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Fuzzy-Affine-Model Based Sampled-Data Filtering Design for Stochastic Nonlinear Systems

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Abstract—This paper addresses the sampled-data piecewise affine (PWA) filter design problem for Itô stochastic nonlinear systems represented by Takagi-Sugeno fuzzy affine models. An input delay method is used to describe the sample-and-hold behavior of the measurement output. Based on a novel piecewise quadratic Lyapunov-Krasovskii functional, some new results on the robust sampled-data PWA filtering design are proposed through a linearization procedure by using some convexification techniques. Simulation studies on a tunnel diode circuit system and an inverted pendulum system are given to illustrate the effectiveness of the proposed method.

Index Terms—Sampled-data; filtering design; stochastic nonlinear systems; convex optimization.

I. INTRODUCTION

During the past decades, increasing research attention has been focused on stochastic nonlinear systems, whose dynamical properties are mathematically characterized via Itô stochastic differential equations [1], [2]. In practical applications, various real-world processes intrinsically contain stochastic behaviors, such as those in mechanics, biology, chemistry, microelectronics, and economics [3]–[6]. As there exists a growing modeling and analysis demand for stochastic features, more recently, a substantial body of valuable works on analysis and design for stochastic systems have been reported [7]–[11]. To mention a few, the authors in [10] proposed a robust $\mathcal{L}_2$-$\mathcal{L}_\infty$ full order filter design scheme for Itô stochastic systems subject to time-varying delay. In [11], by a periodically intermittent control method, the exponential stabilization problem for Itô stochastic Markovian jumping systems was investigated.

On another research front, the methods relying on T-S fuzzy models have been extensively employed in steering of complicated processes with high nonlinearities during the past decades [12]–[21]. It has been proved that T-S fuzzy model can serve as a powerful tool to approximate nonlinear systems to arbitrary degrees of accuracy within any convex compact set [22]–[24]. Due to the universal approximation competence of T-S fuzzy models, the filter design issue for stochastic nonlinear systems via Itô-type stochastic T-S fuzzy models has been a hot research topic, and many elegant results have been obtained [25]–[27]. For instance, the authors in [26] investigated the robust $\mathcal{H}_\infty$ filtering problem for Itô stochastic fuzzy time-delay systems, and both the delay-independent and delay-dependent methods were proposed.

Sampled-data filtering issue has received increasing attention since many control applications and signal processing for analog processes are implemented by digital computers. In recent years, various schemes to handle the sampled-data filtering/control problems have been proposed, for instance, the input delay approach [28], [29] and the discrete-time approach [30], [31]. Specifically, the input delay approach is more powerful to handle the uncertain system or varying sampling periods over the discrete-time approach [32]. Generally speaking, the sampled-data information is firstly converted into a delayed measurement output, and then the stability analysis for the sampled-data filtering error systems is conducted via some relaxed inequality techniques. Consequently, great efforts have been devoted to the sampled-data filtering design problem by virtue of the input delay approach [28], [32]–[34]. To mention a few, the authors in [28] addressed the sampled-data $\mathcal{H}_\infty$ filtering issue for linear systems by an input-output method. Based on the Wirtinger’s inequality, the authors in [33] investigated the event-based $\mathcal{H}_\infty$ filter synthesis problem for sampled-data systems by using some linear matrix inequality techniques. In [34], the $\mathcal{H}_\infty$ filtering issue for Markovian jumping sampled-data systems was investigated.

More recently, there have also been some results on sampled-data filtering design for T-S fuzzy systems [35], [36]. In [35], a sampled-data $\mathcal{H}_\infty$ filter was designed for T-S fuzzy systems subject to interval time-varying state delay. The authors in [36] addressed the sampled-data filtering issue for T-S fuzzy neural networks. Nevertheless, the sampled-data filtering design results in [35], [36] were proposed merely for T-S fuzzy systems with linear local models through a common quadratic Lyapunov function, while the T-S fuzzy affine dynamic models are with substantially improved function approximation competence [13], [14], [39]. Furthermore, these results were basically obtained for deterministic T-S fuzzy systems without considering the stochastic properties. To the authors’ best knowledge, there have been few results on sampled-data filtering design for fuzzy-affine-model-based Itô stochastic nonlinear systems based on piecewise Lyapunov functionals, which motivates this study.

This work focuses on the sampled-data filtering design
problem for Itô stochastic T-S fuzzy affine systems. Through a novel piecewise Lyapunov-Krasovskii functional (PLKF) and an elegant integral inequality, the robust $H_\infty$ performance for the filtering error system is analyzed, and the sampled-data piecewise affine (PWA) filter synthesis results are proposed via some convexification procedures. Simulation studies on a tunnel diode circuit system and an inverted pendulum system are given to verify the effectiveness of the proposed method.

**Notations.** $\text{sym}\{S\}$ is short for $S+S^T$. $\text{diag}\{\cdot\}$ refers to a block diagonal matrix. $\mathbb{Z}_+$ represents the set of non-negative integers. $\mathbb{E}\{\cdot\}$ stands for the mathematical expectation.

II. PRELIMINARIES

Consider an Itô stochastic T-S fuzzy affine system characterized by fuzzy IF-THEN rules as follows.

**Plant Rule** $R^l$: IF $\theta_l(x(t))$ is $\mathcal{F}_l^\phi$, THEN

$$
\begin{align}
&dx(t) = \left[(A_1(x) + \Delta A_1(x)) x(t) + a(x) + \Delta a(x) + B_1 w(t)\right] dt + \left[A_2(x) + \Delta A_2(x) + B_2 w(t)\right] d\varpi(t) \\
y(t) = Cx(t) + Dw(t) \\
z(t) = L_l x(t)
\end{align}
$$

where $R^l$ denotes the $l$-th fuzzy rule; $\mathcal{F}_l^\phi$ refers to fuzzy sets; $r$ represents the number of inference rules; $\theta_l(x(t)) := [\theta_1(x(t)), \ldots, \theta_r(x(t))]$ represent the premise variables; $x(t) \in \mathbb{R}^{n_x}$ denotes the system state; $w(t) \in \mathbb{R}^{n_w}$ represents the external disturbance and $w(t) \in L_2[0, \infty)$; $y(t) \in \mathbb{R}^{n_y}$ represents the system measurement output; $z(t) \in \mathbb{R}^{n_z}$ represents the regulated output; $\varpi(t)$ refers to a one-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with a natural filtration $\{\mathcal{F}_t\}_{t \in \mathbb{R}}$ subject to $\mathbb{E}\{d\varpi(t)\} = 0$ and $\mathbb{E}\{d\varpi^2(t)\} = dt$; $(A_1, A_2, a, B_1, B_2)$ refers to the $l$-th local affine model of the system; $\Delta A_1$ and $\Delta a$ denote the uncertainty terms satisfying

$$
\begin{bmatrix}
\Delta A_1 & \Delta a
\end{bmatrix} = U_l \Delta(t) \begin{bmatrix}
W_1 & W_2
\end{bmatrix}, l \in \mathcal{L}
$$

with $W_1, W_2$, and $W_1$ being known real-valued matrices. $\Delta(t) \in \mathbb{R}^{n_1 \times n_2}$ represents unknown time-varying matrices satisfying

$$
\Delta_t(t) \Delta(t) \leq I, l \in \mathcal{L}.
$$

Denote $\mu_l[\theta(x(t))]$ as the normalized membership function (MF),

$$
\mu_l[\theta(x(t))] := \frac{\prod_{\phi=1}^{\phi=\varphi} \mu_{\phi}[\theta_{\phi}(x(t))]}{\prod_{\phi=1}^{\phi=\varphi} \mu_{\phi}[\theta_{\phi}(x(t))]} \geq 0
$$

with $\mu_{\phi}[\theta_{\phi}(x(t))]$ being the grade of membership of $\theta_{\phi}(x(t))$ in $\mathcal{F}_{\phi}^\phi$. For brevity, denote $\mu_l := \mu_l[\theta(x(t))]$.

Via a center-average defuzzifier, product inference, and singleton fuzzifier, we have the following Itô stochastic T-S fuzzy affine model,

$$
\begin{align}
&dx(t) = \left[(A_1(x) + \Delta A_1(x)) x(t) + a(x) + \Delta a(x) + B_1 w(t)\right] dt + \left[A_2(x) + \Delta A_2(x) + B_2 w(t)\right] d\varpi(t) \\
y(t) = Cx(t) + Dw(t) \\
z(t) = L_l x(t)
\end{align}
$$

where

$$
\begin{align}
A_1(x) &= \sum_{l=1}^{r} \mu_l A_1(x), \quad \Delta A_1(x) = \sum_{l=1}^{r} \mu_l \Delta A_1(x), \\
a(x) &= \sum_{l=1}^{r} \mu_l a_l, \quad \Delta a(x) = \sum_{l=1}^{r} \mu_l \Delta a_l, \\
B_1(x) &= \sum_{l=1}^{r} \mu_l B_1(x), \quad A_2(x) = \sum_{l=1}^{r} \mu_l A_2(x), \\
B_2(x) &= \sum_{l=1}^{r} \mu_l B_2(x), \quad L_l(x) = \sum_{l=1}^{r} \mu_l L_l(x)
\end{align}
$$

Since a polyhedral decomposition of the system state-space is induced by fuzzy rules and MFs, one can deem the global model in (5) as a convex combination of several local models within individual subspaces. According to [14], the premise variable space is divided into crisp subspaces and fuzzy subspaces. The crisp subspace is the region with only one rule, that is, $\mu_l = 1$ for some $l$, and all other MFs are zero. The fuzzy subspaces are subject to $0 < \mu_l < 1$ and the system dynamics are characterized by a convex combination of some local models.

Specify the indices of subspaces $\mathcal{I} = \mathcal{I}_0 \cup \mathcal{I}_1$, and $\mathcal{I}_0$ includes the index set of subspaces covering the origin, while $\mathcal{I}_1$ denotes the index set of subspaces without the origin. The following section is introduced to characterize the indices in each subspace $S_i$

$$
\mathcal{N}(i) := \{m \vert \mu_m[x(t)] > 0, m \in \mathcal{L}, x(t) \in S_i, i \in \mathcal{I}\}.
$$

Similar to [14], with (7), the system (5) is formulated as

$$
\begin{align}
&dx(t) = \left[(A_{i1} + \Delta A_{i1}) x(t) + a_i + \Delta a_i + B_{i1} w(t)\right] dt + \left[A_{i2} x(t) + B_{i2} w(t)\right] d\varpi(t) \\
y(t) = Cx(t) + Dw(t) \\
z(t) = L_i x(t), \quad x(t) \in S_i, i \in \mathcal{I}
\end{align}
$$

where

$$
\begin{align}
A_{i1} &:= \sum_{m \in \mathcal{N}(i)} \mu_m A_{m1}, \quad \Delta A_{i1} := \sum_{m \in \mathcal{N}(i)} \mu_m \Delta A_{m1}, \\
a_i &:= \sum_{m \in \mathcal{N}(i)} \mu_m a_m, \quad \Delta a_i := \sum_{m \in \mathcal{N}(i)} \mu_m \Delta a_m, \\
B_{i1} &:= \sum_{m \in \mathcal{N}(i)} \mu_m B_{m1}, \quad A_{i2} := \sum_{m \in \mathcal{N}(i)} \mu_m A_{m2}, \\
B_{i2} &:= \sum_{m \in \mathcal{N}(i)} \mu_m B_{m2}, \quad L_i := \sum_{m \in \mathcal{N}(i)} \mu_m L_m
\end{align}
$$

with $\sum_{m \in \mathcal{N}(i)} \mu_m \vert \theta(x(t)) \vert = 1, 0 < \mu_m \vert \theta(x(t)) \vert \leq 1$.

In this work, merely the sampled measurement output of $y(t)$ is assumed to be available for filtering design purpose. Specifically, the sampled measurement outputs $y(t_j), j \in \mathbb{Z}_+$ at the sampling instant $t_j$ are kept constant by a zero-order hold (ZOAH), and

$$
0 = t_0 < t_1 < t_2 < \cdots < t_j < \cdots
$$

with $\lim_{j \to \infty} t_j = \infty$. 


Then for system (1), design a sampled-data piecewise affine (PWA) filter as

\[
\begin{align*}
\dot{x}(t) &= [A_{fi}\hat{x}(t) + a_{fi}]dt + B_{fi}y(t_j)dt \\
\dot{z}(t) &= C_{fi}\hat{x}(t), \quad t \in [t_j, t_{j+1}), \quad i \in \mathcal{I}
\end{align*}
\]  (11)

where \(\hat{x}(t) \in \mathbb{R}^{n_f}\) is the filter states, and \(n_f = n_x\) for the full-order filter and \(n_f < n_x\) for the reduced-order case. \(\dot{z}(t) \in \mathbb{R}^{n_z}\) is the estimation of \(z(t)\). \(A_{fi} \in \mathbb{R}^{n_f \times n_f}, B_{fi} \in \mathbb{R}^{n_f \times n_w}, C_{fi} \in \mathbb{R}^{n_z \times n_f},\) and \(a_{fi} \in \mathbb{R}^{n_f \times 1},\) for \(i \in \mathcal{I},\) are filter gains to be determined, and \(a_{fi} = 0\) when \(i \in I_0.\)

Denote

\[d(t) = t - t_j \leq h, \quad t \in [t_j, t_{j+1}).\]  (12)

Associating the system (8) with the filter (11), the filtering error system is derived as,

\[
\begin{align*}
\dot{z}(t) &= \tilde{A}_i\tilde{z}(t) + \tilde{A}_di\tilde{x}(t - d(t)) + \tilde{B}_i\tilde{w}(t)dt \\
\dot{\hat{x}}(t) &= \tilde{A}_i\hat{x}(t) + \tilde{B}_i\hat{w}(t)d\omega(t)
\end{align*}
\]  (13)

where

\[\tilde{A}_i = \begin{bmatrix} A_{i1} + \Delta A_{i1} & 0 \\ 0 & A_{fi} \end{bmatrix}, \quad \tilde{A}_{di} = \begin{bmatrix} -B_{fi}C & \tilde{B}_i \\ \tilde{B}_i & 0 \end{bmatrix}, \quad \tilde{B}_i = \begin{bmatrix} B_{i1} \\ 0 \end{bmatrix}, \quad \tilde{L}_i = \begin{bmatrix} L_i - C_{fi} \end{bmatrix}, \quad \tilde{J}_i = \begin{bmatrix} I_{n_x} \end{bmatrix},\]

if \(i \in \mathcal{I},\)

\[\tilde{A}_i = \begin{bmatrix} A_{i1} + \Delta A_{i1} & 0 \\ 0 & A_{fi} \end{bmatrix}, \quad \tilde{A}_{di} = \begin{bmatrix} -B_{fi}C & \tilde{B}_i \\ \tilde{B}_i & 0 \end{bmatrix}, \quad \tilde{B}_i = \begin{bmatrix} B_{i1} \\ 0 \end{bmatrix}, \quad \tilde{L}_i = \begin{bmatrix} L_i - C_{fi} \end{bmatrix}, \quad \tilde{J}_i = \begin{bmatrix} I_{n_x} \end{bmatrix},\]

if \(i \in I_0.\)

This work will design a PWA filter relying on the sampled measurement output for hő stochastic fuzzy system (1). For a prescribed disturbance attenuation level \(\gamma > 0\), the resulting PWA filtering error system is guaranteed to be stochastically stable with \(\tilde{w}(t) \equiv 0.\) It is also satisfied that

\[
\mathbb{E}\left\{\int_0^\infty \|\tilde{z}(t)\|^2dt\right\} < \gamma^2 \mathbb{E}\left\{\int_0^\infty \|\tilde{w}(t)\|^2dt\right\}
\]  (15)

under zero initial condition for nonzero \(\tilde{w}(t) \in L_2[0, \infty),\) where \(\|\tilde{z}(t)\| = \sqrt{\mathbb{E}\{\tilde{z}(t)\|^2\}}, \|\tilde{w}(t)\| = \sqrt{\mathbb{E}\{\tilde{w}(t)\|^2\}},\)

III. MAIN RESULTS

By a novel piecewise Lyapunov-Krasovskii functional (PLKF), this section will propose some new stochastic stability analysis results for the filtering error system (13), and an elegant integral inequality is employed to tackle the coupling issues in the quadratic crossing terms, such that the conservatism can be further reduced. Then the sampled-data PWA filter synthesis results will be derived under a convex optimization framework.

A. Stochastic Stability Analysis via PLKF

In this subsection, the stochastic stability of the filtering error system will be analyzed by constructing a novel PLKF.

To guarantee the PLKF continuous among boundaries of the subspace, according to [13, 14], construct matrices \(\tilde{F}_i = [f_i \quad F_i], \quad i \in \mathcal{I},\) with \(f_i = 0\) for \(i \in I_0\) to depict the boundary among the subspaces,

\[
\tilde{F}_i \begin{bmatrix} 1 \\ x(t) \end{bmatrix} = \tilde{F}_s \begin{bmatrix} 1 \\ x(t) \end{bmatrix}, \quad x(t) \in S_i \bigcap S_s, \quad i, s \in \mathcal{I} \quad \text{(16)}
\]

For further conservatism reduction, we will also adopt the S-procedure by establishing matrices \(\tilde{G}_i = [g_i \quad G_i],\) \(i \in \mathcal{I},\) with \(g_i = 0\) for \(i \in I_0\) and

\[
\tilde{G}_i \begin{bmatrix} 1 \\ x(t) \end{bmatrix} \succeq 0, \quad x(t) \in S_i, \quad i \in \mathcal{I} \quad \text{(17)}
\]

with \(\succeq\) indicating each entry being nonnegative.

Considering the system (13), for notational convenience, denote

\[
\begin{align*}
\tilde{f}(t) &= \tilde{A}_i\tilde{z}(t) + \tilde{A}_{di}\tilde{x}(t - d(t)) + \tilde{B}_i\tilde{w}(t) \\
\tilde{g}(t) &= \tilde{A}_i\tilde{z}(t) + \tilde{B}_i\tilde{w}(t),
\end{align*}
\]  (18)

and then the system (13) is reformulated as

\[
\begin{align*}
\dot{\tilde{z}}(t) &= \tilde{A}_i\tilde{z}(t) + \tilde{B}_i\tilde{w}(t), \quad \tilde{z}(t) \in \tilde{L}_i\tilde{x}(t), \quad \tilde{z}(t) \in \tilde{J}_i, \quad \tilde{z}(t) \in \tilde{J}_i \quad \text{(19)}
\end{align*}
\]

Theorem 3.1. For a given disturbance attenuation level \(\gamma > 0,\) the system (13) is stochastically stable, if matrices \(0 < Q = \tilde{Q}_T \in \mathbb{R}^{n_x \times n_x}, 0 < M = \tilde{M}_T \in \mathbb{R}^{n_z \times n_z}, X = \tilde{X}_T, Y = \tilde{Y}_T \succeq 0, T = \tilde{T}_T \succeq 0, V_i \in \mathbb{R}^{n_2 + 2n_f + 2n_w} \times (n_z + n_f), \tilde{N} \in \mathbb{R}^{(n_z + n_2 + 2n_w + 2n_w) \times n_z},\)

\[
\begin{align*}
&f_i = 0, \quad f_i \in I_0, \quad V_i \in \mathbb{R}^{(n_2 + 2n_f + 2n_w + 2n_w) \times (n_z + n_f)}, \tilde{N} \in \mathbb{R}^{(n_2 + 2n_f + 2n_w + 2n_w) \times n_z}, \quad \text{for } i \in \mathcal{I}, \quad \text{exist such that the conditions in (20)-(21) hold,}
\end{align*}
\]

\[
\tilde{F}_iT\tilde{F}_i - \tilde{G}_iT\tilde{Y}_iG_i \succ 0, \quad i \in \mathcal{I}, \quad \text{(20)}
\]

\[
\text{Sym}\{e_2^T\tilde{F}_iT\tilde{F}_i1 + \tilde{N}\Sigma + V_i\mathcal{A}_i\} + (\tilde{A}_i\tilde{J}_i e_2 + \tilde{B}_i\tilde{J}_i e_2)\tilde{G}_iT\tilde{F}_i(\tilde{A}_i\tilde{J}_i e_2 + \tilde{B}_i\tilde{J}_i e_2) + (1 - \sigma(n))\tilde{M}\tilde{J}_i e_2 - \tilde{e}_2^T\tilde{Q}_e e_2 + \sigma(n)\tilde{h}\tilde{N}\text{diag}\{\tilde{M}, 3\tilde{M}, 5\tilde{M}, 7\tilde{M}\}^{-1}\tilde{N}^T + \tilde{e}_2^T\tilde{J}_i\tilde{G}_iT\tilde{J}_i e_2 + \tilde{L}_i\tilde{L}_i - \gamma^2\tilde{e}_2^T\tilde{e}_2 < 0, \quad n = 1, 2, \quad i \in \mathcal{I} \quad \text{(21)}
\]

where \(\tilde{A}_i\) and \(\tilde{B}_i\) are given as in (14), and

\[
\begin{align*}
\tilde{F}_i &= \text{diag}(\tilde{F}_i, I_{n_x}), \quad \tilde{G}_i = \text{diag}(\tilde{G}_i, I_{n_x}); \\
J_2 &= \begin{bmatrix} I_{n_w} & 0 \end{bmatrix},
\end{align*}
\]  (22)
are generally bilinear matrix inequalities (BMIs), which
augmented terms
filtering error system (13) have been proposed in Theorem
Remark 3.2.
\[ \begin{aligned}
\Sigma &= \begin{bmatrix}
\epsilon_4^T & \epsilon_4 - \epsilon_5^T & \epsilon_4 - 3\epsilon_5^T + 2\epsilon_6^T \\
\epsilon_4^T & -6\epsilon_5^T + 10\epsilon_6^T - 20\epsilon_7^T
\end{bmatrix}^T,
\end{aligned} \\
\tilde{A}_i &= \begin{bmatrix}
-I_{(n_x+n_f)} & \tilde{A}_{i1} & \tilde{A}_{di} & 0_{(n_x+n_f)\times 4n_x} & \tilde{B}_{i1}
\end{bmatrix},
\end{aligned} \\
\tilde{L}_i &= \begin{bmatrix}
0_{n_x\times (n_x+n_f)} & \tilde{L}_{i1} & 0_{n_x\times (5n_x+2n_w)}
\end{bmatrix},
\end{aligned} \\
e_1 &= \begin{bmatrix}
I_{(n_x+n_f)} & 0_{(n_x+n_f)\times (6n_x+n_f+2n_w)}
\end{bmatrix},
\end{aligned} \\
e_2 &= \begin{bmatrix}
0_{(n_x+n_f)\times (n_x+n_f)} & I_{(n_x+n_f)}
\end{bmatrix},
\end{aligned} \\
e_s &= \begin{bmatrix}
0\cdots 0 & I_{(s-1)\times s} & \cdots & 0
\end{bmatrix},
\end{aligned} \\
\tilde{J}_0 &= \begin{bmatrix}
I_{n_x} & 0_{n_x\times n_f}
\end{bmatrix},
\tilde{J}_1 &= \begin{bmatrix}
0 & I_{n_x}
\end{bmatrix},
\end{aligned} \\
\text{if } i \in \mathcal{T}_0,
\end{aligned} \\
A_i &= \begin{bmatrix}
-I_{(1+n_x+n_f)} & A_{i1} & A_{di}
\end{bmatrix},
\end{aligned} \\
\tilde{L}_i &= \begin{bmatrix}
0_{n_x\times (1+n_x+n_f)} & \tilde{L}_{i1} & 0_{n_x\times (5n_x+2n_w)}
\end{bmatrix},
\end{aligned} \\
e_1 &= \begin{bmatrix}
I_{(1+n_x+n_f)} & 0_{(1+n_x+n_f)\times (1+6n_x+n_f+2n_w)}
\end{bmatrix},
\end{aligned} \\
e_2 &= \begin{bmatrix}
0_{(1+n_x+n_f)\times (1+n_x+n_f)} & I_{(1+n_x+n_f)}
\end{bmatrix},
\end{aligned} \\
e_s &= \begin{bmatrix}
0\cdots 0 & I_{(s-1)\times s} & \cdots & 0
\end{bmatrix},
\end{aligned} \\
\tilde{J}_0 &= \begin{bmatrix}
I_{1+n_x} & 0_{(1+n_x)\times n_f}
\end{bmatrix},
\tilde{J}_1 &= \begin{bmatrix}
0 & I_{n_x}
\end{bmatrix},
\end{aligned} \\
\text{if } i \in \mathcal{T}_1
\end{aligned} \\
with h given in (12), and
\[ \varrho(s) = \begin{cases}
  n_x, s = 3, \ldots, 7, \\
  2n_w, s = 8.
\end{cases} \tag{23} \]

Proof: See Appendix B.

Remark 3.1. By using a new PLKF-based method and an elegant integral inequality given in Lemma 2, new sufficient conditions for the stochastic stability analysis of the filtering error system (13) have been proposed in Theorem 3.1. Note that associated with \( N \in \mathbb{R}^{(2+7n_x+2n_f+2n_w)\times 4n_x} \), for \( i \in \mathcal{D}_0, \mathcal{N} \in \mathbb{R}^{(2+7n_x+2n_f+2n_w)\times 4n_x} \), for \( i \in \mathcal{T}_1 \), the augmented terms \( \int_{t-d(t)}^{\alpha} x^T(s)ds, \int_{t-d(t)}^{\alpha} x^T(s)d\sigma(s), \) \( \int_{t-d(t)}^{\beta} x^T(s)ds, \int_{t-d(t)}^{\beta} x^T(s)d\sigma(s), \) and \( \int_{t-d(t)}^{\beta} x^T(s)ds \) d\( \beta(s) \) d\( \sigma(s) \) d\( \alpha(s) \) d\( \sigma(s) \) involve more information of time-delays in the PLKF in (47), and the analysis conservatism can be further reduced. These features distinguish our results from those given in [25]–[27].

Remark 3.2. Note that the stochastic stability and robust performance analysis results given in Theorem 3.1 are actually delay-dependent. It is also noted that as auxiliary matrix multipliers \( \tilde{V}_i \) are introduced, the Lyapunov matrices in Theorem 3.1 are not involved in any product with the filter gains. It will be shown in the next subsection that this feature enables one to synthesize the sampled-data filter based on PLKF and the corresponding results are also expected to be less conservative due to the increased freedom from piecewise Lyapunov matrices and auxiliary matrix multipliers. These points also distinguish our results from the existing ones in [35], [36].

Remark 3.3. It is noted that when the filter gains \( A_{fi}, a_{fi}, B_{fi} \) and \( C_{fi} \) are unknown, the conditions given in Theorem 3.1 are generally bilinear matrix inequalities (BMIs), which are difficult to be solved. Moreover, in Theorem 3.1, the filter gains and system matrices are coupled with auxiliary matrix multipliers and the parameter uncertainties are also involved, and the linearization of the conditions given in Theorem 3.1 is a challenging problem.

B. Sampled-Data PWA Filter Design

In this subsection, it will be shown that the coupling issues existing in the filter gains with auxiliary matrix multipliers can be solved by a convexification procedure. The sampled-data PWA filter design result is summarized in the following theorem.

Theorem 3.2. For the fuzzy stochastic system (1), the system (13) is stochastically stable in an \( \mathcal{H}_\infty = \gamma \) setup, if matrices \( 0 < Q = Q^T \in \mathbb{R}^{n_x\times n_x}, 0 < M = M^T \in \mathbb{R}^{n_x\times n_x}, X = X^T, Y_i = Y_i^T \geq 0, T_i = T_i^T \geq 0, V_i \in \mathbb{R}^{n_x\times n_x}, \tilde{A}_fi, \tilde{B}_fi, \tilde{C}_fi \in \mathbb{R}^{n_x\times n_x}, i \in \mathcal{I}, N \in \mathbb{R}^{(7n_x+2n_f+2n_w)\times 4n_x}, \tilde{V}_i \in \mathbb{R}^{(2+7n_x+2n_f+2n_w)\times 4n_x}, \) for \( i \in \mathcal{D}_0, \tilde{a}_fi \in \mathbb{R}^{n_x\times n_f}, \tilde{N} \in \mathbb{R}^{(2+7n_x+2n_f+2n_w)\times 4n_x}, \tilde{V}_i \in \mathbb{R}^{(2+7n_x+2n_f+2n_w)\times (1+n_x)}, \) for \( i \in \mathcal{T}_1, \) and scalars \( \varepsilon_i > 0, i \in \mathcal{I} \), exist such that (20) and the following inequality holds,

\[
[ \begin{array}{cccc}
A_1 & \frac{\sigma(n)hN}{\sqrt{\mathcal{M}}} & 0 & 0 \\
0 & -\Lambda_2 & 0 & 0 \\
0 & 0 & -I & 0 \\
0 & 0 & 0 & -\varepsilon_i I
\end{array} ]_{n \times n} < 0, \tag{24}
\]

where

\[
\begin{aligned}
A_1 &= \text{sym}\{e_2^T \tilde{F}_1^T \tilde{X}_1 + \tilde{X}_1 + \tilde{N} \Sigma + \tilde{V}_i \tilde{A}_i \tilde{m} + \tilde{H} \tilde{A}_fi + (A_{m2}J_3e_2 + B_{m2}J_2e_8)^T \tilde{F}_1^T \tilde{X}_1 \tilde{F}_1e_4 + (A_{m2}J_3e_2 + B_{m2}J_2e_8)^T + (1 - \sigma(n))he_2^T \tilde{F}_1^T M \tilde{J}_{1e_2} - \frac{c}{2} \tilde{Q}e_3
\end{aligned}
\]
\[
\begin{align*}
\bar{W}_m &= \begin{bmatrix} 0_{n_x \times (1+n_x+n_f)} & W_{m2} \\ W_{m1} & 0_{n_x \times (5n_x+n_f+2n_w)} \end{bmatrix}, \\
\bar{L}_{im} &= \begin{bmatrix} 0_{n_x \times (1+n_x+n_f)} & 0_{n_x \times 1} & L_{m} \\ -C_{fi} & 0_{n_x \times (5n_x+2n_w)} \end{bmatrix}, \\
\bar{A}_{fi} &= \begin{bmatrix} 0_{n_f \times (1+n_x+n_f)} & -V_i & \bar{a}_{fi} & 0_{n_f \times n_x} & A_{fi} \\ B_{fi} C & 0_{n_f \times (4n_x+n_f)} & B_{fi} D \end{bmatrix}, \\
\bar{H} &= \begin{bmatrix} 0_{n_f \times 1} & H_{m}^T & \bar{I}_{n_f} & 0_{n_f \times 1} & H^T \end{bmatrix}, \\
\bar{J}_3 &= \begin{bmatrix} 0_{1 \times n_x} \\ I_{n_x} \end{bmatrix},
\end{align*}
\]
with \(H_1, H_2 \in \mathbb{R}^{n_x \times n_f}\) being arbitrary matrices, \(v\) being a scalar, and \(\bar{J}_1\) and \(\bar{J}_2\) are given in (14), and \(\Sigma\) is given in (22).

In addition, the filter gains can be obtained by
\[
\begin{align*}
A_{fi} &= V_i^{-1} \bar{a}_{fi}, \quad i \in \mathcal{I}_1, \\
A_{fi} &= V_i^{-1} \bar{A}_{fi}, \quad B_{fi} = V_i^{-1} \bar{B}_{fi}, \quad i \in \mathcal{I}.
\end{align*}
\]

Proof: Extracting the fuzzy MFs, the following inequality indicates (21),
\[
\begin{align*}
\text{Sym}(e_2^T F_i^T X \bar{F}_i e_1 + \bar{N} \Sigma + \bar{V}_i (A_{im} + \Delta A_{im})) \\
+ (A_{m2} \bar{J}_1 e_2 + B_{m2} J_2 e_8) F_i^T X \bar{F}_i (A_{m2} \bar{J}_1 e_2 + B_{m2} J_2 e_8) \\
+ (1 - \sigma(n)) e_2^T J_1^T M J_1 e_2 - e_3^T Q e_3 \\
+ (1 - \sigma(n)) h \bar{N} \bar{D} (M, 3M, 5M, 7M) -1 N^T \\
e_2^T J_1^T G^T T_i G_i \bar{J}_0 e_2 + J^T \bar{L}_{im} \bar{J}_0 e_2 - \gamma^2 e_8 e_8 < 0,
\end{align*}
\]
where
\[
\begin{align*}
A_{im} + \Delta A_{im} &= \begin{bmatrix} -1_{n_x} & 0 & A_{m1} + \Delta A_{m1} & 0 & 0 \\ 0 & -1_{n_f} & 0 & A_{fi} & B_{fi} C \\ 0 & 0 & 0_{n_x \times 4n_x} & B_{m1} & 0 \\ 0 & 0 & 0_{n_f \times n_x} & 0 & B_{fi} D \end{bmatrix},
\end{align*}
\]
\[
\begin{align*}
\bar{A}_{m2} &= \begin{bmatrix} A_{m2} \\ 0 \end{bmatrix}, \quad B_{m2} = \begin{bmatrix} B_{m2} \end{bmatrix},
\end{align*}
\]
\[
\begin{align*}
\bar{L}_{im} &= \begin{bmatrix} 0_{n_x \times (n_x+n_f)} & L_{m} \\ -C_{fi} & 0_{n_x \times (5n_x+2n_w)} \end{bmatrix}, \\
\bar{A}_{im} + \Delta A_{im} &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1_{n_x} & a_m + \Delta a_m & A_{m1} + \Delta A_{m1} & 0 \\ 0 & 0 & 0 & 0 & 0_{n_x \times 4n_x} \\ 0 & 0 & 0 & 0_{n_f \times n_x} & B_{m1} \\ 0 & 0 & 0 & 0_{n_f \times 4n_x} & B_{m1} \\ A_{fi} & B_{fi} C & 0_{n_f \times 4n_x} & 0 & B_{fi} D \end{bmatrix},
\end{align*}
\]
\[
\begin{align*}
\bar{A}_{m2} &= \begin{bmatrix} 0_{n_x \times (1+n_x+n_f)} & 0_{n_x \times 1} & L_{m} \\ -C_{fi} & 0_{n_x \times (5n_x+2n_w)} \end{bmatrix},
\end{align*}
\]
where
\[
\begin{align*}
\bar{W}_m &= \begin{bmatrix} 0_{n_x \times 2} \times (1+n_x+n_f) & W_{m2} \\ W_{m1} & 0_{n_x \times (5n_x+n_f+2n_w)} \end{bmatrix}, \\
\bar{L}_{im} &= \begin{bmatrix} 0_{n_x \times (1+n_x+n_f)} & 0_{n_x \times 1} & L_{m} \\ -C_{fi} & 0_{n_x \times (5n_x+2n_w)} \end{bmatrix},
\end{align*}
\]
By inspecting the structure of system matrices in (28), the filter gains do not appear in the first row of \(\bar{A}_{im} + \Delta A_{im}\) for \(i \in \mathcal{I}_0\), or the first two-rows of \(\bar{A}_{im} + \Delta A_{im}\) for \(i \in \mathcal{I}_1\). Aiming to deal with the condition (27) under a convex optimization framework with less conservatism, specify the slack variables \(\tilde{V}_i\) in (27) as
\[
\begin{align*}
\tilde{V}_i &= \begin{bmatrix} \bar{V}_i^1 & \bar{V}_i^2 \end{bmatrix}, \\
\bar{V}_i^2 &= \begin{bmatrix} V_i^T H_i^T & V_i^T & 0_{n_f \times (4n_x+n_f+2n_w)} \end{bmatrix}^T, \\
\bar{V}_i &= \begin{bmatrix} 0_{n_f \times 1} & V_i^T H_i^T & V_i^2^T H_i^T \\ 0_{n_f \times 1} & V_i^T & 0_{n_f \times (4n_x+n_f+2n_w)} \end{bmatrix},
\end{align*}
\]
where \(\tilde{V}_i^1 \in \mathbb{R}^{(7n_x+2n_f+2n_w) \times n_x}\), for \(i \in \mathcal{I}_0\), \(\tilde{V}_i^1 \in \mathbb{R}^{(2+7n_x+2n_f+2n_w) \times (1+n_x)}\), for \(i \in \mathcal{I}_1\), \(V_i \in \mathbb{R}^{n_f \times n_f}\), and \(\bar{H}_1, H_2 \in \mathbb{R}^{n_x \times n_f}\) are arbitrary matrices, and \(v\) is a scalar. Define
\[
\bar{a}_{fi} = V_i a_{fi}, \quad \bar{A}_{fi} = V_i A_{fi}, \quad \bar{B}_{fi} = V_i B_{fi}.
\]
Based on (29)-(30) and using Lemma 3 given in the Appendix A to tackle the parameter uncertainties presented in (2), the subsequent inequality indicates (27) for scalars \(\varepsilon_i > 0\), \(i \in \mathcal{I}\) relying on Schur complement,
\[
\begin{align*}
\bar{Y}_1 &= \begin{bmatrix} \sqrt{\sigma(n)} \bar{N} & \bar{L}^T \end{bmatrix} + \begin{bmatrix} -\text{diag} \{ M, 3M, 5M, 7M \} \end{bmatrix} \begin{bmatrix} 0 & \bar{L} \end{bmatrix}^T, \\
&= \begin{bmatrix} 0 & \bar{L} \end{bmatrix}^T \begin{bmatrix} -\bar{L}^T \end{bmatrix} \begin{bmatrix} 0 & \bar{L} \end{bmatrix}, \\
&= \begin{bmatrix} -\bar{L}^T \end{bmatrix} \begin{bmatrix} 0 & \bar{L} \end{bmatrix} \begin{bmatrix} -\bar{L} \end{bmatrix} + \begin{bmatrix} -\bar{L} \end{bmatrix}^T \begin{bmatrix} -\bar{L} \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix}, \\
&\leq 0, \\
\tilde{W}_m &= \begin{bmatrix} 0_{n_x \times 2} \times (1+n_x+n_f) & W_{m2} \\ W_{m1} & 0_{n_x \times (5n_x+n_f+2n_w)} \end{bmatrix}, \\
\tilde{L}_{im} &= \begin{bmatrix} 0_{n_x \times (1+n_x+n_f)} & 0_{n_x \times 1} & L_{m} \\ -C_{fi} & 0_{n_x \times (5n_x+2n_w)} \end{bmatrix},
\end{align*}
\]
A software. Inequalities (LMI) under a convex optimization framework, uncertainty terms also appear in other system matrices. It is worth mentioning that the design approach developed in this paper is robustified by convexification procedures, where \( A \) is calculated within Step 1, solve the LMI problems given in (24), (25), (26) and (27). The proof is completed.

**Remark 3.4.** Note that new delay-dependent \( n_f \)-order filter design results are provided in Theorem 3.2 through some convexification procedures, where \( n_f = n_x \) is for the full-order filter and \( n_f < n_x \) is for the reduced-order case.

**Remark 3.5.** It is noted that the filter design results proposed in Theorem 3.2 are presented in terms of linear matrix inequalities (LMIs) under a convex optimization framework, which can be solved efficiently via commercially available software.

**Remark 3.6.** Detailed procedures on calculating the gains of the sampled-data filter are shown as follows.

**Step 1.** Based on the space partitions and normalized MFs of the stochastic fuzzy system (1), compute the matrices \( F_i \) and \( G_i \) by (16)-(17) for Theorem 3.2.

**Step 2.** For given robust \( \mathcal{H}_\infty \) performance \( \gamma \), scalar \( v \) and matrices \( H_1 \) and \( H_2 \), associated with the parameter matrices calculated within Step 1, solve the LMI problems given in Theorem 3.2 over the matrix variables \( X, M, Q, Y_i, T_i, V_i, A_{fi}, \tilde{A}_{fi}, B_{fi}, N_i, \), and scalars \( \varepsilon \) for \( i \in \mathcal{T} \).

**Step 3.** The filter gains \( C_i \) can be obtained directly in Step 2. Relying on the matrices \( V_i, \tilde{A}_i, \tilde{A}_{fi}, \) and \( B_{fi} \) obtained in Step 2, the filter gains \( A_{fi}, a_{fi}, \) and \( B_{fi} \) can be calculated via (26).

**Remark 3.7.** Notice that in Theorem 3.2, when the characteristic matrices \( \hat{F}_i \equiv \left[ \begin{array}{cc} 0_{n_x \times 1} & \mathbf{I}_{n_x} \end{array} \right] \) of the filtering design results are instantly obtained through a common Lyapunov-Krasovskii functional (CLKF) as \( V_i(\tilde{x}, t) = \tilde{x}^T(t)F_i^T(XF_i)\tilde{x}(t) = \chi^T(t)\chi(t) \), and \( \chi(t) = [x(t) \tilde{x}(t)]^T \), and \( V_i(\tilde{x}, t), V_3(\tilde{x}, t) \) are defined the same as in (47). Consequently, the CLKF is basically a special case of the more general PLKFs, which also distinguish the proposed sampled-data filtering design results in this paper from the existing CLKF-based ones in [35], [36]. In the next section, it will be illustrated that the PLKF-based sampled-data filtering design results are generally less conservative than those based on a CLKF.

**Remark 3.8.** To avoid unnecessarily complicated notations, in this work we only consider the parameter uncertainty terms appearing in the matrices \( A_i \) and \( a_i \). However, it is also worth mentioning that the design approach developed in this paper can be easily generalized to the case when the parameter uncertainty terms also appear in other system matrices.

### IV. Simulation Studies

**Example 4.1.** Consider the sampled-data filtering issue of a tunnel diode circuit system [37] shown in Fig. 1, which is interfered with \( \text{It} \) stochastic processes.

![Fig. 1. Tunnel diode circuit system](image)

The dynamical properties of the circuit system is characterized by the subsequent stochastic equations,

\[
\begin{align*}
\frac{dx_1(t)}{dt} &= \frac{1}{C_{\text{cap}}}(\frac{1}{R_D}x_1 + x_2) \, dt \\
\frac{dx_2(t)}{dt} &= \frac{1}{L}(-x_1 - R_2x_2 + w(t)) \, dt \\
&\quad + (0.1x_1 - 0.2x_2 + w(t))d\bar{w}(t)
\end{align*}
\]

where \( x_1 = v_c \) and \( x_2 = I_L \); \( \bar{w}(t) \) represents a one-dimensional Wiener process; \( w(t) \) represents the external disturbance; \( R = 5 \Omega \) is the resistor parameter; \( C_{\text{cap}} = 0.05F \) is the capacitor parameter; \( L = 0.2H \) is the inductor parameter. It is assumed that \( \frac{I_D}{R_D} = \frac{I_D}{v_c} = \eta_1 + \eta_2v_c^2 \), where \( \eta_1 = 0.002 \) and \( \eta_2 = 0.01 \).

Linearizing the nonlinear circuit system (33) around \( (\pm 3, 0)^T \) and \( (0, 0)^T \), then system (33) is described by an \( \text{It} \) stochastic T-S fuzzy affine model with three rules as:

**Plant Rule** \( R^i \): IF \( x_1(t) \) is \( F^i \), THEN

\[
\begin{align*}
\frac{dx(t)}{dt} &= [(A_{11} + \Delta A_{11})x(t) + a_1 + \Delta a_1 + B_{11}w(t)] \, dt \\
&\quad + [A_{12}x(t) + B_{12}w(t)]d\bar{w}(t) \\
y(t) &= Cx(t) + Dw(t) \\
z(t) &= L_i x(t), \quad i \in \mathcal{L} = \{1, \ldots, r\}
\end{align*}
\]

where the system parameters are

\[
\begin{align*}
A_{11} &= A_{31} = \begin{bmatrix} -3.64 & 20 \\ -5 & -25 \end{bmatrix}, \\
A_{21} &= \begin{bmatrix} -0.04 & 20 \\ -5 & -25 \end{bmatrix}, \\
a_1 &= \begin{bmatrix} -5.4 \\ 0 \end{bmatrix}, a_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, a_3 = \begin{bmatrix} 5.4 \\ 0 \end{bmatrix}, \\
B_{11} &= \begin{bmatrix} 0 \\ 5 \end{bmatrix}, B_{12} = \begin{bmatrix} 0.1 \\ -0.2 \end{bmatrix}, B_{22} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\
C &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, D = 5, L_i = \begin{bmatrix} 2 & 1 \end{bmatrix}, i \in \{1, 2, 3\}
\end{align*}
\]

and the parameter uncertainties \( \Delta A_{11} \) and \( \Delta a_i \) are given in the form of (2) with

\[
\begin{align*}
U_i &= \begin{bmatrix} 0.1 & 0 \end{bmatrix}, \\
W_{1i} &= \begin{bmatrix} 0.2 & 0 \end{bmatrix}, W_{2i} = 0.05, \quad i \in \{1, 2, 3\}.
\end{align*}
\]
The fuzzy sets are “$\mathcal{F}_1^1$” is “about $-3$”, and “$\mathcal{F}_1^2$” is “about 0”, and “$\mathcal{F}_1^3$” is “about $3$”. The normalized MFs are presented in Fig. 2 with $d_1 = 1$ and $d_2 = 4$, which are given in Appendix C in detail. The premise variable space is divided as,

$$
S_1 = \{x \in \mathbb{R}^2 | -d_2 \leq x_1 < -d_1\},
S_2 = \{x \in \mathbb{R}^2 | -d_1 \leq x_1 \leq d_1\},
S_3 = \{x \in \mathbb{R}^2 | d_1 < x_1 \leq d_2\},
$$

(37)

We attempt to synthesize a full-order sampled-data PWA filter (11) to ensure the filtering error system to be stochastically stable with robust performance $\gamma$. Assume that the measurement output is in sampled-data form with given maximum sampling interval $h = 0.2s$. Furthermore, the existence of the affine terms $a_l + \Delta a_l$ also introduce much difficulty for the sampled-data filter synthesis. Thus, the methods given in [25]–[27] are not applicable. Given $\gamma = 1$ and using Theorem 3.2 with $H_1 = H_2 = I_2$ and $p = 1$, one can obtain feasible solutions, and the full-order filter gains and Lyapunov matrices are

$$
A_{f1} = \begin{bmatrix}
-4.6580 & 3.2923 \\
-0.0794 & -10.6118 \\
0.2673 & -0.6699
\end{bmatrix}, a_{f1} = \begin{bmatrix}
12.9090 \\
-7.4220 \\
\end{bmatrix},
$$

$$
B_{f1} = \begin{bmatrix}
0.5497 \\
-1.2532 \\
\end{bmatrix}, C_{f1} = \begin{bmatrix}
0.0168 & 0.0185
\end{bmatrix},
$$

$$
A_{f2} = \begin{bmatrix}
-2.5462 & 4.2916 \\
0.1958 & -8.7411
\end{bmatrix},
$$

$$
B_{f2} = \begin{bmatrix}
0.3235 \\
-0.4709
\end{bmatrix}, C_{f2} = \begin{bmatrix}
-0.0009 & 0.0043
\end{bmatrix},
$$

(38)

and

$$
X = \begin{bmatrix}
-0.3126 & 0.0148 & -0.2000 \\
0.0148 & -0.6204 & 0.3035 \\
-0.2000 & 0.3035 & 0.1911 \\
0.1377 & -0.1703 & 0.1185 \\
-0.0266 & 0.0493 & 0.1147 \\
0.0061 & 0.0127 & 0.0441 \\
0.1377 & -0.0266 & 0.0061 \\
-0.1703 & 0.0493 & 0.0127 \\
0.1185 & 0.1147 & 0.0441 \\
0.3929 & 0.0619 & 0.1554 \\
0.0619 & 0.1864 & 0.0940 \\
0.1554 & 0.0940 & 0.2037
\end{bmatrix},
$$

$$
Q = \begin{bmatrix}
12.1547 & 0.1054 \\
0.1054 & 12.1333
\end{bmatrix},
$$

$$
M = \begin{bmatrix}
1.6824 & -0.3906 \\
-0.3906 & 7.9503
\end{bmatrix}.
$$

Notice that when using Theorem 3.2 with PLKF in (47), one can calculate $\bar{G}_i$ and $\bar{F}_i$ by

$$
\begin{bmatrix}
G_1 \\
G_2 \\
G_3
\end{bmatrix} = \begin{bmatrix}
g_1(t) \\
g_2(t) \\
g_3(t)
\end{bmatrix},
$$

and the parameter uncertainties $\Delta A_{l1}$ and $\Delta a_{l1}$ are given in the sequel, we consider a numerical example.

**Example 4.2.** Consider a stochastic fuzzy affine system (1) involving three rules and

$$
A_{11} = \begin{bmatrix}
-1.5 & 0.5 \\
0 & -1
\end{bmatrix}, A_{21} = \begin{bmatrix}
1.2 & 0.1 \\
0.2 & -1.5
\end{bmatrix},
$$

$$
A_{31} = \begin{bmatrix}
0.1 & -0.5 \\
0 & -0.3
\end{bmatrix},
$$

$$
a_1 = \begin{bmatrix}
0 \\
0
\end{bmatrix}, a_2 = \begin{bmatrix}
0 \\
0
\end{bmatrix}, a_3 = \begin{bmatrix}
0 \\
0.3
\end{bmatrix},
$$

$$
A_{12} = \begin{bmatrix}
0 & 0.1 \\
0 & 0.2
\end{bmatrix}, A_{22} = \begin{bmatrix}
0.5 & 1 \\
1 & 0
\end{bmatrix}, C = \begin{bmatrix}
1 & 0
\end{bmatrix},
$$

$$
D = 2, L = \begin{bmatrix}
3 \\
1.5
\end{bmatrix}, l \in \{1, 2, 3\},
$$

(40)

and the parameter uncertainties $\Delta A_{l1}$ and $\Delta a_{l1}$ are given in the sequel, we consider a numerical example.

$$
\begin{bmatrix}
U_l = \begin{bmatrix}
0.15 & 0
\end{bmatrix}^T \\
W_{l1} = \begin{bmatrix}
0.1 & 0
\end{bmatrix}, W_{l2} = 0.1, l \in \{1, 2, 3\},
\end{bmatrix}
$$

(41)

The MFs are given in Fig. 2 with $d_1 = 10, d_2 = 100$. The premise variable space is divided into three regions as in (37). The characteristic matrices $\bar{G}_i$ and $\bar{F}_i$ can also be calculated via (39) with $n_x = 2$ and $\vartheta = \begin{bmatrix}
1 \\
0
\end{bmatrix}$.
We attempt to design a full-order/reduced-order sampled-data PWA filter (11) for the Itô stochastic T-S fuzzy affine system (1) in an $\mathcal{H}_\infty = \gamma$ framework. Note that in Theorem 3.2, when the matrices $\tilde{F}_i \equiv \begin{bmatrix} 0_{n_x \times 1} & I_{n_x} \end{bmatrix}$, the filtering design results relying on a CLKF can be instantly obtained. By Theorem 3.2, a detailed comparison of the robust $\mathcal{H}_\infty$ performance $\gamma_{\text{min}}$ for the full-order/reduced-order sampled-data PWA filters is given in Table I. With observation of Table I, one could easily conclude that the PLKF-based filtering design results are generally less conservative than the CLKF-based ones. It has been also shown that the full-order and reduced-order sampled-data filter design results can be obtained in a unified framework.

In the following, we consider another benchmark example on sampled-data filtering design of an inverted pendulum system interfered with Itô stochastic processes.

**Example 4.3.** The dynamical properties of the pendulum...
system are characterized by the following stochastic equations,

\[
\begin{cases}
    dx_1(t) = x_2(t) dt \\
    dx_2(t) = -g \sin(x_1) - \frac{2b}{m_1} x_2 - \frac{a m_1 \sin(2x_2)}{2} - \frac{a \cos(x_1)}{2} \\
    + (0.2x_2 - 0.1x_1 + w(t)) d\varpi(t)
\end{cases}
\]

where \(x_1\) and \(x_2\), respectively, stand for the angle of the pendulum from the vertical and the angular velocity. \(\varpi(t)\) represents a one-dimensional Wiener process; \(b = 1.5\text{Nm/s}\) refers to the damping coefficient of the pendulum around the pivot; \(g = 9.8\text{m/s}^2\) represents the gravity constant; \(a = \frac{1}{M+m}\); \(m = 2.5\text{kg}\) refers to the mass of the pendulum; \(M = 7.5\text{kg}\) represents the mass of the cart; \(2l = 1.5\text{m}\) is the length of the pendulum; \(w(t)\) represents the external disturbance.

Linearizing the nonlinear pendulum (42) around \((\pm \frac{\pi}{3}, 0)^T\) and \((0, 0)^T\), then system (42) is described by an Itô stochastic T-S fuzzy affine model with three rules as,

**Plant Rule \(R_i\):** IF \(x_1(t)\) is \(\mathcal{F}_i\), THEN

\[
\begin{align*}
    dx(t) &= [(A_{1i} + \Delta A_{1i}) x(t) + a_i + \Delta a_i + B_{1i} w(t)] dt \\
    &\quad + [A_{12i} x(t) + B_{12i} w(t)] d\varpi(t) \\
    y(t) &= Cx(t) + Dw(t) \\
    z(t) &= L_i x(t), \quad i \in \mathcal{L} = \{1, 2, 3\}
\end{align*}
\]

where the system parameters are

\[
\begin{align*}
    A_{11} &= A_{31} = \begin{bmatrix} 0 & 1 \\ -5.8512 & -4.6753 \end{bmatrix}, \\
    A_{21} &= \begin{bmatrix} 0 & 1 \\ -17.2941 & -5.2941 \end{bmatrix}, \\
    A_1 &= \begin{bmatrix} 0 & 7.0992 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & -7.0992 \\ 0 & 0 \end{bmatrix}, \\
    B_{11} &= B_{31} = \begin{bmatrix} 0 & 0 \\ 0.0390 & 0 \end{bmatrix}, B_{21} = \begin{bmatrix} 0 & 0 \\ 0 & -0.0882 \end{bmatrix}. 
\end{align*}
\]
matrices are feasible solutions, and the full-order filter gains and Lyapunov \( \gamma \) of exogenous disturbance \( H \) and \( \vartheta \) matrices is divided into three regions as in (37). The characteristic matrices \( \tilde{G}_i \) and \( F_i \) can also be calculated via (39) with \( n_x = 2 \) and \( \vartheta = [1, 0] \).

We attempt to synthesize a full-order sampled-data PWA filter (11) to ensure the filtering error system to be stochastically stable with robust performance \( \gamma \). Given \( \gamma = 1 \) and using Theorem 3.2 with \( H_1 = H_2 = I_3 \) and \( \vartheta = 1 \), one can obtain feasible solutions, and the full-order filter gains and Lyapunov matrices are

\[
\begin{align*}
A_{f1} &= \begin{bmatrix}
-1.2212 & 0.3926 \\
-0.2559 & -3.0500
\end{bmatrix}, \\
B_{f1} &= \begin{bmatrix}
0.9368 \\
-0.6221
\end{bmatrix}, \\
C_{f1} &= \begin{bmatrix}
-0.0498 \\
0.4666
\end{bmatrix}, \\
A_{f2} &= \begin{bmatrix}
-1.1994 & -4.1328
\end{bmatrix}, \\
B_{f2} &= \begin{bmatrix}
0.9933 \\
-0.7720
\end{bmatrix}, \\
C_{f2} &= \begin{bmatrix}
-0.0037 \\
2.947
\end{bmatrix}, \\
A_{f3} &= \begin{bmatrix}
1.0020 \\
-0.9139
\end{bmatrix}, \\
B_{f3} &= \begin{bmatrix}
0.7710 \\
-0.9662
\end{bmatrix}, \\
C_{f3} &= \begin{bmatrix}
2.4614 \\
2.7937
\end{bmatrix}, \\
F_{f1} &= \begin{bmatrix}
-0.2559 & -3.0500 \\
-0.6221 & -0.5666
\end{bmatrix}, \\
F_{f2} &= \begin{bmatrix}
0.9933 & -0.7720
\end{bmatrix}, \\
F_{f3} &= \begin{bmatrix}
0.7710 & 2.4614
\end{bmatrix}, \\
\rho &= \begin{bmatrix}
0.0037 \\
0.0037
\end{bmatrix}, \\
Q &= \begin{bmatrix}
8.5565 & 0.2995 \\
0.2995 & 5.4228
\end{bmatrix}, \quad M = \begin{bmatrix}
5.0199 & -0.0433 \\
-0.0433 & 3.5322
\end{bmatrix}
\end{align*}
\]

To cope with the real-world environment, the system measurement outputs are also set with injection of noises. Under initial condition \( x_0 = \frac{\pi}{3} \) and \( \tilde{x}_0 = \begin{bmatrix} 0 & 0 \end{bmatrix}^T \) and exogenous disturbance \( w(t) = 12e^{-0.5t} \sin(2\pi t) \), by using the sampled-data PWA filter in (11), Fig. 5(a) and Fig. 5(b) show the system and filter states and the sampled measurement output, respectively. Fig. 5(c) shows the estimation error \( \tilde{x}(t) \). Fig. 5(d) shows the time response of piecewise Lyapunov function \( V(t, \tilde{x}, t) = \tilde{x}^T(t)F(t)X(t)F(t)\tilde{x}(t) \).

V. CONCLUSIONS

This paper has addressed the problem of robust sampled-data PWA filtering design for Itô stochastic nonlinear systems through T-S fuzzy affine models. Via an input delay approach, the filtering error system is formulated into a stochastic PWA system with time-varying delay. By utilizing a PLKF-based scheme combined with an integral inequality, two new theorems on sampled-data PWA filtering analysis and design are proposed under a convex optimization framework. The main advantages and features of the proposed analysis and synthesis results over the existing ones have also been discussed after theorems.

To verify the effectiveness of the developed approach, three simulation examples are presented. Among these simulation cases, two benchmark examples on a tunnel diode circuit system and an inverted pendulum system are provided. A numerical example is also given to show the less conservatism of the sampled-data filtering design results relying on PLKFs over those relying on a CLKF. It has been also shown that the full-order and reduced-order filters can be synthesized in a unified framework.

In addition, it is also noted that in this paper only the robust \( \mathcal{H}_\infty \) performance index is considered in the sampled-data filtering design. In future work, we will further study the sampled-data filtering design by considering the robust \( \mathcal{H}_\infty \) performance index [40] or \( l_1 \)-gain index [41] for other type of external disturbances. Another future work is to study the sampled-data fuzzy filtering design for Itô stochastic nonlinear systems with sensor faults [42].

VI. APPENDIX

A. Useful Lemmas

**Lemma 1** [26]. Denote \( \xi(t) \) as a \( p \)-dimensional Itô process on \( t \geq 0 \) with the following stochastic differential system,

\[
d\xi(t) = f(t)dt + g(t)d\xi(t)
\]

where \( f(t) \in \mathbb{R}^p, g(t) \in \mathbb{R}^{p \times q} \). Define \( V(\xi(t), t) \in C^{2,1} \) with \( C^{2,1}(\mathbb{R}^p \times \mathbb{R}^q, \mathbb{R}^q) \) denoting the family of all real-valued functions \( V(\xi(t), t) \) defined on \( \mathbb{R}^p \times \mathbb{R}^q \) such that the aforementioned functions are continuously twice differentiable in \( \xi \) and \( t \). Then \( V(\xi(t), t) \) represents a real-valued Itô process with its stochastic differential given as

\[
dV(\xi, t) = \mathcal{L}V(\xi, t)dt + V_\xi(\xi, t)g(t)d\xi(t)
\]

where \( \mathcal{L} \) stands for the differential operator, and

\[
\begin{align*}
\frac{\partial V(\xi, t)}{\partial t} &= \frac{\partial V(\xi, t)}{\partial \xi} f(t), \\
V_\xi(\xi, t) &= \frac{\partial V(\xi, t)}{\partial \xi} g(t) \quad \text{for } x \in \mathbb{R}^p \times \mathbb{R}^q.
\end{align*}
\]

**Lemma 2** [38]. \( \rho \) denotes a differentiable function:

\[
[t_1, t_2] \rightarrow \mathbb{R}^p. \quad \text{For a positive definite matrix } 0 < M \in \mathbb{R}^{p \times p} \text{ and matrix } N \in \mathbb{R}^{5p \times 4p}, \text{ the subsequent inequality holds,}
\]

\[
-\int_{t_1}^{t_2} \rho^T(s)Q \rho(s)ds \leq \xi^T \Omega \xi
\]
where $d = t_2 - t_1$, and
\[
\begin{aligned}
&\Omega = d \cdot \mathbf{N} \text{diag}\{M, 3M, 5M, 7M\}^{-1} N^T + \text{Sym}\{N \cdot \Pi\}, \\
&\Pi = \left[ \begin{array}{c} \Pi^1_T \\
\Pi^2_T \\
\Pi^3_T \\
\Pi^4_T \\
\end{array} \right]^T, \\
&\xi = \left[ \begin{array}{c} \rho^T(t_2) \\\n\rho^T(t_1) \\\n\frac{1}{2} \int_{t_1}^{t_2} \rho^T(s)ds \\\n\end{array} \right], \\
&\Pi_1 = e_1 - e_2, \Pi_2 = e_1 + e_2 - 2e_3, \\
&\Pi_3 = e_1 - e_2 + 6e_3 - 6e_4, \\
&\Pi_4 = e_1 + e_2 - 12e_3 + 30e_4 - 20e_5, \\
&e_n = \left[ \begin{array}{c} 0_{p \times (n-1)} \\
I_p \\
0_{p \times (5-n)p} \end{array} \right], \quad n = 1, \ldots, 5.
\end{aligned}
\]

\textbf{Lemma 3} [14]. With real matrices $E = E^T$, $N$, $\Delta(t)$, and $F$, for all $\|\Delta(t)\| \leq I$, the following inequality
\[
E + \text{Sym}\{N\Delta(t)F\} < 0
\]
holds if and only if
\[
E + \varepsilon N^T N + \varepsilon^{-1} F^T F < 0
\]
holds for some parameter $\varepsilon > 0$.

\section*{B. Proof of Theorem 3.1}

Consider the PLKF being continuous across the boundary of the subspaces as
\[
V(\hat{x}, t) = V_1(\hat{x}, t) + V_2(\hat{x}, t) + V_3(\hat{x}, t)
\]
where
\[
\begin{aligned}
V_1(\hat{x}, t) &= \hat{x}^T(\hat{x}, t) \hat{F}_1^T X \hat{F}_1 \hat{x}(t), \\
V_2(\hat{x}, t) &= (h - d(t)) \int_{t-d(t)}^{t} \hat{x}(s) \hat{J}^T \hat{J} \hat{x}(s) ds \\
V_3(\hat{x}, t) &= (h - d(t)) \int_{t-d(t)}^{t} \hat{x}^T(t - d(t)) Q(\hat{x} - t - d(t)), \\
\end{aligned}
\]
and $X = X^T$, $\hat{F}_1 = \text{diag}\{\hat{F}_1, I_{n_f}\}$, $0 < Q = Q^T \in \mathbb{R}^{n_x \times n_x}$, $0 < M = M^T \in \mathbb{R}^{n_x \times n_x}$. Define
\[
\begin{aligned}
\zeta(t) &= \left[ \begin{array}{c} f^T(t) \\
f^T(t - d(t)) \\
\int_{t-d(t)}^{t} \hat{x}^T(s) ds \\
\int_{t-d(t)}^{t} \hat{x}^T(t - d(t)) \hat{x}(s) ds \\
\int_{t-d(t)}^{t} \hat{x}^T(t - d(t)) \hat{x}(s) ds da \\
\int_{t-d(t)}^{t} \hat{x}^T(t - d(t)) \hat{x}(s) ds da db \\
\end{array} \right]_T, \\
e_1 &= \left[ \begin{array}{c} I_{n_x+n_f} \\
0_{(n_x+n_f) \times (6n_x+n_f+2n_w)} \end{array} \right], \\
e_2 &= \left[ \begin{array}{c} 0_{(n_x+n_f) \times (n_x+n_f)} \\
I_{n_x+n_f} \end{array} \right].
\end{aligned}
\]
Obviously, $f(t) = e_1 \zeta(t)$, $\hat{x}(t) = e_2 \zeta(t)$, $x(t) = \hat{J}_1 e_1 \zeta(t)$, $x(t - d(t)) = e_3 \zeta(t)$, and $\tilde{w}(t) = e_4 \zeta(t)$.

The inequality (20) ensures the positive definiteness of PLKF in each subspace. Based on the PLKF given in (47)-(48) with Itô stochastic feature $E\{V_\xi(x, t) g(t) d\xi(t)\} = 0$, if $E\{\mathcal{L}V(\hat{x}, t)\} + E\{\tilde{z}^T(t) \tilde{w}(t)\} - \gamma_0 \tilde{w}(t) \tilde{w}(t) < 0$, (50)

The system (13) is stochastically stable with robust $\mathcal{H}_\infty$ performance $\gamma$.

Along the system (19) and based on Lemma 1, $\mathcal{L}V_1(\hat{x}, t)$ is formulated as
\[
\mathcal{L}V_1(\hat{x}, t) = 2 \hat{x}^T(t) \hat{F}_1^T X \hat{F}_1 f(t) + g^T(t) \hat{F}_1^T X \hat{F}_1 g(t) = \zeta^T(t) \left( 2 \epsilon_2^T \hat{F}_1^T X \hat{F}_1 e_1 + (\hat{A}_{i2} \hat{J}_1 e_2 + \hat{B}_{i2} J e_8) \right) \zeta(t).
\]

By Lemma 1, $\mathcal{L}V_2(\hat{x}, t)$ can be derived as,
\[
\mathcal{L}V_2(\hat{x}, t) = - \int_{t-d(t)}^{t} \hat{x}^T(s) \hat{J}^T \hat{J} \hat{x}(s) ds + (h - d(t)) \hat{x}^T(t) \hat{J}^T \hat{J} \hat{x}(t).
\]
Using Lemma 2 in the Appendix A to handle the first term in the right hand side of (52) yields,
\[
- \int_{t-d(t)}^{t} \hat{x}^T(s) \hat{J}^T \hat{J} \hat{x}(s) ds \leq \zeta^T(t) \left( \text{Sym}\{N \hat{\Sigma}\} + d(t) N \text{diag}\{M, 3M, 5M, 7M\}^{-1} N^T \right) \zeta(t)
\]
\[
\text{where } \hat{\Sigma} \in \mathbb{R}^{(7n_x+2n_f+2n_w) \times 4n_x}, \text{ for } i \in I_0, \quad \hat{\Sigma} \in \mathbb{R}^{(2+7n_x+2n_f+2n_w) \times 4n_x}, \text{ for } i \in I_1, \text{ and }
\]
\[
e_1 = \left[ \begin{array}{c} 0_{n_x+n_f} \end{array} \right], \\
e_2 = \left[ \begin{array}{c} 0_{(n_x+n_f) \times (6n_x+n_f+2n_w)} \\
I_{n_x+n_f} \end{array} \right].
\]

Based on (52)-(54), one has
\[
\mathcal{L}V_3(\hat{x}, t) \leq \zeta^T(t) \left( (h - d(t)) e_2^T \hat{J}^T \hat{J} e_2 + \text{Sym}\{N \hat{\Sigma}\} + d(t) N \text{diag}\{M, 3M, 5M, 7M\}^{-1} N^T \right) \zeta(t).
\]
Computing $\mathcal{L}V_3(\hat{x}, t)$ along the system (19), one has
\[
\mathcal{L}V_3(\hat{x}, t) = - \hat{x}^T(t - d(t)) Q x(t - d(t)) = -\zeta^T(t) e_3^T Q e_3(t).
\]
Considering the first equation of system (13), for arbitrary matrices $\hat{\V}_i, i \in I$, one has,
\[
0 = 2 \hat{\V}_i \left\{ - f(t) + \hat{A}_{i1} \hat{x}(t) + \hat{A}_{di} \hat{J}_1 \hat{x}(t - d(t)) + \hat{B}_{i1} \tilde{w}(t) \right\} = 2 \hat{\V}_i \hat{A}_i \zeta(t)
\]
where
\[
\hat{A}_i = \left[ \begin{array}{c} -I_{n_x+n_f} \\
\hat{A}_{i1} \\
\hat{A}_{di} \\
0_{(n_x+n_f) \times 4n_x} \end{array} \right], \\
\hat{A}_i = \left[ \begin{array}{c} -I_{1+n_x+n_f} \\
\hat{A}_{i1} \\
\hat{A}_{di} \\
0_{(1+n_x+n_f) \times 4n_x} \end{array} \right], \quad \text{if } i \in I_0,
\]
\[
\hat{A}_i = \left[ \begin{array}{c} -I_{1+n_x+n_f} \\
\hat{A}_{i1} \\
\hat{A}_{di} \\
0_{(1+n_x+n_f) \times 4n_x} \end{array} \right], \quad \text{if } i \in I_1.
\]
Based on (16), adopting the S-procedure with considering (51), and (55)-(57), the subsequent inequality indicates (50),
\[
\zeta^T(t) \Gamma_i(d(t)) \zeta(t) < 0, \quad i \in \mathcal{I}
\] (59)
where
\[
\Gamma_i(d(t)) = SYM \left\{ d^T \tilde{F}_i^T X \tilde{F}_i \epsilon_1 + \tilde{N} \Sigma + \tilde{V}_i \tilde{A}_i \right\} + (A_{12} J_i \epsilon_2 + B_{12} J_i \epsilon_3)^T \tilde{F}_i^T X \tilde{F}_i (A_{12} J_i \epsilon_2 + B_{12} J_i \epsilon_3) + (h - d(t))^T J_i^T M J_i \epsilon_2 - e_i^T Q e_i + d(t) N \text{diag} \left[ M, 3M, 5M, 7M \right]^{-1} \tilde{N}^T e_i + e_i^T J_i^T G_i^T \tilde{G}_i J_i \epsilon_2 + L_i^T L_i \epsilon_3 - \gamma^2 e_i^T e_i
\]
\[
\tilde{L}_i = \left[ \begin{array}{c} 0_{n_x \times (n_x + n_j)} \quad \tilde{L}_i \quad 0_{n_x \times (5n_x + 2n_u)} \end{array} \right], \quad i \in \mathcal{I}_0,
\]
\[
\tilde{J}_i = \left[ \begin{array}{c} 0_{n_x \times (1 + n_x + n_j)} \quad \tilde{L}_i \quad 0_{n_x \times (5n_x + 2n_u)} \end{array} \right], \quad i \in \mathcal{I}_1
\] (60)
with $T_i \geq 0$, $i \in \mathcal{I}$.

Then the following inequality implies (59)
\[
\Gamma_i(d(t)) < 0, \quad i \in \mathcal{I}
\] (61)
which indicates that the system (13) is stochastically stable with robust $H_\infty$ performance $\gamma$.

Notice that a time-varying delay $d(t)$ exists in the condition (61) and satisfies
\[
0 \leq d(t) \leq h
\] (62)
which indicates that $d(t)$ can be constructed as,
\[
d(t) = \lambda(t) \cdot 0 + (1 - \lambda(t)) h
\] (63)
with $0 \leq \lambda(t) \leq 1$.

As $d(t)$ in (63) is a linear function of the variable $\lambda(t)$, then it can be seen that (61) holds for $\lambda(t) = 1$ and $\lambda(t) = 0$, respectively, which indicates (21). The proof is accomplished.

C. MFs in Fig. 1

The normalized MFs in Fig. 1 are given as follows. When $x_1(t) \leq -d_2$: $\mu_1(t) = 1$, $\mu_2(t) = 0$, $\mu_3(t) = 0$.

When $-d_2 \leq x_1(t) \leq -d_1$:
\[
\begin{align*}
\mu_1(t) &= 1 - \mu_2(t), \\
\mu_2(t) &= \left\{ \begin{array}{ll}
2 \left( \frac{x_1(t) + d_2}{d_2 - d_1} \right)^2, & -d_2 \leq x_1(t) \leq -\frac{d_1 - d_2}{2}, \\
1 - 2 \left( \frac{x_1(t) + d_1}{d_2 - d_1} \right)^2, & -\frac{d_1 - d_2}{2} \leq x_1(t) \leq -d_1,
\end{array} \right.
\end{align*}
\]
$\mu_3(t) = 0$.

When $-d_1 \leq x_1(t) \leq d_1$: $\mu_1(t) = 0$, $\mu_2(t) = 1$, $\mu_3(t) = 0$.

When $d_1 \leq x_1(t) \leq d_2$:
\[
\begin{align*}
\mu_1(t) &= 0, \\
\mu_2(t) &= 1 - \mu_3(t), \\
\mu_3(t) &= \left\{ \begin{array}{ll}
2 \left( \frac{x_1(t) - d_1}{d_1 - d_2} \right)^2, & d_1 \leq x_1(t) \leq \frac{d_1 + d_2}{2}, \\
1 - 2 \left( \frac{x_1(t) - d_1}{d_1 - d_2} \right)^2, & \frac{d_1 + d_2}{2} \leq x_1(t) \leq d_2.
\end{array} \right.
\end{align*}
\]
When $x_1(t) \geq d_2$: $\mu_1(t) = 0$, $\mu_2(t) = 0$, $\mu_3(t) = 1$.


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