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A Polynomial-Membership-Function Approach for Stability Analysis of Fuzzy Systems

Wen-Bo Xie*, *Member, IEEE*, Song Sang, Hak-Keung Lam, *Fellow, IEEE*, and Jian Zhang

Abstract—For the stability analysis of a polynomial fuzzy system, a new polynomial-membership-function approach is proposed to reduce the conservatism. Based on a state feedback closed-loop system, a polynomials fitting method is utilized and an improved membership functions transformation technique are proposed to approximate the membership functions of fuzzy system. Then the membership function dependent polynomials form stability conditions are derived. The obtained polynomial membership functions and approximation errors will be involved in the stability analysis process. Based on the Sum-of-Squares optimization technique, the polynomials conditions can be directly solved. Finally, by several numerical and practical examples, the conservatism reduction effects are shown by comparisons with existing methods.

Index Terms—polynomial fuzzy system, Sum-of-Squares, polynomial membership functions, conservatism.

I. INTRODUCTION

SINCE the T-S fuzzy system has been proposed by Takagi and Sugeno in [1], the T-S system based control method has become a very effective solution in nonlinear system research [2]. According to the results in last several decades, one can find that the main constraint for fuzzy systems' applications is the conservatism in stability analysis: a general solution must be achieved for several different sub-linear systems. For example, with the common Lyapunov function methods [3], [4], a single positive definite matrix needs to be found to match a series of stability conditions. For such methods, the information of membership functions (MFs) is not utilized at all. With the development of fuzzy system stability analysis, fuzzy Lyapunov functions [5]–[9], piecewise Lyapunov functions [10], [11], nonquadratic Lyapunov functions [12]–[14], non-differentiable Lyapunov functions [15] and many other methods such as [16]–[18] are proposed. These approaches can provide more design freedoms or involve the membership functions information. Therefore, great advances have been achieved on conservatism reductions. And in nowadays, how to involve more information from membership functions became a hot topic.

Among the various membership function dependent results in the stability analysis, MFs approximation method is one of

the most successful approaches to involve MFs information [19]–[24]. This method is using the linear/staircase or other functions to approximate the membership functions. Based on the fuzzy Lyapunov function, the approximation functions are involved into the stability analysis to reduce the conservatism. This approach requires piecewise division processing of the membership functions. With more piecewise sections obtained, more MFs information will be incorporated and certain conservatism could be reduced. But meanwhile, in the convex optimization solving process for the stability conditions, the computational burden of linear matrix inequalities (LMIs) are becoming unacceptable.

Compared with LMI form stability condition, polynomial form conditions can describe more general cases. In other words, a polynomial fuzzy system which has wider applicable situations is an extension of the traditional T-S system as described in [25], [26] and references therein. With the development of Sum-of-squares (SOS) convex optimization method, polynomial approaches have attracted much attention in recent years for the advantage of relaxing conservatism: polynomial fuzzy model is established and SOS approach is adopted to solve the stability analysis in [27]. A Taylor series polynomial fuzzy modelling method is proposed in [28]. For the output feedback control structure: a polynomial observer-controller synthesis approach is designed for continuous fuzzy system in [29], discrete form can be found in [30], unmeasured premise variables case is also investigated in [31]–[33]. SOS based methods are also suitable for other control and stability synthesis problems: guaranteed cost control methods for polynomial fuzzy system can be found in [34], [35]. The stability analysis of time-varying delay polynomial fuzzy system is given in [36]. Robust control and stability analysis of polynomial fuzzy system are studied in [37]–[40].

For the SOS based stability synthesis research, switching polynomial Lyapunov function [41], piecewise polynomial Lyapunov function [42] approaches are designed to further reduce the conservatism. The advantages of parallel distributed compensation (PDC) for polynomial fuzzy system are illustrated by a categorized discussion in [43], non-PDC method is given in [44]. To utilize MFs information in SOS based stability analysis, piecewise membership functions approximation method is used in [19] for reducing conservatism. In [45], a polynomial variables method for involving the MFs in stability analysis has been proposed. But how to involve the more detailed membership functions information is still an open problem.

In this paper, a high order polynomial-membership-functions (PLMFs) method is proposed to incorporate more detailed MFs information and further reduce the conservatism.

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With a state feedback closed-loop polynomial fuzzy system, polynomial fitting method is adopted to approximate the membership functions of the polynomial fuzzy system. The polynomial form stability conditions which can be solved directly by SOS technique are obtained. In the approximation process, a new membership functions transformation approach with an improved auxiliary function is designed to overcome the global optimization property of SOS tools. The approximation PLMFs, upper and lower approximation errors are involved in the stability analysis process. Based on a numerical example and a suspended floater control system, the conservatism reduction effects are compared with existing methods.

The rest of the paper is organized as follows. Section 2 introduces the polynomial fuzzy system and the research problem. Section 3 gives the main results. Section 4 carries out the simulation analysis. Finally, some closing remarks are presented in Section 5.

Notations: Throughout the paper, $\mathbb{R}^n$ denotes the n -dimensional Euclidean space, I is an identity matrix, and 0 denotes a zero matrix or a zero scalar. For a matrix X , X^T and X^{-1} denote its transpose and inverse matrix, respectively. In the situations without confusions, f will be used to denote a function or a variable $f(\cdot)$, $\frac{\partial f}{\partial x}$ denotes the partial derivative of function f with respect to x .

II. PROBLEM STATEMENT

Consider a polynomial fuzzy system:

Rule i : IF $z_1(x(t))$ is M_{i1} ... and $z_p(x(t))$ is M_{ip} THEN

$$\dot{x}(t) = A_i(x(t))\hat{x}(t) + B_i(x(t))u(t)$$

where M_{ij} are fuzzy sets; $i = 1, 2, \dots, l$ with l denoting the number of fuzzy rules; $j = 1, 2, \dots, p$ with p denoting the number of premise variables; $z(x(t)) = [z_1(x(t)) \dots z_p(x(t))]^T \in \mathbb{R}^p$ is the premise variable; $x(t) \in \mathbb{R}^n$ is the system state; $u(t) \in \mathbb{R}^m$ is the control input; $\hat{x} = [\hat{x}_1 \dots \hat{x}_N]^T \in \mathbb{R}^N$ is a vector of monomials in $x(t)$ with assumption $\hat{x}(t) = 0$ if $x(t) = 0$; $A_i(x(t)) \in \mathbb{R}^{n \times N}$ and $B_i(x(t)) \in \mathbb{R}^{n \times m}$ are system parameter matrices, where $i = 1, 2, \dots, l$. The system dynamics can be defined as:

$$\dot{x}(t) = \sum_{i=1}^l h_i(z(x(t))) [A_i(x(t))\hat{x}(t) + B_i(x(t))u(t)] \quad (1)$$
 where:

$$\sum_{i=1}^l h_i(z(x(t))) = 1,$$

$$h_i(z(x(t))) = \frac{\prod_{j=1}^p M_{ij}(z_j(x(t)))}{\sum_{i=1}^l \prod_{j=1}^p M_{ij}(z_j(x(t)))} \geq 0$$

for all i , $h_i(z(x(t)))$ is the normalized grade of membership, and $M_{ij}(z_j(x(t)))$ represents the grade of membership of $z_j(x(t))$ corresponding to the fuzzy set M_{ij} . In the subsequent sections, we will use h_i and z to denote $h_i(z(x(t)))$ and $z(x(t))$. Also, time variable t will be omitted. With a classical fuzzy controller:

$$u = \sum_{i=1}^l h_i K_i(x) \hat{x}. \quad (2)$$

The closed-loop system can be written as:

$$\dot{x} = \sum_{i=1}^l \sum_{j=1}^l h_i h_j (A_i - (x) + B_i(x) K_j(x)) \hat{x}. \quad (3)$$

Assumption 1: Assume that the membership functions h_i is differentiable.

Remark 1: As most polynomial fuzzy systems are derived from practical engineering systems by the sector nonlinearity modelling process, the membership functions are designed with the product of the differentiable nonlinear terms in original systems, so Assumption 1 is reasonable.

Assumption 2: Assume that the premise variable z is a function of system state x as $z = f(x)$, and $f(x)$ is differentiable with respect to time t and state x . The membership functions can be written as $h_i(z) = h_i(f(x)) = g_i(x)$.

For the closed-loop system (3), a less conservative stability analysis method by a polynomial membership functions approximation method will be investigated in the consequent sections.

III. MAIN RESULTS

A. Polynomial membership function

Consider the piecewise function concept in [23] for approximating the membership functions, one has:

$$\sum_{i=1}^l g_i(x) = \sum_{i=1}^l \hat{g}_i(x) = 1 \quad (4)$$

where $0 \leq g_i \leq 1$ denotes the original system membership functions, $\hat{g}_i$ denotes the polynomial membership functions. The approximation errors are defined as:

$$\Delta g_i = g_i - \hat{g}_i \quad (5)$$

and the following inequalities hold,

$$\Delta g_i \leq \Delta g_i \leq \Delta \bar{g}_i \quad (6)$$

where Δg_i and $\Delta \bar{g}_i$ denote the lower and upper limits of the approximation errors. The polynomial membership functions $\hat{g}_i$ are designed according to the mature polynomial fitting theories as in [46], [47]. For example, a nonlinear function $\sin(x)$ for $x \in [0, 2\pi]$, can be approximated by a polynomial function with different orders (1st, 3rd, 6th-order from left to right) as shown in Fig.1:

Fig. 1 Polynomial fitting with different orders

According to the Lemma 1 in [23], the following Lemma can be obtained:

Lemma 1: With the polynomial form membership function approximation , the following equations hold,

$$\sum_{i=1}^l \hat{g}_i = 1, \sum_{i=1}^l \Delta g_i = 0. \quad (7)$$

B. Local High Precision Approximation Method

Different from [48], a local polynomial approximation method with higher precision and less conservatism is proposed in this section. Note that the SOSTOOLS can only solve sum-of-square problems, to guarantee the existence of the convex problems' solutions, the square root of functions g_i are obtained as $g_i^{0.5}$. Then the polynomial fitting method obtained functions $\hat{g}_{i0.5}$ are used to approximate $g_i^{0.5}$ by predefining the polynomial order. Finally, the MF polynomial approximation result is $\hat{g}_{i0.5}^2$. This method can guarantee that the functions $\hat{g}_{i0.5}^2$ are in SOS form. And with the polynomial order increasing, the approximation precision also gets better.

However, the shapes of g_i may be complex. For a wide operation range, approximation of g_i with only one polynomials may be not accurate. To hurdle these problems, the operating domain of membership functions should be partitioned into subregions. In each subregion, one can find a proper approximated polynomial.

For the SOSTOOLS, global solutions will be searched during its optimization process. So the above sub-regions method may induce more conservatism by using the SOSTOOLS. For it is required that the conditions are in SOS form when the premise variables are not in the sub-regions of operation domain. To overcome this issue, a local valid polynomial approximation method will be given here to reduce the conservatism. Considering a fractional function as:

$$\alpha_k(x) = a/(b(x+d)^s + c) \quad (8)$$

where $s > 0$ is an even integer number and $a > 0, b > 0, c > 0, k = 1, \dots, q$, q is the regions division number. It is easy to find that $\alpha_k > 0$. For the k -th divided sub-region of premise variables, the local approximation process can be described as follows:

Step 1, Calculating $(g_i/\alpha_k)^{0.5}$;

Step 2, Using polynomial fitting method to approximate $(g_i/\alpha_k)^{0.5}$, the obtained approximation functions are $\hat{g}_{ik0.5}$;

Step 3, The original functions approximated results are derived as $\hat{g}_{ik} = (\hat{g}_{ik0.5})^2 \alpha_k$.

C. Stability Analysis

A membership functions dependent stability analysis approach is designed in the following theorem.

Theorem 1: The closed-loop system (3) is asymptotically stable for all $i, j \in \{1, \dots, l\}$ and $k \in \{1, \dots, q\}$ if there exist polynomial matrix $N(\tilde{x}) = N^T(\tilde{x}) \in \mathbb{R}^{n \times n}$, $T_{ijk}(x) = T_{ijk}^T(x) \in \mathbb{R}^{n \times n}$, $M_i(\tilde{x}) \in \mathbb{R}^{m \times n}$, $W_k(x) = W_k^T(x) \in \mathbb{R}^{n \times n}$ vector $v \in \mathbb{R}^n$, polynomial scalars $\epsilon_1(x) > 0$, $\epsilon_{2ijk}(x) > 0$, $\epsilon_{3ijk}(x) > 0$ and $\epsilon_{4k}(x) > 0$ making the following SOS conditions hold:

$$v^T [N(\tilde{x}) - \epsilon_1(x)I] v \text{ is SOS} \quad (9)$$

$$v^T [T_{ijk}(x) - \epsilon_{2ijk}(x)I] v \text{ is SOS} \quad (10)$$

$$v^T [\Phi_{ij}(x) + W_k(x) + T_{ijk}(x) - \epsilon_{3ijk}(x)I] v \text{ is SOS} \quad (11)$$

$$\begin{aligned} & -v^T \left\{ \sum_{i=1}^l \sum_{j=1}^l [\hat{f}_{ijk}(x) \Phi_{ij}(x) + \Delta \bar{f}_{ijk}(\Phi_{ij}(x) + W_k(x) \right. \\ & \left. + T_{ij}(x)) - \Delta f_{ijk} T_{ijk}(x)] + \epsilon_{4k}(x)I \right\} v \text{ is SOS} \quad (12) \\ & \Phi_{ij}(x) = \Psi(x) A_i(x) N(\tilde{x}) + \Psi(x) B_i(x) M_j(\tilde{x}) + \\ & N(\tilde{x}) A_i^T(x) \Psi^T(x) + M_j^T(\tilde{x}) B_i^T(x) \Psi^T(x) - \\ & \sum_{k=1}^l \frac{\partial N(\tilde{x})}{\partial x_k} A_i^k(x) \hat{x}. \end{aligned}$$

For $\tilde{x}$ is a subset of x , so we use $\Psi(x)$ to replace $\Psi(x, \tilde{x})$. $\Delta \bar{f}_{ijk} \geq 0$ and $\Delta f_{ijk} \leq 0$ are predefined constant scalars satisfying $\Delta \bar{f}_{ijk} \leq \Delta f_{ijk} \leq \Delta \bar{f}_{ijk}$.

Proof. $\tilde{x}$ is the state variables corresponding to the columns of $B = 0$. Given a polynomial Lyapunov function as $V(x) = \hat{x}^T(x) P(\tilde{x}) \hat{x}(x)$, let $A_i^k(x)$ denotes the k th row of $A_i(x)$, $L = l_1, l_2, \dots, l_m$ denote the row indices of B_i whose corresponding row is equal to zero, when $k \in L$:

$$\dot{\hat{x}}_k = \sum_{i=1}^l h_i A_i^k(x) \hat{x} \quad (13)$$

to avoid the involvement of u in the derivatives of V , the property of $P(\tilde{x})$ can be given and for $k \notin L$:

$$\frac{\partial P(\tilde{x})}{\partial x_k} = 0. \quad (14)$$

Define $f_{ij} = h_i h_j$, from the conditions (13)-(14), the derivative of $V(x)$ can be described as

$$\begin{aligned} \dot{V}(x) &= \dot{\hat{x}}^T P(\tilde{x}) \hat{x} + \hat{x}^T P(\tilde{x}) \dot{\hat{x}} + \hat{x}^T \dot{P}(\tilde{x}) \hat{x} \\ &= \sum_{i=1}^l \sum_{j=1}^l f_{ij} \hat{x}^T [P(\tilde{x}) \Psi(x) A_i(x) + A_i^T(x) \Psi^T(x) P(\tilde{x}) \\ &+ P(\tilde{x}) \Psi(x) B_i(x) K_j(x) + K_j^T(x) B_i^T(x) \Psi^T(x) P(\tilde{x}) \\ &+ \sum_{k=1}^l \frac{\partial P(\tilde{x})}{\partial x_k} A_i^k(x) \hat{x}] \hat{x}. \end{aligned} \quad (15)$$

where $\Psi(x) \in \mathbb{R}^{N \times n}$ denotes a polynomial matrix as:

$$\Psi(x) = \begin{bmatrix} \partial \hat{x}_1 / \partial x_1 & \cdots & \partial \hat{x}_1 / \partial x_n \\ \vdots & \ddots & \vdots \\ \partial \hat{x}_N / \partial x_1 & \cdots & \partial \hat{x}_N / \partial x_n \end{bmatrix}$$

Define $y(x) = P(\tilde{x}) \hat{x}$, one has:

$$\dot{V}(x) = \sum_{i=1}^l \sum_{j=1}^l f_{ij} y^T(x) \Phi_{ij}(x) y(x) \quad (16)$$

Based on the local polynomial approximation method, $\hat{f}_{ijk}$ can be derived. And considering $f_{ij}(z_k) = \hat{f}_{ijk} + \Delta f_{ijk}$, $\sum_{i=1}^l \sum_{j=1}^l \Delta f_{ijk} = 0$ and conditions (10)-(11), one has:

$$\begin{aligned} \sum_{i=1}^l \sum_{j=1}^l \Delta f_{ijk} \Phi_{ij}(x) &= \sum_{i=1}^l \sum_{j=1}^l \Delta f_{ijk} (\Phi_{ij}(x) + W_k(x)) \\ &= \sum_{i=1}^l \sum_{j=1}^l [\Delta f_{ijk} (\Phi_{ij}(x) + W_k(x) \\ &+ T_{ijk}(x)) - \Delta f_{ijk} T_{ijk}(x)] \end{aligned}$$

With the properties $\Delta f_{ijk} (\Phi_{ij}(x) + W_k(x) + T_{ijk}(x)) \leq \Delta \bar{f}_{ijk} (\Phi_{ij}(x) + W_k(x) + T_{ijk}(x))$ and $-\Delta f_{ijk} T_{ijk}(x) \leq -\Delta \bar{f}_{ijk} T_{ijk}(x)$, the following inequality can be obtained:

$$\begin{aligned} \sum_{i=1}^l \sum_{j=1}^l \Delta f_{ijk} \Phi_{ij}(x) &\leq \sum_{i=1}^l \sum_{j=1}^l [\Delta \bar{f}_{ijk} (\Phi_{ij}(x) + W_k(x) \\ &+ T_{ijk}(x)) - \Delta \bar{f}_{ijk} T_{ijk}(x)] \quad (17) \end{aligned}$$

where $W_k(x)$ and $T_{ijk}(x)$ are slack matrices. Pre- and post-multiply (15) by $N(\tilde{x}) = P^{-1}(\tilde{x})$ and its transpose respectively. And considering conditions (9)-(11), define $M_j(\tilde{x}) =$

$K_j N(\tilde{x})$, an equivalent condition for guaranteeing the negativity of $\dot{V}$ can be obtained as:

$$\begin{aligned} \sum_{i=1}^l \sum_{j=1}^l f_{ij}(z_k) \Phi_{ij}(x) &= \sum_{i=1}^l \sum_{j=1}^l [\hat{f}_{ijk} \Phi_{ij}(x) + \Delta f_{ijk}(\Phi_{ij}(x) \\ &\quad + W_k(x) + T_{ijk}(x)) - \Delta \bar{f}_{ijk} T_{ijk}(x)] \\ &\leq \sum_{i=1}^l \sum_{j=1}^l [\hat{f}_{ijk} \Phi_{ij}(x) + \Delta \bar{f}_{ijk}(\Phi_{ij}(x) \\ &\quad + W_k(x) + T_{ijk}(x)) - \Delta \bar{f}_{ijk} T_{ijk}(x)] \end{aligned} \quad (18)$$

where z_k is the premise variable in the k -th sub-region, and $z = \bigcup_{k=1}^q z_k$. Then the proof is completed. $\square$

Remark 2: α_k is usually chosen in fractional form, so it is a challenge to solve such SOS problems. For this problem, we find another way to obtain conservatism relaxed non-fractional form conditions for (12) in Theorem 1 as:

$$\begin{aligned} &-v^T \left\{ \sum_{i=1}^l \sum_{j=1}^l [\hat{f}_{ijk} \alpha_k^{den} \Phi_{ij}(x) + \Delta \bar{f}_{ijk} \alpha_k^{den}(\Phi_{ij}(x) \right. \\ &\quad \left. + W(x) + T_{ij}(x)) - \alpha_k^{den} \Delta \bar{f}_{ijk} T_{ij}(x)] + \epsilon_4(x) I \right\} v \\ &= -v^T \left\{ \sum_{i=1}^l \sum_{j=1}^l [\hat{f}_{ijk} \alpha_k^{den} \Phi_{ij}(x) - \alpha_k^{num} (\Delta \bar{f}_{ijk0.5})^2 T_{ij}(x) \right. \\ &\quad \left. (\Delta \bar{f}_{ijk0.5})^2 \alpha_k^{num} (\Phi_{ij}(x) + W(x) + T_{ij}(x))] + \epsilon_4(x) I \right\} v \end{aligned} \quad (19)$$

where α_k^{num} denotes the numerator of α_k , and α_k^{den} denotes the denominator of α_k . $\Delta \bar{f}_{ijk0.5}$ and $\Delta \bar{f}_{ijk0.5}$ are the lower and upper limits of $\Delta \bar{f}_{ijk0.5}$, which is the approximation of $(\Delta f_{ijk}/\alpha_k)^{0.5}$. One can easily find that (19) can be solved by the SOSTOOLS software. So the above transformation gives an alternative for solving the SOS conditions. Further, a more relaxed form can be given as follows:

$$\begin{aligned} &v^T \kappa_{ik}(x) v \text{ is SOS} \\ &-v^T \left\{ \sum_{i=1}^l \sum_{j=1}^l [\hat{f}_{ijk} \alpha_k^{den} \Phi_{ij}(x) - \alpha_k^{num} (\Delta \bar{f}_{ijk0.5})^2 T_{ij}(x) \right. \\ &\quad \left. + (\Delta \bar{f}_{ijk0.5})^2 \alpha_k^{num} (\Phi_{ij}(x) + W(x) + T_{ij}(x))] + \epsilon_4(x) I \right. \\ &\quad \left. - L_k(x) \right\} v \text{ is SOS} \end{aligned} \quad (21)$$

where scalars $\kappa_{ik}(x) > 0$ exist in $L_k(x)$. If $L_k(x) < 0$ in the premise variable operation domain, conditions (19) can be deduced by the above two relaxed conditions. Compared with [48], an improved $L_k(x)$ design method will be given in the following Remark.

Remark 3: A new $L_k(x)$ form is proposed as:

$$L_k(x) = \sum_{i=1}^{\delta/2} \sum_{j=1}^p \kappa_{ik}(x) v_{ik}(x_j) \quad (22)$$

where:

$$\begin{aligned} v_{ik}(x_j) &= k_{ij1} \left(x_j - \frac{x_{jk \min} + x_{jk \max}}{2} \right)^{2i} - k_{ijk2} \\ k_{ijk2} &= k_{ij1} \left(\frac{x_{jk \min} + x_{jk \max}}{2} \right)^{2i} \end{aligned}$$

k_{ij1} is predefined constant, x_{jk} denotes the j -th element of x on the k -th sub-region. The operating domain for the k -th sub-region is $x_{\min ik} \leq x_i \leq x_{\max ik}$, $j = 1, \dots, p$; $k = 1, \dots, q$; $i = 1, \dots, \delta/2$, δ is the highest order of approximated

polynomial terms, which is an even number. Define a sub-region $s_k = \bigcap (x_{\min jk} \leq x_j \leq x_{\max jk})$, it is easy to find that $L_k(x) < 0$ when $x \in s_k$; $L_k(x) > 0$ when $x \notin s_k$ for $p = 1$. If $p > 1$, for most cases of $x \notin s_k$, $L_k(x) > 0$ will hold. This will guarantee that the inequalities (19) are sufficient conditions for (19), and may reduce certain conservatism for most cases. $L_k(x)$ may be negative for the nearby region outside s_k , one can see that happens from Figs.2-3 for $p = 2$.

An illustrative example is given for $p = 2$ and $q = 1$, as the premise variable is $x = [x_1^T \ x_2^T]^T$. Designing $L_1(x) = v_{11}(x_1) + v_{11}(x_2)$, $v_{11}(x_1) = (x_1 + 1)^2 - 0.1$, $v_{11}(x_2) = (x_2 + 1)^2 - 0.4$. In Fig.2, the dark-colored surface denotes $L_1(x) > 0$, the light-colored circle section denotes the region x for $L_1(x) < 0$. The projection of the ellipse on $x_1 - x_2$ plane is shown in figure 3, which is drawn as red line. The blue rectangle region denotes the exactly operation domain of premise variables. One can reckon that the red ellipse is an approximated region for the blue rectangle.

Fig. 2 Negative values region for $L_1(x)$

Fig. 3 Premise variables region for $L_1(x) < 0$

Remark 4: The calculation complexity comparisons with some existing methods are analyzed in Table 1. Theorem 1 in [23], and Theorem 2 in [48] both need very small piecewise sections, so the corresponding division number q is very large. Instead, the polynomial order can be increased to achieve better approximation precision by the method in this paper. Therefore, q can be chosen relative small for Theorem 1.

Table 1 Computational complexity comparisons

Methods	Number of stability conditions	Number of decision variables
Theorem 1 in this paper	$1 + q + ql + 2ql^2$ SOS	$\frac{n(n+1)}{2}(1 + q + ql^2) + mnl$
Theorem 2 in [48]	$2 + q + 4ql^2$ SOS	$\frac{n(n+1)}{2}(1 + 2ql^2) + mnl$
Theorem 1 (state feedback case) in [23]	$2(2^p ql + l^2 + l)$ LMI	$\frac{1}{2}n(n+1)(l^2 + 2l + 1)$

Remark 5: For this proposed α_k and L_k design methods, α_k can attenuate the values when approximated polynomial functions are outside the premise variable operation regions. And L_k can relax the global SOSTOOLS solving results when premise variable is outside its operation region. These two effects will further reduce the conservatism of SOS stability synthesis method, which is verified in the simulation results section.

Remark 6: Though Theorem 1 is based on the state feedback control system (3), the proposed membership function dependent stability analysis approach can also be applied in output feedback, filter synthesis or any other related areas for polynomial fuzzy systems.

Remark 7: In Theorem 1, the maximal and minimal approximation errors $\Delta \bar{f}_{ijk}$ and $\underline{f}_{ijk}$ are the maximum/minimum values of Δf_{ijk} for $\forall z$, so the state variables will not influence the values of $\Delta \bar{f}_{ijk}$ and $\underline{f}_{ijk}$. Also, the maximum/minimum values are calculated numerically. Even for large number of fuzzy rules, they can be obtained conveniently. Same conclusions can be derived for $\Delta \bar{f}_{ijk0.5}$ and $\Delta \underline{f}_{ijk0.5}$ in Remark 2.

Remark 8: If the membership functions of system (1) are in polynomial form, the proposed methods will exactly involve all the MFs information. Even for the continuous but non-differentiable membership functions, the proposed methods are still available, but larger approximation errors may be incorporated.

Remark 9: The polynomial approximation functions $\hat{f}_{ijk}$ may not stay in $[0, 1]$, so conditions in Theorem 1 and Remark 3 are sufficient conditions for guaranteeing the negativeness of $\dot{V}$.

Remark 10: It is easy to see that with the order of polynomials increasing, the approximation effects become better, the maximal and minimal approximation errors will be smaller, so as the conservatism. Accordingly, the order of variables in SOS solving process increases, with more calculation effort, the conservatism can also be reduced.

Remark 11: The above proposed methods can be applied to T-S and polynomial fuzzy systems to reduce conservatism. But when considering the original nonlinear system, many different types of fuzzy system can be derived by various modelling methods. And by which kind of models, the best conservative results can be obtained are not considered in this paper. Tensor product approaches such as [49]–[51] may help one to solve this issue, which will be discussed in future work.

To further extend Theorem 1, a robust case will be illustrated in the subsequent sections. Based on the closed-loop system (3), with parameter matrices $E_i \in \mathbb{R}^{n \times r}$, a polynomial

fuzzy system with external disturbance $\omega \in \mathbb{R}^r$ can be given as:

$$\dot{x} = \sum_{i=1}^l \sum_{j=1}^l [h_i h_j (A_i(x) + B_i(x) K_j(x)) \hat{x} + E_i \omega] \quad (23)$$

Corollary 1: The closed-loop system (23) is robust asymptotically stable for all $i, j \in \{1, \dots, l\}$, $k \in \{1, \dots, q\}$ if there exist polynomial matrices $N(\tilde{x}) = N^T(\tilde{x}) \in \mathbb{R}^{n \times n}$, $T_{ijk}(x) = T_{ijk}^T(x) \in \mathbb{R}^{(n+r) \times (n+r)}$, $M_i(\tilde{x}) \in \mathbb{R}^{m \times n}$, $W_k(x) = W_k^T(x) \in \mathbb{R}^{(n+r) \times (n+r)}$, $\gamma > 0 \in \mathbb{R}$, $v \in \mathbb{R}$, $v_1 \in \mathbb{R}^n$, $v_2 \in \mathbb{R}^{(n+r)}$, polynomial scalars $\epsilon_1(x) > 0$, $\epsilon_{2ijk}(x) > 0$, $\epsilon_{3ijk}(x) > 0$ and $\epsilon_{4k}(x) > 0$ making the following conditions hold:

$$\text{minimise } \gamma$$

subject to

$$v^T \gamma v \text{ is SOS} \quad (24)$$

$$v^T \kappa_{ik}(x) v \text{ is SOS} \quad (25)$$

$$v_1^T [N(\tilde{x}) - \epsilon_1(x) I] v_1 \text{ is SOS} \quad (26)$$

$$v_2^T [T_{ijk}(x) - \epsilon_{2ijk}(x) I] v_2 \text{ is SOS} \quad (27)$$

$$v_2^T [\Theta_{ij}(x) + W_k(x) + T_{ijk}(x) - \epsilon_{3ijk}(x) I] v_2 \text{ is SOS} \quad (28)$$

$$\begin{aligned} & - v_2^T \left\{ \sum_{i=1}^l \sum_{j=1}^l [\alpha_k^{den} \hat{f}_{ij}(x) \Theta_{ij}(x) + (\Delta \bar{f}_{ijk0.5})^2 \alpha_k^{num} (\Theta_{ij}(x) \right. \\ & \left. + W_k(x) + T_{ijk}(x)) - \alpha_k^{num} (\Delta \underline{f}_{ijk0.5})^2 T_{ijk}(x)] + \epsilon_{4k}(x) I \right. \\ & \left. - L_k(x) \right\} v_2 \text{ is SOS} \end{aligned} \quad (29)$$

where $\Phi_{ij}(x) = \Psi(x) A_i(x) N(\tilde{x}) + \Psi(x) B_i(x) M_j(\tilde{x}) + N(\tilde{x}) A_i^T(x) \Psi(x)^T + M_j^T(\tilde{x}) B_i^T(x) \Psi(x)^T - \sum_{k=1}^l \frac{\partial N(\tilde{x})}{\partial x_k} A_i^k(x) \hat{x}$, $\Theta_{ij}(x) = \begin{bmatrix} \Phi_{ij}(x) & E_i \\ * & -\gamma^2 I \end{bmatrix}$, $\Delta \bar{f}_{ijk0.5} \geq 0$ and $\Delta \underline{f}_{ijk0.5} \leq 0$ are predefined constant scalars satisfying $\Delta \underline{f}_{ijk0.5} \leq \Delta f_{ijk0.5} \leq \Delta \bar{f}_{ijk0.5}$, where $*$ denotes the transposed element for symmetric positions. And L_k is described in Remark 3.

Proof. The proof is omitted since it can be directly obtained from Theorem 1 and a classical H_∞ approach as Theorem 1 in [52].

IV. SIMULATION RESULTS

Example 1. Consider a T-S fuzzy system (special case of polynomial fuzzy system) with two rules:

$$A_1 = \begin{bmatrix} 1.59 & -7.29 \\ 0.01 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} -a & -4.33 \\ 0 & 0.05 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T, B_2 = \begin{bmatrix} -b + 6 & -1 \end{bmatrix}^T$$

where a and b are constant parameters, the membership functions are defined as

$$\begin{cases} h_1(x_1) = 1 - 1/(1 + e^{-(x_1+4)}) \\ h_2(x_1) = 1 - h_1(x_1) \end{cases}$$

where $x \in [-10, 10]$. First, one piecewise section is chosen as $q = 1$. Local regional function α_1 can be chosen as:

$$\alpha_1 = 10/(10 + 10^{-13} x_1^{16})$$

Define $f_{ij} = h_i h_j$ for $x \in [-10, 10]$, and by the approximation procedure steps 1-3, $(f_{ij}/\alpha_1)^{0.5}$ are approximated by the ‘polyfit’ function through Matlab R2016a. The polynomial

order is set as 7. The obtained 14-order polynomial approximated membership functions $\hat{f}_{ij1}$ are as follows:

$$\begin{aligned}\hat{f}_{111,0.5} = & (-2.866 \times 10^{-7}x_1^7 + 1.142 \times 10^{-6}x_1^6 \\ & + 5.547 \times 10^{-5}x_1^5 - 0.0003x_1^4 - 0.003x_1^3 \\ & + 0.020x_1^2 - 0.025x_1 - 0.003)^2\end{aligned}$$

$$\begin{aligned}\hat{f}_{121,0.5} = & (4.916 \times 10^{-8}x_1^7 + 1.686 \times 10^{-6}x_1^6 \\ & - 1.910 \times 10^{-5}x_1^5 - 0.0003x_1^4 + 0.002x_1^3 \\ & + 0.008x_1^2 - 0.086x_1 + 0.158)^2\end{aligned}$$

$$\hat{f}_{211,0.5} = \hat{f}_{121,0.5}, \hat{f}_{221,0.5} = 1 - \hat{f}_{111,0.5} - \hat{f}_{121,0.5} - \hat{f}_{211,0.5}$$

Then, for $x_1 \in [-\infty, \infty]$, we choose a wide range, for example $x_1 \in [-300, 300]$, to show the detailed attenuation effects of α_1 . The membership functions f_{ij}/α_1 are described by $\hat{f}_{ij1}$ given in Fig.4. The membership functions f_{ij} are described by $\hat{f}_{ij1}\alpha_1$ given in Fig.5. The approximation precision are shown in Fig.6, and the maximal/minimal approximation errors $\Delta\hat{f}_{ijk0.5}$ and $\Delta\underline{f}_{ijk0.5}$ can be calculated numerically as given in Table.2. The function $L_1(x)$ is chosen as $L_1(x) = \kappa_1[(0.1x_1^2 - 10) + (10^{-5}x_1^6 - 10)]$. For $q = 1$, The comparisons between Theorem 1 and Remark 2 is given in Fig.7, what shows the conservatism reduction effects of $L_k(x)$ function.

Considering the partition case as $q = 2$, for $x \in [-10, 0]$, local regional function α_1 can be chosen as:

$$\alpha_1 = 10/(10 + 10^{-11}(x_1 + 5)^{16})$$

And for $x_1 \in [0, 10]$, local regional function α_2 is:

$$\alpha_2 = 10/(10 + 10^{-11}(x_1 - 5)^{16})$$

By a similar process as $q = 1$ and same parameter setting, the maximal and minimal approximation errors can be calculated numerically as given in Tables.3-4. Fig.8 shows the stability regions comparison for $q = 1$ and $q = 2$ without L_k . One can see that more segmentations can further reduce approximation errors and conservativeness. Approximated polynomial functions are omitted here due to the space limitation.

Table 2 The maximal/minimal approximation errors for $q = 1$

max	min
$\Delta\bar{f}_{111,0.5} = 0.1156$	$\Delta\underline{f}_{111,0.5} = -0.0569$
$\Delta\bar{f}_{121,0.5} = \Delta\bar{f}_{211,0.5} = 0.0552$	$\Delta\underline{f}_{121,0.5} = \Delta\underline{f}_{211,0.5} = -0.0293$
$\Delta\bar{f}_{221,0.5} = 0.0927$	$\Delta\underline{f}_{221,0.5} = -0.0730$

Table 3 The maximal/minimal approximation errors for $q = 2$ (α_1)

max	min
$\Delta\bar{f}_{112,0.5} = 0.0294$	$\Delta\underline{f}_{112,0.5} = -0.0176$
$\Delta\bar{f}_{122,0.5} = \Delta\bar{f}_{212,0.5} = 0.0038$	$\Delta\underline{f}_{122,0.5} = \Delta\underline{f}_{212,0.5} = -0.0033$
$\Delta\bar{f}_{222,0.5} = 0.0073$	$\Delta\underline{f}_{222,0.5} = -0.0086$

Table 4 The maximal/minimal approximation errors for $q = 2$ (α_2)

max	min
$\Delta\bar{f}_{112,0.5} = 7.8540 \times 10^{-6}$	$\Delta\underline{f}_{112,0.5} = -0.0004$
$\Delta\bar{f}_{122,0.5} = \Delta\bar{f}_{212,0.5} = 0.0003$	$\Delta\underline{f}_{122,0.5} = \Delta\underline{f}_{212,0.5} = -0.0002$
$\Delta\bar{f}_{222,0.5} = 0.0165$	$\Delta\underline{f}_{222,0.5} = -0.0099$

Fig. 4 Polynomial membership functions approximation for $q = 1$

Fig. 5 Polynomial membership functions approximation for $q = 1$ ($x_1 \in [-300, 300]$)

Fig. 6 Polynomial membership functions approximation for $q = 1$ ($x_1 \in [-10, 10]$)

Fig. 7 Conservatism comparison with $L_1(x)$ or without $L_1(x)$ for $q = 1$ of Theorem 1

Fig. 8 Conservatism comparison between $q = 1$ and $q = 2$ without $L_k(x)$

Fig. 9 Conservatism comparison for Theorem 1 of this paper (o), Theorem 1 in [23] (x) and Theorem 1 in [48] (□)

Stable regions comparisons with existing works are shown in Fig.9. Theorem 1 (with Remark 3) of this paper (denoted by o) when $q = 1$ with $L_1 = \kappa_1[(0.1x_1^2 - 10) + (10^{-5}x_1^6 - 10)]$.

Parameters for Theorem 1 in [23] (denoted by $\times$) (are chosen as $|\dot{x}_1| = 10$, $q = 20$). Parameters for Theorem 1 in [48] are set as $q = 1$, and $L_1(x_1) = (x_1 + 10)(x_1 + 5)$. It is obviously that Theorem 1 of this paper can effectively reduce the conservatism in stability analysis.

After the comparisons, system state trajectories are checked for each case. For case $q = 1$ with $L_k(x)$, $a = 12$ and $b = 13$ are chosen. For case $q = 2$ without $L_k(x)$ case, $a = 12$ and $b = 15$ are chosen. According to the format of B_1 , B_2 and the conditions (13)-(14), P can only be set as a 0-order constant matrix. By setting the SOS variables as 0-order in solving process, the gain matrices are calculated via SOSTOOLS given in [53] through Matlab R2016a as $q = 1$ with L_k :

$$K_1 = [-1.8162, 7.1822], K_2 = [-1.8231, 6.6777],$$

$$P = \begin{bmatrix} 0.7556 & -0.4292 \\ -0.4292 & 22.3216 \end{bmatrix}$$

and $q = 2$ without L_k :

$$K_1 = [-1.6122, 5.5710], K_2 = [-1.6525, 5.4646],$$

$$P = \begin{bmatrix} 0.8070 & -1.5537 \\ -1.5537 & 28.5120 \end{bmatrix}$$

Behaviors in $x_1 - x_2$ plane for the above situations are shown in Figs.10-11, respectively. $\circ$ denotes four different initial state as $(\pm 10, \pm 10)$, and $*$ denotes the equilibrium point.

Fig. 10 Behaviors in $x_1 - x_2$ plane for $k = 1$ without $L_k(x)$

Fig. 11 Behaviors in $x_1 - x_2$ plane for $k = 2$ with $L_k(x)$

Example 2. A two-rule polynomial fuzzy model is adopted here to double-check the conservatism reduction effects:

$$A_1 = \begin{bmatrix} 1.59 + 2.45 \times x_1 & -7.29 \\ 0.01 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -a - 2.7 \times x_1^2 & -4.33 \\ 0 & 0.05 - x_1^2 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 + 0.37 \times x_1 \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} -b + 6 + 1.55 \times x_1^2 \\ -1 \end{bmatrix}$$

where a and b are constant parameters. For $x \in [-10, 10]$, the membership functions are given the same as Example 1. Stable regions comparisons with Theorem 1 in [48] are shown in Fig.12. Results of Theorem 1 in this paper are denoted by $\circ$, parameters are set as $q = 2$, L_k is not used. No feasible solution is found for Theorem 1 in [48] in the given regions, $q = 2$, $L_1(x_1) = (x_1 + 10)x_1$ and $L_2(x_1) = x_1(x_1 - 10)$. It is obviously that Theorem 1 of this paper can effectively reduce the conservatism in stability analysis.

Fig. 12 Conservatism comparison for Theorem 1 of this paper ($\circ$) and Theorem 1 in [48] ($\times$)

Fig. 13 A side view of suspended floater system.

For $a = 12$ and $b = 7.8$, system state trajectories for initial state $x_0 = [\pm 10 \pm 10]^T$ are given in Fig. 13. $\circ$ denotes four different initial state as , and $*$ denotes the equilibrium point. By setting the SOS variables as 0-order in solving process,

the gain matrices are calculated via SOSTOOLS:

$$K_1 = [-6.5719, 0.0070], K_2 = [-6.5697, 0.0076],$$

$$P = \begin{bmatrix} 5.4844 \times 10^7 & 0.6233 \times 10^7 \\ 0.6233 \times 10^7 & 3.5378 \times 10^7 \end{bmatrix}$$

Example 3. A braking control task of suspended floater following control system, as shown in Fig.14, is adopted here to verify the effectiveness of the proposed method. Jet propulsion force $\omega = 1N$ which is seen as disturbance will act for 1 second when the floater system starts running. And the control target is to stabilize the floater. Considering the fuzzy system investigated in [54], the operating region is set as $\theta \in [-60^\circ, 60^\circ]$.

Fig. 14 A side view of suspended floater system.

By a sector nonlinearity modelling approach, the system parameter matrices are given as follows:

$$A_1 = A_3 = \begin{bmatrix} 0 & 1 & 0 \\ -6.6637 & -7.8307 \times 10^{-6} & 1.1111 \times 10^{-5} \\ 1.8911 & 2.2222 \times 10^{-6} & -1.6667 \times 10^{-5} \end{bmatrix}$$

$$A_2 = A_4 = \begin{bmatrix} 0 & 1 & 0 \\ -8.0578 & -7.8307 \times 10^{-6} & 1.1111 \times 10^{-5} \\ 2.2867 & 2.2222 \times 10^{-6} & -1.6667 \times 10^{-5} \end{bmatrix}$$

$$B_1 = B_2 = B_3 = B_4 = \begin{bmatrix} 0 \\ -0.0022 \\ 0.0033 \end{bmatrix},$$

$$E_1 = E_2 = \begin{bmatrix} 0 \\ 0.0059 \\ -0.0017 \end{bmatrix}, E_3 = E_4 = \begin{bmatrix} 0 \\ 0.0117 \\ -0.0033 \end{bmatrix}$$

while the membership functions are described as

$$\begin{cases} h_1(\theta) = \frac{1 - \cos \theta}{0.5} \times \frac{1 - \frac{\sin \theta}{0.173}}{0.173} \\ h_2(\theta) = \frac{1 - \cos \theta}{0.5} \times \frac{\frac{\sin \theta}{0.173} - 0.827}{0.173} \\ h_3(\theta) = \frac{\cos \theta - 0.5}{0.5} \times \frac{1 - \frac{\sin \theta}{0.173}}{0.173} \\ h_4(\theta) = \frac{\cos \theta - 0.5}{0.5} \times \frac{\frac{\sin \theta}{0.173} - 0.827}{0.173} \end{cases}$$

According to Corollary 1, control gain matrices are calculated and listed as:

$$K_1 = [11.70 \quad 74.27 \quad -19.62]$$

$$K_2 = [6.70 \quad 45.94 \quad -12.11]$$

$$K_3 = [11.24 \quad 104.41 \quad -13.55]$$

$$K_4 = [7.16 \quad 64.94 \quad -8.34]$$

Robust performance index comparison with zero initial state is presented in Table. 5. For both the transformed system state and external disturbance are adopted in Theorem 1 of [23], so the comparison can only be carried out between Corollary 1 in this paper and an extension of robust stability form of Theorem

1 in [48]. The control performance is shown in Figure 15, it is easy to find that the control errors are converging to zero.

Table 5 H_∞ index comparison

Corollary 1 in this paper	$\gamma_{\min} = 0.00976$
Theorem 1 in [48]	$\gamma_{\min} = 0.01412$

Fig. 15 System control effects

V. CONCLUSIONS

The conservatism problem in stability analysis for polynomial fuzzy system is investigated in this paper, with the proposed polynomial membership functions approximation method, the MFs information is more effectively involved in the stability analysis, based on the SOS optimization method, less conservative results for fuzzy systems are obtained. In simulation section, it is shown that the conservatism is greatly reduced by the proposed method. The method could be used to reduce the conservatism of polynomial fuzzy control systems and enlarge its applications in real world engineering. In future study, TP transformation approach which may further reduce the conservatism will be considered to reform the related SOS conditions. How to involve monomials in the positive matrix of Lyapunov function is another important issue to be handled.

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