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Fuzzy Affine Model-Based Sliding Mode Control for Discrete-Time Nonlinear Two-Dimensional Systems via Output Feedback

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Abstract—This work investigates the issue of output feedback sliding mode control (SMC) for nonlinear two-dimensional systems by Takagi-Sugeno fuzzy affine models. Via combining with the sliding surface, the sliding mode dynamical properties are depicted by a singular piecewise affine system. Through piecewise quadratic Lyapunov functions, new stability and robust performance analysis of the sliding motion are carried out. An output feedback dynamic SMC design approach is developed to guarantee that the system states can converge to a neighborhood of the sliding surface. Simulation studies are given to verify the validity of the proposed scheme.

Index Terms—Fuzzy control; nonlinear two-dimensional systems; output feedback; sliding mode control; convex optimization.

I. INTRODUCTION

A variety of real-world processes or systems intrinsically contain multi-dimensional (m-D) features, for example, thermal processes, image deblurring, and gas absorption [1]–[3]. As a distinct and common type of m-D systems, two-dimensional (2-D) systems usually possess highly complicated nonlinear properties. In past decades, growing research attention has been focused on 2-D systems and many elegant results have been reported [4]–[9]. Particularly, [4] investigated the robust control issue for discrete-time 2-D systems on the basis of Roesser models. The authors in [6] addressed the fault detection issue for 2-D Markov jumping systems.

As a typical class of model-based fuzzy system, Takagi-Sugeno (T-S) fuzzy models have appealed much attention in recent years [10]–[15]. In any convex compact set, T-S fuzzy models have been proved to be general function approximators with enhanced approximation competence to smooth nonlinear plants to arbitrary degrees of accuracy [16]–[23]. Thus, much attention has been also concentrated on 2-D fuzzy systems

recently [24]–[28]. Specifically, in [27], the state feedback control issue for 2-D time-delay T-S fuzzy systems was studied via Roesser models.

Sliding mode control (SMC) has attracted increasing research attention from academic communities to industrial processes, as it possesses elegant characteristics, namely, high convergence speed, good transient performance, and strong robustness [29]–[39]. The sliding mode control contains two main design steps, that is, an appropriate sliding surface (or known as switching surface) firstly is designed wherein prescribed performance is satisfied when the system states remaining on the sliding surface and moving to the origin. Hereafter, a reaching law is organized to force the system states to converge towards the sliding surface [40], [41]. Due to the appealing features of SMC, much effort has been devoted to Markov jump systems [42], [43], descriptor systems [44], delta operator systems [45], fuzzy systems [46], [47] and so on. More recently, some advanced works on fuzzy output feedback SMC have also appeared [48], [49]. Due to the practical engineering and theoretical significance of SMC, a variety of SMC results were also proposed for 2-D systems [50]–[54]. Specifically, *without considering the external disturbance*, [50] studied the SMC problem for discrete-time two-dimensional systems through Roesser models. [53] investigated the SMC issue for 2-D systems with exogenous disturbances via *state feedback*.

Note that the prevailing SMC results on 2-D systems in [50]–[53] were generally attained via the full state feedback approach. Nonetheless, in practice, the system full state variables are not always accessible. Additionally, all these results in [50]–[54] were basically attained under a common quadratic Lyapunov function framework, which tend to have conservatism. Generally, T-S fuzzy affine models inherently have more powerful function approximation capability [17]. All the aforementioned issues indicate that the output feedback SMC for nonlinear 2-D systems by T-S fuzzy affine models still remain challenging and significant, which motivates this work.

This study aims to synthesize a robust output feedback dynamic sliding mode controller for discrete-time nonlinear 2-D systems subject to external disturbances via T-S fuzzy affine models. By associating with sliding surface function, the sliding mode dynamical properties are described by a singular piecewise affine system. Through piecewise quadratic Lyapunov functions (PQLFs), novel stability and robust performance analysis on the sliding motion are carried out, and

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the gain of the sliding surface can be attained via a convex optimization process. A new output feedback dynamic SMC synthesis scheme is developed to guarantee that the system states can be forced into a neighborhood of the sliding surface. Simulation studies are carried out to verify the effectiveness of the developed approach. Main contributions of this work are: 1) A singular piecewise affine system scheme is proposed to tackle the couplings between the control input matrices and the sliding surface gain, and the stability and robust performance analysis results are proposed under a unified convex optimization setup; 2) Relying on system measurement output, the designed output feedback sliding mode controller is suggested to be more applicable; 3) The designed controller is in a dynamic form, both matched and mismatched external disturbances can be handled; 4) The conservatism of the stability and robust performance analysis of sliding motion is further relaxed via adopting PQLFs.

II. PRELIMINARIES

A. Model Description

In the following, consider a discrete-time nonlinear 2-D system characterized via the 2-D T-S fuzzy affine model with r fuzzy IF-THEN rules as,

Plant Rule $\mathcal{R}^l$: IF $\zeta_1(x(i, j))$ is $\mathcal{F}_1^l$ and $\dots$ and $\zeta_\varphi(x(i, j))$ is $\mathcal{F}_\varphi^l$, THEN

$$\begin{cases} x^+(i, j) = (A_l + \Delta A_l) x(i, j) + a_l + \Delta a_l \\ \quad + B_l u(i, j) + D_l w(i, j) \\ y(i, j) = C x(i, j) \\ z(i, j) = L_l x(i, j) + N_l u(i, j), \quad l \in \mathcal{L} = \{1, \dots, r\} \end{cases} \quad (1)$$

where $\mathcal{F}_\phi^l$ ($\phi = 1, \dots, \varphi$) stand for fuzzy sets; $\mathcal{R}^l$ is the l -th fuzzy inference rule; $\zeta(x(i, j)) := [\zeta_1(x(i, j)), \dots, \zeta_\varphi(x(i, j))]$ denote the measurable premise variables; r refers to the number of inference rules; $x(i, j) = [x^h(i, j) \quad x^v(i, j)]^T$, $x^+(i, j) = [x^h(i+1, j) \quad x^v(i, j+1)]^T$, $x^h(i, j) \in \mathbb{R}^{n_x}$ and $x^v(i, j) \in \mathbb{R}^{n_v}$ are the horizontal states and vertical states, respectively, and $n_x = n_x^h + n_x^v$; $y(i, j) \in \mathbb{R}^{n_y}$ stands for the measurement output; $w(i, j) \in l_2\{[0, \infty), [0, \infty)\}$ stands for the exogenous disturbance with $w(i, j) \in \mathbb{R}^{n_w}$; $z(i, j) \in \mathbb{R}^{n_z}$ denotes the regulated output; $u(i, j) \in \mathbb{R}^{n_u}$ represents the control input; ΔA_l and Δa_l represent the uncertainty terms satisfying

$$\begin{bmatrix} \Delta A_l & \Delta a_l \end{bmatrix} = H_{l1} \Delta_l(i, j) \begin{bmatrix} H_{l2} & H_{l3} \end{bmatrix}, l \in \mathcal{L} \quad (2)$$

with H_{l1} , H_{l2} , and H_{l3} being known matrices. $\Delta_l(i, j) \in \mathbb{R}^{s_1 \times s_2}$ represent unknown matrices satisfying

$$\|\Delta_l(i, j)\| \leq \mathbf{I}, \quad l \in \mathcal{L}. \quad (3)$$

Denote $\mu_l[\zeta(x(i, j))] = \mu_l[x^h(i, j), x^v(i, j)]$ as the normalized membership function (MF),

$$\mu_l[\zeta(x(i, j))] := \frac{\prod_{\phi=1}^{\varphi} \mu_{\phi}[\zeta_{\phi}(x(i, j))]}{\sum_{\nu=1}^r \prod_{\phi=1}^{\varphi} \mu_{\nu}[\zeta_{\phi}(x(i, j))]} \geq 0, \quad \sum_{l=1}^r \mu_l[\zeta(x(i, j))] = 1 \quad (4)$$

with $\mu_{\phi}[\zeta_{\phi}(x(i, j))]$ being the grade of membership of $\zeta_{\phi}(x(i, j))$ in $\mathcal{F}_{\phi}^l$. For brevity, we use $\mu_l := \mu_l[\zeta(x(i, j))]$.

By a product inference, center-average defuzzifier, and singleton fuzzifier, formulate the system (1) as,

$$\begin{cases} x^+(i, j) = (A(\mu) + \Delta A(\mu)) x(i, j) + a(\mu) + \Delta a(\mu) \\ \quad + B(\mu)u(i, j) + D(\mu)w(i, j) \\ y(i, j) = Cx(i, j) \\ z(i, j) = L(\mu)x(i, j) + N(\mu)u(i, j) \end{cases} \quad (5)$$

with

$$\begin{cases} A(\mu) = \sum_{l=1}^r \mu_l A_l, \Delta A(\mu) = \sum_{l=1}^r \mu_l \Delta A_l, \\ a(\mu) = \sum_{l=1}^r \mu_l a_l, \Delta a(\mu) = \sum_{l=1}^r \mu_l \Delta a_l, \\ B(\mu) = \sum_{l=1}^r \mu_l B_l, D(\mu) = \sum_{l=1}^r \mu_l D_l, \\ L(\mu) = \sum_{l=1}^r \mu_l L_l, N(\mu) = \sum_{l=1}^r \mu_l N_l. \end{cases} \quad (6)$$

Since MFs and fuzzy rules induce the polyhedral decomposition of the system state-space, along the lines in [17], one can divide the premise variable space into crisp subspaces (for some l , $\mu_l = 1$) and fuzzy subspaces (for some local models, $0 < \mu_l < 1$).

The indices of subspaces can be categorized as $\mathcal{I} = \mathcal{I}_0 \cup \mathcal{I}_1$, where $\mathcal{I}_1$ is the index set of subspaces not covering the origin, while $\mathcal{I}_0$ includes the index set of subspaces containing the origin. To describe the indices within each region S_i , introduce the subsequent set

$$\mathcal{N}(n) := \{m | \mu_m > 0, m \in \mathcal{L}, x(i, j) \in S_n, n \in \mathcal{I}\}. \quad (7)$$

Relying on (7), reformulate the system (5) as

$$\begin{cases} x^+(i, j) = (\mathcal{A}_n + \Delta \mathcal{A}_n) x(i, j) + a_n + \Delta a_n \\ \quad + \mathcal{B}_n u(i, j) + \mathcal{D}_n w(i, j) \\ y(i, j) = Cx(i, j) \\ z(i, j) = \mathcal{L}_n x(i, j) + \mathcal{N}_n u(i, j), \quad x(i, j) \in S_n, n \in \mathcal{I} \end{cases} \quad (8)$$

where

$$\begin{cases} \mathcal{A}_n := \sum_{m \in \mathcal{N}(n)} \mu_m A_m, \Delta \mathcal{A}_n := \sum_{m \in \mathcal{N}(n)} \mu_m \Delta A_m, \\ a_n := \sum_{m \in \mathcal{N}(n)} \mu_m a_m, \Delta a_n := \sum_{m \in \mathcal{N}(n)} \mu_m \Delta a_m, \\ \mathcal{B}_n := \sum_{m \in \mathcal{N}(n)} \mu_m B_m, \mathcal{D}_n := \sum_{m \in \mathcal{N}(n)} \mu_m D_m, \\ \mathcal{L}_n := \sum_{m \in \mathcal{N}(n)} \mu_m L_m, \mathcal{N}_n := \sum_{m \in \mathcal{N}(n)} \mu_m N_m \end{cases} \quad (9)$$

where $0 < \mu_m \leq 1$, $\sum_{m \in \mathcal{N}(n)} \mu_m = 1$.

This paper attempts to propose an output feedback dynamic SMC design approach for 2-D fuzzy-affine system (1), and drive the system states into a neighborhood of the sliding surface. A new existence criterion for the sliding surface will be derived through PQLFs. For a prescribed disturbance attenuation level γ , the sliding motion can be guaranteed to be asymptotically stable when $w(i, j) \equiv 0$, and

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} z^T(i, j) z(i, j) < \gamma^2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} w^T(i, j) w(i, j) \quad (10)$$

for $0 \neq w(i, j) \in l_2\{[0, \infty), [0, \infty)\}$ with zero boundary conditions $x^h(0, j) = 0$ and $x^v(i, 0) = 0$.

B. Sliding Surface Design

Synthesize a new sliding surface as,

$$s(i, j) = Ky(i, j) + u(i, j) \quad (11)$$

where $K \in \mathbb{R}^{n_u \times n_v}$ is the sliding surface gain to be determined.

Similar to [50], [53], it is satisfied that $s(i, j) = 0$ for the ideal quasi sliding motion. Then one has

$$u(i, j) = -KCx(i, j). \quad (12)$$

Based on system (8) and (12), one can derive the sliding motion as

$$\begin{cases} x^+(i, j) = (\mathcal{A}_n + \Delta\mathcal{A}_n)x(i, j) + \mathbf{a}_n + \Delta\mathbf{a}_n \\ \quad + \mathcal{B}_n u(i, j) + \mathcal{D}_n w(i, j) \\ 0 \times u^+(i, j) = -KCx(i, j) - u(i, j) \\ z(i, j) = \mathcal{L}_n x(i, j) + \mathcal{N}_n u(i, j), n \in \mathcal{I}. \end{cases} \quad (13)$$

Reformulate system (13) as

$$\begin{cases} E\bar{x}^+(i, j) = (\bar{\mathcal{A}}_n + \Delta\bar{\mathcal{A}}_n)\bar{x}(i, j) + \bar{\mathbf{a}}_n + \Delta\bar{\mathbf{a}}_n \\ \quad + \bar{\mathcal{D}}_n w(i, j) \\ z(i, j) = \bar{\mathcal{L}}_n \bar{x}(i, j), n \in \mathcal{I} \end{cases} \quad (14)$$

where

$$\begin{cases} E = \text{diag}\{\mathbf{I}_{n_x}, \mathbf{0}_{n_u}\}, \\ \bar{x}^+(i, j) = \begin{bmatrix} x^+(i, j) \\ u^+(i, j) \end{bmatrix}, \bar{x}(i, j) = \begin{bmatrix} x(i, j) \\ u(i, j) \end{bmatrix}, \\ \bar{\mathcal{A}}_n + \Delta\bar{\mathcal{A}}_n = \begin{bmatrix} \mathcal{A}_n + \Delta\mathcal{A}_n & \mathcal{B}_n \\ -KC & -\mathbf{I} \end{bmatrix}, \\ \bar{\mathbf{a}}_n + \Delta\bar{\mathbf{a}}_n = \begin{bmatrix} \mathbf{a}_n + \Delta\mathbf{a}_n \\ \mathbf{0} \end{bmatrix}, \\ \bar{\mathcal{D}}_n = [\mathcal{D}_n^T \quad \mathbf{0}]^T, \bar{\mathcal{L}}_n = [\mathcal{L}_n \quad \mathcal{N}_n]. \end{cases} \quad (15)$$

By augmenting the sliding surface function (11) with the system (8), the singular system (14) can fully characterize the dynamical features of the sliding motion. Then the existence criterion for the sliding surface will be derived. For 2-D system (1), an output feedback dynamic SMC synthesis approach will be also proposed.

III. MAIN RESULTS

A. Stability and Performance Analysis of Sliding Motion

Firstly, introduce the subsequent set Ω to describe the subspace transitions as,

$$\Omega := \{(n, k) | x(i, j) \in S_n, x^+(i, j) \in S_k, n, k \in \mathcal{I}\}. \quad (16)$$

In this paper, according to [17], assume that an ellipsoid $\mathcal{E}_i$ can be adopted to outer approximate each polyhedral subspace S_n in (16) as

$$S_n \subseteq \mathcal{E}_n, \mathcal{E}_n := \{x | \|Q_n x + q_n\| \leq 1\}. \quad (17)$$

The aforementioned outer approximation is useful especially when S_n are slab subspaces. For this situation, parameters Q_n and q_n in (17) are ensured to exist, and the covering is

precise, namely, $\mathcal{E}_n \subseteq S_n$ and $S_n \subseteq \mathcal{E}_n$. In particular, when S_n are slabs with

$$S_n = \{x | \alpha_n \leq v_n^T x \leq \beta_n\}, n \in \mathcal{I} \quad (18)$$

where $v_n \in \mathbb{R}^{n_x}$, $\{\alpha_n, \beta_n\} \in \mathbb{R}$, one can adopt a degenerate ellipsoid shown as in (17) to exactly characterize each subspace S_n with

$$Q_n = \frac{2v_n^T}{\beta_n - \alpha_n}, q_n = -\frac{\beta_n + \alpha_n}{\beta_n - \alpha_n}. \quad (19)$$

Relying on (19), for each subspace, the following relationship holds,

$$\begin{bmatrix} 1 \\ x(i, j) \end{bmatrix}^T \begin{bmatrix} q_n^T q_n - 1 & \star \\ Q_n^T q_n & Q_n^T Q_n \end{bmatrix} \begin{bmatrix} 1 \\ x(i, j) \end{bmatrix} \leq 0, n \in \mathcal{I}. \quad (20)$$

Theorem 3.1. For a robust $\mathcal{H}_\infty$ performance γ , the system (14) is asymptotically stable, if matrices $0 < P_{n1}^h = P_{n1}^{hT} \in \mathbb{R}^{n_x \times n_x}$, $0 < P_{n1}^v = P_{n1}^{vT} \in \mathbb{R}^{n_v \times n_v}$, $P_{n2} \in \mathbb{R}^{n_u \times n_x}$, $P_{n3} \in \mathbb{R}^{n_u \times n_u}$, $n \in \mathcal{I}$, $\bar{K} \in \mathbb{R}^{n_u \times n_v}$, and $\{V_{n1}, W_{n1}\} \in \mathbb{R}^{n_x \times n_x}$, $\{V_{n2}, W_{n2}\} \in \mathbb{R}^{n_u \times n_x}$, $V_0 \in \mathbb{R}^{n_u \times n_u}$, and scalars $\{\varepsilon_n > 0, n \in \mathcal{I}\}$, $\{\rho_n < 0, n \in \mathcal{I}_1\}$ exist, such that the subsequent conditions hold,

$$\begin{bmatrix} \Psi_{11} & \star & \star & \star \\ \Psi_{21} - W_n^T & P_k - V_n - V_n^T & \star & \star \\ \Psi_{31} & \Psi_{32} & -\gamma^2 \mathbf{I} & \star \\ \Psi_{41} & \Psi_{42} & \mathbf{0} & -\varepsilon_n \mathbf{I} \end{bmatrix} < 0, \quad n \in \mathcal{I}_0, m \in \mathcal{N}(n), \quad (21)$$

$$\begin{bmatrix} \rho_n(q_n^T q_n - 1) + \varepsilon_n H_{m3}^T H_{m3} \\ \rho_n \bar{Q}_n^T q_n + \Psi_{10} \\ \Psi_{20} \\ \mathbf{0} \\ \mathbf{0} \\ \rho_n \bar{Q}_n^T \bar{Q}_n + \Psi_{11} & \star & \star & \star \\ \Psi_{21} - W_n^T & P_k - V_n - V_n^T & \star & \star \\ \Psi_{31} & \Psi_{32} & -\gamma^2 \mathbf{I} & \star \\ \Psi_{41} & \Psi_{42} & \mathbf{0} & -\varepsilon_n \mathbf{I} \end{bmatrix} < 0, \quad n \in \mathcal{I}_1, m \in \mathcal{N}(n) \quad (22)$$

where the notation $\star$ represents the terms induced by symme-

try, and

$$\left\{ \begin{array}{l} \Psi_{11} = \text{Sym} \left\{ \begin{bmatrix} W_{n1} \\ W_{n2} \end{bmatrix} \begin{bmatrix} A_m & B_m \end{bmatrix} \right. \\ \left. - \begin{bmatrix} J_2 \\ \varrho_2 \mathbf{I} \end{bmatrix} \begin{bmatrix} \bar{K}C & V_0 \end{bmatrix} \right\} + \varepsilon_n \begin{bmatrix} H_{m2}^T \\ \mathbf{0} \end{bmatrix} \quad (\star) \\ -E^T P_n E + \bar{L}_m^T \bar{L}_m, \\ \Psi_{21} = \begin{bmatrix} V_{n1} \\ V_{n2} \end{bmatrix} \begin{bmatrix} A_m & B_m \end{bmatrix} - \begin{bmatrix} J_1 \\ \varrho_1 \mathbf{I} \end{bmatrix} \begin{bmatrix} \bar{K}C & V_0 \end{bmatrix}, \\ \Psi_{31} = D_m^T \begin{bmatrix} W_{n1}^T & W_{n2}^T \end{bmatrix}, \\ \Psi_{32} = D_m^T \begin{bmatrix} V_{n1}^T & V_{n2}^T \end{bmatrix}, \\ \Psi_{41} = H_{m1}^T \begin{bmatrix} W_{n1}^T & W_{n2}^T \end{bmatrix}, \\ \Psi_{42} = H_{m1}^T \begin{bmatrix} V_{n1}^T & V_{n2}^T \end{bmatrix}, \\ \Psi_{10} = \begin{bmatrix} W_{n1} \\ W_{n2} \end{bmatrix} a_m + \varepsilon_n \begin{bmatrix} H_{m2}^T \\ \mathbf{0} \end{bmatrix} H_{m3}, \\ \Psi_{20} = \begin{bmatrix} V_{n1} \\ V_{n2} \end{bmatrix} a_m, P_n = \begin{bmatrix} \text{diag}\{P_{n1}^h, P_{n1}^v\} & \star \\ P_{n2} & P_{n3} \end{bmatrix}, \\ V_n = \begin{bmatrix} V_{n1} & J_1 V_0 \\ V_{n2} & \varrho_1 V_0 \end{bmatrix}, W_n = \begin{bmatrix} W_{n1} & J_2 V_0 \\ W_{n2} & \varrho_2 V_0 \end{bmatrix}, \\ \bar{L}_m = \begin{bmatrix} L_m & N_m \end{bmatrix}, \bar{Q}_n = \begin{bmatrix} Q_n & \mathbf{0}_{1 \times n_u} \end{bmatrix} \end{array} \right. \quad (23)$$

and $\text{Sym}\{\mathbb{N}\}$ is short for $\mathbb{N} + \mathbb{N}^T$, and $\{J_1, J_2\} \in \mathbb{R}^{n_x \times n_u}$ are arbitrary matrices, ϱ_1, ϱ_2 being constant scalars.

Additionally, the sliding surface gain can be calculated by

$$K = V_0^{-1} \bar{K}. \quad (24)$$

Proof: Notice that the condition (21) for $n \in \mathcal{I}_0$ denotes a special case of the condition (22) for $n \in \mathcal{I}_1$. In sequence, only the more complicated case for $n \in \mathcal{I}_1$ is proved without loss of generality. Construct a PQLF as,

$$V(\bar{x}(i, j), i, j) = \bar{x}^T(i, j) E^T P_n E \bar{x}(i, j) \quad (25)$$

with

$$P_n = \begin{bmatrix} \text{diag}\{P_{n1}^h, P_{n1}^v\} & \star \\ P_{n2} & P_{n3} \end{bmatrix}, \quad n \in \mathcal{I} \quad (26)$$

and $0 < P_{n1}^h = P_{n1}^{hT} \in \mathbb{R}^{n_x^h \times n_x^h}$, $0 < P_{n1}^v = P_{n1}^{vT} \in \mathbb{R}^{n_x^v \times n_x^v}$, $P_{n2} \in \mathbb{R}^{n_u \times n_x}$, $P_{n3} \in \mathbb{R}^{n_u \times n_u}$.

In view of (25), if

$$\begin{aligned} & (\bar{x}^+(i, j))^T E^T P_k E \bar{x}^+(i, j) \\ & - \bar{x}^T(i, j) E^T P_n E \bar{x}(i, j) + z^T(i, j) z(i, j) \\ & - \gamma^2 w^T(i, j) w(i, j) < 0, \quad (n, k) \in \Omega, \end{aligned} \quad (27)$$

the system (14) is asymptotically stable satisfying $\mathcal{H}_\infty$ performance γ .

Denote

$$\xi(i, j) = \begin{bmatrix} 1 & \bar{x}^T(i, j) & (E\bar{x}^+(i, j))^T & w^T(i, j) \end{bmatrix}^T \quad (28)$$

and then rewrite the inequality (27) as,

$$\xi^T(i, j) \begin{bmatrix} 0 & \star & \star & \star \\ \mathbf{0} & \bar{L}_n^T \bar{L}_n - E^T P_n E & \star & \star \\ \mathbf{0} & \mathbf{0} & P_k & \star \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\gamma^2 \mathbf{I} \end{bmatrix} \xi(i, j) < 0, \quad (n, k) \in \Omega. \quad (29)$$

Adopting the S-procedure relying on the partition information in (20), the subsequent inequality implies (29) for $\rho_n < 0$, $n \in \mathcal{I}_1$,

$$\xi^T(i, j) \Sigma \xi(i, j) < 0, \quad n \in \mathcal{I}_1, (n, k) \in \Omega \quad (30)$$

where

$$\Sigma = \begin{bmatrix} \rho_n(q_n^T q_n - 1) & \star & \star & \star \\ \rho_n \bar{Q}_n^T q_n & \Sigma_{11} & \star & \star \\ \mathbf{0} & \mathbf{0} & P_k & \star \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\gamma^2 \mathbf{I} \end{bmatrix}, \quad (31)$$

$$\begin{aligned} \Sigma_{11} &= \rho_n \bar{Q}_n^T \bar{Q}_n + \bar{L}_n^T \bar{L}_n - E^T P_n E, \\ \bar{Q}_n &= \begin{bmatrix} Q_n & \mathbf{0}_{1 \times n_u} \end{bmatrix}. \end{aligned}$$

Based on system (14), it is satisfied for any matrices $\mathcal{G}_n$, $n \in \mathcal{I}$ that

$$\begin{aligned} 0 &= 2\mathcal{G}_n \{ \bar{a}_n + \Delta \bar{a}_n + (\bar{A}_n + \Delta \bar{A}_n) \bar{x}(i, j) + \bar{D}_n w(i, j) \\ &\quad - E \bar{x}^+(i, j) \} \\ &= 2\mathcal{G}_n \begin{bmatrix} \bar{a}_n + \Delta \bar{a}_n & \bar{A}_n + \Delta \bar{A}_n & -\mathbf{I} & \bar{D}_n \end{bmatrix} \xi(i, j). \end{aligned} \quad (32)$$

Combining (30) with (32), one has,

$$\xi^T(i, j) \{ \Sigma + \text{Sym}\{\mathcal{G}_n \mathbb{A}_n\} \} \xi(i, j) < 0, \quad n \in \mathcal{I}_1, (n, k) \in \Omega \quad (33)$$

where $\mathbb{A}_n = \begin{bmatrix} \bar{a}_n + \Delta \bar{a}_n & \bar{A}_n + \Delta \bar{A}_n & -\mathbf{I} & \bar{D}_n \end{bmatrix}$.

It follows from (33) that the following inequality implies (27),

$$\Sigma + \text{Sym}\{\mathcal{G}_n \mathbb{A}_n\} < 0, \quad n \in \mathcal{I}_1, (n, k) \in \Omega. \quad (34)$$

Expanding the MFs in (34) yields,

$$\begin{bmatrix} \rho_n(q_n^T q_n - 1) & \star & \star & \star \\ \rho_n \bar{Q}_n^T q_n & \Xi_{11} & \star & \star \\ \mathbf{0} & \mathbf{0} & P_k & \star \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\gamma^2 \mathbf{I} \end{bmatrix} + \text{Sym}\{\mathcal{G}_n \mathcal{A}_m\} < 0, \quad m \in \mathcal{N}(n), n \in \mathcal{I}_1, (n, k) \in \Omega \quad (35)$$

where

$$\begin{cases} \mathcal{A}_m = \begin{bmatrix} \bar{a}_m + \Delta \bar{a}_m & \bar{A}_m + \Delta \bar{A}_m & -\mathbf{I} & \bar{D}_m \end{bmatrix}, \\ \bar{A}_m + \Delta \bar{A}_m = \begin{bmatrix} A_m + \Delta A_m & B_m \\ -KC & -\mathbf{I} \end{bmatrix}, \\ \bar{a}_m + \Delta \bar{a}_m = \begin{bmatrix} a_m + \Delta a_m \\ \mathbf{0} \end{bmatrix}, \\ \Xi_{11} = \rho_n \bar{Q}_n^T \bar{Q}_n + \bar{L}_m^T \bar{L}_m - E^T P_n E, \\ \bar{L}_m = \begin{bmatrix} L_m & N_m \end{bmatrix}. \end{cases} \quad (36)$$

Notice that the first row of $\bar{a}_m + \Delta \bar{a}_m$ and $\bar{A}_m + \Delta \bar{A}_m$ does not contain the sliding surface gain K . For numerical convexification, specify $\mathcal{G}_n = \begin{bmatrix} \mathbf{0} & W_n^T & V_n^T & \mathbf{0} \end{bmatrix}^T$, wherein the slack variable matrices V_n and W_n are given as,

$$V_n = \begin{bmatrix} V_{n1} & J_1 V_0 \\ V_{n2} & \varrho_1 V_0 \end{bmatrix}, W_n = \begin{bmatrix} W_{n1} & J_2 V_0 \\ W_{n2} & \varrho_2 V_0 \end{bmatrix}, \quad (37)$$

with $\varrho_1 \neq 0$ and $\varrho_2 \neq 0$ being scalar parameters, and $\{J_1, J_2\} \in \mathbb{R}^{n_x \times n_u}$ are arbitrary matrices.

Define

$$\bar{K} = V_0 K. \quad (38)$$

Considering (37)-(38), the inequality (35) can be derived as,

$$\begin{bmatrix} \rho_n(q_n^T q_n - 1) & \star & \star & \star \\ \varrho_n \bar{Q}_n^T q_n + \Lambda_{10} & \Lambda_{11} & \star & \star \\ \Lambda_{20} & \Lambda_{21} - W_n^T & \Lambda_{22} & \star \\ \mathbf{0} & \Lambda_{31} & \Lambda_{32} & -\gamma^2 \mathbf{I} \end{bmatrix} < 0, \quad (39)$$

$m \in \mathcal{N}(n), n \in \mathcal{I}_1, (n, k) \in \Omega$

where

$$\begin{cases} \Lambda_{10} = \begin{bmatrix} W_{n1} \\ W_{n2} \end{bmatrix} (a_m + \Delta a_m), \\ \Lambda_{11} = \rho_n \bar{Q}_n^T \bar{Q}_n - E^T P_n E + \bar{L}_m^T \bar{L}_m \\ \quad + \text{Sym} \left\{ \begin{bmatrix} W_{n1} \\ W_{n2} \end{bmatrix} \begin{bmatrix} A_m + \Delta A_m & B_m \end{bmatrix} \right. \\ \quad \left. - \begin{bmatrix} J_2 \\ \varrho_2 \mathbf{I} \end{bmatrix} \begin{bmatrix} \bar{K}C & V_0 \end{bmatrix} \right\}, \\ \Lambda_{20} = \begin{bmatrix} V_{n1} \\ V_{n2} \end{bmatrix} (a_m + \Delta a_m), \\ \Lambda_{21} = \begin{bmatrix} V_{n1} \\ V_{n2} \end{bmatrix} \begin{bmatrix} A_m + \Delta A_m & B_m \end{bmatrix} \\ \quad - \begin{bmatrix} J_1 \\ \varrho_1 \mathbf{I} \end{bmatrix} \begin{bmatrix} \bar{K}C & V_0 \end{bmatrix}, \\ \Lambda_{22} = P_k - V_n - V_n^T, \\ \Lambda_{31} = D_m^T \begin{bmatrix} W_{n1}^T & W_{n2}^T \end{bmatrix}, \\ \Lambda_{32} = D_m^T \begin{bmatrix} V_{n1}^T & V_{n2}^T \end{bmatrix}. \end{cases} \quad (40)$$

Utilizing Lemma 1 to tackle the uncertainty terms in (39), then for scalars $\varepsilon_n > 0$, the inequality (22) implies (39) by Schur complement.

Additionally, the conditions in (21)-(22) imply $\varrho_1 V_0 + \varrho_1 V_0^T - P_{k3} > 0$, which indicates that V_0 is nonsingular. Consequently, one can compute the sliding surface gain through (24). This ends the proof. ■

Remark 3.1. Via augmenting the control input with system states, a singular piecewise affine system method is proposed to handle the coupling problem between the system matrices and the sliding surface gain. By means of introducing auxiliary matrix multipliers, stability and performance analysis for the sliding motion can be carried out by using some convexification techniques.

Remark 3.2. Note that due to the introduction of the auxiliary matrix multipliers W_n and V_n , the sliding surface gain is also not coupled with the Lyapunov matrices in Theorem 3.1. The conservatism has been also further reduced in light of the increased freedoms introduced by the auxiliary matrix multipliers. The aforementioned points have distinguished our results from the prevalent ones in [50]–[54].

Remark 3.3. Due to the application of the singular piecewise affine system method and the introduction of auxiliary matrix multipliers, the stability and performance analysis results proposed in Theorem 3.1 are derived in terms of linear matrix inequalities under a convex optimization framework, which can be solved efficiently by commercially available software. It is noted that analysis and design conservatism can be also further reduced due to the increased freedom from the slack variables.

B. Sliding Mode Controller Synthesis

Similar to [41], we introduce a sliding patch Υ ,

$$\Upsilon := \{x(i, j) \in \mathbb{R}^{n_x} : \|x(i, j)\| \leq \lambda\} \quad (41)$$

with $\lambda > 0$.

Then an output feedback dynamic sliding mode controller is designed as

$$u(i+1, j+1) = u(i, j) - \sum_{l=1}^r \mu_l K C B_l u(i, j) - (\sigma + \vartheta(i, j)) \text{sgn}(s(i, j)) \quad (42)$$

where $s(i, j) = Ky(i, j) + u(i, j)$ is given in (11), and $\sigma > 0$ is a given scalar, and

$$\vartheta(i, j) = \sum_{l=1}^r \mu_l \{ (\|K C A_l\| + \|K C H_{l1}\| \cdot \|H_{l2}\| + \|K C\|) \lambda + \|K C a_l\| + \|K C H_{l1}\| \cdot \|H_{l3}\| + \|K C D_l\| \eta(i, j) \} \quad (43)$$

with $\eta(i, j)$ representing the uniform upper-bound of $w(i, j)$.

Remark 3.4. Due to the controller (42) being in a dynamic form, no singular valuable decomposition is needed for the control input matrices B_l for sliding mode controller synthesis purpose as in [50]. Additionally, the exogenous disturbance input matrices D_l can be the same as or different from the control input matrices B_l , which indicates the exogenous disturbance can appear in matched or mismatched form. In light of these perspectives, the developed dynamic SMC design scheme is capable of handling both mismatched and matched disturbances.

Theorem 3.2. In view of the patch Υ (41) with the sliding surface (11), the sliding mode controller (42) can force the system states of system (1) into a neighborhood of the sliding surface.

Proof: Consider the Lyapunov function,

$$V(s(i, j), i, j) = \frac{1}{2} s^T(i, j) s(i, j). \quad (44)$$

The incremental $\Delta V(s(i, j), i, j)$ is

$$\begin{aligned} \Delta V(s(i, j), i, j) &= V(s^+(i, j), i+1, j+1) - V(s(i, j), i, j) \\ &= s^T(i, j) \Delta s(i, j) + \Gamma \end{aligned} \quad (45)$$

with $\Delta s(i, j) = s^+(i, j) - s(i, j)$, $s^+(i, j) = K C x^+(i, j) + u(i+1, j+1)$, $\Gamma = \frac{1}{2} \Delta s^T(i, j) \Delta s(i, j)$.

Considering the controller (42) and noticing the sliding

patch Υ given as in (41), one has

$$\begin{aligned}
& \Delta V(s(i, j), i, j) \\
&= s^T(i, j) \{KCx^+(i, j) - KCx(i, j) \\
&\quad + u(i+1, j+1) - u(i, j)\} + \Gamma \\
&= s^T(i, j) \left\{ \sum_{l=1}^r \mu_l KC [(A_l + \Delta A_l - \mathbf{I})x(i, j) + a_l + \Delta a_l \right. \\
&\quad \left. + B_l u(i, j) + D_l w(i, j)] + \Delta u(i, j) \right\} + \Gamma \\
&= s^T(i, j) \sum_{l=1}^r \mu_l KC \left\{ (A_l + \Delta A_l - \mathbf{I})x(i, j) + a_l + \Delta a_l \right. \\
&\quad \left. + D_l w(i, j) \right\} - \vartheta(i, j) \|s(i, j)\| - \sigma \|s(i, j)\| + \Gamma \\
&\leq \|s(i, j)\| \sum_{l=1}^r \mu_l \left\{ (\|KCA_l\| + \|KC\Delta A_l\| + \|KC\|) \lambda \right. \\
&\quad \left. + \|KCa_l\| + \|KC\Delta a_l\| + \|KCD_l\| \eta(i, j) \right\} \\
&\quad - \vartheta(i, j) \|s(i, j)\| - \sigma \|s(i, j)\| + \Gamma \\
&\leq \|s(i, j)\| \sum_{l=1}^r \mu_l \left\{ (\|KCA_l\| + \|KCH_{l1}\| \cdot \|H_{l2}\| + \|KC\|) \lambda \right. \\
&\quad \left. + \|KCa_l\| + \|KCH_{l1}\| \cdot \|H_{l3}\| + \|KCD_l\| \eta(i, j) \right\} \\
&\quad - \vartheta(i, j) \|s(i, j)\| - \sigma \|s(i, j)\| + \Gamma
\end{aligned} \tag{46}$$

with $\Delta u(i, j) = u(i+1, j+1) - u(i, j)$.

Notice that

$$\begin{aligned}
\vartheta(i, j) &= \sum_{l=1}^r \mu_l \{ (\|KCA_l\| + \|KCH_{l1}\| \cdot \|H_{l2}\| + \|KC\|) \lambda \\
&\quad + \|KCa_l\| + \|KCH_{l1}\| \cdot \|H_{l3}\| + \|KCD_l\| \eta(i, j) \}
\end{aligned} \tag{47}$$

and then we have

$$\Delta V(s(i, j), i, j) \leq -\sigma \|s(i, j)\| + \Gamma. \tag{48}$$

Notice that

$$\begin{aligned}
& \Delta s(i, j) = KCx^+(i, j) - KCx(i, j) + \Delta u(i, j) \\
&\leq \sum_{l=1}^r \mu_l \{ (\|KCA_l\| + \|KCH_{l1}\| \cdot \|H_{l2}\| + \|KC\|) \lambda \\
&\quad + \|KCa_l\| + \|KCH_{l1}\| \cdot \|H_{l3}\| + \|KCD_l\| \eta(i, j) \} \\
&\quad - \vartheta(i, j) \operatorname{sgn}(s(i, j)) - \sigma \operatorname{sgn}(s(i, j)).
\end{aligned} \tag{49}$$

Then one can see that

$$\|\Delta s(i, j)\| \leq \sigma + 2\vartheta(i, j) \tag{50}$$

which means that $\|\Delta s(i, j)\|$ is bounded.

By tuning the parameter σ , $\Delta V(s(i, j), i, j) < 0$ can be ensured. Thus, the states of the 2-D fuzzy system (1) can be forced into a neighbourhood of the sliding surface and maintained hereafter by the controller (42) within the patch Υ . The proof is completed. ■

Remark 3.5. As discussed within Theorem 3.2, the convergence of the sliding surface (11) can be guaranteed by the developed output feedback SMC design approach, which also proposes a more general method to investigate the output feedback SMC issue for both matched and mismatched exogenous disturbance cases. The above-mentioned properties also distinguish the obtained results in this work from the prevalent ones in [50]–[54].

IV. SIMULATION STUDIES

Example 4.1. Recall a nonlinear water stream heating, gas absorption, and air drying process, whose dynamical features are described via the Darboux equation as,

$$\frac{\partial^2 T(x, t)}{\partial x \partial t} = b_1 \frac{\partial T(x, t)}{\partial t} + b_2 \frac{\partial T(x, t)}{\partial x} + b_0 T(x, t) + c_0 u(x, t) \tag{51}$$

where $b_1 = -3$, $b_2 = -1$, $c_0 = 1$ are constant parameters; $T(x, t)$ denotes the temperature within the reactant concentration; $b_0 = \cos T(x, t)$ represents a function at state space $x \in [0, x_f]$ and time $t \in [0, \infty)$; $u(x, t)$ represents the spatially uniform temperature within the jacket.

Along the lines in [2], [50], denote $R(x, t) = \frac{\partial T(x, t)}{\partial t} - b_2 T(x, t)$. Considering the external disturbance $w(x, t)$, the system (51) can be rewritten as,

$$\begin{aligned}
\begin{bmatrix} \frac{\partial R(x, t)}{\partial t} \\ \frac{\partial T(x, t)}{\partial t} \end{bmatrix} &= \begin{bmatrix} b_1 & b_1 b_2 + b_0 \\ 1 & b_2 \end{bmatrix} \begin{bmatrix} R(x, t) \\ T(x, t) \end{bmatrix} \\
&\quad + \begin{bmatrix} c_0 \\ 0 \end{bmatrix} u(x, t) + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} w(x, t).
\end{aligned} \tag{52}$$

Define $W(i, j) = [W_1^T(i, j) \ W_2^T(i, j)]^T$, where $W_1(i, j) = R(i\Delta x, j\Delta t)$, $W_2(i, j) = T(i\Delta x, j\Delta t)$, and $w(i, j) = w(i\Delta x, j\Delta t)$, $u(i, j) = u(i\Delta x, j\Delta t)$. Linearize system (52) at $(0, \pm \frac{\pi}{3})^T$ and $(0, 0)^T$, and discretize the nonlinear process (52) with the vertical and horizontal sampling periods $\Delta t = 0.1$ and $\Delta x = 0.1$. Then the system (52) is depicted based on the 2-D Takagi-Sugeno fuzzy affine model as follows.

Plant Rule $\mathcal{R}^l$: IF $W_2(i, j)$ is $\mathcal{F}_{l1}$, THEN

$$\begin{cases} W^+(i, j) = (A_l + \Delta A_l) W(i, j) + a_l + \Delta a_l \\ \quad + B_l u(i, j) + D_l w(i, j) \\ y(i, j) = CW(i, j), \\ z(i, j) = L_l W(i, j) + N_l u(i, j), \quad l \in \{1, 2, 3\} \end{cases} \tag{53}$$

where i and j denote the vertical and horizontal directions for the discrete-time fuzzy 2-D process (53), and

$$\begin{cases} A_1 = \begin{bmatrix} 0.7 & 0.4 \\ 0.1 & 0.9 \end{bmatrix}, A_2 = A_3 = \begin{bmatrix} 0.7 & 0.2593 \\ 0.1 & 0.9 \end{bmatrix}, \\ a_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, a_2 = \begin{bmatrix} -0.0950 \\ 0 \end{bmatrix}, a_3 = \begin{bmatrix} 0.0950 \\ 0 \end{bmatrix}, \\ B_l = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}, D_l = \begin{bmatrix} 0.05 \\ 0 \end{bmatrix}, \\ C = \begin{bmatrix} 0 & 1 \end{bmatrix}, L_l = \begin{bmatrix} 1 & 1 \end{bmatrix}, N_l = 0.5, \quad l \in \{1, 2, 3\} \end{cases} \tag{54}$$

and the uncertainty terms ΔA_l and Δa_l are given in the form of (2) with

$$\begin{cases} \Delta_l(i, j) = \sin(i) \cos(j), \\ H_{l1} = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}, H_{l2} = \begin{bmatrix} 0 & 0.1 \end{bmatrix}, \\ H_{l3} = 0.1, \quad l \in \{1, 2, 3\}. \end{cases} \tag{55}$$

The fuzzy MFs are demonstrated in Fig. 1 wherein $d_1 = \frac{\pi}{3}$ and $d_2 = \frac{\pi}{2}$. It is shown in Fig. 1 that,

$$\begin{cases} S_1 = \{W(i, j) \in \mathbb{R}^2 \mid -d_2 \leq W_2(i, j) \leq -d_1\}, \\ S_2 = \{W(i, j) \in \mathbb{R}^2 \mid -d_1 \leq W_2(i, j) \leq d_1\}, \\ S_3 = \{W(i, j) \in \mathbb{R}^2 \mid d_1 \leq W_2(i, j) \leq d_2\}. \end{cases} \tag{56}$$

Fig. 1. Fuzzy MFs

The above-mentioned three subspaces can be precisely described via degenerate ellipsoids in (17) with

$$\begin{cases} Q_1 = Q_3 = \begin{bmatrix} 0 & \frac{2}{d_2 - d_1} \end{bmatrix}, \\ Q_2 = \begin{bmatrix} 0 & \frac{1}{d_1} \end{bmatrix}, \\ q_1 = \frac{d_1 + d_2}{d_2 - d_1}, q_2 = 0, q_3 = \frac{d_1 + d_2}{d_1 - d_2}. \end{cases} \quad (57)$$

The aim is to synthesize a sliding mode controller (42) to force the system states of the 2-D process (51) into a neighbourhood of the sliding surface (11) and guarantee the asymptotical stability of the sliding motion with disturbance attenuation level γ . Assume that the system states are not fully accessible in this example. Additionally, the external disturbance is in a mismatched form. Furthermore, the affine terms $a_l + \Delta a_l$ also introduce much difficulty into the stability and performance analysis problem for the sliding motion. Thus, the prevailing SMC methods in [48]–[54] are not applicable for this case. Nevertheless, by adopting Theorem 3.1 with $\varrho_1 = \varrho_2 = 1$, $J_1 = J_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$, one can obtain the $\mathcal{H}_\infty$ performance index $\gamma_{\min} = 0.6251$ with sliding surface gain $K = 2.7940$. When it is assumed that $D_l = B_l = \begin{bmatrix} 0.1 & 0 \end{bmatrix}^T$, the external disturbance is in matched form. With the same conditions, one can obtain the $\mathcal{H}_\infty$ performance index $\gamma_{\min} = 1.2505$ with sliding surface gain $K = 2.8135$.

To illustrate the effectiveness of the proposed approach, simulations are conducted. Specifically, with initial states $W(0, j) = W(i, 0) = \begin{bmatrix} 0.5 & -0.5 \end{bmatrix}^T$, $0 \leq i \leq 10$, $0 \leq j \leq 10$, $W(0, j) = W(i, 0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$, $i > 10$, $j > 10$, and nonlinear disturbance function $w(i, j) = 50e^{-0.1i} \sin(0.5i) + 5e^{-2j} \sin(0.5j)$, adopting the sliding mode controller (42) with $\sigma = 0.2$, the system state trajectories and control input are, respectively, demonstrated in Fig. 2 and Fig. 3.

Furthermore, for some practical scenarios, the measured outputs are affected by measurement noises inevitably. Under the same exogenous disturbance and initial system states, the system state trajectories are demonstrated in Fig. 4.

(a) State response of $R(i, j)$ (b) State response of $T(i, j)$

Fig. 2. Response of the closed-loop system states without affection of measurement noises

Fig. 3. Response of the control input in Example 4.1 without affection of measurement noises

(a) State response of $R(i, j)$ (b) State response of $T(i, j)$

Fig. 4. Response of the closed-loop system states affected by measurement noises

Example 4.2. Recall a 2-D fuzzy system (1) with

$$\left\{ \begin{array}{l} A_1 = \begin{bmatrix} 0.5 & 0.3 \\ 0.2 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 0.9 & 0.5 \\ -0.1 & 0.8 \end{bmatrix}, \\ A_3 = \begin{bmatrix} 0.2 & 0.5 \\ 1 & 1 \end{bmatrix}, a_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, -a_2 = a_3 = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}, \\ B_1 = \begin{bmatrix} 1.5 \\ 1.2 \end{bmatrix}, B_2 = \begin{bmatrix} 2 \\ 1.5 \end{bmatrix}, B_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ D_l = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \\ L_l = \begin{bmatrix} 1 & 0 \end{bmatrix}, N_l = 1, l \in \{1, 2, 3\} \end{array} \right. \quad (58)$$

and the uncertainty terms ΔA_l and Δa_l are given in the form of (2) with

$$\left\{ \begin{array}{l} \Delta_l(i, j) = \sin(i) \cos(j), \\ H_{l1} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, H_{l2} = \begin{bmatrix} 0 & 0.1 \end{bmatrix}, \\ H_{l3} = 0.05, l \in \{1, 2, 3\}. \end{array} \right. \quad (59)$$

The three fuzzy MFs are also demonstrated in Fig. 1 where-in $d_1 = 70$ and $d_2 = 700$, and the state space is decomposed

into three subspaces as in (56). The aim is to synthesize a controller (42) to force the 2-D fuzzy system states into a neighbourhood of the sliding surface (11) and guarantee the asymptotical stability of the sliding motion with disturbance attenuation level γ . The system states are also assumed to be not fully accessible in this example. By inspecting the system matrices given in (58), the control input matrices are different for each local models, and the disturbance is mismatched. Therefore, the prevalent SMC design schemes in [48]–[54] are still not applicable for this case. Fortunately, choosing $\varrho_1 = \varrho_2 = 1$, $J_1 = J_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ and using Theorem 3.1, sliding surface gain can be calculated as $K = 0.3571$ with $\mathcal{H}_\infty$ performance index $\gamma_{\min} = 5.7575$. Note that the PQLFs P_n can reduce to a common quadratic Lyapunov function (CQLF) as $P_n \equiv P = \begin{bmatrix} \text{diag}\{P_1^h, P_1^v\} & \star \\ P_2 & P_3 \end{bmatrix}$. Using Theorem 3.1 based on a CQLF (also with the same slack variables), sliding surface gain can be calculated as $K = 0.3556$ with $\mathcal{H}_\infty$ performance index $\gamma_{\min} = 6.0693$, which can be concluded that the PQLF-based method brings less conservatism than the CQLF-based one.

In light of the simulation results, it is clearly shown that the developed control design approach is valid. The designed sliding mode controller can handle both matched and mismatched disturbances.

V. CONCLUSIONS

By Takagi-Sugeno fuzzy affine models, this paper investigates the output feedback SMC issue of nonlinear 2-D systems. Based on a 2-D PQLF-based scheme, stability and performance analysis of the sliding motion is conducted via a convex optimization procedure. A new output feedback dynamic SMC method is proposed to guarantee the system state trajectories to converge into a neighborhood of the sliding surface. Simulation studies are given to illustrate the effectiveness of the proposed approach. Considering the actuator failure situations widely exist in many practical systems, one of our future works is to study the fault tolerant output feedback SMC issue for fuzzy 2-D systems with actuator failures.

VI. APPENDIX

Lemma 1 [17]. Given real matrices $Q = Q^T$, W , U , and $\Delta(t)$, for all $\|\Delta(t)\| \leq \mathbf{I}$, the subsequent inequality holds

$$Q + \text{Sym}\{W\Delta(t)U\} < 0$$

if and only if for some parameter $\varepsilon > 0$,

$$Q + \varepsilon WW^T + \varepsilon^{-1}U^T U < 0$$

holds.

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