ADELIC EULER SYSTEMS FOR $\mathbb{G}_m$

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ABSTRACT. We define a notion of adelic Euler systems for $\mathbb{G}_m$ over arbitrary number fields and prove that all such systems over $\mathbb{Q}$ are cyclotomic in nature. We deduce that all Euler systems for $\mathbb{G}_m$ over $\mathbb{Q}$ are cyclotomic, as has been conjectured by Coleman, if and only if they validate an analogue of Leopoldt's Conjecture.

1. Introduction

1.1. **Background and main results.** Since its introduction by Thaine, and by Kolyvagin, in the late 1980s the theory of Euler systems has played a key role in the proof of many celebrated results in arithmetic.

However, to apply the general theory in any given setting, one must have available an explicit Euler system and it seems extremely difficult to find such systems.

This note is concerned with a classical instance of this problem. To be specific, we write $\mathrm{ES}_K(\mathbb{G}_m)$ for the collection of Euler systems associated to the multiplicative group $\mathbb{G}_m$ over a number field K. This set is naturally a module over the ring $R_K := \varprojlim_E \mathbb{Z}[\mathrm{Gal}(E/K)]$, where E runs over all finite ramified abelian extensions of K (inside a fixed algebraic closure) and the inverse limit is taken with respect to the natural restriction maps.

In general one knows almost nothing about $\mathrm{ES}_K(\mathbb{G}_m)$. For example, the only known elements of $\mathrm{ES}_\mathbb{Q}(\mathbb{G}_m)$ are those generated over $R_\mathbb{Q}$ by the 'cyclotomic' Euler system η^{cyc} and an explicit, and very elementary, family of systems of order two (cf. Remark 2.9). In fact, Coleman has conjectured that these elements should generate *all* of $\mathrm{ES}_\mathbb{Q}(\mathbb{G}_m)$ over $R_\mathbb{Q}$ but this conjecture has proved to be very difficult to resolve (cf. [1] and the references contained therein).

For each prime number p there is also an associated, but essentially distinct, notion of an Euler system for the p-adic representation $\mathbb{Z}_p(1)$ over K. The collection $\mathrm{ES}_K(\mathbb{Z}_p(1))$ of such systems is a module over the pro-p completion of R_K and there exists a canonical diagonal 'restriction' homomorphism of R_K-modules

$$\phi_K : \mathrm{ES}_K(\mathbb{G}_m)_{\mathrm{tf}} \to \prod_p \mathrm{ES}_K(\mathbb{Z}_p(1)),$$

where $\mathrm{ES}_K(\mathbb{G}_m)_{\mathrm{tf}}$ denotes the quotient of $\mathrm{ES}_K(\mathbb{G}_m)$ by its torsion subgroup and in the direct product p runs over all primes (for details see Lemma 2.7).

The nature of ϕ_K is important but has apparently never been investigated. For example, it seems likely that the kernel of ϕ_K is very small (as can be proved if K is either $\mathbb{Q}$ or

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imaginary quadratic - see Proposition 2.8) and hence that ϕ_K is useful for the study of $\mathrm{ES}_K(\mathbb{G}_m)$.

A key idea of this note is then the explicit definition of a closed subgroup $\mathrm{AES}_K(\mathbb{G}_m)$ of $\prod_p \mathrm{ES}_K(\mathbb{Z}_p(1))$ that contains the image of ϕ_K and is naturally a module over the algebra $\varprojlim_{n\in\mathbb{N}} R_K/n \cong \varprojlim_E \widehat{\mathbb{Z}}[\mathrm{Gal}(E/K)]$, where $\widehat{\mathbb{Z}}$ denotes the profinite completion of $\mathbb{Z}$. Elements of $\mathrm{AES}_K(\mathbb{G}_m)$ are intermediate between Euler systems for $\mathbb{G}_m$ and for $\mathbb{Z}_p(1)$ (for all primes p) and will, for convenience, be referred to as 'adelic Euler systems' for $\mathbb{G}_m$ over K (since $\widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is the ring of finite adeles over $\mathbb{Q}$).

The essential advantage of $\mathrm{AES}_K(\mathbb{G}_m)$ is that it seems much more amenable to explicit study than does $\mathrm{ES}_K(\mathbb{G}_m)$. For example, by adapting techniques developed in [1] we are able to prove $\phi_{\mathbb{Q}}(\eta^{\mathrm{cyc}})$ generates $\mathrm{AES}_{\mathbb{Q}}(\mathbb{G}_m)$ over the profinite completion of $R_{\mathbb{Q}}$. This result identifies $\mathrm{ES}_{\mathbb{Q}}(\mathbb{G}_m)_{\mathrm{tf}}$ as a dense subset of a space in which all elements are cyclotomic, implies $\mathrm{AES}_{\mathbb{Q}}(\mathbb{G}_m)$ is saturated in $\prod_p \mathrm{ES}_{\mathbb{Q}}(\mathbb{Z}_p(1))$ and also leads to an explicit conceptual characterization of $R_{\mathbb{Q}} \cdot \eta^{\mathrm{cyc}}$ within $\mathrm{ES}_{\mathbb{Q}}(\mathbb{G}_m)$.

The latter observation implies, in particular, that Coleman's Conjecture is valid if and only if the profinite completion of the homomorphism $\phi_{\mathbb{Q}}$ is injective, thereby showing that Coleman's Conjecture can be seen as an analogue of Leopoldt's Conjecture for the module of classical Euler systems over $\mathbb{Q}$.

The basic contents of this note is as follows. In §2 we review the explicit definitions of $\mathrm{ES}_K(\mathbb{G}_m)$ and $\mathrm{ES}_K(\mathbb{Z}_p(1))$, introduce the map ϕ_K and establish some of its basic properties. In §3 we define $\mathrm{AES}_K(\mathbb{G}_m)$, show that it contains $\mathrm{im}(\phi_K)$ and explicitly describe $\mathrm{AES}_{\mathbb{Q}}(\mathbb{G}_m)$. In §4 we derive consequences of the latter result for the module $\mathrm{ES}_{\mathbb{Q}}(\mathbb{G}_m)$ and, in particular, prove precise versions of the results discussed above.

1.2. **General notation.** For the reader's convenience, we fix some general notation.

1.2.1. For a Galois extension of fields F/E we abbreviate $\mathrm{Gal}(F/E)$ to $G_{F/E}$ and write $\mathrm{N}_{F/E}$ for the field-theoretic norm map $F^\times \to E^\times$.

Let K be a number field. Fix an algebraic closure K^c of K and write Ω_K for the set of all finite ramified abelian extensions of K in K^c.

We write $S_\infty(K)$ for the set of archimedean places of K and $S_p(K)$ for each prime p for the set of p-adic places of K.

For any non-archimedean place v of K we write σ_v for the *inverse* Frobenius automorphism of v in the maximal abelian extension of K in K^c in which v does not ramify.

We write $\mathcal{O}_K$ for the ring of integers of K, U_K for the unit group of $\mathcal{O}_K$ and μ_K for the (finite) torsion subgroup of U_K.

We shall usually use exponential notation to indicate the action of a commutative ring Λ on a multiplicative group U, so that the image of an element u of U under the action of an element λ of Λ is written as u^λ. However we caution the reader that, for typographic simplicity, we shall also occasionally write either $\lambda(u)$ or $\lambda \cdot u$ in place of u^λ.

1.2.2. Let A be an abelian group. We write A_{tor} for the torsion subgroup of A and abbreviate the quotient group A/A_{tor} to A_{tf}. For a homomorphism $\theta : A \to A'$ of abelian groups we denote the induced homomorphism $A_{\mathrm{tf}} \to A'_{\mathrm{tf}}$ by θ_{tf}.

We write $\widehat{A}$ for the profinite completion $\varprojlim_n A/nA$ of A. For each prime p we write $\widehat{A}^p$ for the pro-p completion of A. We note that if A is finitely generated, then $\widehat{A}$ and $\widehat{A}^p$ respectively identify with the tensor products $A\otimes_{\mathbb{Z}}\widehat{\mathbb{Z}}$ and $A_p := A\otimes_{\mathbb{Z}}\mathbb{Z}_p$.

We consider $\widehat{A}$ (resp. $\widehat{A}^p$) as a topological $\widehat{\mathbb{Z}}$ (resp. $\mathbb{Z}_p$) module by endowing it with the inverse limit topology. We note, furthermore, that if A is a $\mathbb{Z}$-algebra then one can verify that $\widehat{A}$ (resp. $\widehat{A}^p$) is naturally a $\widehat{\mathbb{Z}}$ (resp. $\mathbb{Z}_p$) algebra.

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2. Euler systems

In this section we introduce a natural notion of adelic Euler systems for $\mathbb{G}_m$ over number fields and relate it to the classical notion of Euler system.

2.1. For E in Ω_K we set

$$S(E) := S_\infty(K)\cup S_{\mathrm{r}}(E/K),$$

where $S_{\mathrm{r}}(E/K)$ denotes the finite set of places of K that ramify in E. We then write U'_E for the subgroup of $E^\times$ comprising elements that are integral at all places of E outside $S(E)$.

Definition 2.1. *An Euler system over K for $\mathbb{G}_m$ is a collection*

$$u=(u_E)_E\in\prod_{E\in\Omega_K}U'_E$$

with the property that for every E and E' in Ω_K with $E'\subset E$ one has

$$\mathrm{N}_{E/E'}(u_E)=\left(\prod_{v\in S(E)\setminus S(E')}(1-\sigma_v)\right)\cdot u_{E'} \qquad (1)$$

in $U'_{E'}$. We write $\mathrm{ES}_K(\mathbb{G}_m)$ for the R_K-module of all such systems.

Example 2.2. (The cyclotomic Euler system) Fix an embedding $\mathbb{Q}^c\hookrightarrow\mathbb{C}$ and so regard $\mathbb{Q}^c$ as a subfield of $\mathbb{C}$. Then if E belongs to $\Omega_{\mathbb{Q}}$ we set

$$\eta_E^{\mathrm{cyc}} := \mathrm{N}_{\mathbb{Q}(e^{2\pi i/m})/E}(1-e^{2\pi i/m})\in E^\times,$$

where $m=m_E$ is the conductor of E (so $E\subseteq\mathbb{Q}(e^{2\pi i/m})$). The collection $\eta^{\mathrm{cyc}} := (\eta_E^{\mathrm{cyc}})_E$ belongs to $\mathrm{ES}_{\mathbb{Q}}(\mathbb{G}_m)$.

Remark 2.3. If K is either $\mathbb{Q}$ or imaginary quadratic, then $\mathrm{ES}_K(\mathbb{G}_m)$ has an alternative interpretation. We focus on the case $K=\mathbb{Q}$, with analogous details for imaginary quadratic fields given by Coates in [2]. We fix a family $\{\zeta_n\}_{n\in\mathbb{N}}$ of compatible primitive n^{th}-roots of unity in $\mathbb{Q}^c$ and for each $n\in\mathbb{N}$ set $\mathbb{Q}(n) := \mathbb{Q}(\zeta_n)$. Then, with μ^* denoting the set of non-trivial roots of unity in $\mathbb{Q}^c$, a 'circular distribution' is a $G_{\mathbb{Q}^c/\mathbb{Q}}$-equivariant function $f:\mu^*\to\mathbb{Q}^{c\times}$ with the property that $\prod_{\zeta^a=\varepsilon}f(\zeta)=f(\varepsilon)$ for all natural numbers a and all ε in μ^*. For any such f there exists a unique element u_f of $\mathrm{ES}_{\mathbb{Q}}(\mathbb{G}_m)$ with $u_{f,\mathbb{Q}(m)}=f(\zeta_m)$ for all $m>1$, and it is straightforward to check that the assignment $f\mapsto u_f$ defines an isomorphism between the module of circular distributions and $\mathrm{ES}_{\mathbb{Q}}(\mathbb{G}_m)$.

The next result shows that the distribution relation (1) restricts the values of an Euler system.

For each prime p and each E in Ω_K we write U_E^p for the subgroup of U_E comprising elements that are units at all places outside $S_\infty(K) \cup S_p(K)$.

Lemma 2.4. *The following claims are valid for every u in $\mathrm{ES}_K(\mathbb{G}_m)$ and every E in Ω_K.*
(i) *If $S(E)$ contains $S_p(K)$ for some prime p, then $u_E \in U_E^p$.*
(ii) *If $S(E)$ contains $S_p(K)$ for two distinct primes p, then $u_E \in U_E$.*

Proof. If $S_p(K) \subset S(E)$, then as E' ranges over the finite extensions of E in its cyclotomic $\mathbb{Z}_p$-extension, one has $S(E') = S(E)$ and so the distribution relation (1) implies the elements $(u_{E'})_{E'}$ are norm-compatible. The containment $u_E \in U_E^p$ in claim (i) then follows easily from this fact, just as in the proof of [9, Lem. 2.2].

Claim (ii) is an immediate consequence of claim (i) since for any distinct primes p and q one has $U_E^p \cap U_E^q = U_E$. □

2.2. For each E in Ω_K we set

$$S(E,p) := S(E) \cup S_p(K).$$

The following definition is motivated by the approach of Rubin in [8, Def. 2.1.1]. For the convenience of the reader we recall that for each prime p and abelian group M we write M_p for the $\mathbb{Z}_p$-module $M \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

Definition 2.5. *An Euler system over K for the p-adic representation $\mathbb{Z}_p(1)$ is a collection*

$$u = (u_E)_E \in \prod_{E \in \Omega_K} U'_{E,p}$$

with the property that for every E and E' in Ω_K with $E' \subset E$ one has

$$\mathrm{N}_{E/E'}(u_E) = \left(\prod_{v \in S(E) \setminus S(E',p)} (1 - \sigma_v) \right) \cdot u_{E'} \tag{2}$$

in $U'_{E',p}$. We write $\mathrm{ES}_K(\mathbb{Z}_p(1))$ for the $\widehat{R_K}^p$-module comprising all such systems.

For each $E \in \Omega_K$ and rational prime p we equip $U'_{E,p}$ with its canonical profinite topology. In this way $\mathrm{ES}_K(\mathbb{Z}_p(1))$ becomes a topological space by endowing it with the subspace topology of the product topology on $\prod_{E \in \Omega_K} U'_{E,p}$. One can then verify that this topology induces a natural structure of a topological $\widehat{R_K}^p$-module on $\mathrm{ES}_K(\mathbb{Z}_p(1))$.

If E is any field for which $S(E)$ does not contain $S_p(K)$, then the extension F of E generated by any p-power root of unity of large enough order belongs to Ω_K and is such that $S(F) = S(E,p)$. In this case, the distribution relation (2) implies $\eta_E = \mathrm{N}_{F/E}(\eta_F)$ for every η in $\mathrm{ES}_K(\mathbb{Z}_p(1))$. It follows that Euler systems for $\mathbb{Z}_p(1)$ are uniquely determined by their values at fields E with $S_p(K) \subset S(E)$ and this is a crucial difference to Euler systems for $\mathbb{G}_m$. This accounts, for example, for the difference between the following result and the fact recalled in Remark 2.9 below.

Lemma 2.6. $\mathrm{ES}_K(\mathbb{Z}_p(1))$ *is torsion-free.*

Proof. It is enough to show that if u is any element of $\mathrm{ES}_K(\mathbb{Z}_p(1))$ with $u^p = 1$, then $u = 1$.

We first show that $u_E = 1$ for every E in Ω_K with $S_p(K) \subseteq S_{\mathrm{r}}(E/K)$. We write E_1 for the first layer in the cyclotomic $\mathbb{Z}_p$-extension of E. If $u_E \neq 1$, then E contains a primitive p-th root of unity and so u_{E_1} is fixed by the action of $G_{E_1/E}$. The distribution relation (2) therefore implies that $u_E = \mathrm{N}_{E_1/E}(u_{E_1}) = (u_{E_1})^p = 1$, which is a contradiction. Hence one has $u_E = 1$, as claimed.

If now E is any field such that $S_p(K) \not\subseteq S_{\mathrm{r}}(E/K)$, then we can choose a field E' in Ω_K that contains E and is such that $S(E') = S(E,p)$. For such a field E' one has $u_{E'} = 1$ (by the above argument) and so the relation (2) implies $u_E = \mathrm{N}_{E'/E}(u_{E'}) = 1$, as required. □

2.3. For each natural number m we write μ_m for the subgroup of $\mathbb{Q}^{c,\times}$ that is generated by ζ_m. We also write $\widehat{\mathbb{Z}}(1)$ for the inverse limit of the groups μ_m with respect to the transition morphisms $\mu_m \to \mu_{m'}$ for each divisor m' of m that are given by raising to the power m/m'.

We set

$$\mathrm{ES}_K(\widehat{\mathbb{Z}}(1)) := \prod_p \mathrm{ES}_K(\mathbb{Z}_p(1)),$$

where the direct product is taken over all primes p and regarded as a topological $\widehat{R_K}$-submodule of $\prod_{E\in\Omega_K} \widehat{U'_E}$ in the obvious way. We note that this module is related to $\mathrm{ES}_K(\mathbb{G}_m)$ by means of the canonical 'adelification' homomorphism that is described in the following result.

Lemma 2.7. *For E in Ω_K we obtain an element of $\widehat{\mathbb{Z}}[G_{E/K}] = \prod_p \mathbb{Z}_p[G_{E/K}]$ by setting*

$$\epsilon_E := \Big(\prod_{v\in S(E,p)\setminus S(E)} (1-\sigma_v)\Big)_p.$$

Then the assignment $u \mapsto (\epsilon_E \cdot u_E)_E$ defines a homomorphism of R_K-modules

$$\phi_K : \mathrm{ES}_K(\mathbb{G}_m) \to \mathrm{ES}_K(\widehat{\mathbb{Z}}(1)).$$

Proof. This follows by an easy, and explicit, comparison of the distribution relation (1) with that of (2) for all primes p. □

It seems likely that the kernel of ϕ_K is in all cases very small. The next result verifies that this expectation is true in the most significant cases.

To motivate this result, we recall that Euler systems for p-adic representations are of most interest in the case that the core rank of the associated unramified Selmer structure (in the sense of [7, Def. 5.1]) is one. Given this fact, and the explicit computation of [7, Th. 5.4], Euler systems for $\mathbb{Z}_p(1)$ are of most interest in the case that $|S_\infty(K)| = 1$ (so that K is either $\mathbb{Q}$ or an imaginary quadratic field).

Proposition 2.8. *If $|S_\infty(K)| = 1$, then the following claims are valid.*

(i) $\ker(\phi_K) = \mathrm{ES}_K(\mathbb{G}_m)_{\mathrm{tor}}$.

(ii) *If η is an element of $\mathrm{ES}_K(\mathbb{G}_m)_{\mathrm{tor}}$, then for every E in Ω_K, the element η_E belongs to the Hilbert class field of K. In particular, the order of η divides* 12.

Proof. Fix an Euler system $\eta \in \mathrm{ES}_K(\mathbb{G}_m)$ that belongs to $\ker(\phi_K)$.

Then, in view of Lemma 2.6, the first claim will follow if we can demonstrate the existence of a natural number n such that $\eta^n = 1$.

To this end we note that, for every prime p and field $E \in \Omega_K$ such that $S_p(K) \subseteq S(E)$, the image of η_E in $U'_{E,p}$ is trivial and so η_E is a root of unity of order prime to p.

We assume first that $K = \mathbb{Q}$. In this case, for every $E \in \Omega_{\mathbb{Q}}$ a prime ℓ divides the conductor of E if and only if $S_\ell(\mathbb{Q}) \subseteq S(E)$. In particular, the above observation implies that η_E is a root of unity of order prime to the conductor of E. The field $\mathbb{Q}(\eta_E)$ is therefore an unramified extension of $\mathbb{Q}$ so that $\eta_E \in \{\pm 1\}$ and hence $\eta_E^2 = 1$. This proves both claims (i) and (ii) in this case.

In the remainder of the argument, we can therefore assume that K is an imaginary quadratic field. In this case we fix a field $E \in \Omega_K$ and a non-archimedean place v of K that ramifies in E and write p for the residue characteristic of v. We claim that η_E is a root of unity of order prime to p.

If, firstly, v is the unique p-adic place of K, then $S_p(K) = \{v\} \subseteq S(E)$ and so this claim follows directly from the observation made above. We therefore assume that $S(E)$ does not contain $S_p(K)$ and hence that $S_p(K) = \{v, v'\}$ with $v' \neq v$.

We write K_∞ for the unique $\mathbb{Z}_p$-extension of K that is ramified only at v. For each field $F \in \Omega_K$, we then set $F_\infty := FK_\infty$ and, for each natural number n, we write F_n/F for the unique subfield of F_∞/F that has degree p^n over F.

We now fix a field F that contains E and is such that $S(F) = S(E) \cup \{v'\}$. Then for each n the set $S(F_n)$ contains $S_p(K)$ and so, by the argument above, the element η_E is a root of unity of order prime to p. In addition, since $S_{\mathrm{r}}(F_n/F)$ does not contain $S_\ell(F)$ for any prime $\ell \neq p$, the prime-to-p torsion subgroups of the groups $F_n^\times$ are of bounded order and so there exists a natural number t that is prime to p and such that $\eta_{F_n}^t = 1$ for all n. Since $v \in S(E)$, the given distribution relations then imply that for each n one has

$$(1 - \sigma_{v'}) \cdot (\eta_{E_n}^t) = \mathrm{N}_{F_n/E_n}(\eta_{F_n}^t) = 1.$$

Hence, since $\sigma_{v'}$ generates an open subgroup of $\mathrm{Gal}(E_\infty/E)$, there exists a natural number m such that $\eta_{E_n}^t \in E_m^\times$ for all $n > m$. For all such n the distribution relations then further imply that

$$\eta_E^t = \mathrm{N}_{E_n/E}(\eta_{E_n}^t) = \mathrm{N}_{E_m/E}(\eta_{E_n}^t)^{p^{n-m}} \in (E^\times)^{p^{n-m}}.$$

In particular, since this containment is true for all $n > m$, the element η_E^t, and hence also η_E itself, is a root of unity of order prime to p, as claimed.

At this stage we know that, for each field E in Ω_K, the element η_E is a root of unity whose order is coprime to the residue characteristic of every (non-archimedean) place of K that ramifies in E. This implies that the order of η_E is prime to the conductor of E and hence that the extension $K(\eta_E)/K$ is unramified.

It follows that η_E is a torsion element in the multiplicative group of the Hilbert class field $K(1)$ of K. Since it is well-known (and easily shown) that the order of the torsion subgroup of $K(1)^\times$ divides 12, we have now proved claims (i) and (ii) for the field K. □

Remark 2.9. Proposition 2.8(ii) implies $\mathrm{ES}_{\mathbb{Q}}(\mathbb{G}_m)_{\mathrm{tor}}$ is a vector space over the field of two elements. This space is known to be of uncountably infinite dimension, with an explicit basis given by the collection of 'Coleman distributions', as proved by Seo in [10].

3. Adelic Euler systems

3.1. For E in Ω_K we write $\Sigma_{E/K}$ for the set of primes ℓ with $S_\ell(K) \subset S(E)$ and then define a $G_{E/K}$-submodule of U'_E by setting

$$V_E := \begin{cases} U_E, & \text{if } |\Sigma_{E/K}| > 1, \\ U_E^p, & \text{if } \Sigma_{E/K} = \{p\}, \\ \{0\}, & \text{if } \Sigma_{E/K} = \varnothing. \end{cases}$$

For every prime ℓ we write $\pi_{E,\ell}$ for the projection map $\widehat{U'_E} \to U'_{E,\ell}$ and $\iota_{E,\ell}$ for the localization map $U'_E \to U'_{E,\ell}$.

Definition 3.1. *A strict adelic Euler system for $\mathbb{G}_m$ over K is an element u of $\mathrm{ES}_K(\widehat{\mathbb{Z}}(1))$ with the property that for every E in Ω_K the element u_E belongs to the subgroup of $\widehat{U'_E}$ that is defined by the pull-back diagram*

$$\begin{array}{ccc} & & \prod_{\ell\in\Sigma_{E/K}} \iota_{E,\ell}(V_E) \\ & & \big\downarrow (\iota_{E,\ell})_\ell \\ \widehat{U'_E} & \xrightarrow{(\pi_{E,\ell})_\ell} & \prod_{\ell\in\Sigma_{E/K}} U'_{E,\ell}. \end{array} \tag{3}$$

We write $\mathrm{AES}^{\mathrm{str}}_{\mathrm{K}}(\mathbb{G}_\mathrm{m})$ for the collection of all such systems and we endow it with the subspace topology of $\mathrm{ES}_K(\widehat{\mathbb{Z}}(1))$.

An adelic Euler system for $\mathbb{G}_m$ is an element of the topological closure $\mathrm{AES}_K(\mathbb{G}_m)$ in $\mathrm{ES}_K(\widehat{\mathbb{Z}}(1))$ of $\mathrm{AES}^{\mathrm{str}}_{\mathrm{K}}(\mathbb{G}_\mathrm{m})$.

The basic properties of this construction are described by the following result.

Lemma 3.2. $\mathrm{AES}_K(\mathbb{G}_m)$ *is an $\widehat{R_K}$-submodule of* $\mathrm{ES}_{\mathrm{K}}(\widehat{\mathbb{Z}}(1))$ *and contains the image of ϕ_K.*

Proof. For each field E the pullback diagram (3) defines a $G_{E/K}$-submodule of $\widehat{U'_E}$ and so $\mathrm{AES}^{\mathrm{str}}_K(\mathbb{G}_m)$ is an R_K-submodule of $\mathrm{ES}_K(\widehat{\mathbb{Z}}(1))$. Thus, since R_K is dense in $\widehat{R_K}$ and the natural function $R_K \times \mathrm{AES}^{\mathrm{str}}_K(\mathbb{G}_m) \to \mathrm{ES}_K(\widehat{\mathbb{Z}}(1))$ is continuous, the $\widehat{R_K}$-module generated by $\mathrm{AES}_K(\mathbb{G}_m)$ is contained in the topological closure of $\mathrm{AES}^{\mathrm{str}}_K(\mathbb{G}_m)$ and hence in $\mathrm{AES}_K(\mathbb{G}_m)$. It follows that $\mathrm{AES}_K(\mathbb{G}_m)$ is an $\widehat{R_K}$-submodule of $\mathrm{ES}_K(\widehat{\mathbb{Z}}(1))$, as claimed.

To prove the second claim it is enough to prove that the image under ϕ_K of any u in $\mathrm{ES}_K(\mathbb{G}_m)$ is a strict adelic Euler system, or equivalently that for every E in Ω_K the element $\phi_K(u)_E$ belongs to the submodule of $\widehat{U'_E}$ defined by the pullback (3). This follows from Lemma 2.4 and the fact that if $S_\ell(K) \subseteq S(E)$, then the explicit definition of ϕ_K implies that $\pi_{E,\ell}(\phi_K(u)_E)$ is equal to $\iota_{E,\ell}(u_E)$. □

3.2. In this section we adapt techniques and results of Seo and the first author in [1] to determine the structure of the $\widehat{R_K}$-module $\mathrm{AES}_K(\mathbb{G}_m)$ in the case that $K = \mathbb{Q}$.

In order to state the main result we regard $\mathbb{Q}^c$ as a subfield of $\mathbb{C}$ (as in Example 2.2) and write τ both for the element of $G_{\mathbb{Q}^c/\mathbb{Q}}$ and the associated element of $R_\mathbb{Q}$ that is induced by complex conjugation. We also use the fact that $\widehat{\mathbb{Z}}(1)$ is naturally a module over $\widehat{R_\mathbb{Q}}$.

Theorem 3.3. *The $\widehat{R_\mathbb{Q}}$-module $\mathrm{AES}_\mathbb{Q}(\mathbb{G}_m)$ of adelic Euler systems over $\mathbb{Q}$ is cyclic, with generator $\phi_\mathbb{Q}(\eta^{\mathrm{cyc}})$. In particular, the image of $\phi_\mathbb{Q}$ is dense in $\mathrm{AES}_\mathbb{Q}(\mathbb{G}_m)$ and there exists a canonical short exact sequence of $\widehat{R_\mathbb{Q}}$-modules*

$$0 \to \widehat{\mathbb{Z}}(1) \to \mathrm{AES}_\mathbb{Q}(\mathbb{G}_m) \to \widehat{R_\mathbb{Q}}(1+\tau) \to 0.$$

Remark 3.4. This result relies on the existence of the Euler system η^{cyc}. However, as a possible generalization, one could ask whether $\mathrm{im}(\phi_K)$ is dense in $\mathrm{AES}_K(\mathbb{G}_m)$ for all K?

The proof of Theorem 3.3 will occupy the rest of this section. For brevity we shall in the sequel often omit the subscript $\mathbb{Q}$ from each of the notations $\mathrm{ES}_\mathbb{Q}(\mathbb{G}_m), \mathrm{AES}_\mathbb{Q}(\mathbb{G}_m), R_\mathbb{Q}$, $\widehat{R_\mathbb{Q}}$, $\phi_\mathbb{Q}$ and $\Omega_\mathbb{Q}$.

As a first step, we note Proposition 2.8(i) combines with Lemma 3.2 to imply that ϕ gives rise to a commutative diagram of tautological short exact sequences of R-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{AES}(\mathbb{G}_m)^{\tau=-1} & \xrightarrow{\subset} & \mathrm{AES}(\mathbb{G}_m) & \xrightarrow{\eta\mapsto\eta^{1+\tau}} & \mathrm{AES}(\mathbb{G}_m)^{1+\tau} & \longrightarrow & 0 \\ & & \uparrow & & \phi\uparrow & & \uparrow & & \\ 0 & \longrightarrow & (\mathrm{ES}(\mathbb{G}_m)_{\mathrm{tf}})^{\tau=-1} & \xrightarrow{\subset} & \mathrm{ES}(\mathbb{G}_m)_{\mathrm{tf}} & \xrightarrow{\eta\mapsto\eta^{1+\tau}} & (\mathrm{ES}(\mathbb{G}_m)_{\mathrm{tf}})^{1+\tau} & \longrightarrow & 0 \end{array} \tag{4}$$

in which the first two vertical arrows are injective.

In addition, since $\mathrm{AES}(\mathbb{G}_m)$ is a module over $\widehat{R}$ it decomposes as a direct product over all primes p of $\widehat{R}^p$-modules that we denote by $\mathrm{AES}_p(\mathbb{G}_m)$. In order to proceed we now require the following result, the proof of which we defer until after completing the proof of the present claim.

Proposition 3.5. *The following claims are valid:*

(i) *For each rational prime p the assignment $1+\tau \mapsto \phi(\eta^{\mathrm{cyc}})^{1+\tau}$ induces an isomorphism of $\widehat{R}^p$-modules between $\widehat{R}^p(1+\tau)$ and $\mathrm{AES}_p(\mathbb{G}_m)^{1+\tau}$.*

(ii) *The homomorphisms*

$$(\mathrm{ES}(\mathbb{G}_m)_{\mathrm{tf}})^{\tau=-1} \xrightarrow{\phi} \mathrm{AES}(\mathbb{G}_m)^{\tau=-1} \to \mathrm{ES}(\widehat{\mathbb{Z}}(1))^{\tau=-1}$$

are both bijective.

By using the result of Proposition 3.5(i) for all primes p, we can therefore deduce that the assignment $1+\tau \mapsto \phi(\eta^{\mathrm{cyc}})^{1+\tau}$ induces an isomorphism of $\widehat{R}$-modules of the form

$$\widehat{R}(1+\tau) = \prod_p \widehat{R}^p(1+\tau) \cong \prod_p \mathrm{AES}_p(\mathbb{G}_m)^{1+\tau} = \mathrm{AES}(\mathbb{G}_m)^{1+\tau}. \tag{5}$$

Given the exactness of the upper row of (4), it follows that, for every η in $\mathrm{AES}(\mathbb{G}_m)$, there exists an element $\lambda = \lambda_\eta$ of $\widehat{R}$ such that $\eta \cdot \phi(\eta^{\mathrm{cyc}})^\lambda$ belongs to $\mathrm{AES}(\mathbb{G}_m)^{\tau=-1}$.

Hence, if we can prove that the $\widehat{R}$-module $\mathrm{AES}(\mathbb{G}_m)^{\tau=-1}$ is generated by $\phi(\eta^{\mathrm{cyc}})^{1-\tau}$ and naturally isomorphic to $\widehat{\mathbb{Z}}(1)$, then both of the claimed results will follow (with the required exact sequence obtained by combining the isomorphism (5) with the upper row of (4)).

The key point now is to note that, after taking account of the identification described in Remark 2.3, claims (i) and (ii) of [1, Thm. 4.1] imply that $(\mathrm{ES}(\mathbb{G}_m)_{\mathrm{tf}})^{\tau=-1}$ is generated over R by $(\eta^{\mathrm{cyc}})^{1-\tau}$ and is an $\widehat{R}$-module that is naturally isomorphic to $\widehat{\mathbb{Z}}(1)$.

The result now follows upon appealing to the bijectivity of the first vertical homomorphism in (4) that is proved in Proposition 3.5(ii).

Proof of Proposition 3.5(i). We write $\mathbb{N}^*$ for the set of natural numbers that are not congruent to 2 modulo 4. For n in $\mathbb{N}^*$ we set $R_n := \mathbb{Z}[G_{\mathbb{Q}(n)/\mathbb{Q}}]$ and $U'_n := U'_{\mathbb{Q}(n)}$ and write $D(n)$ for the R_n-submodule of U'_n generated by $1-\zeta_n$.

For each natural number m we write $\mathbb{N}(m)$ for the subset of $\mathbb{N}^*$ comprising multiples of m. We then define $\mathcal{V}_p^{\mathrm{d}}$ to be the subgroup of $\prod_{m\in\mathbb{N}(p)} D(m)_p^{1+\tau}$ comprising all elements $(x_m)_m$ with the property that for any m in $\mathbb{N}(p)$ and any prime ℓ such that $m\ell$ belongs to $\mathbb{N}^*$ one has

$$\mathrm{N}_{\mathbb{Q}(m\ell)/\mathbb{Q}(m)}(x_{m\ell}) = \begin{cases} x_m, & \text{if } \ell \text{ divides } m,\\ x_m^{1-\sigma_\ell}, & \text{otherwise.}\end{cases}$$

Then $\phi(\eta^{\mathrm{cyc}})^{1+\tau}$ determines an element $\varepsilon_p^{\mathrm{cyc}}$ of $\mathcal{V}_p^{\mathrm{d}}$ and, by [1, Prop. 5.3(i)], the assignment $(1+\tau)\mapsto\varepsilon_p^{\mathrm{cyc}}$ induces an isomorphism of $\widehat{R}^p$-modules between $\widehat{R}^p(1+\tau)$ and $\mathcal{V}_p^{\mathrm{d}}$.

To deduce the claimed result from this fact we claim it is enough to prove the following: if for each n in $\mathbb{N}^*$ one writes $\overline{D(n)}$ for the image of $D(n)$ under the diagonal map into $\prod_{\ell\mid n} U'_{n,\ell}$, then for every strict adelic Euler system u one has

$$u_n \in \mathbb{Z}_{(n)}\otimes_{\mathbb{Z}}\overline{D(n)} \subset \prod_{\ell\mid n} U'_{n,\ell}, \tag{6}$$

where we set $u_n := u_{\mathbb{Q}(n)}$ and write $\mathbb{Z}_{(n)}$ for the intersection over all prime divisors ℓ of n of the localization $\mathbb{Z}_{(\ell)}$ of $\mathbb{Z}$ at ℓ.

Indeed, if we assume for the moment that this is valid, then the collection $(u_{n,p}^{1+\tau})_{n\in\mathbb{N}(p)}$ belongs to $\mathcal{V}_p^{\mathrm{d}}$ and so the above observation implies that the projection of $u^{1+\tau}$ to $\mathrm{AES}_p(\mathbb{G}_m)$ is contained in $\widehat{R}^p\cdot\phi(\eta^{\mathrm{cyc}})^{1+\tau}$. Since $\widehat{R}^p\cdot\phi(\eta^{\mathrm{cyc}})^{1+\tau}$ is a quotient of the compact space $\widehat{R}^p$ it is itself compact. Moreover, $\mathrm{ES}(\mathbb{Z}_p(1))$ is a Hausdorff space and so we must have that $\widehat{R}^p\cdot\phi(\eta^{\mathrm{cyc}})^{1+\tau}$ is closed inside $\mathrm{ES}(\mathbb{Z}_p(1))$. These two facts taken together imply that $\widehat{R}^p\cdot\phi(\eta^{\mathrm{cyc}})^{1+\tau}$ contains $\mathrm{AES}_p(\mathbb{G}_m)^{1+\tau}$.

On the other hand, $\mathrm{AES}_p(\mathbb{G}_m)$ contains $\widehat{R}^p\cdot\phi(\eta^{\mathrm{cyc}})$ since $\phi(\eta^{\mathrm{cyc}})$ is an adelic Euler system. Hence one has $\mathrm{AES}_p(\mathbb{G}_m)^{1+\tau} = \widehat{R}^p\cdot\phi(\eta^{\mathrm{cyc}})^{1+\tau}$ and this implies the stated result since the modules $\widehat{R}^p\cdot\phi(\eta^{\mathrm{cyc}})^{1+\tau}$ and $\widehat{R}^p\cdot\varepsilon_p^{\mathrm{cyc}} = \mathcal{V}_p^{\mathrm{d}}$ are isomorphic.

Turning now to the containment (6) we note that, since no prime divisor of n is invertible in $\mathbb{Z}_{(n)}$, the argument of Seo in [10, Th. 2.4] implies it is enough to prove that for all n one has $u_n\in\mathbb{Z}_{(n)}\otimes_{\mathbb{Z}} C'(n)$. Here $C'(n)$ denotes the image inside $\prod_{\ell\mid n} U'_{n,\ell}$ of the R_n-module generated by the elements $1-\zeta_m$ as m runs over all divisors of n that are greater than one.

In addition, as $n \not\equiv 2 \pmod 4$, the element $-\zeta_n = (1-\zeta_n)^{1-\tau}$ of $C'(n)$ is a generator of the torsion subgroup W_n of $\mathbb{Q}(n)^\times$ and so it is enough to prove that for every n the image of u_n in $\mathbb{Q}\otimes_\mathbb{Z}\mathbb{Q}(n)^\times$ belongs to $\mathbb{Z}_{(n)}\otimes_\mathbb{Z} C'(n)_{\rm tf}$. Since $\mathbb{Z}_{(n)}\otimes_\mathbb{Z} C'(n)_{\rm tf}$ is the intersection of $\mathbb{Z}_{(p)}\otimes_\mathbb{Z} C'(n)_{\rm tf}$ as p runs over all primes that divide n we are thereby reduced to proving that for each such p one has in $U'_{n,p}$ a containment

$$u_n \in C'(n)_p. \tag{7}$$

We shall prove this by adapting an argument used in [1, §3.3, §3.4] (which itself relies heavily on results of Greither in [6]).

We set $p^* = p$ if p is odd and $p^* = 4$ if $p = 2$. For n in $\mathbb{N}(p^*)$ we define $\mathcal{E}_n$ to be the $\widehat{R}^p$-submodule of $\mathrm{ES}(\mathbb{Z}_p(1))$ comprising elements η with the property that for every non-negative integer t the element η_{np^t} belongs to the submodule $U_{np^t,p}$ of $U'_{np^t,p}$. We then define an $R_{np^t,p}$-submodule of $U_{np^t,p}$ by setting

$$\mathcal{E}(np^t)_p := C(np^t)_p + R_{np^t,p}\cdot\{\eta_{\mathbb{Q}(np^t)} : \eta\in\mathcal{E}_n\},$$

where $C(np^t)_p$ denotes the module $U_{np^t,p}\cap C'(np^t)_p$ of p-adic cyclotomic units.

Now the approach of [1, §3.2.2] shows that each element η in $\mathcal{E}_n$ gives rise to a function from the set of natural numbers congruent to one modulo n to the pro-p completion of $\mathbb{Q}^{c,\times}$ that satisfies (the natural p-adic analogues of) each of the properties $(\mathrm{ES}_1), (\mathrm{ES}_2), (\mathrm{ES}_3)$ and $(\mathrm{ES}_4)_p$ that are listed in loc. cit. One can therefore mimic the argument proving [1, Prop. 3.3] to prove the following result.

Proposition 3.6. *For any n in $\mathbb{N}(p^*)$ the natural inclusion map*

$$\varprojlim_{a\geq 0} C(np^a)_p \to \varprojlim_{a\geq 0}\mathcal{E}(np^a)_p \tag{8}$$

is bijective, where in both cases the transition maps are induced by the norms $\mathrm{N}_{\mathbb{Q}(np^{a+1})/\mathbb{Q}(np^a)}$.

We can now use this fact to derive the required inclusion (7). If, firstly, n is divisible by p^* but is not a power of p, then by the defining property of strict adelic Euler systems, each element $u_{\mathbb{Q}(np^a)}$ belongs to $U_{np^a,p}$ and so $(u_{\mathbb{Q}(np^a)})_a$ belongs to $\varprojlim_{a\geq 0}\mathcal{E}(np^a)_p$. From the bijectivity of (8) we can therefore deduce that u_n belongs to $C(n)_p$ and hence to $C'(n)_p$, as required.

On the other hand, if n is a power of p which is divisible by p^*, then the containment (7) can also be deduced from the bijectivity of (8) in the following way. Set $a_p := 2$ if $p = 2$ and $a_p = 0$ if $p \neq 2$. Choose an automorphism γ of $G_{\mathbb{Q}^c/\mathbb{Q}(p^{a_p})}$ that projects to give a topological generator of $G_{\mathbb{Q}(p^\infty)/\mathbb{Q}(p^{a_p})}$ and define the strict adelic Euler system $u_\gamma = u^{\gamma-1}$.

If for any $a\in\mathbb{N}$ we write Y_{np^a} for the free abelian group on the set of p-adic primes of $\mathbb{Q}(np^a)$ then we have an exact sequence of R_{np^a}-modules

$$0\to U_{np^a}\to U'_{np^a}\to Y_{np^a}.$$

In particular, one knows that $u_{\gamma,\mathbb{Q}(np^a)} = u_{\mathbb{Q}(np^a)}^{\gamma_{np^a}-1} \in U_{np^a,p}$. Here γ_{np^a} denotes the image of γ inside G_{np^a} where for any $m\in\mathbb{N}$ we set $G_m := \mathrm{Gal}(\mathbb{Q}(m)/\mathbb{Q})$.

As such, the system u_γ belongs to $\mathcal{E}_n$ and so the containment (8) implies that there exists $r_n \in R_{n,p}$ with the property that

$$u_{\mathbb{Q}(n)}^{\gamma_n-1} = (1-\zeta_n)^{r_n}. \tag{9}$$

Assume first that p is odd and write $\mathbb{Q}(n)^+$ for the maximal real subfield of $\mathbb{Q}(n)$. Then, since $\mathrm{N}_{\mathbb{Q}(n)/\mathbb{Q}}(u_{\mathbb{Q}(n)}^{\gamma_n-1}) = 1$ and the annihilator of $(1-\zeta_n)^{1+\tau}$ in $\mathbb{Z}_p[G_{\mathbb{Q}(n)^+/\mathbb{Q}}]$ vanishes (see, for example, Lemma 4.3(ii) below), the equality (9) implies r_n belongs to the kernel of the natural projection map $R_{n,p} = \mathbb{Z}_p[G_n] \to \mathbb{Z}_p[G_{\mathbb{Q}(n)^+/\mathbb{Q}}]$. Since this kernel is equal to $R_{n,p}(\tau-1) \subseteq R_{n,p}(\gamma_n-1)$, one therefore has $r_n = (\gamma_n-1)\cdot r'_n$ for an element r'_n of $R_{n,p}$. Hence, in this case, we are reduced to showing that the quotient

$$x_p := u_{\mathbb{Q}(n)}/(1-\zeta_n)^{r'_n}$$

is an element of $C'(n)_p$.

In a similar way, if $p=2$ (so $p^{a_p}=4$), then (9) implies that

$$1 = \mathrm{N}_{\mathbb{Q}(n)/\mathbb{Q}(4)}((1-\zeta_n)^{r_n}) = (1-\zeta_4)^{r_n}.$$

Since the annihilator of $1-\zeta_4$ in $R_{4,2}$ is generated over $\mathbb{Z}_2$ by the image of $4(1-\tau)$, and γ_n generates $G_{\mathbb{Q}(n)/\mathbb{Q}(4)}$, this equality implies there exist elements c of $\mathbb{Z}_2$ and r'_n of $R_{n,2}$ such that $r_n = 4c(1-\tau) + r'_n(\gamma_n-1)$ in $R_{n,2}$. In addition, there exists an odd integer c' such that $(-\zeta_n)^4 = (-\zeta_n)^{c'(\gamma_n-1)}$ (as γ_n generates $G_{\mathbb{Q}(n)/\mathbb{Q}(4)}$), and therefore

$$\begin{aligned}(1-\zeta_n)^{r_n} &= (1-\zeta_n)^{4c(1-\tau)}(1-\zeta_n)^{r'_n(\gamma_n-1)}\\ &= (-\zeta_n)^{4c}(1-\zeta_n)^{r'_n(\gamma_n-1)} = ((-\zeta_n)^{c'c}(1-\zeta_n)^{r'_n})^{\gamma_n-1}.\end{aligned}$$

In particular, since $-\zeta_n = (1-\zeta_n)^{1-\tau}$ belongs to $C'(n)$, we are again reduced to showing that the element

$$x_2 := u_{\mathbb{Q}(n)}/\big((-\zeta_n)^{c'c}(1-\zeta_n)^{r'_n}\big)$$

belongs to $C'(n)_p$.

We observe next that, for every p, the equality (9) implies that x_p is fixed by γ_n and so belongs to $\mathbb{Q}(p^{a_p})$. In addition, the definition of the adelic system u implies that x is an element of the pro-p completion of the group of p-units of $\mathbb{Q}(p^{a_p})$.

Hence, if $p \neq 2$ then $x = \pm p^b$ for some $b \in \mathbb{Z}_p$ and it only remains to observe that

$$-1 = (1-\zeta_n)^{n(1-\tau)}, \quad \text{and} \quad p = (1-\zeta_n)^{\sum_{g\in G_n} g}$$

are both elements of $C'(n)$.

On the other hand, if $p=2$ we may assume, without loss of generality that $\zeta_4 = i$ so that there exist $a, b \in \mathbb{Z}_2$ such that $x = \zeta_4^a(1-\zeta_4)^b$. Then, since $\zeta_4 = (1-\zeta_4)^{3(1-\tau)}$ and $1-\zeta_4$ both belong to $C'(n)$, it is therefore clear that $x \in C'(n)_p$, as required.

This completes the proof of the stated result. □

Proof of Proposition 3.5(ii). Since the injectivity of the given composite map is implied by that of the second vertical homomorphism in the diagram (4) it is thus enough to fix an arbitrary system η in $\mathrm{ES}(\widehat{\mathbb{Z}}(1))$ with $\eta^{1+\tau} = 1$ and show how to construct an explicit element of $(\mathrm{ES}(\mathbb{G}_m)_{\mathrm{tf}})^{\tau=-1}$ that ϕ sends to η.

At the outset we observe that, since $\eta^{1+\tau} = 1$, the argument of [1, Lem. 4.3] implies that for every field E in Ω the element η_E of $\widehat{U_E}$ has finite order. The system η therefore defines an element η' of the submodule $\prod_{E\in\Omega}\mu_E$ of $\prod_{E\in\Omega}U_E$ and we claim that η' belongs to $\mathrm{ES}(\mathbb{G}_m)$.

To establish this it is enough to show that for every prime p the projection η'_p of η' to the submodule $\prod_{E\in\Omega}\mu_{E,p}$ of $\prod_{E\in\Omega}U_E$ belongs to $\mathrm{ES}(\mathbb{G}_m)$. It is then clearly enough to verify that η'_p satisfies the necessary distribution relation

$$\mathrm{N}_{E/E'}(\eta'_{p,E}) = \left(\prod_{v\in S(E)\setminus S(E')}(1-\sigma_v)\right)\eta'_{p,E'} \tag{10}$$

for all fields $E'\subseteq E$ for which the group $\mu_{E',p}$ is not trivial.

If, firstly, p is odd then, since we are assuming E' contains a non-trivial p-th root of unity, it is ramified at p (so $S_p(\mathbb{Q})\subset S(E')$) and hence, in this case, η'_p validates (1) since it is assumed to satisfy the distribution relation (2).

If $2\notin S(E)\setminus S(E')$ then the claim for $p=2$ can once again be deduced from the relation (2). We are therefore left to consider the case that $p=2$ and $2\in S(E)\setminus S(E')$. In this case $\mu_{2,E'}=\{\pm 1\}$ and so we must show that $\mathrm{N}_{E/E'}(\eta'_{2,E})=1$. To do this, we write m and m' for the conductors of E and E'. Then m' is odd, whilst m is divisible by $4m'$, and so $\mathrm{N}_{E/E'}(\eta'_{2,E})$ is equal to

$$\mathrm{N}_{E/E'}(\mathrm{N}_{\mathbb{Q}(m)/E}(\eta'_{2,\mathbb{Q}(m)})) = \mathrm{N}_{\mathbb{Q}(m')/E'}(\mathrm{N}_{\mathbb{Q}(4m')/\mathbb{Q}(m')}(\mathrm{N}_{\mathbb{Q}(m)/\mathbb{Q}(4m')}(\eta'_{2,\mathbb{Q}(m)}))).$$

Since $\mathrm{N}_{\mathbb{Q}(m)/\mathbb{Q}(4m')}(\eta'_{2,\mathbb{Q}(m)})\in\mu_4$ and μ_4 is contained in $\ker(\mathrm{N}_{\mathbb{Q}(4m')/\mathbb{Q}(m')})$, one therefore has $\mathrm{N}_{E/E'}(\eta'_{2,E})=1$, as required.

At this stage we know that η' defines an element of $\mathrm{ES}(\mathbb{G}_m)$ that is annihilated by $1+\tau$ and so projects to give an element η'' of $(\mathrm{ES}(\mathbb{G}_m)_{\mathrm{tf}})^{\tau=-1}$.

By using the explicit definition of the map $\phi(=\phi_{\mathbb{Q}})$ that is given in the proof of Lemma 2.7, it is then also straightforward to check that ϕ sends η'' to η as required to complete the proof of Proposition 3.5(ii) and therefore also of Theorem 3.3. □

We end this section by deriving an interesting consequence of Theorem 3.3.

Corollary 3.7. $\mathrm{AES}(\mathbb{G}_m)$ *is a saturated subgroup of* $\mathrm{ES}(\widehat{\mathbb{Z}}(1))$.

Proof. For convenience we set

$$\overline{\mathcal{E}} := \mathrm{AES}(\mathbb{G}_m),\ \widetilde{\mathcal{E}} := \mathrm{ES}(\widehat{\mathbb{Z}}(1)),\ \overline{\mathcal{E}}^+ := \mathrm{AES}(\mathbb{G}_m)^{1+\tau} \text{ and } \widetilde{\mathcal{E}}^+ := \mathrm{ES}(\widehat{\mathbb{Z}}(1))^{1+\tau}. \tag{11}$$

Then, since $\widetilde{\mathcal{E}}$ is torsion-free (by Lemma 2.6), the given claim is valid if and only if for each natural number n one has $\overline{\mathcal{E}}\cap\widetilde{\mathcal{E}}^n = \overline{\mathcal{E}}^n$.

Now Proposition 3.5(ii) implies that the $\widehat{R}$-module $\widetilde{\mathcal{E}}^{\tau=-1}$ is generated by $\phi(\eta^{\mathrm{cyc}})^{1-\tau}$ and hence canonically isomorphic to $\widehat{\mathbb{Z}}(1)$. Since $\widehat{\mathbb{Z}}(1), \overline{\mathcal{E}}$ and $\widetilde{\mathcal{E}}$ are all torsion-free, there is therefore for each n an exact commutative diagram of $\widehat{R}$-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{\mathbb{Z}}(1)/n & \xrightarrow{\subset} & \widetilde{\mathcal{E}}/n & \xrightarrow{x\mapsto x^{1+\tau}} & \widetilde{\mathcal{E}}^{+}/n & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \widehat{\mathbb{Z}}(1)/n & \xrightarrow{\subset} & \overline{\mathcal{E}}/n & \xrightarrow{x\mapsto x^{1+\tau}} & \overline{\mathcal{E}}^{+}/n & \longrightarrow & 0 \end{array}$$

in which the two vertical arrows are induced by the natural inclusion $\overline{\mathcal{E}} \subseteq \widetilde{\mathcal{E}}$.

We are required to prove that the first vertical arrow in this exact diagram is injective. The diagram therefore implies, firstly, that it is enough to prove the second vertical arrow is injective, or equivalently that

$$\overline{\mathcal{E}}^{+} \cap (\widetilde{\mathcal{E}}^{+})^{n} = (\overline{\mathcal{E}}^{+})^{n}, \tag{12}$$

and then, secondly, that any element of the left hand side of this claimed equality is of the form $u^{1+\tau}$ for an element u of $\overline{\mathcal{E}} \cap (\widetilde{\mathcal{E}})^{n}$.

Next we note that, since $\widetilde{\mathcal{E}}$ is torsion-free, an induction on the number of prime factors of n reduces us to proving (12) in the case that n is equal to a prime p. Writing $\overline{\mathcal{E}}_p$ and $\widetilde{\mathcal{E}}_p$ for the pro-p components $\mathrm{AES}_p(\mathbb{G}_m)$ and $\mathrm{ES}(\mathbb{Z}_p(1))$ of $\overline{\mathcal{E}}$ and $\widetilde{\mathcal{E}}$, it is therefore enough to prove that if u is any given element of $\overline{\mathcal{E}}_p \cap (\widetilde{\mathcal{E}}_p)^p$, then $u^{1+\tau}$ belongs to $(\overline{\mathcal{E}}_p^{1+\tau})^p$.

To show this we write ε for the projection of $\phi(\eta^{\mathrm{cyc}})$ to $\overline{\mathcal{E}}_p$ and for each L in Ω and each element x of $U'_{L,p}$ we let $\langle x\rangle$ denote the $\mathbb{Z}_p[G_{L/\mathbb{Q}}]$-module generated by x.

Then the first assertion of Theorem 3.3 implies that $u_L \in \langle \varepsilon_L\rangle$ for every L in Ω. We claim that it is enough to demonstrate the existence of a cofinal (with respect to inclusion) set of fields L in Ω that are of conductor divisible by p and such that

$$u_L^{1+\tau} \in \langle \varepsilon_L^{(1+\tau)p}\rangle. \tag{13}$$

Indeed, if this is the case then, since any system in $\overline{\mathcal{E}}_p$ is uniquely determined by its restriction to any such subset of Ω, the explicit isomorphism in Proposition 3.5(i) implies that $u^{1+\tau} \in (\overline{\mathcal{E}}_p^{1+\tau})^p$.

It is moreover enough to show that the containment (13) is true for every field L of the form $\mathbb{Q}(n)$ where the integer n is divisible by p^2 (so that, in particular, if $p=2$ then n is divisible by 4) and also by at least one other prime.

To prove this we fix such an n and use the notation of the proof of Proposition 3.5. Now, by assumption, one has $u=\eta^p$ for some $\eta \in \widetilde{\mathcal{E}}_p$. In addition, since n is not a prime power, for every non-negative integer t the equality

$$\eta_{np^t}^{2p} = \eta_{np^t}^{p(1-\tau)} \cdot \eta_{np^t}^{p(1+\tau)} = (\eta_{np^t}^{1-\tau})^p \cdot u_{np^t}^{1+\tau}$$

combines with the fact $\eta_{np^t}^{1-\tau}$ is a root of unity to imply that η_{np^t} belongs to $U_{np^t,p}$. The bijectivity of (8) therefore implies that $\eta_n \in C(n)_p$ and hence that

$$u_n = \eta_n^p \in \langle \varepsilon_n\rangle \cap (C(n)_p)^p.$$

We next recall that the result [5, Lem. 4.4] of Conrad implies that the quotient of $U_{n,p,\mathrm{tor}} + C(n)_p$ by $U_{n,p,\mathrm{tor}} + \langle \varepsilon_n\rangle$ is torsion-free. From the last displayed containment, we can therefore deduce that $u_n \in (U_{n,p,\mathrm{tor}} + \langle\varepsilon_n\rangle)^p$ and hence also that

$$u_n^{1+\tau} \in (U_{n,p,\mathrm{tor}} + \langle\varepsilon_n\rangle)^{p(1+\tau)} = (\langle\varepsilon_n\rangle)^{p(1+\tau)} = \langle\varepsilon_n^{(1+\tau)p}\rangle,$$

as required. □

4. Euler systems over $\mathbb{Q}$

In this section we derive a consequence of Theorem 3.3 concerning the module of classical Euler systems $\mathrm{ES}(\mathbb{G}_m)$ over $\mathbb{Q}$.

4.1. We abbreviate the restriction homomorphism $\phi_{\mathbb{Q}}$ constructed in Lemma 2.7 to ϕ and then, by a slight abuse of notation, write

$$\widehat{\phi} : \widehat{\mathrm{ES}(\mathbb{G}_m)}_{\mathrm{tf}} \to \mathrm{ES}(\widehat{\mathbb{Z}}(1))$$

for the homomorphism of $\widehat{R}$-modules that is induced by ϕ_{tf} upon passing to profinite completions, and noting that $\mathrm{ES}(\widehat{\mathbb{Z}}(1))$ identifies with its profinite completion.

Theorem 4.1.

(i) $\widehat{\mathrm{ES}(\mathbb{G}_m)}_{\mathrm{tf}} = \ker(\widehat{\phi}) \oplus \widehat{R} \cdot \eta^{\mathrm{cyc}}$.

(ii) $\mathrm{ES}(\mathbb{G}_m) = \mathrm{ES}(\mathbb{G}_m)_{\mathrm{tor}} + R \cdot \eta^{\mathrm{cyc}}$ *if and only if* $\widehat{\phi}$ *is injective.*

Remark 4.2. Taking account of Remark 2.3, the discussion of [1, §3] shows that the equality in Theorem 4.1(ii) is equivalent to a conjecture of Robert Coleman (originating in [3, 4]) concerning the module of circular distributions. On the other hand, the injectivity of $\widehat{\phi}$ can be seen as an analogue of Leopoldt's Conjecture for the module of Euler systems.

The proof of Theorem 4.1 occupies the rest of §4. In this argument we shall continue to use the abbreviations in (11) and also now set

$$\mathcal{C} := R \cdot \eta^{\mathrm{cyc}},\ \mathcal{E} := \mathrm{ES}(\mathbb{G}_m)_{\mathrm{tf}},\ \mathcal{C}^+ := R \cdot (\eta^{\mathrm{cyc}})^{1+\tau} \ \text{ and } \ \mathcal{E}^+ := \mathrm{ES}(\mathbb{G}_m)_{\mathrm{tf}}^{1+\tau}.$$

In addition, we abbreviate the set of fields $\Omega_{\mathbb{Q}}$ to Ω.

4.2. In this subsection we prove an important preliminary result.

For each L in Ω we write L^+ for its maximal real subfield and set $G_L^+ := G_{L^+/\mathbb{Q}}$. We then denote by R_L and R_L^+ the group rings $\mathbb{Z}[G_L]$ and $\mathbb{Z}[G_L^+]$ respectively. In the sequel we will often view, without further explicit reference, elements of $R_L(1+\tau)$ as elements of R_L^+ in the natural way.

We also write I_L for the annihilator in $\mathbb{Z}[G_L^+]$ of the element $(\eta_L^{\mathrm{cyc}})^{1+\tau}$ of $L^{+,\times}$. We remark that since $\widehat{\mathbb{Z}}$ is a flat $\mathbb{Z}$-module there is a canonical identification of $\widehat{I_L}$ with the $\widehat{\mathbb{Z}}[G_L^+]$-annihilator of $(\eta_L^{\mathrm{cyc}})^{1+\tau}$ inside $\widehat{U'_L}$.

We first record the following Lemma concerning the ideal I_L which will prove useful in the sequel:

Lemma 4.3. *For each L there exists an explicitly described idempotent e_L of $\mathbb{Q}[G_L^+]$ that has all of the following properties.*

(i) *I_L is equal to the set $\{x \in \mathbb{Z}[G_L^+] \mid e_L \cdot x = 0\}$.*

(ii) *If $\psi : G_L \to \mathbb{Q}^{c,\times}$ is any homomorphism such that $e_\psi \cdot e_L \neq 0$, then ψ is trivial on the decomposition group of at least one prime divisor of the conductor of L.*

Proof. The claims follow directly from [1, Lem. 2.4] and the explicit description of e_L given in [1, (6)]. □

The following Proposition is the main result of this subsection and will play a key role in the proof of Theorem 4.1.

Proposition 4.4. *The quotient* $\mathrm{ES}(\mathbb{G}_m)/R\cdot\eta^{\mathrm{cyc}}$ *has no non-zero divisible submodule.*

Proof. At the outset we observe that $\mathrm{ES}(\mathbb{G}_m)_{\mathrm{tor}}$ has exponent 2 (by Remark 2.9) and thus cannot have any non-zero divisible subquotients.

Moreover, the result of [1, Thm. 4.1(iii)] implies that the R-module homomorphism

$$\mathcal{E}/\mathcal{C}\to\mathcal{E}^+/\mathcal{C}^+ \tag{14}$$

induced by multiplication by $1+\tau$ is injective. We are thus reduced to showing that $\mathcal{E}^+/\mathcal{C}^+$ has no non-zero divisible submodules.

We write $\mathcal{D}$ for the full-preimage in $\mathcal{E}^+$ of the maximal divisible submodule of $\mathcal{E}^+/\mathcal{C}^+$ and note that it suffices to show that $\mathcal{D}$ is equal to $\mathcal{C}^+$. To this end, observe that since $\mathcal{E}^+/\mathcal{C}^+$ is $\mathbb{Z}$-torsion-free (as a consequence of [1, Th. 1.2]), $\mathcal{D}$ is equal to $\bigcap_{n\geq 1}\mathcal{C}^+\cdot\mathcal{E}^{+,n}$ and hence to the kernel of the natural map $\mathcal{E}^+\to\widehat{\mathcal{E}^+/\mathcal{C}^+}$.

The natural map $\mathcal{E}^+\to\widehat{\mathcal{E}^+}$ is injective and we use this to identify $\mathcal{E}^+$ with a subgroup of $\widehat{\mathcal{E}^+}$. In addition, since $\mathcal{E}^+/\mathcal{C}^+$ is torsion-free, the natural sequence $0\to\widehat{\mathcal{C}^+}\to\widehat{\mathcal{E}^+}\to\widehat{\mathcal{E}^+/\mathcal{C}^+}\to 0$ is exact. It follows that $\mathcal{D}$ is equal to $\mathcal{E}^+\cap\widehat{\mathcal{C}^+}$ (with the intersection taking place in $\widehat{\mathcal{E}^+}$) and so we are reduced to proving that

$$\mathcal{E}^+\cap\widehat{\mathcal{C}^+}=\mathcal{C}^+. \tag{15}$$

Set $\eta:=\eta^{\mathrm{cyc}}$. Then any element of $\widehat{\mathcal{C}^+}$ is of the form η^λ with λ in $\widehat{R}(1+\tau)$. Such an element η^λ belongs to $\mathcal{E}^+$, and hence to $\mathcal{D}$, if and only if $\eta_L^{\lambda_L}$ belongs to the submodule $L^\times$ of $\widehat{L^\times}$ for every L in Ω. In this case, therefore, [1, Rem. 2.6] implies the existence of a natural number n_L such that $(\eta_L^{\lambda_L})^{n_L}$ belongs to the R_L^+-module $\mathcal{C}_L^+$ that is generated by η_L. Thus, since $\eta_L^{\lambda_L}$ belongs to $\widehat{\mathcal{C}_L^+}$ and the quotient $\widehat{\mathcal{C}_L^+}/\mathcal{C}_L^+\cong\mathcal{C}_L^+\otimes_{\mathbb{Z}}\widehat{\mathbb{Z}}/\mathbb{Z}$ is uniquely divisible, it follows that $\eta_L^{\lambda_L}$ belongs to $\mathcal{C}_L^+$ and hence that there exists an element r_L of R_L^+ such that $\lambda_L-r_L\in\widehat{I_L}$.

It therefore remains to show that

$$\widehat{R}(1+\tau)\cap\prod_L(R_L+\widehat{I_L})=R(1+\tau).$$

To show this it suffices to demonstrate that if $\lambda=(\lambda_L)_L$ is any element of $\widehat{R}(1+\tau)$ with the property that $\lambda_L\in R_L+\widehat{I_L}$ for every L in Ω, then in fact one has $\lambda_L\in\mathbb{Q}[G_L^+]$ for every L. To prove this we shall argue by induction on the number of prime factors of the conductor f_L of L.

If, firstly, f_L is a prime power, then the results of Lemma 4.3 imply that I_L vanishes and so the given assumptions imply that λ_L belongs to R_L.

Now assume to be given $n\in\mathbb{N}$ and suppose that for every field $L\in\Omega$ such that f_L is divisible by at most n primes, one has that $\lambda_L\in\mathbb{Q}[G_L^+]$. Fix a field $F\in\Omega$ such that f_F is divisible by $n+1$ primes. Observe that, since $\lambda \in \widehat{R}(1+\tau)$, we have a decomposition

$$\lambda_F = 1 \cdot \lambda_F = \sum_{\psi} e_\psi \lambda_F = \sum_{\psi} e_\psi \lambda_{F_\psi} \tag{16}$$

where the sums range over all homomorphisms $\psi : G_F^+ \to \mathbb{Q}^{c,\times}$ and F_ψ denotes the fixed field of $\ker(\psi)$ in F.

Fix such a homomorphism ψ. If f_{F_ψ} is divisible by at most n primes then, by hypothesis, one has $\lambda_{F_\psi} \in \mathbb{Q}[G_F^+]$. On the other hand, if f_{F_ψ} is divisible by $n+1$ primes then one has

$$e_\psi \lambda_{F_\psi} = e_\psi \lambda_F = e_\psi (r_F + i_F) = e_\psi r_F$$

where $r_F \in R_F$ and $i_F \in \widehat{I_F}$. Here the final equality follows by combining the results in Lemma 4.3 together with the fact that, in this case, ψ cannot be trivial on the decomposition group associated to any of the prime divisors of the conductor of L.

This fact, taken together with the decomposition (16), finishes the proof of the claim and thus the Proposition. □

4.3. We are now ready to prove Theorem 4.1(i).

At the outset we claim that $\mathrm{im}(\widehat{\phi})$ is equal to $\widehat{R} \cdot \phi(\eta^{\mathrm{cyc}})$. Indeed, since $\mathrm{ES}(\widehat{\mathbb{Z}}(1))$ has no non-zero divisible subgroups (as it is contained in a product of finitely generated $\widehat{\mathbb{Z}}$-modules), the same is true of $\mathrm{im}(\phi)$. On the other hand, Lemma 2.6 implies that we may pass to profinite completions to deduce that $\widehat{\mathrm{im}(\phi)} = \mathrm{im}(\widehat{\phi})$. These facts taken together imply that $R \cdot \phi(\eta^{\mathrm{cyc}})$ is contained in $\mathrm{im}(\phi) \subseteq \mathrm{im}(\widehat{\phi})$. Since $\widehat{\phi}$ is a homomorphism of $\widehat{R}$-modules it then follows that $\widehat{R} \cdot \phi(\eta^{\mathrm{cyc}})$ is contained in $\mathrm{im}(\widehat{\phi})$. The claim now follows immediately upon noting that $\mathrm{im}(\widehat{\phi}) \subseteq \widehat{R} \cdot \phi(\eta^{\mathrm{cyc}})$ as a result of Theorem 3.3.

To prove Theorem 4.1(i), therefore, it suffices to exhibit a section for the induced surjective homomorphism $\widehat{\phi} : \widehat{\mathcal{E}} \to \widehat{R} \cdot \phi(\eta^{\mathrm{cyc}})$.

It is thus enough to show that the assignment $\phi(\eta^{\mathrm{cyc}}) \mapsto \eta^{\mathrm{cyc}}$ extends to gives a well-defined homomorphism of $\widehat{R}$-modules $\widehat{R} \cdot \phi(\eta^{\mathrm{cyc}}) \to \widehat{\mathcal{C}} \subseteq \widehat{\mathcal{E}}$, or equivalently, that every element of $\widehat{R}$ that annihilates $\phi(\eta^{\mathrm{cyc}})$ must also annihilate the element η^{cyc} of $\widehat{\mathcal{E}}$.

To do this, first observe that if x is any element of an $\widehat{R}$-module for which there exists an exact sequence of $\widehat{R}$-modules of the form

$$0 \to \widehat{\mathbb{Z}}(1) \to \widehat{R} \cdot x \to \widehat{R}(1+\tau) \to 0 \tag{17}$$

then, since $\mathbb{Q} \otimes_{\mathbb{Z}} \widehat{R}(1+\tau)$ is a projective $\mathbb{Q} \otimes_{\mathbb{Z}} \widehat{R}$-module, one has

$$\mathrm{Ann}_{\widehat{R}}(x) = \mathrm{Ann}_{\widehat{R}}(\mathbb{Q} \otimes_{\mathbb{Z}} \widehat{R} \cdot x) = \mathrm{Ann}_{\widehat{R}}(\mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}(1)) \cap \mathrm{Ann}_{\widehat{R}}(\mathbb{Q} \otimes_{\mathbb{Z}} \widehat{R}(1+\tau)). \tag{18}$$

Now recall from [1, Thm. 1.2] that there is an exact sequence

$$0 \to \widehat{\mathbb{Z}}(1) \to R \cdot \eta^{\mathrm{cyc}} \to R(1+\tau) \to 0.$$

By passing to profinite completions and noting that $R(1+\tau)$ is torsion-free, we obtain an exact sequence of the form (17) with $x = \eta^{\mathrm{cyc}}$.

Hence, by applying the equality (18) to both this exact sequence and that of Theorem 3.3 we can deduce that $\phi(\eta^{\mathrm{cyc}})$ and η have the same annihilators in $\widehat{R}$, as required to complete the proof of claim (i).

Turning to Theorem 4.1(ii), we note that ϕ factors as a composite homomorphism

$$\mathcal{E} \xrightarrow{\phi'} \overline{\mathcal{E}} \xrightarrow{\subseteq} \mathrm{ES}(\widehat{\mathbb{Z}}(1))$$

(by Lemma 3.2) and that the displayed exact sequence in Theorem 3.3 implies that the natural map from $\overline{\mathcal{E}}$ to its profinite completion is bijective. There is therefore a commutative diagram of $\widehat{R}$-modules

(19)
$$\begin{array}{ccccc} \widehat{\mathcal{E}} & \xrightarrow{\widehat{\phi'}} & \overline{\mathcal{E}} & \xrightarrow{\subseteq} & \mathrm{ES}(\widehat{\mathbb{Z}}(1)) \\ {\scriptstyle \iota}\uparrow & \nearrow_{\kappa} & & & \\ \widehat{\mathcal{C}} & & & & \end{array}$$

in which the composite horizontal map is $\widehat{\phi}$, ι is the natural inclusion and κ is the restriction of $\widehat{\phi}'$. Now, the natural identification $\widehat{\mathcal{C}} = \widehat{R} \cdot \eta^{\mathrm{cyc}}$ combines with Theorem 3.3 to imply that κ is bijective. In particular, the commutativity of this diagram induces an isomorphism

$$\ker(\widehat{\phi}) = \ker(\widehat{\phi}') \cong \mathrm{cok}(\iota) = \widehat{\mathcal{E}}/\widehat{\mathcal{C}} \cong \widehat{(\mathcal{E}/\mathcal{C})},$$

where the last isomorphism was observed in the proof of Proposition 4.4. It follows that $\widehat{\phi}$ is injective if and only if $\widehat{(\mathcal{E}/\mathcal{C})}$ vanishes, or equivalently, by virtue of Proposition 4.4, that $\mathcal{E} = \mathcal{C}$.

This completes the proof of Theorem 4.1.

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