New admissibility and admissibilization criteria for nonlinear discrete-time singular systems by switched fuzzy models

Jian Chen, Jinpeng Yu, Hak-Keung Lam, Fellow, IEEE

Abstract—Admissibility analysis and control synthesis for nonlinear discrete-time singular systems are considered in this paper. With regard to the type-1 and interval type-2 fuzzy singular systems, the partition of membership functions and scale transform are imposed, and new switched fuzzy systems which are equivalent to the original systems are established. A relaxed stability criterion is derived to ensure the admissibility of the system by using the piecewise Lyapunov function and singular value decomposition. Moreover, two classes of switched controllers are designed for the systems. One is for type-1 systems and the membership functions are consistent with those of the systems. The other can be applied to both of the fuzzy systems by introducing linear membership functions in each subregion. Two criteria are obtained to guarantee that the closed-loop systems are admissible. Several illustrative examples are provided to show the effectiveness of the developed methods.

Keywords: Type-1 fuzzy system, Interval type-2 fuzzy system, discrete-time singular system, admissibility, switched fuzzy controller.

I. INTRODUCTION

In practical systems, nonlinearities are very common [1]–[4]. Takagi and Sugeno [5] present the T-S fuzzy model which can express a wide class of nonlinear plants [6], [7]. The type-1 (T1) fuzzy models are originally proposed to approximate the nonlinear systems. The stability and stabilization of such models have been extensively investigated in recent years [8]. The well-known approach, named parallel distributed compensation (PDC) [5], [9], [10], is popular for controller design. It is conservative that the stability/stabilization conditions are described by a set of linear matrix inequalities (LMIs) [11]. The systems can be verified by finding a common positive definite matrix required in the Lyapunov function approach for all rules. Some results have been derived to reduce the conservatism. For continuous systems, a fuzzy weighting-dependent approach [12] is presented for $H_\infty$ filter design. In [13], [14], the line-integral function is used in fuzzy Lyapunov functions for a class of T-S fuzzy systems. For discrete-time systems, the Kronecker-product approach [15] and the piecewise Lyapunov function (PLF) [16] are proposed. More recently, relaxed conditions for controller design using delayed nonquadratic Lyapunov functions [17] and augmented multi-indexed matrix approach [18] have been proposed. In order to dispose of the parameter uncertainties in real-world application, the interval type-2 (IT2) fuzzy models are presented as an adequate methodology. The stabilization of IT2 systems can not be achieved by the PDC approach. Some new approaches are proposed by considering the information contained in the footprint of uncertainty [19]. The stability conditions can be further relaxed by considering the membership-function-dependent analysis. One approach is to adopt a piecewise-linear membership functions (PLMF) [20]–[24]. By the PLMF, the information of MFs can be good used and the relaxed conditions are obtained, although it requires that the MFs must be one-dimensional functions of the states.

For the controller design of fuzzy systems, if a single controller cannot achieve the control purpose, the switched controller can be considered. In [25], the authors divide the state space into two kinds of regions according to the MFs, and design the piecewise affine controller for continuous-time systems. In [26], [27], a switching polynomial fuzzy control scheme is proposed by using regional membership function information and switching Lyapunov function to facilitate the controller design. In [28], [29], more advanced switching polynomial fuzzy control schemes are proposed. Nevertheless, there are still several restrictions on the form of MFs.

Singular systems, also known as differential-algebraic or descriptor systems, have been an active field of research during the last few decades. The study of singular systems is much more complicated than state-space ones because two new aspects named normality and impulses (or causality for discrete-time systems) should be considered [30]. A great deal of results of admissibility and admissibilization have been obtained [31]. For discrete-time singular systems, the method in [32] studies the $H_\infty$ control problems via dynamic feedback controller. The results in [33], [34] give the bounded real lemma to study the controller design. In [35], the filtering and dissipative control are studied. A sufficient and necessary condition of $H_\infty$ controller design is proposed in [15]. In [36], [37], fuzzy singular models are introduced, and the robust stability is investigated. The $H_\infty$ filter design for T-S fuzzy systems is studied in [38], [39]. However, the results are usually implemented via the PDC approaches which are rather conservative. So far, the state feedback controller design for discrete-time singular fuzzy systems has not yet been fully
studied.

In this paper, we present a switched control scheme for discrete-time singular fuzzy systems (DSFS). We divided the operating space of MFs into several subregions. By scale transform in each subregion, a switched fuzzy system is obtained. The system is switched according to the given switching rule. Relaxed admissibility and admissibilization conditions are derived in the form of LMIs to guarantee the system to be admissible. Comparing with the existing achievements, the main contributions of the obtained results can be summed up as follows.

- By using the operating space partition and scale transform, the T1 and IT2 DSFS are transformed into switched models. The information of MFs can be exactly preserved in the switched models. There is no restriction on the MFs and it has a broader range of applications.
- For T1 DSFS, a fuzzy controller with the MFs similar to the systems is introduced. By using the PDC approach, the admissibility condition can be obtained. For T1 and IT2 DSFS, we present a linear MF based fuzzy controller. The admissibility of the closed-loop systems can be ensured by examining the admissibility conditions at each endpoint of the subregions.
- Thanks to the introduction of singular value decomposition, PLF, and the new auxiliary matrices, the obtained admissibility and admissibilization conditions are less conservative.

**Notation:** The notations are standard throughout this paper. \( I \) and \( 0 \) refer to the identity matrix and zero matrix, respectively. The superscript \(-1\) is the matrix inverse and \( T \) is the matrix transpose. * represents the symmetric matrix of the related matrix.

II. Problem Statement and Preliminaries

The DSFS representing the nonlinear systems are considered as follows.

\[
E x(k + 1) = \sum_{i=1}^{s} \psi_i(\theta(k)) (A_i x(k) + B_i u(k)),
\]

where \( x(k) = (x_1(k),...,x_n(k))^T \in \mathbb{R}^n, u(k) = (u_1(k),...,u_p(k))^T \in \mathbb{R}^p \) are the system state and the input of the system, respectively. The matrix \( E \) may be singular, and is assumed that \( \text{rank}(E) = r \leq n \). \( A_i \) and \( B_i \) are known constant system parameters with appropriate dimensions. \( s \) is the number of fuzzy rules. \( \psi_i(\theta(k)) \) is a measurable function of the \( k \). \( \psi_i(\theta(k)), i = 1,2,...,s \), are the normalized MFs with the characteristics of \( \psi_i(\theta(k)) \geq 0 \) for all \( i \), and \( \sum_{i=1}^{s} \psi_i(\theta(k)) = 1 \).

For T1 DSFS, the MFs \( \psi_i(\theta(k)) \) are completely known and certain. For IT2 DSFS, the MFs are uncertain and only the upper and lower MFs are given as \( \psi_{UL}(\theta(k)) \) and \( \psi_{UL}(\theta(k)) \), respectively. For \( i = 1,2,...,s \), we have

\[
\psi_{UL}(\theta(k)) \leq \psi_i(\theta(k)) \leq \psi_{UL}(\theta(k)),
\]

and

\[
\psi_i(\theta(k)) = \psi_{UL}(\theta(k)) \psi_{UL}(\theta(k)) + \psi_{UL}(\theta(k)) \psi_{UL}(\theta(k)),
\]

where \( \psi_{UL}, \psi_{UL} \in [0,1] \) and \( \psi_{UL}(\theta(k)) + \psi_{UL}(\theta(k)) = 1 \).

For the discrete-time singular system:

\[
E x(k + 1) = A x(k),
\]

the following definition will be adopted.

**Definition 1:** [40] The pair \((E,A)\) is said to be

(i) regular if \( \text{det}(zE - A) \) is not identically zero.

(ii) causal if \( \text{deg}(zE - A) = \text{rank}(E) \).

(iii) stable if for any scalar \( \epsilon > 0 \), there exists a scalar \( \delta(\epsilon) > 0 \) such that the solution \( x(k) \) to system (4) satisfies \( \|x(k)\| \leq \epsilon \) for any \( k \geq 0 \), moreover \( \lim_{k \to \infty} x(k) = 0 \).

System (4) is said to be admissible if the pair \((E,A)\) is regular, causal and stable.

Moreover, for the pair \((E,\Sigma_{i=1}^{s} \psi_i(\theta(k)) A_i)\), there must exist invertible matrices \(G\), and \(N\), such that

\[
\tilde{E} := GEN = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{A}_i = GA_i N = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},
\]

\[
\tilde{A} = G \sum_{i=1}^{s} \psi_i(\theta(k)) A_i N = \begin{bmatrix} \sum_{i=1}^{s} \psi_i(\theta(k)) GA_{11} N & \sum_{i=1}^{s} \psi_i(\theta(k)) GA_{12} N \\ \sum_{i=1}^{s} \psi_i(\theta(k)) GA_{21} N & \sum_{i=1}^{s} \psi_i(\theta(k)) GA_{22} N \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},
\]

\[
\tilde{x}(k) = \begin{bmatrix} \tilde{x}_1(k) \\ \tilde{x}_2(k) \end{bmatrix} = N^{-1} x(k),
\]

where \( A_{11} \in \mathbb{R}^{r \times r} \). If \( \text{det}(A_{22}) \neq 0 \), system (1) can be equivalently transformed into

\[
\tilde{x}_1(k + 1) = A_{11} \tilde{x}_1(k) + A_{12} \tilde{x}_2(k)
\]

\[
= (A_{11} - A_{12} A_{22}^{-1} A_{21}) \tilde{x}_1(k),
\]

\[
0 = A_{21} \tilde{x}_1(k) + A_{22} \tilde{x}_2(k).
\]

One can claim that system (1) with \( u(k) = 0 \) is admissible if and only if \( \text{det}(A_{22}) \neq 0 \) and system (6) is stable.

Next, we will give a useful lemma to end this section.

**Lemma 1:** [40] Let \( \Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \), where \( \Phi \) and \( \Phi_{ij}, i,j = 1,2 \), are any real matrices with appropriate dimensions such that \( \Phi_{22} \) is invertible and \( \Phi + \Phi^T < 0 \). Then we have

\[
\Phi_{11} + \Phi_{12}^T - \Phi_{22} \Phi_{12} - \Phi_{21} \Phi_{12}^T \Phi_{22} \Phi_{12} < 0.
\]

III. Admissibility Analysis

In this section, we will investigate the admissibility of the DSFS (1) with \( u(k) = 0 \). For brevity, the sequence \( k \) associated with the variables is omitted in the absence of ambiguity, e.g., \( x(k) \) and \( \psi_i(\theta(k)) \) are denoted as \( x \) and \( \psi_i(\theta) \), respectively.

A. The switched fuzzy system

To facilitate the analysis, DSFS (1) is transformed into a switched model. The transformation includes three steps.
1) Partition of membership functions: The membership functions of DSFS (1) are \( w_i(\theta) \), \( i = 1, 2, ..., s \). According to the solution space of \( w_i(\theta) \), the operating space of \( w_i(\theta) \) is divided into several subregions. The solution space of \( w_i(\theta) \) is accordingly partitioned into several subregions.

For T1 DSFS, the supremum and infimum of MFs in each subregion are \( w_{ij}^{\text{inf}} \) and \( w_{ij}^{\text{sup}} \), \( j = 1, 2, ..., q \), respectively. For IT2 DSFS, we can not obtain the precise supremum and infimum due to the uncertainties of the MFs. Therefore, \( w_{ij}^{\text{inf}} \) and \( w_{ij}^{\text{sup}} \) can be set as the supremum of \( w_i(\theta) \) and the infimum of \( w_i(\theta) \) in subregion \( j \), respectively.

**Remark 1:** The spatial partition is designed according to the solution space of \( w_i(\theta) \). One advantage of the method is that there is no restriction on the MFs which can be arbitrary dimensions. The subregions can be unconnected intervals, rectangular regions or cuboid regions, etc. So it has a broader range of applications.

2) Scale transform of membership functions: Define the new membership functions as

\[
\bar{w}_{ij}(\theta) = \frac{w_{ij}^{\text{inf}} + w_{ij}^{\text{sup}}}{2} - \frac{w_{ij}^{\text{inf}}}{2} + \lambda_j (w_{ij}(\theta) - \frac{w_{ij}^{\text{inf}} + w_{ij}^{\text{sup}}}{2}),
\]

where \( \lambda_j = \frac{w_{ij}^{\text{inf}} - w_{ij}^{\text{sup}}}{w_{ij}^{\text{inf}} - w_{ij}^{\text{sup}}} \). \( w_{ij}^{\text{inf}} \) and \( w_{ij}^{\text{sup}} \) are the supremum and infimum of \( w_{ij}(\theta) \), respectively. \( w_{ij}(\theta) = w_i(\theta) \) if \( \theta \) is in subregion \( j \). \( \bar{w}_{ij}(\theta) \) is the new MFs which must satisfy

\[
\bar{w}_{ij}(\theta) \geq 0,
\]

\[
\sum_{i=1}^s \bar{w}_{ij}(\theta) = 1, i = 1, 2, ..., s, j = 1, 2, ..., q.
\]

If \( \lambda_j \) is given, the supremum and infimum of \( \bar{w}_{ij}(\theta) \) are constrained by the following relation.

\[
w_{ij}^{\text{sup}} = \lambda_j (w_{ij}^{\text{inf}} - w_{ij}^{\text{inf}}) + w_{ij}^{\text{inf}}.
\]

**Remark 2:** The scalars \( \lambda_j, w_{ij}^{\text{inf}}, w_{ij}^{\text{inf}}, i = 1, 2, ..., s, j \in \Lambda \) need to be prescribed according to Eqs. (9)-(11). It can be seen that Eqs. (9)-(11) have \( 2i + 1 \) equations, \( i \) inequalities, and \( 3i \) variables and usually have many solutions. We want to find the maximum value of \( \lambda_j \). A simplicial calculating method is given to obtain one of the solutions.

- Substituting (11) into (9), we have

\[
\bar{w}_{ij}(\theta) = w_{ij}^{\text{inf}} - \lambda_j w_{ij}^{\text{inf}} + \lambda_j w_i(\theta).
\]

According to \( \sum_{i=1}^s \bar{w}_{ij}(\theta) = \sum_{i=1}^s w_i(\theta) = 1 \), we have

\[
\lambda_j = \frac{1 - \sum_{i=1}^s w_{ij}^{\text{inf}}}{1 - \sum_{i=1}^s w_{ij}^{\text{inf}}}.
\]

- According to \( \bar{w}_{ij}(\theta) \geq 0 \) in (10), the infimum is fixed as \( w_{ij}^{\text{inf}} = 0 \).
- The maximum value of \( \lambda_j \) is obtained by (13).
- According to (11), the supremum \( w_{ij}^{\text{sup}} \) is obtained.
- By (9) and the obtained solutions of \( \lambda_j, w_{ij}^{\text{inf}}, w_{ij}^{\text{sup}}, \bar{w}_{ij}(\theta) \) is uniquely determined.

3) Establishment of the switched T-S fuzzy system: From (9), we have

\[
w_{ij}(\theta) = \frac{w_{ij}^{\text{sup}} + w_{ij}^{\text{inf}}}{2} + \frac{1}{\lambda_j} (\bar{w}_{ij}(\theta) - \frac{w_{ij}^{\text{sup}} + w_{ij}^{\text{inf}}}{2}).
\]

Substituting (14) into (1), we obtain that

\[
E\bar{x}(k+1) = \sum_{i=1}^s w_{ij}(\theta) (A_i \bar{x}(k) + B_i u(k))
\]

\[
= \sum_{i=1}^s \left( \frac{w_{ij}^{\text{sup}} + w_{ij}^{\text{inf}}}{2} + \frac{1}{\lambda_j} (\bar{w}_{ij}(\theta) - \frac{w_{ij}^{\text{sup}} + w_{ij}^{\text{inf}}}{2}) \right) (A_i \bar{x}(k) + B_i u(k))
\]

\[
= \sum_{i=1}^s \bar{w}_{ij}(\theta) (\tilde{A}_{ij} \bar{x}(k) + \tilde{B}_{ij} u(k)),
\]

where

\[
\tilde{A}_{ij} = \frac{1}{\lambda_j} A_i + \sum_{i=1}^s \left( \frac{w_{ij}^{\text{sup}} + w_{ij}^{\text{inf}}}{2} \right) A_i,
\]

\[
\tilde{B}_{ij} = \frac{1}{\lambda_j} B_i + \sum_{i=1}^s \left( \frac{w_{ij}^{\text{sup}} + w_{ij}^{\text{inf}}}{2} \right) B_i.
\]

Let \( \tilde{A}_{ij} = G \tilde{A}_{ij} N \), and \( \tilde{B}_{ij} = G \tilde{B}_{ij} N \), where \( G, N \in \mathbb{R}^{n \times n} \) are the invertible matrices. The following lemma is obtained.

**Lemma 2:** The following system is an equivalent representation of system (1).

\[
\tilde{E}\bar{x}(k+1) = \sum_{i=1}^s \bar{w}_{ij}(\theta) (\tilde{A}_{ij} \bar{x}(k) + \tilde{B}_{ij} u(k)),
\]

where \( \tilde{u}(k) = N^{-1} u(k) \).

Proof: The left and right sides of (17) are multiplied by \( G^{-1} \) and \( N^{-1} \), respectively, system (15) is obtained.

Substituting (9) and (16) into system (15), we can obtain that system (17) is equivalent to system (1).

The process \( j = j(k), k \geq 0 \), taking values in a finite set \( \Lambda = \{1, 2, ..., q\} \), governs the switching among different systems modes with

\[
j(k+1) = bj(k) = a, a, b \in \Lambda,
\]

\[
\text{if } \theta(k) \text{ in subregion } a,
\]

\[
\text{and } \theta(k+1) \text{ in subregion } b.
\]

**Remark 3:** System (15) is a switched fuzzy system with \( q \) system modes. In each mode, the parameters \( \tilde{A}_{ij}, \tilde{B}_{ij} \) are the linear combination of \( A_i, B_i, \) respectively. This operation will reduce the difference between the parameters. It is helpful for the admissibility analysis of fuzzy system.
B. Admissibility of the switched fuzzy system

Theorem 1: If there exist symmetric matrices $P_{ij} > 0$, matrices $H_{ij}$, $R_{ij}$, $S_{ij}$, $i = 1,2, ..., s$, $j \in \Lambda$, such that the following conditions hold, system (15) with $u(k) = 0$ is admissible.

$$\Gamma_{ij} \leq 0,$$
$$\frac{2}{s-1} \Gamma_{ij} + \Gamma_{ij'k} + \Gamma_{ij'k} \leq 0,$$

for $i, l, \nu = 1,2, ..., s$, $j, k \in \Lambda$, (19)

where $\Gamma_{ij} = \begin{bmatrix} -H_{ij} - H_{ij}^T + P_{ij} \ast & \Phi_{ij} \\ \Phi_{ij}^T \ast & \Phi_{ij} \end{bmatrix}$, $\Phi_{ij} = \begin{bmatrix} -P_{ij} \ast \\ 0 \ast \\ 0 \ast \\ 0 \ast \\ 0 \ast \end{bmatrix} + A_{ij} \begin{bmatrix} 0 \ast \\ 0 \ast S_{ij} \ast \\ 0 \ast S_{ij} \ast \end{bmatrix} A_{ij}^T$.

Proof: For the study of admissibility, the following Lyapunov function candidate is employed:

$$V(k) = \bar{x}_T(k)P(k)\bar{x}_1(k).$$

The forward difference of $V(k)$ is shown as follows.

$$V(k+1) - V(k) = \bar{x}_T(k+1)P(k+1)\bar{x}_1(k+1) - \bar{x}_T(k)P(k)\bar{x}_1(k)$$

$$= \bar{x}_T(k)[(A_{ij} - A_{ij}A_{ij}^T A_{ij})^T P(k+1) - P(k)]\bar{x}_1(k)$$

< 0. (21)

From (19), we have

$$\begin{bmatrix} -H_{ij} - H_{ij}^T + P_{ij} \ast & \Phi_{ij} \\ \Phi_{ij}^T \ast & \Phi_{ij} \end{bmatrix}$$

< 0, (22)

where $P_{ij} = \sum_{i=1}^{s} w_{ij}(\theta(k)) P_{ij}$, $\theta(k) \in [\theta_{j-1}, \theta_j]$, $j \in \Lambda$, $P(k+1) = \sum_{l=1}^{s} w_{j,l}(\theta(k+1)) P_{jl}$, $\theta(k+1) \in [\theta_{j-1}, \theta_j]$, $\kappa \in \Lambda$, $\Phi_{jk} = \sum_{i=1}^{s} \sum_{l=1}^{s} w_{ij}(\theta(k)) w_{jl}(\theta(k)+1) \Phi_{ij} \Phi_{lj}$, $A_{ij} = \sum_{i=1}^{s} \sum_{l=1}^{s} w_{ij}(\theta(k)) \bar{A}_{ij}(\theta(k)) \bar{A}_{ij}(\theta(k)+1) \Phi_{ij} \Phi_{jl}$, $H_{ij} = \sum_{i=1}^{s} \sum_{l=1}^{s} \bar{w}_{ij}(\theta(k)) H_{ij}(\theta(k)) R_{ij}(\theta(k)) R_{ij}(\theta(k)+1)$, $R_{ij}(\theta(k)) R_{ij}(\theta(k)+1)$, $\bar{A}_{ij}(\theta(k)) = \sum_{i=1}^{s} \sum_{l=1}^{s} \bar{w}_{ij}(\theta(k)) \bar{A}_{ij}(\theta(k)) \bar{A}_{ij}(\theta(k)+1) \Phi_{ij} \Phi_{jl}$.

Let $W = \begin{bmatrix} -H_{ij}^T(k) + \frac{1}{2} P_{ij} - P_{ij} \ast & 0 \\ \Phi_{ij}^T \ast & \Phi_{ij} \end{bmatrix}$. (23)

From (22), we obtain that

$$W + W^T < 0.$$ (24)

From (23), we have $A_{ij} \sum_{i=1}^{s} \sum_{l=1}^{s} \bar{w}_{ij}(\theta(k)) S_{ij} + \sum_{i=1}^{s} \sum_{l=1}^{s} \bar{w}_{ij}(\theta(k)) S_{ij}^T A_{ij}^T < 0$, which implies that the system is regular and causal.

By Lemma 1, (23) can ensure the following inequality:

$$\begin{bmatrix} -H_{ij} - H_{ij}^T + P_{ij} \ast & \Phi_{ij} \\ \Phi_{ij}^T \ast & \Phi_{ij} \end{bmatrix}$$

< 0, (24)

where $P(k) = P_{ij}(k)$ in subregion $j$.

Noting that $(H_{ij}(k) - P(k))^T (H_{ij}(k) - P(k)) \geq 0$, which implies

$$H_{ij}^T(k)^T P^{-1}(k) H_{ij}(k) \geq H_{ij}(k) + H_{ij}^T(k) - P(k).$$ (25)

By (24)-(25), we obtain that

$$\begin{bmatrix} -H_{ij}^T(k) P(k) H_{ij}(k) \ast & \Phi_{ij} \\ \Phi_{ij}^T \ast & \Phi_{ij} \end{bmatrix}$$

< 0, (26)

where $P(k) = P^{-1}(k)$. $P(k) > 0$ if $P_{ij} > 0$. Left- right-multiplying (26) by $[H_{ij}^T 0 I]$ and its transpose, we obtain that

$$-P(k) \ast A_{ij} + A_{ij} A_{ij}^{-1} A_{ij} - P^{-1}(k+1) \ast < 0.$$ (27)

By Schur complement, (27) can guarantee that (21) holds. This shows that the system is stable.

Remark 4: To reduce the conservatism, the global admissible problem for system (15) can be investigated by a PLF. The Lyapunov matrix is $P(k) = P^{-1}(k)$, $\theta(k) \in [\theta_{j-1}, \theta_j]$, $j \in \Lambda$. If some conditions are imposed on $\theta(k)$, the number of inequalities in Theorem 1 can be further reduced. For example, if $|\theta(k+1) - \theta(k)| \leq \max_{1 \leq j \leq k} |\theta_j - \theta_{j-1}|$, the value of $\kappa$ can be chosen as

$$\kappa = \begin{cases} 1,2, & j = 1, \\
q - 1, & j = q, \end{cases}$$ (28)

IV. SWITCHED CONTROLLER DESIGN

Next, we will design the switched controller for the switched system (15). Firstly, let us consider a controller for T1 DSFS as

$$u(k) = \sum_{i=1}^{s} w_{ij}(\theta(k)) F_{ij}(\theta(k)).$$ (29)

Then, for T1 and T2 DSFS, the controller can be designed as

$$u(k) = \sum_{i=1}^{s} w_{ij}(\theta(k)) K_{ij}(\theta(k)).$$ (30)

the process $j = j(k)$, $k \geq 0$, is declared in (18). $w_{ij}(\theta(k))$ are linear functions of $\theta(k)$ in subregion $j$. Applying these controllers to system (15) yields the following closed-loop systems:

$$E(x(k+1)) = \sum_{i=1}^{s} \sum_{l=1}^{s} w_{ij}(\theta(k)) w_{ij}(\theta(k)) (\bar{A}_{ij} + \bar{B}_{ij} F_{ij}) x(k).$$ (31)

Remark 5: Two classes of fuzzy controllers are considered for DSFS. For T1 DSFS, a switched fuzzy controller in which the MFs are similar to those of the systems is applied. Aiming at the circumstance that the MFs are uncertain, it is the first time that we present a switched fuzzy controller with linear MFs in each subregion of $\theta(k)$. With such a controller, the stability analysis and controller design will become easier.

We have the following results.

Theorem 2: If there exist symmetric matrices $P_{ij} > 0$, matrices $H_{ij}$, $R_{ij}$, $S_{ij}$, $Z_{ij}$, $i = 1,2, ..., s$, $j \in \Lambda$, such that the following conditions hold, system (30) is admissible.

$$\Gamma_{ij} < 0,$$
$$\frac{2}{s-1} \Gamma_{ij} + \Gamma_{ij'k} + \Gamma_{ij'k} < 0,$$

for $i, l, \nu = 1,2, ..., s$, $j, k \in \Lambda$. (32)
where \( \bar{\Gamma}_{ij} \) = 
\[
\begin{bmatrix}
-\frac{H_{ij}}{T_i} & \frac{P_j}{T_i} \\
\frac{H_{ij}^T}{T_i} & \frac{R_j}{T_i}
\end{bmatrix} + \bar{B}_{ij}Z_{ij}\Omega^T
\]
\( \Phi_{ij} = \begin{bmatrix} -P_k & 0 \\ 0 & -S_j \end{bmatrix} + \bar{A}_ij \begin{bmatrix} 0 & 0 \\ 0 & S_j \end{bmatrix} + \bar{B}_{ij}Z_{ij}\Phi + \begin{bmatrix} 0 \\ 0 \end{bmatrix} S_j^T \Phi(32).
\]

Proof: The conditions (33) can guarantee that the following inequality holds [20]:
\[
\begin{bmatrix}
-\frac{H_{ij}}{T_i} - \frac{H_{ij}^T}{T_i} + \frac{P_j}{T_i} \\
\frac{H_{ij}^T}{T_i} & \frac{R_j}{T_i}
\end{bmatrix} + \bar{B}_{ij}Z_{ij}\Omega^T \Phi_{ij} < 0,
\]
where \( P_j(k) = \sum_{i=1}^c m_{ij}(\theta)P_{ij}, \Phi_{ij} = \begin{bmatrix} -P_k & 0 \\ 0 & -S_j \end{bmatrix} + \bar{A}_ij \begin{bmatrix} 0 & 0 \\ 0 & S_j \end{bmatrix} + \bar{B}_{ij}Z_{ij}\Phi + \begin{bmatrix} 0 \\ 0 \end{bmatrix} S_j^T \Phi(34).
\]

Remark 6: Theorem 2 does not impose any restriction on the auxiliary matrices. All the results are presented in a set of LMIs, which facilitates the design of the controllers. Furthermore, compared with the method in [34], our results use less auxiliary matrices, which can require fewer decision variables.

Remark 7: It can be concluded that the admissibility and admissibilization criteria could be further relaxed by increasing the number of modes \( q \), which will be illustrated by simulation examples in the following section. But the need to pay attention to is that the computational complexity would increase with increasing \( q \). An appropriate \( q \) is not too large, but also meets our control purpose. To achieve flexibility, the mode number of the switched controller could be less than \( q \) by merging some subregions of \( \theta(k) \). That is to say, we use the switched controller or not, it depends on whether or not it satisfies the control purpose.

Theorem 3: If there exist symmetric matrices \( P_{ij} > 0 \), matrices \( H_{ij}, R_{ij}, S_{ij}, Z_{ij}, i = 1, 2, \ldots, c, j \in \Lambda \), such that the following conditions hold for \( i = 1, 2, \ldots, s, j, k \in \Lambda \), system (31) is admissible.

\[
\begin{bmatrix}
-\frac{H_{ij}}{T_i} - \frac{H_{ij}^T}{T_i} + \frac{P_j}{T_i} \\
\frac{H_{ij}^T}{T_i} & \frac{R_j}{T_i}
\end{bmatrix} + \bar{B}_{ij}Z_{ij}\Omega^T \Phi_{ij} < 0,
\]
where \( \Phi_{ij} = \begin{bmatrix} -P_k & 0 \\ 0 & -S_j \end{bmatrix} + \bar{A}_ij \begin{bmatrix} 0 & 0 \\ 0 & S_j \end{bmatrix} + \bar{B}_{ij}Z_{ij}\Phi + \begin{bmatrix} 0 \\ 0 \end{bmatrix} S_j^T \Phi(33).
\]

Proof: By replacing \( \bar{A}_ij \) with \( \bar{A}_ij + \bar{B}_{ij}F_{ij}N, \) and let \( Z_{ij} = F_{ij}N \sum_{i=1}^c \bar{w}_i(\theta)R_{ij} \) in Theorem 1, there holds the inequality (32).

Remark 8: For IT2 DSFS, the PDC approach can not be used to deal with the stabilization problems and the obtained criteria are often conservative. The MF-based methods can reduce the conservatism [41]. The switched controller (29) has linear MFs in each subregion. This kind of MFs can be expressed by the linear superposition of the endpoints of MFs \( m_{ij}(\theta^{(j)}(k)) \) [20], and \( m_{ij}(\theta^{(j)}(k)) \) are constants. The stabilization conditions can be achieved by examining the conditions at finite endpoints. Thus, both of the analysis difficulty and the conservatism can be reduced.

V. SIMULATION EXAMPLES

Three examples are presented to illustrate the effectiveness of the method. Example 1 considers the stability analysis of a practical example. The stabilization and admissibilization problems are solved in Examples 2-3.

Example 1: Consider the following stirred tank reactor model [42]:
\[
\begin{cases}
\frac{dx}{dt} = \frac{1}{T_c} \left[ (C_{Af} - C_A) - k_0 \exp(-E/R)C_A \right] + \frac{1}{T} \left[ (T_j - T) + \frac{\Delta C}{\Delta T} k_0 \exp(-E/R)C_A \right] \end{cases}
\]
where \( C_A \) is the concentration of \( A \) in the reactor, \( T \) is the reactor temperature. Other notations are suitably referred to [42]. Denote the nonzero equilibrium as \( \{ C_{A}^T, T_{eq} \} \). Define \( x = [C_A - C_{A}^T, T - T_{eq}]^T \). Denote the bounds on \( x_2 \) as \( x_2 \leq x_2 \leq \bar{x}_2 \) (\( \bar{x}_2 = T_{eq} - T_{eq} = T_{eq} - T_{eq} \)). Using \( x_2 \) as the premise variable of the T-S fuzzy model. Moreover, define
\[
\varphi_1(x_2) = k_0 \exp(-E/R \bar{x}_2 + T_{eq}),
\]
\[
\varphi_2(x_2) = k_0 \left[ \exp(-E/R \bar{x}_2 + T_{eq}) - \exp(-E/R \bar{x}_2 + T_{eq}) \right] C_A^T \bar{x}_2,
\]
\[
\varphi_0 = |\varphi_1(x_2) + \varphi_1(\bar{x}_2)|/2, \varphi_2(x_2) = |\varphi_2(x_2) + \varphi_2(\bar{x}_2)|/2,
\]
where \( \varphi_1(x_2) = \varphi_1(x_2) - \varphi_0, q_1(x_2) = \varphi_1(x_2) - \varphi_2(x_2) = \varphi_2(x_2) - \varphi_0 \).
Choosing $C_A^q = 0.5 \text{mol/l}, T^q = 350K, T^l = 340K$, $T_u = 360K$, $q = 100l/min$, $\rho = 1000g/l$, $C_P = 0.239J/(gK)$, $C_A = 1mol/l$, $T_f = 350K$, $V = 100l$, $\Delta H = -1.2 \times 104J/mol$, $E/R = 8750K$, $k_0 = 7.2 \times 1010\text{min}^{-1}$, $UA = 5 \times 104J/(\text{min}K)$. By discretizing the continuous-time T-S fuzzy model with sampling period $T_s = 0.05\text{min}$, the required discrete-time fuzzy model is obtained. The parameters are:

$$
A_1 = \begin{bmatrix}
0.8613 & -0.0019 \\
6.9390 & 0.9905
\end{bmatrix}, 
A_2 = \begin{bmatrix}
1.0136 & -0.0019 \\
-0.7085 & 0.9905
\end{bmatrix},
$$

$$
A_3 = \begin{bmatrix}
0.9374 & -0.0031 \\
3.1152 & 1.0510
\end{bmatrix}, 
A_4 = \begin{bmatrix}
0.9374 & -0.0007 \\
3.1152 & 0.9300
\end{bmatrix}.
$$

**Fig. 1.** The state response of Example 1

Fig. 1 shows the state responses of the open-loop system with the initial states $x_1(0) = -1$ and $x_2(0) = 3$. It can be seen that the above system is stable, while this fact cannot be verified by the stability conditions in [16]–[18], [37].

By our method, the operating space of MFs $x_2 \in [-10, 10]$ is partitioned into eight subregions as $-10 < -5.5 < -3 < 0.5 < 2.5 < 5 < 6.5 < 8.5 < 10$. By (9)-(11), let $\bar{w_{ij}} = 0$ and $\bar{w_{ijsup}}$ is obtained. The scales $\lambda_j, j = 1, 2, \ldots, 8$, in each subregion are calculated as $6.96, 10.46, 6.37, 9.66, 6.87, 10.3, 7.06, 8.59$. The new MFs $\bar{w_{ij}}(\theta)$ are shown in Fig. 2. $\bar{A}_{ij}$ is solved by Equation (16). By Theorem 1, the system can be proved to be stable, and the Lyapunov matrices are $P_{ij} = \begin{bmatrix}
0.1790 & -2.2460 \\
-2.2460 & 229.8862
\end{bmatrix}$, $P_{3j} = \begin{bmatrix}
0.1776 & -2.2019 \\
-2.1368 & 227.1519
\end{bmatrix}$, $P_{3j} = \begin{bmatrix}
0.1776 & -2.2019 \\
-2.1368 & 227.1519
\end{bmatrix}$, $P_{4j} = \begin{bmatrix}
0.1781 & -2.2147 \\
-2.2147 & 229.8119
\end{bmatrix}$, $j = 1, 2, \ldots, 8$.

**Example 2:** Consider the two-rule discrete-time fuzzy system that is borrowed from [17], [18] with the following local matrices:

$$
A_1 = \begin{bmatrix}
1 & -b \\
-1 & -0.5
\end{bmatrix}, 
B_1 = \begin{bmatrix}
5 + b \\
2b
\end{bmatrix},
$$

$$
A_2 = \begin{bmatrix}
1 & b \\
-1 & -0.5
\end{bmatrix}, 
B_2 = \begin{bmatrix}
5 - b \\
-2b
\end{bmatrix},
$$

where $b$ is a real-valued parameter, and the MFs are $w_1(x_1(k)) = \frac{b + x_1(k)}{2b}, w_2(x_1(k)) = \frac{b - x_1(k)}{2b}$.

The goal is to be able to stabilize the system for the parameter $b$ having a value as large as possible. The following values have been given in recent literatures which are shown in Table I.

<table>
<thead>
<tr>
<th>Literatures</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[43]</td>
<td>1.7669</td>
</tr>
<tr>
<td>[18]</td>
<td>1.81</td>
</tr>
<tr>
<td>[44]</td>
<td>1.82</td>
</tr>
<tr>
<td>[17]</td>
<td>1.95</td>
</tr>
<tr>
<td>Theorem 2 (4 subregions)</td>
<td>1.86</td>
</tr>
<tr>
<td>Theorem 2 (8 subregions)</td>
<td>2.6</td>
</tr>
</tbody>
</table>

Fig. 2. $w_1(x_2)$ and $\bar{w_{ij}}(x_2), i = 1, 2, 3, 4, \Lambda = \{1, 2, \ldots, 8\}$, of Example 1
Next, the IT2 fuzzy system is considered. The upper and lower MFs are given as $w_{1U}(x_1(k)) = \frac{0.025x_1^2(k)}{b^2} + 0.97x_2(k) + 0.54$, $w_{1L}(x_1(k)) = \frac{0.025x_1^2(k)}{b^2} + 0.97x_1(k) + 0.46$, $w_{2U}(x_1(k)) = 1 - w_{1L}(x_1(k))$, $w_{2L}(x_1(k)) = 1 - w_{1U}(x_1(k))$. The operating space of $x_1$ is also partitioned into four subregions as $\frac{b}{2}(j - 3) \leq x_1 \leq \frac{b}{2}(j - 2)$. Let $b = 1.77$, the scales are $\lambda_1 = \lambda_4 = 3.4335$, $\lambda_2 = \lambda_3 = 3.1621$, $\Lambda = \{1, 2, 3, 4\}$. The new MFs are illustrated in Fig. 4. The controller is chosen as $u(k) = \sum_{i=2}^{2} m_{ij}(x_1(k))k_{ij}x(k)$, and $m_{ij}(x_1(k)) = \frac{k_{ij}x_1(k)}{2b}$, $m_{2j}(x_1(k)) = 1 - m_1(x_1(k))$. The endpoints are $x_{ij}^{(d)}(k) \in \left\{\frac{b}{2}(j-3), \frac{b}{2}(j-2)\right\}$ and the endpoints of MFs can be calculated as

$$m_{11}(-d) = 0, m_{11}(-d/2) = 0.25, m_{12}(-d/2) = 0.25, m_{12}(0) = 0.5, m_{13}(0) = 0.5, m_{13}(d/2) = 0.75, m_{14}(d/2) = 0.75, m_{14}(d) = 1, m_{21}(-d) = 1, m_{21}(-d/2) = 0.75, m_{22}(0) = 0.5, m_{22}(d/2) = 0.25, m_{24}(d/2) = 0.25, m_{24}(d) = 0.$$
By the above controller gains, let 
\[ u(k) = \sum_{i=1}^{2} m_{ij}(x_{1}(k))Z_{ij} \left( \sum_{i=1}^{2} m_{ij}(x_{1}(k))H_{ij} \right)^{-1} \]
the state responses of the closed-loop system with the initial states 
\[ x_{1}(0) = 1 \text{ and } x_{2}(0) = 0 \] are shown in Fig. 5. From Fig. 5, we can find that the linear MF-based control design is feasible and effective.

**Example 3:** We use the truck-trailer model formulated by Ichihashi [45] in this simulation.

\[
x_{1}(k + 1) = x_{1}(k) + v \cdot t/L \cdot u(k),
\]
\[
x_{2}(k) = x_{1}(k) - x_{3}(k),
\]
\[
x_{3}(k + 1) = x_{3}(k) + v \cdot t/L \cdot \sin(x_{2}(k)),
\]
\[
x_{4}(k + 1) = x_{4}(k) + v \cdot t \cdot \cos(x_{2}(k)) \cdot \sin\{x_{3}(k + 1) + x_{3}(k)\}/2,
\]

where \( x_{i}(k), i = 1, 2, 3, 4, \) represent the angle of truck, angle difference between truck and trailer, angle of trailer, and vertical position of rear end of trailer, respectively. \( l \) is the length of truck, \( L \) is the length of trailer, \( t \) is sampling time, and \( v \) is the constant speed of backing up. In this paper, \( l = 2.8m, L = 5.5m, v = -1.0m/s, \) and \( t = 2.0s. \)

Let \( \theta(k) = x_{3}(k) + v \cdot t/2L \cdot x_{2}(k), \) and assume \(-\pi < \theta(k) < \pi, \) the MFs are defined as follows:

\[
w_{1}(\theta(k)) = \begin{cases} 
\frac{\sin(\theta(k)) + 0.001\theta(k)}{1.00020(k)}, & \text{if } \theta(k) \neq 0 \\
0.9960, & \text{if } \theta(k) = 0
\end{cases}
\]
\[
w_{2}(\theta(k)) = 1 - h_{1}(\theta(k)).
\]

The T-S fuzzy model that represents the nonlinear system is as follows:

**Plant Rule 1:** If \( \theta(k) \) is \( h_{1} \)

Then \( E x(k + 1) = A_{1}x(k) + Bu(k), \)

**Plant Rule 2:** If \( \theta(k) \) is \( h_{2} \)

Then \( E x(k + 1) = A_{2}x(k) + Bu(k), \)

where \( E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \)

\[
A_{1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & v \cdot t/L & 1 & 0 \\ 0 & 1.001v^{2} \cdot t^{2}/(2L) & 1.001v \cdot t & 1 \end{bmatrix},
\]

\[
A_{2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & v \cdot t/L & 1 & 0 \\ 0 & -0.001 \cdot v^{2} \cdot t^{2}/(2L) & -0.001 \cdot v \cdot t & 1 \end{bmatrix},
\]

\[ B = [v \cdot t/l \ 0 \ 0 \ 0]^{T}. \]

The methods in [15], [20], [36], [37] cannot find a feasible controller for the above fuzzy model. By our method, the partition is \( \frac{\pi}{4} \leq w_{ij}(\theta) \leq \frac{\pi}{4}, j = 1, 2, 3, 4. \)

The operating space of \( \theta(k) \) is divided into four subregions as \([\pi, -2.4865] \cup [2.4865, \pi], [-2.4865, -1.8955] \cup [1.8955, 2.4865], [-1.8955, -1.2618] \cup [1.2618, 1.8955], \) and \([-1.2618, 1.2618]. \) The scales are \( \lambda_{j} = 4, \Lambda = \{1, 2, 3, 4\}. \) By scale transform, we obtain the new MFs which are shown in Fig. 6.

According to Theorem 2, the switched controller is obtained. And the gains \( F_{ij} = Z_{ij} \left[ \begin{array}{cc} H_{j}^{-1}(k) & 0 \\ -S_{j}^{-1}(k)R_{j}(k)H_{j}^{-1}(k) & S_{j}^{-1}(k) \end{array} \right] \) can be calculated by:

\[
Z_{11} = \begin{bmatrix} 702.5853 & 84.6114 & -68.8639 & 164.7983 \end{bmatrix},
\]
\[
Z_{12} = \begin{bmatrix} 711.3543 & 98.7325 & -147.7115 & 174.6692 \end{bmatrix},
\]
\[
Z_{13} = \begin{bmatrix} 713.5568 & 104.6012 & -201.6732 & 187.7205 \end{bmatrix},
\]
\[
Z_{14} = \begin{bmatrix} 720.8015 & 109.8849 & -239.7839 & 194.4601 \end{bmatrix},
\]
\[
Z_{21} = \begin{bmatrix} 729.2609 & 93.6032 & -0.0016 & 157.1062 \end{bmatrix},
\]
\[
Z_{22} = \begin{bmatrix} 708.8209 & 95.3996 & -108.1797 & 169.7289 \end{bmatrix},
\]
\[
Z_{23} = \begin{bmatrix} 708.0350 & 99.5581 & -169.9740 & 183.7333 \end{bmatrix},
\]
\[
Z_{24} = \begin{bmatrix} 714.4116 & 104.1515 & -215.1884 & 190.9879 \end{bmatrix},
\]

R_{11} = [350.0925 111.1428 – 30.4450],
R_{12} = [353.8919 113.9881 – 51.4587],
R_{13} = [354.9713 110.9866 – 78.4434],
R_{14} = [357.1576 108.3022 – 92.9007],
R_{21} = [363.4972 112.1715 – 0.0106],
R_{22} = [352.6616 108.3878 – 46.0958],
R_{23} = [352.2905 105.6992 – 75.1870],
R_{24} = [353.9464 103.4517 – 90.7499],
S_{11} = 124.4053, S_{12} = 120.7418, S_{13} = 116.6355,
S_{14} = 113.4292, S_{21} = 126.0050, S_{22} = 122.6555,
S_{23} = 118.8503, S_{24} = 115.9096.

where 

\[ S_j(k) = \sum_{i=1}^{s} \tilde{w}_{ij}(\theta)S_{ij}, \]

\[ R_j(k) = \sum_{i=1}^{s} \tilde{w}_{ij}(\theta)R_{ij}, \]

\[ H_j(k) = \sum_{i=1}^{s} \tilde{w}_{ij}(\theta)H_{ij}. \]

By the above controller gains, and let \( x(0) = [0 \ \frac{\pi}{2} \ - \frac{\pi}{2} \ 10]^T \), the control responses are illustrated in Fig. 7, and the switching mode is shown in Fig. 8, where the admissibility is achieved.

From Fig. 7, it is obvious that the switched controller method is effective.

VI. CONCLUSION

In this paper, we focus on the admissible analysis and controller design for T1 and IT2 DSFS. To facilitate the analysis, the operating space of the membership functions are partitioned into several subregions. By applying the partition and scale transform, the fuzzy systems are transformed into switched systems with a switching rule that specifies the switching. By employing the piecewise Lyapunov function, and auxiliary matrices, new sufficient conditions for the admissibility is derived. Then, the conditions are extended to check the admissibilization for the closed-loop systems. Numerical examples demonstrate the effectiveness and advantage of the proposed method. There still exist some other interesting problems that need to be addressed, such as the fault-tolerant control for multi-agent singular fuzzy systems by combining a new reinforcement learning algorithm [46], and the extension of our developed approaches to the finite-time [47], [48] or fixed-time [49] control problems with disturbance uncertainties [50], which deserve further investigations.

REFERENCES


Jian Chen received the B.Sc. degree in mathematics from Dalian University of Technology, Dalian, China, in 2001, the M.Sc. degree in system engineering from Shandong University, Jinan, China, in 2005 and the Ph.D. degree from the Institute of Complexity Science, Qingdao University, Qingdao, China, in 2017. She is currently an associate professor at the School of Information and Control Engineering, Qingdao University of Technology, Qingdao. Her research interests include the analysis and fuzzy control for complex nonlinear systems.

Jinpeng Yu received the B.Sc. degree in automation of Qingdao University, Qingdao, China, in 2002, the M.Sc. degree in system engineering of Shandong University, Jinan, China, in 2005 and the Ph.D. degree from the Institute of Complexity Science, Qingdao University, Qingdao, China, in 2011. He is currently a Distinguished Professor at the School of Automation, Qingdao University. He is a recipient of the Shandong Province Taishan Scholar Special Project Fund and Shandong Province Fund for Outstanding Young Scholars. His research interests include electrical energy conversion and motor control, applied nonlinear control and intelligent systems.

Hak-Keung Lam received the B.Eng. (Hons.) and Ph.D. degrees from the Department of Electronic and Information Engineering, The Hong Kong Polytechnic University, Hong Kong, in 1995 and 2000, respectively. During the period of 2000 and 2005, he worked with the Department of Electronic and Information Engineering at The Hong Kong Polytechnic University as Post-Doctoral Fellow and Research Fellow respectively. He joined as a Lecturer at Kings College London in 2005 and is currently a Reader. His current research interests include intelligent control, computational intelligence and machine learning.

He has served as a program committee member, international advisory board member, invited session chair and publication chair for various international conferences and a reviewer for various books, international journals and international conferences. He is an associate editor for IEEE Transactions on Fuzzy Systems, IEEE Transactions on Circuits and Systems II: Express Briefs, IET Control Theory and Applications, International Journal of Fuzzy Systems, Neurocomputing and Nonlinear Dynamics; and guest editor for a number of international journals. He is on the editorial board of Journal of Intelligent Learning Systems and Applications, Journal of Applied Mathematics, Mathematical Problems in Engineering, Modelling and Simulation in Engineering, Annual Review of Chaos Theory, Bifurcations and Dynamical System, The Open Cybernetics and Systemics Journal, Cogent Engineering and International Journal of Sensors, Wireless Communications and Control.