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STOCHASTIC CONTROL REPRESENTATIONS FOR PENALIZED BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS*

GECHUN LIANG†

Abstract. This paper shows that penalized backward stochastic differential equation (BSDE), which is often used to approximate and solve the corresponding reflected BSDE, admits both optimal stopping representation and optimal control representation. The new feature of the optimal stopping representation is that the player is allowed to stop at exogenous Poisson arrival times. The convergence rate of the penalized BSDE then follows from the optimal stopping representation. The paper then applies to two classes of equations, namely multidimensional reflected BSDE and reflected BSDE with a constraint on the hedging part, and gives stochastic control representations for their corresponding penalized equations.

Key words. Reflected BSDE, Penalized BSDE, Optimal stopping, Optimal control, Optimal switching, Regime switching

AMS subject classifications. 60H10, 60G40, 93E20.

1. Introduction. El Karoui et al [8] introduced penalized backward stochastic differential equation (*penalized BSDE* for short) to solve reflected backward stochastic differential equation (*reflected BSDE* for short), and they showed that the solution of a reflected BSDE corresponds to the value of a nonlinear optimal stopping time problem. In this paper, our main result is to show that the solution of the associated penalized BSDE also corresponds to the value of some nonlinear optimal stopping time problem, and the parameter λ appearing in the penalized equation is nothing but the intensity of some exogenous Poisson process.

Let $(W_t)_{t\geq 0}$ be a d-dimensional standard Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ satisfying the *usual conditions*, i.e. the filtration $\mathbb{F}$ is right continuous and complete. In El Karoui et al [8], the authors introduced the following reflected BSDE

$$Y_t = \xi + \int_t^T f_s(Y_s, Z_s)ds + \int_t^T dK_s - \int_t^T Z_s dW_s \tag{1.1}$$

under the constraints

$$\text{(Dominating Condition)}: \quad Y_t \geq S_t \text{ for } t \in [0,T],$$

$$\text{(Skorohod Condition)}: \quad \int_0^T (Y_t - S_t)dK_t = 0 \text{ for } K \text{ continuous and increasing,}$$

where the terminal data ξ, the driver $f_s(y,z)$, and the obstacle $(S_t)_{0\leq t\leq T}$ are the given data for the equation. A solution to the reflected BSDE (1.1) is a triplet of $\mathbb{F}$-adapted processes (Y, Z, K), where Z is a kind of *hedging* process, and K is a kind of *local time* process. The equation (1.1) corresponds to a backward Skorohod problem, which in turn gives the *local time* process K a Skorohod representation. See Qian and Xu [27] in this direction.

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†Department of Mathematics, King's College London, London, WC2R 2LS, U.K.. Email adress: gechun.liang@kcl.ac.uk.

On the other hand, as shown in [8], (1.1) also has an interesting interpretation in the sense that its solution is the value of a nonlinear optimal stopping time problem: For any time $t \in [0,T]$, the value of the following optimal stopping time problem

$$y_t = \operatorname*{ess\,sup}_{\tau \in \mathcal{R}(t)} \mathbf{E}\left[\int_t^{\tau \wedge T} f_s(Y_s, Z_s)ds + S_\tau \mathbf{1}_{\{\tau < T\}} + \xi \mathbf{1}_{\{\tau \geq T\}} | \mathcal{F}_t\right], \tag{1.2}$$

where the control set $\mathcal{R}(t)$ is defined as

$$\mathcal{R}(t) = \{\mathbb{F}\text{-stopping time } \tau \text{ for } t \leq \tau \leq T\},$$

is given by the solution to the reflected BSDE (1.1): $y_t = Y_t$ *a.s.*. The optimal stopping time is given by $\tau_t^* = \inf\{s \geq t : Y_s = S_s\} \wedge T$. The nonlinear optimal stopping problem (1.2) is closely related to pricing and hedging American options as shown in El Karoui et al [9].

One way to solve the reflected BSDE (1.1) is to iterate the solution of the corresponding backward Skorohod problem by Picard iteration. The other way, which seems more commonly used in the literature, is to approximate the *local time* process K by

$$K_t^\lambda = \int_0^t \lambda \max\{0, S_s - Y_s^\lambda\}ds,$$

where (Y^λ, Z^λ) is the solution of the following penalized BSDE

$$Y_t^\lambda = \xi + \int_t^T f_s(Y_s^\lambda, Z_s^\lambda)ds + \int_t^T \lambda \max\{0, S_s - Y_s^\lambda\}ds - \int_t^T Z_s^\lambda dW_s. \tag{1.3}$$

Under Assumption 1.1 introduced below, El Karoui et al [8] proved that Y^λ is increasing in λ, and

$$\lim_{\lambda \uparrow \infty} \mathbf{E}\left[\sup_{t\in[0,T]} |Y_t^\lambda - Y_t|^2 + \int_0^T |Z_t^\lambda - Z_t|^2 dt + \sup_{t\in[0,T]} |K_t^\lambda - K_t|^2\right] = 0. \tag{1.4}$$

Our aim is to give stochastic control representations for the penalized BSDE (1.3). Our main result is to prove that the penalized BSDE (1.3) also admits an optimal stopping representation, which will in turn converge to the original optimal stopping time problem (1.2) with convergence rate $\frac{1}{\lambda}$ (see (1.6) and (3.1) below).

We impose the following standard assumption on the data set (ξ, f, S) as in El Karoui [8], so that both (1.1) and (1.3) admit unique solutions.

Assumption 1.1.

- *The terminal data ξ is $\mathbb{L}^2$-square integrable:* $\mathbf{E}[|\xi|^2] < \infty$;
- *The driver $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is uniformly Lipschitz continuous:*

$$|f_t(y,z) - f_t(\bar{y},\bar{z})| \leq C(|y-\bar{y}| + |z-\bar{z}|) \text{ a.s. for some } C > 0,$$

  with $f_t(0,0)$ being $\mathbb{F}$-adapted and $\mathbb{H}^2$-square integrable: $\mathbf{E}\left[\int_0^T |f_t(0,0)|^2 dt\right] < \infty$*;*
- *The obstacle process S is a continuous $\mathbb{F}$-adapted process, and uniformly square integrable:* $\mathbf{E}\left[\sup_{t\in[0,T]} |S_t|^2\right] < \infty$.

In fact, the above conditions could be relaxed. See, for example, Peng and Xu [25] and Lepeltier and Xu [19] extending to RCLL obstacles, and Kobylanski et al [15] and Bayraktar and Song [1] among others extending to the driver $f_s(y,z)$ with quadratic growth in z. However, we only stick with the above standard assumption in this paper. Under the above standard assumption, we have the following representation which is the main result of this paper.

Let $\{T_n\}_{n\geq 0}$ be the arrival times of an independent Poisson process with intensity λ and minimal augmented filtration $\{\mathcal{H}_t\}_{t\geq 0}$. Define $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ and $\mathbb{G} = \{\mathcal{G}_t\}_{t\geq 0}$. Since $T_0 = 0$ and $T_\infty = \infty$, there exists an integer-valued random variable $M < \infty$ such that $T_M \leq T < T_{M+1}$, i.e. $M(\omega) = \sum_{n\geq 0} n\mathbf{1}_{\{T_n(\omega)\leq T<T_{n+1}(\omega)\}}$.

THEOREM 1.2. *Suppose that Assumption 1.1 holds. Denote (Y^λ, Z^λ) as the unique solution to the penalized BSDE (1.3). For any integer $i \geq 1$, define the control set $\mathcal{R}_{T_i}(\lambda)$ as*

$$\mathcal{R}_{T_i}(\lambda) = \{\mathbb{G}\text{-stopping time } \tau \text{ for } \tau(\omega) = T_N(\omega) \text{ where } i \leq N \leq M+1.\}$$

Then conditional on $\{T_{i-1} \leq t < T_i\}$, the value of the following optimal stopping time problem

$$y_t^\lambda = \operatorname*{ess\,sup}_{\tau\in\mathcal{R}_{T_i}(\lambda)} \mathbf{E}\left[\int_t^{\tau\wedge T} f_s(Y_s^\lambda, Z_s^\lambda)ds + S_\tau \mathbf{1}_{\{\tau<T\}} + \xi\mathbf{1}_{\{\tau\geq T\}}|\mathcal{G}_t\right] \tag{1.5}$$

is given by the solution to the penalized BSDE (1.3): $y_t^\lambda = Y_t^\lambda$ a.s.. The optimal stopping time is given by $\tau_{T_i}^* = \inf\{T_N \geq T_i : Y_{T_N}^\lambda \leq S_{T_N}\} \wedge T_{M+1}$.

Note that on $\{T_{i-1} < t < T_i\}$, there exists an $\mathcal{F}_t$-measurable random variable $\tilde{y}_t^\lambda$ such that $\tilde{y}_t^\lambda = y_t^\lambda$, so y_t^λ can also be regarded as $\mathcal{F}_t$-measurable in this situation. On the other hand, the subscript T_i in $\mathcal{R}_{T_i}(\lambda)$ represents the smallest stopping time that is allowed to choose, and λ represents the intensity of the underlying Poisson process.

There are two new features of the optimal stopping time problem (1.5): First, there is a control constraint in the sense that only stopping at Poisson arrival times is allowed; Secondly, the player is not allowed to stop at the initial starting time t. By the convergence (1.4) and Theorem 1.2, the values of the two optimal stopping time problems (1.2) and (1.5) are related by

$$\lim_{\lambda\uparrow\infty} \mathbf{E}\left[\sup_{t\in[0,T]} |y_t^\lambda - y_t|^2\right] = 0. \tag{1.6}$$

Moreover, by using the optimal stopping representation (1.5), we will further establish the convergence rate of (1.6) in Section 3.

The above optimal stopping with Poisson random intervention times was firstly introduced by Dupuis and Wang [7] (generalized by Lempa [17] recently), where they used it to model perpetual American options in a Markovian setting. Since the state space is one dimensional and the time horizon is infinite, they did not even need to introduce any penalized equation. Instead, they worked out two ordinary differential equations (*ODE* for short) defined in continuity region and stopping region respectively. Recently, Liang et al [21] established a connection between such kind of optimal stopping with Poisson random intervention times and dynamic bank run problems. In a Markovian setting, Dai et al [6] intuitively showed that the penalty method for their optimal stopping time problem is closely related to some intensity framework. However, they did not introduce any stochastic control interpretation for their penalty method.

The paper is organized as follows: Theorem 1.2 is proved in Section 2. Then we provide four applications of the optimal stopping representation (1.5) in the following sections. In Section 3 we give the convergence rate of the penalized BSDE (1.3) in a Markovian setting. We also give an optimal control representation for (1.3) in the sense of randomized stopping in Section 4. Then in Section 5, we apply to multidimensional reflected (oblique) BSDE, and give two optimal switching representations for the associated multidimensional penalized BSDEs, one of which is closely related to BSDE with regime switching. In Section 6, we apply to reflected BSDE with a convex constraint on Z (*constrained reflected BSDE* for short), and give an optimal control/optimal stopping representation for the associated penalized BSDE. Finally, Section 7 concludes.

2. Proof of Theorem 1.2. The optimal stopping time problem (1.5) has a constraint on its control set, i.e. the optimal stopping time must be chosen from the arrival times $\{T_n\}_{n\geq 0}$ of the underlying Poisson process. Given the arrival time T_n, by defining pre-T_n σ-field

$$\mathcal{G}_{T_n}=\left\{A\in\bigvee_{s\geq 0}\mathcal{G}_s : A\cap\{T_n\leq s\}\in\mathcal{G}_s \text{ for } s\geq 0\right\}$$

and denoting $\tilde{\mathbb{G}}=\{\mathcal{G}_{T_n}\}_{n\geq 0}$, it is obvious that the problem (1.5) is equivalent to the following discrete optimal stopping time problem (where the control constraint does not appear): Conditional on $\{T_{i-1}\leq t<T_i\}$,

$$y_t^{\lambda}=\operatorname*{ess\,sup}_{N\in\mathcal{N}_i(\lambda)}\mathbf{E}\left[\int_t^{T_N\wedge T}f_s(Y_s^{\lambda},Z_s^{\lambda})ds+S_{T_N}\mathbf{1}_{\{T_N<T\}}+\xi\mathbf{1}_{\{T_N\geq T\}}|\mathcal{G}_t\right], \tag{2.1}$$

where

$$\mathcal{N}_i(\lambda)=\left\{\tilde{\mathbb{G}}\text{-stopping time } N \text{ for } i\leq N\leq M+1.\right\}$$

Once again, the subscript i in $\mathcal{N}_i(\lambda)$ represents the smallest stopping time that is allowed to choose, and λ represents the intensity of the underlying filtration $\tilde{\mathbb{G}}$. Note that (2.1) is a discrete optimal stopping problem, as the player is allowed to stop at a sequence of integers $i,i+1,\ldots,M+1$. The optimal stopping time is then some integer-valued random variable N_i^* such that $N_i^*=\inf\{N\geq i: Y_{T_N}^{\lambda}\leq S_{T_N}\}\wedge(M+1)$. In the following, we will work on the optimal stopping time problem with the form (2.1).

2.1. Representation for Linear Case. In this section, we consider the case where the driver $f_s(y,z)$ is independent of (y,z), and simply write it as f_s in such a situation. Note that the corresponding reflected BSDE (1.1) becomes linear, and so is the optimal stopping representation (1.2).

LEMMA 2.1. *Suppose that Assumption 1.1 holds, and that $f_s(y,z)=f_s$. Then conditional on $\{T_{i-1}\leq t<T_i\}$, the solution of the penalized BSDE (1.3) is the unique solution of the following recursive equation*

$$Y_t^{\lambda}=\mathbf{E}\left[\int_t^{T_i\wedge T}f_sds+\max\left\{S_{T_i},Y_{T_i}^{\lambda}\right\}\mathbf{1}_{\{T_i\leq T\}}+\xi\mathbf{1}_{\{T_i>T\}}|\mathcal{G}_t\right]. \tag{2.2}$$

Proof. We introduce the dual equation for the penalized BSDE (1.3),

$$\alpha_t = 1 - \int_0^t \lambda \alpha_s ds, \text{ for } t \in [0, T].$$

Applying Itô's formula to $\alpha_t Y_t^\lambda$, we obtain

$$\alpha_t Y_t^\lambda = \alpha_T Y_T^\lambda + \int_t^T \alpha_s \left(f_s + \lambda \max\{S_s, Y_s^\lambda\}\right) ds - \int_t^T \alpha_s Z_s^\lambda dW_s,$$

so that

$$\begin{aligned} Y_t^\lambda &= \frac{\alpha_T}{\alpha_t}\xi + \int_t^T \frac{\alpha_s}{\alpha_{T_n}} \left(f_s + \lambda \max\left\{S_s, Y_s^\lambda\right\}\right) ds - \int_t^T \frac{\alpha_s}{\alpha_t} Z_s^\lambda dW_s \\ &= \mathbf{E}\left[e^{-\lambda(T-t)}\xi + \int_t^T e^{-\lambda(s-T_n)} \left(f_s + \lambda \max\left\{S_s, Y_s^\lambda\right\}\right) ds | \mathcal{F}_t \right]. \end{aligned} \tag{2.3}$$

Next, conditional on $\{T_{i-1} < t < T_i\}$, we use the conditional density $\lambda e^{-\lambda(x-t)}dx$ of $T_i - t$ to calculate (2.3):

$$\begin{aligned} \mathbf{E}\left[\int_t^{T_i \wedge T} f_s ds | \mathcal{G}_t \right] &= \mathbf{E}\left[\int_t^{T_i \wedge T} f_s ds | \mathcal{F}_t \right] \\ &= \mathbf{E}\left[e^{-\lambda(T-t)} \int_t^T f_s ds + \int_t^T \lambda e^{-\lambda(x-t)} (\int_t^x f_u du) dx | \mathcal{F}_t \right] \\ &= \mathbf{E}\left[\int_t^T e^{-\lambda(s-t)} f_s ds | \mathcal{F}_t \right], \end{aligned}$$

where we used integration by parts in the second equality. Similarly, we have that

$$\begin{aligned} &\mathbf{E}\left[\max\left\{S_{T_i}, Y_{T_i}^\lambda\right\} \mathbf{1}_{\{T_i \le T\}} + \xi \mathbf{1}_{\{T_i > T\}} | \mathcal{G}_t \right] \\ &= \mathbf{E}\left[\int_t^T \lambda e^{-\lambda(s-t)} \max\left\{S_s, Y_s^\lambda\right\} ds + e^{-\lambda(T-t)}\xi | \mathcal{F}_t \right]. \end{aligned}$$

Hence, we obtain (2.2) on $\{T_{i-1} < t < T_i\}$ by plugging the above two expressions into (2.3).

It is similar to obtain (2.2) on T_{i-1}:

$$Y_{T_{i-1}}^\lambda = \mathbf{E}\left[\int_{T_{i-1}}^{T_i \wedge T} f_s ds + \max\left\{S_{T_i}, Y_{T_i}^\lambda\right\} \mathbf{1}_{\{T_i \le T\}} + \xi \mathbf{1}_{\{T_i > T\}} | \mathcal{G}_{T_{i-1}} \right].$$

Since the recursive equation (2.2) obviously admits a unique solution, Y_t^λ is then the unique solution to (2.2). □

As a direct consequence of Lemma 2.1, if we define $\widehat{Y}^\lambda = \max\left\{S, Y^\lambda\right\}$, then $\widehat{Y}^\lambda$ satisfies the following recursive equation: For $1 \le i \le M+1$,

$$\widehat{Y}_{T_{i-1}}^\lambda = \max\left\{ S_{T_{i-1}}, \mathbf{E}\left[\int_{T_{i-1}}^{T_i \wedge T} f_s ds + \widehat{Y}_{T_i}^\lambda \mathbf{1}_{\{T_i \le T\}} + \xi \mathbf{1}_{\{T_i > T\}} | \mathcal{G}_{T_{i-1}} \right] \right\}, \tag{2.4}$$

which admits a unique solution, as we can calculate its solution backwards in a recursive way.

In the following, we show that $\widehat{Y}^{\lambda}_{T_{i-1}}$ is the value of another optimal stopping problem. Introduce an auxiliary optimal stopping problem associated with (2.1):

$$\widehat{y}^{\lambda}_{T_{i-1}} = \operatorname*{ess\,sup}_{N\in\mathcal{N}_{i-1}(\lambda)} \mathbf{E}\left[\int_{T_{i-1}}^{T_N\wedge T} f_s ds + S_{T_N}\mathbf{1}_{\{T_N<T\}} + \xi\mathbf{1}_{\{T_N\geq T\}}|\mathcal{G}_{T_{i-1}}\right], \tag{2.5}$$

where

$$\mathcal{N}_{i-1}(\lambda) = \left\{\tilde{\mathbb{G}}\text{-stopping time } N \text{ for } i-1\leq N\leq M+1.\right\}$$

The difference between (2.5) and (2.1) starting from T_{i-1} is that the former is allowed to stop at the initial starting time T_{i-1}, while the latter not.

Lemma 2.2. *Suppose that Assumption 1.1 holds, and that $f_s(y,z)=f_s$. For any integer $1\leq i\leq M+1$, the value $\widehat{y}^{\lambda}_{T_{i-1}}$ of the auxiliary optimal stopping time problem (2.5) satisfies the recursive equation (2.4):*

$$\widehat{y}^{\lambda}_{T_{i-1}} = \max\left\{S_{T_{i-1}}, \mathbf{E}\left[\int_{T_{i-1}}^{T_i\wedge T} f_s ds + \widehat{y}^{\lambda}_{T_i}\mathbf{1}_{\{T_i<T\}} + \xi\mathbf{1}_{\{T_i\geq T\}}|\mathcal{G}_{T_{i-1}}\right]\right\}.$$

The optimal stopping time is given by $\widehat{N}^*_{i-1} = \inf\{N\geq i-1 : \widehat{y}^{\lambda}_{T_N}\leq S_{T_N}\}\wedge(M+1)$. Hence, $\widehat{Y}^{\lambda}_{T_{i-1}} = \widehat{y}^{\lambda}_{T_{i-1}}$ a.s..

Proof. Define the following processes

$$\bar{y}^{\lambda}_t = \widehat{y}^{\lambda}_t + \int_0^t f_s ds;$$
$$\bar{S}_t = S_t + \int_0^t f_s ds;$$
$$\bar{\xi} = \xi + \int_0^T f_s ds.$$

Since $T_M\leq T<T_{M+1}$, the auxiliary optimal stopping problem (2.5) is equivalent to

$$\begin{aligned}\bar{y}^{\lambda}_{T_{i-1}} &= \operatorname*{ess\,sup}_{N\in\mathcal{N}_{i-1}(\lambda)} \mathbf{E}\left[\bar{S}_{T_N}\mathbf{1}_{\{T_N<T\}} + \bar{\xi}\mathbf{1}_{\{T_N\geq T\}}|\mathcal{G}_{T_{i-1}}\right]\\ &= \operatorname*{ess\,sup}_{N\in\mathcal{N}_{i-1}(\lambda)} \mathbf{E}\left[\bar{S}_{T_N}\mathbf{1}_{\{i-1\leq N\leq M\}} + \bar{\xi}\mathbf{1}_{\{N=M+1\}}|\mathcal{G}_{T_{i-1}}\right].\end{aligned}$$

We claim that

$$\begin{cases} \bar{y}^{\lambda}_{T_M} = \max\left\{\bar{S}_{T_M}, \mathbf{E}\left[\bar{\xi}|\mathcal{G}_{T_M}\right]\right\}, \\ \bar{y}^{\lambda}_{T_n} = \max\left\{\bar{S}_{T_n}, \mathbf{E}\left[\bar{y}^{\lambda}_{T_{n+1}}|\mathcal{G}_{T_n}\right]\right\}, \quad \text{for } i-1\leq n\leq M-1. \end{cases} \tag{2.6}$$

If (2.6) holds, then

$$\begin{aligned}\bar{y}^{\lambda}_{T_{i-1}} &= \max\left\{\bar{S}_{T_{i-1}}, \mathbf{E}\left[\bar{y}^{\lambda}_{T_i}\mathbf{1}_{\{i\leq M\}} + \bar{\xi}\mathbf{1}_{\{i>M\}}|\mathcal{G}_{T_{i-1}}\right]\right\}\\ &= \max\left\{\bar{S}_{T_{i-1}}, \mathbf{E}\left[\bar{y}^{\lambda}_{T_i}\mathbf{1}_{\{T_i<T\}} + \bar{\xi}\mathbf{1}_{\{T_i\geq T\}}|\mathcal{G}_{T_{i-1}}\right]\right\},\end{aligned}$$

which is the recursive equation (2.4) if we express the above equation in terms of $\widehat{y}^{\lambda}$, S and ξ.

Therefore, in order to complete the proof, we only need to show (2.6). Indeed, for $n=M$,

$$\begin{aligned}\bar{y}^{\lambda}_{T_M} &= \operatorname*{ess\,sup}_{N\in\mathcal{N}_M(\lambda)} \mathbf{E}\left[\bar{S}_{T_N}\mathbf{1}_{\{N=M\}}+\bar{\xi}\mathbf{1}_{\{N=M+1\}}|\mathcal{G}_{T_M}\right]\\ &=\max\left\{\bar{S}_{T_M},\mathbf{E}\left[\bar{\xi}|\mathcal{G}_{T_M}\right]\right\}.\end{aligned}$$

In general, for $i-1\leq n\leq M-1$,

$$\begin{aligned}\bar{y}^{\lambda}_{T_n} &= \operatorname*{ess\,sup}_{N\in\mathcal{N}_n(\lambda)} \mathbf{E}\left[\bar{S}_{T_N}\mathbf{1}_{\{n\leq N\leq M\}}+\bar{\xi}\mathbf{1}_{\{N=M+1\}}|\mathcal{G}_{T_n}\right]\\ &= \operatorname*{ess\,sup}_{N\in\mathcal{N}_n(\lambda)} \mathbf{E}\left[\mathbf{E}\left[\bar{S}_{T_N}\mathbf{1}_{\{n\leq N\leq M\}}+\bar{\xi}\mathbf{1}_{\{N=M+1\}}|\mathcal{G}_{T_{n+1}}\right]|\mathcal{G}_{T_n}\right]\\ &= \operatorname*{ess\,sup}_{N\in\mathcal{N}_n(\lambda)} \mathbf{E}\left[\bar{S}_{T_N}\mathbf{1}_{\{N=n\}}+\mathbf{E}\left[\bar{S}_{T_N}\mathbf{1}_{\{n+1\leq N\leq M\}}+\bar{\xi}\mathbf{1}_{\{N=M+1\}}|\mathcal{G}_{T_{n+1}}\right]|\mathcal{G}_{T_n}\right]\\ &=\max\left\{\bar{S}_{T_n},\mathbf{E}\left[\bar{y}^{\lambda}_{T_{n+1}}|\mathcal{G}_{T_n}\right]\right\}.\end{aligned}$$

Finally, we prove that $\widehat{N}^*_{i-1}$ is indeed the optimal stopping time for the auxiliary optimal stopping problem (2.5). For this, it suffices to show that $\widehat{y}^{\lambda}_{T_{m\wedge\widehat{N}^*_{i-1}}}$ for $m\geq i-1$ is a $\tilde{\mathbb{G}}$-martingale:

$$\begin{aligned}\mathbf{E}\left[\widehat{y}^{\lambda}_{T_{(m+1)\wedge\widehat{N}^*_{i-1}}}|\mathcal{G}_{T_m}\right] &= \mathbf{E}\left[\left(\sum_{j=i-1}^{m}\mathbf{1}_{\{\widehat{N}^*_{i-1}=j\}}+\mathbf{1}_{\{\widehat{N}^*_{i-1}\geq m+1\}}\right)\widehat{y}^{\lambda}_{T_{(m+1)\wedge\widehat{N}^*_{i-1}}}|\mathcal{G}_{T_m}\right]\\ &=\mathbf{E}\left[\sum_{j=i-1}^{m}\mathbf{1}_{\{\widehat{N}^*_{i-1}=j\}}\widehat{y}^{\lambda}_{T_j}+\mathbf{1}_{\{\widehat{N}^*_{i-1}>m\}}\widehat{y}^{\lambda}_{T_{m+1}}|\mathcal{G}_{T_m}\right]\\ &=\sum_{j=i-1}^{m}\mathbf{1}_{\{\widehat{N}^*_{i-1}=j\}}\widehat{y}^{\lambda}_{T_j}+\mathbf{1}_{\{\widehat{N}^*_{i-1}>m\}}\mathbf{E}\left[\widehat{y}^{\lambda}_{T_{m+1}}|\mathcal{G}_{T_m}\right]\\ &=\sum_{j=i-1}^{m}\mathbf{1}_{\{\widehat{N}^*_{i-1}=j\}}\widehat{y}^{\lambda}_{T_j}+\mathbf{1}_{\{\widehat{N}^*_{i-1}>m\}}\widehat{y}^{\lambda}_{T_m}=\widehat{y}^{\lambda}_{T_{m\wedge\widehat{N}^*_{i-1}}},\end{aligned}$$

where we used the definition of $\widehat{N}^*_{i-1}$ is the second last equality, and the proof is complete. □

We are now in a position to prove the linear situation of Theorem 1.2. From Lemma 2.1 and the definition of $\widehat{Y}^{\lambda}$, conditional on $\{T_{i-1}\leq t<T_i\}$,

$$Y^{\lambda}_t=\mathbf{E}\left[\int_t^{T_i\wedge T}f_s ds+\widehat{Y}^{\lambda}_{T_i}\mathbf{1}_{\{T_i<T\}}+\xi\mathbf{1}_{\{T_i\geq T\}}|\mathcal{G}_t\right].$$

Thanks to Lemma 2.2, $\widehat{Y}^{\lambda}_{T_i}=\widehat{y}^{\lambda}_{T_i}$, which is the value of the auxiliary optimal stopping problem (2.5) starting from T_i. Hence, for any $\tilde{\mathbb{G}}$-stopping time $N \in \mathcal{N}_i(\lambda)$,

$$\begin{aligned}
Y_t^{\lambda} &= \mathbf{E}\left[\int_t^{T_i\wedge T} f_s ds + \widehat{y}_{T_i}^{\lambda}\mathbf{1}_{\{T_i<T\}} + \xi\mathbf{1}_{\{T_i\geq T\}}|\mathcal{G}_t\right] \\
&\geq \mathbf{E}\left[\int_t^{T_i\wedge T} f_s ds + \mathbf{E}\left[\int_{T_i}^{T_N\wedge T} f_s ds + S_{T_N}\mathbf{1}_{\{T_N<T\}} + \xi\mathbf{1}_{\{T_N\geq T\}}|\mathcal{G}_{T_i}\right]\mathbf{1}_{\{T_i<T\}}\right. \\
&\quad \left. + \xi\mathbf{1}_{\{T_i\geq T\}}|\mathcal{G}_t\right] \\
&= \mathbf{E}\left[\int_t^{T_i\wedge T} f_s ds + \left(\int_{T_i}^{T_N\wedge T} f_s ds\right)\mathbf{1}_{\{T_i<T\}} + S_{T_N}\mathbf{1}_{\{T_N<T,T_i<T\}}\right. \\
&\quad \left. + \xi\left(\mathbf{1}_{\{T_N\geq T,\ T_i<T\}} + \mathbf{1}_{\{T_i\geq T\}}\right)|\mathcal{G}_t\right] \\
&= \mathbf{E}\left[\int_t^{T_N\wedge T} f_s ds + S_{T_N}\mathbf{1}_{\{T_N<T\}} + \xi\mathbf{1}_{\{T_N\geq T\}}|\mathcal{G}_t\right],
\end{aligned}$$

where we used the fact that $\{T_i \geq T\} \subset \{T_N \geq T\}$ in the last two equalities. By taking the supremum over $N \in \mathcal{N}_i(\lambda)$, we obtain that $Y_t^{\lambda} \geq y_t^{\lambda}$.

We now choose $N = \widehat{N}_i^*$, where $\widehat{N}_i^*$ is the optimal stopping time for $\widehat{y}_{T_i}^{\lambda}$ given in Lemma 2.2, to get the reverse inequality. Indeed,

$$\begin{aligned}
Y_t^{\lambda} &= \mathbf{E}\left[\int_t^{T_i\wedge T} f_s ds + \widehat{y}_{T_i}^{\lambda}\mathbf{1}_{\{T_i<T\}} + \xi\mathbf{1}_{\{T_i\geq T\}}|\mathcal{G}_t\right] \\
&= \mathbf{E}\left[\int_t^{T_i\wedge T} f_s ds + \mathbf{E}\left[\int_{T_i}^{T_{\widehat{N}_i^*}\wedge T} f_s ds + S_{T_{\widehat{N}_i^*}}\mathbf{1}_{\{T_{\widehat{N}_i^*}<T\}} + \xi\mathbf{1}_{\{T_{\widehat{N}_i^*}\geq T\}}|\mathcal{G}_{T_i}\right]\mathbf{1}_{\{T_i<T\}}\right. \\
&\quad \left. + \xi\mathbf{1}_{\{T_i\geq T\}}|\mathcal{G}_t\right] \\
&= \mathbf{E}\left[\int_t^{T_i\wedge T} f_s ds + \left(\int_{T_i}^{T_{\widehat{N}_i^*}\wedge T} f_s ds\right)\mathbf{1}_{\{T_i<T\}} + S_{T_{\widehat{N}_i^*}}\mathbf{1}_{\{T_{\widehat{N}_i^*}<T,T_i<T\}}\right. \\
&\quad \left. + \xi\left(\mathbf{1}_{\{T_{\widehat{N}_i^*}\geq T,\ T_i<T\}} + \mathbf{1}_{\{T_i\geq T\}}\right)|\mathcal{G}_t\right] \\
&= \mathbf{E}\left[\int_t^{T_{\widehat{N}_i^*}\wedge T} f_s ds + S_{T_{\widehat{N}_i^*}}\mathbf{1}_{\{T_{\widehat{N}_i^*}<T\}} + \xi\mathbf{1}_{\{T_{\widehat{N}_i^*}\geq T\}}|\mathcal{G}_t\right] \leq y_t^{\lambda}.
\end{aligned}$$

Hence, $Y_t^{\lambda} = y_t^{\lambda}$, and the optimal stopping time is $\widehat{N}_i^*$, which is just N_i^* defined at the beginning of Section 2,

$$\begin{aligned}
\widehat{N}_i^* &= \inf\{N \geq i : \widehat{y}_{T_N}^{\lambda} \leq S_{T_N}\} \wedge (M+1) \\
&= \inf\{N \geq i : \widehat{Y}_{T_N}^{\lambda} \leq S_{T_N}\} \wedge (M+1) \\
&= \inf\{N \geq i : Y_{T_N}^{\lambda} \leq S_{T_N}\} \wedge (M+1) = N_i^*.
\end{aligned}$$

2.2. Representation for Nonlinear Case. In this section, we extend the optimal stopping representation to the nonlinear case, and complete the proof of Theorem 1.2.

Denote $(Y^{\lambda}, Z^{\lambda})$ as the unique solution to the penalized BSDE (1.3). Consider the optimal stopping time problem (2.1) conditional on $\{T_{i-1} \leq t < T_i\}$:

$$y_t^\lambda = \operatorname*{ess\,sup}_{N \in \mathcal{N}_i(\lambda)} \mathbf{E}\left[\int_t^{T_N \wedge T} f_s(Y_s^\lambda, Z_s^\lambda) ds + S_{T_N} \mathbf{1}_{\{T_N < T\}} + \xi \mathbf{1}_{\{T_N \geq T\}} | \mathcal{G}_t\right].$$

From Section 2.1, $y_t^\lambda = \tilde{Y}_t^\lambda$ admits the following BSDE representation

$$\tilde{Y}_t^\lambda = \xi + \int_t^T f_s(Y_s^\lambda, Z_s^\lambda) ds + \int_t^T \lambda \max\{0, S_s - \tilde{Y}_s^\lambda\} ds - \int_t^T \tilde{Z}_t^\lambda dW_s.$$

On the other hand, (Y^λ, Z^λ) satisfies the penalized BSDE (1.3)

$$Y_t^\lambda = \xi + \int_t^T f_s(Y_s^\lambda, Z_s^\lambda) ds + \int_t^T \lambda \max\{0, S_s - Y_s^\lambda\} ds - \int_t^T Z_s^\lambda dW_s.$$

Define

$$\delta Y_t^\lambda = \tilde{Y}_t^\lambda - Y_t^\lambda; \ \delta Z_t^\lambda = \tilde{Z}_t^\lambda - Z_t^\lambda.$$

Then $(\delta Y^\lambda, \delta Z^\lambda)$ satisfies the following linear BSDE

$$\delta Y_t^\lambda = \int_t^T \lambda \beta_s \delta Y_s^\lambda ds - \int_t^T \delta Z_s^\lambda dW_s \tag{2.7}$$

with

$$\beta_s = \frac{\max\{0, S_s - \tilde{Y}_s^\lambda\} - \max\{0, S_s - Y_s^\lambda\}}{\delta Y_s^\lambda} \times \mathbf{1}_{\{\delta Y_s^\lambda \neq 0\}}.$$

Obviously, $|\beta_s| \leq 1$, so BSDE (2.7) admits a unique solution (see for example [10] for the proof). On the other hand, $\delta Y_t^\lambda = \delta Z_t^\lambda = 0$ is one obvious solution to BSDE (2.7). Therefore, we conclude that $\tilde{Y}_t^\lambda = Y_t^\lambda$ *a.s.*, which proves Theorem 1.2.

We conclude this section by reformulating the optimal stopping representation (1.5) as the following remark, which will be used in Section 3.

Remark 2.3. *Suppose that Assumption 1.1 holds. Then for any integer $i \geq 1$, conditional on $\{T_{i-1} \wedge T \leq t < T_i \wedge T\}$, the solution to the penalized BSDE (1.3) is the value of the optimal stopping time (1.5): $Y_t^\lambda = y_t^\lambda$ a.s.. Moreover, the value y_t^λ satisfies the recursive equation:*

$$\begin{aligned} y_t^\lambda &= \mathbf{E}\left[\int_t^{T_i \wedge T} f_s(Y_s^\lambda, Z_s^\lambda) ds + \max\left\{S_{T_i}, y_{T_i}^\lambda\right\} \mathbf{1}_{\{T_i \leq T\}} + \xi \mathbf{1}_{\{T_i > T\}} | \mathcal{G}_t\right] \\ &= \mathbf{E}\left[\int_t^{T_i \wedge T} f_s(Y_s^\lambda, Z_s^\lambda) ds + \max\left\{S_{T_i}, y_{T_i}^\lambda\right\} \mathbf{1}_{\{T_i \leq T\}} + \xi \mathbf{1}_{\{T_i > T\}} | \mathcal{F}_t\right]. \end{aligned}$$

3. Application I: Convergence Rate of Penalized BSDE. The penalization method only provides the convergence of the solution $(Y^\lambda, Z^\lambda, K^\lambda)$ of the penalized BSDE (1.3) to the solution (Y, Z, K) of the reflected BSDE (1.1), but without any convergence rate, because the proof of the convergence is based on compactness arguments. What is even worse is that the penalized BSDE (1.3) does not provide an efficient numerical algorithm, as the Lipschitz constant of the driver depends on

λ which will explode when $\lambda \uparrow \infty$. Actually, it is still an open question on how to numerically approximate the corresponding penalized BSDE (1.3) with an even fixed (but large) intensity λ (see Page 26 in [3]).

Thanks to our optimal stopping representation, the penalized BSDE (1.3) is nothing but a random time discretization of the optimal stopping representation for the corresponding reflected BSDE (1.1), where the time is discretized by Poisson arrival times. On the other hand, it has been known the convergence rate of the fixed time discretization of the optimal stopping representation for (1.1), so called the Bermudan approximation in [2] and [23]. Hence, it is plausible to obtain the convergence rate of the penalized BSDE (1.3), or equivalently, the convergence rate of the optimal stopping representation (1.5).

Assumption 3.1.

- *The terminal data ξ, the driver $f_s(y,z)$ and the obstacle S satisfy Assumption 1.1;*
- *Moreover, the driver $f_s(y,z) = f(X_s,y,z)$, the terminal date $\xi = g(X_T)$ for $g(\cdot)$ being Lipschitz continuous, and the obstacle process $S_s = l(X_s)$ for $l(\cdot) \in C^2$, where X is a diffusion process with enough regality.*

We refer to [2, 3, 23] for more detail assumptions on the diffusion X. In the following, we improve the convergence (1.6) by giving its convergence rate.

Proposition 3.2. *Suppose that Assumption 3.1 holds. Then for any integer $M \geq 1$, the value of the optimal stopping time problem (1.5) will converge to the value of (1.2) with the following rate:*

$$\mathbf{E}\left[\sup_{t\in[0,T_M\wedge T]}\mathbf{E}\left[|y_t^\lambda - y_t|^2\right]\right] \leq \frac{C}{\lambda}. \tag{3.1}$$

for some constant C.

Proof. For any $M \geq 1$, Theorem 1.2 and Remark 2.3 imply that

$$y_t^\lambda = \mathbf{E}\left[\int_t^{T_i\wedge T} f_s(Y_s^\lambda, Z_s^\lambda)ds + \hat{y}_{T_i}^\lambda \mathbf{1}_{\{T_i\leq T\}} + g(X_T)\mathbf{1}_{\{T_i>T\}}|\mathcal{F}_t\right]$$

conditional on $t \in [T_{i-1}\wedge T, T_i\wedge T)$, where $\hat{y}_{T_i}^\lambda = \max\left\{l(X_{T_i}), y_{T_i}^\lambda\right\}$ for $1 \leq i \leq M$. This is exactly the Bermudan approximation of the optimal stopping time problem (1.2) if we condition on $\vee_{t\geq 0}\mathcal{H}_t$. Hence, by a similar argument as in Proposition 3.1 of [3] (see also Section 4 of [2] and Section 3 of [23]), conditional on $\vee_{t\geq 0}\mathcal{H}_t$, we obtain that

$$\sup_{t\in[0,T_M\wedge T]}\mathbf{E}\left[|y_t^\lambda - y_t|^2\right] \leq \max_{1\leq i\leq M}(T_i\wedge T - T_{i-1}\wedge T) \leq \max_{1\leq i\leq M}(T_i - T_{i-1}), \tag{3.2}$$

and moreover,

$$\begin{aligned}
\mathbf{E}\left[\sup_{t\in[0,T_M\wedge T]}\mathbf{E}\left[|y_t^\lambda - y_t|^2\right]\right] &= \mathbf{E}\left[\mathbf{E}\left[\sup_{t\in[0,T_M\wedge T]}\mathbf{E}\left[|y_t^\lambda - y_t|^2\right]\middle| \vee_{t\geq 0}\mathcal{H}_t\right]\right] \\
&\leq \mathbf{E}\left[\mathbf{E}\left[\max_{1\leq i\leq M}(T_i - T_{i-1})\middle| \vee_{t\geq 0}\mathcal{H}_t\right]\right] \\
&= \mathbf{E}\left[\max_{1\leq i\leq M}(T_i - T_{i-1})\right].
\end{aligned}$$

The conclusion then follows by observing that $(T_{i-1}-T_i)$ is exponentially distributed with parameter λ and that

$$\begin{aligned}\mathbf{E}\left[\max_{1\leq i\leq M}(T_i-T_{i-1})\right] &= \int_0^\infty \mathbf{P}(\max_{1\leq i\leq M}(T_i-T_{i-1})>x)dx\\ &= \int_0^\infty (1-(1-e^{-\lambda x})^M)dx\\ &= \frac{1}{\lambda}\int_0^1 \frac{1-u^M}{1-u}du = \frac{1}{\lambda}\sum_{i=1}^M \left(\frac{1}{i}\right).\end{aligned}$$

□

REMARK 3.3. *Thanks to the optimal stopping representation (1.5), it is also possible to obtain a numerical algorithm to solve the penalized BSDE (1.3), where the parameter λ is hidden in the Poisson arrival times $\{T_i\}_{i\geq 1}$, and we only need to numerically solve the BSDE with the standard driver $f(x,y,z)$ instead of $f(x,y,z)+\lambda\max\{0,l(x)-y\}$:*

$$\begin{aligned}Y_t^\lambda = &\ \max\left\{l(X_{T_i}),Y_{T_i}^\lambda\right\}\mathbf{1}_{\{T_i\leq T\}}+g(X_T)\mathbf{1}_{\{T_i>T\}}\\ &+\int_t^{T_i\wedge T} f(X_s,Y_s^\lambda,Z_s^\lambda)ds-\int_t^{T_i\wedge T} Z_s^\lambda dW_s\end{aligned}$$

on $\{T_{i-1}\leq t<T_i\}$. Since the numerical approximation is of independent interest, we will leave it for future research.

4. Application II: Randomized Stopping and Optimal Control Representation. Krylov in [16] showed that optimal stopping for controlled diffusion processes can always be transformed to optimal control by using randomized stopping. See also Gyöngy and Siska [11] for its recent development. In this section, our aim is to give optimal control interpretations of both the reflected BSDE (1.1) and the penalized BSDE (1.3).

Let us first recall the basic idea of Krylov's randomized stopping. For simplicity, we only consider the linear case $f_s(y,z)=f_s$. For any fixed time $t\in[0,T]$, consider a nonnegative control process $(r_s)_{s\geq t}$. Let the payoff functional $\int_t^\cdot f_s ds+S_\cdot$ stop with intensity $r_s\Delta$ in an infinitesimal interval $(s,s+\Delta)$. Then the probability that stopping does not occur before time s is

$$e^{-\int_t^s r_u du}.$$

The probability that stopping does not occur before time s and does occur in the infinitesimal interval $(s,s+\Delta)$ is

$$e^{-\int_t^s r_u du}r_s\Delta.$$

Therefore, the payoff functional associated with the control process r from $[t,T]$ is given by

$$\int_t^T(\int_t^s f_u du+S_s)e^{-\int_t^s r_u du}r_s ds+(\int_t^T f_u du+\xi)e^{-\int_t^T r_u du},$$

where the first term is the payoff if stopping does occur before time T, and the second term corresponds to the payoff if stopping does not occur in the time interval $[t,T]$.

By applying integration by parts, the payoff functional is further simplified to

$$\int_t^T (f_s + r_s S_s) e^{-\int_t^s r_u du} + e^{-\int_t^T r_u du}\xi.$$

We have the following optimal control representation for the penalized BSDE (1.3):

PROPOSITION 4.1. *Suppose that Assumption 1.1 holds. Denote (Y^λ, Z^λ) as the unique solution to the penalized BSDE (1.3). For any fixed time $t \in [0,T]$, define the control set $\mathcal{A}(t,\lambda)$ as*

$$\mathcal{A}(t,\lambda) = \{\mathbb{F}\text{-}adapted\ process\ (r_s)_{s\geq t} :\ r_s = 0\ or\ \lambda\}.$$

Then the value of the following optimal control problem

$$y_t^\lambda = \operatorname*{ess\,sup}_{r\in\mathcal{A}(t,\lambda)} \mathbf{E}\left[\int_t^T (f_s(Y_s^\lambda, Z_s^\lambda) + r_s S_s) e^{-\int_t^s r_u du} ds + e^{-\int_t^T r_u du}\xi | \mathcal{F}_t\right] \tag{4.1}$$

is given by the solution to the penalized BSDE (1.3): $y_t^\lambda = Y_t^\lambda$ a.s. for $t \in [0,T]$. The optimal control is given by $r_s^* = \lambda \mathbf{1}_{\{Y_s^\lambda \leq S_s\}}$ for $s \geq t$.

Proof. We only consider the linear case $f_s(y,z) = f_s$. The proof for the nonlinear case $f_s(y,z)$ is the same as the one in Section 2.2.

First, similar to Lemma 2.1, it is easy to show that the following expected payoff process associated with any given control $r \in \mathcal{A}(t,\lambda)$:

$$y_t^\lambda(r) = \mathbf{E}\left[\int_t^T (f_s + r_s S_s) e^{-\int_t^s r_u du} + e^{-\int_t^T r_u du}\xi | \mathcal{F}_t\right]$$

is the unique solution to the following linear BSDE

$$y_t^\lambda(r) = \xi + \int_t^T \left\{f_s + r_s(S_s - y_s^\lambda(r))\right\} ds - \int_t^T z_s^\lambda(r) dW_s.$$

Note that the control r only appears in the driver. For any control $r \in \mathcal{A}(t,\lambda)$, we have

$$f_s + r_s(S_s - y_s^\lambda(r)) \leq f_s + \lambda \max\{0, S_s - y_s^\lambda(r)\},$$

and for $r_s = \lambda \mathbf{1}_{\{y_s^\lambda(r) \leq S_s\}}$, we obtain the equality

$$f_s + \lambda \mathbf{1}_{\{y_s^\lambda(r)\leq S_s\}}(S_s - y_s^\lambda(r)) = f_s + \lambda \max\{0, S_s - y_s^\lambda(r)\}.$$

By the BSDE comparison theorem (see for example [10]), $y_t^\lambda(r) \leq Y_t^\lambda$ for any $r \in \mathcal{A}(t,\lambda)$, where Y^λ is the solution to the penalized BSDE (1.3):

$$Y_t^\lambda = \xi + \int_t^T \left\{f_s + \lambda \max\{0, S_s - Y_s^\lambda\}\right\} ds - \int_t^T Z_s^\lambda dW_s.$$

and $y_t^\lambda(r^*) = Y_t^\lambda$ for $r_s^* = \lambda \mathbf{1}_{\{Y_s^\lambda \leq S_s\}}$. Since $y_t^\lambda = \operatorname{ess\,sup}_{r\in\mathcal{A}(t,\lambda)} y_t^\lambda(r)$, we conclude that $y_t^\lambda = Y_t^\lambda$ *a.s.* for $t \in [0,T]$, and the optimal control is r_s^* for $s \geq t$. □

REMARK 4.2. *The optimal control representation for the reflected BSDE (1.1) is the same as (4.1) except that the control set is changed to $\mathcal{A}(t) = \cup_\lambda \mathcal{A}(t,\lambda)$. As* *shown in Krylov [16] for the diffusion case, the value of the following optimal control problem*

$$y_t = \operatorname*{ess\,sup}_{r\in\mathcal{A}(t)} \mathbf{E}\left[\int_t^T (f_s(Y_s^\lambda, Z_s^\lambda) + r_s S_s)e^{-\int_t^s r_u du} ds + e^{-\int_t^T r_u du}\xi|\mathcal{F}_t\right] \tag{4.2}$$

is given by the solution to the reflected BSDE (1.1): $y_t = Y_t$ a.s. for $t \in [0,T]$.

5. Application III: Multidimensional Reflected BSDE and Regime Switching. Multidimensional reflected BSDE was firstly introduced by Hamadène and Jeanblanc [12], where they used its solution to characterize the value of an optimal switching problem, in particular in the setting of power plant management. The related equation was solved by Hu and Tang [14] using the penalty method, and by Hamadène and Zhang [13] using the iterated optimal stopping time method. See also Chassagneux et al [4] for its recent development. A multidimensional reflected BSDE is a d-dimensional system, where each component $1 \le i \le d$ representing regime i,

$$Y_t^i = \xi^i + \int_t^T f_s^i(Y_s, Z_s)ds + \int_t^T dK_s^i - \int_t^T Z_s^i dW_s \tag{5.1}$$

under the constraints

$$\text{(Dominating Condition)}: \quad Y_t^i \ge \mathcal{M}Y_t^i \text{ for } t \in [0,T],$$

$$\text{(Skorohod Condition)}: \quad \int_0^T (Y_t^i - \mathcal{M}Y_t^i)dK_t^i = 0 \text{ for } K^i \text{ continuous and increasing,}$$

where the impulse term $\mathcal{M}Y_t^i$ is given by

$$\mathcal{M}Y_t^i = \max_{j\neq i}\{Y_t^j - C_t^{i,j}\}$$

representing the payoff of switching to regime j from regime i. The terminal data ξ^i, the driver $f_s^i(y,z)$ and the switching cost $(C_s^{i,j})_{0\le s\le T}$ are the given data. Different from one-dimensional reflected BSDE whose solution must stay above an obstacle process, the solution of the multidimensional reflected BSDE (5.1) evolves in the random closed and convex set

$$\left\{y \in \mathbb{R}^d : y^i \ge \max_{j\neq i}\{y^j - C_t^{i,j}\}\right\}.$$

The following standard assumption on the data set $(\xi^i, f^i, C^{i,j})$ is imposed.

ASSUMPTION 5.1.

- *The terminal data ξ^i and the driver $f_s^i(y,z)$ satisfy Assumption 1.1;*
- *The switching cost $(C^{ij})_{1\le i,j\le d}$ is a bounded $\mathbb{F}$-adapted process satisfying (i) $C_t^{ii} = 0$; (ii) $\inf_{t\in[0,T]} C_t^{ij} \ge C > 0$ for $i \ne j$; and (iii) $\inf_{t\in[0,T]} C_t^{ij} + C_t^{jl} - C_t^{il} \ge C > 0$ for $i \ne j \ne l$.*

In Hu and Tang [14], they further assume that $f_s^i(y,z) = f_s^i(y^i, z^i)$ so that (5.1) admits a solution. This condition was relaxed in Hamadène and Zhang [13] and Chassagneux et al [4], where the driver is even allowed to be coupled in y, i.e. having the form $f_s^i(y, z^i)$. However, it is still an open problem for the case of the fully coupled driver $f_s^i(y,z)$.

Under Assumption 5.1 with the decoupled driver $f^i_s(y,z) = f^i_s(y^i, z^i)$, Hu and Tang [14] proved that the solution to the multidimensional reflected BSDE (5.1) corresponds to the value of an optimal switching problem. Indeed, introduce the control set $\mathcal{K}_i(t)$ as

$$\mathcal{K}_i(t) = \left\{ \mathbb{F}\text{-adapted process } (u_s)_{s\geq t}: \ u_s = \alpha_0 \mathbf{1}_{[t,\tau_1]}(s) + \sum_{k\geq 1} \alpha_k \mathbf{1}_{(\tau_k,\tau_{k+1}]}(s) \right\},$$

where

- $(\tau_k)_{k\geq 1}$ is an increasing sequence of $\mathbb{F}$-stopping times valued in $[t,T]$ with $\tau_M \leq T < \tau_{M+1}$ for some integer-valued random variable $M < \infty$.
- $(\alpha_k)_{k\geq 0}$ is a sequence of random variables valued in $\{1,\cdots,d\}$ such that α_k is $\mathcal{F}_{\tau_k}$-measurable, and $\alpha_0 = i$.

Then the value of the following optimal switching problem

$$y^i_t = \operatorname*{ess\,sup}_{u\in\mathcal{K}_i(t)} \mathbf{E}\left[\int_t^T f^{u_s}_s(Y_s,Z_s)ds + \xi^{u_T} - \sum_{k\geq 1} C^{\alpha_{k-1},\alpha_k}_{\tau_k} \mathbf{1}_{\{t<\tau_k<T\}} |\mathcal{F}_t\right] \tag{5.2}$$

is given by the solution to the multidimensional reflected BSDE (5.1): $y^i_t = Y^i_t$ *a.s.* for $t\in[0,T]$. The optimal switching strategy is given as follows: $\tau^*_0 = t$, $\alpha^*_0 = i$ and for $k\geq 0$,

$$\tau^*_{k+1} = \inf\left\{ s > \tau^*_k : Y^{\alpha^*_k}_s \leq \mathcal{M} Y^{\alpha^*_k}_s \right\} \wedge T, \tag{5.3}$$

where

$$\alpha^*_{k+1} = \operatorname*{arg\,max}_{j\neq\alpha^*_k} \left\{ Y^j_{\tau^*_{k+1}} - C^{\alpha^*_k, j}_{\tau^*_{k+1}} \right\}.$$

Hence, the optimal switching strategy at any time $s\geq t$ is

$$u^*_s = i\mathbf{1}_{[t,\tau^*_1]}(s) + \sum_{k=1}^{M^*} \alpha^*_k \mathbf{1}_{(\tau^*_k,\tau^*_{k+1}]}(s),$$

where $M^* \leq M$ is some integer-valued random variable such that $\tau^*_{M^*} \leq T < \tau^*_{M^*+1}$.

On the other hand, Hu and Tang [14] introduced the following multidimensional penalized BSDE to approximate and solve the multidimensional reflected BSDE (5.1):

$$Y^{i,\lambda}_t = \xi^i + \int_t^T f^i_s(Y^\lambda_s, Z^\lambda_s)ds + \int_t^T \lambda \max\{0, \mathcal{M}Y^{i,\lambda}_s - Y^{i,\lambda}_s\}ds - \int_t^T Z^{i,\lambda}_s dW_s, \tag{5.4}$$

and they proved that under Assumption 5.1 with $f^i_s(y,z) = f^i_s(y^i,z^i)$, $Y^{i,\lambda}$ is increasing in λ, and

$$\lim_{\lambda\uparrow\infty} \mathbf{E}\left[\sup_{t\in[0,T]} |Y^{i,\lambda}_t - Y^i_t|^2 + \int_0^T |Z^{i,\lambda}_t - Z^i_t|^2 dt + \sup_{t\in[0,T]} |K^{i,\lambda}_t - K^i_t|^2 \right] = 0. \tag{5.5}$$

However, the solvability of (5.4) does not rely on the assumption that $f^i_s(y,z) = f^i_s(y^i,z^i)$. Our aim is therefore to give a stochastic control interpretation of the multidimensional penalized BSDE (5.4) with the coupled driver $f_s^i(y,z)$, so we are at least one step closer to solve the general optimal switching problem with the coupled driver $f_s^i(y,z)$ is some sense.

Recall that $\{T_n\}_{n\geq 0}$ are the arrival times of the underlying Poisson process with intensity λ, $\mathbb{G}=\{\mathcal{G}_s\}_{s\geq 0}$ with $\mathcal{G}_s=\mathcal{F}_s\vee\mathcal{H}_s$, and $M<\infty$ is some integer-valued random variable such that $T_M\leq T<T_{M+1}$.

Proposition 5.2. *Suppose that Assumption 5.1 holds. Denote $(Y^{i,\lambda},Z^{i,\lambda})$ as the unique solution to the multidimensional penalized BSDE (5.4). For any integer $n\geq 1$, conditional on $\{T_{n-1}\leq t<T_n\}$, define the control set $\mathcal{K}_i(\lambda,t)$ as*

$$\begin{aligned}\mathcal{K}_i(t,\lambda)=\Big\{&\mathbb{G}\text{-adapted process } (u_s)_{s\geq t}: u_s=\alpha_{n-1}\mathbf{1}_{[t,\tau_n]}(s)+\sum_{k\geq n}\alpha_k\mathbf{1}_{(\tau_k,\tau_{k+1}]}(s),\\ &\text{where } \tau_k(\omega)=T_k(\omega) \text{ for } n\leq k\leq M+1, \text{ and}\\ &\alpha_k\in\mathcal{G}_{T_k} \text{ valued in } \{1,\cdots,d\} \text{ with } \alpha_{n-1}=i.\Big\}\end{aligned}$$

Then the value of the following optimal switching problem

$$y_t^{i,\lambda}=\operatorname*{ess\,sup}_{u\in\mathcal{K}_i(t,\lambda)}\mathbf{E}\left[\int_t^T f_s^{u_s}(Y_s^\lambda,Z_s^\lambda)ds+\xi^{u_T}-\sum_{k\geq n}C_{\tau_k}^{\alpha_{k-1},\alpha_k}\mathbf{1}_{\{t<\tau_k<T\}}|\mathcal{G}_t\right] \tag{5.6}$$

is given by the solution of the multidimensional penalized BSDE (5.4): $y_t^{i,\lambda}=Y_t^{i,\lambda}$ a.s.. The optimal switching strategy for (5.6) is given as follows: $\tau_{n-1}^*=t$, $\alpha_{n-1}^*=i$ and for $k\geq n-1$,

$$\tau_{k+1}^*=\inf\left\{T_N>\tau_k^*:Y_{T_N}^{\alpha_k^*,\lambda}\leq\mathcal{M}Y_{T_N}^{\alpha_k^*,\lambda}\right\}\wedge T_{M+1}, \tag{5.7}$$

where

$$\alpha_{k+1}^*=\operatorname*{arg\,max}_{j\neq\alpha_k^*}\left\{Y_{\tau_{k+1}^*}^{j,\lambda}-C_{\tau_{k+1}^*}^{\alpha_k^*,j}\right\}.$$

Hence, the optimal switching strategy at any time $s\geq t$ is

$$u_s^*=i\mathbf{1}_{[t,\tau_n^*]}(s)+\sum_{k=n}^{M^*}\alpha_k^*\mathbf{1}_{(\tau_k^*,\tau_{k+1}^*]}(s),$$

where $M^*\leq M$ is some integer-valued random variable such that $\tau_{M^*}^*\leq T<\tau_{M^*+1}^*$.

Proof. For any integer $n\geq 1$ and $1\leq i\leq d$, we introduce the following auxiliary optimal stopping time problem on $\{T_{n-1}\leq t<T_n\}$:

$$\tilde{y}_t^{i,\lambda}=\operatorname*{ess\,sup}_{\tau\in\mathcal{R}_{T_n}(\lambda)}\mathbf{E}\left[\int_t^{\tau\wedge T}f_s^i(Y_s^\lambda,Z_s^\lambda)ds+\mathcal{M}Y_\tau^{i,\lambda}\mathbf{1}_{\{\tau<T\}}+\xi^i\mathbf{1}_{\{\tau\geq T\}}|\mathcal{G}_t\right]. \tag{5.8}$$

From Theorem 1.2 (and Remark 2.3), we know that its value is given by $\tilde{y}_t^{i,\lambda}=Y_t^{i,\lambda}$ *a.s.*, and the optimal stopping time is given by

$$\tau_{T_n}^*=\inf\left\{T_N\geq T_n:Y_{T_N}^{i,\lambda}\leq\mathcal{M}Y_{T_N}^{i,\lambda}\right\}\wedge T_{M+1}.$$

Now for any switching strategy $u \in \mathcal{K}_i(t,\lambda)$ with the form

$$u_s = i\mathbf{1}_{[t,T_n]}(s) + \sum_{k=n}^{M} \alpha_k \mathbf{1}_{(T_k,T_{k+1}]}(s),$$

we consider the auxiliary optimal stopping problem (5.8) stopping at the Poisson arrival time T_n, and switching to α_n,

$$\tilde{y}_t^{i,\lambda} \geq \mathbf{E}\left[\int_t^{T_n\wedge T} f_s^i(Y_s^\lambda, Z_s^\lambda)ds + (Y_{T_n}^{\alpha_n,\lambda} - C_{T_n}^{i,\alpha_n})\mathbf{1}_{\{T_n<T\}} + \xi^i \mathbf{1}_{\{T_n\geq T\}}|\mathcal{G}_t\right]. \tag{5.9}$$

From Theorem 1.2, $Y_{T_n}^{\alpha_n,\lambda}$ is the value of the optimal stopping problem (5.8) starting from T_n. We consider such an optimal stopping problem stopping at the Poisson arrival time T_{n+1}, and switching to α_{n+1},

(5.10)

$$\begin{aligned} &Y_{T_n}^{\alpha_n,\lambda} = \tilde{y}_{T_n}^{\alpha_n,\lambda} \\ &\geq \mathbf{E}\left[\int_{T_n}^{T_{n+1}\wedge T} f_s^{\alpha_n}(Y_s^\lambda, Z_s^\lambda)ds + (Y_{T_{n+1}}^{\alpha_{n+1},\lambda} - C_{T_{n+1}}^{\alpha_n,\alpha_{n+1}})\mathbf{1}_{\{T_{n+1}<T\}} + \xi^{\alpha_n}\mathbf{1}_{\{T_{n+1}\geq T\}}|\mathcal{G}_{T_n}\right]. \end{aligned}$$

By plugging (5.10) into (5.9), we have

$$\begin{aligned} \tilde{y}_t^{i,\lambda} \geq\ & \mathbf{E}\Bigg[\int_t^{T_n\wedge T} f_s^i(Y_s^\lambda, Z_s^\lambda)ds + \int_{T_n\wedge T}^{T_{n+1}\wedge T} f_s^{\alpha_n}(Y_s^\lambda, Z_s^\lambda)ds - C_{T_n}^{i,\alpha_n}\mathbf{1}_{\{T_n<T\}} \\ & -C_{T_{n+1}}^{\alpha_n,\alpha_{n+1}}\mathbf{1}_{\{T_{n+1}<T\}} + \xi^i\mathbf{1}_{\{T_n\geq T\}} + \xi^{\alpha_n}\mathbf{1}_{\{T_n<T\leq T_{n+1}\}} + Y_{T_{n+1}}^{\alpha_{n+1},\lambda}\mathbf{1}_{\{T_{n+1}<T\}}|\mathcal{G}_t\Bigg]. \end{aligned}$$

We repeat the above procedure M times, and obtain

$$\begin{aligned} \tilde{y}_t^{i,\lambda} \geq \mathbf{E}\Bigg[&\int_t^{T_n\wedge T} f_s^i(Y_s^\lambda, Z_s^\lambda)ds + \xi^i\mathbf{1}_{\{T_n\geq T\}} \\ &+\sum_{k=n}^{M}\left(\int_{T_k\wedge T}^{T_{k+1}\wedge T} f_s^{\alpha_k}(Y_s^\lambda, Z_s^\lambda)ds - C_{T_k}^{\alpha_k-1,\alpha_k}\mathbf{1}_{\{T_k<T\}} + \xi^{\alpha_k}\mathbf{1}_{\{T_k<T\leq T_{k+1}\}}\right)|\mathcal{G}_t\Bigg]. \end{aligned}$$

Since $T_M \leq T < T_{M+1}$, the above inequality is further simplified to

$$\tilde{y}_t^{i,\lambda} \geq \mathbf{E}\left[\int_t^T f_s^{u_s}(Y_s^\lambda, Z_s^\lambda)ds + \xi^{u_T} - \sum_{k\geq n} C_{T_k}^{\alpha_{k-1},\alpha_k}\mathbf{1}_{\{t<T_k<T\}}|\mathcal{G}_t\right].$$

By taking the supremum over $u \in \mathcal{K}_i(t,\lambda)$ and using Theorem 1.2 once again, we prove that on $\{T_{n-1}\leq t<T_n\}$,

$$Y_t^{i,\lambda} = \tilde{y}_t^{i,\lambda} \geq y_t^{i,\lambda}.$$

To prove the reverse inequality, we take the switching strategy $u = u^*$. From Theorem 1.2 (and Remark 2.3), τ_n^* is the optimal stopping time for (5.8). By the definition of α_n^*,

$$\mathcal{M}Y_{\tau_n^*}^{i,\lambda} = \max_{j\neq i}\{Y_{\tau_n^*}^{j,\lambda} - C_{\tau_n^*}^{i,j}\} = Y_{\tau_n^*}^{\alpha_n^*,\lambda} - C_{\tau_n^*}^{i,\alpha_n^*}.$$

Therefore,

$$(5.11)\quad \tilde{y}_t^{i,\lambda} = \mathbf{E}\left[\int_t^{\tau_n^*\wedge T} f_s^i(Y_s^\lambda, Z_s^\lambda)ds + \left(Y_{\tau_n^*}^{\alpha_n^*,\lambda} - C_{\tau_n^*}^{i,\alpha_n^*}\right)\mathbf{1}_{\{\tau_n^*<T\}} + \xi^i \mathbf{1}_{\{\tau_n^*\geq T\}}|\mathcal{G}_t\right].$$

Similarly, τ_{n+1}^* is the optimal stopping time for (5.8) starting from τ_n^*, and $Y_{\tau_n^*}^{\alpha_n^*,\lambda} = \tilde{y}_{\tau_n^*}^{\alpha_n^*,\lambda}$. By the definition of α_{n+1}^*,

$$\mathcal{M}Y_{\tau_{n+1}^*}^{\alpha_n^*,\lambda} = \max_{j\neq\alpha_n^*}\left\{Y_{\tau_{n+1}^*}^{j,\lambda} - C_{\tau_{n+1}^*}^{\alpha_n^*,j}\right\} = Y_{\tau_{n+1}^*}^{\alpha_{n+1}^*,\lambda} - C_{\tau_{n+1}^*}^{\alpha_n^*,\alpha_{n+1}^*}.$$

Hence,

(5.12)

$$\begin{aligned} Y_{\tau_n^*}^{\alpha_n^*,\lambda} &= \tilde{y}_{\tau_n^*}^{\alpha_n^*,\lambda} \\ &= \mathbf{E}\left[\int_{\tau_n^*}^{\tau_{n+1}^*\wedge T} f_s^{\alpha_n^*}(Y_s^\lambda, Z_s^\lambda)ds + \left(Y_{\tau_{n+1}^*}^{\alpha_{n+1}^*,\lambda} - C_{\tau_{n+1}^*}^{\alpha_n^*,\alpha_{n+1}^*}\right)\mathbf{1}_{\{\tau_{n+1}^*<T\}} + \xi^{\alpha_n^*}\mathbf{1}_{\{\tau_{n+1}^*\geq T\}}|\mathcal{G}_{\tau_n^*}\right]. \end{aligned}$$

Plugging (5.12) into (5.11) gives us

$$\begin{aligned} \tilde{y}_t^{i,\lambda} = \mathbf{E}\Bigg[&\int_t^{\tau_n^*\wedge T} f_s^i(Y_s^\lambda, Z_s^\lambda)ds + \int_{\tau_n^*\wedge T}^{\tau_{n+1}^*\wedge T} f_s^{\alpha_n^*}(Y_s^\lambda, Z_s^\lambda)ds - C_{\tau_n^*}^{i,\alpha_n^*}\mathbf{1}_{\{\tau_n^*<T\}} \\ &-C_{\tau_{n+1}^*}^{\alpha_n^*,\alpha_{n+1}^*}\mathbf{1}_{\{\tau_{n+1}^*<T\}} + \xi^i\mathbf{1}_{\{\tau_n^*\geq T\}} + \xi^{\alpha_n}\mathbf{1}_{\{\tau_n^*<T\leq\tau_{n+1}^*\}} + Y_{\tau_{n+1}^*}^{\alpha_{n+1}^*,\lambda}\mathbf{1}_{\{\tau_{n+1}^*<T\}}|\mathcal{G}_t\Bigg]. \end{aligned}$$

We repeat the above procedure M^* times, and obtain

$$\begin{aligned} \tilde{y}_t^{i,\lambda} &= \mathbf{E}\Bigg[\int_t^{\tau_n^*\wedge T} f_s^i(Y_s^\lambda, Z_s^\lambda)ds + \xi^i\mathbf{1}_{\{\tau_n^*\geq T\}} \\ &\quad+\sum_{k=n}^{M^*}\left(\int_{\tau_k^*\wedge T}^{\tau_{k+1}^*\wedge T} f_s^{\alpha_k^*}(Y_s^\lambda, Z_s^\lambda)ds - C_{\tau_k^*}^{\alpha_{k-1}^*,\alpha_k^*}\mathbf{1}_{\{\tau_k^*<T\}} + \xi^{\alpha_k^*}\mathbf{1}_{\{\tau_k^*<T\leq\tau_{k+1}^*\}}\right)|\mathcal{G}_t\Bigg] \\ &= \mathbf{E}\left[\int_t^T f_s^{u_s^*}(Y_s^\lambda, Z_s^\lambda)ds + \xi^{u_T^*} - \sum_{k\geq n} C_{\tau_k^*}^{\alpha_{k-1}^*,\alpha_k^*}\mathbf{1}_{\{t<\tau_k^*<T\}}|\mathcal{G}_t\right] \leq y_t^{i,\lambda}. \end{aligned}$$

Theorem 1.2 then implies that

$$Y_t^{i,\lambda} = \tilde{y}_t^{i,\lambda} \leq y_t^{i,\lambda},$$

and u^* is the optimal switching strategy. ∎

REMARK 5.3. *The optimal switching representation (5.6) of the multidimensional penalized BSDE (5.4) has a natural economic application to the menu cost model of Stokey [28], which allows the occasional arrival of opportunities to adjust without paying the fixed cost, and those opportunities are modeled as Poisson arrivals. See also [22] for an extension to an infinite horizon BSDE setting with the analysis of the corresponding free boundaries in the sense of Ly Vath and Pham [18].*

The other commonly used penalization scheme for the multidimensional reflected BSDE (5.1) is the following equation:
(5.13)

$$Y_t^{i,\lambda}=\xi^i+\int_t^T f_s^i(Y_s^\lambda,Z_s^\lambda)ds+\int_t^T\sum_{j=1}^d\lambda\max\{0,Y_s^{j,\lambda}-C_s^{i,j}-Y_s^{i,\lambda}\}ds-\int_t^T Z_s^{i,\lambda}dW_s,$$

and we still have the convergence (5.5) as shown in [14].

In the following, we show that (5.13) is closely related to the BSDE with regime switching on a Markov chain. Regime switching on Markov chains has been found useful in many applications as shown in [30] and [31]. Its application in BSDE can be found in a recent work [29] among others.

Define a Markov chain $(X_t)_{t\geq 0}$ with state space $\{1,2,\dots,d\}$, and its Q-matrix: $q_{ij}=\lambda$ if $i\neq j$, and $q_{ij}=-(d-1)\lambda$ if $i=j$. The jump times are denoted as $\{T_n\}_{n\geq 1}$. At each jump time T_n, the player has the right to choose if switching from the current state or not, and if she switches, a cost $C_{T_n}^{i,j}$ incurs if the Markov chain jumps from the state i to j.

For any integer $n\geq 1$, conditional on $\{T_{n-1}\leq t<T_n\}$, define the following control set:

$$\mathcal{K}_i(t,Q)=\Big\{\mathbb{G}\text{-adapted process }(u_s)_{s\geq t}:u_s=X_{\tau_{n-1}}\mathbf{1}_{[t,\tau_n]}(s)+\sum_{k\geq n}X_{\tau_k}\mathbf{1}_{(\tau_k,\tau_{k+1}]}(s),$$
$$\text{where }(\tau_k)_{k\geq n}\text{ chosen from }T_N\text{ for }n\leq N\leq M+1,\ \text{and }X_{\tau_{n-1}}=i.\Big\}$$

PROPOSITION 5.4. *Suppose that Assumption 5.1 holds. Denote $(Y^{i,\lambda},Z^{i,\lambda})$ as the unique solution to the multidimensional penalized BSDE (5.13). Then the value of the following optimal switching problem*

$$y_t^{i,\lambda}=\operatorname*{ess\,sup}_{u\in\mathcal{K}_i(t,Q)}\mathbf{E}\left[\int_t^T f_s^{u_s}(Y_s^\lambda,Z_s^\lambda)ds+\xi^{u_T}-\sum_{k\geq n}C_{\tau_k}^{X_{\tau_{k-1}},X_{\tau_k}}\mathbf{1}_{\{t<\tau_k<T\}}|\mathcal{G}_t\right]\tag{5.14}$$

is given by the solution of the multidimensional penalized BSDE (5.13): $y_t^{i,\lambda}=Y_t^{i,\lambda}$ a.s.. The optimal switching strategy for (5.6) is given as follows: $\tau_{n-1}^*=t$, and for $k\geq n-1$,

$$\tau_{k+1}^*=\inf\left\{T_N>\tau_k^*:Y_{T_N}^{X_{\tau_k^*},\lambda}\leq Y_{T_N}^{X_{T_N},\lambda}-C_{T_N}^{X_{\tau_k^*},X_{T_N}}\right\}\wedge T_{M+1}.\tag{5.15}$$

Hence, the optimal switching strategy at any time $s\geq t$ is

$$u_s^*=i\mathbf{1}_{[t,\tau_n^*]}(s)+\sum_{k=n}^{M^*}X_{\tau_k^*}\mathbf{1}_{(\tau_k^*,\tau_{k+1}^*]}(s),$$

where $M^*\leq M$ is some integer-valued random variable such that $\tau_{M^*}^*\leq T<\tau_{M^*+1}^*$.

Proof. We first rewrite (5.13) in terms of q_{ij} as follows,

$$Y_t^{i,\lambda}=\xi^i+\int_t^T f_s^i(Y_s^\lambda,Z_s^\lambda)ds+\int_t^T\sum_{j=1}^d q_{ij}\max\{0,Y_s^{j,\lambda}-C_s^{i,j}-Y_s^{i,\lambda}\}ds-\int_t^T Z_s^{i,\lambda}dW_s.$$

Then similar to Lemma 2.1, we have that

$$
\begin{aligned}
Y_t^{i,\lambda} =&\mathbf{E}\left[\int_t^{T_n\wedge T} f_s^i(Y_s^\lambda, Z_s^\lambda)ds\right. \\
&\left.+\max\left\{Y_{T_n}^{X_{T_n},\lambda}-C_{T_n}^{i,X_{T_n}}, Y_{T_n}^{i,\lambda}\right\}\mathbf{1}_{\{T_n\leq T\}}+\xi^i\mathbf{1}_{\{T_n>T\}}|\mathcal{G}_t\right].
\end{aligned}
\tag{5.16}
$$

conditional on $\{T_{n-1}\leq t<T_n\}$. From Theorem 1.2, $Y_t^{i,\lambda}$ is the value of the following optimal stopping time problem:

$$
\operatorname*{ess\,sup}_{\tau\in\mathcal{R}_{T_n}(Q)}\mathbf{E}\left[\int_t^{\tau\wedge T} f_s^i(Y_s^\lambda, Z_s^\lambda)ds+\left(Y_\tau^{X_\tau,\lambda}-C_\tau^{i,X_\tau}\right)\mathbf{1}_{\{\tau<T\}}+\xi^i\mathbf{1}_{\{\tau\geq T\}}|\mathcal{G}_t\right],
\tag{5.17}
$$

where

$$
\mathcal{R}_{T_n}(Q)=\{\mathbb{G}\text{-stopping time } \tau \text{ for } \tau(\omega)=T_N(\omega) \text{ where } n\leq N\leq M+1.\}
$$

with the optimal stopping time given by

$$
\tau_{T_n}^*=\inf\left\{T_N\geq T_n: Y_{T_N}^{i,\lambda}\leq Y_{T_N}^{X_{T_N},\lambda}-C_{T_N}^{i,X_{T_N}}\right\}\wedge T_{M+1}.
$$

The rest of the proof is then similar to that of Proposition 5.2, so we omit it. □

Remark 5.5. *If we compare between the optimal switching representations (5.6) and (5.14), the former only allows the player to choose the switching regimes on a sequence of Poisson arrival times, while the latter only allows the player to choose the switching times with the regimes following a Markov chain.*

6. Application VI: Constrained Reflected BSDE. In Cvitanic et al [5], the authors introduced a new class of BSDEs with a convex constraint on the *hedging* process Z, and solved the equation using the stochastic control method[1]. Their equation was further generalized by Peng [24], and in particular, by Peng and Xu [26] to reflected BSDE with a general constraint on Z (*constrained reflected BSDE* for short), where the monotonic limit theorem was introduced in order to show the associated penalized equation converges to the constrained reflected BSDE. A constraint reflected BSDE has the form

$$
Y_t=\xi+\int_t^T f_s(Y_s,Z_s)ds+\int_t^T dK_s^Y+\int_t^T dK_s^Z-\int_t^T Z_s dW_s
\tag{6.1}
$$

under the constraints

$$
\begin{aligned}
&\text{(Dominating Condition)}: && Y_t\geq S_t \text{ for } t\in[0,T],\\
&\text{(Skorohod Condition)}: && \int_0^T (Y_t-S_t)dK_t^Y=0 \text{ for } K^Y \text{ continuous and increasing},\\
&\text{(Hedging Constraint)}: && Z_t\in\Gamma \text{ for } t\in[0,T].
\end{aligned}
$$

The terminal data ξ, the driver $f_s(y,z)$, the obstacle $(S_t)_{0\leq t\leq T}$, and the constraint set $\Gamma\subset\mathbb{R}^d$ are the given data. A solution to the constrained reflected BSDE (6.1) is a quadruple of $\mathbb{F}$-adapted processes (Y,Z,K^Y,K^Z), where K^Y is used to pushed

[1] I would like to thank Ioannis Karatzas for the suggestion of this section.

up the solution Y in order to satisfy the dominating condition, and K^Z (RCLL and increasing) is used to enforce the solution Z staying in the constraint set Γ.

The following standard assumption on the data set (ξ, f, S, Γ) is imposed as in Peng and Xu [26], so (6.1) admits a smallest solution (Y, Z, K^Y, K^Z), in the sense that if $(\overline{Y}, \overline{Z}, \overline{K^Y}, \overline{K^Z})$ is another solution to (6.1), then $\overline{Y}_t \geq Y_t$ *a.s.* for $t \in [0, T]$.

Assumption 6.1.

- *The terminal data ξ, the driver $f_s(y,z)$, and the obstacle S satisfy Assumption 1.1;*
- *The set Γ is a closed and convex set in $\mathbb{R}^d$ including the origin;*
- *There exists at least one solution $(\overline{Y}, \overline{Z}, \overline{K^Y}, \overline{K^Z})$ to (6.1).*

When the driver $f_s(y,z)$ is independent of (y,z), denoted as f_s in such a situation, Cvitanic et al [5] gave a stochastic control representation for the solution of the constrained reflected BSDE (6.1). Indeed, define the control set $\mathcal{D}(t)$ as

$$\mathcal{D}(t) = \bigcup_{m \geq 1} \left\{ \mathbb{F}\text{-adapted process } (\nu_s)_{s \geq t} : \ \mathbb{H}^2\text{-square integrable, valued in } \Gamma^*, \right.$$
$$\left. \text{and } |\nu_s| \leq m \text{ for } s \in [t, T] \right\}.$$

The valued set Γ^* is defined as follows: Given the closed and convex set Γ, define its support function $\delta^*_\Gamma(\cdot)$ as the convex dual of the characteristic function $\delta_\Gamma(\cdot)$ of Γ,

$$\delta^*_\Gamma(z) = \sup_{\bar{z} \in \mathbb{R}^d} \left\{ \bar{z} \cdot z - \delta_\Gamma(\bar{z}) \right\},$$

which is bounded on compact subsets of the barrier cone Γ^*,

$$\Gamma^* = \left\{ z \in \mathbb{R}^d : \delta^*_\Gamma(z) < \infty \right\}.$$

Given $\nu \in \mathcal{D}(t)$, define an equivalent probability measure $\mathbf{P}^\nu$ as

$$\frac{d\mathbf{P}^\nu}{d\mathbf{P}} = \exp\left\{ \int_0^\cdot \nu_s dW_s - \frac{1}{2} \int_0^\cdot |\nu_s|^2 ds \right\}.$$

Then the value of the following stochastic control problem

$$y_t = \operatorname*{ess\,sup}_{\tau \in \mathcal{R}(t), \nu \in \mathcal{D}(t)} \mathbf{E}^{\mathbf{P}^\nu} \left[\int_t^{\tau \wedge T} [f_s - \delta^*_\Gamma(\nu_s)] ds + S_\tau \mathbf{1}_{\{\tau < T\}} + \xi \mathbf{1}_{\{\tau \geq T\}} | \mathcal{F}_t \right] \tag{6.2}$$

is given by the solution to the constrained reflected BSDE (6.1) with the driver f_s: $y_t = Y_t$ *a.s.* for $t \in [0, T]$.

On the other hand, (6.1) can be solved by approximating two "local time" processes K^Y and K^Z by

$$K_t^{Y,\lambda} = \int_0^t \lambda \max\{0, S_s - Y_s^{(\lambda,m)}\} ds$$

and

$$K_t^{Z,m} = \int_0^t m \times \text{dist}_\Gamma(Z_s^{(\lambda,m)}) ds = \int_0^t m \times \inf_{z \in \Gamma} |z - Z_s^{(\lambda,m)}| ds$$

respectively, where $(Y^{(\lambda,m)}, Z^{(\lambda,m)})$ is the solution of the following constrained penalized BSDE

(6.3)
$$\begin{aligned} Y_t^{(\lambda,m)} = \xi + \int_t^T & f_s(Y_s^{(\lambda,m)}, Z_s^{\lambda,m}) + \lambda \max\{0, S_s - Y_s^{(\lambda,m)}\} + m \times \mathrm{dist}_\Gamma(Z_s^{(\lambda,m)}) ds \\ & - \int_t^T Z_s^{(\lambda,m)} dW_s. \end{aligned}$$

Peng and Xu [26] proved that the solution $(Y^{(\lambda,m)}, Z^{(\lambda,m)}, K^{Y,\lambda}, K^{Z,m})$ converges to the smallest solution (Y, Z, K^Y, K^Z) of the constrained reflected BSDE (6.1) in the sense of monotonic limit theorem as $\lambda, m \uparrow \infty$.

Our aim in this section is to give a stochastic control representation of the constrained penalized BSDE (6.3), which has a similar structure to the stochastic control representation (6.2).

PROPOSITION 6.2. *Suppose that Assumption 6.1 holds. Denote $(Y^{(\lambda,m)}, Z^{(\lambda,m)})$ as the unique solution to the constrained penalized BSDE (6.3). For any $t \in [0,T]$, define the control set $\mathcal{D}(t,m)$ as*

$$\begin{aligned} \mathcal{D}(t,m) = \big\{ & \mathbb{F}\text{-adapted process } (\nu_s)_{s\geq t} : \ \mathbb{H}^2\text{-square integrable, valued in } \Gamma^* \\ & \text{and } |\nu_s| \leq m \text{ for } s \in [t,T] \big\}, \end{aligned}$$

and for any integer $i \geq 1$, the control set $\mathcal{R}_{T_i}(\lambda)$ as in Theorem 1.2. Then conditional on $\{T_i \leq t < T_i\}$, the value of the following stochastic control problem

$$\begin{aligned} y_t^{(\lambda,m)} = \operatorname*{ess\,sup}_{\tau \in \mathcal{R}_{T_i}(\lambda), \nu \in \mathcal{D}(t,m)} \mathbf{E}^{\mathbf{P}^\nu} \Big[& \int_t^{\tau \wedge T} [f_s(Y_s^{(\lambda,m)}, Z_s^{(\lambda,m)}) - \delta_\Gamma^*(\nu_s)] ds \\ & + S_\tau \mathbf{1}_{\{\tau < T\}} + \xi \mathbf{1}_{\{\tau \geq T\}} | \mathcal{G}_t \Big] \end{aligned} \tag{6.4}$$

is given by the solution to the constrained penalized BSDE (6.3): $y_t^{(\lambda,m)} = Y_t^{(\lambda,m)}$ a.s.. The optimal stopping time $\tau_{T_i}^* \in \mathcal{R}_{T_i}(\lambda)$ is given by

$$\tau_{T_i}^* = \inf\{T_N \geq T_i : Y_{T_N}^{(\lambda,m)} \leq S_{T_N}\} \wedge T_{M+1}, \tag{6.5}$$

and the optimal control $v^* \in \mathcal{D}(t,m)$ is the solution of the following algebraic equation

$$m \times \mathit{dist}_\Gamma(Z_s^{(\lambda,m)}) = Z_s^{(\lambda,m)} \cdot v_s^* - \delta_\Gamma^*(v_s^*), \ \text{for a.e. } (s,\omega) \in [t,T] \times \Omega. \tag{6.6}$$

Proof. We only consider the linear case $f_s(y,z) = f_s$, as the proof for the nonlinear case $f_s(y,z)$ is the same as the one in Section 2.2.

First, we remark that if $v \in \mathcal{D}(t,m)$, in particular $|v_s| \leq m$, then the support function $\delta_\Gamma^*(v_s)$ has the convex dual representation

$$\delta_\Gamma^*(v_s) = \sup_{z \in \mathbb{R}^d} \{ z \cdot v_s - m \times \mathrm{dist}_\Gamma(z) \}, \ \text{for } a.e. \ (s,\omega) \in [t,T] \times \Omega.$$

See Lemma 3.1 in [5] for the proof. Intuitively, it means that we use $m \times \mathrm{dist}_\Gamma(\cdot)$ to approximate the characteristic function $\delta_\Gamma(\cdot)$. Moreover, as shown in [5], since $m \times \mathrm{dist}_\Gamma(\cdot)$ is convex,

$$m \times \mathrm{dist}_\Gamma(Z_s^{(\lambda,m)}) = \sup_{\nu \in \mathcal{D}(t,m)} \{ Z_s^{(\lambda,m)} \cdot \nu_s - \delta_\Gamma^*(\nu_s) \}, \ \text{for } a.e. \ (s,\omega) \in [t,T] \times \Omega,$$

and there exists $v^* \in \mathcal{D}(t,m)$ solving the algebraic equation (6.6).

Now for any control $\nu \in \mathcal{D}(t,m)$, we rewrite (6.3) as

(6.7)
$$\begin{aligned} Y_t^{(\lambda,m)} &= \xi + \int_t^T \left[f_s + \lambda \max\{0, S_s - Y_s^{(\lambda,m)}\} - \delta_\Gamma^*(\nu_s) + Z_s^{(\lambda,m)} \cdot \nu_s \right] ds \\ &\quad + \int_t^T \left[m \times \mathrm{dist}_\Gamma(Z_s^{(\lambda,m)}) - Z_s^{(\lambda,m)} \cdot \nu_s + \delta_\Gamma^*(\nu_s) \right] ds - \int_t^T Z_s^{(\lambda,m)} dW_s. \end{aligned}$$

Since

$$m \times \mathrm{dist}_\Gamma(Z_s^{(\lambda,m)}) - Z_s^{(\lambda,m)} \cdot \nu_s + \delta_\Gamma^*(\nu_s) \geq 0$$

for any $\nu \in \mathcal{D}(t,m)$, from the BSDE comparison theorem, $Y_t^{(\lambda,m)} \geq Y_t^{(\lambda,m)}(\nu)$, where $Y^{(\lambda,m)}(\nu)$ is the solution of the following BSDE

$$\begin{aligned} Y_t^{(\lambda,m)}(\nu) &= \xi + \int_t^T \left[f_s + \lambda \max\{0, S_s - Y_s^{(\lambda,m)}(\nu)\} - \delta_\Gamma^*(\nu_s) + Z_s^{(\lambda,m)}(\nu) \cdot \nu_s \right] ds \\ &\quad - \int_t^T Z_s^{(\lambda,m)}(\nu) dW_s, \end{aligned}$$

or equivalently, under the probability measure $\mathbf{P}^\nu$,

$$Y_t^{(\lambda,m)}(\nu) = \xi + \int_t^T \left[f_s + \lambda \max\{0, S_s - Y_s^{(\lambda,m)}(\nu)\} - \delta_\Gamma^*(\nu_s) \right] ds - \int_t^T Z_s^{(\lambda,m)}(\nu) dW_s^\nu,$$

where $W_s^\nu = W_s - \int_0^s v_u du$ for $s \geq 0$ is the Brownian motion under the probability measure $\mathbf{P}^\nu$.

From Theorem 1.2, we know that conditional on $\{T_{i-1} \leq t < T_i\}$, $Y_t^{(\lambda,m)}(\nu) \geq y_t^{(\lambda,m)}(\tau,\nu)$ for any stopping time $\tau \in \mathcal{R}_{T_i}(\lambda)$, where

$$y_t^{(\lambda,m)}(\tau,\nu) = \mathbf{E}^{\mathbf{P}^\nu} \left[\int_t^{\tau \wedge T} [f_s - \delta_\Gamma^*(\nu_s)] ds + S_\tau \mathbf{1}_{\{\tau < T\}} + \xi \mathbf{1}_{\{\tau \geq T\}} | \mathcal{G}_t \right]. \tag{6.8}$$

Hence, $Y_t^{(\lambda,m)} \geq y_t^{(\lambda,m)}(\tau,\nu)$. Taking the supremum over $\tau \in \mathcal{R}_{T_i}(\lambda)$ and $\nu \in \mathcal{D}(t,m)$ gives us $Y_t^{(\lambda,m)} \geq y_t^{(\lambda,m)}$ on $\{T_{i-1} \leq t < T_i\}$.

Next, we choose $\nu = \nu^*$ and $\tau = \tau_{T_i}^*$ to get the reverse inequality. Indeed, for v^* solving (6.6), $Y_t^{(\lambda,m)} = Y_t^{(\lambda,m)}(\nu^*)$. Moreover, if we choose $\nu = \nu^*$ and $\tau = \tau_{T_i}^*$, we get

$$\tau_{T_i}^* = \inf\{T_n \geq T_i : Y_{T_n}^{(\lambda,m)} \leq S_{T_n}\} \wedge T_{M+1} = \inf\{T_n \geq T_i : Y_{T_n}^{(\lambda,m)}(\nu^*) \leq S_{T_n}\} \wedge T_{M+1}.$$

From Theorem 1.2, $Y_t^{(\lambda,m)}(\nu^*) = y_t^{(\lambda,m)}(\tau_{T_i}^*, \nu^*) \leq y_t^{(\lambda,m)}$. Therefore, $Y_t^{(\lambda,m)} = y_t^{(\lambda,m)}$ on $\{T_{i-1} \leq t < T_i\}$, and $(\nu^*, \tau_{T_i}^*)$ are the optimal control and optimal stopping time of (6.4) respectively. □

7. Conclusion. In this paper, we find the stochastic control representations of (multidimensional, constrained) reflected BSDEs and associated penalized BSDEs, which are summarized in the following table. The main feature of the related optimal

Table 7.1
Stochastic Control Representations of Reflected BSDEs and Penalized BSDEs

	Stochastic control representations
Reflected BSDE	(1.2) with $\tau \in \mathcal{R}(t)$/(4.2) with $r \in \mathcal{A}(t)$
Penalized BSDE	(1.5) with $\tau \in \mathcal{R}_{T_i}(\lambda)$/ (4.1) with $r \in \mathcal{A}(t,\lambda)$
Multidimensional Reflected BSDE	(5.2) with $u \in \mathcal{K}_i(t)$
Multidimensional Penalized BSDE	(5.6) with $u \in \mathcal{K}_i(t,\lambda)$/(5.14) with $u \in \mathcal{K}_i(t,Q)$
Constrained Reflected BSDE	(6.2) with $\tau \in \mathcal{R}(t)$ and $\nu \in \mathcal{D}(t)$
Constrained Penalized BSDE	(6.4) with $\tau \in \mathcal{R}_{T_i}(t,\lambda)$ and $\nu \in \mathcal{D}(t,m)$

stopping representation is that the player only stops at arrival times of some exogenous Poisson process.

Finally, it seems that the only existing representation result for penalized BSDE was given by Lepeltier and Xu in [19] and [20][2], where they found a connection between penalized BSDE and a standard optimal stopping problem with modified obstacle $\min\{S_t, Y_t^\lambda\}$. Our represent results are different, and seem more natural: Penalized BSDE is nothing but a random time discretization of the optimal stopping representation for the corresponding reflected BSDE, where the time is discretized by Poisson arrival times.

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