Static Output-Feedback Tracking Control For Positive Polynomial Fuzzy Systems

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Abstract—Nonlinear positive control system can be found in many real-world applications but the positivity requirements lead to challenges in system analysis and control design. In this paper, we approach the problem by fuzzy-model-based control techniques and overcome some challenges including transforming the non-convexity conditions when both positive and stability conditions exist into convexity conditions that can be solved by the convex programming techniques. This paper focuses on the static output-feedback tracking control issue of positive polynomial fuzzy-model-based (PPFMB) systems. The purpose of the tracking control is to design an appropriate static output feedback polynomial fuzzy controller which can drive the system states of the nonlinear plant to follow those of a stable reference model subject to an $H_\infty$ performance. The concept of imperfectly matched premises is employed to enhance the design and implementation flexibility. To circumvent the problem of non-convex stability conditions, an approach is employed to transform the non-convex stability conditions into convex ones by introducing a novel Scalar Implantation Transformation (SIT) technique. Besides, the partition approximation of membership functions with local information of membership functions is used to promote stability analysis and synthesis of controllers. The positive and relaxed stability conditions for static output-feedback tracking control with $H_\infty$ performance being taken into account are obtained in terms of sum-of-squares (SOS). Finally, a simulation example is presented to verify the effectiveness of the proposed tracking control approach.

Index Terms—PPFMB, Static Output-Feedback Tracking Control, SIT, Imperfectly Matched Premises, Membership-Function-Dependent Stability Analysis, $H_\infty$ Performance.

I. INTRODUCTION

In the real world, there is a special type of practical systems, called positive systems, which has the inherent constraint that the state variables are limited to be positive (or at least non-negative) under any non-negative initial conditions [1]–[3]. It demonstrates a wide range of practical applications such as economic systems, population systems, biological systems that the state variables of these systems represent the economic profit and loss, population quantity, biological quantity, etc, which have no practical significance when these quantities are negative. In recent years, many achievements have been made in the research of positive linear systems, for example, positive observer and dynamic output feedback controller design [4], state feedback controller design [5], filter design [6], etc. It is also noticed that the control problems of positive nonlinear systems were also investigated in the works in [7]–[9]. Since most practical plants are nonlinear systems, the research of positive nonlinear systems demonstrates important practical significance.

Nonlinear control and system analysis are challenging tasks due to the system nonlinearity is not easy to deal with in a systematic way. In the literature, many control design methods have been proposed for nonlinear systems, such as fuzzy control [10]–[13], sliding mode control [14] and adaptive control [15]–[17]. Among them, fuzzy-model-based (FMB) control [18]–[21] is one of the popular approaches for the analysis and control of nonlinear systems. Under the FMB control paradigm, the Takagi-Sugeno (T-S) fuzzy model [22]–[25] plays an important role to describe the system dynamics of the nonlinear system. The T-S fuzzy model describes the global behavior of the nonlinear system by taking a family of local linear sub-systems and then fuzzily blended together smoothly through the membership functions. Owing to the T-S FMB control system has a good model structure to support the control design grounded on a strict mathematical basis, the stability analysis and control synthesis of T-S FMB control system can be guided by systematic methods. In [26], T-S FMB control technique was used to handle positive nonlinear system for the first time. At present, the research on the positive T-S FMB control systems have been applied in various fields, such as stability analysis for positive T-S FMB continuous time systems with time delay [27], observer-based control [28], filter design [29], etc.

Although a large number of research achievements on positive T-S FMB system have been made in recent years, the research on tracking control positive T-S FMB systems is relatively limited. In fact, the tracking control problem is often encountered in many nonlinear control applications, thereby, it is essential to investigate the tracking control problem for positive T-S FMB systems. In the fuzzy tracking control problems, the control objective is to design a fuzzy controller, which is not only used to stabilize the nonlinear system [30], but also to drive the states of the nonlinear plant represented by the fuzzy model as close as possible to those of the stable reference model [31], so it is agreed that the tracking control problem is more challenging than the stabilization...
problem. In the past few decades, many research results have been obtained for various tracking control problems such as output-feedback tracking control [31–34], state-feedback tracking control [35] and sampled-data tracking control [36], [37]. However, the existing results are mainly for general T-S fuzzy systems instead of positive T-S fuzzy systems. In the literature, there has been very few achievement in the design of tracking control under positive conditions, thereby, it is of far-reaching significance to study the tracking control for positive fuzzy systems. When considering that some system state variables are inaccessible, the design of observer and the use of output feedback are the main methods of research. The work in [7] is to design an observer for positive T-S fuzzy model, in [38] is to employ static output feedback control for nonlinear polynomial fuzzy systems. In this paper, we consider the use of the output variables to fulfill the tracking control objective, where the feedback compensation of the control can be realized using only the system outputs instead of full state information, which makes the control systems more flexible and more convenient to be applied to real applications. However, for static output feedback control, non-convexity is usually caused when the stability conditions of closed-loop positive systems are analyzed. Due to the difficulty in transforming non-convex stability conditions into convex stability conditions [39], the research results of static output feedback control for positive T-S fuzzy systems are relatively limited at present.

Recently, the T-S fuzzy model is generalized to the polynomial fuzzy model in which sub-systems are represented by polynomial variables in some published works [31], [40]. Relatively speaking, the polynomial fuzzy model can effectively describe a wider class of nonlinear systems than the T-S fuzzy model. Moreover, polynomial fuzzy model has better global approximation ability, which is helpful to reduce the number of fuzzy rules compared with the T-S fuzzy model. However, due to the existence of polynomials, the linear matrix-inequality (LMI) method can no longer be used to study the stability analysis. Fortunately, using the advanced SOS toolbox, the stability conditions of PFM control system can be given in the forms of sum-of-squares (SOS) [41]. Although the polynomial fuzzy model can more effectively describe a class of nonlinear systems, the stability conditions of the positive polynomial fuzzy system are still relatively conservative due to the lack of membership function information (Membership-Function-Independent (MFI)) is considered in the analysis.

In order to obtain more relaxed stability analysis results, the pioneering work by H. K. Lam et.al. [42] underpinned the membership-function-dependent (MFD) analysis that the information of the membership functions is used in the stability analysis. And further advanced MFD techniques using the boundary information and approximate membership functions (staircase, piecewise linear and Taylor series membership functions) were proposed by H. K. Lam et.al. and his co-workers [19] which offer the most relaxed MFD stability analysis results in the literature [43]. Furthermore, to improve the design and implementation flexibility, the novel concept of imperfect premise matching [19], [44], [45] was initiated by H. K. Lam et.al. for the first time which proposed that the fuzzy model and the fuzzy controller do not need to share the same premise membership functions and/or number of rules which removed the longstanding constraints required in the well-known parallel distributed compensation (PDC) technique. In this paper, various piecewise information including membership functions, the sub-domain and the output-state boundary will be brought into stability analysis that will contribute to the relaxation of stability conditions. The obtained MFD SOS-based positivity and stability conditions will be obtained to support the control synthesis under the consideration of system positivity and stability.

Having mentioned and reviewed the previous related works, to the authors’ best knowledge, as far as static output-feedback tracking control for positive polynomial fuzzy system is concerned, the work of designing a static output-feedback polynomial fuzzy controller to drive the states of positive polynomial fuzzy model to track those of a stable positive reference model that optimizes the tracking performance has not been analyzed.

To sum up, in order to meet the requirements of tracking control for nonlinear positive systems, we are motivated to analyze and study this problem from a theoretical perspective.

To distinguish the proposed work from the previously published work [31], [32], the main contributions of this paper lies in the following aspects: 1) Due to the introduction of the output matrix, when the quadratic Lyapunov function candidate is used for analysis, the positivity conditions and stability conditions obtained are in non-convex forms that cannot be solved directly by convex programming techniques. In order to transform non-convex conditions into convex conditions, a novel Scalar Implantation Transformation (SIT) technique is proposed in this paper. 2) The nonlinear tracking control which extends the T-S FMB control techniques to PPFMB one is studied in this paper for the first time. 3) In order to improve the tracking control performance, this paper adopts the MFD techniques by extracting and introducing membership function information into the tracking performance conditions resulting in more relaxed stability conditions. Further, this paper considers dividing the whole operation domain of the premise variable into some connected sub-domains, in each sub-domain, all upper bounds and lower bounds of both product terms of membership functions and approximate error terms of membership functions are obtained, and all of them are introduced into the stability analysis together with output states boundary information in each sub-domain.

The rest of this paper are organized as follows: In Section II, notations and preliminaries are introduced, polynomial fuzzy system representing the nonlinear system, static output feedback polynomial fuzzy controller and the stable positive polynomial reference model are presented. In Section III, based on the Lyapunov stability theory, the SOS-based positive and stability conditions are obtained using the MFI and MFD approaches. In Section IV, a simulation example is presented to verify the effectiveness of the proposed control strategy. In Section V, a conclusion is drawn.

II. NOTATIONS AND PRELIMINARIES

Standard notations and fundamental technical concepts of positive polynomial fuzzy model, positive polynomial ref-
reference model and static output feedback polynomial fuzzy controller used in this paper are introduced in this section.

A. Notation

Throughout this paper, the following notations are adopted [31]. The monomial in \( x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \) is a function of the form \( x_1^{d_1}(t), x_2^{d_2}(t), \ldots, x_n^{d_n}(t) \), where \( d_k, k \in \{1, 2, \ldots, n\} \) is a nonnegative integer. The degree of a monomial is defined as \( d = \sum_{k=1}^{n} d_k \). \( p(x(t)) \) is a polynomial if it can be expressed as a finite linear combination of monomials with real coefficients. If a polynomial \( p(x(t)) \) can be expressed as \( p(x(t)) = \sum_{j=1}^{m} q_j(x(t))^2 \), where \( m \) is a non-zero positive integer and \( q_j(x(t)) \) is a polynomial for all \( j \), then it means that the polynomial \( p(x(t)) \) is an SOS and \( p(x(t)) \geq 0 \). For a matrix \( H \in \mathbb{R}^{m \times m} \), where \( h_{rs} \) is the element of \( r \)-th row and \( s \)-th column of \( H \), the expressions of \( H \succ 0, H \succeq 0, H \prec 0, \) and \( H \preceq 0 \) mean that each element \( h_{rs} \) is positive, non-negative, negative and non-positive, respectively. If a matrix \( A \), its off diagonal elements are all non-negative, that is \( A = (a_{rs}), r, s \in \{1, 2, \ldots, n\} \), where \( a_{rs} \) denotes the \( (r, s) \)-th element of \( A \) and \( a_{rs} \geq 0 \) for all \( r \neq s \), then it is called a Metzler matrix [46]. The transpose of a real matrix \( R \) is defined as \( R^T \). The expression \( Q(x) = \text{diag}\{x_1, x_2, \ldots, x_n\} \) means that the matrix \( Q(x) \) is a diagonal matrix whose diagonal elements are \( x_1, x_2, \ldots, x_n \). \( \mathbb{R}_+^n \) stands for the nonnegative orthant in Euclidean space. With the third-party MATLAB toolbox SOSTOOLS, SOS programmes can be solved [47].

B. Positive Polynomial Fuzzy Model

A positive polynomial fuzzy model with \( p \) rules is employed to represent the dynamics behaviour of the nonlinear plant. We define the \( i \)-th rule of the positive polynomial fuzzy model for the nonlinear plant as follows:

**Rule i:**

IF \( f_i(x(t)) = M_i^x \) AND \( \ldots \) AND \( f_p(x(t)) = M_p^x \)

THEN \( \dot{x}(t) = A_i(x(t))x(t) + B_i(x(t))u(t) \),

\( x(t) = \psi(t), \) \hspace{1cm} (1)

where, for \( l \in \{1, 2, \ldots, \Psi\}, \Psi \) is a positive integer, and \( i \in \{1, 2, \ldots, p\}, M_i^x \) is the fuzzy set of the \( i \)-th rule corresponding to the function \( f_i(x(t)): f_i(x(t)) \) is the premise variable; \( \psi(t) \) is the vector valued initial function; \( A_i(x(t)) \in \mathbb{R}^{n \times n} \) and \( B_i(x(t)) \in \mathbb{R}^{n \times m} \) are polynomial system and input matrices respectively; \( x(t) \in \mathbb{R}^{n \times 1} \) is the system state vector; \( u(t) \in \mathbb{R}_+^{n \times 1} \) is the input vector.

The system dynamics and its output are described by

\[
\dot{x}(t) = \sum_{i=1}^{p} w_i(x(t))(A_i(x(t))x(t) + B_i(x(t))u(t)),
\]

\[
y(t) = Cx(t),
\] \hspace{1cm} (2)

where

\[
w_i(x(t)) = \frac{\prod_{l=1}^{\Psi} \mu_{M_i^x}(f_l(x(t)))}{\sum_{l=1}^{p} \prod_{l=1}^{\Psi} \mu_{M_i^x}(f_l(x(t)))} \in [0, 1], \forall i,
\]

\[
\sum_{i=1}^{p} w_i(x(t)) = 1, w_i(x(t)) \geq 0, \forall i,
\] \hspace{1cm} (3)

\( w_i(x(t)) \) is the normalized grade of membership; \( \mu_{M_i^x}(f_l(x(t))) \) is the grade of membership corresponding to the fuzzy term \( M_i \); \( y(t) \in \mathbb{R}^{1 \times 1} \) is the output vector; \( C = [c_1, c_2, \ldots, c_n] \in \mathbb{R}^{n \times 1} \) is an output matrix, where \( c_s \in \mathbb{R}_+^{1 \times 1}, s \in \{1, 2, \ldots, n\} \) is the \( s \)-th column vector of matrix \( C \).

**Remark 3:** The output matrix \( C \) is considered as a constant matrix here, which is conservative to some extent, but the output matrix \( C \) is not limited to be constant matrix. If the output equation in (4) is nonlinear [21], that is, \( y(t) = \sum_{i=1}^{p} h_i(x(t))x(t) \), when \( r = 1 \), it will be simplified to the form in this paper, so nonlinear output equations may be less conservative. Since this is not easy to be realized, it is only conceptually be formulated like this but the analysis is challenging and more efforts are needed for this problem.

**Definition 1** ([46]): A nonlinear system is deemed to be positive if the initial condition \( \psi(\cdot) \not\equiv 0 \) holds and the corresponding state trajectory \( x(t) \not\equiv 0 \) for all \( t \geq 0 \) is satisfied.

**Lemma 1** ([48]): A system is said to be controlled positive if the polynomial system, output and input matrices satisfy the conditions that \( A_i(x(t)) \) is a Metzler matrix, \( C \not\equiv 0 \) and \( B_i(x(t)) \not\equiv 0, \forall i \) when \( u(t) \not\equiv 0 \).

**Assumption 1:** It is assumed that there exists an exact polynomial fuzzy model to represent the nonlinear system, i.e., the polynomial fuzzy model is constructed by sector nonlinearity method [49].

C. Positive Polynomial Reference model

A stable positive polynomial reference model is defined as follows

\[
\dot{x}_r(t) = A_r(x_r(t))x_r(t) + B_r(x_r(t))r(t),
\] \hspace{1cm} (6)

\[
y_r(t) = Cx_r(t),
\] \hspace{1cm} (7)

where \( A_r(x_r(t)) \in \mathbb{R}^{n \times n} \) and \( B_r(x_r(t)) \in \mathbb{R}^{n \times m} \) are polynomial system and input matrices respectively; \( x_r(t) \in \mathbb{R}^{n \times 1} \) is the state vector of the reference model, which needs to be followed by the positive polynomial fuzzy model (3); \( r(t) \in \mathbb{R}^{n \times 1} \) is the reference input vector; \( y_r(t) \in \mathbb{R}_+^{n \times 1} \) is the output vector of the reference model.

**Remark 2:** Subject to Lemma 1, the reference model is of positive, if the following conditions are met: \( A_r(x_r(t)) \) is a Metzler matrix, \( C \not\equiv 0 \) and \( B_r(x_r(t)) \not\equiv 0 \) when \( r(t) \not\equiv 0 \).

**Remark 3:** The reference model is not limited to the form of positive polynomial system, it can be extended to a positive polynomial fuzzy model. However, when positive polynomial fuzzy model is used, the analysis has to be revised accordingly. This regardless of which reference model is chosen, it is required to make sure that the reference model is stable.
D. Static Output Feedback Polynomial Fuzzy Controller

In order to drive the states of the nonlinear plant represented by the positive polynomial fuzzy model (3) to track those in the reference model (6), a static output feedback polynomial fuzzy controller is employed. The error between the system states and reference states is defined as follows

\[ e(t) = x(t) - x_r(t). \]  
(8)

From (4), (7) and (8), the output error is defined as follows

\[ e_y(t) = y(t) - y_r(t) = Ce(t). \]  
(9)

A static output feedback polynomial fuzzy controller is described by the following c rules, the j-th rule of the polynomial fuzzy controller is presented as the following format

Rule \( j \):

IF \( g_l(y(t)) \) is \( N^j_l \) AND \( \cdots \) AND \( g_M(y(t)) \) is \( N^j_M \)

THEN \( u(t) = F_j(h(t))v_{e_y}(t) + G_j(h(t))v_{y_r}(t) \),
(10)

where, for \( l \in \{1,2,\ldots,M\} \), \( \Omega \) is a positive integer, \( j \in \{1,2,\ldots,c\} \). \( N^j_l \) is the fuzzy set of the \( j \)-th rule corresponding to the function \( g_l(y(t)) \); \( g_l(y(t)) \) is the premise variable. Define \( h(t) = (y(t),y_r(t)) \) and a user-selected non-zero constant vector \( v \in \mathbb{R}^{1 \times l} \) that satisfies \( vC = \mathcal{C} \succeq 0 \) and \( \mathcal{C} \in \mathbb{R}^{1 \times n} \). \( F_j(h(t)) \in \mathbb{R}^{m \times 1} \) and \( G_j(h(t)) \in \mathbb{R}^{m \times 1} \) are the static output polynomial feedback gains to be determined. The static output feedback polynomial fuzzy controller is defined as follows

\[ u(t) = \sum_{j=1}^{c} m_j(y(t))(F_j(h(t))v_{e_y}(t) + G_j(h(t))v_{y_r}(t)), \]  
(11)

where

\[ m_j(y(t)) = \prod_{l=1}^{M} \mu_{N^j_l}(g_l(y(t))) \]  
\[ \sum_{k=1}^{M} \prod_{l=1}^{M} \mu_{N^j_l}(g_l(y(t))) \in [0, 1], \forall j, \]

\[ \sum_{j=1}^{c} m_j(y(t)) = 1, m_j(y(t)) \geq 0, \forall j, \]  
(12)

\[ \mu_{N^j_l}(g_l(y(t))) \] is the grade of membership corresponding to the fuzzy term \( N^j_l \); \( m_j(y(t)) \) is the normalized grade of membership.

III. Stability Analysis

The objective of tracking control is to design a static output feedback polynomial fuzzy controller (11) to drive the states of the nonlinear plant represented by the positive polynomial fuzzy model (3), (4) as close as possible to those of the stable positive polynomial reference model (6), (7) and satisfy the tracking performance under the premise that the static output feedback PPFMB control system remains positive and stable.

In the following analysis, we will employ \( H_\infty \) index to describe the degree of tracking error \( e(t) \), and to improve the performance of the tracking control system in the sense of \( H_\infty \) performance. An appropriate static output feedback polynomial fuzzy controller can effectively reduce the state error \( e(t) \) (as small as possible) between the states of the positive polynomial fuzzy model and reference system, which means that \( H_\infty \) control performance can be improved.

For brevity, in the following analysis, the time \( t \) associated with the variables is dropped for the situation without ambiguity, \( x(t), u(t) \) and \( e(t) \) are denoted as \( x, u \) and \( e \), respectively. Furthermore, \( u_i(x(t)) \) and \( m_j(y(t)) \) are denoted as \( u_i \) and \( m_j \), respectively.

A. Basic Stability Analysis With \( H_\infty \) Performance

Considering the positive polynomial fuzzy model (3) and the static output feedback polynomial fuzzy controller (11), by using the properties of the membership functions (5) and (12), i.e., \( \sum_{i=1}^{p} \sum_{j=1}^{c} w_i m_j = 1 \), we get the static output feedback PPFMB control system as follows

\[ \dot{x} = \sum_{i=1}^{p} \sum_{j=1}^{c} w_i m_j \left( A_i(x) u_i + B_i(x)(F_j(h)vC)\right) \]  
(13)

From (7), (8) and (9), (13) is reformulated as follows

\[ \dot{x} = \sum_{i=1}^{p} \sum_{j=1}^{c} w_i m_j \left( A_i(x) + B_i(x)F_j(h)vC \right) x + B_i(x)(G_j(h) - F_j(h)vC)x_r. \]  
(14)

To ensure the positivity of PPFMB control system (14), by Lemma 1, it is required that \( A_i(x) + B_i(x)F_j(h)vC \) is a Metzler matrix and \( B_i(x)(G_j(h) - F_j(h)vC \succeq 0, \) for all \( i \) and \( j \), which are represented element-wise as follows

\[ a_{irs}(x) + b_{irs}(x)F_j(h)vC \geq 0, \]  
(15)

For further analysis, we consider the state error vector (8), from (6) and (14), the dynamics equation of \( \dot{e} \) is obtained as follows

\[ \dot{e} = \dot{x} - \dot{x}_r \]  
(16)

\[ = \sum_{i=1}^{p} \sum_{j=1}^{c} w_i m_j \left( A_i(x) + B_i(x)F_j(h)vC \right) e + \sum_{i=1}^{p} \sum_{j=1}^{c} w_i m_j \left( A_i(x) - A_i(x_r) + B_i(x)G_j(h)vC \right) x_r \]  

In the following analysis, due to the existence of non-convex issues in both positive conditions and stability analysis, some mathematical tricks are employed to convert non-convex stability conditions into convex ones.

Inspired by the literature [31], a nonsingular transformation matrix \( \Gamma \in \mathbb{R}^{n \times n} \) is adopted as follows

\[ \Gamma = \left[ C^T \mathcal{C} C^T \right]^{-1} \text{ortc}(C^T), \]  
(17)

where \( \text{ortc}(C^T) \in \mathbb{R}^{n \times (n-1)} \) denotes the orthogonal complement of \( C^T \), from (17), it is acquired that

\[ \Gamma \mathcal{C} = \left[ \begin{array}{c} \mathcal{C}^T \\ \mathcal{C}^T \end{array} \right], \]  
(18)
where $1$ is a constant scalar and $0_{n-1}$ is a all-zero row vector of $n - 1$ elements.

Define $0 < X = X^T \in \mathbb{R}^{n \times n}$. We choose
\[ X = \begin{bmatrix} x_{11} & 0 \\ \ast & X_{22} \end{bmatrix}, \] (19)
where $x_{11} \in \mathbb{R}^{1 \times 1}$, $X_{22} \in \mathbb{R}^{(n-1) \times (n-1)}$.

The formulated conditions are non-convex that convex programming techniques cannot be directly to find a feasible solution. In the following, a novel Scalar Implantation Transformation (SIT) technique is proposed to approximate the non-convex conditions with convex ones. Define a new scalar to be determined that satisfies the following conditions
\[ h_1 = CTX(C\Gamma)^T = x_{11}. \] (20)

The polynomial feedback gains are defined as follows
\[ F_j(h)h_1 = M_j(h), \quad G_j(h)h_1 = N_j(h), \] (21)
where $M_j(h) \in \mathbb{R}^{m \times 1}$ and $G_j(h) \in \mathbb{R}^{m \times 1}$ are polynomial matrices.

According to (14), (20) and (21), multiply $h_1$ to the left side of (15) to obtain the following: $a_{ir}(x)h_1 + b_{ir}(x)M_j(h)v_c$ and $b_{ir}(x)(N_j(h) - M_j(h))v_c$, respectively.

Lemma 2 ([50]): According to Lemma 1, the system (14) is controlled positive for $\psi(\cdot) \geq 0$, if the following conditions are met: $a_{ir}(x)h_1 + b_{ir}(x)M_j(h)v_c \geq 0$, $r \neq s$, for all $i$ and $j$; $b_{ir}(x)(N_j(h) - M_j(h))v_c \geq 0$, for all $i$ and $j$.

In order to analyze the stability conditions by applying Lyapunov stability theory, referring to (16), with (18), (20) and (21), the derivative of state error $\dot{e}$ in (16) can be reformulated as follows
\[ \dot{e} = \sum_{i=1}^{p} \sum_{j=1}^{c} w_{ij}m_j(A_i(x)\Gamma X + B_i(x)M_j(h)C\Gamma)X^{-1}\Gamma^{-1}e + \sum_{i=1}^{p} \sum_{j=1}^{c} w_{ij}m_j(A_i(x)\Gamma X - A_r(x_r)\Gamma X + B_i(x)N_j(h)C\Gamma)X^{-1}\Gamma^{-1}x_r - B_r(x_r)r \]
\[ = \sum_{i=1}^{p} \sum_{j=1}^{c} w_{ij}m_j \Phi_{ij}(x, x_r), \] (22)
where $\Phi_{ij}(x, x_r) = [\Phi_{ij}^{(1)}(x, x_r), \Phi_{ij}^{(2)}(x, x_r), \Phi_{ij}^{(3)}(x, x_r)]$.

\[ \Phi_{ij}^{(1)}(x, x_r) = A_i(x)\Gamma X, \quad \Phi_{ij}^{(2)}(x, x_r) = B_i(x)M_j(h)C\Gamma; \]
\[ \Phi_{ij}^{(3)}(x, x_r) = -B_r(x_r), \quad \text{and} \quad Z^T = [Z_1^T Z_2^T Z_3^T]. \]

The system stability of the error system (22) is studied. The following Lyapunov function candidate is chosen to investigate the basic stability conditions
\[ V(t) = \Lambda^T(t)X^{-1}A(t). \] (23)

In order to facilitate the stability analysis, in the following, we define
\[ \Lambda(t) = \Gamma^{-1}e, \] (24)

\[ L_1 = [I_n 0 0], \quad L_2 = [0 I_n 0], \quad L_3 = [0 0 I_m]. \] (25)

Based on the Lyapunov stability theory, if $V(t) > 0$ (equality holds when $e = 0$) and $\dot{V}(t) < 0$, then we can conclude that the error system in (22) is asymptotically stable. From the above analysis, it is obvious that $V(t)$ is a positive function, and $\dot{V}(t)$ can be obtained in the following form
\[ \dot{V}(t) = \dot{A}^T(t)X^{-1}A(t) + A^T(t)X^{-1}\dot{A}(t) \]
\[ = \sum_{i=1}^{p} \sum_{j=1}^{c} w_{ij}m_j(Z^T(\Phi_{ij}^{(1)}(x, x_r)(\Gamma^{-1})^T L_1 + L_1^T(\Gamma^{-1} \Phi_{ij}^{(2)}(x, x_r))Z) \]
\[ = \sum_{i=1}^{p} \sum_{j=1}^{c} w_{ij}m_j Z^T \Xi(x, x_r) Z, \] (26)

where $\Xi(x, x_r) = \Phi_{ij}^{(1)}(x, x_r)(\Gamma^{-1})^T L_1 + L_1^T(\Gamma^{-1} \Phi_{ij}^{(2)}(x, x_r))$.

Hereinafter, the $H_{\infty}$ performance will be used for the analysis to address the tracking performance. Since the reference model (6), (7) is a stable positive system, the reference state $x_r$ and the reference input $r$ are bounded. In the following analysis, this property will be employed to construct the following $H_{\infty}$ performance.

Define a new variable
\[ \Omega = \dot{V} + Z_1^T Z_1 - \sigma_1 Z_2^T Z_2 - \sigma_2 Z_3^T Z_3, \] (27)
where $\sigma_1$ and $\sigma_2$ are scalars to be determined.

From (27), we further obtain the following form
\[ \dot{V}(t) = \Omega - Z_1^T Z_1 + \sigma_1 Z_2^T Z_2 + \sigma_2 Z_3^T Z_3. \] (28)

Considering
\[ \Omega < 0, \] (29)
from (28), we have
\[ \dot{V}(t) \leq -Z_1^T Z_1 + \sigma_1 Z_2^T Z_2 + \sigma_2 Z_3^T Z_3. \] (30)

Define $t_f$ as the terminal time of control and taking integration on both sides of (30) with respect to time $t$ [31], [51], as follows
\[ \int_0^{t_f} \dot{V}(t)dt \leq \int_0^{t_f} (-Z_1^T Z_1 + \sigma_1 Z_2^T Z_2 + \sigma_2 Z_3^T Z_3)dt \]
\[ \rightarrow V(t_f) - V(0) \leq \int_0^{t_f} (-Z_1^T Z_1 + \sigma_1 Z_2^T Z_2 + \sigma_2 Z_3^T Z_3)dt \]
\[ \Rightarrow V(t_f) \leq \int_0^{t_f} (-Z_1^T Z_1 + \sigma_1 Z_2^T Z_2 + \sigma_2 Z_3^T Z_3)dt. \] (31)

Thus, the $H_{\infty}$ performance [31] regarding to tracking error can be expressed as
\[ \int_0^{t_f} (\sigma_1 Z_2^T Z_2 + \sigma_2 Z_3^T Z_3)dt \leq 1. \] (32)

According to (32), the system tracking error $e$ can be attenuated to the prescribed level by decreasing the two scalars $\sigma_1$ and $\sigma_2$ that will be determined.

Remark 4: In order to drive the nonlinear system formulated by the positive polynomial fuzzy model (3), (4) as far as
possible to track the state trajectory of the stable positive
reference system (6), (7) with satisfying the $H_\infty$ performance
in (32), the inequality (29) is necessary to be satisfied.

From (25), (26) and (27), we have
\[
\Omega = \dot{V} + \sum_{i=1}^{p} \sum_{j=1}^{c} w_{ij} Z^T \Theta(x, x_r) Z + (L_1) Z^T (L_1 Z) - \sigma_1 (L_2 Z)^T (L_2 Z) - \sigma_2 (L_3 Z)^T (L_3 Z) = \sum_{i=1}^{p} \sum_{j=1}^{c} w_{ij} Z^T \Theta(x, x_r) Z,
\]
where $\Theta(x, x_r) = \Xi(x, x_r) + L_1^T L_1 - \sigma_1 L_2^T L_2 - \sigma_2 L_3^T L_3$, and can be expressed in matrix form as follows
\[
\Theta(x, x_r) = \begin{bmatrix}
\phi_{11} & \phi_{12} & \phi_{13} \\
* & \phi_{22} & \phi_{23} \\
* & * & \phi_{33}
\end{bmatrix},
\]
(33)

where $\phi_{11} = \Gamma^{-1} \Phi_{1}(x, x_r) + (\Gamma^{-1} \Phi_{1}(x, x_r))^T + I$; $\phi_{12} = \Gamma^{-1} \Phi_{2}(x, x_r)$; $\phi_{13} = \Gamma^{-1} \Phi_{3}(x, x_r)$; $\phi_{22} = -\sigma_1 I$; $\phi_{23} = 0$; $\phi_{33} = -\sigma_2 I$.

The inequality (29) holds, if (33) < 0 is satisfied, which

Combined with Lemma 2, the positivity and stability analy-

**Theorem 1**: A PPFMB tracking control system that satisfies

Lemma 2 is considered, which is formed by a nonlinear plant

**Remark 5**: Since $h_1 = x_{11}$, and the decision matrix

variables $X$ of form (19) satisfies $X > 0$, then the scalar $h_1 > 0$

must be guaranteed: $a_{ir}(x) h_1 + b_{ir}(x) M_j(h)v C_s; \forall r \neq s, i, j$; $v^T (b_{ir}(x)/(N_j(h) - M_j(h))v C_s - e_2(x, x_r)) I v$ is used to ensure that the static output feedback PPFMB control system (14) satisfies Lemma 2, which means that the PPFMB control system (14) is a positive system.

**B. Membership-Function-Dependent Relaxed Stability Condi-

In the previous section, the stability conditions in Theorem

1 are MFI, in which the information of membership functions

is not engaged and thus it usually leads to conservative results.

In this section, the sub-domain information, the output-

state boundary information and the approximate function of

$w_i(x) m_j(y)$ using MFD analysis techniques are considered for relaxing the stability analysis results.

Since $w_i(x)$ and $m_j(y)$ are determined by different vari-

ables, i.e. $x$ and $y$, respectively, therefore, when using the

MFD stability analysis technique, it is difficult to analyze and

relax the stability conditions. As the stability condition (29)

must be satisfied for all the product term $w_i(x)m_j(y)$ and both

$x$ and $y$ are continuous variables, there is an infinite number

of stability conditions, consequently, it is not practical to find a feasible solution. In this section, we propose a stability analysis method which approximates the infinite number of stability conditions by finite ones.

In the analysis, the membership functions will first be

approximated by polynomial functions [52], [53]. In general,

in order to obtain a better approximation, a higher order

approximate polynomial function may be required in most

papers, which will mean a higher computational burden

is needed to find a feasible solution. Hence, in this paper, we

first divide the whole operation domain of the premise variable

into $D$ connected sub-domain, each of which is defined as $s_r,

$\tau \in \{1, 2, \ldots, D\}$, and then in each sub-domain $s_r$, we use the

nonlinear least squares data fitting method to obtain the relatively

simpler lower degrees polynomial function $h_{ijs}(x, y)$

and use it as the polynomial approximation function of

the product term $(w_i(x) m_j(y))_{s_r}$.

**Remark 6**: The function $h_{ijs}(x, y)$ can be a constant, linear function, or polynomial function of $x$ (because $y = Cx$, $m_j(y)$ is actually a function of $x$), so $h_{ijs}(x, y)$ can be rewritten as $h_{ijs}(x)$. However, the more complex the approximation function is, the stronger the computational power and the heavier the computational burden will be, therefore, it is important to make a trade-off between the relaxation and the computational burden when the stability conditions are analyzed.

To introduce state boundary information [19] of $x$ to relax the stability conditions, a scalar function $f_{s_r}(x)$ with the following form is defined to indicate whether $x$ is in the active/inactive sub-domains,

\[
f_{s_r}(x) = (x - x_{\text{min}, s_r})^T \Gamma (x_{\text{max}, s_r} - x),
\]

\[
\begin{cases}
(f_r = 1, e_r = 1) & \forall x \in s_r, \\
(f_r = 0, e_r = 0) & \forall x \notin s_r,
\end{cases}
\]

(36)
where $E = \text{diag}\{e_1, e_2, \ldots, e_n\} \in \mathbb{R}^{n \times n}$, $e_r \in \{0, 1\}$. If the state boundary information of $x$ is contained in the operating sub-domain $s_r$, then $f_r = 1$ and $e_r = 1$ are satisfied, meanwhile the scalar function will satisfy $f_s(x) \geq 0$, otherwise $f_s = 0$ and $e_r = 0$ are satisfied and the scalar function will satisfy $f_s(x) < 0$.

Associated with the approximation functions $h_ijs_r(x)$, the constant approximation error is defined as follows

$$\Delta h_{ijs_r}(x) = w_i(x)m_j(y) - h_ijs_r(x), \forall i, j, s_r,$$  \hspace{1cm} (37)

where $\Delta h_{ijs_r}(x)$ is the error term for all $i, j$ and $s_r$. In order to reduce the conservativeness in the stability conditions obtained by Theorem 1, we consider the information of error term $\Delta h_{ijs_r}(x)$ and the product term of membership functions in each sub-domain $(w_i(x)m_j(y))_{s_r}$, thus, more information of membership functions is given and will be introduced into the stability analysis by considering their upper bounds and lower bounds given as follows

$$\gamma_{ijs_r} \leq \Delta h_{ijs_r}(x) \leq \tau_{ijs_r},$$  \hspace{1cm} $0 \leq \delta_{ijs_r} \leq (w_i(x)m_j(y))_{s_r} \leq \delta_{ijs_r} \leq 1,$$ \hspace{1cm} (38)

where $\gamma_{ijs_r}$ and $\tau_{ijs_r}$ are the upper bound and lowerbound of the error term $\Delta h_{ijs_r}(x)$ in each sub-domain, $\delta_{ijs_r}$ and $\delta_{ijs_r}$ are the lower bound and upper bound of the product term of membership functions $(w_i(x)m_j(y))_{s_r}$ in each sub-domain, all the terms to be determined can effectively help capture the position and shape information of the membership functions.

To facilitate the stability analysis, we introduce the slack polynomial matrices, which satisfies

$$0 < Y_{1ijs_r}(x, x_r) = Y_{1ijs_r}(x, x_r)^T \in \mathbb{R}^{(2n+m) \times (2n+m)},$$

$$0 < Y_{2ijs_r}(x, x_r) = Y_{2ijs_r}(x, x_r)^T \in \mathbb{R}^{(2n+m) \times (2n+m)},$$ \hspace{1cm} (40)

Recalling the property in (39) and (40), the following inequalities can be obtained

$$\left((w_i(x)m_j(y))_{s_r} - \delta_{ijs_r}\right)Y_{1ijs_r}(x, x_r) \geq 0,$$

$$\left(\delta_{ijs_r} - (w_i(x)m_j(y))_{s_r}\right)Y_{2ijs_r}(x, x_r) \geq 0.$$ \hspace{1cm} (41)

Considering (29) in each of these sub-domains, based on the properties given in (36) and (41), (33) can be further relaxed as follows

$$\sum_{i=1}^{p} \sum_{j=1}^{c} w_i m_j \Theta(x, x_r) \leq \sum_{r=1}^{D} \sum_{i=1}^{p} \sum_{j=1}^{c} f_r \left((w_i(x)m_j(y))_{s_r} - \delta_{ijs_r}\right)Y_{1ijs_r}(x, x_r)$$

$$+ \left(\delta_{ijs_r} - (w_i(x)m_j(y))_{s_r}\right)Y_{2ijs_r}(x, x_r)$$

$$= \sum_{r=1}^{D} \sum_{i=1}^{p} \sum_{j=1}^{c} f_r \left((w_i(x)m_j(y))_{s_r} \left(\Theta(x, x_r) + Y_{1ijs_r}(x, x_r)\right) + Y_{1ijs_r}(x, x_r)\right)$$

$$- Y_{2ijs_r}(x, x_r) - \delta_{ijs_r}Y_{1ijs_r}(x, x_r) + \delta_{ijs_r}Y_{2ijs_r}(x, x_r),$$ \hspace{1cm} (42)

In addition, to facilitate the stability analysis, we introduce the slack polynomial matrix, defining $0 < Y_{3ijs_r}(x, x_r) = Y_{3ijs_r}(x, x_r)^T \in \mathbb{R}^{(2n+m) \times (2n+m)}$ and

$$Y_{3ijs_r}(x, x_r) \geq \left(\Theta(x, x_r) + Y_{1ijs_r}(x, x_r) - Y_{2ijs_r}(x, x_r)\right).$$

Then, the following inequalities can be obtained

$$\sum_{r=1}^{D} \sum_{i=1}^{p} \sum_{j=1}^{c} f_r \left((\gamma_{ijs_r} - \tau_{ijs_r})Y_{3ijs_r}(x, x_r) - \delta_{ijs_r}\right)Y_{1ijs_r}(x, x_r)$$

$$+ \delta_{ijs_r}Y_{2ijs_r}(x, x_r) + \left(h_{ijs_r}(x) + \gamma_{ijs_r}\right)$$

$$\left(\Theta(x, x_r) + Y_{1ijs_r}(x, x_r) - Y_{2ijs_r}(x, x_r)\right).$$ \hspace{1cm} (43)

In the following, the output-state boundary information contained in each sub-domain $s_r$ is considered in the stability analysis. From (36), it can be seen that whether the system state $x$ is working in the operating sub-domain $s_r$. Since $y = Cx$, similarly, we have

$$f_s(x) = (y - y_{min,s_r})^T \Theta(y_{max,s_r} - y),$$

$$\begin{cases} f_s(x) \geq 0, & \forall y \in s_r, \\ f_s(x) < 0, & \forall y \notin s_r. \end{cases}$$ \hspace{1cm} (44)

In order to bring the output-state boundary information $f_s(x)$ in (44) to relax the stability condition (43), we define a polynomial slack matrix $P_{s_r}(y)$ that satisfies $0 < P_{s_r}(y) = P_{s_r}(y)^T \in \mathbb{R}^{(2n+m) \times (2n+m)}$, which is considered in (43), we obtain the following form

$$\sum_{r=1}^{D} \sum_{i=1}^{p} \sum_{j=1}^{c} \left(\gamma_{ijs_r} - \tau_{ijs_r}\right)Y_{3ijs_r}(x, x_r) - \delta_{ijs_r}$$

$$Y_{1ijs_r}(x, x_r) + \delta_{ijs_r}Y_{2ijs_r}(x, x_r) + \left(h_{ijs_r}(x) + \gamma_{ijs_r}\right)$$

$$\left(\Theta(x, x_r) + Y_{1ijs_r}(x, x_r) - Y_{2ijs_r}(x, x_r)\right).$$ \hspace{1cm} (45)

In order to satisfy (29), we need (45) to be negative definite in every sub-domain. In this way, the system stability can be guaranteed, and the static output feedback polynomial fuzzy controller defined in (11) is able to drive the states of the nonlinear plant represented by the positive polynomial fuzzy model (3), (4) to follow those of the stable positive polynomial reference model (6), (7) subject to the prescribed $H_\infty$ performance in (32). The MFD stability analysis results are summarized in the following theorem.

**Theorem 2:** Consider a PPFMB tracking control system with satisfying the conditions in Lemma 2, which is formed by a nonlinear plant described by the polynomial fuzzy model (3), (4) and the static output feedback polynomial fuzzy controller (11) connected in a closed loop. Its system states are driven
to follow the state trajectory of the stable reference model (6), (7) subject to the $H_\infty$ performance (32) characterized by the scalars $\sigma_1 > 0$ and $\sigma_2 > 0$, if there exist decision matrix variables $P_{x_s}(y) = P_{x_s}(y)^T \in \mathbb{R}^{(2n+m) \times (2n+m)}$, $Y_{1ij_s}(x, x_r) = Y_{1ij_s}(x, x_r)^T \in \mathbb{R}^{(2n+m) \times (2n+m)}$, $Y_{2ij_s}(x, x_r) = Y_{2ij_s}(x, x_r)^T \in \mathbb{R}^{(2n+m) \times (2n+m)}$, $Y_{3ij_s}(x, x_r) = Y_{3ij_s}(x, x_r)^T \in \mathbb{R}^{(2n+m) \times (2n+m)}$, $X = X^T \in \mathbb{R}^{n \times n}$ in the form of (19) and $M_j(h) \in \mathbb{R}^{m \times 1}$, $N_j(h) \in \mathbb{R}^{m \times 1}$, $j \in \{1, 2, \ldots, c\}$, and a user-chosen non-zero constant vector $v \in \mathbb{R}^{1 \times l}$ with satisfying $vC = \mathbb{C} \geq 0$ such that following the GEVP is feasible:

Minimize the $H_\infty$ to follow the state trajectory of the stable reference model can be obtained by the TP model transformation method [54]–[57].

**Remark 8:** The control design procedure and parameter selection guide mainly include the following steps: 1) Construct a polynomial fuzzy system for the nonlinear system, which are characterized by the number of rules (“p”), membership functions $\psi_i(x)$ and the matrices $A_i(x), B_i(x)$ and $C$. 2) A stable positive polynomial reference model is adopted, by choosing the matrices $A_r(x_r), B_r(x_r)$ and $C$ and vector r. 3) Design a static output feedback polynomial fuzzy controller in the form of (11) by selecting the number of rules (“c”), and membership functions $m_j(y)$. 4) Obtain the approximated membership function $\tilde{h}_{ij_s}(x)$ for $(\psi_i(x)m_j(y))_c$, say, using the nonlinear least squares data fitting method. According to equations (37), (38) and (39) in the paper, all corresponding terms are obtained. 5) Find a feasible solution to the positive stability conditions through numerically solving the relevant theorems using convex programming software.

**IV. SIMULATION EXAMPLE**

A simulation example is given to demonstrate and verify the effectiveness of the proposed SOS-based control conditions in the tracking control strategy. A PPFMB tracking control system is equipped with the static output feedback polynomial fuzzy controller to track the states of the positive polynomial stable reference model. Consider a three-rule positive polynomial fuzzy model with the following matrices $X = [x_1 \ x_2]^T$.

\[
\begin{align*}
A_1(x_1) &= \begin{bmatrix}
-1.18 + x_1^2 + 0.28x_1 & 0.46 \\
6.72 & -13.63
\end{bmatrix}, \\
A_2(x_1) &= \begin{bmatrix}
0.85 + x_1^2 & 1.25 \\
4.59 & -12.6
\end{bmatrix}, \\
A_3(x_1) &= \begin{bmatrix}
-1.51 + x_1^2 & 0.35 \\
7.21 & -10.86
\end{bmatrix}, \\
B_1(x_1) &= \begin{bmatrix}
0.63 + 0.46x_1^2 & 0 \\
0.85 + 0.46x_1^2 + 0.018x_1 & 0.26
\end{bmatrix}, \\
B_2(x_1) &= \begin{bmatrix}
0.42 + 0.46x_1^2 & 0 \\
0 & 0.26
\end{bmatrix}
\end{align*}
\]

$v = [1, 0]$, $C = [1 \ 0]$.

The positive reference model is selected as follows:

\[
A_r = \begin{bmatrix}
-1.2 & 0.45 \\
6.72 & -11.8
\end{bmatrix}, \quad B_r = \begin{bmatrix}
1 \\
0
\end{bmatrix}
\]

and $r(T) = 2.1 + 1.89 \sin(0.57t)$.

In these matrices, we can see that the system matrices $A_i(x_1), i \in \{1, 2, 3\}$ are Metzler matrices; all elements of the input matrices $B_i(x_1), i \in \{1, 2, 3\}$ and the output matrix $C$ are non-negative. $v \in \mathbb{R}^{1 \times l}$ is a user-chosen non-zero constant vector and chosen arbitrarily, which is satisfied with $vC = \mathbb{C} = [1 \ 0] \geq 0$. The reference model (6), (7) is required to be positive by satisfying the following conditions:
In Fig. 1, membership functions of positive polynomial fuzzy model: \( w_1(x_1) \) (left \( z \) shape in solid line), \( w_2(x_1) \) (Gaussian shape in solid line), \( w_3(x_1) \) (right \( s \) shape in solid line). Membership functions of the static output feedback polynomial fuzzy controller: \( m_1(x_1) \) (left \( z \) shape in dotted line), \( m_2(x_1) \) (right \( s \) shape in dotted line).

The system matrix of the reference model \( A_r \) is Metzler matrix; all elements of the input matrix and the output matrix of the reference model \( B_r \) and \( C \) are non-negative, and \( r \in \mathbb{R}_+^n \).

Under the imperfect premise matching design [19, 42, 44, 45], a two-rule static output-feedback polynomial fuzzy controller in the form (11) is used to achieve better tracking control performance. As the output in this example is \( y = Cx = x_1 \), we will use \( x_1 \) instead of \( y \). Considering that the positive polynomial fuzzy model working under \( x_1 \in [0, 20] \), the membership functions of the positive polynomial fuzzy model and static output-feedback polynomial fuzzy controller are selected as follows: \( w_1(x_1) = \frac{1}{1 + e^{-1(x-10)}} \), \( w_2(x_1) = 1 - w_1(x_1) \), \( w_3(x_1) = \frac{1}{1 + e^{-1(x-10)}} \), and \( m_1(x_1) = 1 - \frac{1}{1 + e^{-1(x-10)}} \), \( m_2(x_1) = 1 - m_1(x_1) \), which are shown in Fig. 1.

In this example, the whole operation domain of the premise variable \( x_1 \) is divided into \( D \) connected sub-domains, when Theorem 2 is applied, the number of sub-domains is selected differently, i.e., taking \( D = 7 \) as case 1 and \( D = 15 \) as case 2. For both cases, the boundaries of the operation domain partition is shown in the Table I. By using nonlinear least square data fitting method, in each sub-domain, we obtain the corresponding simpler lower degrees polynomial approximation function \( h_{ij,s}(x) \) of the product term \( w_i(x)m_j(y) \), and according to (37), the error term \( \Delta h_{ij,s}(x) \) can be obtained, meanwhile, the upper bound \( (\gamma_{ij,s}) \) and lower bound \( (\gamma_{ij,s}) \) of the error term in each sub-domain satisfying (38) and the upper bound \( (\delta_{ij,s}) \) and lower bound \( (\delta_{ij,s}) \) of the membership function product terms in each sub-domain satisfying (39) are listed separately in Tables III-X in the Supplemental file. From (36), \( f_s(x_1) \) is expressed in the following form

\[
 f_s(x_1) = (x_1 - x_{min,s})(x_{max,s} - x_1), \quad \text{where the left and right boundaries of } x_1 \text{ in the operating sub-domain } s \text{ are expressed as } x_{min,s} \text{ and } x_{max,s}, \text{ respectively. For example, referring to the region } (0 \leq x_1 \leq 1.5) \text{ of Case 2 in Table I, the left and right boundary values of } x_1 \text{ are 0 and 1.5, respectively, namely } x_{min,s} = 0 \text{ and } x_{max,s} = 1.5 \text{ which gives } f_s(x_1) = (x_1 - 0)(1.5 - x_1). \]

For the purpose of comparing MFI conditions in Theorem 1 with MFD conditions in Theorem 2, both Theorem 1 and Theorem 2 are applied to the same positive polynomial fuzzy model, positive reference model and all other parameters.

Theorem 2 are applied to the same positive polynomial fuzzy model, positive reference model and all other parameters. We used the minimum values of \( \sigma_1 \) and \( \sigma_2 \) to study the tracking performance. To obtain the smallest values of \( \sigma_1 \) and \( \sigma_2 \), the summation of \( \sigma_1 \) and \( \sigma_2 \) can be set as the objective function in SOSTOOLS and further be minimized. By applying MFI conditions in Theorem 1, there is no feasible solution obtained. However, by applying MFD conditions of \( D = 15 \) in Theorem 2, we get the following results:

\[
 X = \begin{bmatrix}
 0.4845 & -0.0007238 \\
 -0.0007238 & 0.406 
\end{bmatrix},
\]

the feedback gains are obtained as \( F_1 = -27.6530, F_2 = -25.3539, G_1 = -2.1825, G_2 = -2.0626 \), the scalar \( h_1 = 0.4854 \), and the minimum values of \( \sigma_1 \) and \( \sigma_2 \) are obtained as 0.1002 and 0.1024, respectively. This reveals that the information of membership functions introduced effectively improves the relaxation of stability conditions.

In the case of \( D = 15 \), the static output feedback polynomial fuzzy controller is applied to tracking control with the initial conditions \( x(0) = [0 \ 0.2]^T \) and \( x_r(0) = [0 \ 0]^T \), the time response simulation results of state response and control signal are shown in Figs. 2-4 under \( \sigma_1 = 0.1002 \) and \( \sigma_2 = 0.1024 \), which indicates that the tracking errors are sufficiently small.

**Remark 9:** In the case of Theorem 2, compared to \( D = 15 \), with all other settings being the same, when the whole operation domain of the premise variable \( x \) is divided into 7 sub-domains \( (D = 7) \), no feasible solution can be found. This indicates that the more number of operation sub-domains is divided, the smaller the approximation error \( \Delta h_{ij,s}(x) \) of membership function is, which makes the stability conditions more easily satisfied.

**Remark 10:** The focus of this paper is to investigate the stability and improve the performance of the tracking control system in the sense of \( H_\infty \) performance with a given positive polynomial fuzzy model. With a given model, it can be shown that when more and more MFs information and state information are used, MFD stability analysis results will be progressively relaxed. It is also worth mentioning that the MFD theorem is developed based on the same Lyapunov function. This is to say that subject to the same fuzzy model, MFD theorems will offer more relaxed stability analysis results and better tracking performance. It also implies that when a better fuzzy model is used, MFD theorems will offer more relaxed analysis results compared with the existing MFI analysis results in the literature.
MFI and MFD frameworks. Based on the Lyapunov stability theory, the basic SOS-based stability conditions of MFI have been obtained. To reduce conservatism, relaxed SOS-based stability conditions of MFD also have been obtained by dividing the whole operation domain of the premise variable $x$ into $D$ connected sub-domain, then, all information including product terms of membership functions, approximation error terms of membership functions and output states boundary are introduced into each sub-domain. A numerical simulation example has been presented to demonstrate the effectiveness of the proposed PPFMB control design and strategy. In the future, as time-delay is often considered in practical control applications, the tracking control system based on time-delay SOS-based method will be studied and more novel MFD methods can be obtained.

### V. Conclusion

The stability and positivity under $H_{\infty}$ performance index of static output-feedback tracking control for PPFMB fuzzy system have been investigated. A static output-feedback polynomial fuzzy controller is designed to ensure the states of the nonlinear plant represented by the positive polynomial fuzzy model as close as possible to those of the stable reference mode, and the $H_{\infty}$ performance index is used to evaluate the tracking control performance. Furthermore, in order to make the control system more flexible and reduce the implementation cost, the concept of imperfect premise matching is employed and the stability analysis is performed under both MFI and MFD frameworks. Based on the Lyapunov stability theory, the basic SOS-based stability conditions of MFI have been obtained. To reduce conservatism, relaxed SOS-based stability conditions of MFD also have been obtained by dividing the whole operation domain of the premise variable $x$ into $D$ connected sub-domain, then, all information including product terms of membership functions, approximation error terms of membership functions and output states boundary are introduced into each sub-domain. A numerical simulation example has been presented to demonstrate the effectiveness of the proposed PPFMB control design and strategy. In the future, as time-delay is often considered in practical control applications, the tracking control system based on time-delay SOS-based method will be studied and more novel MFD methods can be obtained.


