Robust Tracking Control of Interval Type-2 Positive Takagi-Sugeno Fuzzy Systems with External Disturbance

Lining Fu, H. K. Lam, Fellow, IEEE, Fucai Liu, Hongying Zhou and Zhixiong Zhong

Abstract—Nonlinear positive systems are ubiquitous in practical applications. The constraint of positive conditions makes the design of this system still quite challenging even for the existing advanced control theory. In addition, when uncertainty and external disturbance also act on the positive nonlinear system at the same time, it will aggravate the complexity of the problem. In response to this problem, for the first time, this paper proposes a feasible robust tracking control strategy for the positive nonlinear systems with external disturbance and subject to uncertainty. In the proposed control strategy, the positive Takagi-Sugeno (T-S) fuzzy model is used to effectively represent the nonlinearity of the positive system, and the Interval Type-2 (IT2) fuzzy sets are used to deal with the uncertainty. The focus of tracking control is to design a reasonable IT2 fuzzy controller, which can make the states of the plant follow those of the given stable positive reference system as accurately as possible under the $H_{\infty}$ performance. Different from the tracking control research of general systems in the existing literature, in the tracking control analysis of positive systems, the generation of non-convex conditions is inevitable. This paper smoothly removes the obstacles of this problem. In addition, this paper successfully attempts to relax the conservativeness of the basic analysis conditions by improving IT2 membership-function-dependent (IT2MFD) method, thereby improving the tracking effect under $H_{\infty}$ performance. Finally, a detailed simulation example is given to verify the effectiveness of the proposed strategy.

Index Terms—Positive T-S Fuzzy-Model-Based (PTSFB) control systems, Tracking control design, Interval Type-2 (IT2) fuzzy sets, non-convex conditions, $H_{\infty}$ performance, IT2 membership-function-dependent (IT2MFD) method.

I. INTRODUCTION

In the past few years, more and more attention has been devoted to the research of positive systems, which have the natural property of keeping their state vectors in the positive quadrant under any non-negative initial conditions [1]–[3]. And its practical applications are also very prominent, such as, the field of biology: metabolic systems; the field of compartment systems: pharmacokinetics, epidemiology, etc. Different from the general system defined in linear space, the positive system is defined in a conical region [4]. This natural and unique positive constraint makes many analytical techniques that are mature for general systems no longer applicable to positive systems. Therefore, combined with the practicability of the positive system, this kind of special system has aroused the interest of researchers in a wider range, such as reachability and controllability [5]–[7], stability analysis and optimal control [8]–[10]. At the same time, a variety of control strategies have also been derived, such as state feedback control, output feedback control and the design of filters and observers [11]–[14], etc. All these related achievements have played a guiding role in the research of widely used positive nonlinear systems.

Generally speaking, due to the inherent high complexity of nonlinear systems [15]–[22], its research is extremely challenging. In order to design a suitable control strategy for nonlinear systems, the Takagi-Sugeno Fuzzy-Model-Based (TSFMB) control method was successfully selected as an effective candidate due to its rigorous mathematical structure and superb nonlinear processing ability [23]. Therefore, in the current research on positive nonlinear systems, the TSFMB method has been greatly welcomed, such as stability analysis for PTSFMB continuous time systems with time delay [24], observer-based control [25], filter design [26], etc. In [27], T-S FMB control technique was used to handle positive nonlinear system for the first time. However, in the related research of these positive nonlinear systems, the stabilization analysis [28]–[31] occupies most of the market, and the research on the tracking field is really rare.

In fact, the practical application of tracking control is very extensive, such as missile trajectory control [32], aircraft attitude tracking [33] and robot tracking control [34], etc. Compared with the stabilization problem, tracking control research, as another potential research topic, is generally considered to be more challenging [35]–[37]. The reason is that in tracking control, the designed controller not only needs to ensure the stability of the system, but also needs to drive the states of the nonlinear system to follow those of the reference system. As far as tracking control is concerned, in the past research and development, various control strategies have emerged one after another, such as output-feedback tracking control [38]–
[40], state-feedback tracking control [32], [37] and sampled-data tracking control [41]–[43]. However, the current tracking control researches rarely involve positive nonlinear systems. But, as mentioned in the above analysis, the application of positive nonlinear systems is extremely wide. Therefore, the research on tracking control for positive nonlinear systems reflects out of the absolute necessity. In addition, the existence of disturbance is not negligible in practical systems [44], and it has a profound impact on the stability of the system. From a practical point of view, disturbance factors will be considered in the tracking control of the positive system in this paper. Conventionally, to quantify tracking control performance, the robust $H_{\infty}$ performance indicators can be introduced to evaluate and improve the tracking errors [38], [45], [46].

Fuzzy sets are of great significance to the research of fuzzy-model-based (FMB) control system. Similar to nonlinearity, as another natural phenomenon, uncertainty is inevitable in many cases, for instance, the partially unknown parameters, unexpected disturbance in the fuzzy systems, diverse understandings of linguistic variables from different experts, etc. Although the Type-1 fuzzy sets have shown strong ability in dealing with the nonlinearity of the control system, they are not good in dealing with the uncertainty [47]–[51]. In order to solve this problem, the concept of Footprint of Uncertainty (FOU) related to the Type-2 fuzzy sets was introduced into the analysis [50]. Due to the use of Type-2 fuzzy sets, the uncertainty within the FOU can be directly resolved. However, it should also be noted that although the (general) Type-2 fuzzy sets [52]–[55] have the ability to capture uncertainty, they also bring multiple challenges such as more complex solutions and greater computational complexity. To make a compromise, the Interval Type-2 (IT2) [56]–[58] fuzzy sets came into being. They can directly solve the system uncertainty while simultaneously reducing the computational complexity. In order to introduce IT2 fuzzy sets into control strategies of IT2 fuzzy logic systems, the stability analysis and control synthesis of IT2 FMB control systems were investigated for the first time in work [51]. Subsequently, the IT2 fuzzy model was extended and applied to the control synthesis of the positive system [59]. It should be pointed out that although the IT2 fuzzy sets reduce the computational intensity compared to the Type-2 fuzzy sets, they still increase the analysis complexity compared to the general Type-1 fuzzy sets.

In the current research on fuzzy systems, the advantages of membership-function-dependent (MFD) techniques [60]–[62] in promoting the relaxation of system stability analysis are gradually recognized. Piecewise linear membership functions (PLMFs) technology [61], as a highly potential member of MFD technologies, provides great help in extracting original membership functions (MFs) information. At the same time, in continuous system analysis, it can also approximate the infinite number of analysis conditions to finite ones, so that the idea of using MFD techniques to relax the analysis conditions and find a feasible solution is more practical. However, due to the complexity of stability analysis brought by MFs, the research on tracking control of the IT2 positive T-S fuzzy model based (IT2 PTSFMB) with external disturbance effects has never been involved by researchers. Therefore, developing a suitable method to introduce the PLMFs technology into this research field is our motivation for this work.

This paper is dedicated to improving the tracking performance of the IT2 PTSFMB with external disturbance tracking system. To reduce the computational burden and characterize the imperfect matching MFs resulted from parameter uncertainty, the IT2 fuzzy controller is designed under the imperfect premise matching (IPM) concept [63], which means that the fuzzy model and controller can have different premise MFs and number of rules. The main contributions of this paper are summarized as follows: First of all, as far as the authors know, there is no related report work research on the tracking control of IT2 PTSFMB control systems with external disturbance, in this paper, we present the robust tracking control strategy for the IT2 PTSFMB control systems with external disturbance for the first time. The influences of positive constrains, nonlinearity, parameter uncertainties and external disturbance on the system are considered comprehensively. Secondly, due to the uniqueness of the positive system, when the positive conditions and the stability conditions need to be solved at the same time, the generation of some non-convex conditions is inevitable. These non-convex conditions make it impossible to obtain a numerical solution based on the existing technology. By designing the suitable control scheme and applying some mathematical skills, this paper successfully breaks through the obstacles of this problem. At last, a novel IT2 MFD analysis method based on piecewise linear embedding Type-1 MFs (PLETMFs) for IT2 PTSFMB tracking system with external disturbance is proposed. By using this method, more effective original membership function information is extracted and utilized, so that the analysis conditions are relaxed, and the goal of improving tracking performance is achieved. At the same time, this improved IT2 MFD analysis method allows users to freely choose the reference embedded Type-1 MFs (RETMFs), which means that there is no need to carry out the tedious process of finding suitable RETMFs.

The rest of this paper are organized as follows: In Section II, notations used in the paper are introduced, the preliminaries of the IT2 PTSFMB with external disturbance, IT2 fuzzy controller and the stable positive reference model are presented. In Section III, the IT2 PTSFMB with external disturbance tracking control stability analysis is conducted through the Lyapunov stability theory, and the positive and stability conditions are presented using the membership-function-independent (MFI) and MFD approaches. In Section IV, a simulation example is presented to verify the effectiveness of the proposed control strategy. In Section V, a conclusion is drawn.

II. NOTATIONS AND PRELIMINARIES

This section will introduce the standard notations and preliminary knowledge used in the full paper. Among them, the preliminary knowledge includes IT2 positive T-S fuzzy model with external disturbance, stable positive reference model and IT2 fuzzy controller.
A. Notation

The following notations will be used throughout this paper [60]. The \( r \)-th row and \( s \)-th column of matrix \( A \in \mathbb{R}^{n \times m} \) is expressed as \( \lambda_{rs} \), when each element \( \lambda_{rs} \) in the matrix is negative, non-negative, positive and non-positive, the expressions are respectively \( A < 0 \), \( A \geq 0 \), \( A > 0 \) and \( A \leq 0 \). A matrix whose off diagonal elements are all non-negative is called Metzler matrix. The expression \( Q(x) = \text{diag}\{x_1, x_2, \ldots, x_n\} \) means that the matrix \( Q(x) \) is a diagonal matrix whose diagonal elements are \( x_1, x_2, \ldots, x_n \). \( \zeta \) stands for \( \{1, 2, \ldots, \zeta\} \).

B. IT2 Positive T-S Fuzzy Model With External Disturbance

By using the \( q \) rules, the IT2 positive T-S fuzzy model with external disturbance is used to describe the dynamic behavior of nonlinear plant, where the \( i \)-th rule is defined as follows:

Rule \( i : i \in \tilde{Q} \)

IF \( \varphi_1(x(t)) \) \( \text{AND} \) \( \cdots \) \( \text{AND} \) \( \varphi_q(x(t)) \) is \( \tilde{M}_q \) THEN \( \hat{x}(t) = A_i x(t) + B_{ui} u(t) + B_{oi} \omega(t), \quad \hat{x}(t) = \psi(t), \) (1)

where \( x(t) \in \mathbb{R}^{n \times 1} \), \( u(t) \in \mathbb{R}^{m \times 1} \) and \( \omega(t) \in \mathbb{R}^{m \times 1} \) stand for the state, the control input vector and the bounded external disturbance, respectively; \( A_i \in \mathbb{R}^{n \times n}, B_{ui} \in \mathbb{R}^{n \times m} \) and \( B_{oi} \in \mathbb{R}^{n \times m} \) are the system, input and disturbance matrices with appropriate dimensions; \( \tilde{M}_q \) is the fuzzy term of the \( i \)-th rule corresponding to the known function \( \varphi_i(x(t)) \), and \( \varphi_i(x(t)) \) is the premise variable, \( l \in \tilde{\Psi}, \Psi \) is a positive integer; \( \psi(t) \) is the vector valued initial function.

The firing strength of the \( i \)-th rule is represented within the following interval sets:

\[
\begin{align*}
\Psi & = \left[ \prod_{l=1}^{\tilde{Q}} \mu_{\tilde{M}_l}(\varphi_i(x(t))) \right] \left[ \prod_{l=1}^{\Psi} \bar{\mu}_{\tilde{M}_l}(\varphi_i(x(t))) \right] \\
& = \psi_i^L(x(t), x_i), \psi_i^U(x(t), x_i), \end{align*}
\]

where \( 0 \leq \overline{\mu_{\tilde{M}_l}(\varphi_i(x(t)))} \leq 1 \) and \( 0 \leq \underline{\mu_{\tilde{M}_l}(\varphi_i(x(t)))} \leq 1 \) denote the upper and lower grades of membership, respectively. Through the definition of IT2 membership function, \( 0 \leq \psi_i^L(x(t)) \leq \psi_i^U(x(t)) \leq 1 \) can be understood. \( \psi_i(x(t)) \) is obtained as \( \psi_i(x(t)) = \underline{\Lambda}_i(x(t))^T \psi_i^L(x(t)) + \bar{\Lambda}_i(x(t))^T \psi_i^U(x(t)) \), \( \sum_{i=1}^{q} \psi_i(x(t)) = 1, \psi_i(x(t)) \geq 0, \underline{\Lambda}_i(x(t)) \) and \( \bar{\Lambda}_i(x(t)) \) are nonlinear type reduction functions related to uncertainty, which satisfy: \( 0 \leq \underline{\Lambda}_i(x(t)) \leq 1, 0 \leq \bar{\Lambda}_i(x(t)) \leq 1 \) and \( \underline{\Lambda}_i(x(t)) + \bar{\Lambda}_i(x(t)) = 1, \forall i \). Hence, the IT2 positive T-S fuzzy model is described as:

\[
x(t) = \sum_{i=1}^{q} \psi_i(x(t)) \left( A_i x(t) + B_{ui} u(t) + B_{oi} \omega(t) \right), \quad \hat{x}(t) = \psi(t),
\]

\text{Definition 1}: If the initial condition \( \psi(t) \neq 0 \) is maintained and the corresponding state trajectory \( x(t) \neq 0 \) is satisfied at all \( t \geq 0 \), such the system is called the positive system [64].

\text{Lemma 1}: If the following conditions are met: the system matrices \( A_i \) is Metzler matrix, \( B_{ui} \geq 0 \) and \( B_{oi} \geq 0 \); meanwhile, the bounded external disturbance vector \( \omega(t) \geq 0 \), then the system can be guaranteed to be a positive system [29].

\text{Assumption 1}: It is assumed that there exists an accurate T-S fuzzy model (3) constructed by the sector nonlinearity method [65], which is used to represent the nonlinear system.

C. Stable Positive Reference model

A stable positive reference model is defined as follows:

\[
x_r(t) = A_r x_r(t) + B_r r(t), \quad \hat{x}(t) = \psi(t),
\]

where \( A_r \in \mathbb{R}^{n \times n} \) and \( B_r \in \mathbb{R}^{n \times m} \) are the system and input matrices of the matching dimensions, respectively, \( x_r(t) \in \mathbb{R}^{n \times 1} \) is the state matrix, and \( r(t) \in \mathbb{R}^{m \times 1} \) is the input matrix. It must be emphasized that the reference model should be guaranteed to be stable.

\text{Remark 1}: According to Lemma 1, the premise that the reference model is guaranteed to be positive is that the following conditions are met: \( A_r \) is Metzler matrix, \( B_r \geq 0 \) and \( r(t) \neq 0 \).

D. IT2 Fuzzy Controller Design

Through the use of \( c \) rules, IT2 fuzzy controller is designed, where the \( j \)-th rule has the following form:

Rule \( j : j \in \tilde{C} \)

IF \( \varphi_1(x(t)) \) is \( \tilde{N}_j^1 \) \( \text{AND} \) \( \cdots \) \( \text{AND} \) \( \varphi_q(x(t)) \) is \( \tilde{N}_j^q \) THEN \( u(t) = F_j e(t) + G_j x_r(t), \)

where \( e(t) = x(t) - x_r(t) \) represents the error between the system states and the reference states, \( \tilde{N}_j^q \) is an IT2 fuzzy term of the \( j \)-th rule corresponding to the function \( \varphi_j(x(t)) \), \( g_j(x(t)) \) is the premise variable, \( t \in \tilde{\Omega}, \Omega \) is a positive integer. \( F_j \in \mathbb{R}^{m \times n} \) and \( G_j \in \mathbb{R}^{m \times m} \) are the feedback gains to be determined. \( m_j(x(t)) \geq 0 \) is defined as follows:

\[
m_j(x(t)) = \frac{\phi_j(x(t)) m_j^L(x(t)) + \psi_j(x(t)) m_j^U(x(t))}{\sum_{k=1}^{\Omega} (\phi_k(x(t)) m_k^L(x(t)) + \psi_k(x(t)) m_k^U(x(t)))},
\]

where \( m_j^L(x(t)) = \Omega i=1 \mu_{\tilde{N}_j}(g_i(x(t))) \) and \( m_j^U(x(t)) = \Omega i=1 \bar{\mu}_{\tilde{N}_j}(g_i(x(t))) \), in which \( 0 \leq \bar{\mu}_{\tilde{N}_j}(g_i(x(t))) \leq 1 \) and \( 0 \leq \mu_{\tilde{N}_j}(g_i(x(t))) \leq 1 \) denote the upper and lower grades of membership governed by the upper and lower membership functions, respectively. By the definition of IT2 membership functions, the property \( 0 \leq \bar{\mu}_{\tilde{N}_j}(g_i(x(t))) \leq \bar{\mu}_{\tilde{N}_j}(g_i(x(t))) \leq 1 \) holds and further leads to the \( 0 \leq m_j^L(x(t)) \leq m_j^U(x(t)) \leq 1 \) holds for all \( j \) \( 0 \leq \phi_j(x(t)) \leq 1, 0 \leq \psi_j(x(t)) \leq 1 \) and \( \phi_j(x(t)) + \psi_j(x(t)) = 1, \forall j \). \( \phi_j(x(t)) \) and \( \psi_j(x(t)) \) are nonlinear functions to be determined.

After fuzzy blending all the rules together, the IT2 fuzzy controller is represented by:

\[
u(t) = \sum_{j=1}^{c} m_j(x(t))(F_j e(t) + G_j x_r(t)),
\]

where \( m_j(x(t)) \in [0, 1], \sum_{j=1}^{c} m_j(x(t)) = 1, \forall j \).

III. STABILITY ANALYSIS

The goal of tracking control in this paper is to design a suitable IT2 fuzzy controller, and use this controller to make
the states of the controlled system closely follow those of the given stable positive reference model. In order to describe the level of tracking error, the $H_{\infty}$ performance index will be used in the following analysis. For the purpose of simplification, when the notation is clear, the time $t$ in the variable will be omitted, e.g., $x(t)$, $u(t)$ and $e(t)$ are represented as $x$, $u$ and $e$, respectively. Furthermore, $w_i(x(t))$ and $m_j(x(t))$ are represented as $w_i$ and $m_j$, respectively.

A. $MFI$ Stability and Positivity conditions under $H_{\infty}$ performance

Connecting the IT2 PTSFMB system with external disturbance (3) and IT2 fuzzy controller (7), the closed-loop dynamic equation can be obtained as:

$$\dot{x} = \sum_{i=1}^{c} \sum_{j=1}^{c} w_i m_j \left( A_i x + B_{ui} (F_j e + G_j x_r) + B_{ui} \omega \right)$$

$$= \sum_{i=1}^{c} \sum_{j=1}^{c} w_i m_j \left( (A_i + B_{ui} F_j) x + B_{ui} (G_j - F_j) x_r + B_{ui} \omega \right)$$

$$+ B_{ui} \omega \right). \quad (8)$$

In order to ensure that the IT2 PTSFMB with external disturbance system (8) is a positive system, according to Lemma 1, the following conditions need to be met: $A_i + B_{ui} F_j$ is Metzler matrix, $B_{ui} \geq 0$ and $B_{ui} (G_j - F_j) \geq 0$, for $\forall i, j$.

For stability analysis, the error vector $e$ will be considered. The dynamics of $e$ is given as:

$$\dot{e} = \dot{x} - \dot{x}_r$$

$$= \sum_{i=1}^{c} \sum_{j=1}^{c} w_i m_j \left( (A_i + B_{ui} F_j) e + (A_i - A_r) + B_{ui} G_j x_r + B_{ui} \omega - B_r \right). \quad (9)$$

To study the stability of the error system, the following Lyapunov function candidate is selected:

$$V(t) = e^T(t) Pe(t), \quad (10)$$

where $0 < P = P^T \in \mathbb{R}^{n \times n}$, this will ensure that $V(t)$ is a positive function. However, due to the inherent positive constraints of the positive system, in the following analysis, when the stability conditions and positive conditions are satisfied at the same time, all the obtained conditions are non-convex conditions that are not soluble by the convex programming technique. In order to remove the obstacles of these non-convex conditions, the positive definite diagonal matrix $X$ is defined and satisfies $X = P^{-1}$. At the same time, there are the following definitions $F_j = M_j P$ and $G_j = N_j P$, where $M_j \in \mathbb{R}^{n \times n}$ and $N_j \in \mathbb{R}^{m \times n}$.

On the one hand, in order to convert non-convex conditions of positive constraints to convex conditions, the following techniques will be used: Perform the post-multiplying of $X$ on both sides of positive conditions at the same time to guarantee that $(A_i + B_{ui} F_j) X$ is Metzler matrix and $(B_{ui} (G_j - F_j)) X \geq 0$, for $\forall i, j$. That is, the following element form:

$$b_{ui r} \geq 0, \quad \forall i; \quad b_{ui r} (n_j - m_j) \geq 0 \quad \forall i, j;$$

$$a_{i r s} x_{s r} + b_{ui r} m_j \geq 0, \quad r \neq s, \forall i, j. \quad (11)$$

On the other hand, for non-convex problems under stable conditions, the error vector $\dot{e}$ needs to be operated. Its new form is:

$$\dot{e} = \sum_{i=1}^{c} \sum_{j=1}^{c} w_i m_j ((A_i X + B_{ui} M_j) X)^{-1} e + (A_i X - A_r X + B_{ui} N_j + B_{ui} \omega - B_r \right)$$

$$= \sum_{i=1}^{c} \sum_{j=1}^{c} w_i m_j \Xi_{ij}, \quad (12)$$

$$\Xi_{ij} = [\Xi_1 \quad \Xi_2 \quad \Xi_3 \quad \Xi_4], \quad Z^T = [Z_1^T \quad Z_2^T \quad Z_3^T \quad Z_4^T].$$

$\Xi_1 = A_i X + B_{ui} M_j$, $\Xi_2 = A_i X - A_r X + B_{ui} N_j$, $\Xi_3 = B_{ui} \omega$, $\Xi_4 = -B_r$, and $Z_1 = X^{-1} e$, $Z_2 = X^{-1} x_r$, $Z_3 = \omega$, $Z_4 = r$.

In order to obtain stability conditions, the following vectors will be defined: $L_1 = [I_n 0 0 0]$; $L_2 = [0 I_n 0 0]$; $L_3 = [0 0 I_m 0]$ and $L_4 = [0 0 0 I_n]$. According to Lyapunov stability theory, $V(t)$ has the following form:

$$V(t) = e^T(t) Pe(t) + e^T(t) Pe(t)$$

$$= \sum_{i=1}^{c} \sum_{j=1}^{c} w_i m_j (Z_1^T \Xi_{ij}^T Pe + e^T(t) \Xi_{ij} Z)$$

$$= \sum_{i=1}^{c} \sum_{j=1}^{c} w_i m_j Z_1^T \Xi_{ij}^T L_1 + L_1^T (L_1^T) Z. \quad (13)$$

In the following analysis, the tracking performance will be reflected by the $H_{\infty}$ index. Since the reference model is a given stable system, so that $x_r$ and $r$ are bounded, and using this property, the $H_{\infty}$ performance [60] for tracking error will be constructed as:

$$\frac{\int_{0}^{t_f} Z_1^T Z_1 dt - V(0)}{\int_{0}^{t_f} \left( \sigma_1 Z_2^T Z_2 + \sigma_2 Z_3^T Z_3 + \sigma_3 Z_4^T Z_4 \right) dt} \leq 1, \quad (14)$$

where $t_f$ is the terminal time of control, $\sigma_1 > 0$, $\sigma_2 > 0$ and $\sigma_3 > 0$ are the prescribed attenuation levels, $V(0)$ indicates that the initial condition is also considered. It should be pointed out that by reducing the prescribed attenuation levels $\sigma_1$, $\sigma_2$, and $\sigma_3$, the $H_{\infty}$ tracking error can be obtained as small as possible.

In order to facilitate the stability analysis, rewrite (14) as:

$$-V(0) + \int_{0}^{t_f} (Z_1^T Z_1 - \sigma_1 Z_2^T Z_2 - \sigma_2 Z_3^T Z_3 - \sigma_3 Z_4^T Z_4) dt \leq 0$$

Next, based on the always hold of $V(t_f) \geq 0$, the inequality scaling operation is applied to obtain:

$$-V(0) + \int_{0}^{t_f} (Z_1^T Z_1 - \sigma_1 Z_2^T Z_2 - \sigma_2 Z_3^T Z_3 - \sigma_3 Z_4^T Z_4) dt$$

$$= \int_{0}^{t_f} (\dot{V} + Z_1^T Z_1 - \sigma_1 Z_2^T Z_2 - \sigma_2 Z_3^T Z_3 - \sigma_3 Z_4^T Z_4) dt$$
where, \( J_1 \) is denoted as:

\[
J_1 = \int_0^T \left( V + Z_1^T \sigma_1 Z_1 - \sigma_1 Z_1^T Z_2 - \sigma_2 Z_2^T Z_3 - \sigma_3 Z_3^T Z_4 \right) dt
\]

If \( J_1 < 0 \) holds, the \( H_\infty \) tracking performance (14) will be guaranteed. Therefore, considering \( J_1 < 0 \) and substituting (13) into \( J_1 \), the following form can be obtained:

\[
J_1 = \int_0^T \left( \sum_{i=1}^q \sum_{j=1}^c w_i m_j Z^T \Phi_{ij} Z \right) dt,
\]

where \( \Phi_{ij} = \Xi_{ij} L_1 + L_1^T \Xi_{ij} + L_2^T L_1 - \sigma_1 L_2^T L_2 - \sigma_2 L_3^T L_3 - \sigma_3 L_4^T L_4 \).

From (16), if \( \sum_{i=1}^q \sum_{j=1}^c w_i m_j Z^T \Phi_{ij} Z < 0 \) holds, then, the \( H_\infty \) tracking performance can be satisfied.

Combining the above analysis, the design results of the controller in IT2 positive system tracking control are presented by the following theorem.

**Theorem 1:** Consider an IT2 PTSFMB tracking control system, which is a combination of the positive fuzzy model with external disturbance (3) and the IT2 fuzzy controller (7). Given matrices \( A_1, B_i, C_i, A_r \) and \( B_r \) of matching dimensions, and using the controller designed in this paper, then, the \( H_\infty \) tracking performance will be realized by the prescribed attenuation levels \( \sigma_1 > 0, \sigma_2 > 0 \) and \( \sigma_3 > 0 \), if there is a positive definite diagonal matrix \( X \in \mathbb{R}^{m \times n} \) and matrices \( M_j \in \mathbb{R}^{m \times n}, N_j \in \mathbb{R}^{m \times n} \), such that the following GEVP is feasible:

Minimize the \( \sigma_1 + \sigma_2 + \sigma_3 \) subject to

\[
\begin{align*}
\sigma_1 &> 0, \sigma_2 > 0, \sigma_3 > 0; \\
v^T (X - \lambda I) v &\text{ is } \text{SOS}; \\
\alpha_{sir} x_s + b_{uir} m_{js} &\text{ is } \text{SOS}; \forall r \neq s, i, j; \\
b_{uir} (n_j - m_j) &\text{ is } \text{SOS}; \\
-v_1^T (\Phi_{ij} + \epsilon I) v_1 &\text{ is } \text{SOS},
\end{align*}
\]

where \( v \) and \( v_1 \) are arbitrary vectors of matching dimensions, \( \epsilon \) is the user-defined scalar, the vectors \( L_1, L_2, L_3, L_4 \) are as shown in the paper.

According to the above analysis, the feedback gain can be given by \( F_j = M_j X^{-1} \) and \( G_j = N_j X^{-1} \).

**B. IT2 MFD Stability and Positivity Conditions Analysis with \( H_\infty \) Performance**

In the previous subsection, the stability conditions based on the membership-function-independent (MFI) approach are presented. However, due to the independence of the MFs, the analysis results are undoubtedly conservative. Therefore, in order to obtain a more perfect tracking effect, the information of IT2 MFS will be tried to use in this subsection.

This paper novelty proposes to limit the range of the embedded Type-1 membership function (ETMF) by introducing upper and lower membership function information. Within the limited range, the EMTF is freely selected by the user. At the same time, PLMF (explained in detail in [61]) technology is used to approximate the ETMF. Due to the introduction of more effective original MFs information, the analysis conditions are relaxed and better tracking performance is obtained.

As shown in Fig. 1, the area between the upper membership function (UMF) and lower membership function (LMF) is considered to be the FOU, which represents the characteristics of the IT2 MFs [50]. And it can be seen that FOU is composed of an infinite number of EMTFs. Among them, an EMTF selected arbitrarily in Fig. 1 is defined as \( \tilde{w}_1(x(t)) \), and it is given by \( \tilde{w}_1(x(t)) = \tilde{\lambda}_1(x(t)) \tilde{w}_1^T(x(t)) + \tilde{\lambda}_1(x(t)) \tilde{w}_1(x(t)) \) with the arbitrarily predefined functions \( \tilde{\lambda}_1(x(t)) \) and \( \tilde{\lambda}_1(x(t)) \). In this paper, the reference EMTF \( \tilde{w}_1(x(t)) \) is approximated by piecewise linear embedded Type-1 membership function (PLETFM) \( \tilde{w}_1(x(t)) \), which can give a feasible solution through the SOS-based or LMI-based conditions. In this case, the parameter uncertainty of IT2 MFs will be introduced into the analysis conditions through the approximation error and unlike [41], this approximation error in this paper is not only limited to a positive value, but arbitrary. That is to say, as long as \( 0 \leq \tilde{\lambda}_1(x(t)) \leq 1, 0 \leq \tilde{\lambda}_1(x(t)) \leq 1 \) and \( \tilde{\lambda}_1(x(t)) + \tilde{\lambda}_1(x(t)) = 1 \) are satisfied, then \( \tilde{\lambda}_1(x(t)) \) and \( \tilde{\lambda}_1(x(t)) \) can be arbitrary values.

On the basis of global approximation analysis [62], in order to reduce the conservativeness of the analysis conditions, the sub-domains operation method of local approximation is used in this paper. The whole operation domain \( \Theta \) of the premise variable is divided into \( D \) related sub-domains \( \Theta_d, d \in D, D = \bigcup_{i=1}^n \Theta_i \), \( K_i \) is the number of sub-state spaces of \( x_i \), \( p \) represents the number of system variables on which the stability condition depends. In the following analysis, a membership function \( h_{ij}(x) = w_i(x)m_j(x) \) is selected as the RETMF through any given uncertainty parameter, and the \( h_{ij}(x) \) in each sub-domain of the RETMF is defined as \( h_{ij}(x) = (w_i(x)m_j(x))_d \). When using PLMF technology, in order to more easily distinguish whether the approximation is performed in each sub-domain, the state boundary information of each sub-domain is introduced through the scalar function.
where $v_{i\tau_{ij}}(x_{i})$ is called predefined interpolation function and satisfies: $0 \leq v_{i\tau_{ij}}(x_{i}) \leq 1$ and $v_{i\tau_{1d}}(x_{i}) + v_{i\tau_{2d}}(x_{i}) = 1$, $l \in \tilde{p}, \tau_{i} = \{1,2\}$, $\delta_{ij}, \delta_{ij} \in \mathbb{R}^{d}$ represents the constant value of RETM $\hat{h}_{ij}(x)$ at each sampling point $x$. $\hat{h}_{ij}(x)$ is the approximate membership function of RETM $h_{ij}(x)$ in each sub-domain.

Then, in each sub-domain $\Theta_{d}$, the approximation error of IT2 MFs is given by $\Delta_{h_{ij}}(x) = h_{ij}(x) - \hat{h}_{ij}(x)$. UMF is expressed as: $\tilde{h}_{ij}(x) = (\tilde{\eta}_{r}(x)\tilde{\mu}_{d}(x))$, $l$. LMF is expressed as: $h_{ij}(x) = (\tilde{\eta}_{r}(x)\mu_{d}(x))$, $l$, and the approximation error of LMF is expressed as $\Delta_{h_{ij}}(x)$. Since $\hat{h}_{ij}(x) \leq h_{ij}(x) \leq \tilde{h}_{ij}(x)$, the following inequality holds:

$$\hat{h}_{ij}(x) - \hat{h}_{ij}(x) \leq h_{ij}(x) - \hat{h}_{ij}(x) \leq \tilde{h}_{ij}(x) - \hat{h}_{ij}(x) \Rightarrow \Delta_{h_{ij}}(x) \leq \Delta_{h_{ij}}(x) \leq \Delta_{h_{ij}}(x) \leq \Delta_{h_{ij}}(x).$$

(20)

In Theorem 1, the realization of $J_{1} < 0$ is only guaranteed by $\Phi_{ij} < 0$. In fact, the guarantee of $\Phi_{ij} < 0$ is obviously a sufficient and unnecessary condition for the establishment of $\sum_{i=1}^{n} \sum_{j=1}^{m} \omega_{i} \phi_{ij} Z^{T} \Phi_{ij} Z < 0$. This further shows that the complete maintenance of $J_{1} < 0$ will weaken the consensiveness of the stability condition in Theorem 1. Therefore, in order to introduce the original information of IT2 MFs into the stability condition more comprehensively, the following upper and lower boundary conditions in each sub-domain $\Theta_{d}$ will be given:

$$\begin{align*}
&1) \delta_{ij} \leq \Delta_{h_{ij}}(x), \\
&2) \delta_{ij} \leq \Delta_{h_{ij}}(x) \leq \delta_{ij}, \\
&3) \delta_{ij} \leq \Delta_{h_{ij}}(x) \leq \delta_{ij}. \\
&4) \delta_{ij} \leq \Delta_{h_{ij}}(x) \leq \delta_{ij}.
\end{align*}$$

(21)

Remark 2: The use of these boundary information can more accurately restore and introduce the original information of IT2 MFs, making the relaxed stability conditions in this paper closer to the actual stability conditions in the sub-domain $\Theta_{d}$, thereby greatly improving the tracking performance.

Through the above analysis, in order to use the set of inequalities (21) to relax the stability condition, the slack matrix $W_{ij}$ that satisfies $W_{ij} = W_{ij}^{T} > 0$ is introduced to obtain:

$$\Psi_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{m} \omega_{i} \phi_{ij} Z^{T} \Phi_{ij} Z$$

$$\leq \sum_{i=1}^{n} \sum_{j=1}^{m} \omega_{i} \phi_{ij} Z^{T} \Phi_{ij} Z + \sum_{i=1}^{n} \sum_{j=1}^{m} \omega_{i} \phi_{ij} Z^{T} \Phi_{ij} Z$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \omega_{i} \phi_{ij} Z^{T} \Phi_{ij} Z$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \omega_{i} \phi_{ij} Z^{T} \Phi_{ij} Z + \sum_{i=1}^{n} \sum_{j=1}^{m} \omega_{i} \phi_{ij} Z^{T} \Phi_{ij} Z$$

(22)

It can be seen from the above formula that the guarantee of (22) $< 0$ will lead to the satisfaction of $J_{1} < 0$, but due to the continuity of the state variable $x$ in the formula, it is unrealistic to give a feasible solution to the current form. Therefore, we will make use of the conditions in the inequality group (21) to further transform the formula (22) to obtain:

$$\Psi_{ij} \leq \sum_{i=1}^{n} \sum_{j=1}^{m} \omega_{i} \phi_{ij} Z^{T} \Phi_{ij} Z + \sum_{i=1}^{n} \sum_{j=1}^{m} \omega_{i} \phi_{ij} Z^{T} \Phi_{ij} Z$$

(23)

where, these slack matrices $0 < Q_{ij} \leq Q_{ij}^{T}, 0 < Q_{ij} \leq Q_{ij}^{T}, 0 < Q_{ij} \leq Q_{ij}^{T}$. Because of the establishment of equation 3) in the inequality group (21), the following inequality relationship is obtained: $-\delta_{ij} \leq \delta_{ij}$. Thus, $\Psi_{ij}$ can further be obtained:

$$\Psi_{ij} \leq \sum_{i=1}^{n} \sum_{j=1}^{m} \omega_{i} \phi_{ij} Z^{T} \Phi_{ij} Z$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \omega_{i} \phi_{ij} Z^{T} \Phi_{ij} Z + \sum_{i=1}^{n} \sum_{j=1}^{m} \omega_{i} \phi_{ij} Z^{T} \Phi_{ij} Z$$

(24)

For the purpose of considering the boundary information of sub-domain $\Theta_{d}$ in the stability condition, the following sub-domain boundary condition is given:

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \omega_{i} \phi_{ij} Z^{T} \Phi_{ij} Z$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \omega_{i} \phi_{ij} Z^{T} \Phi_{ij} Z + \sum_{i=1}^{n} \sum_{j=1}^{m} \omega_{i} \phi_{ij} Z^{T} \Phi_{ij} Z$$

(25)

where, $x_{l}$ is the state variable of the system state $x_{l}$ in each sub-domain $\Theta_{d}$. And it can be seen that if $x_{l}$ is within the corresponding sub-domain, the above constraint is positive, otherwise it will not be considered. Therefore, $\Psi_{ij}$ can be further relaxed by:

$$\Psi_{ij} \leq \sum_{i=1}^{n} \sum_{j=1}^{m} \omega_{i} \phi_{ij} Z^{T} \Phi_{ij} Z + \sum_{i=1}^{n} \sum_{j=1}^{m} \omega_{i} \phi_{ij} Z^{T} \Phi_{ij} Z$$

(26)
\[ Q_{ijd} + (\delta_{1ijd} - \delta_{2ijd}) Q_{2ijd} + (\delta_{2ijd} - \delta_{2ijd})(Q_{3ijd} + Q_{5ijd}) + (\delta_{iijd} - \delta_{1ijd})(Q_{4ijd}) + (\delta_{1ijd} - \delta_{1ijd})(x - z_{ijd}) \]

\[ W_{ijd} + \sum_{l=1}^{p} (x_l - x_{l_{\text{min}}})(x_{l_{\text{max}}} - x_l)P_{ld}(x) \right) Z. \]  

Due to \( u_{li_d}(x_l) \geq 0 \) for \( \forall l \in \tilde{p}, i \in \{1, 2\}, d \in \tilde{D} \), therefore, the establishment of \( \Psi_{ij} \leq 0 \) only requires the maintenance of \( \sum_{i=1}^{m} \sum_{j=1}^{n} \left\{ \frac{1}{2} (2\delta_{iijd} - \delta_{1ijd}) + (\delta_{1ijd} - \delta_{2ijd})(Q_{3ijd} + Q_{5ijd}) + (\delta_{iijd} - \delta_{1ijd})(Q_{4ijd}) + (\delta_{1ijd} - \delta_{1ijd})(x - z_{ijd}) \] \[ W_{ijd} + \sum_{l=1}^{p} (x_l - x_{l_{\text{min}}})(x_{l_{\text{max}}} - x_l)P_{ld}(x) \right) < 0. \]

The above analysis is for the relaxation of the stability conditions in Theorem 1. Next, the analysis results of IT2 MFs relaxation will be given by the following theorem.

**Theorem 2:** Consider an IT2 PTSFMB tracking control system composed of the positive fuzzy model with external disturbance (3) and the IT2 fuzzy controller (7). Given matrices \( A, B, u, B, u, A, \) and \( B, \) of matching dimensions, and using the controller designed in this paper, then, the \( H_{\infty} \) tracking performance will be realized by the prescribed attenuation levels \( \sigma_1 > 0, \sigma_2 > 0 \) and \( \sigma_3 > 0, \) if there is a positive definite diagonal matrix \( X \in \mathbb{R}^{n \times n}, \) matrices \( M_j \in \mathbb{R}^{m \times n}, N_j \in \mathbb{R}^{m \times n} \) and positive symmetric matrices \( W_{ijd}, Q_{1ijd}, Q_{2ijd}, Q_{3ijd}, Q_{4ijd} \) of matching dimensions, such that the following GEVP is feasible:

Minimize the \( \sigma_1 + \sigma_2 + \sigma_3 \) subject to

\[ \frac{\sigma_1}{\sigma_2} > 0, \frac{\sigma_2}{\sigma_3} > 0, \]

\[ v^T (X - \epsilon I) v \text{ is SOS}; \]

\[ a_{irs}x_{ss} + b_{air}m_{js} \text{ is SOS; } \forall r \neq s, i, j; \]

\[ b_{air}(n_j - m_j) \text{ is SOS}; \]

\[ v^T (W_{ijd} - \epsilon I) v_1 \text{ is SOS; } v^T (Q_{1ijd} - \epsilon I) v_1 \text{ is SOS; } \]

\[ v^T (Q_{2ijd} - \epsilon I) v_1 \text{ is SOS; } v^T (Q_{3ijd} - \epsilon I) v_1 \text{ is SOS; } \]

\[ v^T (Q_{4ijd} - \Phi_{ijd} - \epsilon I) v_1 \] is SOS;

\[ u_{irs}x_{ss} + b_{air}m_{js} \text{ is SOS; } \forall r \neq s, i, j; \]

\[ b_{air}(n_j - m_j) \text{ is SOS; } \]

\[ v^T (W_{ijd} - \epsilon I) v_1 \text{ is SOS; } v^T (Q_{1ijd} - \epsilon I) v_1 \text{ is SOS; } \]

\[ v^T (Q_{2ijd} - \epsilon I) v_1 \text{ is SOS; } v^T (Q_{3ijd} - \epsilon I) v_1 \text{ is SOS; } \]

\[ v^T (Q_{4ijd} - \Phi_{ijd} - \epsilon I) v_1 \text{ is SOS; } \]

\[ v^T (Q_{5ijd} - \Phi_{ijd} - \epsilon I) v_1 \text{ is SOS; } \]

\[ -v^T \left( \sum_{i=1}^{\eta} \sum_{j=1}^{m} \left\{ \frac{1}{2} (2\delta_{iijd} - \delta_{1ijd}) + (\delta_{1ijd} - \delta_{2ijd})(Q_{3ijd} + Q_{5ijd}) + (\delta_{iijd} - \delta_{1ijd})(Q_{4ijd}) + (\delta_{1ijd} - \delta_{1ijd})(x - z_{ijd}) \] \[ W_{ijd} + \sum_{l=1}^{p} (x_l - x_{l_{\text{min}}})(x_{l_{\text{max}}} - x_l)P_{ld}(x) \right) + \epsilon I \right) v_1 \]

is SOS, \( \forall l \in \tilde{p}, i \in \{1, 2\}, d \in \tilde{D}, \)

where \( v \) and \( v_1 \) are arbitrary vectors of matching dimensions, \( e \) is the user-defined scalar, the vectors \( L_1, L_2, L_3, L_4 \) are as shown in the paper. The controller feedback gain is given by:

\[ F_j = M_j X^{-1} \text{ and } G_j = N_j X^{-1}. \]

In the following analysis, the method of analyzing MFD relaxation stability conditions in the previous literature will be introduced. We apply it to the analysis of IT2 MFD relaxation stability conditions to compare the effectiveness of the IT2 MFD relaxation method proposed in this paper.

**C. IT2 MFD Relaxed Stability and Positivity Conditions with Global Approximation**

This section will continue to use the PLMF technique in Theorem 2 to approximate the original MFs, thus, this technique will not be repeated here.

Regarding the relaxation of stability conditions, the idea is exactly the same as in [62], and the following definitions and conditions are given: the approximation error is \( \Delta h_{ij}(x) = h_{ij}(x) - \tilde{h}_{ij}(x), \forall i, j. \) In the global state space \( \Theta, \) the minimum and maximum values of \( \Delta h_{ij}(x) \) are denoted by \( \eta_{ij} \) and \( \theta_{ij}, \) respectively, and \( \delta_{ij} \) as the minimal value of PLETMF \( \tilde{h}_{ij}(x). \) In addition, define positive definite symmetric matrices \( Y_{ij} \) and \( K_{ij} \) with matching dimensions, and the matrix \( Y_{ij} \) satisfies \( Y_{ij} \geq \Phi_{ij} \). Then there are the following conditions:

\[ \Psi_{ij} = \sum_{i=1}^{c} \sum_{j=1}^{m} v_i \epsilon_i Z^T \Phi_{ij} Z \]

\[ \leq \sum_{i=1}^{c} \sum_{j=1}^{m} Z^T \left( (b_{ij}(x) + \eta_{ij}) \Phi_{ij} + (\eta_{ij} - \eta_{ij}) Y_{ij} \right) \]

\[ + \left( \tilde{h}_{ij}(x - \epsilon I) K_{ij} Z \right) \]

\[ = \sum_{d=1}^{D} \sum_{i=1}^{c} \sum_{j=1}^{m} \left( \delta_{ij} x_{ij} \Phi_{ij} + (\eta_{ij} - \eta_{ij}) Y_{ij} \right) \]

\[ + \left( \delta_{ij} x_{ij} \Phi_{ij} + (\eta_{ij} - \eta_{ij}) Y_{ij} \right) K_{ij} Z. \]  

Due to \( u_{li_d}(x_l) \geq 0 \) is independent of the rule \( \sum_{i=1}^{l=1} \sum_{j=1}^{c} \) and it satisfies the following property

\[ \sum_{i=1}^{c} \sum_{j=1}^{m} \left( \frac{1}{2} (2\delta_{iijd} - \delta_{1ijd}) + (\delta_{1ijd} - \delta_{2ijd})(Q_{3ijd} + Q_{5ijd}) + (\delta_{iijd} - \delta_{1ijd})(Q_{4ijd}) + (\delta_{1ijd} - \delta_{1ijd})(x - z_{ijd}) \] \[ W_{ijd} + \sum_{l=1}^{p} (x_l - x_{l_{\text{min}}})(x_{l_{\text{max}}} - x_l)P_{ld}(x) \right) + \epsilon I \right) v_1 \]

is SOS, \( \forall l \in \tilde{p}, i \in \{1, 2\}, d \in \tilde{D}, \)

where \( v \) and \( v_1 \) are arbitrary vectors of matching dimensions, \( e \) is the user-defined scalar, the vectors \( L_1, L_2, L_3, L_4 \) are as shown in the paper. The controller feedback gain is given by:

\[ F_j = M_j X^{-1} \text{ and } G_j = N_j X^{-1}. \]
\( a_{irs}x_{ss} + b_{uir}m_{js} \) is SOS; \( \forall r \neq s, i, j; \)
\( b_{uir}(n_{ij} - m_{ij}) \) is SOS;
\( v_1^T(K_i - \epsilon I)v_1 \) is SOS; \( v_1^T(Y_{ij} - \epsilon I)v_1 \) is SOS;
\( v_1^T(Y_{ij} - \Phi_{ij} - \epsilon I)v_1 \) is SOS;
\[-v_1^T\left( \sum_{i=1}^{n} \sum_{j=1}^{n} \left( (\delta_{ij} - \epsilon_{ij}) \Phi_{ij} + (\eta_{ij} - \eta_{ij}) Y_{ij} \right) \right) \]
\[+ (\delta_{ij} - \epsilon_{ij}) \Phi_{ij} + (\eta_{ij} - \eta_{ij}) Y_{ij} \] is SOS, \( \epsilon \) is the user-defined scalar, the vectors \( L_1, L_2, L_3, L_4 \) are as shown in the paper.

IV. SIMULATION EXAMPLE

In order to verify the effectiveness of the design method, a simulation example of three-rule positive T-S fuzzy model with external disturbance is presented with \( x = [x_1 \ x_2]^T \).

\[
A_1 = \begin{bmatrix} 0.08 & 0.46 \\ 6.72 & -11.63 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1.85 & 1.25 \\ 4.59 & -9.6 \end{bmatrix}, \\
A_3 = \begin{bmatrix} 0.69 & 0.35 \\ 7.21 & -8.86 \end{bmatrix}, \\
B_{u1} = \begin{bmatrix} 3.53 \\ 0.25 \end{bmatrix}, B_{u2} = \begin{bmatrix} 2.42 \\ 0.42 \end{bmatrix}, B_{u3} = \begin{bmatrix} 1.85 \\ 0.63 \end{bmatrix}, \\
B_{\omega1} = \begin{bmatrix} 0.05 \\ 0.01 \end{bmatrix}, B_{\omega2} = \begin{bmatrix} 0.02 \\ 0.02 \end{bmatrix}, B_{\omega3} = \begin{bmatrix} 0.03 \\ 0.01 \end{bmatrix}.
\]

The positive stability reference model is chosen as follows:
\[
A_r = \begin{bmatrix} -4.5 & 1.06 \\ 6.35 & -12.75 \end{bmatrix}, \quad B_r = \begin{bmatrix} 1.75 \\ 0.36 \end{bmatrix}, \\
\begin{cases} 1.3, & 15 \leq t < 30 \\ 0.5, & \text{otherwise} \end{cases}
\]

and \( r(t) = \begin{cases} 0.9 + 0.3 \sin(0.2t), & 30 \leq t < 80 \\ 0, & \text{otherwise} \end{cases} \)

The bounded external disturbance is chosen as \( \omega(t) = \begin{cases} \text{awgn}(1.3 + \sqrt{0.2} \sin(0.5t)), & 25 \leq t \leq 70 \\ 0, & \text{otherwise} \end{cases} \)

where awgn is a function, whose role is to add white Gaussian noise to the signal, that is, \( Y = \text{awgn}(X, \text{SNR}) \) adds white Gaussian noise to \( X \). The power of \( X \) is assumed to be 0 dBW, and the SNR represents the reasonably given Signal-Noise Ratio, in dB, in this example \( \beta = 10 \). The positive disturbance signal and positive reference input signal are shown Fig. 2.

Remark 3: The reference model is chosen according to the control specifications of the applications, i.e., the performance of the closed-loop system is to follow. Also, it is required to make sure that the reference model is positive and stable.

According to Lemma 1, it is obvious that both the IT2 TSFMB system and the stable reference system meet the requirements of the positive system in the given simulation example, which means that they are all positive systems.

The upper and lower MFs of the IT2 PTSFMB system are chosen as \( \overline{\mu}_1(x_1) = 1 - \frac{1}{1 + e^{-1.5(x_1-0.5)^2}}, \overline{\mu}_2(x_1) = 1 - \frac{1}{1 + e^{-1.5(x_1+0.5)^2}}, \overline{\mu}_3(x_1) = 1 - \frac{1}{1 + e^{-1.5(x_1-1)^2}}, \overline{\mu}_4(x_1) = 1 - \frac{1}{1 + e^{-1.5(x_1+1)^2}}, \overline{\mu}_5(x_1) = 1 - \frac{1}{1 + e^{-1.5(x_1-1.5)^2}}, \overline{\mu}_6(x_1) = 1 - \frac{1}{1 + e^{-1.5(x_1+1.5)^2}}, \overline{\mu}_7(x_1) = 1 - \frac{1}{1 + e^{-1.5(x_1-2)^2}}, \overline{\mu}_8(x_1) = 1 - \frac{1}{1 + e^{-1.5(x_1+2)^2}}, \)

\( s_1 = 0.3, s_2 = 0.2 \). The IPM concept is used to enhance the flexibility of controller design. Therefore, a two-rule IT2 fuzzy controller in the form (7) is designed to achieve the tracking control goal, the upper and lower MFs of the controller are chosen as \( \overline{\mu}_1(x_1) = 1 - \frac{1}{1 + e^{-1.5(x_1-0.5)^2}}, \overline{\mu}_2(x_1) = 1 - \frac{1}{1 + e^{-1.5(x_1+0.5)^2}}, \overline{\mu}_3(x_1) = 1 - \frac{1}{1 + e^{-1.5(x_1-1)^2}}, \overline{\mu}_4(x_1) = 1 - \frac{1}{1 + e^{-1.5(x_1+1)^2}}, \overline{\mu}_5(x_1) = 1 - \frac{1}{1 + e^{-1.5(x_1-1.5)^2}}, \overline{\mu}_6(x_1) = 1 - \frac{1}{1 + e^{-1.5(x_1+1.5)^2}}, \overline{\mu}_7(x_1) = 1 - \frac{1}{1 + e^{-1.5(x_1-2)^2}}, \overline{\mu}_8(x_1) = 1 - \frac{1}{1 + e^{-1.5(x_1+2)^2}}, \)

In the simulation example, the uncertainty-related nonlinear type reduction functions embedded in the IT2 membership function are selected as \( \lambda_1(x_1) = (\sin(x_1) + 1)/2, \lambda_2(x_1) = (\cos(x_1) + 1)/2, \lambda_3(x_1) = 1 - \lambda_1(x_1) - \lambda_2(x_1), \lambda_4(x_1) = 1 - \lambda_3(x_1); \) correspondingly, the type reduction functions for the controller are chosen as \( \overline{\mu}_j(x) = \pi_j(x) = 0.5, j = 1, 2, \ldots, 30. \)

The operating domain of the control process is considered as the premise variable \( x_1 \) from 0 to 10. When using PLMF technology, we consider equally dividing the whole operating domain \( \Theta \) into 30 sub-domains (i.e., \( D=30 \)). Thus, the upper and lower boundaries of the \( d \)-th sub-domain of \( \Theta \) are \( \overline{\pi}_d = 0.33d \) and \( \underline{\pi}_d = 0.33(d-1), d = 1, 2, \ldots, 30 \). It should be noted that the whole operation domain \( \Theta \) is not mandatory to be divided equally, and the number of sub-domain divisions is reasonably selected by the user.

In the simulation example, \( \epsilon \) is set to \( \epsilon = 1 \times 10^{-3} \), and in order to ensure fairness, when applying these theorems, the parameter settings in Theorem 1 to 3 are kept the same. By applying the IT2 fuzzy controller for tracking control with the initial conditions \( x(0) = [0.1 \ 0.05] \) and \( x_r(0) = [0.13 \ 0.08] \), the simulation result of the state response is shown in Fig. 3 and Fig. 4, which fully demonstrate the effectiveness of the designed tracking control.

In order to show the advantages of the proposed method, on the one hand, MFI is compared with MFD. Without considering the MFs information and the rest of the configuration is exactly the same, Theorem 1 is used to compare with the proposed Theorem 2.

It is obvious from Fig. 3 and Fig. 4 that under the same bounded external disturbances, the tracking error given by Theorem 1 is the largest, and from the data in Table I, it can also be further seen that whether in terms of the maximum
verify the superiority of adopting MFD method.

On the other hand, under the premise that the MFD method is used, the proposed method will be compared with the previously existing method. That is, Theorem 3 is used to compare with the proposed Theorem 2. Similarly, all remaining configurations are exactly the same.

Under the same bounded external disturbances, it can be clearly seen from Fig. 3 and Fig. 4 and the data in Table I that the tracking effect of Theorem 2 is relatively better, and the terms of the maximum tracking error and the RMSE tracking error are all smaller, all of which can show the advantages of the IT2 MFD relaxation condition method proposed in Theorem 2.

It is worth mentioning that under different forms of reference input signals, whether it is from the terms of the maximum tracking error or the RMSE tracking error, it effectively proves the reliability of the IT2 PTSFMB with external disturbance control strategy proposed in this paper.

### V. Conclusion

Under the $H_{\infty}$ performance, the tracking control design problem of the IT2 PTSFMB control system with external disturbance has been studied. Comprehensive consideration of nonlinearity, positive constraints and parameter uncertainty, and an effective IT2 fuzzy controller has been designed. In the process of design analysis, obstacles like non-convex conditions have also been successfully overcome. In addition, the IT2 MFD analysis method has been improved by introducing the upper and lower bound information of IT2 MFs and the state boundary information of each sub-domain into stability conditions, thereby achieving the purpose of weakening the conservativeness. In the future, the tracking control design for the positive system can be studied from different aspects. For example, the linear copositive Lyapunov function method can be used for analysis instead of the quadratic Lyapunov function method defined in the whole state space, the research on tracking control of the positive system can be extended to event-triggered FMB control, such as the fuzzy observer-based control and adaptive fuzzy control. In addition, it is also an interesting topic on how to relax the analysis conditions by introducing more effective membership function information.

### Table I

<table>
<thead>
<tr>
<th>Theorem</th>
<th>Matrix X</th>
<th>Controller gain $F_j$ and $G_j$</th>
<th>$\sigma_{\min}$</th>
<th>$\epsilon_{\max}$</th>
<th>$\epsilon_{\text{RMSE}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>diag{0.4003,0.3539}</td>
<td>$F_1 = [-10.9226, -0.129511]$</td>
<td>$\sigma_1 = 0.4164$</td>
<td>$\epsilon_{1\text{max}} = 0.0938$</td>
<td>$\epsilon_{1\text{RMSE}} = 0.0639$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$F_2 = [-10.9226, -0.129511]$</td>
<td>$\sigma_2 = 0.008095$</td>
<td>$\epsilon_{2\text{max}} = 0.0497$</td>
<td>$\epsilon_{2\text{RMSE}} = 0.0306$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$G_1 = [-1.77108, -0.26685]$</td>
<td>$\sigma_3 = 0.2962$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$G_2 = [-1.77108, -0.26685]$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>diag{0.5010,0.3122}</td>
<td>$F_1 = [-10.9238, -0.129401]$</td>
<td>$\sigma_1 = 0.3945$</td>
<td>$\epsilon_{1\text{max}} = 0.0436$</td>
<td>$\epsilon_{1\text{RMSE}} = 0.0207$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$F_2 = [-10.9224, -0.129313]$</td>
<td>$\sigma_2 = 0.007517$</td>
<td>$\epsilon_{2\text{max}} = 0.0300$</td>
<td>$\epsilon_{2\text{RMSE}} = 0.0060$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$G_1 = [-1.81419, -0.126183]$</td>
<td>$\sigma_3 = 0.2672$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$G_2 = [-0.603746, -0.125734]$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>diag{0.5380,0.2923}</td>
<td>$F_1 = [-10.9241, -0.129342]$</td>
<td>$\sigma_1 = 0.4345$</td>
<td>$\epsilon_{1\text{max}} = 0.0627$</td>
<td>$\epsilon_{1\text{RMSE}} = 0.0424$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$F_2 = [-10.9241, -0.129283]$</td>
<td>$\sigma_2 = 0.008812$</td>
<td>$\epsilon_{2\text{max}} = 0.0379$</td>
<td>$\epsilon_{2\text{RMSE}} = 0.0183$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$G_1 = [-1.88538, -0.125921]$</td>
<td>$\sigma_3 = 0.2817$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$G_2 = [-1.23082, -0.0502976]$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Fig. 3. State responses during the control process under Theorem 1 to 3.

Fig. 4. Tracking errors during the control process under Theorem 1 to 3.


