Hidden-Markov-Model-Based Asynchronous $H_{\infty}$ Tracking Control of Fuzzy Markov Jump Systems

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Abstract—This paper is concerned with the problem of asynchronous $H_{\infty}$ output tracking control for Takagi-Sugeno fuzzy Markov jump systems. A hidden Markov model is established due to the fact that the modes information of the system may not accurately transmit to the controller, which is used to depict the asynchronous phenomenon between the system modes and controller modes. The packet loss in the communication process is described by a stochastic variable that is subject to Bernoulli distribution. Then, with the help of a novel Lyapunov function, the mode-dependent and fuzzy-basis-dependent stability criteria are derived and the asynchronous control scheme is developed with an $H_{\infty}$ tracking performance. Finally, a numerical example is applied to demonstrate the effectiveness of the proposed approach.

Index Terms—Output tracking control, Markov jump systems, hidden Markov model (HMM), packet loss, asynchronous control.

I. INTRODUCTION

With the rapid development of science and technology, the control systems in modern industry are becoming extremely complicated, which contain more and more nonlinearities and uncertainties. Fuzzy technique has emerged in the case that traditional control methods dependent on building accurate models are difficult to achieve the expected effect. Particularly, Takagi-Sugeno (T-S) fuzzy model can approximate a smooth nonlinear function with arbitrary precision, which provides an effective way for the modeling and optimizing control of complex nonlinear systems. Therefore, there have been numerous results reported on T-S fuzzy model [1]–[4]. On the other hand, most physical plants often experience abrupt changes in structures and parameters induced by component failures or environmental variations, which stimulates the introduction of Markov jump systems (MJSs) [5]–[9]. Considering the mode evolution in T-S fuzzy systems, fuzzy Markov jump systems (FMJSs) have aroused abroad attention among scholars as a class of stochastic hybrid nonlinear systems. For instance, Niu et al. [10] have applied a T-S fuzzy model to describe nonlinear Markov jump singularly perturbed systems, in which both $H_{\infty}$ and passive performance have been taken into account. In [11], an input-output approach for discrete-time FMJSs has been employed to guarantee the stochastic stability. Tao et al. [12] have focused on the reliable dissipativity analysis of T-S FMJSs with the existence of actuator faults.

In addition, the tracking control has been an important research direction in the control field, which aims to make the controlled plant track the specified trajectory and minimize the error between trajectories. It has been widely used in the mechanical arms [13], spacecrafts [14], missiles [15], ship trajectories [16] and so on. Furthermore, it can be seen that many specialists have devoted themselves to the tracking control design for fuzzy systems [17]–[21] and MJSs [22]–[27]. In order to reduce the conservativeness of the results, the output feedback tracking controller has been designed via a membership-function-dependent approach for T-S fuzzy systems with input saturation in [19]. For the flexible-joint robot system, Sun et al. [13] have developed an adaptive fuzzy tracking control method to achieve the desired performance. Zhang et al. have investigated the output tracking control problem for fuzzy systems in [20] and [21], where the network-induced delays have been considered. The sliding mode tracking problem has been addressed for nonlinear MJSs in [24], in which the transition probabilities are partly available and the system models are unknown. The work in [26] has found the solution to the robust $H_{\infty}$ tracking control for the delayed MJSs. A mode-dependent controller containing gain perturbations has been constructed under bounded modes transition rates in [27], which intends to ensure that the output of the considered singular MJSs can track the given reference system well.

It is worth pointing out that the tracking controllers designed in [25]–[27] all based on the assumption that they can keep the same modes with the original systems. Actually, the system modes are hidden to the controller/filter, which causes that it is almost impossible for controllers/filters to have access to the accurate modes information of plants, let alone to synchronize with the operational modes. Consequently, it is crucial to establish asynchronous controllers/filters for MJSs. The asynchronous reliable filter design problem has been discussed for continuous-time nonlinear MJSs in [28], which relies on a hidden Markov model. In [29], Wu et al. have put forward to an asynchronous passive control scheme for MJSs. As for FMJSs, researchers in [30] have adopted a quantizer to study the guaranteed cost control problem. The issue of asynchronous dissipative control of FMJSs has been settled...
in [31], and the achieved results can be applied to the $H_\infty$ and passivity problem. Note that the aforementioned results in [28]–[31] all have used a hidden Markov model (HMM) to depict the asynchronous phenomenon between the system modes and controller modes. The function of the HMM is to link the controller with the system through a conditional probability matrix, in which the conditional probabilities reflect the asynchronization degree. Moreover, these works in [28]–[31] all have required the conditional probabilities of the HMM to be completely known. However, in fact, the obstacle of obtaining the total conditional probabilities is not easy to overcome. Accordingly, it is meaningful to establish a novel frame, in which partial conditional probabilities are known beforehand. To the best of the authors’ knowledge, in the existing literature, the problem of asynchronous tracking control for T-S fuzzy-model-based MJSs has not been solved, which inspires this work.

This paper develops an asynchronous tracking control method for T-S FMJSs, in which the conditional probabilities in the HMM are assumed to be partial unknown. The main contributions of this paper are summarized as follows.

1) Motivated by the discussion in anteriority [24]–[27], it is the first time that the asynchronous $H_\infty$ output tracking controller is designed for T-S fuzzy-model-based MJSs via the HMM.

2) Compared with [28]–[31], the proposed HMM is more general and practical, in which the conditional probabilities are assumed to be partly available.

3) The achieved results also cover the case that the conditional probability matrix associated with the HMM is completely accessible.

The remaining parts of this paper are organized as follows. Section II formulates the problem under investigation. In Section III, the stability criteria for the FMJSs are obtained and the asynchronous fuzzy controller is constructed to ensure the system’s tracking performance. A numerical example is given to verify the priority of the proposed approach in Section IV. Finally, the paper is concluded in Section V.

Notations: $\mathbb{R}^n$ and $\mathbb{R}^{m \times n}$ represent the $n$-dimensional Euclidean space and the set of all $m \times n$ real matrices, respectively. $P > 0$ ($P < 0$) means that the matrix $P$ is symmetric and positive (negative)-define. $\text{diag}\{\cdot\}$ is a block-diagonal matrix. “$T$” represents the matrix’s transposition and “$^\circ$” is an ellipsis as the symmetry in a matrix. $\mathbb{P}\{\cdot\}$ denotes the probability. $\mathbb{E}\{\cdot\}$ stands for the mathematical expectation. $\lambda_{\text{min}}(\cdot)$ refers to the minimum eigenvalue of a matrix.

II. PROBLEM FORMULATION AND PRELIMINARIES

A. The system description

Consider a fuzzy Markov jump system described by the following IF-THEN rules.

Plant rule $i$: IF $\eta_i(k)$ is $v_{i1}$, $\eta_2(k)$ is $v_{i2}$, ..., and $\eta_N(k)$ is $v_{iN}$, THEN

$$
\begin{align*}
& x(k + 1) = A_{\sigma_k,i}x(k) + B_{\sigma_k,i}u(k) + E_{\sigma_k,i}\omega(k) \\
& z(k) = C_{\sigma_k,i}x(k) + D_{\sigma_k,i}u(k) + F_{\sigma_k,i}\omega(k)
\end{align*}
$$

where $i \in \mathbb{I} = \{1, 2, \ldots, r\}$, $r$ is the number of the fuzzy rules; $\eta_j(k)$ ($j = 1, 2, \ldots, N$) are the premise variables and $\{v_{ij}\}$ is a fuzzy set; $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$ and $z(k) \in \mathbb{R}^p$ represent the system state vector, control input and output vector, respectively. $\omega(k) \in \mathbb{R}^p$ is the external disturbance belonging to $l_2[0, +\infty)$. $A_{\sigma_k,i}$, $B_{\sigma_k,i}$, $C_{\sigma_k,i}$, $E_{\sigma_k,i}$, $F_{\sigma_k,i}$ are known real matrices. $\sigma_k \in \mathcal{L} = \{1, 2, \ldots, L\}$ denotes the system mode, which is used to describe the jump phenomenon. The transition probability matrix $\Pi = \{\pi_{mn}\}$ is determined by

$$
\text{Prob}(\sigma_{k+1} = n|\sigma_k = m) = \pi_{mn}, \quad m, n \in \mathcal{L}
$$

where $\pi_{mn}$ is the transition probability of the mode $\sigma_k$ and is subject to $0 \leq \pi_{mn} \leq 1$, $\sum_{n=1}^{L} \pi_{mn} = 1$.

By means of the T-S fuzzy approach, the inferred system with $\sigma_k = m$ is written as

$$
\begin{align*}
& x(k + 1) = A_{mh}x(k) + B_{mh}u(k) + E_{mh}\omega(k) \\
& z(k) = C_{mh}x(k) + D_{mh}u(k) + F_{mh}\omega(k)
\end{align*}
$$

where

$\begin{align*}
& h_i(\eta(k)) = \sum_{i=1}^{r} v_i(\eta(k)) \cdot v_{ij}(\eta_j(k)) = \prod_{j=1}^{N} v_{ij}(\eta_j(k)) \\
& A_{mh} = \sum_{i=1}^{r} h_i(\eta(k))A_{mi,i}, \quad B_{mh} = \sum_{i=1}^{r} h_i(\eta(k))B_{mi,i} \\
& C_{mh} = \sum_{i=1}^{r} h_i(\eta(k))C_{mi,i}, \quad D_{mh} = \sum_{i=1}^{r} h_i(\eta(k))D_{mi,i} \\
& E_{mh} = \sum_{i=1}^{r} h_i(\eta(k))E_{mi,i}, \quad F_{mh} = \sum_{i=1}^{r} h_i(\eta(k))F_{mi,i}
\end{align*}$

$\eta(k) = [\eta_1(k), \eta_2(k), \ldots, \eta_N(k)]$. $v_{ij}(\eta_j(k))$ is the grade of membership $\eta_j(k)$ in $v_{ij}$. And $h_i(\eta(k))$ represents the normalized membership function of rule $i$, for all $i \in \mathbb{I}$, $h_i(\eta(k)) \geq 0$ and $\sum_{i=1}^{r} h_i(\eta(k)) = 1$. For notation simplicity, $h_i$ stands for $h_i(\eta(k))$ in the subsequent sections.

In order to implement the tracking of the desired output signal, choose the following reference model

$$
\begin{align*}
x_r(k + 1) = A_rx_r(k) + B_ru(k) \\
z_r(k) = C_rx_r(k)
\end{align*}
$$

where $r(k) \in \mathbb{R}^q$ is the bounded reference input. $x_r(k) \in \mathbb{R}^n$, $z_r(k) \in \mathbb{R}^p$ are the state vector and output vector, respectively. $A_r$, $B_r$ and $C_r$ are known real matrices with appropriate dimensions, and $A_r$ is a Hurwitz matrix.
B. Asynchronous fuzzy controller

In view of the parallel distributed compensation (PDC) method, the asynchronous fuzzy controller is designed as follows:

Controller rule i: IF \( \eta_1(k) \) is \( v_{11} \), \( \eta_2(k) \) is \( v_{12} \), ..., and \( \eta_n(k) \) is \( v_{1n} \), THEN
\[
 u_c(k) = K\theta_{1,j}x_c(k)
\]
(5)

where \( K\theta_{1,j} \in \mathbb{R}^{n \times n} \) is the controller gain matrix to be determined. \( x_c(k) \in \mathbb{R}^m \) and \( u_c(k) \in \mathbb{R}^m \) are the input and output of the controller, respectively. \( \eta_k \) plays a role similar to \( \sigma_k \), which represents the mode of the controller and takes value in the set \( \mathcal{M} = \{1, 2, \ldots, M\} \). The conditional probability matrix \( \Lambda = \\{\lambda_{mt}\} \) gives the probabilities of \( \eta_k \) replacing \( \sigma_k \) with
\[
 \text{Prob}\{\eta_k = t|\sigma_m = k\} = \lambda_{mt}, \quad m \in \mathcal{L}, \quad t \in \mathcal{M}
\]
(6)

where \( 0 \leq \lambda_{mt} \leq 1 \), \( \sum_{t=1}^{M} \lambda_{mt} = 1 \).

In addition, the information of matrix \( \Lambda \) may not be completely accessed, i.e., some probabilities are partial unknown. For instance, when \( L = 3 \) and \( M = 4 \), matrix \( \Lambda \) may be as
\[
 \Lambda = \begin{bmatrix}
 \lambda_{11} & ? & ? & ? \\
 \lambda_{21} & ? & \lambda_{23} & ? \\
 ? & \lambda_{32} & ? & \lambda_{34}
\end{bmatrix}
\]

where “?” represents the inaccessible elements. To analyze conveniently, denote \( \mathcal{M}_c = \mathcal{M}_c^{(m)} + \mathcal{M}_c^{(m)} \) with
\[
 \mathcal{M}_c^{(m)} = \{t: m_{it} \text{ is known}\},
\]
\[
 \mathcal{M}_c^{(m)} = \{t: m_{it} \text{ is unknown}\}.
\]
(7)

Moreover, if \( \mathcal{M}_c^{(m)} \neq \emptyset \) and \( \mathcal{M}_c^{(m)} \neq \emptyset \), it can be further described as
\[
 \mathcal{M}_c^{(m)} = (K_1^{(m)}, \ldots, K_s^{(m)}), \quad 1 \leq s \leq L,
\]
\[
 \mathcal{M}_c^{(m)} = (U_1^{(m)}, \ldots, U_{M}^{(m)}), \quad 1 \leq v \leq M
\]
(8)

where \( K_s^{(m)} \) is the \( s \)th known element with the index \( K_s^{(m)} \) in the \( m \)th row of matrix \( \Lambda \), and correspondingly, \( U_v^{(m)} \) is the \( v \)th unknown element with the index \( U_v^{(m)} \).

Then, the control law (5) with \( \eta_k = t \) can be inferred as
\[
 u_c(k) = K_{th}x_c(k)
\]
(9)

where
\[
 K_{th} = \sum_{i=1}^{r} h_i K_{t,i}
\]

Remark 1. Different from the emphasis on the incomplete known transition probabilities in [32]–[35], the focus of our study lies in that the conditional probability matrix \( \Lambda \) in the HMM contains partial unknown elements. In subsequent section, the obtained results will include the case as previously mentioned results in [28]–[31].

C. Communication links

Owing to communication failures, the problem of packet loss often inevitably occurs during data transmission. Introduce two mutually independent Bernoulli-distributed variables \( \alpha(k), \beta(k) \in \{0, 1\} \), which are employed to depict the packet dropout phenomena in two channels: the sensor to the controller and the controller to the actuator, respectively. Hence, the input of the controller and the output of the actuator can be expressed as
\[
 \begin{align*}
 x_r(k) &= \alpha(k)(x(k) - x_r(k)) \\
 u(k) &= \beta(k)u_c(k)
\end{align*}
\]
(10)

Define \( \gamma(k) = \alpha(k)\beta(k) \). Clearly, it can be found that \( \gamma(k) = 1 \) holds only when \( \alpha(k) = \beta(k) = 1 \), which implies that the data transmits successfully. Otherwise, \( \gamma(k) = 0 \) means the packet loss. It is assumed that \( \gamma(k) \) obeys the following probability distribution:
\[
 \begin{align*}
 \text{Prob}\{\gamma(k) = 1\} &= E\{\gamma(k)\} = E\{\gamma^2(k)\} = \gamma \\
 \text{Prob}\{\gamma(k) = 0\} &= 1 - \gamma
\end{align*}
\]
(11)

where \( \gamma \) denotes the packet arriving rate. From (9) and (10), one has
\[
 u(k) = \gamma(k)K_{th}(x(k) - x_r(k))
\]
(12)

Combining (3), (4) with (12), the following augmented system with \( e(k) = z(k) - x_r(k) \) can be obtained
\[
 \begin{align*}
 \dot{x}(k + 1) &= A_{th}x(k) + \tilde{E}_{th}\tilde{w}(k) \\
 e(k) &= C_{th}x(k) + \tilde{F}_{th}\tilde{w}(k)
\end{align*}
\]
(13)

where \( \tilde{w}(k) = \gamma(k) - \gamma \)
\[
 \tilde{x}(k) = [x^T(k) \ x_r^T(k)]^T, \quad \tilde{w}(k) = [\omega^T(k) \ r^T(k)]^T
\]
\[
 A_{th} = A_1 + \gamma(k)A_2, \quad C_{th} = C_1 + \gamma(k)C_2
\]
\[
 A_{1mth} = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j A_{1mtij}, \quad A_{2mth} = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j A_{2mtij}
\]
\[
 C_{1mth} = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j C_{1mtij}, \quad C_{2mth} = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j C_{2mtij}
\]

Definition 1 [36]: The system (13) is said to be stochastically stable with \( \tilde{w} = 0 \), if for every zero initial condition \( \tilde{x}(0), \sigma_0 \in \mathcal{L}, \tilde{\sigma}_0 \in \mathcal{M} \), the following inequality holds:
\[
 E\left\{\sum_{k=0}^{\infty} \|\tilde{x}(k)\|^2 |\tilde{x}(0), \sigma_0, \tilde{\sigma}_0\right\} < \infty.
\]
(14)

This paper aims to derive an asynchronous fuzzy controller in (9) such that the augmented system (13) can meet the following two conditions simultaneously.

i) The system (13) keeps stochastically stable under \( \tilde{w} = 0 \).

ii) The system (13) achieves an \( H_{\infty} \) output tracking perfor-
mance index $\rho$, i.e., for all nonzero $\bar{\omega} \in L_2[0, +\infty)$ and under zero initial condition,

$$E \left\{ \sum_{k=0}^{\infty} e^T(k)e(k) \right\} \leq \rho^2 \sum_{k=0}^{\infty} \bar{\omega}^T(k)\bar{\omega}(k).$$  \hfill (15)

### III. Main Results

In this section, the $H_\infty$ output tracking performance analysis for the system (13) will be presented and the asynchronous control strategy will be further developed.

**Theorem 1.** Given scalars $\rho > 0$, $\gamma \in [0,1]$ and gain matrices $K_i(j \in I)$, the system (13) is stochastically stable and achieves the prescribed $H_\infty$ output tracking performance $\rho$, if there exist matrices $P_{mi} > 0$, $Q_{mti} > 0$ such that for $\forall m \in \mathcal{L}$, $\forall t \in \mathcal{M}$ and $\forall f, i \in I$, the following inequalities hold

$$\sum_{t \in \mathcal{M}_k^{(m)}} \lambda_{mt} Q_{mti} < P_{mi} \quad \sum_{t \in \mathcal{M}_k^{(m)}} Q_{mti} < P_{mi}$$  \hfill (16)

$$\Phi_{mtfii} < 0 \quad \Phi_{mtfij} < 0, \quad i < j$$  \hfill (17)

where

$$\Phi_{mtfij} = \begin{bmatrix} -Q_{mti} & 0 & \tilde{C}_{mtij}^T & \Phi_{14} \\ * & -\rho^2 I & \tilde{F}_{mti}^T & \Phi_{24} \\ * & * & -I & \Phi_{34} \\ * & * & * & -\Phi_{44} \end{bmatrix}$$

$$\Phi_{14} = \begin{bmatrix} \sqrt{\pi_{m1}} \tilde{P}_{1,f} \tilde{A}_{mtij} \\ \sqrt{\pi_{m2}} \tilde{P}_{2,f} \tilde{A}_{mtij} \\ \cdots \\ \sqrt{\pi_{mL}} \tilde{P}_{L,f} \tilde{A}_{mtij} \end{bmatrix}, \quad \Phi_{24} = \begin{bmatrix} \sqrt{\pi_{m1}} \tilde{P}_{1,f} \tilde{E}_{mi} \\ \sqrt{\pi_{m2}} \tilde{P}_{2,f} \tilde{E}_{mi} \\ \cdots \\ \sqrt{\pi_{mL}} \tilde{P}_{L,f} \tilde{E}_{mi} \end{bmatrix}, \quad \Phi_{34} = \begin{bmatrix} \sqrt{\pi_{m1}} \tilde{P}_{1,i} \tilde{A}_{mtij} \\ \sqrt{\pi_{m2}} \tilde{P}_{2,i} \tilde{A}_{mtij} \\ \cdots \\ \sqrt{\pi_{mL}} \tilde{P}_{L,i} \tilde{A}_{mtij} \end{bmatrix}, \quad \Phi_{44} = \begin{bmatrix} \sqrt{\pi_{m1}} \tilde{P}_{1,i} \tilde{E}_{mi} \\ \sqrt{\pi_{m2}} \tilde{P}_{2,i} \tilde{E}_{mi} \\ \cdots \\ \sqrt{\pi_{mL}} \tilde{P}_{L,i} \tilde{E}_{mi} \end{bmatrix}$$

**Proof:** Choose the following Lyapunov function for the system (13):

$$V(k) = \bar{x}^T(k)P_{mh}\bar{x}(k)$$  \hfill (19)

where

$$P_{mh} = \sum_{i=1}^{r} h_i P_{mi}, \quad P_{mi} = \text{diag} \{P_{1mi}, P_{2mi}\}.$$  \hfill (20)

Firstly, when $\bar{\omega}(k) = 0$, one has

$$E \{ \Delta V(k) \} = \sum_{i=1}^{r} h_i \Delta \bar{V}(k)$$

$$= \sum_{i=1}^{r} h_i \left\{ \sum_{k \in \mathcal{M}} \lambda_{mt} \tilde{A}_{mth}^T \tilde{P}_{mth} + \tilde{A}_{mth} - P_{mth} \right\} \bar{x}(k)$$

$$\geq \sum_{i=1}^{r} h_i \left\{ \sum_{k \in \mathcal{M}} \lambda_{mt} \tilde{A}_{mth}^T \tilde{P}_{mth} + \tilde{A}_{mth} - P_{mth} \right\} \bar{x}(k),$$  \hfill (21)

where

$$\tilde{P}_{mth} = \text{diag} \{P_{mth}, P_{mth}\}, \quad \tilde{A}_{mth} = \sum_{i=1}^{r} h_i h_j \tilde{A}_{mthij}$$

$$P_{mth} = \sum_{n=1}^{L} \pi_{mn} P_{mth}, \quad P_{mth} = \sum_{f=1}^{r} h_f^T P_{nf}, \quad h^+ = h_{k+1}.$$

From (16), it leads to

$$\sum_{t \in \mathcal{L}_k^{(m)}} \lambda_{mt} Q_{mti} < P_{mh}, \quad \sum_{t \in \mathcal{L}_k^{(m)}} Q_{mti} < P_{mh},$$  \hfill (22)

where

$$Q_{mti} = \sum_{i=1}^{r} h_i Q_{mti}, \quad Q_{mti} = \text{diag} \{Q_{1mti}, Q_{2mti}\}.$$  \hfill (23)

Combining (17), (18) with the system (13), the following relationship is true

$$\Phi_{mt} = \begin{bmatrix} -Q_{mt} & 0 & \tilde{C}_{mt}^T & \Phi_{14} \\ * & -\rho^2 I & \tilde{F}_{mt}^T & \Phi_{24} \\ * & * & -I & \Phi_{34} \\ * & * & * & -\Phi_{44} \end{bmatrix}$$

$$\Phi_{14} = \begin{bmatrix} \sqrt{\pi_{m1}} \tilde{P}_{1h} \tilde{A}_{mth} \\ \sqrt{\pi_{m2}} \tilde{P}_{2h} \tilde{A}_{mth} \\ \cdots \\ \sqrt{\pi_{mL}} \tilde{P}_{Lh} \tilde{A}_{mth} \end{bmatrix}, \quad \Phi_{24} = \begin{bmatrix} \sqrt{\pi_{m1}} \tilde{P}_{1h} \tilde{E}_{mi} \\ \sqrt{\pi_{m2}} \tilde{P}_{2h} \tilde{E}_{mi} \\ \cdots \\ \sqrt{\pi_{mL}} \tilde{P}_{Lh} \tilde{E}_{mi} \end{bmatrix}, \quad \Phi_{34} = \begin{bmatrix} \sqrt{\pi_{m1}} \tilde{P}_{1i} \tilde{A}_{mthij} \\ \sqrt{\pi_{m2}} \tilde{P}_{2i} \tilde{A}_{mthij} \\ \cdots \\ \sqrt{\pi_{mL}} \tilde{P}_{Li} \tilde{A}_{mthij} \end{bmatrix}, \quad \Phi_{44} = \begin{bmatrix} \sqrt{\pi_{m1}} \tilde{P}_{1i} \tilde{E}_{mi} \\ \sqrt{\pi_{m2}} \tilde{P}_{2i} \tilde{E}_{mi} \\ \cdots \\ \sqrt{\pi_{mL}} \tilde{P}_{Li} \tilde{E}_{mi} \end{bmatrix}.$$  \hfill (24)

Due to $P_{mth} = \sum_{i=1}^{r} h_i^T P_{nf}$, $P_{nf} > 0$, by using Schur complement, it follows that

$$\tilde{A}_{mth}^T \tilde{P}_{mth} + \tilde{A}_{mth} - Q_{mth} < 0$$  \hfill (25)

and

$$\Xi_{mth}^T \tilde{P}_{mth} \Xi_{mth}^T + \Xi_{mth}^T \Xi_{mth}^T + \Xi_{mth}^T < 0$$  \hfill (26)

where

$$\Xi_{mth} = \begin{bmatrix} \tilde{A}_{mth} & \tilde{E}_{mi} \\ * & -Q_{mth} \end{bmatrix}, \quad \Xi_{mth} = \begin{bmatrix} \tilde{C}_{mth} & \tilde{F}_{mi} \\ * & -\rho^2 I \end{bmatrix}.$$  \hfill (27)

From (21), (23) and (24), it can be derived that

$$\sum_{t \in \mathcal{M}} \lambda_{mt} \tilde{A}_{mth}^T \tilde{P}_{mth} + \tilde{A}_{mth} < P_{mth}$$  \hfill (28)
\[
\sum_{t \in M} \lambda_{mt} (\Xi_{mth}^T \hat{P}_{mh} + \Xi_{mth}^2 + \Xi_{mth}^3 + \hat{\Xi}_{mth}^3) < 0 \quad (26)
\]
where
\[
\hat{\Xi}_{mth}^3 = \text{diag} \{ -P_{mh}, -\rho^2 I \}
\]

According to (25), it yields
\[
E \{ \Delta V(k) \} < -\lambda_{\min} \left( - \sum_{t \in M} \lambda_{mt} \hat{A}_{mth}^T \hat{P}_{mh} + \hat{A}_{mth} + P_{mh} \right) \hat{x}(k) \hat{x}(k) < 0
\]
(27)
where
\[
\mu = \inf \left\{ \lambda_{\min} \left( - \sum_{t \in M} \lambda_{mt} \hat{A}_{mth}^T \hat{P}_{mh} + \hat{A}_{mth} + P_{mh} \right) \right\}.
\]

Then, it concludes that
\[
E \left\{ \sum_{k=0}^{\infty} \| \hat{x}(k) \|^2 | \hat{x}(0), \sigma_0, \theta_0 \right\} < \frac{1}{\mu} E \{ V(0) - V(\infty) \} < \frac{1}{\mu} E \{ V(0) \} < \infty
\]
which implies that the system (13) is stochastically stable.

In addition, regarding the tracking performance, one can obtain
\[
E \{ \Delta V(k) + e^T(k) e(k) - \rho^2 \hat{\omega}^T(k) \hat{\omega}(k) \} = \xi^T(k) \left( \sum_{t \in M} \lambda_{mt} (\Xi_{mth}^T \hat{P}_{mh} + \Xi_{mth}^2 + \Xi_{mth}^3 + \hat{\Xi}_{mth}^3) \right) \xi(k)
\]
where
\[
\xi(k) = \begin{bmatrix} \hat{x}(k) \\ \hat{\omega}(k) \end{bmatrix}.
\]
Recalling (26), it holds that
\[
\sum_{k=0}^{\infty} E \{ \Delta V(k) + e^T(k) e(k) - \rho^2 \hat{\omega}^T(k) \hat{\omega}(k) \} < 0
\]
(30)
Under the zero initial condition, it can be inferred that
\[
\sum_{k=0}^{\infty} E \{ e^T(k) e(k) \} < \rho^2 \hat{\omega}^T(k) \hat{\omega}(k). \quad (31)
\]
This completes the proof.

The following theorem gives the solutions to the controller gains for the system (13).

**Theorem 2.** Given scalars \( \rho > 0 \), \( \gamma \in [0, 1] \) and gain matrices \( K_{ij} (j \in \mathbb{I}) \), the system (13) is stochastically stable and achieves the prescribed \( H_{\infty} \) \( \infty \) tracking performance \( \rho \), if there exist matrices \( \bar{P}_{mi} > 0 \), \( \bar{Q}_{mti} > 0 \), \( X_t \) and \( Y_{tj} \) such that for \( \forall m \in \mathcal{L}, \forall t \in \mathcal{M} \) and \( \forall f, i, j \in \mathbb{I} \), the following inequalities hold
\[
\begin{bmatrix} -\bar{P}_{mi} & P_{mi} \\ \ast & -Q_{mti} \end{bmatrix} < 0
\]
(32)
Moreover, the controller gains \( K_{ij} \) in (8) can be obtained by
\[
K_{ij} = Y_{tj} X_t^{-1}
\]
(35)
where if \( \mathcal{M}_{\mathcal{M}(m)} \neq \emptyset \), \( \mathcal{M}_{\mathcal{M}(m)} \neq \emptyset \),
\[
\begin{align*}
\mathbf{P}_{mi} & \triangleq \bar{P}_{mi} \begin{bmatrix} \sqrt{\pi_{11}} \hat{Q}_{mti}^{11} & \sqrt{\pi_{21}} \hat{Q}_{mti}^{21} & \cdots & \sqrt{\pi_{M1}} \hat{Q}_{mti}^{M1} \\ \sqrt{\pi_{12}} \hat{Q}_{mti}^{12} & \sqrt{\pi_{22}} \hat{Q}_{mti}^{22} & \cdots & \sqrt{\pi_{M2}} \hat{Q}_{mti}^{M2} \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{\pi_{1M}} \hat{Q}_{mti}^{1M} & \sqrt{\pi_{2M}} \hat{Q}_{mti}^{2M} & \cdots & \sqrt{\pi_{MM}} \hat{Q}_{mti}^{MM} \end{bmatrix} \\
\mathbf{Q}_{mti} & \triangleq \begin{bmatrix} \hat{Q}_{mti}^{11} & \hat{Q}_{mti}^{21} & \cdots & \hat{Q}_{mti}^{1M} \\ \hat{Q}_{mti}^{21} & \hat{Q}_{mti}^{22} & \cdots & \hat{Q}_{mti}^{2M} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{Q}_{mti}^{M1} & \hat{Q}_{mti}^{M2} & \cdots & \hat{Q}_{mti}^{MM} \end{bmatrix}
\end{align*}
\]
(33)
\[
\begin{align*}
\mathbf{P}_{mtfij} & = \begin{bmatrix} \Psi_{mtfii} & 0 & \Psi_{mtfij} & \Psi_{mtfji} \\ 0 & -\rho^2 I & F_{mti} & \Phi_{mtij} \\ \ast & \ast & -I & 0 \\
\ast & \ast & \ast & -\Phi_{mtij} \end{bmatrix} < 0
\end{align*}
\]
(34)
(35)

Proof: Since \( \hat{Q}_{mti} > 0 \), it can be obtained via Schur complement
\[
\sum_{t \in M} \lambda_{mt} \bar{P}_{mi} \hat{Q}_{mti}^{-1} \bar{P}_{mi}^T < \bar{P}_{mi}
\]
(36)
and
\[
\sum_{t \in M} \bar{P}_{mi} \hat{Q}_{mti}^{-1} \bar{P}_{mi}^T < \bar{P}_{mi}
\]
(37)
where \( \bar{P}_{mi} = P_{mi}^{-1} \), \( \bar{Q}_{mti} = Q_{mti}^{-1} \).
Pre- and post-multiplying \( \bar{P}_{mi} \) simultaneously to (36) and (37), one can get (16), which indicates that the result in (32) is consistent with (16).
Pre- and post-multiplying (33) by \( \text{diag} \{ I, I, \Psi_{mtfii}^{-1} \} \) and its transpose, it leads to
\[
\Psi_{mtfii} < 0
\]
\[
\Psi_{14} = \begin{bmatrix}
\sqrt{\rho_{mi}} \hat{P}_{mi} \hat{A}_{m_i} & \sqrt{\rho_{m2}} \hat{P}_{m2} \hat{A}_{m2} \\
\sqrt{\rho_{m3}} \hat{P}_{m3} \hat{A}_{m3} & \ldots \\
\sqrt{\rho_{mL}} \hat{P}_{mL} \hat{A}_{mL}
\end{bmatrix}, \quad \Psi_{24} = \begin{bmatrix}
\sqrt{\rho_{m1}} \hat{P}_{m1} \hat{E}_{mi} & \sqrt{\rho_{m2}} \hat{P}_{m2} \hat{E}_{mi} \\
\sqrt{\rho_{m3}} \hat{P}_{m3} \hat{E}_{m3} & \ldots \\
\sqrt{\rho_{mL}} \hat{P}_{mL} \hat{E}_{mL}
\end{bmatrix}.
\]

Notice that \(\bar{Q}_{m_i} > 0\), hence the following inequality holds
\[
\left( Q^{-1}_{m_i} - \chi^T \right) Q_{m_i} \left( Q^{-1}_{m_i} - \chi \right) > 0,
\]
which leads to
\[
-\chi^T Q_{m_i} \chi_i < \bar{Q}_{m_i} - \chi_i^T - \chi_i
\]
where \(\chi_i = \text{diag} \{ X_i, X_i \}\).

Further, substituting \(\Psi_{11}\) for \(-\chi_i^T Q_{m_i} \chi_i\), it can be found that
\[
\Psi''_{mfi} < 0
\]
where
\[
\Psi''_{mfi} = \begin{bmatrix}
-\chi^T Q_{m_i} \chi_i & 0 & 0 \\
* & -\rho^2 I & \bar{C}_m^T \\
* & * & -I & 0
\end{bmatrix} \begin{bmatrix}
\Psi_{11}^T \\
\Psi_{12}^T \\
\Psi_{12}^T
\end{bmatrix}.
\]

Pre- and post-multiply (41) by \(\Delta \{ \chi_i^{-T}, I, I, I \} \) and \(\Delta \{ \chi_i^{-T}, I, I, I \} \), respectively. (17) can be attained, which means that (33) can be established from (17). Using the similar way, (34) is available from (18). This completes the proof.

**Remark 2.** In Theorem 2, the design method of the controller is provided under the prescribed performance index \(\rho\). The tracking control issue is transformed into the a convex optimization problem, in which the minimal value of \(\rho\) and controller gains can be acquired by:
\[
\min \rho
\]
\[
s.t. \quad (32) - (34) \text{ with } \rho = \rho^2.
\]

Then \(\rho^* = \sqrt{\rho}\) is regarded as the optimal tracking performance.

**Remark 3.** It is noted that Theorem 2 gives a sufficient condition that ensures the system (13) is stochastically stable with an \(H_\infty\) output tracking performance \(\rho\), in which the conditional probabilities are partial accessible. The results in Theorem 2 can be extended to the case where the conditional probabilities are completely known, which is shown in the following corollary.

**Corollary 1.** Given scalars \(\rho > 0, \gamma \in [0,1]\) and gain matrices \(K_j (j \in \mathbb{I})\), the system (13) is stochastically stable and achieves the prescribed \(H_\infty\) output tracking performance \(\rho\), if there exist matrices \(\bar{P}_{mi} > 0, \bar{Q}_{m_i} > 0, X_i \) and \(Y_{ij}\) such that for \(\forall m \in \mathbb{L}, \forall t \in \mathbb{M}\) and \(\forall f, i, j \in \mathbb{I}\), the following inequalities hold
\[
\begin{bmatrix}
-\bar{P}_{m_i} & \bar{P}_{m_i} \\
* & -\bar{Q}_{m_i}
\end{bmatrix} < 0
\]
\[
\Psi_{mfi} < 0
\]
\[
\Psi_{mfi} + \Psi_{mtj} < 0, \quad i < j
\]
Moreover, the controller gains \(K_j\) in (8) can be obtained by
\[
K_j = Y_j X_i^{-1}
\]
and the definitions of other parameters are the same as those in Theorem 2.

**IV. SIMULATION EXAMPLE**

In this section, a numerical example is given to illustrate the effectiveness of the achieved theoretical results.

The model parameters of the fuzzy system (3) are considered as follows \((r = 2, L = 2)\):
\[
A_{11} = \begin{bmatrix} 0.14 & 0.3 \\ 0.7 & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} -0.14 & 0.3 \\ 0.7 & 0 \end{bmatrix}, \\
A_{21} = \begin{bmatrix} 0.14 & 0.3 \\ 0.7 & 0 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} -0.14 & 0.3 \\ 0.7 & 0 \end{bmatrix}, \\
B_{11} = B_{12} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, \quad B_{21} = B_{22} = \begin{bmatrix} 0.4 \\ 1 \end{bmatrix}, \\
E_{11} = E_{12} = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \quad E_{21} = E_{22} = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, \\
C_{11} = C_{21} = \begin{bmatrix} 1 \\ 0.1 \end{bmatrix}, \quad C_{12} = C_{22} = \begin{bmatrix} 1 \\ 0.05 \end{bmatrix}, \\
D_{11} = D_{12} = 0.1, \quad D_{21} = D_{22} = 0.3, \\
F_{11} = F_{12} = 0.1, \quad F_{21} = F_{22} = 0.2.
\]

The reference model is given by
\[
A_r = \begin{bmatrix} -0.1 \\ -0.1 \end{bmatrix}, \quad B_r = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C_r = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix}.
\]

Here, the fuzzy basis functions are assumed as:
\[
h_1(x_1(k)) = 0.5|\sin(x_1(k)) + \cos(x_1(k))|, \\
h_2(x_1(k)) = 1 - h_1(x_1(k)).
\]

It is supposed that the operational modes of the system (3) obey the following transition probability matrix:
\[
\Pi = \begin{bmatrix} 0.4 & 0.6 \\ 0.35 & 0.65 \end{bmatrix}
\]

And the modes of the controller are determined by the following conditional probability matrices in two cases \((M = 3)\):
\[
A^{(1)} = \begin{bmatrix} 0.4 & 0.2 & 0.4 \\ 0.4 & 0.1 & 0.5 \end{bmatrix}, \quad A^{(2)} = \begin{bmatrix} 0.4 & ? & ? \\ 0.4 & ? & ? \end{bmatrix}.
\]

In this example, the external disturbance and the reference input are set as \(\omega = 0.1\cos(2k), \quad r(k) = 0.5\sin(0.3k)\), respectively. Let the packet arriving rate \(\gamma = 0.8\), the controller gains in two cases can be computed via Corollary 1 and Theorem 2, respectively, which is shown in Table I.

The initial states are chosen as \(x(0) = [0.5 \quad -0.1]^T, \quad x_r(0) = [0.2 \quad 0.1]^T\). The modes evolution of the system and the controller are presented in Fig. 2. From Fig. 3 and Fig. 5, it can be seen that the system outputs \(z(k)\) in two cases track the reference signals \(z_r(k)\) well under the asynchronous control laws. And the overall tracking effect of Case 2 is not
Table I: Controller gain matrices ($\gamma = 0.8$)

<table>
<thead>
<tr>
<th>$t$</th>
<th>$j$</th>
<th>$K_{t,j}$ (Case 1)</th>
<th>$K_{t,j}$ (Case 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>$[-0.2192 - 0.3244]$</td>
<td>$[-0.4401 - 0.1739]$</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>$[-0.1975 - 0.3149]$</td>
<td>$[-0.3078 - 0.2564]$</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>$[-0.2232 - 0.3301]$</td>
<td>$[-0.1556 - 0.2978]$</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>$[-0.1642 - 0.2141]$</td>
<td>$[-0.3889 - 0.1095]$</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>$[-0.1297 - 0.2226]$</td>
<td>$[-0.3332 - 0.1103]$</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>$[-0.1720 - 0.2153]$</td>
<td>$[-0.2250 - 0.1455]$</td>
</tr>
</tbody>
</table>

inferior to that of Case 1, although the partial probability information of Case 2 is unknown. Fig. 4 and Fig. 6 show the corresponding output tracking errors of two cases, which mean that our designed approach is feasible and effective.

Next, it is worthwhile to investigate how the value of $\gamma$ affects the optimal tracking performance $\rho^*$. Let $\gamma$ takes different values, the values of $\rho^*$ in two cases are listed in Tables II and III, respectively. It can be easily found that the smaller $\gamma$ is, the bigger $\rho^*$ is. And when the value of $\gamma$ is the same in both cases, $\rho^*$ of Case 2 is a relatively larger value compared with $\rho^*$ of Case 1. To further clarify the impact of $\gamma$ on the system performance, the outputs curves for two cases under $\gamma = 1, 0.5, 0.1$ are plotted in Fig. 7 and Fig. 8, respectively. It is evident that smaller packet arriving rate $\gamma$ results in poorer tracking performance of the system, which is consistent with the objective actuality.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>1</th>
<th>0.8</th>
<th>0.5</th>
<th>0.2</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho^*$</td>
<td>0.8426</td>
<td>0.8944</td>
<td>1.0050</td>
<td>1.1832</td>
<td>1.2042</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>1</th>
<th>0.8</th>
<th>0.5</th>
<th>0.2</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho^*$</td>
<td>1.3416</td>
<td>1.4283</td>
<td>1.6941</td>
<td>2.2136</td>
<td>2.5100</td>
</tr>
</tbody>
</table>
V. CONCLUSION

The issue of the asynchronous $H_\infty$ output tracking control has been investigated for a class of discrete-time FMJSs, which takes the random packet dropout into consideration. Sufficient conditions have been acquired to ensure that the resulting system is stochastically stable with an $H_\infty$ tracking performance index via the Lyapunov function. The asynchronous controller has also been derived based on the HMM approach, in which both cases are covered: partial known and completely known conditional probabilities. The validity of the method proposed in this paper has been illustrated by a simulation example. It is necessary to emphasize that this paper has only discussed the incomplete information in the HMM, the problem of partly unknown transition probabilities has not been involved. Accordingly, how to extend the obtained results to a more general case including partial information on both transition probabilities and conditional probabilities, which deserves further development.

REFERENCES


