Cohomology of Kähler Manifolds with $c_1 = 0$

S. M. Salamon

To Professor Calabi on his 70th birthday

Introduction

A more informal title of this report might be ‘Betti, Hodge and Chern numbers’, without an implied order of preference. To begin with, let $M$ be a compact connected Kähler surface, so that the real dimension of $M$ is 4. The Betti numbers of $M$ are given in terms of Hodge numbers by the usual relations

$$b_1 = h^{1,0} + h^{0,1} = 2h^{0,1}$$
$$b_2 = h^{2,0} + h^{1,1} + h^{0,2} = h^{1,1} + 2h^{0,2}.$$

Noether’s formula states that

$$\langle \frac{1}{12}(c_1^2 + c_2), [M] \rangle = 1 - h^{0,1} + h^{0,2},$$

and the left-hand side is by definition the Todd genus of $M$. Moreover, the Euler characteristic $e(M)$ is equal to either side of the equation

$$\langle c_2, [M] \rangle = 2 - 2b_1 + b_2.$$

If the first Chern class $c_1$ of $M$ vanishes over $\mathbb{Z}$, then the canonical bundle of $M$ is trivial, and $h^{0,2} = 1$. As a consequence, we obtain the formula

$$0 = 22 - 4b_1 - b_2. \quad (1)$$

In these circumstances, the classification of complex surfaces implies that $M$ must in fact be a torus $T$ ($b_1 = 4$, $b_2 = 6$) or a K3 surface $K$ ($b_1 = 0$, $b_2 = 22$). We may construct $K$, at least up to diffeomorphism, by resolving the 16 singular points of the orbifold $T/\mathbb{Z}_2$, where $\mathbb{Z}_2$ is generated by the mapping $x \mapsto -x$. Observe that the right-hand side of (1) assumes the value 16 for $T/\mathbb{Z}_2$ ($b_1 = 0$, $b_2 = 6$); in fact, we shall see that the formula has a significance which transcends the examples.

We move on to consider the case in which $M$ is now a Kähler manifold of complex dimension 4. The introduction to [16] contains a formula which (after slight rearrangement) bears a striking similarity to (1). Namely, if $\chi^p$ denotes the alternating sum $\sum_{q=0}^4 (-1)^q h^{p,q}$ of Hodge numbers of $M$ then

$$\langle c_1 c_3, [M] \rangle = 22\chi^0 - 4\chi^1 - \chi^2. \quad (2)$$

This equation is generalized by Theorem 2, which asserts that on any almost complex $n$-dimensional manifold the characteristic number $\langle c_1 c_{n-1}, [M] \rangle$ can be expressed in terms of the integers $\chi^p$,
$0 \leq p \leq \lfloor n/2 \rfloor$. This is a simple corollary of the Riemann-Roch theorem that was derived in [40] (an expanded version of this article) by differentiating a suitable K-theoretical expression. The author subsequently discovered that the corollary had been used earlier by Narasimhan and Ramanan [32], in a context discussed briefly at the end of Section 3. In any case, we deduce below that if $n = 2m$ is even and $c_1 = 0$ then the ‘mirror symmetry’

$$h^{p,q} \leftrightarrow h^{n-p,q}, \quad 0 \leq p, q \leq n,$$

(3)

reverses the sign of a certain linear combination $\Phi$ of Betti numbers, defined in Theorem 1. This complements the obvious fact that the operation (3) acts as $(-1)^n$ on the Euler characteristic $\chi$, and adds weight to the view that (3) has some significance in higher dimensions.

Compact hyper-Kähler manifolds constitute a class of Kähler manifolds with $c_1 = 0$ and whose Hodge numbers are invariant by the symmetry (3). It follows from above that their Betti numbers are subject to the equation $\Phi = 0$, of which (1) is a special case. This provides an effective means to prove the non-existence of hyper-Kähler metrics on particular manifolds. The name ‘hyper-Kähler’ was introduced by Calabi [8, 9], who gave the first non-trivial examples of these metrics, and highlighted their importance prior to their appearance on symplectic quotients and on moduli spaces of anti-self-dual connections [23, 26, 27, 31]. Although our ‘hyper-Kähler constraint’ $\Phi = 0$ is ultimately derived from the Atiyah-Singer Index Theorem, it is in some ways an independent result and may be expected to have a more direct proof. Quite why it respects functorial properties of hyper-Kähler manifolds is briefly explained in Section 3, with reference to the now standard examples of Beauville [4]; more details are in [40]. The remainder of Section 3 is of a different character, as we discuss the topology of a selection of topical examples which fall outside the scope of the title, but which nonetheless are suggestive of a more extensive theory linking geometrical structures and cohomology.
1. Background and main results

Throughout, \( M \) denotes a compact oriented manifold of real dimension \( d \). Recall that \( M \) is Kähler if it possesses both a Riemannian metric \( g \) and an orthogonal complex structure \( J \) for which the resulting 2-form
\[
\omega(X,Y) = g(JX,Y), \quad X,Y \in TM,
\] (4)
is closed. We shall always denote the complex dimension of \( M \) by \( n \), so that \( d = 2n \). We denote the \( k \)-th Chern class of (the holomorphic tangent bundle of) \( M \) by \( c_k \), and we shall generally regard it as an element of \( H^{2k}(M, \mathbb{R}) \). In particular, the first Chern class \( c_1 \) of \( M \) is represented by \( 1/2\pi \) times the Ricci tensor (converted into a 2-form by \( J \) in analogy to (4)). If \( c_1 \) vanishes over \( \mathbb{R} \), then Yau’s proof of the Calabi conjecture implies that \( M \) admits a Kähler metric with zero Ricci tensor [44, 6]. This implies that the holonomy group \( \text{Hol}(M) \) of the Levi Civita connection is contained in the subgroup \( SU(n) \) of \( SO(d) \).

For example, a quartic hypersurface in \( \mathbb{CP}^3 \) is a K3 surface which is manifestly Kähler and has \( c_1 = 0 \); it must therefore admit a Kähler (‘Calabi-Yau’) metric with holonomy group equal to \( SU(2) \). The latter is best identified with the group \( \text{Sp}(1) \) of unit quaternions. More generally, the Cheeger-Gromoll splitting theorem implies that a compact Kähler manifold with \( c_1 = 0 \) has a finite holomorphic covering of the form
\[
T^{2k} \times X_1 \times \cdots \times X_r \times Y_1 \times \cdots \times Y_s,
\] (5)
where \( T^{2k} \cong \mathbb{C}^k/\mathbb{Z}^{2k} \) is a complex torus, \( X_i \) and \( Y_j \) are simply-connected, and
\[
\text{Hol}(X_i) = SU(n_i), \quad \text{Hol}(Y_j) = \text{Sp}(m_j)
\]
with \( m_j \geq 1 \) and \( n_i \geq 3 \). Further details may be found in [4, 6, 7, 28].

A Riemannian manifold \( M \) of real dimension \( d = 4m \) which, like the \( Y_j \) or their products, has \( \text{Hol}(M) \subseteq \text{Sp}(m) \) is said to be hyper-Kähler. Such a manifold can be characterized by the existence of a a triple of orthogonal almost complex structures \( J_1, J_2, J_3 \) satisfying the algebraic condition \( J_1J_2 = J_3 = -J_2J_1 \) and such that the associated 2-forms \( \omega_1, \omega_2, \omega_3 \) defined by (4) are all closed. The latter condition (in contrast to the case of a single almost complex structure) implies that each \( J_i \) gives rise to a complex, and therefore Kähler, structure [23, 37]. A hyper-Kähler manifold \( M \) actually possesses a continuous family of complex structures, namely \( \sum_{i=1}^3 a_iJ_i \) with \( \sum_{i=1}^3 (a_i)^2 = 1 \), parameterized by \( S^2 \) or \( \mathbb{CP}^1 \). The resulting deformation or twistor space \( M \times \mathbb{CP}^1 \) was emphasized from the start in Calabi’s study [9]. A hyper-Kähler manifold \( M \) is irreducible if and only if \( \text{Hol}(M) = \text{Sp}(m) \); in this case \( h^{2,0} = 1 \). By contrast, the manifolds \( X_i \) in (5) have \( h^{2,0} = 0 \).

We shall denote the Poincaré polynomial of the compact oriented \( d \)-manifold \( M \) by \( P(M; t) \), or by \( P(t) \) if the latter causes no confusion. Thus, if the Betti numbers of \( M \) are denoted by \( b_j \), \( 0 \leq j \leq d \), then
\[
P(t) = \sum_{j=0}^d b_j t^j = \sum_{j=0}^d b_j t^{d-j}.
\] (6)
The alternating sum \( P(-1) \) equals the Euler characteristic \( e(M) \); our first main result concerns an analogous expression.
Theorem 1. Let $M$ be a compact oriented manifold of real dimension $d = 4m$. Set
\[ \Phi(M) = 6P''(-1) + \frac{1}{2} d(5 - 3d)P(-1). \]
If $M$ has a hyper-Kähler metric then $\Phi(M) = 0$.

If $m = 1$ then $\Phi(M)$ equals twice the right-hand side of (1); for $m = 2$ we obtain
\[ \frac{1}{4} \Phi(M) = 46 - 25b_1 + 10b_2 - b_3 - b_4. \] (7)

A proof of Theorem 1 based upon index theory for the Dirac operator was sketched in [38]. Below, we shall derive it from a corresponding result for almost complex manifolds, Theorem 2 below, by representing $\Phi$ as a suitable characteristic number.

There is an elementary reason why the first derivative $P'(−1)$ is not needed in the definition of $\Phi(M)$. For from (3),
\[ P'(−1) = - \sum_{j=1}^{d} (-1)^j j b_j = - \sum_{j=1}^{d} (-1)^j (d - j) b_j, \]
and adding the last two members,
\[ 2P'(−1) = -dP(−1). \] (8)

Using this equation, Theorem 1 may be expressed in the equivalent form
\[ me(M) = 6 \sum_{j=0}^{2m-1} (-1)^j (2m - j)^2 b_j. \] (9)

It is well known that on a Kähler manifold, $b_j \equiv 0 \mod 2$ if $j$ is odd; similarly on a hyper-Kähler manifold, $b_j \equiv 0 \mod 4$ if $j$ is odd [43, 14]. The next result combines this fact with (9).

Corollary. Let $M$ be a compact hyper-Kähler $4m$-manifold. Then $me(M) \equiv 0 \mod 24$, and in particular $e(M) \equiv 0 \mod 2$ if $m \not\equiv 0 \mod 8$.

Now let $M$ be a compact almost complex manifold of real dimension $2n$. The choice of an almost Hermitian metric on $M$ enables one to define the formal adjoint $\overline{\partial}^* = -* \overline{\partial}$ of the $\overline{\partial}$ operator. There is then an elliptic differential operator
\[ \bigoplus_{q \text{ even}} \Omega^{p,q} \overline{\partial} + \bigoplus_{q \text{ odd}} \Omega^{p,q}, \]
whose index is denoted by $\chi^p$ in the notation of [22]. The next results are valid in this general setting, although when the almost complex structure on $M$ is integrable, $\chi^p$ is more conveniently defined as $\sum_{q=0}^{n} (-1)^q h^{p,q}$, where $h^{p,q}$ is the dimension of the corresponding Dolbeault cohomology space or equivalently of the Čech space $H^q(M, \mathcal{O}(\wedge^p T^*))$.

In all cases there is the ‘Serre duality’ relation
\[ \chi^{n-p} = (-1)^n \chi^p, \] (10)
and we set
\[\chi(t) = \sum_{p=0}^{n} \chi^p t^p = (-1)^n \sum_{p=0}^{n} \chi^{n-p} t^p, \tag{11}\]
which is often denoted by \(\chi_t\). For example,
\[\chi(-1) = \sum_{p=0}^{n} (-1)^p \chi^p = \sum_{j=0}^{2n} (-1)^j b_j \tag{12}\]
coincides with \(P(-1) = e(M)\). As we shall explain in the next section, the well-known formula
\[\langle c_n, [M] \rangle = \chi(-1) \tag{13}\]
may be regarded as the first of a sequence expressing the coefficients of the polynomial \(\chi(-1 - t)\) (which is in some ways more natural than \(\chi(t)\)) in terms of Chern numbers. The quadratic term yields

**Theorem 2.** Let \(M\) be a compact almost complex manifold of real dimension \(2n\). Then
\[\langle c_1c_{n-1}, [M] \rangle = 6\chi''(-1) + \frac{1}{2}n(5-3n)\chi(-1).\]

An equivalent version of the formula can be found at the end of \[32\]. If the complex dimension \(n\) is odd then (12) and (13) imply immediately that \(e(M)\) is even. The following are slightly less obvious consequences of Theorem 2 implicit in the theory of SU cobordism described in \[11\].

**Corollary.** Let \(M\) be a compact almost complex manifold of real dimension \(2n\) with \(c_1 = 0\). Then (i) \(n e(M) \equiv 0 \mod 3\), and (ii) if \(n \equiv 2 \mod 4\) then \(e(M) \equiv 0 \mod 2\).

As regards (i), which is also follows from \[21\], we remark that when \(n = 3\) there are some familiar manifolds with \(3\) admitting non-integrable almost complex structures, for instance \(S^6\) and \(\mathbb{CP}^3\). If \(M\) is simply-connected and \(c_1 = 0\) then the structure group of \(M\) reduces to \(SU(n)\) and lifts to \(Spin(2n)\), and (ii) also follows from Ochanine’s theorem \[34\]. The latter states that a compact oriented smooth spin manifold of real dimension \(d \equiv 4 \mod 8\) has signature divisible by 16 (and therefore \(e(M)\) divisible by 2). Other divisibility properties of the Chern numbers appear in the next section, and related results on Chern (or rather Segre) numbers can also be found in \[1\] and references therein.

### 2. Proofs and generalizations

In this section, \(T\) denotes the holomorphic tangent bundle of an almost complex manifold of real dimension \(2n\). We give the total Chern class of \(T\) a formal factorization
\[c(T) = \prod_{i=1}^{n} (1 + x_i),\]
so that the Chern classes of \( T \) may be regarded as the elementary symmetric polynomials in the symbols \( x_1, \ldots, x_n \). The Chern character of \( T \) is then given by

\[
\text{ch}(T) = \sum_{i=1}^{n} e^{x_i} = n + s_1 + \frac{1}{2!} s_2 + \frac{1}{3!} s_3 + \cdots
\]  

(14)

where \( s_k = \sum_{i=1}^{n} x_i^k \). We shall in fact need \( \text{ch}(\wedge^p T^*) \), which is formed by replacing the \( n \) elements \( x_i \) by the \( \binom{n}{p} \) elements \(- (x_{i_1} + x_{i_2} + \cdots + x_{i_p})\), \( i_1 < i_2 < \cdots < i_p \), which are the weights of the representation defining \( \wedge^p T^* \). Recall also that the Todd class of \( T \) is given by

\[
\text{td}(T) = \prod_{i=1}^{n} \frac{x_i}{1 - e^{-x_i}} = 1 + \frac{1}{2} c_1 + \frac{1}{12} (c_1^2 + c_2) + \frac{1}{24} c_1 c_3 + \cdots
\]

With the above preliminaries, the general form of the Hirzebruch-Riemann-Roch theorem [3, 20] implies, making use of (11), that

\[
\chi(t) = (-1)^n \sum_{p=0}^{n} t^p \langle \text{ch}(\wedge^{n-p} T^*) \text{td}(T), [M] \rangle.
\]

(15)

\[
= (-1)^n \langle \text{ch}(V(t)) \text{td}(T), [M] \rangle,
\]

(16)

where

\[
V(t) = \sum_{p=0}^{n} t^p \wedge^{n-p} T^*
\]

is regarded as an element of \( K(M)[t] \). Using the exterior power operation of K-theory, we may write

\[
V(-1) = \wedge^n (T^* - \underline{C}),
\]

where \( \underline{C} \) denotes a trivial line bundle. Moreover, \( V(-1) \) may be differentiated with respect to \( t \) to obtain analogous expressions

\[
V'(-1) = \wedge^{n-1} (T^* - \underline{C}^2),
\]

\[
V''(-1) = 2 \wedge^{n-2} (T^* - \underline{C}^3),
\]

where \( \underline{C}^k \) denotes a trivial line bundle with fibre \( \mathbb{C}^k \).

Suppose for a moment that \( T^* \) contains \( \underline{C}^3 \) as a subbundle, so that \( T^* - \underline{C}^3 \) is a genuine complex vector bundle of rank \( n - 3 \). This effectively corresponds to the case in which \( c_k = 0 \) for \( k > n - 3 \). Then \( V''(-1) \) is zero, merely by virtue of its dimension. Accordingly, we may deduce that in general \( \text{ch}(V''(-1)) \) belongs to the ideal \( \langle c_{n-2}, c_{n-1}, c_n \rangle \) generated by the ‘top three’ Chern classes, and the proof of Theorem 2 is completed by the more precise

**Lemma.** \((-1)^n \text{ch}(V''(-1)) \text{td}(T) = 2c_{n-2} + (n-1)c_{n-1} + \frac{1}{12} (2c_1 c_{n-1} + n(3n-5)c_n).\)

This equation is derived from the formal factorization

\[
(-1)^n \text{ch}(V(-1-t)) \text{td}(T) = \prod_{i=1}^{n} \left( x_i + t \frac{x_i}{1 - e^{-x_i}} \right).
\]

(17)
The coefficient of $t^2$ in (17) is the sum of \( \binom{n}{2} \) terms, one of which is
\[
[1 + \frac{1}{2}(x_1 + x_2) + \frac{1}{12}(x_1^2 + 3x_1x_2 + x_2^2)]x_3x_4 \cdots x_n.
\] (18)

On the other hand, $c_1c_{n-1}$ gives rise to a sum of $n$ terms, one of which is
\[
(x_1 + \cdots + x_n)x_2 \cdots x_n = x_1x_2 \cdots x_n + x_2^2x_3 \cdots x_n + \cdots
\] (19)
The proof of the lemma is completed from a comparison of (18) and (19).

We may rewrite (17) in the form
\[
\chi(-1 - t) = \langle K^*_n(t), [M] \rangle,
\]
or equivalently
\[
\chi^{(k)}(-1) = (-1)^k k! \langle K^*_n, [M] \rangle,
\] (20)
where
\[
K^*_n(t) = K^*_{n,0} + K^*_{n,1}t + K^*_{n,2}t^2 + \cdots
\]
denotes the component of either side of (17) in $H^{2n}(M, \mathbb{R})$ (we use the notation of (11) in which the bullet indicates a class of top degree). A generalization of the above lemma asserts that both $K^*_{n,2k}$ and $K^*_{n,2k+1}$ belong to
\[H^{2n}(M, \mathbb{R}) \cap \langle c_{n-2k+1}, c_{n-2k+2}, \ldots, c_n \rangle,\]
with the exception of $K^*_{n,0}$ which equals $c_n$. In fact $K^*_{n,2k+1}$ is a linear combination of $K^*_{n,2j}$ for $0 \leq j \leq k$, and we have the following explicit formulae.

**Lemma.**

\[
2^{23}K^*_{n,2} = c_1c_{n-1} + \frac{1}{2}n(3n - 5)c_n,
\]
\[
2^{4325}K^*_{n,4} = [-c_1^3 + 3c_1c_2 - 3c_3]c_{n-3} + [c_1^2 + 3c_2]c_{n-2} + \frac{1}{2}(15n^2 - 85n + 108)c_1c_{n-1}
+ \frac{1}{8}n(15n^3 - 150n^2 + 485n - 502)c_n
\]
\[
2^{533517}K^*_{n,6} = [c_1^5 - 5c_1^3c_2 + 5c_1c_2^2 + 5c_2^3c_3 - 5c_2c_3 - 5c_1c_4 + 5c_5]c_{n-5}
+ \frac{1}{2}(-2c_1^4 + c_1^2c_2 + 10c_2^2 - c_1c_3 - 20c_4)c_{n-4}
+ \frac{1}{4}[-(21n^2 - 203n + 472)c_1c_3
+ (63n^2 - 609n + 1430)c_1c_2
- (63n^2 - 609n + 1388)c_3]c_{n-3}
+ \frac{1}{16}[(21n^2 - 203n + 472)c_1^2
+ (63n^2 - 609n + 1408)c_2]c_{n-2}
+ \frac{1}{256}[105n^4 - 1890n^3 + 12131n^2 - 32242n + 28800]c_1c_{n-1}
+ \frac{1}{96}n[63n^5 - 1575n^4 + 15435n^3 - 73801n^2 + 171150n - 152696]c_n.
\]

We see from (20) that, as $k$ increases, $\chi^{(k)}(-1)$ involves progressively more Chern numbers, and only if $k = n$ is even do we obtain an expression
\[
\chi^{(n)}(-1) = n! \chi^n = (-1)^n n! \chi^0
\] (21)
(a multiple of the Todd genus) in which the term $c_1^n$ appears. Taking $k = 4$ in (20),
Theorem 3. Let $M$ be a compact almost complex manifold of real dimension $2n$ with $c_1 = 0$. Then
\[
(c_2c_{n-2} - c_3c_{n-3}, [M]) = 10 \chi^{(iv)}(-1) - \frac{1}{24} n(15n^3 - 150n^2 + 485n - 502) \chi(-1).
\]

For example, when $n = 4$ and $c_1 = 0$, this is equivalent to the fact that the top Todd class reduces to $(3c_2^2 - c_4)/720$.

Corollary. Let $M$ be a compact almost complex manifold of real dimension $2n$ with $c_1 = 0$. Then
\[
2n c_n + c_2c_{n-2} - c_3c_{n-3} \equiv 0 \mod 5.
\]

More generally, on any almost complex $2n$-manifold, the indicated summands of Newton’s formula
\[
\sum_{p,q=0}^{n} (-1)^{p+q}h_{p,q} = 0
\]
are individually zero modulo $k + 1$ if if $k + 1 \geq 3$ is prime \cite{21,40}.

Let us now turn attention to the derivation of Theorem 1 from Theorem 2. First suppose that $M$ is a compact hyper-Kähler manifold of real dimension $d = 2n = 4m$. If we fix a compatible complex structure $J_1$ then the closed 2-form $\eta = \omega_2 + i\omega_3$ is holomorphic, and $\eta^n$ is nowhere zero. Wedging by $\eta^n$ is known to induce an isomorphism $H^{p,q} \rightarrow H^{n-p,q}$ between the appropriate Dolbeault cohomology spaces, which implies that the Hodge numbers of a hyper-Kähler manifold are invariant by the mirror symmetry \cite{3}; this was proved by Fujiki \cite{14}. Also, the $(n,0)$-form $\eta^n$ trivializes the canonical bundle of $M$, so that $c_1 = 0$.

The Betti and Hodge numbers of the Kähler manifold $M$ of real dimension $2n$ are related by the formula $P(t) = H(t,t)$, where
\[
H(x,y) = \sum_{p,q=0}^{n} h^{p,q}x^py^q
\]
is the so-called Hodge polynomial. Because $H(x,y)$ is symmetric in $x$ and $y$, we have
\[
P''(-1) = H_{xx}(-1, -1) + 2H_{xy}(-1, -1) + H_{yy}(-1, -1)
\]
\[
= 2(H_{xx}(-1, -1) + H_{xy}(-1, -1)).
\]
Firstly, $H(t, -1) = \chi(t)$, so
\[
H_{xx}(-1, -1) = \chi''(-1).
\]
Secondly,
\[
H_{xy}(-1, -1) = \sum_{p,q=0}^{n} (-1)^{p+q}pqh^{p,q}
\]
\[
= -\frac{1}{2} n\chi'(-1) + \frac{1}{2} \sum_{p,q=0}^{n} (-1)^{p+q}[pq - (n-p)q]h^{p,q}
\]
\[
= \frac{1}{4} n^2\chi(-1) + \frac{1}{2} \sum_{p,q=0}^{n} (-1)^{p+q}pq(h^{p,q} - h^{n-p,q}).
\]
The last equality uses an analogue of (8) that follows from (10).

Combining the above equations, we obtain
Lemma. $\Phi(M) = 2\left[6\chi''(-1) + \frac{1}{2}n(5 - 3n)\chi(-1)\right] + 6 \sum_{p,q=0}^{n} (-1)^{p+q}pq(h^{p,q} - h^{n-p,q})$.

From Theorem 2 we deduce that when $c_1c_{n-1} = 0$, the symmetry (3) reverses the sign of $\Phi$. The latter is therefore zero in the hyper-Kähler case.

3. Examples and remarks

If $M$ and $N$ are both hyper-Kähler, then their Riemannian product $M \times N$ admits an obvious hyper-Kähler structure. We must therefore understand why the constraint of Theorem 1 is preserved by the process of taking products.

First suppose that $M$ is an almost complex manifold of real dimension $2n$ with $e(M) \neq 0$. If we set

$$\gamma(M) = \frac{\langle c_1c_{n-1},[M]\rangle}{\langle c_n,[M]\rangle},$$

the identity $c(M \times N) = c(M)c(N)$ for the total Chern class implies that

$$\gamma(M \times N) = \gamma(M) + \gamma(N). \quad (22)$$

It follows from Theorem 2 that the quantity

$$\frac{6\chi''(-1) + \frac{1}{2}n(5 - 3n)\chi(-1)}{\chi(-1)},$$

and therefore

$$\psi = \frac{4\chi''(-1)}{\chi(-1)} - n^2,$$

also satisfies (22) in place of $\gamma$. When $M$ has even real dimension $d$, the same must be true of

$$\phi = \frac{4P''(-1)}{P(-1)} - d^2 \quad (23)$$

(also defined in [38]), since the coefficients of $P(t)$ satisfy (10) with $d$ in place of $n$.

In fact the additivity of $\phi$ is as elementary as that of $\gamma$, and can deduced from the observation that $\phi$ is proportional to the coefficient of $t^2$ in the formal power series

$$\log P(-1 + t) = \log e(M) - \frac{1}{2}dt + \frac{1}{8}\phi t^2 + \frac{1}{24}(3\phi + 2d)t^3 + \cdots$$

and similar remarks apply to $\psi$. Referring to Theorem 1, and still assuming that $e(M) \neq 0$, we have $\Phi(M) = (3\phi(M) + 5d)e(M)/2$. Thus, $\Phi(M)$ is zero if and only if

$$\frac{\phi(M)}{\dim(M)} = -\frac{5}{3},$$

and we may think of the right-hand side as a ‘coupling constant’ for hyper-Kähler manifolds.
Hilbert schemes of points. Let $S$ be a compact complex surface, and let $S^{(m)}$ denote its $m$-fold symmetric product obtained by quotienting the Cartesian product by the group of permutations. There is a resolution

$$\varepsilon: S^{[m]} \longrightarrow S^{(m)}$$

in which $S^{[m]}$ is the Hilbert scheme of closed 0-dimensional subschemes of length $m$ on $S$, and is a smooth complex $2m$-dimensional manifold. Each non-trivial fibre $\varepsilon^{-1}(x)$ is a product $(V_2)^{\alpha_2} \times \cdots \times (V_m)^{\alpha_m}$, where $V_i = \text{Hilb}^i(\mathbb{C}[x, y])$ is the scheme that parameterizes ideals in $\mathbb{C}[x, y]$ of colength $i$, and $\alpha_i$ denotes the number of $i$-tuples of points that have coalesced in $x \in S^{(m)}$ [14, 18].

Following an example of Fujiki [13], Beauville has proved [1] that if $S$ has a holomorphic symplectic structure then so does $S^{(m)}$ for all $m \geq 2$; its holomorphic 2-form is induced from a natural one on $S^{(m)}$. It follows that if $S$ admits a hyper-Kähler metric then so does $S^{[m]}$ for any $m \geq 2$, and we may apply this construction when $S$ is a K3 surface or a torus. If $S = T$ is a torus, then $T^{[m]}$ is not locally irreducible and has a $4^m$-fold covering of the form (5) with $k = 2$ and unique non-flat factor $Y_1$ of real dimension $4m - 4$. The latter can also be viewed as a submanifold of $T^{[m]}$ and is denoted $K_{m-1}$ in [8] and $A^{[m]}$ in [17]; when $m = 2$ it is merely the Kummer surface associated to $T$.

For any manifold $S$, the Betti numbers of $S^{(m)}$ were computed by Macdonald [30]. When $S$ has real dimension 4, we have

$$\sum_{m \geq 0} P(S^{(m)}; t)x^m = \frac{(1 + tx)^{b_1}(1 + t^3x)^{b_3}}{(1 - x)^{b_0}(1 - t^2x)^{b_2}(1 - t^4x)^{b_4}}.$$  \hspace{2cm} (24)

The form of the right-hand side indicates its generalization to higher dimensions, though in the present context we have $b_1 = b_3$ and we may assume that $b_0 = 1 = b_4$. Deeper results of Göttsche and Soergel [16, 17] give the Betti numbers of $S^{[m]}$, at least when $S$ is a projective surface, and building on (24), we have

$$P(S^{[m]}; t) = \prod_{\alpha \geq 0} P(S^{(\alpha)}; t)t^{2m - 2\alpha_1}.$$  \hspace{2cm} (25)

The sum is over all partitions $\alpha$ of $m$, each of which is uniquely determined by $m$ non-negative integers $\alpha_1, \ldots, \alpha_m$ with the property that $m = \sum_{i=1}^{\alpha_m} i\alpha_i$.

Given that the Betti numbers of $K^{[2]}$ and $K_2$ both satisfy the constraint $\Phi = 0$ given in (3), one may deduce independently of (25) that

$$P(K^{[2]}; t) = 1 + 23t^2 + 276t^4 + 23t^6 + t^8,$$
$$P(K_2; t) = 1 + 7t^2 + 8t^3 + 108t^4 + 8t^5 + 7t^6 + t^8,$$

and the Euler characteristics are

$$e(K^{[2]}) = 324, \hspace{2cm} e(K_2) = 108.$$  

One needs (25) and a related formula in [13] to treat the higher-dimensional cases, and it is amusing to note that

$$e(K^{[8]}) = 30178575, \hspace{2cm} e(K_8) = 9477$$

are both odd. As predicted by Theorem 1, we have

$$\Phi(K^{[m]}) = 0 = \Phi(K_m)$$  \hspace{2cm} (26)
for all \( m \geq 1 \); by contrast, \( \Phi(K^{(m)}) = 24m(m - 1)/25 \). Underlying (20) is the fact that for any projective surface \( S \) with \( \epsilon(S) \neq 0 \), one has

\[
\phi(S^{[m]}) = m \phi(S), \\
\psi(S^{[m]}) = m \psi(S). \tag{27}
\]

In particular, the second formula is encoded in the Hodge polynomial of \( S^{[m]} \) computed in \([17]\) and \([18]\); further details can be found in \([20]\). We emphasize that the equations (27) have taken us outside the realm of manifolds with \( c_1 = 0 \), and we shall now take the opportunity to move further afield.

**Other holonomy groups.** Let \( M \) be a compact oriented manifold of real dimension \( d = 2n \). The constituents \( P(M; -1) \) and \( P''(M; -1) \) of \( \Phi(M) \) are both zero if and only if \( (1 + t)^4 \) is a factor of \( P(M; t) \). This is certainly the case if \( k \geq 2 \) in the decomposition (23). Now suppose that \( M = N \times S^1 \), so that

\[ P(M; t) = P(N; t)(1 + t), \]

and the Euler characteristic \( P(N; -1) \) is zero. Then \( \Phi(M) = 0 \) if and only if \( P(N; t) \) is divisible by \((1 + t)^3\), which is equivalent to the equation \( P'(N; -1) = 0 \). Whilst these observations are elementary and of little interest in the case in which \( M \) is hyper-Kähler, it seems that when \( d = 8 \) both \( \Phi(M) \) and

\[ P'(N, -1) = -b_3 + 3b_2 - 5b_1 + 7 \tag{28} \]

are natural quantities to consider in the context of other holonomy reductions. Now, if \( M \) is an irreducible Riemannian \( d \)-manifold with zero Ricci tensor whose holonomy group \( \text{Hol}(M) \) is a proper subgroup of \( \text{SO}(d) \), then Berger’s theorem \([5, 6]\) implies that exactly one of the following situations occurs: (i) \( M \) is Kähler and \( \text{Hol}(M) \subseteq \text{SU}(d/2) \), (ii) \( d = 7 \) and \( \text{Hol}(M) \cong G_2 \), or (iii) \( d = 8 \) and \( \text{Hol}(M) \cong \text{Spin}(7) \).

Various examples of compact 7-manifolds of type (ii) have now been described by Joyce \([24]\) by smoothing orbifolds of the form \( T^7/\Gamma \), where \( \Gamma \) is a finite group acting on \( T^7 \) preserving the flat \( G_2 \)-structure. The latter is characterized by an invariant 3-form \( \varphi = \sum_{i=1}^7 \varphi_i \) on \( \mathbb{R}^7 \) which is the sum of simple 3-forms \( \varphi_i \), \( 1 \leq i \leq 7 \). The first example announced by Joyce actually had Betti numbers

\[ b_1 = 0, \quad b_2 = 12, \quad b_3 = 43, \tag{29} \]

so that (25) vanishes. In this case, \( \Gamma \cong (C_2)^3 \) is a abelian group of affine transformations preserving \( \varphi \), and each element of order 2 acts trivially on the 3-dimensional subspace of \( \mathbb{R}^3 \) determined by exactly one of the \( \varphi_i \); moreover \( T^7/\Gamma \) has Betti numbers \( b_1 = 0 = b_2, b_1 = 7 \) so that \( P'(T^7/\Gamma; -1) = 0 \). Topologically, the smoothing process replaces each of 12 singular 3-tori by \( T^3 \times S^2 \); each replacement adds 1 to the second Betti number and 3 to the third, thereby preserving (28). Other examples turned out to be less respectful of (28), although at the orbifold level for any \( \Gamma \) it is necessarily the case that \( P'(T^7/\Gamma; -1) \geq 0 \).

If \( N \) has type (ii) above, then the holonomy of the product metric on \( M = N \times S^1 \) is certainly a subgroup of \( \text{Spin}(7) \). Now, manifolds with holonomy contained in \( \text{Spin}(7) \) generalize those with holonomy equal to \( \text{SU}(4) \), and at the same time the two structure groups have much in common. For example, the equation \( \langle 4p_2 - p_1^2, [M] \rangle = 8c(M) \), which is valid for any almost complex 8-manifold \( M \) with \( c_1 = 0 \) (since then \( p_1 = -2c_2 \) and \( p_2 = 2c_1 + c_2^2 \)), is in fact also satisfied in the \( \text{Spin}(7) \) case, essentially because \( \text{SU}(4) \) and \( \text{Spin}(7) \) share a maximal torus \([18]\). Given that \( \text{SU}(4) \) mirror
symmetry changes the sign of $\Phi$, in addition to fixing $b_3$ and $2b_2 + b_4$, it is curious to see what can be said about these quantities in the Spin(7) case. For example, according to Joyce, there is a compact 8-manifold with holonomy equal to Spin(7) whose Betti numbers

$$b_1 = 0, \quad b_2 = 12, \quad b_3 = 16, \quad b_4 = 150$$

satisfy the constraint $\Phi = 0$ (see (7)). In this case, some standard index theory yields in addition

$$b_4^{-} = 3b_2 + 7 = 43$$

(cf. (29)), where $b_4^-$ is the dimension of the space of harmonic anti-self-dual 4-forms $\alpha$ (i.e., those satisfying $\ast \alpha = -\alpha$).

The index theory calculations in Section 2 grew out of similar ones for the class of quaternion-Kähler manifolds, which are characterized by the condition $\text{Hol}(M) \subseteq \text{Sp}(d/4)\text{Sp}(1)$. Unless $\text{Hol}(M) \subseteq \text{Sp}(d/4) \subseteq \text{SU}(d/2)$ (the hyper-Kähler case), there is no compatible Kähler structure and the Ricci tensor is non-zero. Nevertheless the Betti numbers of a compact quaternion-Kähler manifold $M$ of positive scalar curvature are also subject to a linear relation analogous to Theorem 1. It is known that the odd Betti numbers of $M$ are zero, and that if $d = 4m$ then the integers

$$\beta_{2k} = b_{2k}(M) - b_{2k-4}(M), \quad 0 \leq k \leq m,$$

(where $b_j = 0$ for $j < 0$) are non-negative [14]. These ‘primitive Betti numbers’ are subject to their own remarkable constraint

$$\sum_{k=1}^{m} k(m + 1 - k)(m + 1 - 2k)\beta_{2k} = 0,$$

(31)

which is an equivalent but neater form of the relation on the ordinary Betti numbers of $M$ that appears in [22, 39].

The character of (31) no doubt reflects its validity for certain symmetric spaces which are the obvious (and conceivably only) candidates for $M$. For instance, if $m = 7$ then (31) becomes

$$7\beta_2 + 8\beta_4 + 5\beta_6 = 5\beta_{10} + 8\beta_{12} + 7\beta_{14},$$

and is satisfied not only by the projective space $\mathbb{HP}^7$ ($\beta_{2k} = 0$), the Grassmannians $Gr_2(\mathbb{C}^9)$ ($\beta_{2k} \equiv 1$) and $Gr_4(\mathbb{R}^{11})$ ($\beta_{4k} \equiv 1, \beta_{4k-2} \equiv 0$), but also in a striking way by the exceptional space $F_4/\text{Sp}(3)\text{Sp}(1)$ ($\beta_{2k} = 0$ for $k \neq 4, \beta_8 = 1$). The arithmetic is more intriguing for the $E$-type quaternion-Kähler symmetric spaces, since their primitive Betti numbers satisfy the equation $\beta_{2m-2k} = \beta_{2k}$ which is not obviously compatible with the $k \leftrightarrow m + 1 - k$ invariance of (31)! The resulting theory is closely related to that of the signature and other elliptic genera of homogeneous spaces, which is the subject of [22].

A natural notion of self-dual connection for vector bundles over hyper-Kähler and quaternion-Kähler manifolds has long been known. This theory extends in a natural way to any Riemannian manifold with reduced holonomy, by requiring that the curvature of the connection take values in a simple summand of the holonomy algebra. For each such algebra, the resulting problem has an well-defined elliptic complex whose first cohomology group parameterizes infinitesimal deformations of the connection [16]. Investigation of the corresponding moduli spaces can be expected to provide further interplay between index theory and the topology of the manifolds discussed above.
An example with $c_1 > 0$. We conclude with an example of a Kähler manifold with ample anticanonical bundle. Namely, let $\mathcal{M}_g$ denote the moduli space of stable rank 2 vector bundles $V$ over a Riemann surface of genus $g \geq 2$ with $\mathbb{C}V$ isomorphic to a fixed line bundle of degree one. Then $\mathcal{M}_g$ is a smooth Fano manifold of complex dimension $n = 3g - 3$ and index 2, its first Chern class $c_1$ being twice a positive integral class $[33]$. The Betti numbers of $\mathcal{M}_g$ were first given by Newstead [33], and subsequently determined by a variety of methods [3, 11, 19]. In particular, Atiyah and Bott include a comparison of their own equivariant Morse theory methods with the number-theoretic approaches, and also explain how $\mathcal{M}_g$ arises from an infinite-dimensional symplectic quotient construction. The Poincaré polynomial of $\mathcal{M}_g$ is

$$P(t) = \frac{(1 + t^3)^{2g} - t^{2g}(1 + t)^{2g}}{(1 - t^2)(1 - t^4)} = (1 + t)^{2g-2} \sum_{i=0}^{g-1} (1 - t + t^2)^i t^{2g-2-2i}.$$ 

In particular, for all $g$,

$$b_2(\mathcal{M}_g) = 1, \quad b_3(\mathcal{M}_g) = 2g,$$

the sum $P(1)$ of the Betti numbers equals $2^{2g-2}g$, and the Euler characteristic $P(-1)$ is zero.

Because of the property $P(i)(-1) = 0$, $i \leq 2g - 1$, the simply-connected manifold $\mathcal{M}_g$ obviously satisfies the constraint $\Phi(\mathcal{M}_g) = 0$ of Theorem 1. This leads one to consider the polynomial $\chi_i = \chi(t)$ which can in theory be computed from a knowledge of the Chern classes $c_k$ of $\mathcal{M}_g$. Newstead and Ramanan made a number of conjectures concerning the characteristic classes of $\mathcal{M}_g$ which have been subsequently proved. In particular, $c_k = 0$ for $k > 2g - 2$ [13], and the Pontrjagin ring (which is known to be generated solely by $p_1$) vanishes in degrees $4g$ and above [22, 32]. We shall illustrate consequences of these facts in the relatively simple case $g = 3$ which is nonetheless indicative of the general situation.

The relations $c_5 = 0 = c_6$ on $\mathcal{M}_3$ imply immediately that

$$\chi(-1 + t) = at^4 + bt^5 + ct^6,$$

where $a, b, c \in \mathbb{Z}$. From [21], $c = \chi^0$ is the Todd genus, and equals 1 because $c_1 > 0$. Furthermore, $a = \chi^{(iv)}(-1)/4!$ is given by

$$\chi^{(iv)}(\mathcal{M}_3) = \frac{1}{720} \left[ (-c_3^3 + 3c_1c_2 - 3c_3)c_3 + (c_1^3 + 3c_2)c_4 \right],$$

which is seen to equal 4, for example using expressions for the Chern classes in [35]. Consequently,

$$\chi(t) = (1 + t)^4((b + 5) + (b + 2)t + t^2),$$

and from [10] we deduce that $b = -4$. We thereby arrive at a special case of the more general result

$$\chi(t) = (1 + t)^{2g-2}(1 - t)^{g-1} \quad \text{(32)}$$

which was proved in [32], and could also be derived from a knowledge of the Hodge polynomial of $\mathcal{M}_g$. Given that $b_2(\mathcal{M}_g) = 1$, $e(\mathcal{M}_g) = 0$, and $c_1 \neq 0$, the vanishing of $c_{3g-2}$ may be deduced from that of $\chi''(-1)$, although the vanishing of $c_k$ for $2g - 2 < k < 3g - 2$ appears to be altogether deeper. Finally, observe that [12] implies that the signature $\chi(1)$ of $\mathcal{M}_g$ is zero for all $g$, which is consistent with the vanishing of all Pontrjagin numbers.

Acknowledgments. The author thanks S. Donaldson, F. Hirzebruch, and M. Thaddeus for useful conversations. He is also grateful to R. Jung who computed the formula for $K^*_{6,6}$ in Section 2, and to D. Joyce for information provided in Section 3.
References