = -\frac{1}{2} + \frac{1}{\log x}$. At $s = 0$ we collect a simple pole with residue equal to $\log L(1, \text{Sym}^2 f)$. For $x \geq 2$, using (5.5) the top horizontal contour is bounded by

$$\begin{aligned} &\ll \frac{1}{x \log(|t| + x)} \int_{-\frac{1}{2} + \frac{1}{\log x} + ix \log(|t|+x)}^{1+ix \log(|t|+x)} |\log L(s+1, \text{Sym}^2 f)| |x^s| |ds| \\ &\ll \frac{(\log x) \cdot \log(|t| + x \log(|t| + x))}{x \log(|t| + x)} \int_{-1/2}^1 x^\sigma d\sigma \ll 1. \end{aligned}$$

Similarly, the bottom horizontal contour also contributes $O(1)$. Hence, we conclude that for $x \geq 2$

$$(5.6) \quad \sum_{p \leq x} \frac{\lambda_f(p^2)}{p} = \log L(1, \text{Sym}^2 f) + O\left(1 + \frac{\log x}{x^{1/2}} \int_{-x \log(|t|+x)}^{x \log(|t|+x)} \frac{\log(|t| + |\tau|)}{1 + |\tau|} d\tau\right).$$

To get (5.2), use $\lambda_f(p^2) = \lambda_f(p)^2 - 1$ in the above equation and note that for $x \geq (\log |t|)^{2+\varepsilon}$ the error term is $O(1)$. To establish (5.3), choose $x = (\log |t|)^3$ in (5.6) (so the error term is $O(1)$), and use the bound $\lambda_f(p^2) = \lambda_f(p)^2 - 1 \geq -1$ to get that the LHS of (5.6) is at least $-\log \log \log |t| + O(1)$. \square

For the proof of both Lemma 5.2 and Lemma 5.1 we will use a theorem of Chandee [3] which is a generalization of an inequality first obtained by Soundararajan for the Riemann zeta-function in [48].

Lemma 5.4. *Assume the Generalized Riemann Hypothesis. Let f be an even, arithmetically normalized Hecke Maaß eigencuspform with eigenvalue $-(\frac{1}{4} + t^2)$. Let d be a fundamental discriminant. Then for $x \geq 100$,*

$$\log L\left(\frac{1}{2}, f \otimes \chi_d\right) \leq \sum_{p^n \leq x} \frac{\chi_d(p^n)(\alpha_p^n + \beta_p^n)}{np^{n(1/2+1/\log x)}} \cdot \frac{\log(x/p^n)}{\log x} + 3 \cdot \frac{\log |dt|}{\log x}$$

where α_p, β_p are the Satake parameters.

Proof. See Theorem 2.1 of [3]. □

Remark 3. Note that $\alpha_p^n + \beta_p^n \in \mathbb{R}$ for all p and $n \geq 1$, regardless of the truth of the Ramanujan-Petersson conjecture. Since $\lambda_f(p) = \alpha_p + \beta_p$ and $\alpha_p\beta_p = 1$, it follows that α_p and β_p are roots of $x^2 - \lambda_f(p)x + 1$. If $|\lambda_f(p)| > 2$ then the discriminant of this polynomial is positive, hence its roots α_p, β_p are real, and therefore $\alpha_p^n + \beta_p^n$ is real. On the other hand if $|\lambda_f(p)| \leq 2$ then the roots are complex, lie on the unit circle and are conjugates of each other, and hence $\alpha_p^n + \beta_p^n = 2 \cos(n\theta_p)$ for some $\theta_p \in \mathbb{R}$.

Additionally, the proofs of Lemma 5.2 and Lemma 5.1 will require the following very weak estimate for the second moment of $L(s, f \otimes \chi_d)$. This relies on the following lemma.

Lemma 5.5. *Let $\varepsilon > 0$. Assume the Generalized Riemann Hypothesis. Let f be an even, arithmetically normalized Hecke Maaß eigencuspform with eigenvalue $-(\frac{1}{4} + t^2)$. Then uniformly in $|t|^\varepsilon \leq X$ and $x \leq X$*

$$\sum_{p \leq x} \frac{\lambda_f(p)^2}{p} \ll_\varepsilon (\log X)^{1/3}.$$

Proof. If $x \leq (\log |t|)^{32/23}$ then we apply the Kim-Sarnak bound $|\lambda_f(p)| \leq p^{7/64} + p^{-7/64}$ to see that

$$\sum_{p \leq x} \frac{\lambda_f(p)^2}{p} \ll x^{7/32+o(1)} \ll (\log |t|)^{7/23+o(1)} \ll_\varepsilon (\log X)^{1/3}.$$

If $x > (\log |t|)^{32/23}$ then by (5.6) and using Li's bound (5.1) we see that

$$\begin{aligned} \sum_{p \leq x} \frac{\lambda_f(p)^2}{p} &= \sum_{p \leq x} \frac{1 + \lambda_f(p)}{p} \ll \log \log x + (\log |t|)^{1/8+o(1)} + x^{-1/2+o(1)} (\log |t|)^{1+o(1)} \\ &\ll \log \log x + (\log |t|)^{7/23+o(1)} \ll_\varepsilon (\log X)^{1/3} \end{aligned}$$

for $x \leq X$, as needed. □

Lemma 5.6. *Let $\varepsilon > 0$. Assume the Generalized Riemann Hypothesis. Let f be an even, arithmetically normalized Hecke Maaß eigencusp form with eigenvalue $-(\frac{1}{4} + t^2)$. Then, uniformly in $|t|^\varepsilon \leq X$*

$$\sum_{|d| \leq X} L(\tfrac{1}{2}, f \otimes \chi_d)^2 \ll_\varepsilon X \exp((\log X)^{1/3+\varepsilon})$$

where the summation is over fundamental discriminants.

Proof. Let

$$\mathcal{S}(X, \mathcal{V}) = \{|d| \leq X : \log L(\tfrac{1}{2}, f \otimes \chi_d) > \mathcal{V}\},$$

where we interpret $\log L(\tfrac{1}{2}, f \otimes \chi_d)$ as $-\infty$ if $L(\tfrac{1}{2}, f \otimes \chi_d) = 0$. Observe that

$$\sum_{|d| \leq X} L(\tfrac{1}{2}, f \otimes \chi_d) = \int_{\mathbb{R}} e^{\mathcal{V}} \#\mathcal{S}(X, \mathcal{V}) d\mathcal{V}.$$

Thus, it will be enough to show that for $(\log X)^{1/3+\varepsilon} < \mathcal{V}$

$$(5.7) \quad \#\mathcal{S}(X, \mathcal{V}) \ll_{\varepsilon} X \exp(-c\mathcal{V} \log \mathcal{V}),$$

where $c > 0$ depends at most on ε . Note that this bound is vacuous once $\mathcal{V} > \frac{4}{\varepsilon} \log X / \log \log X$ since choosing $x = (\log |d|/t)$ in Lemma 5.4 shows that $\log L(\tfrac{1}{2}, f \otimes \chi_d) \leq \frac{4}{\varepsilon} \log X / \log \log X$, since $|t| \leq X^{1/\varepsilon}$ and $|d| \leq X$.

To obtain (5.7) take $x = X^{1/(\varepsilon^2 \mathcal{V})}$. By Lemma 5.4

$$\log L(\tfrac{1}{2}, f \otimes \chi_d) \leq \sum_{p^n \leq x} \frac{\chi_d(p^n)(\alpha_p^n + \beta_p^n)}{np^{n(\frac{1}{2}+1/\log x)}} \cdot \frac{\log(x/p^n)}{\log x} + 4\varepsilon \mathcal{V}.$$

In the sum over the prime powers, p^n , the terms with $n \geq 3$ contribute $O(1)$. Using Lemma 5.5 the contribution to the above sum from the squares of primes $p^2 \leq x$ is

$$\leq \sum_{p \leq \sqrt{x}} \frac{\lambda_f(p)^2 + 2}{p} \ll_{\varepsilon} (\log X)^{1/3}.$$

We conclude that if $\log L(\tfrac{1}{2}, f \otimes \chi_d) > \mathcal{V}$ then

$$\frac{1}{2} \mathcal{V} < \sum_{p \leq x} \frac{\lambda_f(p) \chi_d(p) \log x/p}{p^{\frac{1}{2}+1/\log x} \log x}.$$

Using Markov's inequality and Lemma 4.3 it follows that for any integer $k \leq \frac{\varepsilon^2 \mathcal{V}}{4}$ the number of fundamental discriminants d for which this holds is

$$(5.8) \quad \begin{aligned} &\ll \frac{2^{2k}}{\mathcal{V}^{2k}} \sum_{|d| \leq X} \left(\sum_{p \leq x} \frac{\lambda_f(p) \chi_d(p) \log x/p}{p^{\frac{1}{2}+1/\log x} \log x} \right)^{2k} \ll \frac{2^{2k}}{\mathcal{V}^{2k}} \cdot X \cdot \frac{(2k)!}{2^k k!} \left(\sum_{p \leq x} \frac{\lambda_f(p)^2}{p} \right)^k \\ &\ll X \cdot \left(\frac{Ck}{\mathcal{V}^2} (\log X)^{1/3} \right)^k \end{aligned}$$

where $C > 0$ depends at most on ε and in the last step we used Lemma 5.5 to bound the sum over primes. Taking $k = \lfloor \frac{\varepsilon^2 \mathcal{V}}{4} \rfloor$, we see that since $\mathcal{V} > (\log X)^{1/3+\varepsilon}$, there exists $c > 0$, which depends at most on ε , such that the RHS of (5.8) is $\ll X e^{-c\mathcal{V} \log \mathcal{V}}$ which establishes (5.7), thus giving the lemma. \square

5.3. **The Proof of Lemma 5.1.** Let \mathbb{Z}^b denote the set of fundamental discriminants and for $a, b \in \mathbb{N}$ and $\ell \in \mathbb{Z}$ let

$$\mathcal{N}(X) = \{(d_1, d_2) \in \mathbb{Z}^b \times \mathbb{Z}^b : a|d_1| \leq X, ad_1 = bd_2 + \ell\}.$$

Also define

$$(5.9) \quad \mu(f, X) = -\log \log X + \log L(1, \text{Sym}^2 f),$$

$$\sigma^2(f, X) = 2 \log \log X + 2 \log L(1, \text{Sym}^2 f).$$

Lemma 5.7. *Let $\varepsilon > 0$. Let $0 \neq |\ell| \leq \log X$ and $1 \leq a, b \leq (\log X)^{100}$. Also let*

$$(5.10) \quad I_{1,\varepsilon}(X) = (\varepsilon \log \log X, \varepsilon \sigma^2(f, X) \log \sigma(f, X)]$$

and

$$(5.11) \quad I_{2,\varepsilon}(X) = (\varepsilon \sigma^2(f, X) \log \sigma(f, X), 2\sqrt{\log X}].$$

Then for $x = X^{1/(4\varepsilon\mathcal{V})}$ and $X \geq |t|^\varepsilon$

$$(5.12) \quad \# \left\{ (d_1, d_2) \in \mathcal{N}(X) : \left| \sum_{\substack{2 < p \leq x \\ p \nmid ab}} \frac{(\chi_{d_1}(p) + \chi_{d_2}(p)) \lambda_f(p) \log x/p}{p^{\frac{1}{2}+1/\log x} \log x} \right| \geq \mathcal{V} \right\} \\ \ll \frac{X}{[a, b]} \begin{cases} \exp\left(-\frac{\mathcal{V}(1-2\varepsilon)^2}{2\sigma^2(f, X)}\right) & \text{if } \mathcal{V} \in I_{1,\varepsilon}(X), \\ \exp(-c\mathcal{V} \log \frac{\mathcal{V}}{\sigma^2(f, X)}) & \text{if } \mathcal{V} \in I_{2,\varepsilon}(X), \end{cases}$$

where $c > 0$ depends on at most ε .

Proof. Let $z = x^{1/\log \log X}$. It suffices to bound the number of $(d_1, d_2) \in \mathcal{N}(X)$ such that

$$(5.13) \quad \left| \sum_{\substack{2 < p \leq z \\ p \nmid ab}} \frac{(\chi_{d_1}(p) + \chi_{d_2}(p)) \lambda_f(p) \log x/p}{p^{\frac{1}{2}+1/\log x} \log x} \right| \geq \mathcal{V}(1 - \varepsilon)$$

or

$$(5.14) \quad \left| \sum_{\substack{z < p \leq x \\ p \nmid ab}} \frac{(\chi_{d_1}(p) + \chi_{d_2}(p)) \lambda_f(p) \log x/p}{p^{\frac{1}{2}+1/\log x} \log x} \right| \geq \varepsilon \mathcal{V}.$$

Using Markov's inequality and Lemma 4.1 we see that the number of $(d_1, d_2) \in \mathcal{N}(X)$ for which (5.13) holds is bounded by, for any $k \leq \varepsilon \mathcal{V} \log \log X$,

$$(5.15) \quad \frac{1}{(\mathcal{V}(1-\varepsilon))^{2k}} \sum_{\substack{d_1, d_2 \\ a|d_1 \leq X \\ ad_1 = bd_2 + \ell}} \left(\sum_{\substack{2 < p \leq z \\ p \nmid ab}} \frac{(\chi_{d_1}(p) + \chi_{d_2}(p)) \lambda_f(p) \log x/p}{p^{\frac{1}{2} + 1/\log x} \log x} \right)^{2k} \\ \ll \frac{X(2k)!}{[a, b](2(\mathcal{V}(1-\varepsilon))^2)^k k!} \left(2 \sum_{p \leq z} \frac{\lambda_f(p)^2}{p} + O\left(1 + \sum_{p|\ell} \frac{\lambda_f(p)^2}{p}\right) \right)^k.$$

Note that the use of Lemma 4.1 is justified (taking $a(p) = \lambda_f(p) \frac{\log x/p}{p^{1/\log x \log x}}$ for $p \neq 2$ and $a(2) = 0$); to see this, first observe for $|\ell| \leq \log X$

$$\sum_{p|\ell} \frac{\lambda_f(p)^2}{p} \ll \sum_{p|\ell} p^{-2\delta} \ll_\varepsilon (\log \log X)^{1-2\delta}$$

using the bound $|\lambda_f(p)| \ll p^{1/2-\delta}$ (where importantly we know that $\delta > 0$ following for example the works of [26]). Secondly, since $X \geq |t|^\varepsilon$, $\mathcal{V} \leq 2\sqrt{\log X}$, $\frac{\log x/p}{p^{1/\log x \log x}} \geq \frac{1}{2\sqrt{\varepsilon}}$ for $p \leq \sqrt{x}$, and $z = X^{1/(4\varepsilon \mathcal{V} \log \log X)} > \exp\left(c \frac{\sqrt{\log |t|}}{\log \log |t|}\right)$ where $c > 0$ depends on at most ε , it follows under GRH using (5.2) and (5.3) that

$$B(z) = \sum_{2 < p \leq z} \frac{\lambda_f(p)^2}{p^{1+2/\log x}} \left(\frac{\log x/p}{\log x} \right)^2 \geq \frac{1}{4e} \sum_{2 < p \leq z} \frac{\lambda_f(p)^2}{p} = \frac{1}{4e} (\log \log z + \log L(1, \text{Sym}^2 f) + O(1)) \\ \gg_\varepsilon \log \log z \gg_\varepsilon \log \log X.$$

The two estimates above imply $2B(z) \geq A(\ell)$, and the application of Lemma 4.1 has now been justified. Also, using (5.2) we have

$$2 \sum_{p \leq x} \frac{\lambda_f(p)^2}{p} = 2 \log \log x + 2 \log L(1, \text{Sym}^2 f) + O(1) \leq \sigma^2(f, X)(1 + o(1)),$$

where we also used (5.3) to get $\sigma^2(f, X) \gg_\varepsilon \log \log X$ which is used to deduce the last inequality. Hence, we get that the RHS of (5.15) is

$$(5.16) \quad \ll \frac{X(2k)!}{[a, b](2(\mathcal{V}(1-\varepsilon))^2)^k k!} (\sigma^2(f, X)(1 + o(1)))^k \ll \frac{X}{[a, b]} \left(\frac{2k\sigma^2(f, X)}{e(\mathcal{V}(1-2\varepsilon))^2} \right)^k$$

for any integer $1 \leq k \leq \varepsilon \mathcal{V} \log \log X$.

We now specialize our choice of k . If $\mathcal{V} \leq \varepsilon \sigma^2(f, X) \log \log X$ then we choose $k = \lfloor (\mathcal{V}(1-2\varepsilon))^2 / (2\sigma^2(f, X)) \rfloor$ in (5.15), and in the remaining case we choose

$k = \lfloor \mathcal{V}/2 \rfloor$. This leads to the following bound

$$(5.17) \quad \ll \frac{X}{[a, b]} \begin{cases} \exp\left(\frac{-(\mathcal{V}(1-2\varepsilon))^2}{2\sigma^2(f, X)}\right) & \text{if } \varepsilon \log \log X \leq \mathcal{V} \leq \varepsilon \sigma^2(f, X) \log \log X, \\ \exp\left(-\frac{1}{3}\mathcal{V} \log \frac{\mathcal{V}}{\sigma^2(f, X)}\right) & \text{if } \varepsilon \sigma^2(f, X) \log \log X < \mathcal{V} \leq 2\sqrt{\log X}, \end{cases}$$

for the number of $(d_1, d_2) \in \mathcal{N}(X)$ for which (5.13) holds.

It remains to bound the number $(d_1, d_2) \in \mathcal{N}(X)$ for which (5.14) holds. Recall that since $\mathcal{V} \leq 2\sqrt{\log X}$ it follows that $z = X^{1/(4\varepsilon\mathcal{V} \log \log X)} > (\log |t|)^{2+\varepsilon}$. Hence, under GRH it follows from (5.2) that

$$(5.18) \quad \sum_{z < p \leq x} \frac{\lambda_f(p)^2}{p} = \sum_{p \leq x} \frac{\lambda_f(p)^2}{p} - \sum_{p \leq z} \frac{\lambda_f(p)^2}{p} = \log \frac{\log x}{\log z} + O(1).$$

We will now apply Markov's inequality and Lemma 4.1. In Lemma 4.1 set $a(p) = 0$ for $p < z$ and $a(p) = \lambda_f(p) \frac{\log x/p}{p^{1/\log x} \log x}$ otherwise. This way the conditions of the lemma are easily verified, since $\sum_{p|\ell} a(p)^2/p = 0$ because $|\ell| \leq \log X$ according to our assumption and

$$\sum_{z < p \leq x} \frac{\lambda_f(p)^2}{p^{1+2/\log x}} \left(\frac{\log x/p}{\log x} \right)^2 \geq \frac{1}{4e} \sum_{z < p \leq \sqrt{x}} \frac{\lambda_f(p)^2}{p} = \frac{1}{4e} \log \frac{\log x}{\log z} + O(1) \gg \log \log \log X.$$

Thus applying Markov's inequality, Lemma 4.1, as in (5.15) with $k = \lfloor \varepsilon \mathcal{V} \rfloor$, and then using (5.18) it follows that for $\varepsilon \log \log X \leq \mathcal{V} \leq 2\sqrt{\log X}$, the number of $(d_1, d_2) \in \mathcal{N}(X)$ for which (5.14) holds is

$$\begin{aligned} & \ll \frac{X}{[a, b]} \frac{(2k)!}{(2(\varepsilon\mathcal{V})^2)^k k!} \left(2 \log \frac{\log x}{\log z} + O(1) \right)^k \\ & \ll \frac{X}{[a, b]} \left(\frac{2k}{e(\varepsilon\mathcal{V})^2} \log \log \log X (1 + o(1)) \right)^k \ll \frac{X}{[a, b]} \exp\left(-\frac{1}{2} \cdot \varepsilon \mathcal{V} \log \mathcal{V}\right). \end{aligned}$$

Combining this with (5.17) gives (5.12), thereby completing the proof. \square

Proof of Lemma 5.1. For fundamental discriminants d_1, d_2 write $\mathcal{L}_f(d_1, d_2) = L(\frac{1}{2}, f \otimes \chi_{d_1}) L(\frac{1}{2}, f \otimes \chi_{d_2})$ and set

$$\mathcal{S}(X, \mathcal{V}) = \{(d_1, d_2) \in \mathcal{N}(X) : \log \mathcal{L}_f(d_1, d_2) \geq \mathcal{V}\}.$$

Observe that for $\mathcal{U} = \exp(\sqrt{\log X})$

$$(5.19) \quad \sum_{\substack{d_1, d_2 \\ a|d_1| \leq X \\ ad_1 = bd_2 + \ell \\ \mathcal{L}_f(d_1, d_2) > \mathcal{U}}} (\mathcal{L}_f(d_1, d_2))^{1/2} \leq \mathcal{U}^{-1/2} \sum_{\substack{d_1, d_2 \\ a|d_1| \leq X \\ ad_1 = bd_2 + \ell}} \mathcal{L}_f(d_1, d_2)$$

$$(5.20) \quad \ll_{\varepsilon} X \mathcal{U}^{-1/2} \cdot \exp((\log X)^{1/3+\varepsilon}) \ll X \exp\left(-\frac{1}{4}\sqrt{\log X}\right)$$

by Lemma 5.6, since $\mathcal{L}_f(d_1, d_2) \leq L(\frac{1}{2}, f \otimes \chi_{d_1})^2 + L(\frac{1}{2}, f \otimes \chi_{d_2})^2$. Also,

$$(5.21) \quad \sum_{\substack{d_1, d_2 \\ a|d_1| \leq X \\ ad_1 = bd_2 + \ell \\ \mathcal{L}_f(d_1, d_2) \leq \mathcal{U}}} (\mathcal{L}_f(d_1, d_2))^{1/2} = \frac{1}{2} \cdot \int_{-\infty}^{\sqrt{\log X}} e^{\frac{1}{2} \cdot \mathcal{V}} \#\mathcal{S}(X, \mathcal{V}) d\mathcal{V} \\ \leq \frac{L(1, \text{Sym}^2 f)^{1/2}}{2(\log X)^{1/2}} \int_{-\infty}^{2\sqrt{\log X}} e^{\frac{1}{2} \cdot \mathcal{V}} \#\mathcal{S}(X, \mathcal{V} + \mu(f, X)) d\mathcal{V},$$

where $\mu(f, X)$ is as defined in (5.9). The portion of the integral over $\mathcal{V} \leq \varepsilon \log \log X$ is bounded by

$$\ll \#\mathcal{N}(X) \cdot \frac{L(1, \text{Sym}^2 f)^{1/2}}{(\log X)^{1/2}} \int_{-\infty}^{\varepsilon \log \log X} e^{\frac{1}{2} \cdot \mathcal{V}} d\mathcal{V} \ll \frac{X}{[a, b]} (\log X)^{-1/2+\varepsilon} L(1, \text{Sym}^2 f)^{1/2},$$

which is negligible since $L(1, \text{Sym}^2 f)^{1/2} \ll (\log X)^{\varepsilon} L(1, \text{Sym}^2 f)^{3/4}$ by (5.3).

We will now use Lemma 5.7 to bound the remaining portion of the integral on the RHS of (5.21). By Lemma 5.4

$$\log \mathcal{L}_f(d_1, d_2) \leq \sum_{p^n \leq x} \frac{(\chi_{d_1}(p^n) + \chi_{d_2}(p^n))(\alpha_p^n + \beta_p^n) \log(x/p^n)}{np^{n(\frac{1}{2}+1/\log x)}} + 3 \cdot \frac{\log |d_1 d_2| t^2}{\log x}.$$

In the sum over prime powers the terms with $n \geq 3$ contribute $O(1)$, which is negligible. Using (5.2) the sum over prime squares is for $x \geq (\log |t|)^{2+\varepsilon}$

$$\sum_{p \leq \sqrt{x}} \frac{(\chi_{d_1}(p^2) + \chi_{d_2}(p^2))(\lambda_f^2(p) - 2) \log x / p^2}{2p^{1+2/\log x}} \frac{\log x / p^2}{\log x} \\ \leq \sum_{p \leq \sqrt{x}} \frac{\lambda_f(p)^2}{p} - 2 \sum_{p \leq \sqrt{x}} \frac{1}{p} \left(1 - O\left(\frac{\log p}{\log x}\right)\right) + O\left(\sum_{p|d_1 d_2} \frac{1}{p}\right) \\ = \mu(f, x) + O(\log \log \log(|d_1 d_2| + e^{\varepsilon})).$$

Hence, for $x \geq (\log |t|)^{2+\varepsilon}$

$$\begin{aligned} \log \mathcal{L}_f(d_1, d_2) - \mu(f, x) &\leq \sum_{2 < p \leq x} \frac{(\chi_{d_1}(p) + \chi_{d_2}(p))\lambda_f(p) \log x/p}{p^{\frac{1}{2}+1/\log x} \log x} \\ &\quad + 3 \cdot \frac{\log |d_1 d_2| t^2}{\log x} + O(\log \log \log(|d_1 d_2| + e^e)). \end{aligned}$$

Also,

$$\sum_{p|ab} \frac{|\lambda_f(p)|}{\sqrt{p}} < \omega(ab) = o(\log \log X),$$

since $a, b \leq (\log X)^{100}$. Let $\tilde{\mathcal{V}} = (1 - 25\varepsilon)\mathcal{V}$. Combining the previous two observations and taking $x = X^{1/(4\varepsilon^2\tilde{\mathcal{V}})}$ it follows that

$$(5.22) \quad \log \mathcal{L}_f(d_1, d_2) - \mu(f, X) \leq \sum_{\substack{2 < p \leq x \\ p \nmid ab}} \frac{(\chi_{d_1}(p) + \chi_{d_2}(p))\lambda_f(p) \log x/p}{p^{\frac{1}{2}+1/\log x} \log x} + 25\varepsilon\mathcal{V}.$$

Hence, if the LHS is $\geq \mathcal{V}$ then the sum over primes is $\geq (1 - 25\varepsilon)\mathcal{V} = \tilde{\mathcal{V}}$ so that by applying Lemma 5.7 (with $\tilde{\mathcal{V}}$ in place of \mathcal{V} and ε^2 in place of ε) it follows for $I_{1,\varepsilon^2}(X), I_{2,\varepsilon^2}(X)$ as in (5.10), (5.11) (resp.) that

$$(5.23) \quad \#\mathcal{S}(X, \mathcal{V} + \mu(f, X)) \ll \frac{X}{[a, b]} \cdot \begin{cases} \exp\left(-\frac{(\tilde{\mathcal{V}}(1-2\varepsilon^2))^2}{2\sigma^2(f, X)}\right) & \text{if } \tilde{\mathcal{V}} \in I_{1,\varepsilon^2}(X), \\ \exp(-c\tilde{\mathcal{V}} \log \frac{\tilde{\mathcal{V}}}{\sigma^2(f, X)}) & \text{if } \tilde{\mathcal{V}} \in I_{2,\varepsilon^2}(X), \end{cases}$$

where $c > 0$ depends at most on ε .

To estimate the portion of the integral on the RHS of (5.21) over $I_{1,\varepsilon^2}(X)$ we use (5.23) along with the formula

$$(5.24) \quad \int_{\mathbb{R}} e^{-\alpha x^2 + \beta x} dx = \sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\beta^2}{4\alpha}\right)$$

with $\beta = \frac{1}{2}$ and $\alpha = \frac{((1-2\varepsilon^2)(1-25\varepsilon))^2}{2\sigma^2(f, X)}$. Hence,

$$(5.25) \quad \int_{I_{1,\varepsilon^2}(X)} e^{\frac{1}{2}\cdot\mathcal{V}} \#\mathcal{S}(X, \mathcal{V} + \mu(f, X)) d\mathcal{V} \ll_{\varepsilon} \frac{X}{[a, b]} \sigma(f, X) L(1, \text{Sym}^2 f)^{1/4+51\varepsilon} (\log X)^{1/4+51\varepsilon}.$$

To bound the portion of the integral on the RHS of (5.21) over $I_{2,\varepsilon^2}(X)$ we note that by (5.23) the integrand is $\ll_\varepsilon \frac{X}{[a,b]} \exp(-\frac{c}{2}\mathcal{V}(\log \log \log X))$ in this range, so that

$$\begin{aligned} & \int_{I_{2,\varepsilon^2}(X)} e^{\frac{1}{2}\mathcal{V}} \#\mathcal{S}(X, \mathcal{V} + \mu(f, X)) d\mathcal{V} \\ & \ll_\varepsilon \frac{X}{[a,b]} \int_{\log \log X}^{\infty} \exp(-\frac{c}{2}\mathcal{V}(\log \log \log X)) d\mathcal{V} \ll_\varepsilon \frac{X}{[a,b]} \cdot \frac{1}{(\log X)^{10}}. \end{aligned}$$

Using the above bound and (5.25) in (5.21) along with the bound (5.19) and applying (5.3) completes the proof. \square

5.4. Proof of Lemma 5.2. Given a real number $y \geq 2$, an interval $\mathcal{I} \subset (1, y]$ and an integer m let

$$\mathcal{P}_{\mathcal{I}}(m, y) = \sum_{p \in \mathcal{I}} \frac{\binom{m}{p} \lambda_f(p) \log y/p}{p^{\frac{1}{2}+1/\log y} \log y},$$

where $\binom{m}{p}$ is the Legendre symbol.

Lemma 5.8. *Let $\varepsilon > 0$. Let $I = (1, (\log X)^5]$ and $\mathcal{J}(X) = \log \log X \log \log \log X$. Then for $x = X^{1/(4\varepsilon\mathcal{V})}$ and $X \geq |t|^\varepsilon$*

$$(5.26) \quad \sum_{\substack{|d| \leq X \\ |\mathcal{P}_I(d, X)| \leq \frac{1}{2}(\log X)^{5/6} \\ \mathcal{P}_{(\log X)^5, x}(d, x) \geq \mathcal{V}}} e^{\mathcal{P}_I(d, X)} \ll_\varepsilon L(1, \text{Sym}^2 f)^{1/2} X(\log X)^\varepsilon \begin{cases} \exp\left(-\frac{(\mathcal{V}(1-2\varepsilon))^2}{2 \log \log X}\right) & \text{if } \varepsilon \log \log X \leq \mathcal{V} \leq \varepsilon \mathcal{J}(X), \\ \exp(-c\mathcal{V} \log \mathcal{V}) & \text{if } \varepsilon \mathcal{J}(X) \leq \mathcal{V} \leq (\log X)^{5/6}, \end{cases}$$

where the summation is over fundamental discriminants and $c > 0$ depends at most on ε .

Proof. Let $z = x^{1/\log \log X}$. Also, define $J_1 = ((\log X)^5, z]$ and $J_2 = (z, x]$. If $\mathcal{P}_{((\log X)^5, x]}(d, x) > \mathcal{V}$ then $\mathcal{P}_{J_1}(d, x) > \mathcal{V}(1 - \varepsilon)$ or $\mathcal{P}_{J_2}(d, x) > \varepsilon\mathcal{V}$. Hence, writing $\mathcal{V}_1 = \mathcal{V}(1 - \varepsilon)$ and $\mathcal{V}_2 = \varepsilon\mathcal{V}$ we get by using Markov's inequality that

$$(5.27) \quad \begin{aligned} \sum_{\substack{|d| \leq X \\ |\mathcal{P}_I(d, X)| \leq \frac{1}{2}(\log X)^{5/6} \\ \mathcal{P}_{((\log X)^5, x]}(d, x) \geq \mathcal{V}}} e^{\mathcal{P}_I(d, X)} &\leq \sum_{\substack{|d| \leq X \\ |\mathcal{P}_I(d, X)| \leq \frac{1}{2}(\log X)^{5/6} \\ \mathcal{P}_{J_1}(d, x) \geq \mathcal{V}_1}} e^{\mathcal{P}_I(d, X)} + \sum_{\substack{|d| \leq X \\ |\mathcal{P}_I(d, X)| \leq \frac{1}{2}(\log X)^{5/6} \\ \mathcal{P}_{J_2}(d, x) \geq \mathcal{V}_2}} e^{\mathcal{P}_I(d, X)} \\ &\leq \sum_{|\mathcal{P}_I(d, X)| \leq \frac{1}{2}(\log X)^{5/6}} \left(\left(\frac{\mathcal{P}_{J_1}(d, X)}{\mathcal{V}_1} \right)^{2k_1} + \left(\frac{\mathcal{P}_{J_2}(d, x)}{\mathcal{V}_2} \right)^{2k_2} \right) e^{\mathcal{P}_I(d, X)} \end{aligned}$$

for any integers $k_1, k_2 > 0$.

In the above sum note $|\mathcal{P}_I(d, X)| \leq \frac{1}{2}(\log X)^{5/6}$, so that we can approximate $e^{\mathcal{P}_I(d, X)}$ by the first $O((\log X)^{5/6})$ terms of its Taylor series expansion. To see this, note that for an even integer H and $u \leq H/e^2$ that

$$e^u = \sum_{h=0}^H \frac{u^h}{h!} + \frac{e^\xi}{(H+1)!} u^{H+1} \leq 2 \sum_{h=0}^H \frac{u^h}{h!}$$

where ξ is between u and 0. The inequality is trivial for $u < 0$ and follows from Stirling's formula for $u > 0$. Thus extending the sum on the RHS of (5.27) to all integers $|m| \leq X$ with $|\mathcal{P}_I(m, X)| \leq \frac{1}{2}(\log X)^{5/6}$, adding a Schwartz function F which majorizes $1_{[-1, 1]}$ with \widehat{F} compactly supported in $(-10, 10)$ and setting $M = \frac{1}{2}(\log X)^{5/6}$ we have that the RHS of (5.27) is

$$(5.28) \quad \begin{aligned} &\leq 2 \sum_{i=1}^2 \frac{1}{\mathcal{V}_i^{2k_i}} \sum_{|\mathcal{P}_I(m, X)| \leq M} \sum_{h \leq 2\lfloor e^2 M \rfloor} \frac{1}{h!} \mathcal{P}_I(m, X)^h \cdot \mathcal{P}_{J_i}(m, x)^{2k_i} F\left(\frac{m}{X}\right) \\ &\leq 2 \sum_{i=1}^2 \sum_{h \leq 2\lfloor e^2 M \rfloor} \frac{1}{h!} \frac{1}{\mathcal{V}_i^{2k_i}} \sum_m \mathcal{P}_I(m, X)^h \cdot \mathcal{P}_{J_i}(m, x)^{2k_i} F\left(\frac{m}{X}\right) \end{aligned}$$

where in the last step we used the fact that

$$\sum_{h=0}^H \frac{u^h}{h!} > 0$$

for any $u \in \mathbb{R}$ if H is a positive even integer.

We now assume that $k_1 \leq \varepsilon \mathcal{V} \log \log X$ and $k_2 \leq \varepsilon \mathcal{V}$. Using Lemma 4.3 we bound the RHS of (5.28) to see that it is

$$(5.29) \quad \begin{aligned} &\leq 2\widehat{F}(0)X \sum_{i=1}^2 \sum_{h \leq 2\lfloor e^2 M \rfloor} \frac{1}{h!} \frac{1}{\mathcal{V}_i^{2k_i}} \widetilde{C}(h) \widetilde{C}(2k_i) \left(\sum_{p \in I} \frac{\lambda_f(p)^2}{p} \right)^{h/2} \left(\sum_{p \in J_i} \frac{\lambda_f(p)^2}{p} \right)^{k_i} \\ &\ll X \sum_{i=1,2} \left(\frac{2k_i}{e\mathcal{V}_i^2} \right)^{k_i} \left(\sum_{p \in J_i} \frac{\lambda_f(p)^2}{p} \right)^{k_i} \sum_{h=0}^{\infty} \frac{1}{(2h)!} \cdot \frac{(2h)!}{2^h h!} \left(\sum_{p \in I} \frac{\lambda_f(p)^2}{p} \right)^h. \end{aligned}$$

Since $X \geq |t|^\varepsilon$ and $\mathcal{V} \leq (\log X)^{5/6}$ it follows that $x = X^{1/(4\varepsilon\mathcal{V})} > (\log |t|)^{2+o(1)}$. Also, $(\log X)^5 \geq (\log |t|)^{2+o(1)}$. So by (5.2) the inner sum over I equals $\log L(1, \text{Sym}^2 f) + O(\log \log \log X)$, which implies that the sum over h is $\ll L(1, \text{Sym}^2 f)^{1/2} (\log X)^\varepsilon$. Using (5.2) and noting that $z \geq (\log |t|)^{2+o(1)}$ we get

$$\sum_{p \in J_1} \frac{\lambda_f(p)^2}{p} \leq \log \log X + O(1), \quad \sum_{p \in J_2} \frac{\lambda_f(p)^2}{p} = \log \log \log X + O(1).$$

Hence, applying the previous two observations it follows that the RHS of (5.29) is

$$(5.30) \quad \begin{aligned} &\ll XL(1, \text{Sym}^2 f)^{1/2} (\log X)^\varepsilon \\ &\quad \times \left(\left(\frac{2k_1 \log \log X (1 + o(1))}{e\mathcal{V}_1^2} \right)^{k_1} + \left(\frac{2k_2 \log \log \log X (1 + o(1))}{e\mathcal{V}_2^2} \right)^{k_2} \right). \end{aligned}$$

We will now specialize our choice of k_1, k_2 . First for k_1 , if $\varepsilon \log \log X \leq \mathcal{V} \leq \varepsilon \mathcal{J}(X)$ then choose $k_1 = \lfloor \mathcal{V}_1^2 / (2 \log \log X) \rfloor$. While if $\varepsilon \mathcal{J}(X) \leq \mathcal{V} \leq (\log X)^{5/6}$ then choose $k_1 = \lfloor \varepsilon \mathcal{V} \rfloor$. Secondly, for k_2 set $k_2 = \lfloor \varepsilon \mathcal{V} \rfloor$ for both ranges of \mathcal{V} . Hence, (5.30) (and thus (5.27) as well) is bounded by

$$\ll \begin{cases} L(1, \text{Sym}^2 f)^{1/2} X (\log X)^\varepsilon e^{-\frac{(\mathcal{V}(1-2\varepsilon))^2}{2 \log \log X}} & \text{if } \varepsilon \log \log X < \mathcal{V} < \varepsilon \mathcal{J}(X), \\ L(1, \text{Sym}^2 f)^{1/2} X (\log X)^\varepsilon e^{-(\frac{1}{2}\varepsilon + o(1))\mathcal{V} \log \mathcal{V}} & \text{if } \varepsilon \mathcal{J}(X) \leq \mathcal{V} \leq (\log X)^{5/6}. \end{cases}$$

□

Finally we are ready to prove Lemma 5.2.

Proof of Lemma 5.2. First, by Lemma 5.6 for $\mathcal{U} = e^{\frac{1}{2}(\log X)^{5/6}}$ we have

$$(5.31) \quad \sum_{\substack{|d| \leq X \\ \log L(\frac{1}{2}, f \otimes \chi_d) > \mathcal{U}}} L(\tfrac{1}{2}, f \otimes \chi_d) \ll \mathcal{U}^{-1} \sum_{|d| \leq X} L(\tfrac{1}{2}, f \otimes \chi_d)^2 \ll_\varepsilon X.$$

Secondly, let F be a Schwartz function which majorizes $\mathbf{1}_{[-1,1]}$ with \widehat{F} compactly supported in $(-10, 10)$ and let $k = \lfloor (\log X)^{1/2} \rfloor$. By Lemma 4.3, the number of fundamental discriminants $|d| \leq X$ for which $|\mathcal{P}_I(d, X)| \geq \frac{1}{2}(\log X)^{5/6}$ is

$$(5.32) \quad \begin{aligned} &\leq \frac{4^k}{(\log X)^{5k/3}} \sum_{m \in \mathbb{Z}} |\mathcal{P}_I(m, X)|^{2k} F\left(\frac{m}{X}\right) \ll X \frac{(2k)!}{k!2^k} \cdot \left(\frac{4}{(\log X)^{5/3}} \cdot \sum_{p \in I} \frac{\lambda_f(p)^2}{p} \right)^k \\ &\ll X \left(\frac{Ck(\log X)^{1/3}}{(\log X)^{5/3}} \right)^k \ll_\varepsilon X e^{-\frac{1}{2}(\log X)^{1/2}} \end{aligned}$$

where we used Lemma 5.5 in the second bound and $C > 0$ depends at most on ε . It follows that

$$\sum_{\substack{|d| \leq X \\ |\mathcal{P}_I(d, X)| > \frac{1}{2}(\log X)^{5/6}}} L\left(\frac{1}{2}, f \otimes \chi_d\right) \ll_\varepsilon \left(\sum_{|d| \leq X} L\left(\frac{1}{2}, f \otimes \chi_d\right) \right)^{1/2} \cdot X^{1/2} \exp\left(-\frac{1}{4}(\log X)^{1/2}\right) \ll_\varepsilon X$$

on using Lemma 5.6. Combining this along with (5.31) we have that

$$(5.33) \quad \sum_{|d| \leq X} L\left(\frac{1}{2}, f \otimes \chi_d\right) = \sum_{\substack{|d| \leq X \\ \log L\left(\frac{1}{2}, f \otimes \chi_d\right) \leq \frac{1}{2}(\log X)^{5/6} \\ |\mathcal{P}_I(d, X)| \leq \frac{1}{2}(\log X)^{5/6}}} L\left(\frac{1}{2}, f \otimes \chi_d\right) + O_\varepsilon(X).$$

Define $\widetilde{\mathcal{L}}_f(d, X) = L\left(\frac{1}{2}, f \otimes \chi_d\right) e^{-\mathcal{P}_I(d, X)}$ and

$$\widetilde{\mathcal{S}}(X, \mathcal{V}) = \sum_{\substack{|d| \leq X \\ |\mathcal{P}_I(d, X)| \leq \frac{1}{2}(\log X)^{5/6} \\ \log \widetilde{\mathcal{L}}_f(d, X) > \mathcal{V}}} e^{\mathcal{P}_I(d, X)}.$$

Then

$$(5.34) \quad \begin{aligned} \sum_{\substack{|d| \leq X \\ \log L\left(\frac{1}{2}, f \otimes \chi_d\right) \leq \frac{1}{2}(\log X)^{5/6} \\ |\mathcal{P}_I(d, X)| \leq \frac{1}{2}(\log X)^{5/6}}} L\left(\frac{1}{2}, f \otimes \chi_d\right) &\leq \sum_{\substack{|d| \leq X \\ \log \widetilde{\mathcal{L}}_f(d, X) \leq (\log X)^{5/6} \\ |\mathcal{P}_I(d, X)| \leq \frac{1}{2}(\log X)^{5/6}}} \widetilde{\mathcal{L}}_f(d, X) \cdot e^{\mathcal{P}_I(d, X)} \\ &= \int_{-\infty}^{(\log X)^{5/6}} e^{\mathcal{V}} \cdot \widetilde{\mathcal{S}}(X, \mathcal{V}) d\mathcal{V}. \end{aligned}$$

In particular, it follows from this along with (5.33) that

$$(5.35) \quad \sum_{|d| \leq X} L\left(\frac{1}{2}, f \otimes \chi_d\right) \leq \int_{-\infty}^{(\log X)^{5/6}} e^{\mathcal{V}} \cdot \widetilde{\mathcal{S}}(X, \mathcal{V}) d\mathcal{V} + O_\varepsilon(X).$$

Let $\tilde{\mu}(f, X) = -\frac{1}{2} \log \log X + \frac{1}{2} \log L(1, \text{Sym}^2 f)$. Next note that using (5.2) it follows for $x > (\log |t|)^{2+o(1)}$

$$\begin{aligned} & \sum_{p \leq \sqrt{x}} \frac{\chi_d(p^2)(\lambda_f(p^2) - 2)}{2p^{1+2/\log x}} \cdot \frac{\log x/p^2}{\log x} \\ & \leq \sum_{p \leq \sqrt{x}} \frac{\lambda_f(p)^2}{2p} - \sum_{p \leq \sqrt{x}} \frac{1}{p} \left(1 - O\left(\frac{\log p}{\log x}\right)\right) + O\left(\sum_{p|d} \frac{1}{p}\right) \\ & = \tilde{\mu}(f, x) + O(\log \log \log(|d| + e^\epsilon)). \end{aligned}$$

Also, for $\varepsilon^2 \log \log X \leq \mathcal{V} \leq 2(\log X)^{5/6}$ set $x = X^{1/(4\varepsilon^2(1-26\varepsilon)\mathcal{V})} (> e^{(\log X)^{1/6}})$. Applying Lemma 5.4, bounding the sum over $p^n \leq x$ with $n \geq 3$ as $O(1)$, and using the previous estimate to bound the sum over prime squares it follows that

$$\log \widetilde{\mathcal{L}}_f(d, X) - \tilde{\mu}(f, X) \leq \mathcal{P}_{((\log X)^5, x]}(d, x) + (\mathcal{P}_I(d, x) - \mathcal{P}_I(d, X)) + 25\varepsilon\mathcal{V}.$$

If the LHS of the above inequality is $> \mathcal{V}$ then we must have at least one of the following events occur: $\mathcal{P}_{((\log X)^5, x]}(d, X) > \mathcal{V}(1 - 26\varepsilon)$ or $\mathcal{P}_I(d, x) - \mathcal{P}_I(d, X) > \varepsilon\mathcal{V}$. Let $\mathcal{V}_1 = (1 - 26\varepsilon)\mathcal{V}$ and $\mathcal{V}_2 = \varepsilon\mathcal{V}$. Hence,

$$(5.36) \quad \widetilde{\mathcal{S}}(X, \mathcal{V} + \tilde{\mu}(f, X)) \leq \sum_{\substack{|d| \leq X \\ |\mathcal{P}_I(d, X)| \leq \frac{1}{2}(\log X)^{5/6} \\ \mathcal{P}_{((\log X)^5, x]}(d, x) > \mathcal{V}_1}} e^{\mathcal{P}_I(d, X)} + e^{\frac{1}{2}(\log X)^{5/6}} \sum_{\substack{|d| \leq X \\ \mathcal{P}_I(d, x) - \mathcal{P}_I(d, X) > \mathcal{V}_2}} 1.$$

To bound the second term on the RHS of (5.36) we extend the sum to all integers and include a Schwartz function F as before, then we apply Markov's inequality to see that for any $k \geq 1$ it is

$$(5.37) \quad \leq e^{\frac{1}{2}(\log X)^{5/6}} \sum_{m \in \mathbb{Z}} \left(\frac{\mathcal{P}_I(m, x) - \mathcal{P}_I(m, X)}{\mathcal{V}_2} \right)^{2k} F\left(\frac{m}{X}\right).$$

To estimate (5.37) we note that $|p^{-1/\log x} \frac{\log x/p}{\log x} - p^{-1/\log X} \frac{\log X/p}{\log X}| \leq C\mathcal{V}(\log X)^{-1+o(1)}$ for $p \leq (\log X)^5$, and $C > 0$ depends at most on ε . So applying Lemmas 4.3 and 5.5 with

$$a(p) = \lambda_f(p) \left(p^{-1/\log x} \frac{\log x/p}{\log x} - p^{-1/\log X} \frac{\log X/p}{\log X} \right),$$

for $p \neq 2$ and $a(2) = 0$, we get for $k = \lfloor (\log X)^{6/7} \rfloor$ that (5.37) is

$$(5.38) \quad \begin{aligned} & \ll X \frac{e^{\frac{1}{2}(\log X)^{5/6}}}{\mathcal{V}_2^{2k}} \frac{(2k)!}{2^k k!} \left(\frac{C^2 \mathcal{V}^2}{(\log X)^{2-o(1)}} \sum_{p \in I} \frac{\lambda_f(p)^2}{p} \right)^k \\ & \ll X e^{\frac{1}{2}(\log X)^{5/6}} \left(\frac{C'(\log X)^{6/7+1/3+\varepsilon}}{(\log X)^{2-o(1)}} \right)^{-\frac{1}{2}(\log X)^{6/7}} \ll_\varepsilon X e^{-(\log X)^{6/7}}, \end{aligned}$$

where $C' > 0$ is a constant which depends on at most ε .

In (5.35) making a change of variables then applying (5.36) and (5.38) gives

$$(5.39) \quad \sum_{|d| \leq X} L\left(\frac{1}{2}, f \otimes \chi_d\right) \leq e^{\tilde{\mu}(f, X)} \int_{-\infty}^{2(\log X)^{5/6}} e^{\mathcal{V}} \sum_{\substack{|d| \leq X \\ |\mathcal{P}_I(d, X)| \leq \frac{1}{2}(\log X)^{5/6} \\ \mathcal{P}_{((\log X)^5, x]}(d, x) > \mathcal{V}_1}} e^{\mathcal{P}_I(d, X)} d\mathcal{V} + O_\varepsilon(X),$$

where we have used Lemmas 5.3 and 5.5 to bound $\tilde{\mu}(f, X)$ in the estimation of the error term. Using Lemma 5.8 (with \mathcal{V}_1 in place of \mathcal{V} and ε^2 in place of ε) and arguing as in the proof of Lemma 5.1 (see equations (5.23)-(5.25), in (5.24) take $\beta = 1$, $\alpha = \frac{(1-C''\varepsilon)}{2 \log \log X}$, for $C'' > 0$ a sufficiently large absolute constant) gives

$$\int_{-\infty}^{(\log X)^{5/6}} e^{\mathcal{V}} \sum_{\substack{|d| \leq X \\ |\mathcal{P}_I(d, X)| \leq \frac{1}{2}(\log X)^{5/6} \\ \mathcal{P}_{((\log X)^5, x]}(d, x) > \mathcal{V}_1}} e^{\mathcal{P}_I(d, X)} d\mathcal{V} \ll L(1, \text{Sym}^2 f)^{1/2} X (\log X)^{1/2+2C''\varepsilon}.$$

Applying this estimate in (5.39) and recalling that $\tilde{\mu}(f, X) = -\frac{1}{2} \log \log X + \frac{1}{2} \log L(1, \text{Sym}^2 f)$ completes the proof. \square

6. AVERAGES OF COEFFICIENTS OF HALF-INTEGRAL WEIGHT MAASS FORMS

Let $\{g_j\}$ be a basis of V^+ consisting of simultaneous eigenfunctions of $\Delta_{1/2}$ and of the Hecke operators T_p , $p \neq 2$, and with each g_j normalized so that $\iint_{\Gamma_0(4) \backslash \mathbb{H}} |g_j(z)|^2 d\text{vol}(z) = 1$. Write $g = g_j$ with $\Delta_{1/2} g = -(\frac{1}{4} + t^2)g$. As usual $g(z)$ has a Fourier expansion of the form

$$g(z) = \sum_{n \neq 0} b_{g, \infty}(n) W_{\frac{1}{4} \text{sgn}(n), it} (4\pi |n| y) e(nx), \quad z = x + iy.$$

In this section our goal is to estimate the following averages of the coefficients $b_{g, \infty}$,

$$(6.1) \quad \sum_{X \leq n \leq 2X} |b_{g, \infty}(\pm n)|^2 \quad \text{and} \quad \sum_{X \leq n \leq 2X} |b_{g, \infty}(\pm n) b_{g, \infty}(\pm n + \ell)|.$$

We do this by relating the behavior of the coefficients to central values of L -functions through a Waldspurger type result which follows after collecting the result of Baruch and Mao [1] and a recent result of Duke-Imamoglu-Tóth [7] (which strengthens the earlier work of Biró [2], and Katok-Sarnak [18]). We will postpone the proof of the proposition until the appendix. We also recall the definition of the inner product on $\Gamma_0(N)\backslash\mathbb{H}$

$$\langle f, g \rangle_{\Gamma_0(N)} := \frac{1}{[\mathrm{PSL}_2(\mathbb{Z}) : \Gamma_0(N)]} \iint_{\Gamma_0(N)\backslash\mathbb{H}} f(z) \overline{g(z)} \cdot \frac{dx dy}{y^2}.$$

Proposition 6.1. *Let $g \in V^+$ with $g \neq 0$. Suppose that g is a simultaneous eigenfunction of $\Delta_{1/2}$ and of the Hecke operators T_{p^2} , $p \neq 2$. Then, there exists an even weight 0 Maaß (Hecke normalized eigen-)form f with $\Delta f = -(\frac{1}{4} + (2t)^2)f$ such that for any fundamental discriminant d*

$$\frac{|b_{g,\infty}(d)|^2}{\langle g, g \rangle_{\Gamma_0(4)}} = \frac{L(\frac{1}{2}, f \otimes \chi_d)}{\langle f, f \rangle_{\mathrm{SL}_2(\mathbb{Z})}} \cdot (\pi|d|)^{-1} \left| \Gamma\left(\frac{1}{2} - \frac{\mathrm{sgn}(d)}{4} - it\right) \right|^2.$$

Additionally, for an integer $n = d\delta^2$ with d a fundamental discriminant,

$$|b_{g,\infty}(n)|^2 \ll_{\varepsilon} |b_{g,\infty}(d)|^2 \cdot \delta^{2\theta-2+\varepsilon}$$

for any $\varepsilon > 0$, and where θ is the best currently known exponent towards the Ramanujan conjecture in the case of weight 0 Maaß forms.

Note that in our setting $\langle g, g \rangle_{\Gamma_0(4)} = \frac{1}{6}$ since g is normalized so that

$$\iint_{\Gamma_0(4)\backslash\mathbb{H}} |g(z)|^2 \mathrm{dvol}(z) = 1.$$

In addition since the corresponding form f of weight 0 is Hecke normalized, its L^2 norm is determined by the formula

$$(\cosh 2\pi t) \cdot \langle f, f \rangle_{\mathrm{SL}_2(\mathbb{Z})} \asymp L(1, \mathrm{Sym}^2 f).$$

Combining the above with Proposition 6.1 and a bound towards the Ramanujan conjecture due to Kim and Sarnak [19] we obtain the following corollary.

Corollary 6.1. *For $n = d\delta^2$ with d a fundamental discriminant we have*

$$(6.2) \quad |b_{g,\infty}(n)| \ll_{\varepsilon} \frac{1}{\sqrt{|n|}} \cdot \left(\frac{L(\frac{1}{2}, f \otimes \chi_d)}{L(1, \mathrm{Sym}^2 f)} \right)^{1/2} \cdot |\delta|^{7/64+\varepsilon} \cdot |t|^{-\mathrm{sgn}(n)/4} \cdot e^{\pi|t|/2}$$

for any $\varepsilon > 0$.

The bound of Corollary 6.1 reduces the problem of understanding (6.1) to corresponding questions on L -functions which we have addressed in the previous section. In this way we obtain the following two lemmas.

Lemma 6.1. *Let $\varepsilon > 0$. Assume the Generalized Riemann Hypothesis. Then for $|t|^\varepsilon \leq X$*

$$\sum_{X \leq n \leq 2X} |b_{g,\infty}(\pm n)|^2 \ll_\varepsilon (\log X)^\varepsilon \cdot |t|^{\mp \frac{1}{2}} \cdot e^{\pi|t|}.$$

In addition for $X \leq |t|^\varepsilon$ we have the trivial bound $\ll |t|^{\mp \frac{1}{2} + \varepsilon} \cdot e^{\pi|t|}$.

The trivial bound in Lemma 6.1 is an immediate consequence of Corollary 6.1 and the Generalized Riemann Hypothesis which implies that $L(\frac{1}{2}, f \otimes \chi_d) \ll_\varepsilon |dt|^\varepsilon$ for any $\varepsilon > 0$.

In the next lemma we estimate the second sum in (6.1). The proof builds in part on the argument used to establish Lemma 6.1. We notice however that a trivial application of Cauchy-Schwarz and Lemma 6.1 gives only a bound of size $|t|^{\mp \frac{1}{2}} e^{\pi|t|} (\log X)^\varepsilon$ for the second sum in (6.1) (with $|\ell| \leq X^{1/2}$, say), which does not suffice for our argument.

Lemma 6.2. *Let $\varepsilon > 0$. Assume the Generalized Riemann Hypothesis. Then for $|t|^\varepsilon \leq X$ and $0 \neq |\ell| \leq \log X$*

$$\sum_{X \leq n \leq 2X} |b_{g,\infty}(\pm n) b_{g,\infty}(\pm n + \ell)| \ll_\varepsilon |t|^{\mp \frac{1}{2}} \cdot e^{\pi|t|} \cdot (\log X)^{-25/156 + \varepsilon}.$$

In Lemma 6.2, we expect that the correct exponent on the logarithm is $-\frac{1}{4}$. Note that $\frac{25}{156} = \frac{1}{4} - 0.089\dots$ and that this weaker exponent is due to the lack of the Ramanujan bound for weight 0 Maaß forms. It is however quite possible that with a more involved argument the sharp exponent $-\frac{1}{4}$ can be reached.

Proof of Lemma 6.1. Write $n = d\delta^2$ with d a fundamental discriminant. For simplicity we only consider terms with $d > 0$, as those with $d < 0$ are handled in the same way and satisfy the analogous bound. Let us first prove the “trivial bound”. On the Generalized Riemann Hypothesis we have

$$\frac{L(\frac{1}{2}, f \otimes \chi_d)}{L(1, \text{Sym}^2 f)} \ll |td|^\varepsilon,$$

where we used Lemma 5.3 to bound $L(1, \text{Sym}^2 f)$. Therefore, by the bound of Corollary 6.1

$$\begin{aligned} \sum_{X \leq n \leq 2X} |b_{g,\infty}(n)|^2 &\ll \frac{|t|^{-\frac{1}{2} + \varepsilon} \cdot e^{\pi|t|}}{X} \sum_{X \leq d\delta^2 \leq 2X} |d|^\varepsilon \cdot |\delta|^{14/64 + \varepsilon} \\ &\ll X^\varepsilon \cdot \frac{|t|^{-1/2} e^{\pi|t|}}{X} \sum_{\delta \leq \sqrt{X}} |\delta|^{16/64} \cdot \frac{2X}{\delta^2} \ll X^\varepsilon |t|^{-1/2} e^{\pi|t|}. \end{aligned}$$

When $X \leq |t|$ this of course implies the claimed trivial bound $|t|^{-1/2+\varepsilon} e^{\pi|t|}$.

Now let's consider the case when $X \geq |t|^\varepsilon$, in which case we can do better. Inserting the bound (6.2) we get

$$\sum_{X \leq n \leq 2X} |b_{g,\infty}(n)|^2 \ll \frac{|t|^{-\frac{1}{2}} \cdot e^{\pi|t|}}{X} \sum_{X \leq d\delta^2 \leq 2X} \frac{L(\frac{1}{2}, f \otimes \chi_d)}{L(1, \text{Sym}^2 f)} \cdot \delta^{14/64+\varepsilon}.$$

Pick any $0 < \theta < \frac{1}{2}$. In the sum we split according to $\delta < X^\theta$ or $\delta > X^\theta$. This gives

$$(6.3) \quad \frac{1}{X} \sum_{X \leq d\delta^2 \leq 2X} \frac{L(\frac{1}{2}, f \otimes \chi_d)}{L(1, \text{Sym}^2 f)} \cdot \delta^{14/64+\varepsilon} \ll \frac{1}{X} \sum_{\delta < X^\theta} \delta^{14/64+\varepsilon} \sum_{X/\delta^2 \leq d \leq 2X/\delta^2} \frac{L(\frac{1}{2}, f \otimes \chi_d)}{L(1, \text{Sym}^2 f)} +$$

$$(6.4) \quad + \frac{1}{X} \sum_{\delta > X^\theta} \delta^{14/64+\varepsilon} \sum_{d \leq 2X/\delta^2} \frac{L(\frac{1}{2}, f \otimes \chi_d)}{L(1, \text{Sym}^2 f)}.$$

We now use Lemma 5.2 to bound the sum on the RHS of (6.3) in the following way

$$\frac{1}{X} \sum_{\delta < X^\theta} \delta^{14/64+\varepsilon} \sum_{X/\delta^2 \leq d \leq 2X/\delta^2} \frac{L(\frac{1}{2}, f \otimes \chi_d)}{L(1, \text{Sym}^2 f)} \ll (\log X)^\varepsilon \sum_{\delta < X^\theta} \frac{\delta^{14/64+\varepsilon}}{\delta^2} \ll (\log X)^\varepsilon.$$

To estimate the sum in (6.4) note that the Lindelöf Hypothesis implies $L(\frac{1}{2}, f \otimes \chi_d)/L(1, \text{Sym}^2 f) \ll (|td|)^\varepsilon$. This gives

$$\frac{1}{X} \sum_{\delta > X^\theta} \delta^{14/64+\varepsilon} \sum_{d \leq 2X/\delta^2} \frac{L(\frac{1}{2}, f \otimes \chi_d)}{L(1, \text{Sym}^2 f)} \ll X^\varepsilon \sum_{\delta > X^\theta} \frac{\delta^{14/64+\varepsilon}}{\delta^2} \ll (\log X)^{-100}.$$

Using this estimate in (6.4) completes the proof. \square

Proof of Lemma 6.2. Again let us write, $n = d_0 \delta_0^2$ and $n + \ell = d_1 \delta_1^2$ and as before we will only consider the case $d_0, d_1 > 0$. We first show that the contribution of those n for which $\delta_0 > (\log X)^\kappa$ or $\delta_1 > (\log X)^\kappa$ is negligible, for some $0 < \kappa < 10$ to be fixed later. Indeed, by Cauchy-Schwarz, and Lemma 6.1, the contribution of those integers for which $\delta_0 > (\log X)^\kappa$ is bounded by

$$(6.5) \quad \left(\sum_{\substack{X \leq d\delta^2 \leq 2X \\ \delta > (\log X)^\kappa}} |b_{g,\infty}(d\delta^2)|^2 \right)^{1/2} \cdot \left(\sum_{X/2 \leq n \leq 3X} |b_{g,\infty}(n)|^2 \right)^{1/2}.$$

By Lemma 6.1 the second term is $\ll |t|^{-1/4} \cdot e^{\frac{\pi}{2}|t|} \cdot (\log X)^\varepsilon$. On the other hand using (6.2) and splitting into two ranges $(\log X)^\kappa \leq \delta < X^\theta$ and $X^\theta \leq \delta \leq \sqrt{2X}$, with any

fixed $0 < \theta < \frac{1}{2}$, we see that the sum in the first term of (6.5) is bounded by

$$\begin{aligned} & \frac{|t|^{-\frac{1}{2}} \cdot e^{\pi|t|}}{X} \sum_{(\log X)^\kappa < \delta < X^\theta} \delta^{14/64+\varepsilon} \sum_{X/\delta^2 \leq d \leq 2X/\delta^2} \frac{L(\frac{1}{2}, f \otimes \chi_d)}{L(1, \text{Sym}^2 f)} \\ & + \frac{|t|^{-\frac{1}{2}} \cdot e^{\pi|t|}}{X} \sum_{\delta > X^\theta} \delta^{14/64+\varepsilon} \sum_{X/\delta^2 \leq d \leq 2X/\delta^2} \frac{L(\frac{1}{2}, f \otimes \chi_d)}{L(1, \text{Sym}^2 f)}. \end{aligned}$$

Proceeding as in the proof of Lemma 6.1 we use Lemma 5.2 to bound the first term above and the Lindelöf Hypothesis bound $L(\frac{1}{2}, f \otimes \chi_d) \ll (|dt|)^\varepsilon$ to bound the second term. This gives the following total bound for the sum in the first term of (6.5)

$$\ll |t|^{-\frac{1}{2}} \cdot e^{\pi|t|} \left((\log X)^\varepsilon \sum_{\delta > (\log X)^\kappa} \frac{\delta^{14/64+\varepsilon}}{\delta^2} + (\log X)^{-100} \right) \ll |t|^{-\frac{1}{2}} \cdot e^{\pi|t|} \cdot (\log X)^{-25\kappa/32+\varepsilon}.$$

Combining estimates, we bound (6.5) as $O(|t|^{-1/2} e^{\pi|t|} (\log X)^{-25\kappa/64+\varepsilon})$. Similarly we can bound the contribution of those integers n for which $n + \ell = d_1 \delta_1^2$ with d_1 a fundamental discriminant and $\delta_1 > (\log X)^\kappa$.

Therefore,

$$\begin{aligned} & \sum_{X \leq n \leq 2X} |b_{g,\infty}(n) b_{g,\infty}(n + \ell)| \\ & \ll |t|^{-\frac{1}{2}} \cdot e^{\pi|t|} \left(\frac{(\log X)^{7\kappa/32+\varepsilon}}{X} \sum_{\substack{X/2 \leq d_0 \delta_0^2 \leq 3X \\ d_0 \delta_0^2 - d_1 \delta_1^2 = -\ell \\ 1 \leq \delta_0, \delta_1 \leq (\log X)^\kappa}} \frac{L(\frac{1}{2}, f \otimes \chi_{d_0})^{1/2} L(\frac{1}{2}, f \otimes \chi_{d_1})^{1/2}}{L(1, \text{Sym}^2 f)} \right. \\ & \quad \left. + (\log X)^{-25\kappa/64+\varepsilon} \right). \end{aligned}$$

Applying Lemmas 5.1 and (5.3) the above sum is

$$\ll X (\log X)^{-1/4+\varepsilon} \sum_{1 \leq \delta_0, \delta_1 \leq (\log X)^\kappa} \frac{1}{[\delta_0, \delta_1]^2} \ll X (\log X)^{-1/4+\varepsilon}.$$

We conclude that

$$\sum_{X \leq n \leq 2X} |b_{g,\infty}(n) b_{g,\infty}(n + \ell)| \ll |t|^{-\frac{1}{2}} \cdot e^{\pi|t|} \cdot ((\log X)^{-25\kappa/64} + (\log X)^{7\kappa/32-1/4}) \cdot (\log X)^\varepsilon.$$

Choosing $\kappa = \frac{16}{39}$ we get $\ll |t|^{-\frac{1}{2}} \cdot e^{\pi|t|} (\log X)^{-25/156+\varepsilon}$. \square

7. EXTENDING THE LENGTH OF SUMMATION

As before let g be an element of a basis $\{g_j\}$ of V^+ consisting of simultaneous eigenfunctions of $\Delta_{1/2}$ and T_{p^2} , $p \neq 2$, with each g_j normalized so that $\iint_{\Gamma_0(4) \backslash \mathbb{H}} |g_j(z)|^2 \operatorname{dvol}(z) = 1$ and denote by $b_{g,\infty}(n)$ the Fourier coefficients of g . As usual, we write the eigenvalue of g as $\lambda = -(\frac{1}{4} + t^2)$.

Following an idea of Holowinsky [12] we show in this section that the average of $|b_{g,\infty}(n)|^2$ over integers $n \asymp |t|$ can be related to a corresponding average of $|b_{g,\infty}(n)|^2$ over integers $n \asymp |t|Y$ with Y a parameter whose size roughly depends on the saving in the shifted convolution problem

$$\sum_{n \asymp |t|} b_{g,\infty}(n)b_{g,\infty}(n + \ell).$$

The advantage of such an ‘‘extension’’ of the length of summation will become clear in the next section, where we will show that the average of $|b_{g,\infty}(n)|^2$ can be estimated as soon as we sum slightly more than $|t|$ terms.

Given $\psi \in C_c^\infty(\mathbb{R}_+)$, $\ell \in \mathbb{Z}$ and $t \in \mathbb{R}$ define $\mathcal{S}_Y(\ell, t; \psi)$ as

$$(7.1) \quad \int_0^\infty \psi(yY) \left| \sum_{n \neq 0, -\ell} b_{g,\infty}(n)b_{g,\infty}(n + \ell) W_{\frac{1}{4} \operatorname{sgn}(n), it}(4\pi|n|y) W_{\frac{1}{4} \operatorname{sgn}(n+\ell), it}(4\pi|n + \ell|y) \right| \frac{dy}{y^2}.$$

Lemma 7.1. *Let h, k be two smooth compactly supported functions on \mathbb{R}_+ and with k non-negative. Let $H(s) := \int_0^\infty h(x)x^{s-1}dx$ and $K(s) := \int_0^\infty k(x)x^{s-1}dx$ denote their respective Mellin transforms. Then for $Y \geq 1$*

$$(7.2) \quad K(-1)Y \mathcal{S}_1(0, t; h) = H(-1)(1 + O(Y^{-1/2})) \mathcal{S}_Y(0, t; k) + O(\sqrt{Y}) \\ + O_A\left(\frac{1}{\sqrt{Y}} \sum_{\ell \neq 0} \frac{(\nu(\ell) + 1)2^{\nu(\ell)} d(|\ell|)}{1 + |\ell/Y|^A} \cdot \mathcal{S}_Y(\ell, t; k)\right)$$

for any $A > 0$ where $\nu(\ell)$ is the highest power of two dividing ℓ and \mathcal{S}_Y is as defined in (7.1).

Notice here that when $\ell = 0$ the term appearing inside the absolute value in $\mathcal{S}_Y(0, t; \psi)$ is non-negative, since $W_{\kappa, it}(x) \in \mathbb{R}$ for $|\kappa| < \frac{1}{2}$ and $t \in \mathbb{R}$. The proof of Lemma 7.1 is rather involved so we explain now the general principle behind its proof. Consider the incomplete Eisenstein series

$$(7.3) \quad E(z|h) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4)} h(\operatorname{Im}(\gamma z))$$

and

$$(7.4) \quad E^Y(z|k) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4)} k(Y \cdot \text{Im}(\gamma z)).$$

In the proof of Lemma 7.1 we will evaluate

$$(7.5) \quad \iint_{\Gamma_0(4) \backslash \mathbb{H}} E(z|h) E^Y(z|k) |g(z)|^2 \text{dvol}(z)$$

in two different ways. First, we will express $E^Y(z|k)$ as a Perron integral involving the real analytic Eisenstein series $E(z, s)$, which we recall is defined as

$$E(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4)} (\text{Im}(\gamma z))^s.$$

Shifting contours and collecting the residue from the simple pole of $E(z, s)$ at $s = 1$ will lead to the main term on the left-hand side of (7.2). Second, we will use the unfolding technique with $E^Y(z|k)$ and then expand $E(z|h)$ as a Fourier series. This will lead to the right-hand side of (7.2). The Fourier development is described in the next lemma.

Lemma 7.2. *We have*

$$E(z|h) = a_{0,h}(y) + \sum_{\ell \neq 0} a_{\ell,h}(y) e(\ell x)$$

where

$$a_{\ell,h}(y) = \begin{cases} \frac{H(-1)}{2\pi} + O(\sqrt{y}) & \text{if } \ell = 0, \\ O_A\left(\sqrt{y} \frac{\nu(\ell)+1}{1+|\ell y|^A} 2^{\nu(\ell)} d(|\ell|)\right) & \text{if } \ell \neq 0, \end{cases}$$

and $\nu(\ell)$ is the largest power of 2 dividing ℓ and $d(\cdot)$ is the divisor function.

Proof. The proof closely follows the argument of Holowinsky [12], with appropriate modifications at the prime $p = 2$. Consider the Eisenstein series

$$E(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4)} (\text{Im}(\gamma z))^s.$$

At the cusp at ∞ , $E(z, s)$ has the Fourier expansion

$$E(z, s) = y^s + \varphi(s) y^{1-s} + 2\sqrt{y} \sum_{\ell \neq 0} \varphi(s, \ell) K_{s-\frac{1}{2}}(2\pi|\ell|y) e(n x),$$

here $K_\nu(u)$ is the modified Bessel function of second kind and for $\text{Re}(s) > \frac{1}{2}$

$$\varphi(s) = \pi^{1/2} \frac{\Gamma(s - \frac{1}{2})}{4^{2s} \Gamma(s)} \sum_{n \geq 1} \frac{\phi(4n)}{n^{2s}} = \pi^{1/2} \frac{\Gamma(s - \frac{1}{2})}{4^{2s} \Gamma(s)} \frac{2}{1 - 2^{-2s}} \frac{\zeta(2s - 1)}{\zeta(2s)}$$

and

$$\varphi(s, \ell) = \frac{\pi^s}{4^{2s}\Gamma(s)} |\ell|^{s-\frac{1}{2}} \sum_{n \geq 1} \frac{c_{4n}(\ell)}{n^{2s}} = \frac{\pi^s}{4^{2s}\Gamma(s)} \frac{\mathcal{L}_2(s, \ell)}{\zeta(2s)} \sum_{ab=|\ell|} \left(\frac{a}{b}\right)^{s-\frac{1}{2}}$$

where $c_q(n)$ is a Ramanujan sum (see [22, Section 2.2] and [16, Section 4.2]). Writing $\nu(\ell)$ for the largest power of 2 dividing ℓ , note that $\mathcal{L}_2(s, \ell)$ is given by

$$\mathcal{L}_2(s, \ell) = \frac{1 - 2^{1-2s}}{(1 - 2^{(\nu(\ell)+1)(1-2s)})(1 - 2^{-2s})} \sum_{j=0}^{\nu(\ell)} \frac{c_{2^{j+2}}(\ell)}{2^{2js}} = O((\nu(\ell) + 1)2^{\nu(\ell)})$$

which holds uniformly for $\delta \leq \operatorname{Re}(s) \leq 1$, for any $\delta > 0$.

To obtain the Fourier series expansion of the incomplete Eisenstein series $E(z|h)$ we write

$$E(z|h) = \frac{1}{2\pi i} \int_{(2)} E(z, s) H(-s) ds = a_0(y) + \sum_{\ell \neq 0} a_{\ell, h}(y) e(\ell x).$$

By shifting contours of integration it follows that

$$a_{0, h}(y) = \frac{1}{2\pi i} \int_{(2)} (y^s + \varphi(s)y^{1-s}) H(-s) ds = \frac{H(-1)}{2\pi} + O(\sqrt{y}).$$

To estimate $a_{\ell, h}(y)$ we use the bounds $K_{it}(u) \ll \min(u^{-1/2}e^{-u}, e^{-\frac{\pi}{2}|t|})$ (see [9, Corollary 3.2]) and $H(-s) \ll (1 + |s|)^{-A}$, to get that

$$\begin{aligned} a_{\ell, h}(y) &= \frac{\sqrt{y}}{2\pi i} \int_{(2)} K_{s-\frac{1}{2}}(2\pi|\ell|y) \frac{\pi^s}{4^{2s}\Gamma(s)} \frac{\mathcal{L}_2(s, \ell)}{\zeta(2s)} \sum_{ab=|\ell|} \left(\frac{a}{b}\right)^{s-\frac{1}{2}} H(-s) ds \\ &\ll \sqrt{y}(\nu(\ell) + 1)2^{\nu(\ell)} d(|\ell|) \int_{\mathbb{R}} \left| \frac{K_{i\tau}(2\pi|\ell|y)}{\Gamma(\frac{1}{2} + i\tau)} \right| (1 + |\tau|)^{-A} d\tau \\ &\ll \frac{\sqrt{y}(\nu(\ell) + 1)2^{\nu(\ell)} d(|\ell|)}{1 + |\ell y|^A}, \end{aligned}$$

which gives the claim. \square

We are now ready to prove Lemma 7.1. As promised, in order to arrive at (7.2) we will evaluate (7.5) in two different ways.

7.1. Proof of Lemma 7.1 Step 1: A first evaluation of (7.5). Note that

$$E^Y(z|k) = \frac{1}{2\pi i} \int_{(2)} K(-s) Y^s E(z, s) ds$$

with $K(s) := \int_0^\infty k(x)x^{s-1}dx$. Therefore we can re-write (7.5) as

$$\frac{1}{2\pi i} \int_{(2)} K(-s)Y^s \left(\iint_{\Gamma_0(4)\backslash\mathbb{H}} E(z,s)E(z|h)|g(z)|^2 \frac{dx dy}{y^2} \right) ds.$$

Shifting the contour to $\operatorname{Re}(s) = \frac{1}{2}$ we collect a residue at $s = 1$ coming from $E(z,s)$. The value of the residue of $E(z,s)$ at $s = 1$ is $(\operatorname{vol}(\Gamma_0(4)\backslash\mathbb{H}))^{-1} = (2\pi)^{-1}$. Therefore we get that (7.5) equals

$$(7.6) \quad K(-1) \cdot \frac{Y}{2\pi} \iint_{\Gamma_0(4)\backslash\mathbb{H}} E(z|h)|g(z)|^2 \frac{dx dy}{y^2} + \frac{1}{2\pi i} \int_{(1/2)} K(-s)Y^s \left(\iint_{\Gamma_0(4)\backslash\mathbb{H}} E(z,s)E(z|h)|g(z)|^2 \frac{dx dy}{y^2} \right) ds.$$

In the first term we apply the unfolding method

$$(7.7) \quad \iint_{\Gamma_0(4)\backslash\mathbb{H}} E(z|h)|g(z)|^2 \frac{dx dy}{y^2} = \sum_{n \neq 0} |b_{g,\infty}(n)|^2 \int_0^\infty W_{\frac{1}{4} \operatorname{sgn}(n), it} (4\pi|n|y)^2 h(y) \frac{dy}{y^2}.$$

In the second term, since $E(z|h)$ is compactly supported in $\Gamma_0(4)\backslash\mathbb{H}$, the z variable is restricted to a compact set. Therefore we have the bound $E(z, \frac{1}{2} + iu) \ll (1 + |u|)^N$ for some large N , uniformly in the compact set to which z belongs, with the implied constant depending only on h . Therefore, since $K(-\frac{1}{2} - iu) \ll_A (1 + |u|)^{-A}$ the integral in (7.6) is

$$\ll \sqrt{Y} \iint_{\Gamma_0(4)\backslash\mathbb{H}} |E(z|h)||g(z)|^2 \frac{dx dy}{y^2} \ll \sqrt{Y} \iint_{\Gamma_0(4)\backslash\mathbb{H}} |g(z)|^2 \frac{dx dy}{y^2} = \sqrt{Y}.$$

Combining this with (7.6) and (7.7) we conclude that

$$\begin{aligned} & \iint_{\Gamma_0(4)\backslash\mathbb{H}} E^Y(z|k)E(z|h)|g(z)|^2 \frac{dx dy}{y^2} \\ &= K(-1) \frac{Y}{2\pi} \sum_{n \neq 0} |b_{g,\infty}(n)|^2 \int_0^\infty W_{\frac{1}{4} \operatorname{sgn}(n), it} (4\pi|n|y)^2 h(y) \frac{dy}{y^2} + O(\sqrt{Y}). \end{aligned}$$

This gives the left-hand side of (7.2).

7.2. Proof of Lemma 7.1. Step 2: A second evaluation of (7.5). By the unfolding method (7.5) is equal to

$$\int_0^\infty \int_{-1/2}^{1/2} k(Yy)E(z|h)|g(z)|^2 \frac{dx dy}{y^2}.$$

Expanding $E(z|h)$ into the Fourier series given in Lemma 7.2 we see that the previous equation equals

$$(7.8) \quad \int_0^\infty \int_{-1/2}^{1/2} a_{0,h}(y) k(Yy) |g(z)|^2 \frac{dx dy}{y^2}$$

$$(7.9) \quad + \sum_{\ell \neq 0} \int_0^\infty \int_{-1/2}^{1/2} e(\ell x) k(Yy) a_{\ell,h}(y) |g(z)|^2 \frac{dx dy}{y^2}.$$

We now investigate (7.8) and (7.9). By Lemma 7.2 the contribution of (7.8) equals

$$\frac{H(-1)}{2\pi} \int_0^\infty \int_{-1/2}^{1/2} |g(z)|^2 k(Yy) \frac{dx dy}{y^2} + O\left(\int_0^\infty \int_{-1/2}^{1/2} \sqrt{y} k(Yy) |g(z)|^2 \frac{dx dy}{y^2}\right).$$

Since k is compactly supported we may bound the term \sqrt{y} in the integrand in the second term above as $O(Y^{-1/2})$. It follows from this and the non-negativity of k that the above is equal to

$$(1 + O(Y^{-1/2})) \frac{H(-1)}{2\pi} \int_0^\infty \int_{-1/2}^{1/2} |g(z)|^2 k(Yy) \frac{dx dy}{y^2}.$$

Upon expanding $g(z)$ into a Fourier series we see that the above integral equals

$$\frac{H(-1)}{2\pi} \sum_{n \neq 0} |b_{g,\infty}(n)|^2 \cdot \int_0^\infty k(Yy) W_{\frac{1}{4} \operatorname{sgn}(n), it} (4\pi|n|y)^2 \frac{dy}{y^2}.$$

This gives the main term on the RHS of (7.2).

It remains to estimate (7.9). First, we expand $g(z)$ into a Fourier series once again to see that (7.9) is equal to

$$\sum_{\ell \neq 0} \sum_{n \neq 0, -\ell} b_{g,\infty}(n) b_{g,\infty}(n + \ell) \int_0^\infty k(Yy) a_{\ell,h}(y) W_{\frac{1}{4} \operatorname{sgn}(n), it} (4\pi|n|y) W_{\frac{1}{4} \operatorname{sgn}(n+\ell), it} (4\pi|n + \ell|y) \frac{dy}{y^2}.$$

Moving the sum over n inside, taking absolute values, and applying Lemma 7.2 we see that the above is

$$\ll \sum_{\ell \neq 0} (\nu(\ell) + 1) 2^{\nu(\ell)} d(|\ell|) \int_0^\infty k(yY) \cdot \frac{\sqrt{y}}{1 + |\ell y|^A} \times \\ \left| \sum_{n \neq 0, -\ell} b_{g,\infty}(n) b_{g,\infty}(n + \ell) W_{\frac{1}{4} \operatorname{sgn}(n), it} (4\pi|n|y) W_{\frac{1}{4} \operatorname{sgn}(n), it} (4\pi|n + \ell|y) \right| \frac{dy}{y^2}.$$

Since $k(\cdot)$ is compactly supported, in the integrand we may bound the term $\sqrt{y}/(1 + |\ell y|^A)$ as $O(Y^{-1/2} \cdot (1 + |\ell/Y|^A)^{-1})$. This along with the estimate for (7.8) gives the RHS of (7.2) and completes the proof of Lemma 7.1.

8. THE SUMMATION FORMULA AND PROOF OF QUE

First recall the notation which will be set in place for the duration of this section: Let $\{g_j\}$ be a basis of V^+ consisting of simultaneous eigenfunctions of $\Delta_{1/2}$ and T_{p^2} , $p \neq 2$, with each g_j normalized so that $\iint_{\Gamma_0(4)\backslash\mathbb{H}} |g_j(z)|^2 d\text{vol}(z) = 1$. For simplicity we will write $g = g_j$ and $t = t_j$ so that $\Delta_{1/2}g = -(\frac{1}{4} + t^2)g$. Recall that $t \in \mathbb{R}$ since there are no forms corresponding to exceptional eigenvalues in V^+ .

8.1. Estimates for Whittaker functions. Before establishing the summation formula we first establish the following estimate for Whittaker functions, which we will use repeatedly.

Lemma 8.1. *Let h be a smooth compactly supported function on \mathbb{R}_+ . Then for $Y > 0$*

$$(8.1) \quad \int_{\mathbb{R}} W_{\pm\frac{1}{4},it}(4\pi|n|y)^2 \cdot \frac{h(yY)}{y^2} dy \ll_{\varepsilon} (1 + |t|)^{\pm\frac{1}{2}} \cdot e^{-\pi|t|} \cdot Y \cdot \begin{cases} |n/(tY)|^{1-\varepsilon} & \text{if } |n| \leq |t|Y, \\ |(tY)/n|^{100} & \text{if } |n| > |t|Y, \end{cases}$$

for any $\varepsilon > 0$.

Proof. The proof of (8.1) relies on some computations of Matthes [30, 31] (further refined in a lemma of Luo-Rudnick-Sarnak [28]). Let

$$\mathcal{M}_{\kappa,it}(s) := \int_0^{\infty} W_{\kappa,it}(y)^2 \cdot y^{s-2} dy, \quad \text{Re}(s) > 0.$$

In Lemma 1 of [31], Matthes proves that for $s = \sigma + i\tau$ with $\varepsilon < \sigma \leq 1000$, and $-\frac{1}{2} \leq \kappa < \frac{1}{2}$

$$(8.2) \quad |\mathcal{M}_{\kappa,it}(s)| \leq \mathcal{M}_{\kappa,it}(\sigma) \ll (1 + |t|)^{\sigma-1+2\kappa} e^{-\pi|t|}.$$

In addition if $H(s) := \int_0^{\infty} h(y)y^{s-1}dy$, then we notice that

$$\int_0^{\infty} W_{\kappa,it}(4\pi|n|y)^2 \cdot \frac{h(yY)}{y^2} dy = \frac{1}{2\pi i} \int_{(-\varepsilon)} H(s)Y^{-s}(4\pi|n|)^{s+1} \mathcal{M}_{\kappa,it}(-s) ds.$$

For the proof of the bound for $|n| \leq |t|Y$ we apply the triangle inequality and Matthes bound on the line $\text{Re}(s) = -\varepsilon$, and for the proof of the bound for $|n| > |t|Y$ we shift to $\text{Re}(s) = -101$ and then bound trivially using Matthes's result. \square

8.2. The summation formula. In this section we derive the analogue of a convexity bound for the Dirichlet series with coefficients given by $|b_{g,\infty}(n)|^2$. This allows us to estimate the average of $|b_{g,\infty}(n)|^2$ when we sum slightly more than $|t|$ of these coefficients. To rephrase this in terms of L -functions note that this is analogous to using the approximate functional equation for $L(\frac{1}{2}, f \otimes \chi_d)$ followed by an application

of Poisson summation for χ_d ; as a result we would relate the sum of $L(\frac{1}{2}, f \otimes \chi_d)$ with $|d| \leq X$ to a sum of similar shape but of length $1 + |t|^2/X$, which is shorter when X is slightly larger than $|t|$. From this we see that we expect to gain a saving in the length of the sum after applying Poisson summation only when X exceeds $|t|$. Since we can only extend our sum to have length $|t|(\log |t|)^\eta$ for some small $\eta > 0$, we require a convexity bound which is stronger than what one obtains by using the Phragmen-Lindelöf principle. For general L -functions such convexity bounds have been given by Heath-Brown [11], and a stronger “weakly subconvex” bound was obtained by Soundararajan [50]. The main tool used to prove the convexity bound is the following functional equation.

Lemma 8.2. *Consider*

$$\mathcal{M}_{\kappa, it}(s) = \int_0^\infty \frac{W_{\kappa, it}(y)^2}{y} \cdot y^{s-1} dy \text{ and } R_{\pm, g}(s) = \sum_{n>0} \frac{|b_{g, \infty}(\pm n)|^2}{n^{s-1}}.$$

Let

$$G(s) = \sum_{\pm} R_{\pm, g}(s) \mathcal{M}_{\pm \frac{1}{4}, it}(s)$$

Then $G(s)$ is a meromorphic function, whose only singularity in $\operatorname{Re}(s) > \frac{1}{2}$ is a simple pole at $s = 1$ with residue $(2\pi)^{-1}$. In addition $G(s)$ has no singularities in $\operatorname{Re}(s) < 0$. Finally, the completed Dirichlet series $\tilde{G}(s) := \pi^{-2s} \Gamma(s) \zeta(2s) G(s)$ satisfies the functional equation $\tilde{G}(s) = \tilde{G}(1-s)$ and $s(1-s)\tilde{G}(s)$ is entire.

Proof. The proof of this is nearly identical to the proof given by Kohnen and Zagier [20] for holomorphic half-integral weight forms lying in the Kohnen plus space. For completeness we have included a proof in the appendix. \square

This allows us to prove the following “convexity bound”.

Lemma 8.3. *Let $\varepsilon > 0$. Assume the Generalized Riemann Hypothesis. Let $G(s)$ be as in Lemma 8.2 and $s = \sigma + i\tau$. Then for $\frac{1}{2} < \sigma < 1$*

$$G(\sigma + i\tau) \ll_{\sigma} (\log |t|)^{\varepsilon} \cdot (1 + |\tau|)^{1-\sigma+\varepsilon}.$$

Proof. Let $H(s) = e^{s^2}$, and $\Lambda(s) = s(1-s)\pi^{-2s}\Gamma(s)\zeta(2s)G(s)$. Then according to Lemma 8.2 we have $\Lambda(s) = \Lambda(1-s)$ and $\Lambda(s)$ is an entire function. Therefore by a standard argument, for $\operatorname{Re}(s) > \frac{1}{2}$

$$(8.3) \quad \Lambda(s) = \frac{1}{2\pi i} \int_{(1)} \Lambda(s+w) \frac{H(w)dw}{w} + \frac{1}{2\pi i} \int_{(1)} \Lambda(1-s+w) \frac{H(w)dw}{w}.$$

Define $c_{\pm}(n)$ by

$$\sum_{k^2 \ell = n} \frac{c_{\pm}(n)}{n^s} = \zeta(2s) R_{\pm, g}(s), \quad c_{\pm}(n) := \sum_{k^2 \ell = n} |b_{g, \infty}(\pm \ell)|^2 \ell.$$

Also, let

$$(8.4) \quad \Phi_s(w) = \frac{(s+w)(1-(s+w))\pi^{-2w}\Gamma(s+w)}{s(1-s)} \cdot \frac{H(w)}{w}.$$

Thus, expanding $\zeta(2s)R_{\pm,g}(s)$ as a Dirichlet series we have for $\frac{1}{2} < \sigma < 1$

$$(8.5) \quad \begin{aligned} \zeta(2s)G(s) &= \sum_{\pm} \sum_{n \geq 1} \frac{c(\pm n)}{n^s} \frac{1}{2\pi i} \int_{(1)} n^{-w} \frac{\Phi_s(w)}{\Gamma(s)} \mathcal{M}_{\pm\frac{1}{4},it}(s+w) dw \\ &+ \sum_{\pm} \sum_{n \geq 1} \frac{c(\pm n)}{n^{1-s}} \frac{1}{2\pi i} \int_{(1)} n^{-w} \frac{\Phi_{1-s}(w)}{\Gamma(s)} \mathcal{M}_{\pm\frac{1}{4},it}(1-s+w) dw. \end{aligned}$$

For $s = \sigma + i\tau$ and $w = u + iv$ Stirling's formula gives that uniformly in the range $0 < \varepsilon \leq \sigma \leq A$, $-\sigma + \varepsilon \leq u \leq A$, $|w| \geq \varepsilon$ and $|s-1| \geq \varepsilon$

$$|\Phi_s(w)| \ll |\Gamma(s)| \cdot (1 + |\tau|)^u e^{-v^2/2}.$$

Using this and (8.2) gives

$$(8.6) \quad \begin{aligned} &\int_{(1)} n^{-w} \frac{\Phi_s(w)}{\Gamma(s)} \mathcal{M}_{\pm\frac{1}{4},it}(s+w) dw \\ &\ll \begin{cases} \left| \frac{t}{n} \right|^c (1 + |t|)^{\sigma-1 \pm \frac{1}{2}} e^{-\pi|t|} (1 + |\tau|)^c & \text{for any } \varepsilon \leq c \leq 100, \\ (1 + |t|)^{\sigma-1 \pm \frac{1}{2}} e^{-\pi|t|} \left(1 + \left| \frac{t}{n} \right|^c\right) & \text{for any } -\sigma + \varepsilon \leq c \leq \varepsilon. \end{cases} \end{aligned}$$

Hence, when $|n| > |t|$ we pick $c = 1 - \sigma + \varepsilon$ in (8.6) to get

$$(8.7) \quad \begin{aligned} &\sum_{\pm} \sum_{n \geq 1} \frac{c(\pm n)}{n^s} \frac{1}{2\pi i} \int_{(1)} n^{-w} \frac{\Phi_s(w)}{\Gamma(s)} \mathcal{M}_{\pm\frac{1}{4},it}(s+w) dw \\ &\ll (1 + |\tau|)^{1-\sigma+\varepsilon} |t|^{\sigma-1} e^{-\pi|t|} \left(\sum_{\pm} |t|^{\pm\frac{1}{2}} \left(\sum_{n \leq |t|} \frac{c(\pm n)}{n^\sigma} + \sum_{|t| \leq n} \frac{c(\pm n)}{n^\sigma} \left| \frac{t}{n} \right|^{1-\sigma+\varepsilon} \right) \right). \end{aligned}$$

To estimate the RHS observe that as a consequence of Lemma 6.1 we have under GRH that

$$(8.8) \quad \sum_{X \leq n \leq 2X} c_{\pm}(n) \ll X \cdot (1 + |t|)^{\mp\frac{1}{2}} e^{\pi|t|} \cdot \begin{cases} (\log X)^\varepsilon & \text{if } X \geq |t|^\varepsilon, \\ |t|^\varepsilon & \text{if } X \leq |t|^\varepsilon. \end{cases}$$

Let $\mathcal{J} = 1/(\varepsilon \log 2)$ and apply (8.8) to see that

$$(8.9) \quad \sum_{n \leq |t|} \frac{c(\pm n)}{n^\sigma} \ll \sum_{n \leq |t|^\varepsilon} \frac{c(\pm n)}{n^\sigma} + \frac{1}{|t|^\sigma} \sum_{j=0}^{\mathcal{J}} 2^{j\sigma} \sum_{\frac{|t|}{2^{j+1}} \leq n \leq \frac{|t|}{2^j}} c(\pm n) \\ \ll e^{\pi|t|} |t|^{\mp \frac{1}{2}} (|t|^\varepsilon + |t|^{1-\sigma} (\log |t|)^\varepsilon).$$

Similarly,

$$(8.10) \quad \sum_{|t| \leq n} \frac{c(\pm n)}{n^\sigma} \left| \frac{t}{n} \right|^{1-\sigma+\varepsilon} \ll |t|^{-\sigma} \sum_{j=0}^{\infty} (2^j)^{-(1+\varepsilon)} \sum_{2^j |t| \leq n \leq 2^{j+1} |t|} c(\pm n) \\ \ll e^{\pi|t|} |t|^{1-\sigma \mp \frac{1}{2}} (\log |t|)^\varepsilon.$$

By (8.9) and (8.10) we see that (8.7) is $\ll (1 + |\tau|)^{1-\sigma+\varepsilon} (\log |t|)^\varepsilon$. By a similar argument we have

$$\sum_{\pm} \sum_{n \geq 1} \frac{c(\pm n)}{n^{1-s}} \frac{1}{2\pi i} \int_{(1)} n^{-w} \frac{\Phi_{1-s}(w)}{\Gamma(s)} \mathcal{M}_{\pm \frac{1}{4}, it}(1-s+w) dw \ll (1 + |\tau|)^{1-\sigma+\varepsilon} (\log |t|)^\varepsilon.$$

Using these estimates in (8.5) completes the proof. \square

We are now ready to prove our summation formula.

Lemma 8.4. *Let k be a smooth compactly supported function on \mathbb{R}_+ . Then, for $Y \geq 1$*

$$\sum_{n \neq 0} |b_{g, \infty}(n)|^2 \int_0^\infty W_{\frac{1}{4} \operatorname{sgn}(n), it}(4\pi|n|y)^2 \cdot \frac{k(yY)dy}{y^2} = \frac{YK(-1)}{2\pi} + O_\varepsilon(Y^{1/2+\varepsilon} (\log |t|)^\varepsilon).$$

for any $\varepsilon > 0$.

Proof. Let us keep the notation of Lemma 8.2. Applying Mellin inversion we find that

$$\sum_{n \neq 0} |b_{g, \infty}(n)|^2 \int_0^\infty W_{\frac{1}{4} \operatorname{sgn}(n), it}(4\pi|n|y)^2 \cdot \frac{k(yY)dy}{y^2} \\ = \frac{1}{2\pi i} \int_{(\sigma)} K(-s) Y^s (4\pi)^{1-s} G(s) ds, \quad \sigma > 1.$$

Shifting the contour to $\sigma = \frac{1}{2} + \varepsilon$ we collect a simple pole at $s = 1$ with residue $(2\pi)^{-1} Y K(-1)$, and we get that the above integral equals

$$\frac{1}{2\pi} \cdot K(-1) Y + \frac{1}{2\pi i} \int_{(\frac{1}{2}+\varepsilon)} K(-s) Y^s (4\pi)^{1-s} G(s) ds.$$

Applying Lemma 8.3, the above integral is

$$O\left(Y^{\frac{1}{2}+\varepsilon}(\log |t|)^\varepsilon\right),$$

which gives the claim. \square

8.3. Proof of Quantum Unique Ergodicity. Recall the discussion in Section 2.1 where we saw that Theorem 2 follows from the estimates

$$(8.11) \quad \sum_{n \neq 0} |b_{g,\infty}(n)|^2 \int_0^\infty W_{\frac{1}{4} \operatorname{sgn}(n),it}(4\pi y|n|)^2 h(y) \frac{dy}{y^2} = \frac{1}{\operatorname{vol}(\Gamma_0(4) \backslash \mathbb{H})} \int_0^\infty h(y) \frac{dy}{y^2} + o(1)$$

as $|t| \rightarrow \infty$ and for $\ell \neq 0$

$$\sum_{n \neq 0, -\ell} b_{g,\infty}(n) b_{g,\infty}(n+\ell) \int_0^\infty W_{\frac{1}{4} \operatorname{sgn}(n),it}(4\pi|n|y) W_{\frac{1}{4} \operatorname{sgn}(n+\ell),it}(4\pi|n+\ell|y) h(y) \frac{dy}{y^2} = o(1),$$

as $|t| \rightarrow \infty$. To bound the off-diagonal contribution we note that $W_{\kappa,iu}(y) = W_{\kappa,-iu}(y)$ and that $W_{\kappa,iu}(y) \in \mathbb{R}$. Using the bound $|ab| \leq a^2 + b^2$, we see that to control the off-diagonal it will be enough to show that

$$(8.12) \quad \sum_{n \neq 0, -\ell} |b_{g,\infty}(n) b_{g,\infty}(n+\ell)| \int_0^\infty W_{\frac{1}{4} \operatorname{sgn}(n),it}(4\pi|n|y)^2 \cdot \frac{h(y) dy}{y^2} = o(1)$$

as $|t| \rightarrow \infty$ for $\ell \neq 0$. Thus the proof of the theorem reduces to establishing (8.12) and (8.11).

First let us consider (8.12). Combining Lemma 6.2 and Lemma 8.1 we have

$$(8.13) \quad \begin{aligned} & \sum_{n \neq 0, -\ell} |b_{g,\infty}(n) b_{g,\infty}(n+\ell)| \int_0^\infty W_{\frac{1}{4} \operatorname{sgn}(n),it}(4\pi|n|y)^2 \cdot \frac{h(y) dy}{y^2} \\ & \ll e^{-\pi|t|} \left(\sum_{|n| \leq |t|} |t|^{\operatorname{sgn}(n) \cdot \frac{1}{2}} |b_{g,\infty}(n) b_{g,\infty}(n+\ell)| \left| \frac{n}{t} \right|^{1-\varepsilon} \right. \\ & \quad \left. + \sum_{|n| > |t|} |t|^{\operatorname{sgn}(n) \cdot \frac{1}{2}} |b_{g,\infty}(n) b_{g,\infty}(n+\ell)| \left| \frac{t}{n} \right|^{100} \right) \\ & \ll (\log |t|)^{-25/156+\varepsilon}, \end{aligned}$$

for each fixed $\ell \neq 0$ and in the last step the terms with $|n| \leq |t|^\varepsilon$ are handled trivially while in the other ranges we split up the sums into intervals of the form $\mathcal{I}_j = (|t|/2^{j+1}, |t|/2^j]$, $j \in \mathbb{Z}$, apply Lemma 6.2 to bound the sums over the intervals \mathcal{I}_j , then sum over j (see (8.9) and (8.10) for a similar argument).

The estimation of the LHS of (8.11) is more intricate. Given smooth functions h, k compactly supported on \mathbb{R}_+ denote by $H(s), K(s)$ their respective Mellin transforms. Then for $Y \geq 1$, by Lemma 7.1 the LHS of (8.11) is equal to

$$(8.14) \quad \frac{H(-1)}{K(-1)Y} \cdot (1 + O(Y^{-1/2})) \mathcal{S}_Y(0, t; k) + O(Y^{-1/2}) \\ + O\left(\frac{1}{Y^{3/2}} \sum_{\ell \neq 0} \frac{(\nu(\ell) + 1) 2^{\nu(\ell)} d(|\ell|)}{1 + |\ell/Y|^A} \cdot \mathcal{S}_Y(\ell, t; k)\right)$$

where $\nu(\ell)$ is the exponent of the largest power of 2 dividing ℓ and $\mathcal{S}_Y(\ell, t; k)$ is defined as

$$\int_0^\infty k(yY) \left| \sum_{n \neq 0, -\ell} b_{g,\infty}(n) b_{g,\infty}(n + \ell) W_{\frac{1}{4} \operatorname{sgn}(n), it}(4\pi|n|y) W_{\frac{1}{4} \operatorname{sgn}(n+\ell), it}(4\pi|n + \ell|y) \right| \frac{dy}{y^2}.$$

Moving the absolute value inside and using Lemma 8.1 we have by proceeding as in (8.13) that for $0 < |\ell| \leq \varepsilon \log |t|$

$$|\mathcal{S}_Y(\ell, t; k)| \ll Y e^{-\pi|t|} \left(\sum_{|n| \leq |t|^\varepsilon} |t|^{\operatorname{sgn}(n) \cdot \frac{1}{2}} |b_{g,\infty}(n)|^2 \left| \frac{n}{tY} \right|^{1-\varepsilon} \right. \\ + \sum_{|t|^\varepsilon < |n| \leq |Yt|} |t|^{\operatorname{sgn}(n) \cdot \frac{1}{2}} |b_{g,\infty}(n) b_{g,\infty}(n + \ell)| \left| \frac{n}{tY} \right|^{1-\varepsilon} \\ \left. + \sum_{|n| > |tY|} |t|^{\operatorname{sgn}(n) \cdot \frac{1}{2}} |b_{g,\infty}(n) b_{g,\infty}(n + \ell)| \left| \frac{tY}{n} \right|^{100} \right).$$

By Lemma 6.1 the first sum above is $\ll e^{\pi|t|} Y^{\varepsilon-1} |t|^{-1/2}$. To bound the second and third sums, we argue as before and split the ranges of summation into intervals $(|t|/2^{j+1}, |t|/2^j]$, $j \in \mathbb{Z}$, and use Lemma 6.2, which is valid since $1 \leq |\ell| \leq \varepsilon \log |t|$. It follows that the second and third sums on the RHS of the equation above each contribute $\ll e^{\pi|t|} (\log |t|)^{-25/156+\varepsilon}$. We conclude that for $0 < |\ell| \leq \varepsilon \log |t|$

$$(8.15) \quad |\mathcal{S}_Y(\ell, t; k)| \ll Y(Y^{\varepsilon-1}|t|^{-1/2} + (\log |t|)^{-25/156+\varepsilon}) \ll Y(\log |t|)^{-25/156+\varepsilon}.$$

In addition applying Cauchy-Schwarz and using Lemma 6.1 and Lemma 8.1 we get

$$(8.16) \quad |\mathcal{S}_Y(\ell, t; k)| \ll_\varepsilon Y(\log |t|)^\varepsilon$$

for every ℓ . Choosing $Y = (\log |t|)^\eta$ for some exponent $\eta < 1$ to be specified later and applying (8.15) and (8.16) it follows that the error term in (8.14) is

$$(8.17) \quad \ll Y^{1/2} \cdot (\log |t|)^{-25/156+\varepsilon}.$$

To estimate the main term in (8.14) apply the summation formula in Lemma 8.4 to see that

$$(8.18) \quad \frac{H(-1)}{K(-1)Y} \mathcal{S}_Y(0, t; k) = \frac{H(-1)}{2\pi} + O(Y^{-1/2+\varepsilon}(\log |t|)^\varepsilon).$$

Using (8.17) and (8.18) in (8.14) it follows that

$$\begin{aligned} \sum_{n \neq 0} |b_{g, \infty}(n)|^2 \int_0^\infty W_{\frac{1}{4} \operatorname{sgn}(n), it}(4\pi y |n|)^2 h(y) \frac{dy}{y^2} \\ = \frac{H(-1)}{2\pi} + O(Y^{-1/2+\varepsilon}(\log |t|)^\varepsilon) + O(\sqrt{Y}(\log |t|)^{-25/156+\varepsilon}). \end{aligned}$$

Choosing $Y = (\log |t|)^{25/156}$ the error term is $O((\log |t|)^{-25/312+\varepsilon}) = o(1)$, thereby establishing (8.11). Additionally, we have already shown that the LHS of (8.12) is $O((\log |t|)^{-25/156+\varepsilon})$. Therefore, we have proved Theorem 2.

9. EQUIDISTRIBUTION OF MASS AND ZEROS OF HALF-INTEGRAL WEIGHT HOLOMORPHIC FORMS

9.1. Preliminaries. Let us first recall some results of Kohnen [21] and Kohnen-Zagier [20]. The Kohnen plus space $S_{k+1/2}^+(\Gamma_0(4))$ denotes the subspace of $S_{k+1/2}(\Gamma_0(4))$ of cusp forms whose Fourier coefficients satisfy $c(n) = 0$ unless $(-1)^k n \equiv 0, 1 \pmod{4}$ and has a basis consisting of simultaneous eigenfunctions of the operators T_{p^2} for all odd p (where T_{p^2} is as defined in (1.1)). For such a $g \in S_{k+1/2}^+(\Gamma_0(4))$ with $T_{p^2}g = \lambda_g(p)g$ there exists a Hecke cusp form $f \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$, such that $H_p f = \lambda_g(p)f$ where H_p is the usual Hecke operator for $S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$. By the strong multiplicity one theorem this determines f (up to scalar multiplication). In addition, writing the Fourier expansion of f as

$$f(z) = \sum_{n \geq 1} a(n)e(nz)$$

and normalizing with $a(1) = 1$ the Fourier coefficients of f and g are related by the formula

$$c(|d|\delta^2) = c(|d|) \sum_{e|\delta} \mu(e)e^{k-1} \chi_d(e) a\left(\frac{\delta}{e}\right)$$

where d is a fundamental discriminant with $(-1)^k d > 0$. With this correspondence, one has for $n = \delta^2 |d|$ with d a fundamental discriminant such that $(-1)^k d > 0$

$$(9.1) \quad |c(n)| \ll_\varepsilon n^{\frac{k}{2} - \frac{1}{4}} \delta^\varepsilon (4\pi)^{k/2} \left(\frac{1}{\Gamma(k + \frac{1}{2})}\right)^{1/2} \left(\frac{L(\frac{1}{2}, f \otimes \chi_d)}{L(1, \mathrm{Sym}^2 f)}\right)^{1/2},$$

for any $\varepsilon > 0$, where g has been L^2 -normalized, that is

$$\iint_{\Gamma_0(4)\backslash\mathbb{H}} y^{k+1/2}|g(z)|^2 d\text{vol}(z) = 1,$$

and recall f is arithmetically normalized with $a(1) = 1$ so that

$$\iint_{\text{SL}_2(\mathbb{Z})\backslash\mathbb{H}} y^{2k}|f(z)|^2 d\text{vol}(z) = \frac{(2k-1)!L(1, \text{Sym}^2 f)}{(4\pi)^{2k-1}2\pi^2}.$$

Kohnen-Zagier also showed that the function

$$D(s) = (4\pi)^{-k-\frac{1}{2}}\Gamma(s+k-\frac{1}{2})\sum_{n\geq 1}\frac{|c(n)|^2}{n^{s+k-\frac{1}{2}}}$$

is absolutely convergent for $\text{Re}(s) > 1$, admits a meromorphic continuation to the complex plane with the only singularity in $\text{Re}(s) \geq \frac{1}{2}$ being a simple pole at $s = 1$ with residue $1/(2\pi)$ (note that $D(s)$ may have poles at $s = \rho/2$ for zeros ρ of $\zeta(s)$). Moreover, the completed function $\tilde{D}(s) = \pi^{-2s}\Gamma(s)\zeta(2s)D(s)$ satisfies the functional equation $\tilde{D}(s) = \tilde{D}(1-s)$ and $s(1-s)\tilde{D}(s)$ is an entire function.

Finally we record the simple consequence of Stirling's formula which will be used repeatedly

$$(9.2) \quad \frac{\Gamma(s+k)}{\Gamma(k)} = k^s(1 + O((|s|+1)^2k^{-1}))$$

for $|s| = o(\sqrt{k})$.

9.2. Proof of Theorem 1. It suffices to show that

$$(9.3) \quad \sum_{n\geq 1}|c(n)|^2 \int_0^\infty e^{-4\pi ny}y^{k+\frac{1}{2}}h(y)\frac{dy}{y^2} = \frac{1}{\text{vol}(\Gamma_0(4)\backslash\mathbb{H})} \int_0^\infty h(y)\frac{dy}{y^2} + o(1)$$

as $k \rightarrow \infty$ and for $\ell \neq 0$

$$(9.4) \quad \sum_{n\geq 1}c(n)c(n+\ell) \int_0^\infty e^{-2\pi(2n+\ell)y}y^{k+\frac{1}{2}}h(y)\frac{dy}{y^2} = o(1),$$

as $k \rightarrow \infty$ (see Section 2.1).

To estimate the sums of Fourier coefficients we will proceed in the same way as for the Maaß forms. First, we will use estimates for moments of $L(\frac{1}{2}, f \otimes \chi_d)$ and (9.1) to obtain (9.4). Next, to estimate the main term (9.3) we will extend the length of summation then apply a summation formula, which is obtained through a convexity bound for $D(s)$. This will give (9.3) thereby proving Theorem 1.

9.2.1. *Estimates for moments.* Let f be a weight $2k$ level 1 holomorphic Hecke cusp form. We require analogues of Lemmas 5.1 and 5.2. Here the weight plays the role of the spectral parameter $|t|$. Additionally, since we have Deligne's bound for the Hecke eigenvalues of f it follows from GRH that

$$(9.5) \quad \frac{1}{\log \log k} \ll L(1, \text{Sym}^2 f) \ll \log \log k.$$

Repeating the same argument used in the proof of Lemma 5.1 we have uniformly for $\varepsilon > 0$, $X > k^\varepsilon$ and $1 \leq a, b, |\ell| \leq (\log X)^{100}$ that

$$\sum_{\substack{d_1, d_2 \\ a|d_1| \leq X \\ ad_1 = bd_2 + \ell}} L\left(\frac{1}{2}, f \otimes \chi_{d_1}\right)^{1/2} L\left(\frac{1}{2}, f \otimes \chi_{d_2}\right)^{1/2} \ll_\varepsilon \frac{X}{[a, b]} (\log X)^{-1/4+\varepsilon}$$

and

$$\sum_{|d| \leq X} L\left(\frac{1}{2}, f \otimes \chi_d\right) \ll_\varepsilon X (\log X)^\varepsilon,$$

where the summation in both sums is over fundamental discriminants. In fact, in the holomorphic case the estimate (9.5) allows us to omit entirely the argument used in the proof of Lemma 5.2, where we treated the small primes separately.

From the moments estimates we obtain a bound on sums of Fourier coefficients. Using (9.1) in place of Corollary 6.1 we proceed as in Section 6 to see that for $\varepsilon > 0$ and $X \geq k^\varepsilon$ that

$$(9.6) \quad \sum_{X \leq n \leq 2X} \frac{|c(n)c(n+\ell)|}{(2\pi(2n+\ell))^{k-\frac{1}{2}}} \ll_\varepsilon \frac{X}{\Gamma(k+\frac{1}{2})} \cdot \begin{cases} (\log X)^{-1/4+\varepsilon} & \text{if } 0 \neq |\ell| \leq \log X \\ (\log X)^\varepsilon & \text{if } \ell = 0. \end{cases}$$

Here one uses the elementary bound

$$\left(\frac{\sqrt{n(n+\ell)}}{n+\ell/2} \right)^{k-\frac{1}{2}} \leq 1,$$

which holds for $|\ell| < n/2$. Also, it follows from the Lindelöf bound that the LHS of (9.6) is $\ll (Xk(|\ell|+1))^\varepsilon X/\Gamma(k+\frac{1}{2})$ for all $X \geq 1$ and ℓ .

9.2.2. *Extending the length of summation.* Next, we need an analogue of Lemma 7.1. Let $h, j \in C_c^\infty(\mathbb{R}_+)$, $j(x) \geq 0$, with Mellin transforms $H(s), J(s)$ (resp.). As before, we evaluate

$$(9.7) \quad \iint_{\Gamma_0(4) \backslash \mathbb{H}} E(z|h) E^Y(z|j) y^{k+\frac{1}{2}} |g(z)|^2 \text{dvol}(z)$$

in two different ways. Here $E(z|h)$ and $E^Y(z|j)$ are as defined in (7.3) and (7.4) (resp.). First, we proceed as in Section 7.1 rewriting $E^Y(z|j)$ in terms of $E(z, s)$,

using Mellin inversion, then shifting contours of integration (taking advantage of the analytic continuation of the Eisenstein series) to see that (9.7) equals

$$J(-1) \frac{Y}{2\pi} \sum_{n \geq 1} |c(n)|^2 \int_0^\infty e^{-4\pi n y} \cdot y^{k-\frac{3}{2}} h(y) dy + O(\sqrt{Y}).$$

Next, as in Section 7.2 we use the unfolding technique with $E^Y(z|j)$ and then expand $E(z|h)$ into a Fourier series as given in Lemma 7.2 to see that (9.7) equals

$$\sum_{n,\ell} c(n)c(n+\ell) \int_0^\infty a_{\ell,h}(y) e^{-2\pi(2n+\ell)y} y^{k-\frac{3}{2}} j(yY) dy.$$

Applying the estimates for $a_{\ell,h}(y)$ given in Lemma 7.2 we get that the above equals

$$\frac{H(-1)}{2\pi} \sum_{n \geq 1} |c(n)|^2 \int_0^\infty e^{-4\pi n y} \cdot y^{k-\frac{3}{2}} j(yY) dy \cdot (1 + (Y^{-1/2})) + O(\mathcal{E}_Y)$$

where

$$\mathcal{E}_Y = \frac{1}{\sqrt{Y}} \sum_{\ell \neq 0} \frac{(\nu(\ell) + \ell) 2^{\nu(\ell)} d(|\ell|)}{1 + |\ell/Y|^A} \sum_{n \geq 1} |c(n)c(n+\ell)| \int_0^\infty e^{-2\pi(2n+\ell)y} \cdot y^{k-\frac{3}{2}} j(yY) dy$$

and $\nu(\ell)$ is the largest power of 2 dividing ℓ . This completes our second evaluation of (9.7). Combining the above formulas gives us the extended sum

$$(9.8) \quad \begin{aligned} H(-1) \sum_{n \geq 1} |c(n)|^2 \int_0^\infty e^{-4\pi n y} \cdot y^{k-\frac{3}{2}} j(Yy) dy \cdot (1 + (Y^{-1/2})) \\ = J(-1)Y \sum_{n \geq 1} |c(n)|^2 \int_0^\infty e^{-4\pi n y} \cdot y^{k-\frac{3}{2}} h(y) dy + O(\mathcal{E}_Y) + O(\sqrt{Y}). \end{aligned}$$

9.2.3. *A summation formula.* We will show assuming GRH that for $\varepsilon > 0$ and $\frac{1}{2} < \sigma < 1$

$$(9.9) \quad |D(\sigma + i\tau)| \ll_\varepsilon (1 + |\tau|)^{1-\sigma+\varepsilon} (\log k)^\varepsilon.$$

By a contour integration argument this gives the following summation formula

$$(9.10) \quad \begin{aligned} \sum_{n \geq 1} |c(n)|^2 \int_0^\infty e^{-4\pi n y} \cdot y^{k-\frac{3}{2}} j(Yy) dy &= \frac{1}{2\pi i} \int_{(2)} J(-s) Y^s (4\pi)^{1-s} D(s) ds \\ &= \frac{YJ(-1)}{2\pi} + O_\varepsilon(Y^{1/2+\varepsilon} (\log k)^\varepsilon). \end{aligned}$$

To show (9.9) let

$$\sum_{n \geq 1} \frac{\tilde{c}(n)}{n^s} = \zeta(2s) \sum_{n \geq 1} \frac{|c(n)|^2}{n^{s+k-\frac{1}{2}}}, \quad \tilde{c}(n) = \sum_{m^2 h = n} |c(h)|^2 h^{\frac{1}{2}-k}$$

and note that by (9.6) we have

$$(9.11) \quad \sum_{X \leq n \leq 2X} \tilde{c}(n) \ll_{\varepsilon} \frac{(4\pi)^k X}{\Gamma(k + \frac{1}{2})} \cdot \begin{cases} (\log k)^{\varepsilon} & \text{if } X > k^{\varepsilon}, \\ k^{\varepsilon} & \text{if } X \leq k^{\varepsilon}, \end{cases}$$

for any $\varepsilon > 0$. Also, let $\Phi_s(w)$ be defined as in (8.4). Since $s(1-s)\tilde{D}(s)$ is invariant under $s \rightarrow 1-s$ a standard contour integration argument gives

$$(9.12) \quad \begin{aligned} (4\pi)^{k+1/2} \zeta(2s) D(s) &= \sum_{n \geq 1} \frac{\tilde{c}(n)}{n^s} \frac{1}{2\pi i} \int_{(1)} n^{-w} \Phi_s(w) \frac{\Gamma(k+w+s-\frac{1}{2})}{\Gamma(s)} dw \\ &+ \sum_{n \geq 1} \frac{\tilde{c}(n)}{n^{1-s}} \frac{1}{2\pi i} \int_{(1)} n^{-w} \Phi_{1-s}(w) \frac{\Gamma(k+w-s+\frac{1}{2})}{\Gamma(s)} dw. \end{aligned}$$

Using an analogue of (8.6) (which is obtained by applying (9.2)) along with (9.11) we see that

$$\begin{aligned} &\frac{1}{(4\pi)^{k+1/2}} \sum_{n \geq 1} \frac{\tilde{c}(n)}{n^s} \int_{(1)} n^{-w} \Phi_s(w) \frac{\Gamma(k+s+w-\frac{1}{2})}{\Gamma(s)} dw \\ &\ll_{\varepsilon} (1+|\tau|)^{1-\sigma+\varepsilon} (4\pi)^{-k} k^{\sigma-1} \Gamma(k+\frac{1}{2}) \left(\sum_{n \leq k} \frac{\tilde{c}(n)}{n^{\sigma}} + \sum_{n \geq k} \frac{\tilde{c}(n)}{n^{\sigma}} \left| \frac{k}{n} \right|^{1-\sigma+\varepsilon} \right) \\ &\ll_{\varepsilon} (\log k)^{\varepsilon} (1+|\tau|)^{1-\sigma+\varepsilon}. \end{aligned}$$

Similarly, the second term in (9.12) is $O_{\varepsilon}((1+|\tau|)^{1-\sigma+\varepsilon}(\log k)^{\varepsilon})$. This proves (9.9).

9.2.4. Completion of the proof. First we will establish (9.4). Note that for $h \in C_c^{\infty}(\mathbb{R}_+)$ using Mellin inversion together with (9.2) then applying (9.6) we have that for any $\varepsilon > 0$ and fixed $\ell \neq 0$

$$\begin{aligned} &\sum_{n \geq 1} c(n)c(n+\ell) \int_0^{\infty} e^{-2\pi(2n+\ell)y} y^{k+\frac{1}{2}} h(y) \frac{dy}{y^2} \\ &\ll_{\varepsilon} \Gamma(k-\frac{1}{2}) \sum_{n \geq 1} \frac{|c(n)c(n+\ell)|}{(2\pi(2n+\ell))^{k-\frac{1}{2}}} \left(\left| h\left(\frac{k-\frac{1}{2}}{(2\pi(2n+\ell))}\right) \right| + \frac{1}{k^{1-\varepsilon}} \cdot \frac{1}{1+(n/k)^A} \right) \\ &\ll_{\varepsilon} \frac{\Gamma(k-\frac{1}{2})}{\Gamma(k+\frac{1}{2})} \cdot k(\log k)^{-1/4+\varepsilon} \ll (\log k)^{-1/4+\varepsilon}. \end{aligned}$$

It remains to estimate the main term (9.3) and to accomplish this we will use (9.8). We first estimate \mathcal{E}_Y by choosing $Y = (\log k)^{\eta}$ for some $0 < \eta < 1$ to be chosen later

and proceeding as above we get that

$$\begin{aligned}
\mathcal{E}_Y &= \frac{1}{\sqrt{Y}} \sum_{\ell \neq 0} \sum_{\ell \neq 0} \frac{(\nu(\ell) + \ell) 2^{\nu(\ell)} d(|\ell|)}{1 + |\ell/Y|^A} \sum_{n \geq 1} |c(n)c(n + \ell)| \int_0^\infty j(yY) e^{-2\pi(2n+\ell)y} y^{k-3/2} dy \\
&\ll_\varepsilon \frac{\Gamma(k - \frac{1}{2})}{\sqrt{Y}} \sum_{\ell \neq 0} \frac{(\nu(\ell) + 1) 2^{\nu(\ell)} d(|\ell|)}{1 + |\ell/Y|^A} \sum_{n \geq 1} \frac{|c(n)c(n + \ell)|}{(2\pi(2n + \ell))^{k-\frac{1}{2}}} \\
&\quad \times \left(j \left(\frac{Y(k - \frac{1}{2})}{(2\pi(2n + \ell))} \right) + \frac{1}{k^{1-\varepsilon}} \cdot \frac{1}{1 + (n/(kY))^A} \right) \\
&\ll_\varepsilon Y^{3/2} (\log k)^{-1/4+\varepsilon},
\end{aligned}$$

where to handle the contribution of the terms with $\log k < \ell < k$ we used Cauchy-Schwarz and the second bound in (9.6) and for $\ell > k$ we use instead the trivial bound. Combining this along with (9.8) and (9.10) gives

$$\sum_{n \geq 1} |c(n)|^2 \int_0^\infty e^{-4\pi ny} \cdot y^{k-\frac{3}{2}} h(y) dy = \frac{H(-1)}{2\pi} + O_\varepsilon(Y^{-1/2+\varepsilon} (\log k)^\varepsilon) + O_\varepsilon(\sqrt{Y} (\log k)^{-1/4+\varepsilon}).$$

Taking $Y = (\log k)^{1/4}$ we get

$$\sum_{n \geq 1} |c(n)|^2 \int_0^\infty e^{-4\pi ny} \cdot y^{k-\frac{3}{2}} h(y) dy = \frac{1}{\text{vol}(\Gamma_0(4) \backslash \mathbb{H})} \int_0^\infty h(y) \frac{dy}{y^2} + O_\varepsilon((\log k)^{-1/8+\varepsilon}),$$

thereby establishing (9.3) and completing the proof of Theorem 1.

9.3. Equidistribution of zeros: Proof of Corollary 1.1. For a holomorphic function f on \mathbb{H} let $\text{ord}_\varrho(f)$ denote the order of vanishing of f at ϱ . Also let

$$\Gamma_z = \{\gamma \in \Gamma_0(4) / \{\pm 1\} : \gamma z = z\}$$

be the stabilizer group of z and write $M_{k+1/2}(\Gamma_0(4))$ for the space of weight $k + 1/2$ holomorphic modular forms for $\Gamma_0(4)$. For $g \in M_{k+1/2}(\Gamma_0(4))$ one trivially has $f(z) := g(z)^4 \in M_{4k+2}(\Gamma_0(4))$ (notice that $\left(\left(\frac{-1}{d}\right)^k \bar{\varepsilon}_d \left(\frac{c}{d}\right)^{2k+1}\right)^4 = 1$). The standard valence formula (see [41, 16]) applied to f gives

$$N_f := \sum_{\mathfrak{a}} \text{ord}_{\mathfrak{a}}(f) + \sum_{\varrho \in \Gamma_0(4) \backslash \mathbb{H}} \frac{1}{\#\Gamma_\varrho} \cdot \text{ord}_\varrho(f) = [\text{SL}_2(\mathbb{Z}) : \Gamma_0(4)] \cdot \frac{(4k+2)}{12}$$

where the first sum is over the cusps \mathfrak{a} of $\Gamma_0(4) \backslash \mathbb{H}$ and $\text{ord}_{\mathfrak{a}}(f)$ is the order of vanishing of f at \mathfrak{a} . Since $\text{ord}_\varrho(f) = 4 \text{ord}_\varrho(g)$ we conclude that the total number of weighted $\Gamma_0(4)$ nonequivalent zeros of g , which we denote by N_g , equals

$$N_g = \frac{1}{2} \cdot \left(k + \frac{1}{2}\right)$$

for any modular form $g \in M_{k+1/2}(\Gamma_0(4))$.

Given $\psi \in C_c^\infty(\mathbb{H})$ define

$$\Psi(z) = \sum_{\gamma \in \Gamma_0(4)} \psi(\gamma z),$$

so that $\Psi \in C_c^\infty(\Gamma_0(4)\backslash\mathbb{H})$. Again, for $g \in M_{k+1/2}(\Gamma_0(4))$ we note that $f = g^4 \in M_{4k+2}(\Gamma_0(4))$ so Lemma 2.1 of Rudnick [39] applies to f , from which we deduce that for any $g \in M_{k+1/2}(\Gamma_0(4))$

$$(9.13) \quad \frac{1}{N_g} \sum_{\varrho \in \Gamma_0(4)\backslash\mathbb{H}} \frac{1}{\#\Gamma_\varrho} \cdot \text{ord}_\varrho g(\varrho) \Psi(\varrho) = \frac{1}{\text{vol}(\Gamma_0(4)\backslash\mathbb{H})} \iint_{\Gamma_0(4)\backslash\mathbb{H}} \Psi(z) \frac{dx dy}{y^2} + \frac{2}{(k + \frac{1}{2})\pi} \iint_{\Gamma_0(4)\backslash\mathbb{H}} \log(y^{k+1/2}|g(z)|^2) \Delta_{\mathbb{H}} \Psi(z) \frac{dx dy}{y^2}.$$

The first integral above is our main term and it remains to estimate the second integral (here we will assume additional properties about g). Let $\{g_k\}$ be a sequence of weight $k + 1/2$ cusp forms lying in the Kohnen plus subspace that are also eigenfunctions of T_{p^2} for all $p > 2$. Corollary 1.1 follows from the estimate

$$(9.14) \quad \left| \frac{1}{k} \iint_{\Gamma_0(4)\backslash\mathbb{H}} \log(y^{k+1/2}|g_k(z)|^2) \Delta_{\mathbb{H}} \Psi(z) \frac{dx dy}{y^2} \right| = o(1) \quad (k \rightarrow \infty).$$

Note that we can and will assume that g_k is L^2 -normalized.

To establish (9.14) one can follow either Rudnick's argument (see also the proof of Shiffman-Zelditch [42]) or the argument used to prove Theorem 2.1 of [23]. Both extend to a more general setting and establish (9.14) provided that the following two criteria are met:

- i) Inside compact subsets of $\Gamma_0(4)\backslash\mathbb{H}$ one has $y^{k+1/2}|g_k(z)|^2 \ll k^A$ for some $A > 0$.
- ii) For each hyperbolic ball $B(r, z) \subset \Gamma_0(4)\backslash\mathbb{H}$ there exists a point $z_0 = x_0 + iy_0 \in B(r, z)$ such that given $\varepsilon > 0$ one has for all sufficiently large k that

$$y_0^{k+1/2}|g_k(z_0)|^2 \geq e^{-k\varepsilon}.$$

Following an argument of Iwaniec and Sarnak (see Lemma A.1 of [14]), Rudnick (see Appendix A.2) showed *i*) holds for integral weight holomorphic forms with $A = 1$ and a straightforward adaptation of this method gives *i*) for any $g \in M_{k+1/2}(\Gamma_0(4))$. Moreover, for g_k as above Steiner [51] proved that *i*) holds with $A = \frac{6}{7} + \varepsilon$ unconditionally and with $A = \frac{1}{2} + \varepsilon$ under GRH. To establish *ii*) we apply Theorem 1, which gives a much better lower bound. This is the *only* place where mass equidistribution is used in this argument. Since criteria *i*) and *ii*) hold the methods of either Rudnick

and Shiffman-Zelditch or [23] give (9.14). Thus, by (9.13) and (9.14), along with an approximation argument, we conclude that for a compact subset $\mathcal{D} \subset \Gamma_0(4) \backslash \mathbb{H}$ with boundary measure zero we have as $k \rightarrow \infty$

$$\frac{1}{N_{g_k}} \sum_{\rho \in \mathcal{D}} \frac{1}{\#\Gamma_\rho} \cdot \text{ord}_\rho(g_k) = \frac{\text{vol}(\mathcal{D})}{\text{vol}(\Gamma_0(4) \backslash \mathbb{H})} + o(1)$$

thereby proving Corollary 1.1. Finally, note the above result shows that $\text{ord}_\rho(g_k) = o(k)$, so since $\#\Gamma_z = 1$ for all but $O(1)$ points it follows that we can restate this as

$$\sum_{\substack{\rho \in \mathcal{D} \\ g_k(\rho)=0}} 1 = \frac{k}{2} \cdot \frac{\text{vol}(\mathcal{D})}{\text{vol}(\Gamma_0(4) \backslash \mathbb{H})} + o(k)$$

where in the sum the zeros are counted with multiplicity.

10. APPENDIX: PROOF OF PROPOSITION 6.1

The formula for $|b_{g,\infty}(d)|$ with d a fundamental discriminant follows from the main result of Baruch and Mao, Theorem 1.4 [1]. Given a fundamental discriminant d , consider the d th Shimura lift

$$\text{Sh}_d g(z) = \sqrt{y} \sum_{k \neq 0} 2a_{\text{Sh}_d g}(|k|) K_{2it}(2\pi|k|y) e(kx)$$

where

$$a_{\text{Sh}_d g}(k) = k \sum_{\substack{mn=k \\ m,n>0}} \frac{1}{n^{3/2}} \left(\frac{d}{n}\right) b_{g,\infty}(dm^2).$$

If $\text{Sh}_d g(z)$ is identically equal to zero, then by Möbius inversion $b_{g,\infty}(d\delta^2) = 0$ for all $\delta > 0$ and therefore the bound $|b_{g,\infty}(d\delta^2)|^2 \ll |b_{g,\infty}(d)|^2 \delta^{2\theta-2+\varepsilon}$ is vacuously true for that fundamental discriminant d and all integers $\delta > 0$. On the other hand if $\text{Sh}_d g(z)$ is not identically zero, then following the proof of Proposition 6 in [7] we conclude that $\text{Sh}_d g(z)$ is a weight 0 Maaß form. Therefore there exists a Hecke normalized Hecke Maaß form ϕ of weight zero, such that $\langle \text{Sh}_d g, \phi \rangle \neq 0$ (where $\langle \cdot, \cdot \rangle$ is the Petersson inner product). Consider now the weight 0 Hecke operators H_p . These are self-adjoint for $\langle \cdot, \cdot \rangle$ and commute with the Shimura lift, in the sense that $H_p(\text{Sh}_d g(z)) = \text{Sh}_d(T_{p^2}g(z))$ for all primes $p > 2$. Let λ_p denote the T_{p^2} eigenvalues of g , so that $T_{p^2}g = \lambda_p g$ for all $p > 2$, and denote by $\lambda_\phi(p)$ the Hecke eigenvalues of ϕ , so that $H_p \phi = \lambda_\phi(p) \phi$ for all primes p . Using the just mentioned two facts about the Hecke operators H_p , we find that for all $p > 2$,

$$\lambda_p \langle \text{Sh}_d g, \phi \rangle = \langle \text{Sh}_d(T_{p^2}g), \phi \rangle = \langle H_p(\text{Sh}_d g), \phi \rangle = \langle \text{Sh}_d g, H_p \phi \rangle = \langle \text{Sh}_d g, \phi \rangle \cdot \lambda_\phi(p).$$

Since $\langle \text{Sh}_d g, \phi \rangle \neq 0$ we conclude that $\lambda_p = \lambda_\phi(p)$ for all $p > 2$, hence $T_{p^2} g = \lambda_\phi(p)g$ for all $p > 2$. Taking the Shimura lift on both sides of this equation, and using the relation $H_p(\text{Sh}_d g(z)) = \text{Sh}_d(T_{p^2} g(z))$, we conclude that $H_p(\text{Sh}_d g) = \lambda_\phi(p)\text{Sh}_d g$ for all $p > 2$. Hence by the strong multiplicity one theorem it follows that $\text{Sh}_d g$ is a constant multiple of ϕ . Since ϕ is Hecke normalized we conclude that $\text{Sh}_d g = b_{g,\infty}(d)\phi$, and that by necessity $b_{g,\infty}(d) \neq 0$. Therefore,

$$k \sum_{\substack{mn=k \\ m,n>0}} \frac{1}{n^{3/2}} \left(\frac{d}{n}\right) b_{g,\infty}(dm^2) = b_{g,\infty}(d)\lambda_\phi(k).$$

Using Möbius inversion

$$mb_{g,\infty}(dm^2) = b_{g,\infty}(d) \sum_{k|m} \frac{\mu(k)}{\sqrt{k}} \left(\frac{d}{k}\right) \cdot \lambda_\phi\left(\frac{m}{k}\right)$$

and then applying the bound $|\lambda_\phi(k)| \ll k^{\theta+\varepsilon}$, where θ is the best exponent towards the Ramanujan-Petersson conjecture, gives $|b_{g,\infty}(d\delta^2)|^2 \ll |b_{g,\infty}(d)|^2 \delta^{2\theta-2+\varepsilon}$.

11. APPENDIX: THE FUNCTIONAL EQUATION

As promised we include for completeness the proof of Lemma 8.2. As already pointed out this follows very closely the proof given for holomorphic forms by Kohnen and Zagier [20].

Proof of Lemma 8.2. Let

$$E_\infty^{(4)}(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4)} (\text{Im}(\gamma z))^s$$

be the Eisenstein series for $\Gamma_0(4)$ at the cusp at ∞ and

$$E_\infty(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \text{SL}_2(\mathbb{Z})} (\text{Im}(\gamma z))^s$$

the Eisenstein series for the full modular group. By the unfolding method,

$$(11.1) \quad (4\pi)^{-s+1} G(s) = \iint_{\Gamma_0(4) \backslash \mathbb{H}} |g(z)|^2 E_\infty^{(4)}(z, s) \, d\text{vol}(z).$$

Since $E_\infty^{(4)}(z, s)$ has a simple pole at $s = 1$ with residue $(\text{vol}(\Gamma_0(4) \backslash \mathbb{H}))^{-1} = (2\pi)^{-1}$ and has no other poles for $\text{Re}(s) > \frac{1}{2}$ we conclude that the residue of $G(s)$ at $s = 1$ is

$$\frac{1}{2\pi} \iint_{\Gamma_0(4) \backslash \mathbb{H}} |g(z)|^2 \, d\text{vol}(z) = \frac{1}{2\pi}.$$

We would like to also conclude from the above representation that $G(s)$ has a functional equation relating s to $1 - s$, but this is not completely immediate since $E_\infty^{(4)}(z, 1 - s)$ is a linear combination of Eisenstein series at the cusps of $\Gamma_0(4)$. In fact to establish the functional equation we will use the fact that $g \in V^+$. Recall that V^+ corresponds to the subspace of V with eigenvalue 1 of a certain operator L . Following Katok-Sarnak [18] we will now provide an explicit description of L . Consider the involution

$$\tau_2 F(z) = e^{i\pi/4} \left(\frac{z}{|z|} \right)^{-1/2} F\left(\frac{-1}{4z}\right)$$

and the Hecke operator

$$\sigma F(z) = \frac{\sqrt{2}}{4} \sum_{v \pmod{4}} F\left(\frac{z+v}{4}\right)$$

Then $L = \tau_2 \sigma$. For a $g \in V^+$ we have $Lg = g$, and with the explicit description of L this implies an identity that we will use. First notice that,

$$\sigma g = \sqrt{2}g_0$$

where for $j = 0, 1$,

$$g_j(z) = \sum_{\substack{n \neq 0 \\ n \equiv j \pmod{4}}} b_{g, \infty}(n) W_{\frac{1}{4} \operatorname{sgn}(n), it}(\pi|n|y) e(nx/4).$$

Therefore $Lg = g$ is equivalent to $\sqrt{2}\tau_2 g_0(z) = g(z)$. Since τ_2 is an involution this means that $\sqrt{2}g_0(z) = \tau_2 g(z)$. Hence from $\sqrt{2}\tau_2 g_0(z) = g(z)$ and $\sqrt{2}g_0(z) = \tau_2 g(z)$ we obtain the following two identities

$$\begin{aligned} \sqrt{2}g_0(z) &= e^{\pi i/4} \left(\frac{z}{|z|} \right)^{-1/2} g\left(\frac{-1}{4z}\right) \\ e^{\pi i/4} \left(\frac{z}{|z|} \right)^{-1/2} g_0\left(\frac{-1}{4z}\right) &= \frac{1}{\sqrt{2}}g(z) \end{aligned}$$

In addition, using the Fourier expansion of g and the above two properties we get

$$\begin{aligned} e^{\pi i/4} \left(\frac{z}{|z|} \right)^{-1/2} \cdot g\left(\frac{1}{2} - \frac{1}{4z}\right) &= e^{\pi i/4} \left(\frac{z}{|z|} \right)^{-1/2} \cdot \left(2g_0\left(-\frac{1}{z}\right) - g\left(\frac{-1}{4z}\right) \right) \\ &= \sqrt{2} \left(g\left(\frac{z}{4}\right) - g_0(z) \right) = \sqrt{2}g_1(z) \end{aligned}$$

using in both equations the fact that the coefficients $n \equiv 2, 3 \pmod{4}$ of $g(z)$ are equal to zero. Now since g_0 and g_1 are automorphic forms of half-integral weight for

$\Gamma_0(4)$ (since $\sigma : V \rightarrow V$, $\tau : V \rightarrow V$ and V is a Hilbert space) by the Rankin-Selberg method we get

$$\begin{aligned} \pi^{-s+1}G(s) &= \iint_{\Gamma_0(4)\backslash\mathbb{H}} |g_1(z)|^2 E_\infty^{(4)}(z, s) \, d\text{vol}(z) + \iint_{\Gamma_0(4)\backslash\mathbb{H}} |g_0(z)|^2 E_\infty^{(4)}(z, s) \, d\text{vol}(z) \\ &= \frac{1}{2} \iint_{\Gamma_0(4)\backslash\mathbb{H}} \left| g\left(-\frac{1}{4z}\right) \right|^2 E_\infty^{(4)}(z, s) \, d\text{vol}(z) \\ &\quad + \frac{1}{2} \iint_{\Gamma_0(4)\backslash\mathbb{H}} \left| g\left(\frac{1}{2} - \frac{1}{4z}\right) \right|^2 E_\infty^{(4)}(z, s) \, d\text{vol}(z) \\ &= \frac{1}{2} \iint_{\Gamma_0(4)\backslash\mathbb{H}} |g(z)|^2 \cdot \left(E_\infty^{(4)}\left(\frac{-1}{4z}, s\right) + E_\infty^{(4)}\left(\frac{-1}{4z+2}, s\right) \right) \, d\text{vol}(z). \end{aligned}$$

Since (see for example Kohnen-Zagier, p. 191)

$$E_\infty^{(4)}\left(-\frac{1}{4z}, s\right) + E_\infty^{(4)}\left(-\frac{1}{4z+2}, s\right) = \frac{1}{4^s - 1} (2^s E_\infty(2z, s) - E_\infty(4z, s))$$

we conclude that

$$(11.2) \quad \pi^{-s+1}G(s) = \frac{1}{2} \iint_{\Gamma_0(4)\backslash\mathbb{H}} |g(z)|^2 \cdot \left(\frac{1}{4^s - 1} (2^s E_\infty(2z, s) - E_\infty(4z, s)) \right) \, d\text{vol}(z).$$

But by (11.1) and the identity

$$E_\infty^{(4)}(z, s) = \frac{1}{4^s - 1} (E_\infty(4z, s) - 2^{-s} E(2z, s))$$

we also get

$$(11.3) \quad \pi^{-s+1}G(s) = 4^{s-1} \iint_{\Gamma_0(4)\backslash\mathbb{H}} |g(z)|^2 \cdot \left(\frac{1}{4^s - 1} (E_\infty(4z, s) - 2^{-s} E_\infty(2z, s)) \right) \, d\text{vol}(z).$$

Multiplying both sides of (11.3) by two, then adding (11.2) and dividing both sides of the resulting formula by three, we get

$$\pi^{-s+1}G(s) = \frac{1}{6} \iint_{\Gamma_0(4)\backslash\mathbb{H}} |g(z)|^2 E_\infty(4z, s) \, d\text{vol}(z).$$

Since $\pi^{-s}\Gamma(s)\zeta(2s)E_\infty(4z, s)$ is invariant under $s \mapsto 1 - s$ we conclude that

$$\tilde{G}(s) = \pi^{-2s}\Gamma(s)\zeta(2s)G(s)$$

is invariant under $s \mapsto 1 - s$ as claimed. \square

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DEPARTMENT OF MATHEMATICS, KTH ROYAL INSTITUTE OF TECHNOLOGY, SE-100 44 STOCKHOLM, SWEDEN

Current address: School of Mathematical Sciences, Queen Mary University of London, 327 Mile End Road, London E1 4NS, United Kingdom

E-mail address: sjlester@gmail.com

DEPARTMENT OF MATHEMATICS, MCGILL UNIVERSITY, 845 RUE SHERBROOKE OUEST, MONTRÉAL,
QUÉBEC H3A 0G4, CANADA

Current address: Department of Mathematics, Caltech, 1200 E California Blvd, Pasadena, CA,
91125

E-mail address: maksym.radziwill@gmail.com