Sigma-models in Kac-Moody algebras and M-theory

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Sigma-models in Kac-Moody algebras and M-theory

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Abstract

We discuss the historical evidence for the conjecture that the non-linear realisation of the Kac-Moody algebra $E_8^{++}$, which will be referred to as $E_{11}$, describes the extension of eleven-dimensional supergravity known as M-theory. The algebraic background is presented and some of the consequences of the conjecture are explored. In particular, we present the construction of half-BPS branes using the $E_{11}$ solution generating element with low-level roots before discussing the role of general roots in the solution generating method. The correspondence between roots within $E_{11}$ and brane solutions is used to reproduce the rules for brane configurations which lead to bound and marginal states. Using these rules, we present the embeddings of simply-laced algebras within $E_{11}$ with their supergravity solution interpretation.

The use of non-linear sigma-models with symmetric spaces to describe the hidden symmetries of gravity, as well as extended gravitational theories, is reviewed. Examples include: the original work of Ehlers, the more general construction of axisymmetric stationary solutions and theories which are consistent truncations of eleven-dimensional supergravity. It is shown that these symmetries generate non-linear transformations of solutions, of which many have well-understood physical interpretations. Applications of the target space symmetries are described and used to generate and transform between solutions.

Motivated by the use of null geodesics on symmetric spaces to describe solutions of theories with hidden symmetries, we construct one-dimensional sigma models. These models are built with cosets of normal real forms of the finite, simply-laced algebras and general involution invariant subalgebras. The $\text{SL}(n, \mathbb{R})/\text{SO}(p,q)$ models, with $n \leq 4$, are reproduced before we present our work with general $A_{n(n)}$, $D_{n(n)}$ and $E_{n(n)}$. Solutions are presented for algebras of low rank and used, iteratively, to construct solutions of arbitrary rank. These models are embedded into $A_{n(n)}^{++}$ algebras to generate dual-gravity solutions and $E_{11(11)}$ to generate supergravity solutions with the configurations which are classified in the preliminary material.

A special set of maximal co-dimension $A_{2(2)}$ solutions embedded within $A_{n(n)}^{++}$ are considered which form a telescopic series of finite algebras within $A_{2(2)}^{++}$. These are shown to interpolate between known gravity solutions and exotic objects. We discuss and interpret the gravitational theory extensions which these objects are solutions of. We show that bound states of branes with various algebraic configurations are shown to possess symmetries which are easily described within the sigma model. Several examples are provided and we discuss the role of these symmetries in solution generation.
Acknowledgements

I would like to express my sincere gratitude to my advisor Paul Cook for his patience, motivation, enthusiasm and generosity with his time. Peter West has been an inspiration in my studies and I have appreciated our discussions.

During my studies I have learned a great deal from the other occupants of the PhD office at King’s. In particular, I would like to thank Mehmet Akyol, who emigrated with me from the cold Chesham building, and Moritz Küntzler. It has been a pleasure to discuss our research as well as topics ranging very far from physics and mathematics.

There have been several PhD theses which are broadly within the connection between Kac-Moody algebras and M-theory. While I have referenced relevant sections where appropriate, I cannot properly describe how useful they have been - particularly in my first year. Special thanks are due to Nassiba Tabti and Josef Lindman Hörlund.

Most importantly, I would like to thank my wife Kate and parents, Tim and Brenda, for their encouragement, support and love. Without them this thesis would not exist.
To my beloved parents, Tim and Brenda,  
for their unconditional support.
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In modern high-energy theoretical physics there is a general desire to reconcile theories which describe different phenomena. This is based on the expectation, or hope, that there exists a theory which describes all of the known forces at arbitrary energy and on the great successes in unifying previously separate physical theories. The work of the 19th Century that led to the unification of two forces into electromagnetism heavily influenced later generations of physicists and mathematicians, who have produced the remarkable Standard Model of particle physics. This has not only been a story of triumph in forming a more aesthetically pleasing quantum field theory of electromagnetism as well as the weak and strong nuclear forces, but in predicting phenomena and particles which would not be expected within the separate theories.

The Standard Model does not include gravity, for which our best description is the theory of general relativity. All of the fields of the Standard Model exist within a fixed background spacetime. In general relativity the spacetime is a dynamic manifold whose geometric qualities produce a physical description. General relativity is a classical theory which has been very successful in describing large scale physics, but fails where we would anticipate the need for a quantum theory. When we consider the Friedmann-Robertson-Walker cosmological metric near the moment of the Big Bang we find a divergence in the Ricci scalar [2], for example. There is a unique characteristic length scale, the Planck length, around which quantum corrections are expected to become very large. A natural approach for resolving this issue might be to make a quantum field theory of gravity, but this encounters renomalisability problems [3].

String theory is an area of research which is rooted in the replacement of particles with one-dimensional strings propagating in spacetime. Bosonic string theory is a quantum theory which naturally includes gravity and is therefore a candidate quantum theory of gravity. We
can introduce fermionic fields with supersymmetry in five different, consistent ways. These produce the type I, type IIA, type IIB, heterotic $SO(32)$ and heterotic $E_8 \times E_8$ superstring theories. The discovery of five consistent theories naturally troubled string theorists until it was found that all of the theories are related to each other by transformations known as S and T-dualities. This discovery led to the realisation that each of the five superstring theories provide the description of a limit within an underlying theory known as M-theory \cite{4,5}.

There is no Lagrangian formulation of M-theory and little is known about it directly. Each of the five superstring theories possesses an infinite spectrum of string states which can be identified by their excitations from a ground state. The massless states contain fields that match the content of supergravity theories, which provide a low-energy effective description of the string theories. By studying the supergravity theories we find that string theory naturally includes extended brane solutions which were not expected elements of the initial theory. While there are multiple ten-dimensional supergravity theories associated with the string theories, the eleven-dimensional supergravity which provides the low-energy description of M-theory is unique \cite{6}. It is through the study of this effective theory that a great deal of our knowledge of M-theory has been found. One very useful discovery was finding that when supergravity theories are compactified on tori the dimensionally reduced theory can be described by a coset model with some $G/H$ Lie group symmetry \cite{7,8}.

The study of coset model descriptions of dimensionally reduced theories began with four-dimensional gravity and the discovery of Ehlers’ symmetry \cite{9}. The three-dimensional theory can be written as gravity coupled to a scalar coset action with symmetry $SL(2,\mathbb{R})/SO(2)$. Supergravities with additional field content generally possess larger symmetry groups when compactified and the example of principal interest to us is eleven-dimensional supergravity, whose coset symmetries $G/H$ are listed in chapter 3. The reduction to three dimensions produces a coset with group generated by the normal real form of the largest of the exceptional classical Lie algebras $G = E_{8(8)}$.

When we further reduce the Ehler’s coset with another, commuting Killing vector the symmetry is enlarged to the infinite-dimensional group $G = SL(2,\mathbb{R})^+$, known as the Geroch group, which is the affine extension of the Ehler’s group \cite{10–12}. Analogously, it was conjectured that the reduction of eleven-dimensional supergravity to two dimensions would produce a theory with the infinite-dimensional symmetry of $E_{9(9)}$ \cite{13}. This was later supported by the discovery of $E_{9(9)}$ symmetries on the solutions of two-dimensional $N = 16$ supergravity \cite{14}.

The diffeomorphism algebra which generates the symmetries of general relativity has two subalgebras, the affine and conformal, which together with their arbitrary commutators reform the full diffeomorphism algebra \cite{15}. It was found that by isolating these two subalgebras and constructing the simultaneous non-linear realisation of both one could reproduce general relativity \cite{16}. Building off of this approach, West was able to formulate eleven-dimensional supergravity as a simultaneous non-linear realisation of the conformal group and another, referred to as $G_{11}$, which contains the inhomogenous general linear
CHAPTER 1. INTRODUCTION

The non-linear realisation of $E_{11}$ is formulated using a coset group element whose parameters are the vielbein $e_\mu^m$, the three-form $A_{\mu_1\mu_2\mu_3}$, the six-form $A_{\mu_1...\mu_6}$, the dual graviton $A_{\mu_1...\mu_8,\nu}$ as well as infinitely many more fields. To simplify calculation the infinite dimensional sub-algebra may be consistently truncated to a finite dimensional algebra. The non-linear realisation of the truncated symmetry possesses a group element parameterised by only a subset of the original fields. For example, the consistent truncation to the three fields $e_\mu^m$, $A_{\mu_1\mu_2\mu_3}$ and $A_{\mu_1...\mu_6}$ leads to a non-linear realisation that gives the bosonic sector of eleven dimensional supergravity [17]. This led to the conjecture that the non-linear realisation of $E_{11}$ provides a description of the extension of eleven-dimensional supergravity which is M-theory [18]. This statement has been refined by identifying the $l_1$ representation of $E_{11}$ [19], which provides the extension of the translations in the Poincaré algebra.

One of the topics which has been quite fertile is the connection between roots of $E_{11}$ and solutions of eleven-dimensional supergravity, as well as supergravities obtained through dimensional reduction of the eleven-dimensional theory. It was realised in [20, 21] that the group element appearing in the non-linear realisation of $\mathfrak{sl}(2,\mathbb{R}) \in E_{11}$, parameterised by $e_\mu^m$ and just one other tensor field, could be written in a way which encoded the $\frac{1}{2}$-BPS solutions of eleven-dimensional supergravity as well as those of the type IIA and type IIB string theories. This was put into a formal framework when a Lagrangian for the non-linear realisation of $E_{11}$ was constructed and it was shown that the fields for the $\frac{1}{2}$-BPS branes were exact (truncated) solutions of the equations of motion [22]. Requiring the fields to be solutions of these truncated equations of motion provides justification for the form of the gauge fields which was given as an ansatz before. Previously the Lagrangian for the non-linear realisation of $E_{10}$ had been found in [23] and used to show the appearance of an $E_{10}$ symmetry in the vicinity of a cosmological singularity. The solution to the equations of motion for the $\mathfrak{sl}(2,\mathbb{R}) \in E_{11}$ Lagrangian described a null geodesic on the coset space $\text{SL}(2,\mathbb{R})/\text{SO}(1,1)$. One might wonder why such large truncations do not trivialise the $E_{11}$ symmetry; they do not as the metric for the supergravity solution is determined by the embedding of the particular $\mathfrak{sl}(2,\mathbb{R}) \in E_{11}$. Despite the truncation to a finite dimensional sub-algebra information from the full algebra concerning the embedding of the sub-algebra is retained and used to construct the metric.

This approach was later extended to include marginal bound states of branes [24], for which the isolated algebras were composed of generators whose roots have zero inner product and therefore generate direct products of the $\mathfrak{sl}(2,\mathbb{R})$ algebra. Non-marginal bound states are encoded by $E_{11}$ roots which have non-zero inner product and which form some larger algebra than $\mathfrak{sl}(2,\mathbb{R})$. By truncating the algebra one loses some of the power of the symmetry to identify complex solutions. It is therefore interesting to carry out the non-linear realisation and identify the null geodesic solutions on larger coset groups. Truncations of $E_{11}$ to sub-algebras larger than $\mathfrak{sl}(2,\mathbb{R})$ including $\mathfrak{sl}(3,\mathbb{R})$, $\mathfrak{sl}(4,\mathbb{R})$, $\mathfrak{sl}(5,\mathbb{R})$ and $\mathfrak{so}(4,4)$ lead to coset groups whose null geodesics encode bound states of branes [1, 25–27]. The dyonic
membrane [28] and other bound state solutions were encoded as a group element in [25]. It was subsequently shown in [26] that these solution-encoding group elements could be systematically derived from a Lagrangian formulated on cosets of groups embedded in $E_{11}$. The dyonic membrane in eleven space-time dimensions, for example, is encoded by a null geodesic on the coset $SL(3,\mathbb{R})/SO(1,2)$ [26]. Many further examples of bound states were constructed in this fashion in type IIA and type IIB string theory in [1]. The one-dimensional coset models for the $SL(n,\mathbb{R})/SO(p,q)$ models with $n \leq 4$ were constructed and solved in [26]. This more detailed construction gave the gauge fields as solutions to the equations of motion, rather than an ansatz, and the parameter motion through the bound states is easily interpreted by the subgroup action on the Noether current. There are many non-marginal bound states in ten and eleven-dimensional supergravity theories. Using the rules provided by roots of $E_{11}$ and their associated supergravity solutions a classification of bound states with non-exotic finite-dimensional sub-algebras was given in [1]. The one-dimensional coset models solved in [26] provide descriptions only for the smallest algebras and finding the solutions for algebras of rank $r \geq 4$ (besides the $SL(5,\mathbb{R})$ solved in [1]) remained an open question. Many of these models are not necessarily as interesting to the general community because they contain higher-level roots of $E_{11}$ which are associated with exotic, massive solutions. While there are infinitely many exotic objects, only a few have a well-understood interpretation, such as the $D8$ brane of Romans type IIA massive supergravity and the conjectured $M9$ parent solution in M-theory [29].

Ultimately one might aspire to work with the full non-linear realisation of $E_{11}$. A stepping stone in this direction would be the complete understanding of the solutions described by null geodesics on cosets of affine groups embedded in $E_{11}$. The associated solution would be described by infinitely many parameters and would approach the complexity of the full non-linear realisation of $E_{11}$. Early work on affine cosets in this setting was carried out in [30] where the cosets on $A_{D-2}^{++}$, the over-extension of $SL(D,\mathbb{R})$, were investigated by restricting the algebra to an interesting infinite subset of generators which were argued to correspond to polarised Gowdy cosmologies. A particular embedding of the affine algebra $E_9$, which is a sub-algebra of both $E_{10}$ and $E_{11}$, was investigated in [31], where it was shown that the Weyl reflections of affine $SL(2,\mathbb{R})$ contained within $E_9$ discretely mapped supergravity solutions to exotic supergravity solutions. This infinite dimensional solution generating group was identified as the Geroch group. An affine $\mathfrak{sl}(2,\mathbb{R})$ sub-group within $E_9$ was shown to act similarly on the M2 and M5 branes of M-theory as well as the gravitational sector [31]. The infinite towers of solutions are constructed using the Weyl reflections of $\mathfrak{sl}(2,\mathbb{R})$ which form a discrete sub-algebra of $A_1^{+}$. In light of the recent successes in identifying continuous symmetry groups with bound state solutions of supergravity and string theory, it is timely to investigate whether the discrete solution generation associated with the Weyl reflections of the Geroch group might be extended to a continuous group.

We investigate the class of generalised Kac-Moody algebras denoted $A_{D-3}^{++}$, the very-extended algebra whose non-linear realisation was proposed to describe gravity [32] and
further investigated in [33]. An infinite tower of roots, associated with dual gravitons, was identified within $A_{D-3}^{++} \subset E_{11}$ in [34] and dual actions for each of these dual fields were constructed in [35, 36]. Before we describe our investigation it will be useful to motivate the study of these algebras and explain their connection to $E_{11}$. The Dynkin diagram for this class of algebras is shown in figure 1.0.1. For the case where $D = 4$ the Geroch group $A_1^+$ is manifestly embedded within $A_1^{+++}$. The non-linear realisation of $A_{D-3}^{++}$ is a theory containing only gravitational degrees of freedom. Deletion of node $D-1$ leaves the Dynkin diagram of some $E_n$ or Kac-Moody extension. The $E_n$ series of symmetries that appear upon compactification of eleven dimensional bosonic supergravity all appear in this manner by deleting node $D-1$ from the $A_{D-3}^{++}$ Dynkin diagram to give the hidden $E_n$ symmetry that appears upon dimensional reduction to $12-D$ dimensions as summarised in table 1.0.1. The dimensional reduction of $A_{D-3}^{+++}$ is akin to Kaluza-Klein dimensional reduction: a purely gravitational theory in $D+1$ dimensions whose dimensional reduction gives a gravitational theory and a gauge theory in $D$-dimensions as well as a tower of KK states.

When $D = 12$ the relevant algebra is $A_9^{+++}$ which upon dimensional reduction leaves a theory with a manifest $E_{11}$ symmetry which is conjectured to be M-theory. Moreover $E_8^{+++} \equiv E_{11} \supset A_9^{+++}$, while $E_8^{++} \equiv E_{10} \supset A_8^{++}$ and $E_8^+ \equiv E_9 \supset A_8^+ \supset E_8$ which identifies a sequence of inclusions of infinite algebras terminating with the finite algebra $E_8$:

$$A_9^{+++} \supset E_{11} \supset A_8^{+++} \supset E_{10} \supset A_8^{++} \supset E_9 \supset A_8^+ \supset E_8.$$ (1.1)

The fact that $E_{11}$ lies between $A_9^{+++}$ and $A_8^{+++}$ in this sequence coupled with the expectation that the $A_{D-3}^{+++}$, being gravitational algebras rather than p-form algebras, should provide new ways to investigate $E_{11}$ motivates the present focus on $A_{D-3}^{+++}$ algebras.

### 1.1 Organisation of the thesis

This thesis can be separated into two general parts. Chapters 2-4 provide preliminary background which is necessary to understand the the material of the following chapters. None of these chapters contains original work. In Chapter 2 we review the semi-simple Lie algebras and their real forms which are used to construct coset Lagrangians. We explore some of the symmetry actions for $SL(n, \mathbb{R})$, $n = 2, 3$ before presenting some representation theory of Lie algebras. We extend the classical Lie algebras to define Kac-Moody algebras which will
### Table 1.0.1: The decomposition of $A_{D-3}^{+++}$ containing the $E_{D-1}$ series of algebras.

<table>
<thead>
<tr>
<th>$D$</th>
<th>Dynkin diagram of $A_{D-3}^{+++}$</th>
<th>Following the deletion of node $(D - 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td><img src="image1.png" alt="Dynkin diagram" /></td>
<td><img src="image2.png" alt="Dynkin diagram" /></td>
</tr>
<tr>
<td>11</td>
<td><img src="image3.png" alt="Dynkin diagram" /></td>
<td><img src="image4.png" alt="Dynkin diagram" /></td>
</tr>
<tr>
<td>10</td>
<td><img src="image5.png" alt="Dynkin diagram" /></td>
<td><img src="image6.png" alt="Dynkin diagram" /></td>
</tr>
<tr>
<td>9</td>
<td><img src="image7.png" alt="Dynkin diagram" /></td>
<td><img src="image8.png" alt="Dynkin diagram" /></td>
</tr>
<tr>
<td>8</td>
<td><img src="image9.png" alt="Dynkin diagram" /></td>
<td><img src="image10.png" alt="Dynkin diagram" /></td>
</tr>
<tr>
<td>7</td>
<td><img src="image11.png" alt="Dynkin diagram" /></td>
<td><img src="image12.png" alt="Dynkin diagram" /></td>
</tr>
<tr>
<td>6</td>
<td><img src="image13.png" alt="Dynkin diagram" /></td>
<td><img src="image14.png" alt="Dynkin diagram" /></td>
</tr>
<tr>
<td>5</td>
<td><img src="image15.png" alt="Dynkin diagram" /></td>
<td><img src="image16.png" alt="Dynkin diagram" /></td>
</tr>
<tr>
<td>4</td>
<td><img src="image17.png" alt="Dynkin diagram" /></td>
<td><img src="image18.png" alt="Dynkin diagram" /></td>
</tr>
</tbody>
</table>
be used throughout the thesis. Using the observations in representation theory we decompose two infinite-dimensional Kac-Moody algebras, $E_8^{+++}$ and $A_n^{+++}$, into representations of a finite-dimensional subalgebra. In order to connect the coset Lagrangians with hidden supergravity symmetries we review the machinery of dimensional reduction in Chapter 3. This is applied to four-dimensional gravity and found to produce an $SL(2,\mathbb{R})$ symmetry. Five dimensional gravity reduced to three dimensions is described in two reductions which contain a (de-)coupled scalar in the first reduction to four dimensions. We show how charges are found and modified by group transformations before considering the reduction of eleven-dimensional supergravity. The example of a reduction to type IIA supergravity is shown before reviewing a method of describing the general reduction. A few comments are made on the reduction to two dimensions and infinite-dimensional symmetries. In Chapter 4 we review the construction of non-linear realisations of the conformal algebra and an extension of the affine algebra which was used to find a description of eleven-dimensional supergravity. The solution generating group elements for low levels in $E_{11}$ are given and, after finding fundamental solutions to eleven-dimensional and type II supergravity theories, we show that the group theoretic construction reproduces well-known solutions. The technique of generating solutions from group elements is extended by considering marginal and then non-marginal bound states of branes, with the dyonic membrane provided as an example. We finish the preliminary review with the technical description of the $l_1$ representation of $E_{11}$ which is used to give the full non-linear realisation containing the extension of the translations in the affine algebra.

In Chapter 5 we first review work of other authors in constructing and solving one-dimensional sigma models with $G = SL(n,\mathbb{R})$. We generalise the method of calculating the equations of motion for arbitrary simply-laced Lie algebras. Besides the $SL(n,\mathbb{R})$ series, the $D_{4(4)}$ and $E_{6(6)}$ models with particular involution invariant subalgebras are constructed and the equations of motion are given in the appendix. After providing a solution for the $SL(n,\mathbb{R})$ with an involution invariant $SO(1,n-1)$ we discuss the challenges of the $D_{n(n)}$ and $E_{n(n)}$ models. These are overcome and result in more complicated solutions which are presented for the two worked examples. Generalising these, we find solutions for the simply-laced finite-dimensional classical Lie algebras with arbitrary involution. Chapter 6 includes the application of the solutions of Chapter 5 using the Kac-Moody/Supergravity correspondence. Examples from the low-level roots of $E_{11}$ are reproduced before calculating the complete form of a bound state of branes with $SO(4,4)/(SO(2,2) \times SO(2,2)$ symmetry. Several other configurations are discussed within eleven-dimensional and type IIA supergravity. We finish the chapter with a discussion on bound states of branes which include exotic limits and comments on future work.

In Chapter 7 we focus on a particular sequence of real roots which appear within $A_{D-3}^{+++}$. We review some of the material of Chapter 5 to construct geodesic motion on $SL(2,\mathbb{R})$ and $SL(3,\mathbb{R})$ cosets embedded within $A_{D-3}^{+++}$. The spacetime objects encoded in these one-dimensional models are unsmeread to the maximal mutually transverse dimension and it is
shown that the $SL(3,\mathbb{R})$ bound states are not solutions of the Einstein-Hilbert action in the interpolation between $SL(2,\mathbb{R})$ endpoints which are solutions. We investigate matter-field containing actions, which these objects are solutions to, and discuss the obstruction to the dualisation of these theories into a purely gravitational action. Chapter 7 is based on the paper [37].
The main results presented in this thesis are based on the connections between geodesic motion through symmetric spaces and solutions to theories with coset symmetries. These connections are consequences of the conjectured Kac-Moody symmetry of extended supergravity theories and M-theory. This chapter therefore aims to both provide a review of the cosets of the real forms of semi-simple Lie groups as well as an introduction to some classes of Kac-Moody algebras used in the Kac-Moody/supergravity correspondence. The latter is done for a class of Lorentzian algebras through the decomposition of the infinite-dimensional algebra into finite-dimensional highest weight representations of a finite subalgebra and is used ubiquitously within this thesis.

The material of this chapter is based on numerous texts, including [38–44], and citations are provided for proofs not presented here, as well as for explanations which are drawn directly from other authors.
CHAPTER 2. MATHEMATICAL PRELIMINARIES

2.1 Lie Groups and Algebras

Definition 2.1.1. A topological space $T$ is a set of points $T$ with a set of subsets $U$ satisfying the following three axioms:

1. The null set $\emptyset$ and the set of all points $T$ are elements of $U$.

2. Finite intersections of sets in $U$ are still elements of $U$.

3. Any number of unions of sets in $U$ are still elements of $U$.

Each of the sets $U \in U$ are by definition open subsets of the space $T$. Of the several separability axioms that topologies can be endowed with we will choose the Hausdorff condition:

Definition 2.1.2. A topological space is Hausdorff if, for each each two distinct points $p, q \in T$, there exist sets $U_p \in U$ and $U_q \in U$ such that $p \in U_p$, $q \in U_q$ and $U_p \cap U_q = \emptyset$.

Many definitions which are quite intuitive in analysis have rather unfamiliar definitions within this framework.

Definition 2.1.3. A map $\phi : (T_1, U) \to (T_2, V)$ is continuous if the preimage of any open set $\phi^{-1}(V)$, $V \in V$, is an open set.

We now have enough machinery to give a definition of the object which will provide the foundation of our study of Lie groups.

Definition 2.1.4. A Hausdorff topological space $T$ with a collection $\Phi$ of invertible mappings $\phi_i : U_i \to \mathbb{R}^n$, $\phi_i \in \Phi$, $U_i \in U$ is a differentiable manifold $M$ given that:

1. The open sets $U_i$ cover $T$; $\cup_i U_i = T$.

2. Each $\phi_i$ maps onto an open subset of $\mathbb{R}^n$ and both $\phi_i$ and $\phi_i^{-1}$ are continuous.

3. For any $U_i$ and $U_j$ such that $U_i \cap U_j \neq \emptyset$, the map $\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ is infinitely differentiable.

The maps allow us to (locally) associate coordinates with the points of the topological space. The last condition deals with maps $\phi : \mathbb{R}^n \to \mathbb{R}^n$ for which we can employ standard calculus techniques. One topological quality, compactness, will be used ubiquitously in the identification and classification of Lie groups and algebras, as well as their many cosets:

Theorem 2.1.1 (Heine-Borel). For a subset of a metric space $S$ the following two statements are equivalent:

1. $S$ is compact: every open subcover of $S$ has a finite subcover.

2. $S$ is complete (every Cauchy sequence converges to a point inside the space) and bounded.
Definition 2.1.5. A **Lie group** $G$ is a differentiable manifold with a group operation, $\circ : G \times G \to G$ which also possess an inverse action $I : G \to G$, $g \to g^{-1}$.

Lie groups possess all of the other qualities of groups. An important subset of the Lie groups are those which may be written as matrices.

Definition 2.1.6. A **matrix Lie group** $G$ is a subgroup of set of set of invertible $n \times n$ matrices $\text{GL}(n, \mathbb{C})$ such that any sequence of matrices in $G$, which converges, must converge to either a matrix in $G$ or a non-invertible matrix.

The requirement that each sequence limits in this way is equivalent to requiring that the group be a closed subset of $\text{GL}(n, \mathbb{C})$. However, some groups are not closed in the set of general $n \times n$ matrices, hence the caveat regarding non-invertible limits. While every matrix Lie group is a Lie group, the converse is not true. The usual counterexamples are the universal covers which do not have any faithful linear representation, such as the cover of $\text{SL}(2, \mathbb{R})$.

An essential mathematical object in this thesis is the closely related Lie algebra.

Definition 2.1.7. A **Lie algebra** is a vector space $\mathfrak{g}$ with a bilinear map $\left[\cdot, \cdot\right]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ with the following properties for all $X_i \in \mathfrak{g}$:

1. $\left[X_1, X_2\right] = -\left[X_2, X_1\right]$
2. $\left[X_1, [X_2, X_3]\right] + [X_2, [X_3, X_1]] + [X_3, [X_1, X_2]] = 0$

Matrix Lie groups are directly related Lie algebras through exponentiation. The exponential map for matrices possesses several important qualities which we will make use of.

Proposition 2.1.1. For any $n \times n$ matrix $X$ over $\mathbb{C}$, the matrix exponential $e^X$ is given by the convergent series $e^X = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$ which has the properties:

1. $\det(e^X) = e^{\text{Tr}(X)}$
2. $e^X$ is non-singular
3. $t \to e^{tX}$ is a smooth curve in $\text{GL}(n, \mathbb{C})$ through the identity
4. $\frac{d}{dt}(e^{tX})|_{t=0} = X$

The first property can be easily seen by considering an arbitrary matrix which can be written as $X = YTY^{-1}$ with an upper-triangular matrix $T$. Clearly the propositional statement is true for upper-triangular matrices, so

$$
\det(e^X) = \det(Ye^T Y^{-1}) = \det(e^T) = e^{\text{Tr}(T)} = e^{\text{Tr}(X)}.
$$

(2.1)
The others are easily shown by considering the additive inverse $Y = -X$ and by calculating the $t$ derivatives of $e^{tX}$. An important difference between Lie groups and algebras is that, while there are many Lie groups which are not matrix Lie groups, the analogous statement for Lie algebras is false.

**Theorem 2.1.2 (Ado).** Every finite-dimensional Lie algebra over $\mathbb{F}$ is isomorphic to a subalgebra of $\mathfrak{gl}(n, \mathbb{F})$ for $\mathbb{F}$ either $\mathbb{R}$ or $\mathbb{C}$.

A proof of this theorem can be found in, for example, [41].

While Lie algebras can be thought of as the infinitesimal generators of Lie groups, they are more fundamental objects and will play a central role in this thesis. In order to continue with our aim of classifying the Lie algebras we require a few more definitions.

**Definition 2.1.8.** A subalgebra of a Lie algebra $\mathfrak{g}$ is a subspace $\mathfrak{h}$ such that $[H_1, H_2] \in \mathfrak{h}$ $\forall H_1, H_2 \in \mathfrak{h}$. If $[G, H] \in \mathfrak{h}$ $\forall H \in \mathfrak{h}, G \in \mathfrak{g}$ then the subalgebra is an ideal.

An important consequence of the Jacobi identity is that the Lie bilinear operation acting over any two ideals produces another ideal. The full algebra and the null algebra are (trivial) ideals so that we may define;

$$
\mathfrak{g}^{(0)} = \mathfrak{g}, \quad \mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}], \quad \mathfrak{g}^{(n+1)} = [\mathfrak{g}^{(n)}, \mathfrak{g}^{(n)}],
$$

which generates an ideal subalgebra sequence. If in this sequence we reach any $\mathfrak{g}^{n} = 0$ the algebra $\mathfrak{g}$ is solvable. Another sequence is generated by

$$
\mathfrak{g}_{(0)} = \mathfrak{g}, \quad \mathfrak{g}_{(1)} = [\mathfrak{g}, \mathfrak{g}], \quad \mathfrak{g}_{(n+1)} = [\mathfrak{g}, \mathfrak{g}_{(n)}]
$$

and if we find that some $\mathfrak{g}_{n} = 0$ the algebra $\mathfrak{g}$ is nilpotent. With these definitions we can now identify the essential quality of Lie algebras which will allow us to construct a non-degenerate inner product.

**Definition 2.1.9.** A finite dimensional Lie algebra $\mathfrak{g}$ which has no non-trivial ideals is simple. If the algebra has no non-zero solvable ideals it is semisimple.

### 2.2 Root Spaces and Dynkin Diagrams

In order to classify Lie algebras we will define a canonical construction of bases for semisimple algebras which will allow us to write out the Lie algebra bracket acting on general elements.

There are generally many representations for any given Lie algebra. Since the algebra is itself a vector space we may consider the representation of the algebra on itself, which is known as the adjoint representation. We define this representation as:

$$
\phi_{\text{ad}} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g}), \quad x \rightarrow \text{ad}_X, \quad \text{ad}_X(Y) = [X, Y].
$$
We note that the only non-trivial check for an algebra homomorphism is true, due to the Jacobi identity:

\[ [\text{ad}_{X_1}, \text{ad}_{X_2}](Y) = [X_1, [X_2, Y]] - [X_2, [X_1, Y]] = \text{ad}_{[X_1, X_2]}(Y). \] (2.5)

Using this representation we identify any algebra element \( X \) as diagonalisable if \( \text{ad}_{X}(Y) = \lambda Y \) and we focus on a unique (up to automorphisms of the algebra) subalgebra.

**Definition 2.2.1.** The linear span \( \mathfrak{g}_0 \) of a maximal set of linearly independent \( \text{ad} \)-diagonalisable elements of a Lie algebra \( \mathfrak{g} \) which commute with each other under the Lie bracket is the **Cartan subalgebra**. The dimension of \( \mathfrak{g}_0 \) is the **rank** of \( \mathfrak{g} \).

We label a choice of basis \( \{H_i | i = 1, \ldots, r\} \) with \( r = \dim \mathfrak{g}_0 \). Because each element of the Cartan subalgebra has zero Lie bracket with each other element, \( [H_i, H_j] = 0 \forall i, j \), they are simultaneously diagonalisable. The result is that any element in \( X \in \mathfrak{g} \) is an eigenvector of any element of the adjoint map of a Cartan subalgebra element \( H \in \mathfrak{g}_0 \):

\[ \text{ad}_H(X) = [H, X] = \alpha_X(H)X. \] (2.6)

When we take the action of \( \text{ad}_H \) on some general basis, \( [H, X^\mu] = \sum M^\mu_{\nu} X^\nu, \alpha_X(H) \) it must satisfy the **secular equation**:

\[ \det( M - \alpha_X(H) \mathbb{1}) = 0. \] (2.7)

In order to guarantee solutions to this equation we will require that the field \( F \) is algebraically closed and we therefore take the field to be \( \mathbb{C} \). The eigenvalue \( \alpha_X(H) \) is an element of the field \( \mathbb{C} \) and depends linearly on \( H \), so \( \alpha_X \) is a function \( \alpha_X : \mathfrak{g}_0 \rightarrow \mathbb{C} \) and the vector space of these maps forms the dual space \( \mathfrak{g}_0^* \) to the Cartan subalgebra. Since the eigenvalues are the roots of the characteristic function of \( \text{ad}_H \), we call these \( \alpha \) functions the **roots** of the algebra.

Since the algebra is spanned by elements which are eigenvectors of the Cartan subalgebra under the adjoint representation, we can take the **root space decomposition**

\[ \mathfrak{g} = \mathfrak{g}_0 \bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha, \quad \mathfrak{g}_\alpha = \{X \in \mathfrak{g} | [H, X] = \alpha_X(H)X\}. \] (2.8)

With a particular choice of basis \( \{H^i\} \) for the Cartan subalgebra, the remainder of the algebra can be constructed out of elements \( \{E^\alpha\} \) such that

\[ [H^i, E^\alpha] = \alpha^i E^\alpha, \quad \alpha^i = \alpha(H^i). \] (2.9)

The vector \( \alpha^i \) of eigenvalues with respect to the Cartan basis is called a **root vector**.

We define an inner product over the adjoint representation by using the trace of the linear maps:

\[ \kappa(X_1, X_2) = \text{tr}(\text{ad}_{X_1} \circ \text{ad}_{X_2}). \] (2.10)
CHAPTER 2. MATHEMATICAL PRELIMINARIES

This bilinear map is known as the **Cartan-Killing** form. In general, bilinear forms such as the Killing form allow us to define a map from a vector space to the dual vector space. If we write the pairing \( \langle \cdot, \cdot \rangle : g^*_0 \times g_0 \to \mathbb{C} \) then we define \( \phi : g_0 \to g^*_0 \) by

\[
\langle \phi(X_1), X_2 \rangle = \kappa(X_1, X_2).
\]  

(2.11)

We define the **radical** of a bilinear form as

\[
\text{rad} (\kappa) = \{ X \in g | \kappa(X, Y) = 0 \ \forall \ Y \in g \} \tag{2.12}
\]

and call any vector space with null radical **nondegenerate**. It is clear that the kernel of the map \( \phi \) is the radical of the bilinear form, therefore \( \phi \) is an isomorphism if and only if \( \text{rad} (\kappa) = \emptyset \). We will be focusing on semisimple Lie algebras and a well-known theorem gives us this isomorphism \( \phi \).

**Theorem 2.2.1** (Cartan’s Criterion for Semisimplicity). A Lie algebra \( g \) is semisimple if and only if the Cartan-Killing form \( \kappa \) is nondegenerate.

A proof of this is provided, for example, in [43], where the contrapositive is proven using *Cartan’s Criterion for Solvability*, that \( g \) is solvable iff \([g, g] \subset \text{rad} (\kappa)\). For semisimple Lie algebras we therefore have a dual vector space to the Cartan subalgebra with dimension \( r \) and with an inner product inherited from the Cartan-Killing form. We will identify a root \( \alpha \) with an element \( H^\alpha \) of the Cartan subalgebra, up to normalisation, such that

\[
\alpha(H) = c_\alpha \kappa(H^\alpha, H). \tag{2.13}
\]

The inner product on the root space is then

\[
(\alpha, \beta) = c_\alpha c_\beta \kappa(H^\alpha, H^\beta) = c_\beta \alpha(H^\beta). \tag{2.14}
\]

The root space can be split into two equal components by a hyperplane through the origin. We label one half the positive space and the other negative and label the \( r \) positive roots which are closest to the hyperplane **simple roots**. We take the set of simple roots \( \{ \alpha_i | i = 1, \ldots, r \} \) and each positive root is a linear sum of the simple roots \( \alpha = \sum_{i=1}^r a_i \alpha_i \). For each positive root \( \alpha \) there is one matching negative root \(-\alpha\). We define the **Cartan matrix** \( A \) as the \( r \times r \) matrix with entries:

\[
A_{ij} = 2 \frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}. \tag{2.15}
\]

We will choose to work in the Chevalley-Serre basis and we set the as-yet undetermined normalisation of equation (2.13) will be defined by

\[
c_\alpha = \frac{1}{2} (\alpha, \alpha) = \kappa(E^\alpha, E^{-\alpha})^{-1}, \tag{2.16}
\]
so that the Cartan subalgebra elements for a particular root are

$$H^\alpha = \sum_{i=1}^{r} \frac{2a_i}{(\alpha, \alpha)} H^i$$

and as a result $\alpha(H^\alpha) = 2$. We will also find that $[E^\alpha, E^{-\alpha}] = 2H^\alpha$ and $[H^\alpha, E^{\pm\alpha}] = \pm 2E^{\pm\alpha}$. By labeling all of the Cartan subalgebra elements associated with simple roots $H^{\alpha_i} = H^i$ and the positive/negative elements $E^{\pm\alpha_i} = E_i^{\pm}$, we can write out the Lie brackets for the Chevalley-Serre presentation:

$$[H^i, H^j] = 0 \quad [H^i, E^j_\pm] = \pm A^{ij} E^j_\pm \quad [E^i_+, E^j_-] = \delta_{ij} H^j$$

$$(\text{ad}_{E^i_\pm})^{1-A^{ij}} E^j_\pm = 0.$$  

The first three are equations which we have determined from our choices of normalisations and definitions above while the last is known as the Serre relations. These provide an algorithm for constructing the positive/negative root spaces. The unique information which we need to describe an algebra with this presentation is summarised in the Cartan matrix alone.

In order to classify all of the allowable Cartan matrices we introduce a graph known as a Dynkin diagram. This diagram is defined as a set of $r$ vertices which are labelled with the same $i$-indices as the Cartan matrix. Each $i$ vertex is connected to each other $j$ vertex by $A_{ij}A_{ji}$ lines. Whenever vertices are connected by more than 1 line the simple roots will have different norm values under the Killing form and a distinction in shading is made where the node of the node of the simple root with smaller magnitude is darkened. The classification

Table 2.2.1: Finite dimensional simple Lie algebra Dynkin diagrams with simple roots labeled in black and Coxeter labels in red.
may proceed by proving several rules (see for example [41]) regarding the allowed values of 
\( A_{ij} \), which translates into simple rules regarding the allowable connections.

The metric which is induced on the root space from the Cartan-Killing form is Euclidean
on the real span of the roots; for any root \( \beta \), \((\beta, \beta) \geq 0\) with equality only when \( \beta = 0 \).
The Cartan matrix is, by definition, directly related to the metric on the root space and
consequently must have \( \det(A) > 0 \). We will later relax this condition in order to consider
some infinite dimensional algebras with null and imaginary roots, where \((\beta, \beta) \leq 0 \).

The Serre relations allow us to build all of the positive(negative) generators which are
each associated with a positive(negative) root. In each of these algebras there exists a highest
root \( \Theta \) such that, for any positive root \( \beta \), \( \Theta = \beta + \sum_i a_i \alpha_i \) with positive coefficients \( a_i \). The
Coxeter labels \( a_i \) are the simple root coefficients for the highest root. All of the Dynkin
diagrams for finite semisimple Lie algebras with simple root labels and Coxeter labels are
shown in table 2.2.1. The Cartan matrices for each algebra are easily computed using the
rules described above.

## 2.3 Real Forms

In the preceding section we classified the semisimple Lie algebras using eigenvalue equations
and required the field \( F \) to be algebraically closed. As a consequence we only classified the
complex semisimple Lie algebras. In this section we will present the classification of the real
simple Lie algebras. This turns out to be equivalent to studying automorphic involution
invariant subalgebras.\(^1\)

An involutive automorphism of a Lie algebra \( g \) is an isomorphism \( \Omega : g \to g \) such
that \( \Omega^2 = 1 \). If we write this as an operator expression with operator \( O \), algebra elements
are mapped as \( X \to OX \) and the operator has eigenvalues of \( \pm 1 \):

\[
(O - 1)(O + 1) = 0. \quad (2.20)
\]

This allows us to decompose the algebra in subspaces

\[
g = t \oplus p \text{ such that } \quad O(g) = (+)t \oplus (-)p \quad (2.21)
\]

where we have indicated the eigenvalues of the subspaces. For any \( T \in t \) and \( P \in p \) we find
that

\[
(T, P) = (O(T), O(P)) = -(T, P) = 0, \quad (2.23)
\]

so the subspaces are orthogonal. The subspace \( t \) is closed under commutation;

\[
([T_1, T_2], P) = (O([T_1, T_2]), O(P))) = -([T_1, T_2], P) = 0, \quad (2.24)
\]

\(^1\)For a review of automorphic involutions and the generation of real forms we suggest chapter 9 of [40],
which this section is based on.
and is therefore the involution invariant subalgebra.

The topological qualities of the Lie group and its algebraic properties are closely related. This connection will provide us with elegant methods for classifying real forms which is taken from [40].

The standard normalisation for the inner product over the basis \(\{H^i, E^\alpha\}\) is

\[
(H^i, H^j) = \delta^{ij}, \quad (E^{\beta_1}, E^{\beta_2}) = \delta^{\beta_1+\beta_2,0}.
\]

(2.25)

The first real form that we will consider is the most natural, where we simply take the real span of these algebra elements. This is known as the \textbf{normal real form}. The Cartan-Killing form, as a matrix over this basis, then takes the form:

\[
\begin{pmatrix}
1 & & & & \\
& \ddots & & & \\
& & 0 & 1 & \\
& & 1 & 0 & \\
& & & \ddots & \\
& & & & 0 & 1 \\
& & & & 1 & 0
\end{pmatrix}
\begin{pmatrix}
H^1 \\
\vdots \\
H^n \\
E^\alpha \\
E^{-\alpha} \\
\vdots \\
E^\beta \\
E^{-\beta}
\end{pmatrix}
\]

We define the \textbf{character} \(\chi\) of a real form as the trace of the metric and in the case of the normal form it is clear that this is equal to the rank. In order to make connections between these algebraic qualities and topological properties of the Lie group we require the following theorem.

\textbf{Theorem 2.3.1.} If the Cartan-Killing form of a real form of a Lie algebra is negative definite then the algebra is compact.

Assume that the Cartan-Killing form is negative definite and select the basis \(\{X_i\}\) where \((X_i, X_j) = -\delta_{ij}\). Symmetries of the metric allow us to write

\[
([Y, X_i], X_j) + (X_i, [Y, X_j]) = 0,
\]

(2.26)

for a general algebra element \(Y\). We can see that a matrix realisation of the adjoint representation of \(Y\) is antisymmetric: \(R^i_j(Y) + R^j_i(Y) = 0\). Since this is true for any element of the algebra it is a subalgebra of the orthogonal Lie algebra, which is compact.

The character of the real form gives us a method for calculating how compact it is. Each (positive) negative value in the metric corresponds to a (non)compact generator so the character is given by \(\chi = \text{number of noncompact generators} - \text{number of compact generators}\). By rearranging the basis of the normal form with \(H^i\) and \((E^\alpha \pm E^{-\alpha})/\sqrt{2}\) we find a diagonal metric. By performing the \textit{Weyl unitary trick}, \(H \to iH\) and \((E^\alpha \pm E^{-\alpha})/\sqrt{2} \to i(E^\alpha + E^{-\alpha})/\sqrt{2},\)
we find a metric;

\[
\begin{pmatrix}
-1 & & & \\
& \ddots & & \\
& & -1 & \\
& & & -1 \\
-1 & & & -1 \\
& & & \ddots \\
& & & & -1 \\
\end{pmatrix}
\begin{pmatrix}
iH^1 \\
\vdots \\
iH^n \\
i(E^α + E^{-α})/\sqrt{2} \\
(E^α - E^{-α})/\sqrt{2} \\
\vdots \\
i(E^β + E^{-β})/\sqrt{2} \\
(E^β - E^{-β})/\sqrt{2}
\end{pmatrix}
\]

with character $\chi = -\dim \mathfrak{g}$. Since all of the generators of the algebra are compact this is referred to as the **compact real form**. The involution $\sigma$ which takes $\sigma(H^i) = -H^i$ and $\sigma(E^α) = -E^{-α}$ decomposes the real form of the algebra into eigenspaces with exactly the basis given in the compact form above:

\[
\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}
\]

\[
t = \text{span}_\mathbb{R} \left( E^α - E^{-α} \right)
\]

\[
\mathfrak{p} = \text{span}_\mathbb{R} \left( H^i, E^α + E^{-α} \right).
\]  

The Weyl unitary trick then produces a new algebra $\mathfrak{g}^* = \mathfrak{t} \oplus i\mathfrak{p}$ and allows us to map between the normal and compact real forms.

If we start with the compact form $\mathfrak{g}$ the adjoint representation can be written as matrices for each subspace such that

\[
M(t) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} t \\
M(p) = \begin{pmatrix} 0 & C \\ -C^T & 0 \end{pmatrix} t
\]

where $A$, $B$ and $C$ are real matrices with $A = -A^T$ and $B = -B^T$. There are three classes of automorphisms, of which one is the complex Weyl trick and the other two are given by matrices:

\[
\Omega_{p,q}^1 = \begin{pmatrix} 1_p & 0 \\ 0 & -1_q \end{pmatrix} \\
\Omega_{p,q}^2 = \begin{pmatrix} 0 & 1_p \\ -1_p & 0 \end{pmatrix}
\]  

We take as an example the algebra $A_n$ which has a Dynkin diagram given in table 2.2.1. The positive root space is given by a set of roots $\sum_{i=1}^n a_i \alpha_i$ where the vector $a$ has $a_i$ values of either 0 or 1. We write a generator $E^α = E_{(a_1, \cdots, a_n)}$ and $E^{-α} = F_{(a_1, \cdots, a_n)}$ and present a matrix realisation:

\[
H^i = K^i - K^{i+1} \quad E^{α_i} = K^{i+1}_i,
\]  

where $K^i, j$ is the $(n + 1) \times (n + 1)$ matrix with the $(i, j)$ entry 1 and all others zero. We define the non-simple root generators with the left-positive commutators:

\[
[E^{α_i}, E^{α_{i+1}}] = +E^{α_i + α_{i+1}}
\]

\[
[E(0,\cdots,0,1_a,\cdots,1_b,0,\cdots,0), E(0,\cdots,0,1_{b+1},\cdots,1_c,0,\cdots,0)] = +E(0,\cdots,0,1_a,\cdots,1_c,0,\cdots,0).
\]
Each $F^\alpha = E^{-\alpha}$ so the positive(negative) generators are upper(lower) triangular. These form the basis of $(n+1) \times (n+1)$ traceless matrices. Over the complex numbers this is the Lie algebra $\mathfrak{sl}(n, \mathbb{C})$. By restricting to the real numbers we obtain the normal form $\mathfrak{sl}(n, \mathbb{R})$. The involution which takes $\Omega(H) = -H$ and $\Omega(E^\alpha) = -F^\alpha$ leaves all of the $E^\alpha - F^\alpha$ generators invariant, forming the $t$ subalgebra, while the remaining symmetric traceless generators are all elements of the complement $p$. The maximally-compact involution invariant subalgebra is therefore $\mathfrak{so}(n)$. After acting with the Weyl map:

\[
\begin{align*}
\mathfrak{sl}(n, \mathbb{R}) &= t \oplus p = A \oplus B \\
\downarrow
\end{align*}
\]

\[
\mathfrak{su}(n, \mathbb{C}) = t \oplus i p = A \oplus iB,
\]

with $A$ antisymmetric real and $B$ traceless symmetric real matrices, we find the set of traceless antihermitian matrices which form the Lie algebra of $SU(n, \mathbb{C})$. Other real forms are obtained from the compact form by applying the other involutions. For example, $\Omega^1_{p,q}$ maps $\mathfrak{su}(n, \mathbb{C})$ to the set of matrices which obey $M^\dagger M_{p,q} = -I_{p,q} M$, which is the Lie algebra $\mathfrak{su}(p, q, \mathbb{C})$. We denote the real form with character $\chi$ by $A_{n(\chi)}$, so the normal and compact forms are $A_{n(n)}$ and $A_{n(n-1)}$, respectively. The Dynkin diagram produces the $\mathfrak{so}(2n, \mathbb{C})$ Lie algebra which has the compact real form $D_{n(2n-1)} = \mathfrak{so}(2n, \mathbb{R})$. The $\Omega^1_{p,q}$ involution maps the general antisymmetric matrices into the Lie algebra $\mathfrak{so}(p, q, \mathbb{R})$. The normal form is obtained when $p = q$ and is $D_{n(n)} = \mathfrak{so}(n, n, \mathbb{R})$. The real forms of the exceptional algebras are often listed simply by their character where $E_{n(n)}$ is the normal form and $E_{n(d)}$ the compact form.

Different involutions of Lie algebras generally leave different subalgebras invariant. In the above example we considered an involution which mapped each positive generator as $\Omega(E^\alpha) = -F^\alpha$. For the $\mathfrak{sl}(n, \mathbb{R})$ real form of $A_{n-1}$ this leaves the set of $n \times n$ antisymmetric real matrices, $t = \mathfrak{so}(n, \mathbb{R})$, invariant. A more general involution given by its action on the simple generators $\Omega(E^\alpha) = -\epsilon_i F^{\alpha_i}$ allows us to write the action on any generator as

\[
\Omega(\{E_{a_1, \ldots, a_n}\}) = -\left(\sum_{i=1}^{n} (\epsilon_i)_{a_i}\right) F_{\{a_1, \ldots, a_n\}}.
\]

(2.34)

This provides us with an easy method of finding the invariant subalgebra by counting the number (non)compact generators. For example, with $\epsilon = (-1, +1, \ldots, +1)$ we find the invariant subalgebra $\mathfrak{so}(1, n-1, \mathbb{R})$. Other involutions of this type yield $\mathfrak{so}(p, q, \mathbb{R})$ invariant subalgebras. The involution which leaves the maximal compact subalgebra, in this case $\mathfrak{so}(n, \mathbb{R})$, is known as the Cartan involution.

Another algebra of interest in later work is the normal real form $D_{n(n)} = \mathfrak{so}(n, n, \mathbb{R})$. The Cartan involution, with $\epsilon = (+1, \ldots, +1)$ leaves $\mathfrak{so}(n, \mathbb{R}) \oplus \mathfrak{so}(n, \mathbb{R})$ invariant while more general involutions with arbitrary $\epsilon$ leave some $\mathfrak{so}(p_1, q_1, \mathbb{R}) \oplus \mathfrak{so}(p_2, q_2, \mathbb{R})$ invariant. The Cartan involution invariant subalgebras of the normal forms of $E_{6(6)}$, $E_{7(7)}$ and $E_{8(8)}$ are $\mathfrak{sp}(4)$, $\mathfrak{su}(8)$ and $\mathfrak{so}(16)$, respectively. By altering the signs within $\epsilon$ other invariant
subalgebras can be calculated as required. In practice this is most easily done by counting
the number of (non)-compact generators and referring to a table of characters.

2.4 Coset Lagrangians

The involutions of the previous section led to a decomposition of general Lie algebras into
an invariant subalgebra \( t \) and the complement (not an algebra) \( p \). Due to their eigenvalues
under the involution it is clear that

\[
[t, t] \subset t, \quad [t, p] \subset p, \quad [p, p] \subset t.
\] (2.35)

A consequence of the second relation is that \( p \) is a module under the action of \( \text{ad}_t \). If we con-
sider the Lie groups \( G = \exp(g) \) and \( H = \exp(t) \) the relations of the subspaces (2.35) provide
the definition for a symmetric space \( G/H \). The Lie algebra for this coset is isomorphic
to \( p \). For general Lie algebras we found that there exists a root space decomposition. The Borel subalgebra is defined simply as the direct sum of the Cartan subalgebra with the
positive root generators: \( g_B = h \oplus g_{a^+} = h \oplus n^+ \). We will refer to group elements which are
written as \( g = \exp(h)\exp(n^+) \) as being in the Borel gauge. Selecting this gauge will make
many future computations much more straight-forward.

We may construct coset Lagrangians over some manifold \( \mathcal{M} \) which possess the symmetries
of some symmetric space \( G/H \) by introducing a map \( g : \mathcal{M} \rightarrow G/H \) which we will set in the
Borel gauge:

\[
g(x^\mu) = \exp \left\{ \sum_{i=1}^r \phi_i(x^\mu)H_i \right\} \exp \left\{ \sum_{\alpha} C_\alpha(x^\mu)e_\alpha \right\}.
\] (2.36)

We will be constructing a Lagrangian which should be invariant under global transformations
with some local compensation, so it is natural to use the unique algebra-valued Maurer-
Cartan form \( \omega \) which can be explicitly written for matrix groups as

\[
\omega = dgg^{-1} = \partial_\mu g(x)dx^\mu g^{-1}(x).
\] (2.37)

With our algebra decomposed into eigenspaces of the involution \( \Omega \) we can decompose the
Borel gauged Maurer-Cartan form as \( \omega = \mathcal{P} + \mathcal{Q} \), with

\[
\frac{1}{2}(\omega - \Omega(\omega)) = \mathcal{P} \in \mathfrak{p},
\]

\[
\frac{1}{2}(\omega + \Omega(\omega)) = \mathcal{Q} \in \mathfrak{t}.
\] (2.38)

Transforming \( \omega \) with a coordinate-independent \( g_o \in G \) and local, gauge compensating \( h(x) \in H \) as

\[
g(x) \rightarrow h(x)g(x)g_o
\] (2.39)

we find that both \( \mathcal{P} \) and \( \mathcal{Q} \) transform by conjugation with the subgroup element, while there
is an additional term due to the coordinate dependence of the transformation:

\[
\mathcal{P} \rightarrow h\mathcal{P}h^{-1}, \quad \mathcal{Q} \rightarrow h\mathcal{Q}h^{-1} + dhh^{-1}.
\] (2.40)
We construct a Lagrangian, using the Killing form on the Lie algebra, which is manifestly invariant under these transformations over the manifold \( \mathcal{M} \) with metric \( h^{\mu \nu} \) as

\[
L = \sqrt{h} h^{\mu \nu} \kappa (\mathcal{P}_\mu, \mathcal{P}_\nu).
\]  

(2.41)

In order to calculate the equations of motion we must recall two identities:

\[
e^B \text{d}e^{-B} = - f(\text{ad}_B) \text{d}B = \text{d}B + \frac{1}{2} [B, \text{d}B] + \cdots
\]

\[
e^B \text{A}e^{-B} = \sum_{n=0}^{\infty} \frac{1}{n!} (\text{ad}_B)^n A = A + [B, A] + \cdots
\]  

(2.42)

where the function \( f(z) = z^{-1}(e^z - 1) \). These are consequences of the well-known Baker-Campbell-Hausdorff theorem\(^2\). As the Lagrangian is invariant under all transformations except coordinate-dependent \( \exp(p) \) we consider an infinitesimal transformation \( g \rightarrow e^\epsilon(x) g \) where \( \epsilon(x) \in \mathfrak{p} \). Under this transformation we find that \( \mathcal{P} \rightarrow \mathcal{P} + \text{d} \epsilon + [Q, \epsilon] \) so the equations of motion are

\[
\partial_\mu \left( \sqrt{h} \mathcal{P}^\mu \right) - \sqrt{h} [Q_\mu, \mathcal{P}^\mu] = 0.
\]  

(2.43)

We also note that the equations of motion imply a conserved Noether form,

\[
\mathcal{J}^\mu = \sqrt{h} g^{-1} \mathcal{P}^\mu g,
\]  

(2.44)

where \( \partial_\mu \mathcal{J}^\mu = 0 \) is equivalent to the equations of motion. It is clear that under general coordinate-free transformations \( g_o \) this transforms as \( \mathcal{J}^\mu \rightarrow g_o \mathcal{J}^\mu g_o^{-1} \) and can be used to observe how charges transform under group action.

Another commonly used method for constructing invariant Lagrangians is done through the definition of a unimodular matrix

\[
\mathbf{M} = g^\# g
\]  

(2.45)

where the new operation is the **generalised transpose** which acts on the group through the Lie algebra generators of the group element:

\[
\#: G \rightarrow G \quad g = \exp(\phi(x)X) \rightarrow g^\# = \exp(-\phi(x)\Omega(X))
\]  

(2.46)

When the involution takes \( \Omega(E_\alpha) = -F_\alpha \) these become regular transposes. The Lagrangian is defined in an analogous way

\[
L = \sqrt{h} \kappa (\partial_\mu \mathbf{M}^{-1}, \partial^\mu \mathbf{M})
\]  

(2.47)

and is invariant under the same transformations (2.39). The matrix \( \mathbf{M} \) transforms as

\[
\mathbf{M} \rightarrow g_o^\# \mathbf{M} g_o
\]  

(2.48)

under global \( g_o \in G \) and is invariant under local \( h(x) \) transformations. This formalism will be used at some points to more easily describe the transformations of fields in the coset.

\(^2\)For proofs see any of the referenced texts on Lie groups/algebras.
2.4.1 Example: $\text{SL}(2, \mathbb{R})/\text{SO}(2)$ using $\mathbf{M}$

We take this model as an example because it is quite simple and because it will be used to describe four dimensional gravitational solutions with some symmetries. The generators are conveniently written as:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. $$

The involution which leaves $\mathfrak{so}(2)$ invariant is

$$\Omega(E) = -F, \quad \Omega(H) = -H \quad (2.49)$$

$$S = \frac{1}{2}(E + F), \quad H \in \mathfrak{p} \quad K = \frac{1}{2}(E - F) \in \mathfrak{t} \quad (2.50)$$

A group element for the coset in Borel gauge,

$$g = \exp(\phi(x)H) \exp(C(x)E), \quad (2.51)$$

will give a unimodular matrix

$$\mathbf{M} = g^\# g = e^{CF} e^{\phi H} e^{\phi H} e^{CE}$$

$$= \begin{pmatrix} e^{2\phi} & e^{2\phi}C \\ e^{2\phi}C & e^{2\phi}C^2 + e^{-2\phi} \end{pmatrix}. \quad (2.52)$$

The Lagrangian here will simply be the trace over the product of the matrices:

$$\text{tr} \left( \text{d}\mathbf{M} \text{d}\mathbf{M}^{-1} \right) = 4(d\phi)^2 + e^{4\phi}(dC)^2. \quad (2.53)$$

If we had chosen the involution $\Omega(E) = +F$ which leaves $\mathfrak{t} = \mathfrak{so}(1, 1)$ invariant, $\mathbf{M}$ would be antisymmetric and the $C^2$ term would have an opposite sign. This would result in a Lagrangian with opposite sign for $(dC)^2$. The symmetries of the action are realised by transformations on $\mathbf{M}$, given by equation (2.48), which translate into transformations of the fields which appear in the Lagrangian. By decomposing the general group action as exponentials of the three individual algebra generators two of the transformations are quite simple:

$$\mathbf{M} \rightarrow e^{\lambda_E F} \mathbf{M} e^{\lambda_E E} : \quad C \rightarrow C + \lambda_E \quad (2.54)$$

$$\mathbf{M} \rightarrow e^{\lambda_H H} \mathbf{M} e^{\lambda_H H} : \quad C \rightarrow Ce^{-2\lambda_H}, \quad \phi \rightarrow \phi + 2\lambda_H. \quad (2.55)$$

For the $g_o = \exp(\lambda_F F)$ transformation we define $\phi(x) = 1/2\log N(x)$ so that the fields $C$ and $N$ transform as

$$N \rightarrow \frac{N}{(1 + \lambda_F C)^2 + \lambda_F^2 N^2}$$

$$C \rightarrow \frac{C + \lambda_F C^2 + \lambda_F N^2}{(1 + \lambda_F C)^2 + \lambda_F^2 N^2}. \quad (2.56)$$

The $\lambda_E$ transformation appears as a gauge symmetry of the field $C$ while the $\lambda_H$ acts like a gauge symmetry on $\phi$ but also scales $C$ as necessary. The $\lambda_F$ symmetry acts in a highly non-trivial fashion on the fields and when we find physical interpretations for coset models these will prove the most interesting.
2.4.2 Example: $SL(3, \mathbb{R})/SO(1, 2)$ using $\mathcal{P}$

We summarise a choice of basis for the matrices intuitively with:

$$
\begin{pmatrix}
H_1 & E_{(1,0)} & E_{(1,1)} \\
F_{(1,0)} & H_2 - H_1 & E_{(0,1)} \\
F_{(1,1)} & F_{(0,1)} & -H_2
\end{pmatrix},
$$

(2.57)

and the computations below can easily be performed only with reference to the commutators. The group element in Borel gauge is

$$
g = \exp (\phi_1 H_1 + \phi_2 H_2) \exp \left( C_{(1,0)} E_{(1,0)} + C_{(0,1)} E_{(0,1)} + C_{(1,1)} E_{(1,1)} \right)
$$

(2.58)

and the Maurer-Cartan form is therefore given by

$$
\omega = d\phi_1 H_1 + d\phi_2 H_2 + e^{2\phi_1 - \phi_2} dC_{(1,0)} E_{(1,0)} + e^{2\phi_2 - \phi_1} dC_{(0,1)} E_{(0,1)} \\
+ e^{\phi_1 + \phi_2} (dC_{(1,1)} + 1/2 (C_{(1,0)} dC_{(0,1)} - C_{(0,1)} dC_{(1,0)})) E_{(1,1)}.
$$

(2.59)

Due to the gauge choice $\omega$ does not depend on the involution. For this example we take $\Omega(E_{(1,0)}) = +F_{(1,0)}$, $\Omega(E_{(0,1)}) = -F_{(0,1)}$ which leaves an invariant $t = \mathfrak{so}(1, 2)$ with generators

$$
K_{(1,0)} = \frac{1}{2} (E_{(1,0)} + F_{(1,0)}), \quad K_{(0,1)} = \frac{1}{2} (E_{(0,1)} - F_{(0,1)}), \quad K_{(1,1)} = \frac{1}{2} (E_{(1,1)} + F_{(1,1)}).
$$

The Maurer-Cartan form will now decompose with the same fields of the Maurer-Cartan form appearing in from of each $E_{\alpha}$, $K_{\alpha}$ and $S_{\alpha}$. The choice of involution will then affect the signs which result when taking the inner product. In this case

$$
(\mathcal{P}, \mathcal{P}) = 4(d\phi_1)^2 - 4d\phi_1 d\phi_2 + 4(d\phi_1)^2 - e^{4\phi_1 - 2\phi_2} (dC_{(1,0)})^2 + e^{4\phi_2 - 2\phi_1} (dC_{(0,1)})^2 \\
- e^{2\phi_1 + 2\phi_2} (dC_{(1,1)} + 1/2 (C_{(1,0)} dC_{(0,1)} - C_{(0,1)} dC_{(1,0)}))^2.
$$

(2.60)

In general the Cartan fields will remain the same for each involution while the terms in the Lagrangian associated with each $S_{\alpha}$ in the Maurer-Cartan form will be negative(positive) when $\Omega(E_{\alpha}) = (-)F_{\alpha}$. Note that the restriction of this Lagrangian to the $SL(2, \mathbb{R})/SO(2)$ where only $C_{(0,1)}$ and $\phi_2$ are non-zero is precisely the Lagrangian we found in the previous subsection.

2.5 Some Representation Theory

In order to decompose infinite-dimensional extended algebras in the next section we must review a few facts about finite-dimensional representations, which are homomorphisms of a Lie algebra, $\phi : \mathfrak{g} \rightarrow \text{End}(V)$, into the endomorphisms of some finite-dimensional vector space $V$. Since the Cartan subalgebra elements commute with each other, we may choose a basis for the vector space such that the homomorphism maps the Cartan subalgebra to
simultaneously diagonal endomorphisms. We then define a **weight** \( \lambda \) associated with a **weight space** \( V_\lambda \)

\[
V_\lambda = \{ v \in V \mid (\phi(H) - \lambda(H)) v = 0 \}.
\]

Due to the commutators of the algebra we find that the non-Cartan elements \( E_\alpha^\pm \) act on vectors in the representation to modify the Eigenvalue and as a result they collectively form strings of weights which characterise the representation.

With the collection of roots for a Lie algebra we can naturally define an ordering of weights and find that finite-dimensional representations have a unique **highest weight** \( \Lambda \).

**Theorem 2.5.1** (Theorem of Highest Weight). *Up to equivalence, finite-dimensional irreducible representations \( \phi \) of a Lie algebra \( g \) have a one-to-one correspondence with the highest weights \( \Lambda \).*

A proof can be found in [43]. Given a highest weight and root space of rank \( r \) we define the **Dynkin coefficients** as the \( r \)-many values \( \Lambda_i \), where

\[
\Lambda_i = 2 \frac{\langle \Lambda, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}.
\]

The Dynkin coefficients for each representation are equivalent to some linear combination of simple roots and the successive application of lowering generators produces a string of weights which can be written as linear combinations of roots. A useful relation, which can be found in [38], is that for an arbitrary weight \( M \) the sequence \( M + p\alpha, \ldots, M, \ldots, M - m\alpha \) has limits given by

\[
m - p = 2 \frac{\langle M, \alpha \rangle}{\langle \alpha, \alpha \rangle}.
\]

This can be proven by considering the recursive application of lowering generators to the highest weight of a representation. Using this fact we can more easily find all of the weights of a given highest weight representation. A useful algorithm for finding all of the allowed roots is that the application of simple lowering generators subtracts the corresponding row of the Cartan matrix from the vector of Dynkin labels. As an example consider the \( A_n(n) \) normal real form \( SL(n + 1, \mathbb{R}) \) and introduce a basis of weights \( \lambda_i \) such that \( \langle \lambda_i, \alpha_i \rangle = 1 \).

The representation with highest weight \( \Lambda = \lambda_1 \) gives a full set of weights

\[
\lambda_1, -\lambda_1 + \lambda_2, -\lambda_2 + \lambda_3, \ldots, -\lambda_n
\]

for a total of \( n + 1 \) weights. This is the **fundamental representation** and for \( SL(n + 1, \mathbb{R}) \) is the vector representation in \( n + 1 \) dimensions. We can determine other representations for \( A_n(n) \) by finding their dimensions. The representation with \( \Lambda = \sum_{i=1}^{n} \lambda_i \) has dimension \((n + 1)^2 - 1\), the same as the algebra, and is the adjoint representation. We can easily count the dimension of a few other highest weight representations with \( \Lambda = m\lambda_1 \) and \( \Lambda = \lambda_m \) which are the symmetric and antisymmetric \( m \)-Tensors. More general representations \( \Lambda = \sum a_i \lambda_i \) are given by products of appropriately symmetrised tensors. A few are listed in table 2.5.1.
### Table 2.5.1: Representation of $A_{n-1(n-1)} \equiv SL(n, \mathbb{R})$ arranged by Dynkin coefficients

<table>
<thead>
<tr>
<th>Dynkin Labels of $\Lambda$</th>
<th>Dimension</th>
<th>Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1$</td>
<td>$n$</td>
<td>Vector</td>
</tr>
<tr>
<td>$2\lambda_1$</td>
<td>$n(n+1)/2$</td>
<td>Symmetric 2-Tensor</td>
</tr>
<tr>
<td>$m\lambda_1$</td>
<td>$(n+m-1)_m$</td>
<td>Symmetric $m$-Tensor</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>$n(n-1)/2$</td>
<td>Antisymmetric 2-Tensor</td>
</tr>
<tr>
<td>$\lambda_m$</td>
<td>$(n)_m$</td>
<td>Antisymmetric $m$-Tensor</td>
</tr>
<tr>
<td>$\sum_{i=1}^{n-1} \lambda_i$</td>
<td>$n^2 - 1 = r$</td>
<td>Adjoint</td>
</tr>
</tbody>
</table>

The exceptional Lie algebras $E_n$ do not have interpretations which are as intuitive as the standard $A$, $B$, $C$ and $D$ Lie algebras which are derived from matrix Lie groups. A convenient method for manipulating them is provided by decomposing the algebra with respect to the exceptional node which when removed yields a connected $A_{n-1}$ Dynkin diagram. A general root can be written $\beta = a_n\alpha_n + \sum_{i \neq n} a_i\alpha_i$ with exceptional node $n$ and the inner product of the exceptional root with all other simple roots is $\langle \alpha_n, \alpha_i \rangle = -\delta_{i,\star}$, where $\star$ is the label of the only node connected to the exceptional node. As a result, the $A_{n-1}$ algebra interacts with this root as the weight $-\lambda_\star$. The roots can be broken down in a level decomposition where the set of roots with a given $a_n$ form a representation of $A_{n-1}$ with highest weight $a_n\lambda_\star$.

Let us provide an example, which will be used later in this thesis, by levelling the normal real form $E_{6(6)}$. When $a_6 = 0$ the roots form the $A_5$ algebra which generates the $SL(6, \mathbb{R})$ real form. The exceptional node is connected to the third node in $A_{5(5)}$ and, as we saw above, this corresponds with $\lambda_3$ and the three-form representation of $SL(6, \mathbb{R})$ with dimension 20. There is also one root (the highest, $\Theta$) with $a_6 = 2$ which provides the one-dimensional representation which is dual to the six-form. This comes from the product of the two three-form representations. We may therefore write out the $SL(6, \mathbb{R})$ generators as:

$$H_i = K^i_i - K^{i+1}_{i+1}, \quad E_{a_i} = K^i_{i+1}, \quad F_{a_i} = K^{i+1}_i$$

(2.65)

$$\left[ K^i_j, K^l_m \right] = \delta^l_j K^i_m - \delta^i_m K^l_j.$$  

(2.66)

To this we add antisymmetric forms for the $a_6 \neq 0$ generators:

$$H_6 = -\frac{1}{3} (K^1_1 + K^2_2 + K^3_3) + \frac{2}{3} (K^4_4 + K^5_5 + K^6_6)$$

(2.67)

$$E_{a_6} = R^{456}, \quad F_{a_6}, \quad E_{(1,2,3,2,1,2)} = R^{123456}, \quad F_{(1,2,3,2,1,2)} = R^{123456}$$

(2.68)

$$\left[ K^i_j, R^{a_1a_2a_3} \right] = \delta^a_{i,j} R^{a_2a_3} + \text{sym}, \quad \left[ K^i_j, R_{a_1a_2a_3} \right] = -\delta^i_{a_1} R_{a_2a_3} + \text{sym}$$

(2.69)

$$\left[ R^{a_1a_2a_3}, R_{b_1b_2b_3} \right] = R^{a_1a_2a_3b_1b_2b_3}, \quad \left[ R_{a_1a_2a_3}, R_{b_1b_2b_3} \right] = R_{a_1a_2a_3b_1b_2b_3}$$

(2.70)

$$\left[ R^{a_1a_2a_3}, R_{b_1b_2b_3} \right] = 18 \delta_{a_1a_2a_3}^{[a_1a_2} K^{b_3]}_{b_3] - 2 \delta_{b_1b_2b_3}^{a_1a_2a_3} \sum_{i=1}^{6} K^i_i$$

(2.71)
Each of the non-simple generators can be computed from commutators given above. For sign conventions we will set each $+R^{ijk}$ ordered as $i < j < k$ to be the positive sign generator. For example, $E_{(0,0,1,0,0)} = +R^{356}$ and $E_{(1,2,2,2,1)} = +R^{124}$.

2.6 Extensions and Kac-Moody Algebras

In the previous sections we stipulated that, among other things, Lie algebras must be defined by Cartan matrices with $\det(A) > 0$ and as a result the Cartan-Killing form gives a Euclidean geometry to the root space: each root $\beta$ has $\langle \beta, \beta \rangle > 0$. If we remove this condition we obtain the general class of Kac-Moody algebras. In order to retain some of the qualities of the classical Lie algebras we relax, rather than remove, the condition on the determinant of the Cartan matrix and focus on subsets of the general Kac-Moody algebras. One of the essential properties was the existence of a bilinear, symmetric form with particular invariance. We call a Cartan matrix $A$ symmetrisable if there exists an invertible diagonal matrix $D$ such that

$$A = DS \quad (2.73)$$

for some symmetric matrix $S$. A theorem from Kac [42] gives us the essential structure.

**Theorem 2.6.1.** Let $\mathfrak{g}$ be a Lie algebra with symmetrisable Cartan matrix. There exists a nondegenerate symmetric bilinear form $(\cdot, \cdot) : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{F}$ such that $([X_1, X_2], X_3) = (\{X_2, X_3\}, X_1)$.

This bilinear form naturally acts on the root space but without the restriction on the Cartan matrix there will be, in general, imaginary roots $\beta$ such that $\langle \beta, \beta \rangle \leq 0$.

The class of Kac-Moody algebras known as affine Lie algebras is defined by Cartan matrices which when any one row $i$ and matching column $i$ are deleted yields a new Cartan matrix which has positive determinant. An equivalent definition is that an affine Lie algebra possesses a Dynkin diagram which when any one node and its connections is removed leaves only (potentially disconnected) Dynkin diagrams of finite Lie algebras. The symmetrised Cartan matrix of an affine Lie algebra with rank $r$ will have $r - 1$ positive eigenvalues and one zero eigenvalue. As a result they are called positive semi-definite. The affine Lie algebras have been fully classified.

Another class of Kac-Moody algebras are known as the hyperbolic Kac-Moody algebras and are defined by symmetrised Cartan matrices of rank $r$ which have $r - 1$ positive eigenvalues and one negative eigenvalue. These are called indefinite Cartan matrices. The equivalent statement here is that upon deletion of one node in the Dynkin diagram we retain only finite-dimensional with at most one affine Dynkin diagram. These algebras have also been fully classified [45].
The algebras of the last class that we will discuss are known as Lorentzian Kac-Moody algebras. These are of particular interest to those studying the infinite-dimensional symmetries of maximal supergravities [46] and are defined by Dynkin diagrams which possess one node which when deleted leaves only finite-dimensional with at most one affine Dynkin diagram. It is clear from the definition that every Lorentzian algebra is a hyperbolic algebra. The symmetrised Cartan matrices also have $r - 1$ positive eigenvalues with one negative eigenvalue and are indefinite.

We can construct a variety of Kac-Moody algebras through the extension of the classical Dynkin diagrams with additional nodes and connections. The non-twisted affine Lie algebras can be constructed through the addition of a node with simple root $\alpha_0 = \delta - \Theta$ where $\delta$ is a new null root $(\delta, \delta) = 0$ which is orthogonal to the simple roots of the non-extended algebra $g$ and $\Theta$ is the highest root of $g$. We call the algebra extended with one node $g^\pm$. From the Serre relations we find that

$$\delta - \Theta - \alpha_i \notin \Delta(g^+) \quad \text{and} \quad \delta + \Theta + \alpha_i \quad \text{for} \quad i \neq 0.$$ 

However, the commutators of the Serre relations for addition of multiple $\delta$ do not vanish and we find that for a classical Lie algebra $g$:

$$\Delta(g^+) = \{ \pm \alpha + n\delta \mid \forall \alpha \in \Delta(g), \ n \in \mathbb{Z} \} \cup \{ n\delta \mid n \in \mathbb{Z} \}$$

(2.74)

Since $\delta$ is orthogonal to the simple roots of $g$ we find that

$$(\alpha_0, \alpha_i) = - (\Theta, \alpha_i),$$

(2.75)

so that the extensions of the classical Lie algebras are easily constructed. We already listed the classical algebras with Coexeter labels in table 2.2.1, so finding the non-zero inner products becomes quite simple. For $A_n$ we find that $(\alpha_0, \alpha_i) = - \delta_{i,1} - \delta_{i,n}$, which tells us that the extended Dynkin diagram includes a 0 node which has one connection with both node 1 and node $n$, making a loop of rank $n + 1$. The other extensions are all summarised in table 2.6.1.

In order to continue with our extension programme we introduce a four-dimensional lattice $\mathbb{Z}^{1,1} \times \mathbb{Z}^{1,1}$ which is spanned by the vectors:

$$v_i, \quad i = 1, 2, 3, 4$$

(2.76)

$$(v_i, v_j) = \delta_{i,1} \delta_{j,2} + \delta_{i,3} \delta_{j,4}.$$ 

By taking $v_1 = \delta$ we may construct the affine extension above. We construct the over-extended algebra $g^{++}$ by introducing another root

$$\alpha_{-1} = -v_1 - v_2,$$

(2.77)

which has inner products:

$$(\alpha_{-1}, \alpha_{-1}) = 2, \quad (\alpha_{-1}, \alpha_0) = -1, \quad (\alpha_{-1}, \alpha_i) = 0$$

(2.78)
for all $\alpha_i \in \Delta(g)$. This clearly modifies the Dynkin diagram by adding one node which is singly connected to the affine node only. By introducing a third root we can construct a very-extended algebra $g^{+++}$ where

$$\alpha_{-2} = v_1 - v_3 - v_4,$$

which has inner products:

$$(\alpha_{-2}, \alpha_{-2}) = 2, \quad (\alpha_{-2}, \alpha_{-1}) = -1, \quad (\alpha_{-2}, \alpha_0) = 0, \quad (\alpha_{-2}, \alpha_i) = 0$$

for all $\alpha_i \in \Delta(g)$. The Dynkin diagrams for the triple extension now include the double extension with a new node which has a single connection to the node for $\alpha_{-1}$. All of the Dynkin diagrams for the classical algebras $g$ and their various extensions are summarised in table 2.6.1.

### 2.6.1 Very-extended $E_8^{+++}$

In section 2.5 we found that that $A_n$ representations with various Dynkin coefficients are described by tensor representations with particular symmetries. Simply-laced algebras which have a node associated with simple root $\alpha_*$ which, when removed leaves an $A_n$ Dynkin diagram, can be 'leveled' with respect to this node. By collecting all of roots at a given level $l$ (possessing $a_\ast = l$) we find that they form an $A_n$ tensor representation. We already constructed a basis for the generators of $E_{6(6)}$ in equations (2.65)-(2.72) and here we provide a similar construction for the normal real form of the triple extension of $E_8$. The Dynkin diagram for $E_{11}$ is given in table 2.6.1, although for this section we will use a simpler labeling which is shown in figure 2.6.1. The standard basis for the root space, $\alpha_i$ $i = 1, \ldots, 11$ will prove cumbersome, so we introduce a basis of $e_i$, for $i = 1, \ldots, 11$, with inner product [25]:

$$\langle u, v \rangle = \sum_{i=1}^{11} u_i v_i - \frac{1}{9} \left( \sum_{i=1}^{11} u_i \right) \left( \sum_{i=1}^{11} v_i \right)$$

$$(2.81)$$

$$u = \sum_{i=1}^{11} u_i e_i, \quad v = \sum_{i=1}^{11} v_i e_i.$$  

The roots of $E_{11}$ are then given by

$$\alpha_i = e_i - e_{i+1} \quad i = 1, \ldots, 10$$

$$\alpha_{11} = e_9 + e_{10} + e_{11}.$$  

(2.82)  

(2.83)
Table 2.6.1: Very-extended Dynkin diagrams with extended nodes in teal and the usual labeling of nodes for finite subalgebra table 2.2.1
Figure 2.6.2: Young tableau describing the symmetries of the tensor representation for lowest weight root $\beta = \sum_i b_i e_i$.

As in the case of $E_{6(6)}$, $E_{11(11)}$ can be levelled with respect to the exceptional node $\alpha_{11}$ so that at level $l = 0$ we obtain $A_{10(10)}$. At $l = 1$ we find the three-form representation and at $l = 2$ we find the six-form, both over eleven dimensions. With our new basis the relationship between roots and the levelled generators is quite clear. For the lowest root $\beta = \sum_i b_i e_i$ at some non-zero level construct a Young tableau with eleven rows and place $b_i$ horizontal blocks in the $i$-th row. This diagram 2.6.1 describes the symmetry of the $(\sum_i b_i)$-tensor representation that the root is the lowest weight of. For example,

$$\alpha_{11} = e_9 + e_{10} + e_{11} \quad E_{\alpha_{11}} = R^{91011}_{11}$$

$$\alpha_6 + 2\alpha_7 + 3\alpha_8 + 2\alpha_9 + \alpha_{10} + 2\alpha_{11} = e_6 + e_7 + e_8 + e_9 + e_{10} + e_{11} \quad E_{(0,0,0,0,1,2,3,2,1,2)} = R^{67891011}_{11}$$

are the lowest weight generators at levels 1 and 2. By acting with the $K_{ij}$ we find all of the basis elements of the representation.

With $E_6$ the Serre relations indicated that $a_6 = 2$ for the highest root, but in $E_{11}$ $a_{11}$ is not bounded. At some level $l = n + 1$ the representation is generally given by a subset of the Young tableaux in the tensor product of the $l = n$ representation and the three-form. The tensor products can be decomposed into direct sums of Young tableau which are given by the Littlewood-Richardson rules. The product of two three-forms is:

$$\otimes = \oplus$$

This would suggest that at level $l = 2$ there should be more generators than the six-form. However, the length squared of any root in the extension of simply-laced algebras is bounded by the simple roots. From our $e_i$ construction we find that $\alpha_{11}^2 = (e_9 + e_{10} + e_{11})^2 = 2$ and a six-form $\beta^2 = (e_{i_1} + e_{i_2} + \ldots + e_{i_6})^2 = 2$ while the mixed-symmetry $(5, 1)$-form representation

---

3 We deal with lowest roots as a technicality. This is because $\alpha_{11} = x - \lambda_8$ where $\lambda_8$ is the fundamental weight and $x$ is orthogonal to the $A_{10}$ subalgebra.

4 This can be seen by considering roots as Weyl transformations of the simple roots of a simply-laced
belongs to a root $\gamma$ with $\gamma^2 = (2e_{i_1} + e_{i_2} + e_{i_3} + e_{i_4} + e_{i_5})^2 = 4$. This vector does not exist in the root space of $E_{11}$ and the other mixed-symmetry forms that we could construct at $l = 2$, as seen in equation (2.86), are also eliminated by this argument.

At the next level $l = 3$ we obtain another string of Young tableau in the decomposition

\[
\begin{array}{c}
\otimes \\
\oplus \\
\oplus \\
\oplus
\end{array}
\]

The nine-form lowest weight root has length squared $(\alpha_{(0,0,1,2,3,4,5,6,4,2,3)})^2 = 0$ and the $(8,1)$-form has $(\alpha_{(0,0,1,2,3,4,5,3,1,3)})^2 = 2$. The roots associated with the remainder of the mixed-symmetry tensors in equation (2.87) have length squared $\beta^2 \geq 4$ and are not in the root space.

In general the multiplicity of a root $\beta$ is equal to the dimension of the vector space

\[ g_\beta = \{ X \in g | [H,X] = \alpha(H)X, \forall H \in h \}. \]  

(2.88)

This is equivalent to the sum of the weight multiplicities of the root within the representations in the levelled decomposition times the number of times the representation appears, known as the outer multiplicity. The weight multiplicities can be calculated using the Freudenthal recursion formula \cite{42} and the outer multiplicities can be calculated with knowledge of the root multiplicity. This process has been performed for some Kac-Moody algebras including $E_{11}$, using computers, up to at least $l = 10$ \cite{47}. The vector associated with the nine-form $\alpha_{(0,0,1,2,3,4,5,6,4,2,3)}$ gives us the first example of a zero outer multiplicity representation which indicates a root with zero multiplicity\(^5\). We reproduce a table of roots and their $A_{10}$ representations in table 2.6.2

<table>
<thead>
<tr>
<th>$l$</th>
<th>Dynkin labels $\lambda_i$</th>
<th>Simple roots $a_i$</th>
<th>$\alpha^2$</th>
<th>Dim</th>
<th>Mult</th>
<th>O. Mult</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0 0 0 0 0 0 0 1 0 0</td>
<td>0 0 0 0 0 0 0 0 1</td>
<td>2</td>
<td>165</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0 0 0 0 1 0 0 0 0 0</td>
<td>0 0 0 0 0 1 2 3 2 1</td>
<td>2</td>
<td>462</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>0 1 0 0 0 0 0 0 0 0</td>
<td>0 0 1 2 3 4 5 6 4 2</td>
<td>0</td>
<td>55</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0 0 1 0 0 0 0 0 0 0 1</td>
<td>0 0 0 1 2 3 4 5 3 1</td>
<td>2</td>
<td>1760</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>0 0 0 0 0 0 0 0 0 0 0 1</td>
<td>1 2 3 4 5 6 7 8 5 2</td>
<td>-2</td>
<td>11</td>
<td>46</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>1 0 0 0 0 0 0 0 0 0 0 0</td>
<td>0 1 2 3 4 5 6 7 4 2</td>
<td>0</td>
<td>594</td>
<td>8</td>
<td>0</td>
</tr>
</tbody>
</table>

\(^5\)The fact that this root has zero multiplicity can be derived from the Jacobi identities which require that $[R^{245}, [R^{678}, R^{0101}]] = 0$.\n
We have already discussed the representations which appear at levels $l = 1, 2, 3$ and it appears that the number and dimension of the representations will become rather unmanageable using this method at much higher levels. However, certain patterns of roots can be distinguished. Consider the roots

\[
\alpha(0,0,0,0,1,0,0,0,0,0,0) = (\sum_{i=3}^{11} e_i) + e_9 + e_{10} + e_{11} \quad E = R_{34567891011}^{34567891011} \quad (2.89)
\]

\[
\alpha(0,0,1,0,0,0,0,0,0,0,0) = (\sum_{i=3}^{11} e_i) + e_6 + \ldots + e_{11} \quad E = R_{3\ldots11}^{3\ldots11} \quad (2.90)
\]

\[
\alpha(0,0,1,0,0,0,0,0,0,0,0) = 2(\sum_{i=3}^{11} e_i) + e_{11} - e_3 \quad E = R_{3\ldots11}^{3\ldots11,4\ldots11} \quad (2.91)
\]

which appear at levels $l = 4, 5, 6$ respectively and which all have $\alpha^2 = 2$ and multiplicities of 1. By defining the null vector $\delta = \sum_{i=3}^{11} e_i$, these roots appear in a tower of roots

\[
\alpha_{3k+1} = k\delta + e_9 + e_{10} + e_{11} \quad E_{\alpha_{3k+1}} = R_{3\ldots11}^{3\ldots11,\ldots(3\ldots11)k,91011} \quad (2.92)
\]

\[
\alpha_{3k+2} = k\delta + e_6 + e_7 + \ldots + e_{11} \quad E_{\alpha_{3k+2}} = R_{3\ldots11}^{3\ldots11,\ldots(3\ldots11)k,67891011} \quad (2.93)
\]

\[
\alpha_{3k+3} = (k+1)\delta + e_{11} - e_3 \quad E_{\alpha_{3k+3}} = R_{3\ldots11}^{3\ldots11,\ldots(3\ldots11)k,4\ldots11,11} \quad (2.94)
\]

where $k \in \mathbb{N}$. All of these roots have length squared $\alpha^2 = 2$ and multiplicity 1. The Young
Figure 2.6.3: Dynkin diagram of $A^{+++}_n$

tableaux of the representations given by these highest weights are

$$R_{\alpha_{k+1}} = \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \quad R_{\alpha_{k+2}} = \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \quad R_{\alpha_{k+3}} = \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array}$$

(2.95)

with $k$ columns of nine boxes. It has been noted that these generators, when combined with the $k = 0$ $A_{10}$ generators contain the algebra $E^+_9 \equiv E_9$. This can be seen by restricting to the roots with $a_9 = a_{10} = 0$ and limiting the various form representations to be over the remaining nine dimensions. This fact was essential to the identification of an $E_9$ multiplet of supergravity solutions [31].

### 2.6.2 Very-extended $A^{+++}_n$

Having already described the process for levelling and decomposing very extended algebras as representations of an $A_n$ subalgebra, in this section we present another algebra which will play a central role in this thesis. The $A^{+++}_n$ algebra with Dynkin diagram 2.6.3 can be levelled with respect to node $n$ so that at $l \equiv a_n = 0$ we find the $A_{n+2}$ algebra. At the first level $l = 1$ the $\lambda_3 + \lambda_{n+2}$ Dynkin coefficients are equivalent to $\lambda_n + \lambda_1$ and we obtain the mixed-symmetry $(n, 1)$-form tensor. By using the Littlewood-Richardson rules we can find the forms at level two which contain

$$\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array} \otimes \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array} = \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array} \oplus \cdots$$

(2.96)

with $m = n - 1$. For convenience we introduce another $e_i$, $i = 1, \cdots, n + 3$ basis for this algebra

$$\alpha_i = e_i - e_{i+1} \quad i = 1, \ldots, n + 2$$

(2.97)

$$\alpha_{n+3} = e_4 + e_5 + \cdots + e_{n+2} + 2e_{n+3}.$$  

(2.98)
with inner product

$$\langle u, v \rangle = \frac{1}{n+1} \left( \sum_{i=1}^{n+3} u_i \right) \left( \sum_{i=1}^{n+3} v_i \right)$$

(2.99)

$$u = \sum_{i=1}^{n+3} u_i e_i, \quad v = \sum_{i=1}^{n+3} v_i e_i.$$  

(2.99)

With these inner products the roots associated with the lowest weight representations in equation (2.96) all have length squared $\beta^2 = 2$.

While the algebra contains infinitely many roots and the representations will proliferate at higher levels, we can identify a single representation at each level which has root length squared 2 and multiplicity 1 by taking the multiple tensor products

$$\underbrace{1 \otimes \cdots \otimes 1}_{n \text{-columns}} \otimes \cdots \otimes \underbrace{m \otimes \cdots \otimes m}_{n \text{-columns}}$$

(2.100)

where the number of tensor products on the left is equal to the number of $n$-columns on the right.

In order to provide a more computationally concrete example we consider the special case of $n = 8$. The algebra with Dynkin diagram 2.6.4 has an inner product, given in equation (2.99), which matches exactly with that of $E_{11}$ (2.81). The only difference between the two bases is with $\alpha_{11}(A_8^{+++})$, which incidentally appears at level three in $E_{11}$ as $\alpha(0,0,0,1,2,3,4,5,3,1,3)$. The $(8,1)$-form similarly appears at levels one and three, respectively, so the nested tensor products at level $k$ in $A_8^{+++}$ will form a subspace of level $3k$ in $E_{11}$ and indeed $A_8^{+++} \subset E_{11}$. In our table of roots for $A_8^{+++}$ up to level two note that each of the roots appear at levels three or six of table 2.6.2.

<table>
<thead>
<tr>
<th>$l$</th>
<th>Dynkin labels $\lambda_i$</th>
<th>Simple roots $a_i$</th>
<th>$\alpha^2$</th>
<th>Dim</th>
<th>Mult</th>
<th>O. mult</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0 1 0 0 0 0 0 0 0 0</td>
<td>0 0 1 1 1 1 1 1 1 1</td>
<td>0</td>
<td>55</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0 0 1 0 0 0 0 0 0 1</td>
<td>0 0 0 0 0 0 0 0 0 1</td>
<td>2</td>
<td>1760</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0 0 0 1 0 0 0 0 0 0</td>
<td>1 2 3 2 2 2 2 2 2 2</td>
<td>-4</td>
<td>330</td>
<td>185</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1 0 1 0 0 0 0 0 0 0</td>
<td>0 1 2 2 2 2 2 2 2 2</td>
<td>-2</td>
<td>1485</td>
<td>44</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0 2 0 0 0 0 0 0 0 0</td>
<td>0 0 2 2 2 2 2 2 2 2</td>
<td>0</td>
<td>1210</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0 0 0 0 1 0 0 0 0 1</td>
<td>1 2 3 2 1 1 1 1 1 2</td>
<td>-2</td>
<td>4752</td>
<td>40</td>
<td>1</td>
</tr>
</tbody>
</table>
Table 2.6.3: $A_{10}$ representations in $A_8^{+++}$ with $t \leq 2$. 

<table>
<thead>
<tr>
<th>$n$</th>
<th>Representation</th>
<th>$l$</th>
<th>$A_8^{+++}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1 0 0 1 0 0 0 0 1 0 1 2 1 1 1 1 1 1 2</td>
<td>0</td>
<td>33033</td>
</tr>
<tr>
<td>2</td>
<td>0 0 0 0 0 1 0 0 1 0 1 2 3 2 1 0 0 0 1 2</td>
<td>0</td>
<td>20328</td>
</tr>
<tr>
<td>2</td>
<td>0 0 0 0 0 1 0 0 0 2 1 2 3 2 1 0 0 0 0 0 2</td>
<td>2</td>
<td>25740</td>
</tr>
<tr>
<td>2</td>
<td>0 1 1 0 0 0 0 0 0 1 0 0 1 1 1 1 1 1 1 2</td>
<td>2</td>
<td>57200</td>
</tr>
<tr>
<td>2</td>
<td>1 0 0 1 0 0 0 1 0 1 2 1 0 0 0 0 0 1 2</td>
<td>2</td>
<td>214500</td>
</tr>
</tbody>
</table>


The five consistent supersymmetric string theories contain infinitely many excited states. By isolating the massless states we find a spectrum which contains a variety of \( n \)-forms. For low energies (or equivalently high tension \( T = 2\pi \alpha'^{-1} \)) the massive states become inaccessible and we find a supergravity theory which provides a low-energy effective description. For example, the type IIA string theory contains a symmetric and anti-symmetric two-tensor, one- and three-forms and dilaton as the bosonic fields [48]. The bosonic sector of the associated type IIA supergravity contains precisely these fields. Each of the string theories are related by dualities, and famously S-duality maps type IIA string theory to an eleven-dimensional theory, known as M-theory, with the new coordinate proportional to the string coupling. A reciprocal observation is that the circle reduction of eleven-dimensional supergravity produces the ten-dimensional type IIA supergravity, as we will demonstrate later in section 3.4.1.

While there is no Lagrangian formulation of M-theory, eleven-dimensional supergravity [6] provides the low-energy effective theory. The theory is simpler than the string supergravities...
### Dimensional Reduction

In this section we present a review of the toroidal compactification of supergravity theories which is based on the lecture notes of [49]. First consider the Einstein-Hilbert action in $D$ dimensions with a line element which has been decomposed into a convenient form,

$$ ds^2 = e^{2\alpha\phi} ds^2 + e^{2\beta\phi} (dz + A_\mu dx^\mu)^2, $$

where the $z$ coordinate has been singled out and $\mu$ runs over the $D - 1$ dimensional space. If we give the $z$ coordinate a circular topology with period $2\pi r$ any object which is mapped from the $D$ dimensional space, such as the scalar $\phi$, can be Fourier expanded as

$$ \phi(x^\mu, z) = \sum_{n=-\infty}^{\infty} \phi_n(x^\mu) e^{inz/r}. $$

With a very small length scale for $r$ the mass associated with any but the $n = 0$ Fourier mode puts it at a very high energy. We therefore truncate to the massless modes and all of

<table>
<thead>
<tr>
<th>Dimension</th>
<th>$G$</th>
<th>$H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>$\mathbb{R}$</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>$SL(2, \mathbb{R})$</td>
<td>$U(1)$</td>
</tr>
<tr>
<td>8</td>
<td>$SL(3, \mathbb{R}) \times SL(2, \mathbb{R})$</td>
<td>$SO(3) \times SO(2)$</td>
</tr>
<tr>
<td>7</td>
<td>$SL(5, \mathbb{R})$</td>
<td>$SO(5)$</td>
</tr>
<tr>
<td>6</td>
<td>$SO(5, 5)$</td>
<td>$SO(5) \times SO(5)$</td>
</tr>
<tr>
<td>5</td>
<td>$E_6(6)$</td>
<td>$USp(8)$</td>
</tr>
<tr>
<td>4</td>
<td>$E_7(7)$</td>
<td>$SU(8)$</td>
</tr>
<tr>
<td>3</td>
<td>$E_8(8)$</td>
<td>$SO(16)$</td>
</tr>
</tbody>
</table>

Table 3.0.1: $G/H$ coset symmetries of eleven-dimensional supergravity when reduced on an $n$–torus to $D = 11 - n$ dimensions.

and the bosonic sector contains only gravity with a three-form potential

$$ L = \sqrt{-g} \left( R - \frac{1}{2 \times 4!} F_{\mu_1 \mu_2 \mu_3 \mu_4} F^{\mu_1 \mu_2 \mu_3 \mu_4} \right) $$

$$ + \frac{1}{144} \epsilon_{\mu_1 \mu_2 \mu_3 \mu_4 \nu_1 \nu_2 \nu_3 \nu_4} A^{\mu_1 \mu_2 \mu_3} F^{\nu_1 \nu_2 \nu_3 \nu_4} A^{\rho_1 \rho_2 \rho_3} A^{\mu_4 \nu_4 \rho_4} $$

where $F_{[4]} = dA_{[3]}$. When compactified on an $n$–torus, the theory possesses remarkable ‘hidden’ coset symmetries [7]. These are summarised in table 3. These coset symmetries, their infinite-dimensional extensions and the transformations of their fields will be central to this thesis. We will review aspects of dimensional reduction and provide some examples of the uses of coset models before applying the techniques to eleven-dimensional supergravity and introducing Kac-Moody algebras.

### 3.1 Dimensional Reduction

In this section we present a review of the toroidal compactification of supergravity theories which is based on the lecture notes of [49]. First consider the Einstein-Hilbert action in $D$ dimensions with a line element which has been decomposed into a convenient form,
our fields depend only on the \( D-1 \) coordinates. We may then calculate the \( D \) dimensional Lagrangian density in terms of the \( D-1 \) dimensional fields. With the appropriate choices of the constants,\(^1\)

\[
\beta = \alpha (3 - D) \quad \text{and} \quad \alpha^2 = [2(D - 3)(D - 2)]^{-1},
\]

we obtain an action which leads with the \( D-1 \) gravity Lagrangian and possesses a standard normalisation on the scalar kinetic term;

\[
\sqrt{-\hat{g}} R = \sqrt{-g} \left[ R - \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{4} e^{2(2-D)\alpha \phi} F_{\mu \nu} F^{\mu \nu} \right],
\]

(3.5)

where \( \mathcal{F} = dA \).

The compactified action of (3.5) possesses the \( D-1 \) Einstein-Hilbert term with scalar and vector fields. If we wish to continue compactifying to \( D-2 \) or generally begin with modified gravity theories we must consider the reduction of general anti-symmetric forms. The \( m \)-form \( \hat{F}_{(m)} = d\hat{A}_{(m-1)} \) can be split into components which do or do not contain \( dz \) in their bases. With some convenient definitions;

\[
F_{(m)} = dA_{(m-1)} - dA_{(m-2)} \wedge A, \quad F_{(m-1)} = dA_{(m-2)},
\]

(3.6)

we find that

\[
\hat{F}_{(m)} = dA_{(m-1)} + dA_{(m-2)} \wedge dz = F_{(m)} + F_{(m-1)} (dz + A).
\]

(3.7)

This choice allows us to more easily calculate the field strength with our vielbein, \( \mathcal{e}^a = e^{\alpha \phi} e^a \) and \( \mathcal{e}^z = e^{(3-D)\phi} (dz + A) \), so that

\[
\hat{F}_{(m)} = \frac{1}{m!} \hat{F}_{a_1 \cdots a_m} (\mathcal{e}^{a_1} \wedge \cdots \wedge \mathcal{e}^{a_m}) + \frac{1}{(m-1)!} \hat{F}_{a_1 \cdots a_{m-1} z} (\mathcal{e}^{a_1} \wedge \cdots \wedge \mathcal{e}^{a_{m-1}} \mathcal{e}^z)
\]

(3.8)

\[
= \frac{e^{\alpha \phi}}{m!} \hat{F}_{a_1 \cdots a_m} (e^{a_1} \wedge \cdots \wedge e^{a_m}) + \frac{e^{(m-D+2)\alpha \phi}}{(m-1)!} \hat{F}_{a_1 \cdots a_{m-1} z} (e^{a_1} \wedge \cdots \wedge e^{a_{m-1}} e^z).
\]

The components of the form are therefore

\[
\hat{F}_{a_1 \cdots a_m} = e^{-\alpha \phi} \tilde{F}_{a_1 \cdots a_m}, \quad \hat{F}_{a_1 \cdots a_{m-1} z} = e^{(D-m-2)\alpha \phi} \tilde{F}_{a_1 \cdots a_{m-1} z}
\]

(3.9)

and the form part of a Lagrangian can be reduced as

\[
\frac{\sqrt{-\hat{g}}}{m!} F_{(m)}^2 = \frac{1}{m!} \hat{F}_{(m)}^2 + \frac{e^{2(1-m)\alpha \phi}}{(m-1)!} \hat{F}_{(m-1)}^2 + \frac{e^{2(D-m-1)\alpha \phi}}{(m-1)!} \hat{F}_{(m-1)}^2.
\]

(3.10)

\(^1\)We assume that \( D \geq 4 \) for the purposes of this review.
3.2 Four Dimensional Gravity

Using the machinery of the previous section we take the $D = 4$ Einstein-Hilbert action and compactify over one coordinate. With our constants in equation (3.4) we find that the four dimensional density decomposes as

$$\sqrt{-\hat{g}} \hat{R} = \sqrt{-g} \left[ R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} e^{-2\phi} F_{\mu\nu} F^{\mu\nu} \right].$$

(3.11)

When compactifying to three dimensions our theory possesses, besides the Einstein-Hilbert term, scalar and vector fields. However, in three dimensions we can dualise the vector into a twist scalar potential $\chi$ in order to obtain a purely scalar modification to gravity. If we define the dual scalar field such that

$$d\chi = \star \left( e^{-2\phi} F \right)$$

(3.12)

the action is immediately recognised as a three dimensional gravitational sigma model

$$\sqrt{-\hat{g}} \hat{R} = \sqrt{-g} \left[ R - G_{AB} \partial_\mu \Phi^A \partial^\mu \Phi^B \right],$$

(3.13)

where the target space is the familiar Lagrangian (2.53) of the $SL(2)/SO(2)$ coset

$$G_{AB} d\Phi^A d\Phi^B = \frac{1}{2} \left( d\phi^2 + e^{2\phi} d\chi^2 \right).$$

(3.14)

We found in section 2.4.1 that this Lagrangian has several coset symmetries which correspond to motion in the scalar manifold. The $e^{\lambda H^E}$ transformation $\chi \to \chi + E^\mu x^\mu$ is simply a gauge symmetry on the dual field. With our metric decomposed as

$$ds_4^2 = e^\phi ds_3^2 + e^{-\phi} (dz + A_\mu dx^\mu)^2$$

(3.15)

the $e^{\lambda H^E}$ action does not leave the geometry invariant but acts as a scaling transformation. In order to understand the final $e^{\lambda \tau^F}$ symmetry we redefined $\phi = \log N(x)$ and obtained transformations (2.56) which did not have a simple interpretation. One of the possible four-dimensional solutions with a metric of the form of (3.15) is the Schwarzschild black hole where $ds^2 = -dt^2 A_\mu = 0, e^{-\phi} = 1 + q/r$ and $ds_3^2$ is the flat Euclidean metric. We act on this well-known seed solution with the $e^{\lambda \tau^F}$ transformation to find

$$N' = \frac{N}{1 + \lambda_F N^2}, \quad \chi' = \frac{\lambda_F N^2}{1 + \lambda_F N^2}.$$

(3.16)

With some coordinate transformations and charge definitions,

$$\rho = r \sqrt{1 + \lambda_F^2 + \kappa \lambda_F}, \quad \tau = \frac{\lambda_F q}{\sqrt{1 + \lambda_F^2}}, \quad \kappa = \frac{\lambda_F \sqrt{1 + \lambda_F^2}}{1 + \lambda_F^2},$$

$$f(\rho) = \frac{\rho^2 + Q\rho - \kappa^2}{\rho^2 + \kappa^2}, \quad Q = q \frac{1 - \lambda_F^2}{\sqrt{1 + \lambda_F^2}}.$$
we find another well-known metric, the Taub-NUT:
\[
\begin{align*}
    ds_4^2 &= -f(\rho) \left( dt^2 + 2\kappa \cos \theta d\phi \right)^2 + f^{-1}(\rho) d\rho^2 + \left( \rho^2 + \kappa^2 \right) (d\theta^2 + \sin^2 \theta d\phi^2).
\end{align*}
\]
(3.18)

As we have now shown, the \(SL(2,\mathbb{R})/SO(2)\) symmetric space provides a description of compactified four-dimensional Einstein gravity and, using the coset symmetries, allows us to transform the Schwarzschild solution to the Taub-NUT solution. This symmetry was originally found by Jürgen Ehlers [9] and is known as Ehlers’ symmetry.

### 3.3 Five Dimensional Gravity

By dimensionally reducing five-dimensional gravity in the method described in section 3.1 we obtain an Einstein-Maxwell-Dilaton (EMD) theory in four dimensions. In order to perform a compactification of this four dimensional theory we will then use the machinery of that section to deal with the new gauge field.

Before we continue let us note that we took the constants \(\alpha\) and \(\beta\) in the reduction ansatz to be particular values in order to reproduce canonical normalisations. If we were to consider general non-zero constants we can calculate the target space Lagrangian and attempt to solve the Killing equations. For the four-dimensional EMD theory the most general set of vector fields can possess an \(SL(3,\mathbb{R})\) symmetry, as shown in [50], but this requires a unique choice of constants which is exactly the one we have proposed. If we were to allow \(\alpha = 0\), it can be shown [50] that one of the Killing vectors will commute with all of the others and the symmetry algebra becomes the \(su(2,1)\) of Einstein-Maxwell (EM) with a decoupled scalar. When we do not set \(\alpha = 0\) we find that one of the equations of motion,
\[
4\Box \phi = -3\alpha e^{-3\alpha \phi} F^2,
\]
(3.19)
prohibits us from setting \(\phi = 0\), or eliminating the dilaton field, since this would then imply \(\phi = 0 \Rightarrow F^2 = 0\). In other words, the EM theory is not a consistent truncation of EMD unless the scalar is decoupled.

#### 3.3.1 Five Dimensional Gravity as a \(su(2,1) \times \mathbb{R}\) coset

While we already mentioned the normal and compact real forms of \(A_n\) in (2.33), there is also the algebra \(su(p,q)\), with \(p + q = n + 1\), which can be reached through action of the automorphism generated by \(\Omega_{p,q}^1\). Using the standard \(sl(3)\) generator realisation (2.57) we can construct a basis for \(su(2,1)\) using
\[
\begin{align*}
    \tilde{H}_1 &= H_1 + H_2, \quad \tilde{H}_2 = i(H_1 - H_2) \\
    \tilde{E}_{(1,0)} &= E_{(1,0)} + E_{(0,1)}, \quad \tilde{E}_{(0,1)} = i(E_{(1,0)} - E_{(0,1)}), \quad \tilde{E}_{(1,1)} = iE_{(1,1)} \\
    \tilde{F}_{(1,0)} &= F_{(1,0)} + F_{(0,1)}, \quad \tilde{F}_{(0,1)} = i(F_{(1,0)} + F_{(0,1)}), \quad \tilde{F}_{(1,1)} = iF_{(1,1)}.
\end{align*}
\]

The involution generated by \(\epsilon = (+,+)\) on the \(sl\) generators (which would give \(\mathfrak{g} = so(3)\) in that case) now generates an involution invariant \(\mathfrak{k} = su(2) \oplus u(1)\). Another involution,
CHAPTER 3. SUPERGRAVITY AND COSETS

\( \epsilon = (-, +) \), generates the involution invariant \( \mathfrak{t} = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{u}(1) \). The non-compact coset is relevant for reduction on a time-like circle, as we shall now demonstrate.

After dimensionally reducing the four-dimensional EMD theory with a decoupled scalar \( \alpha = 0 \) we obtain a three-dimensional scalar Lagrangian which, neglecting the decoupled scalar, is

\[
\mathcal{L}_3 = \sqrt{-g_3} \left( R_3 - \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi + \frac{1}{4} e^{-2\phi} F_{\mu \nu} F^{\mu \nu} - \frac{1}{4} e^{-\phi} F_{\mu \nu} F^{\mu \nu} + \frac{1}{2} e^\phi \partial_{\mu} \psi \partial^{\mu} \psi \right). \tag{3.20}
\]

We can dualise all of field strengths into derivatives of scalar fields as

\[
F_{\mu \nu} = \frac{\epsilon^{\mu \nu \lambda} e^{2\phi}}{\sqrt{g_3}} \left( 2(C_{(1,0)} \partial_{\lambda} C_{(0,1)} - C_{(0,1)} \partial_{\lambda} C_{(1,0)}) + \sqrt{2} \partial_{\lambda} C_{(1,1)} \right) \tag{3.22}
\]

\[
F_{\mu \nu} = - \frac{\epsilon^{\mu \nu \lambda} e^\phi}{\sqrt{g_3}} \partial_{\lambda} C_{(1,0)} \tag{3.23}
\]

where we have relabeled \( \psi = C_{(0,1)} \). The rewritten Lagrangian,

\[
\mathcal{L}_3 = \sqrt{-g_3} \left( R_3 - \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi + \frac{1}{2} e^\phi (\partial_{\mu} C_{(1,0)} \partial^{\mu} C_{(1,0)} + \partial_{\mu} C_{(0,1)} \partial^{\mu} C_{(0,1)}) \right) \tag{3.24}
\]

\[
- \frac{1}{2} e^\phi (\partial_{\mu} C_{(1,1)} + \sqrt{2} C_{(1,0)} \partial_{\lambda} C_{(0,1)} - \sqrt{2} C_{(0,1)} \partial_{\mu} C_{(1,0)})^2 \right), \tag{3.25}
\]

can be constructed as a coset Lagrangian (see section 2.4) for \( SU(2,1)/(SL(2,\mathbb{R}) \times U(1)) \) using the Borel-gauge group element

\[
g = e^{\frac{1}{2} \Phi} \hat{H}_1 e^{\sqrt{2} C_{(1,0)} \hat{E}_{(1,0)} + \sqrt{2} C_{(0,1)} \hat{E}_{(0,1)} + \sqrt{2} C_{(1,1)} \hat{E}_{(1,1)}}. \tag{3.26}
\]

Following [51] for the rest of this subsection, we note that the general BPS-solutions to the EM theory are generalised four dimensional Reissner-Nordström black holes [52]

\[
ds^2 = -\frac{r^2 - N^2 - 2Mr + Q^2 + P^2}{r^2 + N^2} (dt + 2N \cos \theta d\phi)^2 \tag{3.27}
\]

\[
+ \frac{r^2 + N^2}{r^2 - N^2 - 2Mr + Q^2 + P^2} dr^2 + (r^2 + N^2)(d\theta^2 + \sin^2 \theta d\phi^2),
\]

\[
A_t = \frac{Qr + NP}{r^2 + N^2}, \quad A_\phi = \frac{2Nqr + P(N^2 - r^2)}{r^2 + N^2} \cos \theta \tag{3.28}
\]

with mass \( M \), NUT \( N \), electric \( Q \) and magnetic charge \( P \). The BPS condition for these solutions is that \( M^2 + N^2 - Q^2 - P^2 = 0 \) and the solution can be written using the scalar fields

\[
\phi(r) = \ln \left( \frac{(r + M)^2 + N^2}{r^2} \right) \quad C_{(1,0)}(r) = \frac{NP + Q(M + r)}{(r + M)^2 + N^2} \tag{3.29}
\]

\[
C_{(0,1)}(r) = \frac{NQ - P(M + r)}{(r + M)^2 + N^2} \quad C_{(1,1)}(r) = \frac{-\sqrt{2}Nr}{(r + M)^2 + N^2}.
\]
Since the group element limits to the identity as $r \to \infty$, the Noether current (2.44) limits as $\lim_{r \to \infty} J = P$. Integrating over spatial infinity with these scalar fields, we find that

$$Q = \int_{S^\infty} J = -MH_1 - \frac{Q}{\sqrt{2}} S_{(1,0)} + \frac{P}{\sqrt{2}} S_{(0,1)} + NS_{(1,1)} \in \mathfrak{p},$$

(3.30)

where the $S_\alpha$ are the involution anti-invariant generators. In this case we find that

$$Q = \begin{pmatrix}
-M & \frac{iP-Q}{\sqrt{2}} & iN \\
\frac{iP+Q}{\sqrt{2}} & 0 & -\frac{iP-Q}{\sqrt{2}} \\
-iN & \frac{-iP+Q}{\sqrt{2}} & M
\end{pmatrix}.$$  

(3.31)

### 3.3.2 Five Dimensional Gravity as a $\mathfrak{sl}(3, \mathbb{R})$ coset

If we take the two coordinates that we are compactifying over to be spacelike and timelike we may decompose the metric, as shown in [53], as

$$ds^2_5 = \lambda_{ab} \left( dx^a + \omega^a_i dx^i \right) \left( dx^b + \omega^b_j dx^j \right) + \tau^{-1} h_{ij} dx^i dx^j$$

(3.32)

where the $a, b$ label the 2 dimensional compactified space, the $i, j$ label the remaining 3 dimensional space and $\tau = -\det \lambda$. All of the fields $\lambda, \omega, h$ are functions of the non-compactified $x^i$. As before, we dualise the vectors in the metric to define the dual scalars

$$\partial^i V_a = -\tau \lambda_{ab} \epsilon^{ijk} \partial_j \omega^b_k.$$  

(3.33)

The unimodular **Maison matrix** can now be defined [53]

$$\chi = \begin{pmatrix}
\lambda_{ab} - \tau^{-1} V_a V_b & \tau^{-1} V_a \\
\tau^{-1} V_b & -\tau^{-1}
\end{pmatrix}$$  

(3.34)

so that the three-dimensional Lagrangian can be written as

$$L = \sqrt{h} \left( R(h) - \frac{1}{4} \text{tr} \left( \chi^{-1} \partial_i \chi \chi^{-1} \partial^i \chi \right) \right).$$  

(3.35)

It is quite possible to write out the fields in the Lagrangian and we will write this out for general $T^2$ reduction in the next section. Here we explore solutions and coset transformations which map through particular solution spaces.

We will follow [54] by identifying well-known solutions and determine how the symmetry transformations of the theory alter charges. Our first observation is that the flat metric is given by $\chi = \text{diag} (-1, 1, -1) = \eta$. If we focus on asymptotically flat solutions we must have $\chi \to \eta$ as $r \to \infty$. Group transformations map $\chi \to g\chi g^T$, so this restricts us to $SO(1, 2)$.

Let us first consider static solutions and, as in [54], write

$$\chi = \eta e^{2A/(x^1)} , \quad A = \begin{pmatrix}
-M - S/\sqrt{3} & -Q & N \\
Q & 2S/\sqrt{3} & P \\
N & -P & M - S/\sqrt{3}
\end{pmatrix}$$

(3.36)
where \( f(x^i) \) is a harmonic function with source, such as \( 1/r \) in three dimensions. If we restrict ourselves to null solutions, where the target space line element is zero, this translates into the condition that \( \text{tr}(A^2) = 0 \) or

\[
M^2 + S^2 = P^2 + Q^2 + N^2, \tag{3.37}
\]

making these extremal static solutions. It is common to avoid the asymptotics which result from NUT charges, so the few examples of these static extremal shown in table 3.3.1 all have \( N = 0 \).

More general rotating stationary solutions should possess an order \( r^{-2} \) moment for the angular momentum charge, so that that if we keep the NUT charge \( N = 0 \) we should find that \( \chi \) is asymptotically

\[
\chi \rightarrow \begin{pmatrix} -1 + \frac{2M+2S/\sqrt{3}}{r} & \frac{2Q}{r} & -\frac{2J\cos\theta}{r^2} \\ \frac{2Q}{r} & 1 + \frac{4S}{r\sqrt{3}} & \frac{2P}{r} \\ -\frac{2J\cos\theta}{r^2} & \frac{2P}{r} & -1 - \frac{2M-2S/\sqrt{3}}{r} \end{pmatrix}, \tag{3.38}
\]

As a seed solution we may consider the Kerr black string with only non-zero charges \( aM = J \) and Maison matrix

\[
\chi = \begin{pmatrix} -1 + \frac{2Mr}{r^2+a^2\cos^2\theta} & -\frac{4M^2a^2\cos^2\theta}{(r^2+a^2\cos^2\theta)(1-\frac{2M}{r^2+a^2\cos^2\theta})} & 0 \\ \frac{2Macos\theta}{(r^2+a^2\cos^2\theta)(1-\frac{2M}{r^2+a^2\cos^2\theta})} & 1 & 0 \\ -\frac{2Macos\theta}{(r^2+a^2\cos^2\theta)(1-\frac{2M}{r^2+a^2\cos^2\theta})} & 0 & -\frac{1}{1-\frac{2M}{r^2+a^2\cos^2\theta}} \end{pmatrix},
\]

which has been written in a helpful form to split the off-diagonal modification of \( \lambda \). The general Rasheed black string with charges \( M, S, Q, P \) and \( J \) (and \( N = 0 \)) can then be constructed by transforming the Maison matrix with the asymptotic-preserving \( SO(1,2) \) transformations [54].

In this construction we have focused on the five-dimensional black string solutions which are asymptotically

\[
ds_5^2 = dz^2 - dt^2 + dr^2 + r^2 \left( d\theta^2 + \sin^2\theta d\phi^2 \right) \quad \Rightarrow \quad \chi = \text{diag}(-1, 1, 1). \tag{3.39}
\]

If we desire the asymptotics which will give black holes rather than strings in five dimensions

\[
ds^2 = -dt^2 + dr^2 + r^2 \left( d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2 \right) \tag{3.40}
\]
it is necessary to consider coordinates $x^a$ which are linear combinations of angular coordinates [55] such as $l(\phi + \psi)$, with some length scale $l$, so that a metric of the form (3.32) can be built with

$$\lambda_{ab} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & l^2 & 0 \\ 0 & 0 & l^2 \end{pmatrix}, \quad \omega^a = (0, l\cos2\theta(d\psi - d\phi))$$

(3.41)

$$ds_3^2 = \frac{l^2}{4l^2} \left(dr^2 + r^2\theta^2 + r^2\sin^2\theta\cos2\theta d\psi^2 - d\phi^2 \right).$$

As a result the Maison matrix will limit to

$$\chi \rightarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$  

(3.42)

The transformation between these two asymptotic metrics is given by a particular $SL(3, \mathbb{R})$ element \( e^\chi \) which lies outside the $SO(1, 2)$ subgroup. This transformation acts generally to modify the asymptotics of the solution by introducing a $KK6$ monopole charge [56]. We can, for example, map black string solutions with event horizon topology $S^2 \times \mathbb{R}$ to black hole solutions with topology $S^3$ [57].

### 3.4 General Supergravity Reduction

After presenting the toroidal compactification of pure Einstein gravity in four and then five dimensions we would naturally like to extend the process for higher dimensional theories and theories with additional field content, such as the EMD theory that we found in four dimensions in the previous section.

We noticed that in three dimensions all of the antisymmetric tensors can be described as either scalars, their derivates or the duals of the derivatives. A key observation is that an action for a two-form can be modified by including a Lagrange multiplier $\lambda$

$$S = \int d^3x \sqrt{-g} \left( \frac{1}{4} e^{a\phi} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \lambda \partial_\rho F_{\mu\nu}\epsilon^{\rho\mu\nu} \right)$$

(3.43)

so that the equations of motion are just the Bianchi identity for a two-form and a redefinition of $F$ as the dual of a scalar:

$$F_{\mu\nu} = e^{-a\phi} \epsilon_{\mu\nu\xi} \partial^\xi \phi$$

(3.44)

After integrating by parts to eliminate the field strength derivative we can substitute in $\phi$ to obtain an equivalent action for the scalar field:

$$S = \int d^3x \sqrt{-g} \left( \frac{1}{2} e^{-a\phi} \partial_\mu \phi \partial^\mu \phi \right).$$

(3.45)

In this way we were able to describe dimensional reduction of gravity from four and five dimensions as three-dimensional scalar cosets. Let us now consider the Lagrangian $T^2$ reduction of gravity, that we did not write down in the previous section, starting in dimension

\[\text{This is the } D \text{ operator of [55], see section 2.3 of that paper for more details.}\]
D. We take our dimension-specific couplings $\alpha_D$ as in equation (3.4) and recall the reduction of forms in equation (3.10) to find that

$$\int d^{D+1}x \sqrt{-g_{D+1}} R_{D+1} = \int d^{D-1}x \sqrt{-g_D} \left( R_D - \frac{1}{2} \partial_\mu \phi D \partial^\mu \phi_D - \frac{1}{4} e^{2(1-D)\alpha_D \phi_D} F_{D\mu\nu} F_D^{\mu\nu} \right)$$

$$= \int d^{D-1}x \sqrt{-g_{D-1}} \left( R_{D-1} - \frac{1}{2} \partial_\mu \phi D \partial^\mu \phi_D - \frac{1}{2} \partial_\mu \phi_{D-1} \partial^\mu \phi_{D-1} \right)$$

$$- \frac{1}{4} e^{2(1-D)\alpha_D \phi_D} e^{-2\alpha_{D-1} \phi_{D-1}} F_{D\mu\nu} F_D^{\mu\nu}$$

$$- \frac{1}{2} e^{2(1-D)\alpha_D \phi_D} e^{2(D-3)\alpha_{D-1} \phi_{D-1}} F_{D\mu} F_D^{\mu}$$

$$- \frac{1}{4} e^{2(2-D)\alpha_{D-1} \phi_{D-1}} F_{D-1\mu\nu} F_{D-1}^{\mu\nu},$$

(3.46)

where we have identified the scalars $\phi$ and field strengths of the KK-vectors $F$ by the dimension that they first appear in. When we further reduce this Lagrangian it is clear that the terms will proliferate and the prefactors of each field will become increasingly complicated. We simplify the process by taking the approach of [32]. In the $d$-dimensional reduction from $D+1$ dimensions we will have a $d$-dimensional vector of scalars $\phi = (\phi_D, \phi_{D-1}, \ldots, \phi_{D-d+1})$ and we can construct a vector of coefficients

$$\alpha = (a_D \alpha_D, a_{D-1} \alpha_{D-1}, \ldots, a_{D-d+1} \alpha_{D-d+1})$$

(3.47)

which can be used to simplify reduced gravity Lagrangians. We will use the basis presented in [32]:

$$\alpha_i = (0, \ldots, 0, 2, 2(2-i-D)\alpha_{D-i+1}, 2(D-i-2)\alpha_{D-i}, 0, \ldots, 0) \quad i = 1, \ldots d.$$  

(3.48)

For our $d = 2$ case $\alpha_1 = (2(1-D)\alpha_D, 2(D-3)\alpha_{D-1})$ and $\alpha_2 = (0, 2(D-2)\alpha_{D-1})$. Using the standard Euclidean inner product over the $d$-dimensional vectors we can then write equation (3.46) as:

$$\int d^{D-1}x \sqrt{-g_{D-1}} (R_{D-1} - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} e^{(\alpha_1 + \alpha_2) \phi} F_{D\mu\nu} F_D^{\mu\nu}$$

$$- \frac{1}{2} e^{\alpha_1 \phi} F_{D\mu} F_D^{\mu} - \frac{1}{4} e^{\alpha_2 \phi} F_{D-1\mu\nu} F_{D-1}^{\mu\nu}).$$

(3.49)

(3.50)

The inner products of the $\alpha_i$ in equation (3.48) are easily found to be

$$\alpha_i \cdot \alpha_i = 4 \quad i \neq d$$

$$\alpha_i \cdot \alpha_{i+1} = -2$$

$$\alpha_i \cdot \alpha_{i+j} = 0 \quad \text{if } j \geq 2.$$  

(3.51)

The last $\alpha$ basis vector to be added in any reduction has the unique norm,

$$\alpha_d \cdot \alpha_d = \frac{2(D-d)}{D-d-1}.$$  

(3.52)
which only matches the uniform value of 4 when $D - d = 2$. This is precisely when the reduction terminates at three dimensions and all of the field content can be described by scalars. The dimensional reduction of gravity from $D + 1$ to $D + 1 - d$ produces an action

$$
\int d^{D+1-d}\sqrt{-g} \left( R_{D+1-d} - \frac{1}{2} \partial_{\mu} \phi \cdot \partial^{\mu} \phi - \frac{1}{2} \left[ \mathcal{L}_{F-\text{scalar}} \right] - \frac{1}{4} \left[ \mathcal{L}_{F-\text{vector}} \right] \right) \tag{3.53}
$$

where the vector component is

$$
\mathcal{L}_{F-\text{vector}} = e^{(\alpha_1+\alpha_2+\ldots+\alpha_d)\phi} F_{D\mu\nu} F_{D\mu\nu} + e^{(\alpha_2+\alpha_3+\ldots+\alpha_d)\phi} F_{D-1\mu\nu} F_{D-1\mu\nu} + \ldots + e^{\alpha_d\phi} F_{D-d+1\mu\nu} F_{D-d+1\mu\nu} \tag{3.54}
$$

The entire collection of $KK$-vectors which are never reduced all contain $\alpha_d$, while the $\mathcal{L}_{F-\text{scalar}}$ part of the action contains precisely all those vectors without $\alpha_d$. We find that, in general, the $KK$-field strength $F_{D-i+1\mu\nu}$ which appears in the $i$-dimensional reduction produces scalar fields which appear at every one of the $i + j$ reductions. These scalar fields have the prefactor $e^{\alpha\phi}$ where

$$
\alpha = \alpha_i + \alpha_{i+1} + \ldots + \alpha_{i+j} \tag{3.55}
$$

We can therefore write the additional scalar action component as

$$
\mathcal{L}_{F-\text{scalar}} = \sum_{i=1}^{k} \sum_{j=1}^{d-1} e^{(\sum_{j=i+1}^{d} \alpha_j)\phi} \psi^{(\sum_{j=i}^{d} \alpha_j)}, \tag{3.56}
$$

where we have implied a $1 - 1$ matching of the coefficients $e^{\alpha\phi}$ and $\phi_\alpha$. It is clear that the $\alpha$ vectors of this scalar action form the positive root space of $A_{d-1}$ and, if we include those that appear in equation (3.54) when $D - d = 2$ we find the full $A_d = A_{D-2}$ positive root space.

There are many theories which contain additional fields that are not simply the product of dimensional reduction of gravity and possess coset symmetries. In [32] the same approach is used to describe gravity coupled to three-form and four-form field strengths. We will focus on the latter as it is the bosonic sector of eleven-dimensional supergravity. The action

$$
\int d^{D+1}x \sqrt{-g_{D+1}} \left( R_{D+1} - \frac{1}{2} F_{\mu\nu\lambda\rho} F_{\mu\nu\lambda\rho} \right) \tag{3.57}
$$

when reduced on $T^d$ $d \geq 3$ will contain, along with many others, the scalar term

$$
\sqrt{-g_{D-d+1}} \frac{1}{2} e^{2(D-5)(\alpha_D \phi_D + \alpha_{D-1} \phi_{D-1} + \alpha_{D-2} \phi_{D-2})} F_\mu F_\mu. \tag{3.58}
$$

Writing these coefficients as a vector

$$
\beta = (2(D-5)\alpha_D \phi_D, 2(D-5)\alpha_{D-1} \phi_{D-1}, 2(D-5)\alpha_{D-2} \phi_{D-2}, 0_4, \ldots, 0_d) \tag{3.59}
$$
it is not a linear combination of the vectors that we would obtain in the reduction of pure gravity. However, it has surprisingly simple inner products with the simple root vectors

$$\beta \cdot \alpha_i = -2\delta_{i,3},$$

$$\beta \cdot \beta = \frac{6(D - 4)}{D - 1}.$$  \hspace{1cm} (3.60)

Taking $\beta$ to be a new basis vector $\beta \equiv \alpha_8$ we must set $D + 1 = 11$ in order to obtain the desired norm. When we reduce this theory on $T^8$ the full set of scalars are multiplied by $e^{\alpha \cdot \phi}$ where the set of $\alpha$ fill out the full positive root space of $E_8$.

3.4.1 Eleven-dimensional to type IIA supergravity

The circle compactification of eleven-dimensional supergravity with Lagrangian (4.43) will proceed just as in the previous sections. The gravitational components will introduce a dilaton and $KK$-vector while the four-form will break down into a ten-dimensional four form and three-form. For notational simplicity we label the eleven-dimensional field strength $F_{11[4]} = G_{[4]}$, while the ten-dimensional fields, whose description comes from section 3.1, are $dA_{[1]} = F_{[2]}, \ dA_{[2]} = F_{[3]}$ and $dA_{[3]} = F_{[4]}$. In the reduction of the Chern-Simons term $F \wedge F \wedge A$ three permutations involve immediate reduction with $A_{[2]}$ while eight involve $F_{[3]}$. The exterior derivative can be swapped by parts as

$$-8\epsilon^\mu_{\nu_1 \cdots \nu_3 \rho_1 \cdots \rho_3} F_{\mu_1 \cdots \mu_3} F_{\nu_1 \cdots \nu_4} A_{\rho_1 \cdots \rho_3} = 6\epsilon^\mu_{\nu_1 \cdots \nu_3} A_{\mu_1 \mu_2} F_{\nu_1 \cdots \nu_4} F_{\rho_1 \cdots \rho_4},$$  \hspace{1cm} (3.61)

noting that the combinatoric factor changes due to the altered form structure. The ten-dimensional action is then;

$$\mathcal{L}_{10D} = \sqrt{-g_{10}} \left( R_{10} - \frac{1}{2} (\partial \phi)^2 - \frac{1}{2 \times 2!} e^{3\phi/2} F_{[2]}^2 - \frac{1}{2 \times 3!} e^{-\phi} F_{[3]}^2 - \frac{1}{2 \times 4!} e^{\phi/2} F_{[4]}^2 \right) + \frac{9}{144^{2}} \epsilon_{\mu_1 \mu_2 \nu_1 \cdots \nu_4 \rho_1 \cdots \rho_4} A_{\mu_1 \mu_2} F_{\nu_1 \cdots \nu_4} F_{\rho_1 \cdots \rho_4},$$  \hspace{1cm} (3.62)

the Lagrangian for IIA supergravity.

3.5 Reduction to 2D and Extended Symmetry

Besides building larger theories, another obvious generalisation of the material in this chapter is to consider the possibility of compactifying to dimensions lower than three. As we observed earlier, four-dimensional gravity with one time-like killing vector was equivalent to a three-dimensional $\sigma$ model with the Ehler’s group $G_E = SL(2, \mathbb{R})$. If we further require axial symmetry, gravity reduces to a two-dimensional theory which is invariant under an extended affine symmetry. This was originally explored, in the context of symmetries for solutions to stationary-axisymmetric gravity, by Robert Geroch [10, 11] and the symmetry group is
known as the Geroch group. The group structure was elucidated later in [12], where the infinite dimensional algebra results from two $SL(2, \mathbb{R})$ subalgebras. The first comes from Ehler’s symmetry while the second is known as the $SL(2, \mathbb{R})$ Matzner-Misner group [58]. These are not connected as a finite Lie algebra, but as the affine extension of $A_1$ as shown in figure 3.5.1, where both off-diagonal entries of the Cartan matrix are $-2$.

Concretely, consider the gravity action which has already been rewritten as a three-dimensional coset model in section 3.2. As we discussed in section 2.4, this action can be written as

$$S_{3D} = \int d^3x \sqrt{-g} \left( R - \frac{1}{2} g_{\mu\nu} (P_{\mu}, P_{\nu}) \right)$$

(3.63)

where we have taken $P$ to be the fields of the involution anti-invariant $p$ in the Maurer-Cartan form of the Ehler’s group. We will now follow the description of the Breitenlohner-Maison construction found in [59], in which the requirement of axial symmetry on the Ehler’s coset leads to an affine symmetry algebra. With axial isometry, we can write any three-dimensional metric as

$$ds^2_{3D} = f^2 ds^2_{2D} + \rho^2 d\phi^2$$

(3.64)

with some function $f$ of the two-dimensional subspace and a function $\rho$ which must be harmonic due to the equations of motion. It is convenient to write the Weyl coordinates $(\rho, z)$ for the two-dimensional subspace using the harmonic conjugate $z$ defined by $dz = \star d\rho$. With complex light-cone coordinates $x^\pm = 1/2(z \mp i\rho)$ the equations of motion can be rewritten as

$$\pm i f^{-1} \partial_\pm f = \frac{\rho}{2} (P_\pm, P_\pm)$$

(3.65)

$$\partial_\mu (\rho P^\mu) - \rho [Q_\mu, P^\mu] = 0.$$  

(3.66)

Now introduce a parameter $t$ and expand a general group element of the Ehler’s coset in $t$

$$g(t) = \gamma^{(0)} + t\gamma^{(1)} + \frac{1}{2} t^2 \gamma^{(2)} + \ldots$$

(3.67)

around a point $g$ such that in the limit $\lim_{t \to 0} g(t) = g$. Breitenlohner and Maison proposed an ansatz for the Maurer-Cartan form which, in the clean notation of Katsimpouri, Kleinschmidt and Virmani

$$\partial_\pm g g^{-1} = \frac{1 \mp i}{1 \pm it} P_\pm + Q_\pm$$

(3.68)
is referred to as the BM linear system. Equivalence between the integrability of the BM system and the equations of motion for the theory is given by two differential equations on \( t \) with two solution branches

\[
t_{\pm} = \frac{1}{\rho} \left( z - w \pm \sqrt{(z - w)^2 + \rho^2} \right)
\]  

(3.69)

with the introduction of an integration constant \( w \). The group element expansion in \( t \) now transforms under a larger set of symmetries

\[
g(t) \rightarrow k(t)g(t)g_0(w)
\]  

(3.70)

where the group elements are now functions of the two newly-introduced parameters. These introduce maps of \( S^1 \) into \( SL(2, \mathbb{R}) \) which produce the loop algebra and with the inclusion of a central element acting on the conformal factor we obtain the full affine symmetry.

The symmetries of the BM system can be used to generate new solutions by mapping group elements under some \( k(t) \) and \( g_0(w) \). This can be done using the unimodular matrix (2.45), through a chain of transformations, but can be computationally challenging due to the required factorisation of a transformed \( M \).

An alternative method for constructing new solutions using the Geroch symmetry was developed by Belinsky and Zakharov [60]. A Lax pair can be constructed such that the integrability condition is equivalent to the equations of motion for the coset and new solutions are obtained using the inverse scattering technique.

This observation that four-dimensional gravity reduced to two dimensions by a pair of commuting Killing vectors produces an affine extension of the \( SL(2, \mathbb{R}) \) symmetry in three dimensions can also be applied to other matter-gravity theories. When reduced on an \( n \)-torus, eleven-dimensional supergravity possesses a coset symmetry as shown in table 3. It was conjectured that the dimensional reduction to \( D \leq 2 \) would result in coset symmetries of the extensions of \( E_8 \) [13]. This was supported by the evidence for the affine extension \( E_8^{+} \) symmetry in the solutions of two-dimensional \( \mathcal{N} = 16 \) supersymmetry [14, 61, 62].
Having explored the (hidden) coset symmetries of (super)gravity theories in the previous chapter, we were led to the conjectured cosets of infinite-dimensional groups which appear in the reduction to two dimensions. For eleven-dimensional supergravity we would consider the extensions of $E_8$ while for $D$-dimensional gravitational theories we extend $A_{D-3}$ [32].

The programme of utilising Lorentzian Kac-Moody algebras in the study supergravities and their extensions has been directed by work in non-linear realisations of $E_{11}$ [18]. This is based on the approach of Borisov and Ogievetsky in describing gravity as a non-linear realisation of two subalgebras of the diffeomorphism algebra. We review the non-linear realisation of $E_{11}$ before reviewing its connections with supergravity solutions and bound states of branes. The $l_1$ representation is also constructed so that we can give the more precise realisation with the inhomogeneous group.
4.1 Diffeomorphism Subalgebras

A differentiable manifold \( \mathcal{M}^d \) of dimension \( d \), as defined in the first chapter, consists of a topological space with an atlas of homeomorphisms to \( \mathbb{R}^d \). These maps provide a system of coordinates for the manifold. These are not unique and we call the smooth homeomorphisms between different coordinate charts diffeomorphisms. Physical theories constructed on manifolds should not depend on any arbitrary set of coordinates and the coordinate invariance of the theory is referred to as the diffeomorphism symmetry.

We define the group of diffeomorphisms of a \( d \)-dimensional manifold that are analytically connected to the identity coordinate transformation to be \( \text{Diff}(d) \). With a set of coordinates \( x_a^\mu \) for some chart we naturally define a basis for the tangent space \( T_x \mathcal{M}^d \) so that a vector field can be expanded as

\[
X(x^\mu) = X^\mu \frac{\partial}{\partial x^\mu}
\]  

(4.1)

and a general diffeomorphism \( \varphi_X \in \text{Diff}(d) \) can be written as

\[
\varphi_X : x^\mu \rightarrow e^{X^\nu \partial_\nu} x^\mu.
\]  

(4.2)

The vector fields have a bilinear antisymmetric structure which satisfies the Jacobi identity and is indeed a Lie group. The generators of the vector space form the Lie algebra \( \mathfrak{diff}_d \).

We follow section 1.3 of [63] and cite Theorem 1.9. This is the group of general linear transformations with translations, not to be confused with the infinite-dimensional extensions of Lie algebras.
By taking the trace of this expression and reinserting the result we find that

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d} g_{\mu\nu} \partial^\lambda \epsilon_\lambda. \quad (4.8)$$

From this point we assume that the infinitesimal transformations are deforming some constant flat metric \( g = \eta_{\rho\sigma} \). We can then apply a combination of derivatives \( \partial^\mu \partial^\nu \) to find that

$$\begin{equation}
(d - 1) \partial^\rho \partial_\mu (\partial^\nu \epsilon_\nu) = 0. \quad (4.9)
\end{equation}$$

By taking linear combinations of derivatives of equation (4.8) we can also find that

$$\begin{equation}
(2 - d) \partial^\rho \partial_\nu \left( \partial^\lambda \epsilon_\lambda \right) = g_{\mu\nu} \partial^\lambda \partial_\lambda (\partial^\rho \epsilon_\rho) \quad (4.10)
\end{equation}$$

so that \( \partial^\rho \partial_\mu (\partial^\rho \epsilon_\mu) = \partial_\mu \partial_\nu (\partial^\rho \epsilon_\rho) = 0 \) when \( d \geq 3 \). The general solution to these differential equations is an \( \epsilon \) which is quadratic in \( x \) with several restrictions which can be found by inputting general \( \epsilon \) into the equations above and their intermediaries. We find that the general \( \epsilon \),

$$\epsilon^\mu = a^\mu + bx^\mu + c^\mu_{\rho
\nu} x^\nu + (d^\mu x^\nu - \frac{2}{d} x^\mu d^\nu) x^\nu, \quad (4.11)$$

with \( c_{\mu\nu} \) antisymmetric, describes translations \( x^\mu \to x^\mu + a^\mu \), dilations \( x^\mu \to bx^\mu \), Lorentz transformations (boosts and rotations based on \( \eta_{\rho\sigma} \)) \( x^\mu \to c^\mu_{\nu} x^\nu \) and the special conformal transformations

$$x^\mu \to \frac{x^\mu + d^\mu x^2}{1 + 2d^\mu x_\mu + b^2 x^2} \quad (4.12)$$

which when expanded to first order in \( d \) give the last term in (4.11). We label the vector fields which generate this diffeomorphisms

**Translations** \( P_\mu = \partial_\mu \quad (4.13) \)

**Dilations** \( D = x^\mu \partial_\mu \quad (4.14) \)

**Lorentz** \( A_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu \quad (4.15) \)

**Special Conformal** \( C_\mu = 2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu \quad (4.16) \)

and all of the non-zero commutators are:

$$[D, P_\mu] = -P_\mu \quad [D, C_\mu] = C_\mu \quad (4.17)$$

$$[K_\mu, P_\nu] = -2 (D\eta_{\mu\nu} + A_{\mu\nu}) \quad (4.18)$$

$$[A_{\mu\nu}, C_\lambda] = \eta_{\lambda\mu} C_\nu - \eta_{\lambda\nu} C_\mu \quad [A_{\mu\nu}, P_\lambda] = \eta_{\lambda\nu} P_\mu - \eta_{\lambda\mu} P_\nu \quad (4.19)$$

$$[A_{\mu\nu}, A_{\lambda\rho}] = \eta_{\rho\lambda} A_{\mu\nu} + \eta_{\rho\nu} A_{\mu\lambda} - \eta_{\mu\nu} A_{\rho\lambda} - \eta_{\mu\lambda} A_{\rho\nu} - \eta_{\nu\lambda} A_{\mu\rho} - \eta_{\nu\rho} A_{\mu\lambda}. \quad (4.20)$$

The affine subalgebra of \( \text{diff}_d \) contains \( \mathfrak{gl}_d \) which can be decomposed into a trace, antisymmetric and symmetric traceless subsets. The first two coincide with the \( D \) and \( A_{\mu\nu} \) generators that we have found in the conformal algebra. The only generators outside the intersection of the conformal and affine algebras are therefore the special conformal \( K_\mu \) and the symmetric traceless \( S_{\mu\nu} \in K_{\mu\nu} \), so the work of Ogievetsky informs us that \( \text{diff}_d \) can be constructed as the closure of the conformal algebra with \( S_{\mu\nu} \) or the closure of the affine algebra with \( K_\mu \).
4.2 Non-linear Realisations of \( \mathfrak{diff} \) Subalgebras

Having broken down the diffeomorphism algebra as the closure of two finite algebras, Borisov and Ogievetsky used the non-linear realisations of both subalgebras to construct joint equivariant connections [16]. In this way gravity can be constructed as a non-linear realisation of symmetry algebras. In this section we will present the extended realisation, based on [18], where \( IGL(11) \) has been enlarged to a subset of \( E_{11} \) with the translation generators. We will later understand how the inhomogeneous generators arise when we consider the \( l_1 \) representation.

We add the three-form and six-form generators \( R^{[3]} \) and \( R^{[6]} \) which add the following non-zero commutators in \( G_{11} \):

\[
\begin{align*}
[K^\mu, R^{\lambda_1 \lambda_2 \lambda_3}] &= \delta_\nu^{\lambda_1} R^{\nu \lambda_2 \lambda_3} + \text{sym} \\
[K^\nu, R^{\lambda_1 \lambda_2 \lambda_3 \lambda_4 \lambda_5 \lambda_6}] &= \delta_\nu^{\lambda_1} R^{\nu \lambda_2 \lambda_3 \lambda_4 \lambda_5 \lambda_6} + \text{sym} \\
[R^{\lambda_1 \lambda_2 \lambda_3}, R^{\lambda_4 \lambda_5 \lambda_6}] &= 2R^{\lambda_1 \lambda_2 \lambda_3 \lambda_4 \lambda_5 \lambda_6}.
\end{align*}
\]

The \( G_{11} \) gauged group element for the non-linear realisation takes the form

\[ g = \exp \left\{ x^\mu P_\mu \right\} \exp \left\{ h_a b K^{ab} \right\} \exp \left\{ \frac{1}{3!} A_{a_1 a_2 a_3} R^{a_1 a_2 a_3} \right\} \exp \left\{ \frac{1}{3!} A_{a_1 \cdots a_6} R^{a_1 \cdots a_6} \right\} \]

where \( x^\mu \) are the 11-dimensional space-time coordinates and each of the other \( h, A^{[3]} \) and \( A^{[6]} \) fields are additional parameters of the group. The Maurer-Cartan form can be expanded using the BCH formula (2.42) as we demonstrate in appendix A. The result, originally found in [17] and reproduced in the appendix using the notation from that paper, is that

\[
\omega = dx^\mu \left( g^{-1} \partial_\mu g - \lambda_\mu \right)
= dx^\mu \left( e_\mu^a P_a + \Omega_{\mu a}^b K^a_b + \frac{1}{3!} R^{a_1 a_2 a_3} \tilde{D}_\mu A_{a_1 a_2 a_3} + \frac{1}{6!} R^{a_1 \cdots a_6} \tilde{D}_\mu A_{a_1 \cdots a_6} \right)
\]

where

\[
\begin{align*}
\epsilon_\mu^a &= \left( e^b \right)^a_{\mu} \\
\Omega_{\mu a}^b &= \partial_\mu \left( e^{-1} \partial e_\mu \sigma^a \right)_b - \lambda_{\mu a}^b \\
\tilde{D}_\mu A_{a_1 a_2 a_3} &= \partial_\mu A_{a_1 a_2 a_3} + \left( e^{-1} \partial e_\mu \sigma^a \right)_{a_1} A_{a_2 a_3} + \text{sym} \\
\tilde{D}_\mu A_{a_1 \cdots a_6} &= \partial_\mu A_{a_1 \cdots a_6} + \left( e^{-1} \partial e_\mu \sigma^a \right)_{a_1} A_{a_2 \cdots a_6} + \text{sym} \\
&\quad - \left( A_{a_1 a_2 a_3} \tilde{D}_{[\mu} A_{a_4 a_5 a_6]} \right).
\end{align*}
\]

Since we will be building the simultaneous realisation of both subalgebras of the diffeomorphism algebra we must perform the analogous calculations with the conformal group. Using the generators of the previous section, a group element, where the local group of the coset is the Lorentz subgroup, is

\[ g = \exp \left\{ x^\mu P_\mu \right\} \exp \left\{ \phi^b C_b \right\} \exp \left\{ \sigma D \right\}, \]
where the fields \( x^\mu \) of the translation generators are the space-time coordinates and the other fields are \( x \)-dependent. As we show in appendix A, the Maurer-Cartan form can be expanded as

\[
\omega = dx^\mu \left( e^\sigma P_\mu + e^{-\sigma} \left( 2\phi_\mu \phi^a - \phi^a \delta_\mu^a + \partial_\mu \phi^a \right) C_a + \left( 2\phi_\mu + \partial_\mu \sigma \right) D + 2\phi^a J_{a\mu} \right).
\] (4.29)

We therefore find conformal covariant derivatives for the space-time fields

\[
\Delta_\mu \phi^a = e^{-2\sigma} \left( 2\phi_\mu \phi^a - \phi^a \delta_\mu^a + \partial_\mu \phi^a \right)
\]

\[
\Delta_\mu \sigma = e^{-\sigma} \left( 2\phi_\mu + \partial_\mu \sigma \right).
\] (4.30) (4.31)

The covariant derivatives for a vector under the Lorentz representation and subsequently the three-form are

\[
\Delta_a A_b = e^{-\sigma} \left( \partial_a A_b + \eta_{ab} (\partial^\tau \sigma) A_c - (\partial_b \sigma) A_a \right)
\]

\[
\Delta_a A_{b_1 b_2 b_3} = e^{-\sigma} \left( \partial_a A_{b_1 b_2 b_3} + \eta_{a[b_1} (\partial^\tau \sigma) A_{c]b_2 b_3} - (\partial_{b_1} \sigma) A_{a[b_2 b_3]} \right).
\] (4.32) (4.33)

Now that we have covariant derivatives for both the conformal and \( G_{11} \) extended affine groups our objective is to produce a simultaneous realisation by placing restrictions on the derivatives and fields. We start by eliminating the independence of \( \phi_a \) and \( \sigma \) by setting \( \Delta_\mu \sigma = 0 \), or equivalently \( \partial_\mu \sigma - 2\phi_\mu = 0 \), and identify the dilation field with the diagonal metric elements:

\[
h_{\mu}^a = \bar{h}_{\mu}^a + \sigma \delta_{\mu}^a, \quad \bar{h}_{\mu}^a = 0 \quad \Rightarrow \quad e_{\mu}^a \equiv \left( e^b \right)_{\mu}^a = \left( e^b \right)_{\mu}^a e^a e^\sigma.
\] (4.34)

We can now substitute the conformal covariant derivative of the three-form (4.33) into the \( G_{11} \) derivative (4.26)

\[
\tilde{D}_a A_{b_1 b_2 b_3} = \tilde{e}_a^b (\Delta_\mu A_{b_1 b_2 b_3} - e^{-\sigma} \eta_{a[b_1} (\partial^\tau \sigma) A_{c]b_2 b_3} - (\partial_{b_1} \sigma) A_{a[b_2 b_3]} - \epsilon^{-1} \partial_\mu \tilde{e}^c_{b_1} A_{c[b_2 b_3]}) \partial_\mu - (\partial_{b_1} \sigma) A_{b_1 b_2 b_3})
\] (4.35)

If we consider the terms which have \( \bar{h}^0 \) dependence we find that there are only the terms proportional to

\[
\eta_{a[b_1} \partial_\mu \sigma A_{c]b_2 b_3]} - \partial_{b_1} \sigma A_{a[b_2 b_3]} - \partial_\mu \sigma A_{b_1 [b_2 b_3]}
\] (4.36)

which prevent equality between \( \tilde{D} A_{[3]} \) and \( \Delta A_{[3]} \). It is clear that when we anti-symmetrise \([ab_1 b_2 b_3]\) this vanishes and in general

\[
\tilde{D}_{[a} A_{b_1 b_2 b_3]} = \Delta_{[a} A_{b_1 b_2 b_3]} \equiv F_{a b_1 b_2 b_3}.
\] (4.37)

Using the same arguments we can construct the conformal and \( G_{11} \) covariant fields strength of the six-form \( A_{[6]} \):

\[
F_{a_1 a_2 \ldots a_7} \equiv \tilde{D}_{[a_1} A_{a_2 \ldots a_7]} = \Delta_{[a_1} A_{a_2 \ldots a_7]}.
\] (4.38)
The equations of motion for the theory must be constructed out of covariant fields and must be covariant under local Lorentz transformations. Combining the new four and seven-forms with the Riemann tensor we find the equations of motion to be

$$F_{a_1\cdots a_4} = \frac{1}{7!}\epsilon_{a_1\cdots a_{11}}F_{a_5\cdots a_{11}}$$

(4.39)

$$R_{\mu
u bc}e^c_{a\mu} - \frac{1}{2}\eta_{ab}R_{\mu
u dc}e^d_{b\nu} = \frac{1}{4}\left(F_{ac_1\cdots c_3}F_{b\epsilon_1\cdots \epsilon_3} - \frac{1}{6}\eta_{ab}F_{c_1\cdots c_4}F^{\epsilon_1\cdots \epsilon_4}\right),$$

(4.40)

which coincide with the equations of motion for eleven-dimensional supergravity in locally flat coordinates, as we will shortly verify.

### 4.3 Solution Generating Elements

Generation of solutions to theories with Kac-Moody coset symmetries is a central topic in this thesis, so we devote a section to the historic identification of the connection between roots of the symmetry algebra and solutions.

It was first shown in [65] (with generalisations in [21]) that for a theory with very-extended Lie algebra $g^{+++}$ coset symmetry, the roots $\beta \in \Delta(g^{+++})$ can be used to construct solutions using the Borel-gauged group element

$$g = \exp\left(-\frac{1}{\beta^2}\log(N)\beta \cdot H\right)\exp((1 - N)E_{\beta}),$$

(4.41)

where the notation $\beta \cdot H$ indicates the Cartan subalgebra element associated with the root and $N$ is a function of the space-time coordinates:

$$N = 1 + \frac{k}{D - 2 - \sum_{i=1}^{D-1}(i-D)(\alpha_i,\beta)}.$$  

(4.42)

It was discovered that solutions could be reproduced for a variety of theories by taking a $gl$ representation for the Cartan elements and identifying the group element $\exp\left(-\frac{1}{\beta^2}\log(N)\beta \cdot H\right)$ with vielbein while adding a gauge field $A_{\beta} = N^{-1} - 1$ with a representation based on the $gl$ identification. The fact that equation (4.41) contains the root length squared of $\beta$ is a reference to the fact that non-simply-laced algebras and Kac-Moody algebras (simply-laced or not, they possess imaginary roots) have been studied\(^3\). We will focus on and provide as an example the connection between $E_{11}$ and eleven-dimensional supergravity. Before applying this solution-generating technique we pause for a quick review of the membrane solutions of eleven-dimensional supergravity.

### The $M_2$ and $M_5$ solutions in supergravity

The action for the bosonic fields of eleven-dimensional supergravity contains the metric and an anti-symmetric three-form gauge field

$$S = \int d^{11}x \left(\sqrt{-g}\left(R - \frac{1}{2 \times 4!}F_4^2\right) + \frac{1}{6}F_4 \wedge F_4 \wedge A_3\right).$$

(4.43)

\[^3\text{Indeed the matter of null roots and supergravity solutions has been studied [27].}\]
where $F_4 = \mathrm{d}A_3$. The last term in the Lagrangian is the Chern-Simons topological term which is required for consistency of the action and is non-trivial for many solutions, but not those that we consider here. The equations of motion for the metric and gauge field are

$$R_{\mu\nu} = \frac{1}{12} \left( F_{\mu\rho\lambda\sigma} F_{\nu}^{\rho\lambda\sigma} - \frac{1}{12} g_{\mu\nu} F_{\rho\sigma\lambda\tau} F^{\rho\sigma\lambda\tau} \right)$$

(4.44)

$$0 = \partial_{\lambda} F^{\lambda\rho\sigma\tau} + c \epsilon_{\mu_1 \cdots \mu_4 \nu_1 \cdots \nu_4} \epsilon_{\rho\sigma\lambda\tau} F_{\mu_1 \cdots \mu_4} F_{\nu_1 \cdots \nu_4}$$

(4.45)

respectively. The constant $c = 2^{-7/3} - 2$ contains combinatoric factors due to the forms and contractions. Using differential forms we can rewrite this as

$$0 = \mathrm{d} \star F_4 + \frac{1}{2} F_4 \wedge F_4.$$  

(4.46)

The additional fermionic content of the supergravity theory will not be necessary for this discussion and the bosonic sector, shown in equation (4.43), is obtained by setting the gravitino field $\psi = 0$ in the full theory. In order to make this consistent with the (super)symmetries of the theory we must also enforce the constraint that the supersymmetry variation with spinor parameter $\epsilon(x)$ of the gravitino vanishes

$$0 = \delta \psi_A |_{\psi = 0} = \tilde{D}_A \epsilon$$

$$0 = \left[ \partial_A + \frac{1}{4} \omega_A^{BC} \Gamma_{BC} - \frac{1}{2 \times 12^2} \left( \Gamma_A^{BCDE} - 8 \delta_A^B \Gamma^{CDE} \right) F_{BCDE} \right] \epsilon.$$  

(4.47)

These are known as the Killing spinor equations and in solving them we are determining the maximal dimension of the space of spinors which are not forced to be zero. As the Killing vectors describe the isometries of the spacetime, the Killing spinors tell us what supersymmetries are left by the solution.

In order to find individual solutions of the equations of motion we follow the $p$-brane ansatz for a $D$-dimensional gravity theory with a $(p+2)$-rank field strength which is reviewed in [67]. The manifest Poincaré invariance of the theory can be broken down to (Poincaré) $(p+1) \times SO(D - p - 1)$ so that we may split the coordinates $x^M = (x^\mu, x^i)$ where $\mu = 1, 2, \ldots, p + 1$ and $i = p + 2, \ldots, D$. We consider a metric ansatz which is manifestly compatible with our suggested symmetry

$$ds^2 = e^{2A(r)} dx^\mu dx^\nu \eta_{\mu\nu} + e^{2B(r)} dx^i dx^j \delta_{ij},$$

(4.48)

where $r = \sqrt{x^i x^j \delta_{ij}}$ is the radial coordinate of the Euclidean $(D - p - 1)$-dimensional space. The solutions contain $(p + 1)$-dimensional hyperplanes which form the worldvolume of a $p$-dimensional space-like surface which is related to $p$-form conserved charges. As a result, they are known as $p$-brane solutions. We will consider two different ansätze for the gauge field which are written in terms of their antisymmetric field strengths, $\mathrm{d}A_{[p+1]}$ and $\star \mathrm{d}A_{[p+1]} = F_{[D-p-2]}$:

Electric ansatz :

$$F_{\mu_1 \cdots \mu_{p+1}} = \epsilon_{\mu_1 \cdots \mu_{p+1}} \partial_i e^{C(r)}$$

(4.49)

Magnetic ansatz :

$$F_{i_1 \cdots i_{D-p-2}} = \epsilon_{i_1 \cdots i_{D-p-2}} x^j \frac{1}{p+1}.$$  

(4.50)

[66] provides an excellent review for these ubiquitous equations (with all of the fermionic detail, as well).
In both constructions the derivatives run over the $SO(D−P−1)$ invariant space and the
‘magnetic’ field strength also lies in this space, while the ‘electric’ field strength occupies the
Poincaré invariant subspace.

A more general consistent truncation contains a scalar field coupled to the field strength,
as discussed in [67]. The gravitational equations of motion become relatively simple due to
the diagonal metric with only $r$ dependence. The differential equations including $A(r)$, $B(r)$,
$C(r)$ and the additional scalar $\phi(r)$ can be solved with the constraint that

$$(p + 1) \partial_r A + (D - p - 3) \partial_r B,$$

as well as a linearity condition on $\phi$ which allows us to write the $\phi$ equation of motion as a
Laplacian

$$\nabla^2 e^{(\pm) \frac{A}{\Delta} \phi} = 0,$$

where $a$ is the scalar coupling, the sign is $+$ for electric and $-$ for magnetic and $\Delta = a^2 + 2(p + 1)(D - p - 3)/(D - 2)$. If we define exp $((\pm) \frac{A}{\Delta} \phi) \equiv H(r)$ and take the point source
solution we can then solve the equations of motion to find

$$ds^2 = H^{-\frac{4}{D-2}} \left( 1 + \frac{q}{r^{D-p-3}} \right) \frac{1}{\Delta} \log H + H^{\frac{4}{D-2}} \left( 1 + \frac{q}{r^{D-p-3}} \right) \frac{1}{\Delta} \log H$$

and the $p = 2$ solution, known as the $M^2$-brane,

$$d s_{M 2}^2 = \left( 1 + \frac{q}{r^6} \right)^{-\frac{1}{2}} \left( 1 + \frac{q}{r^6} \right)^{\frac{1}{2}} \partial_{i}^2 \partial_{j}, \quad \mu = 1, \ldots, 3 \quad i = 1, \ldots, 11$$

and the $p = 5$ solutions, known as the the $M 5$-brane,

$$d s_{M 5}^2 = \left( 1 + \frac{q}{r^6} \right)^{-\frac{1}{2}} \left( 1 + \frac{q}{r^6} \right)^{\frac{1}{2}} \partial_{i}^2 \partial_{j}, \quad \mu = 1, \ldots, 6 \quad i = 1, \ldots, 11.$$
where $\Omega_n$ is the volume of the unit $n$-sphere. There are several different charges that could be found by integrating over various volumes. With any basis $x^i$ we can construct a basis for these charges. For the electric charge $U$ there are a 2-form’s worth of choices $U_{A_1A_2}$ and for the magnetic charge we find $V_{A_1\ldots A_5}$. Taking these to be the space-like coordinates of our solutions (with $x^1 = t$) the only non-zero charges are integrated over the $x^i$ transverse space. As a result only $U_{23} = 3q/2$ and $V_{23456} = 3q/4$ are non-zero. Note that these are proportional to the constant which appears in the harmonic functions $H(r)$.

The Killing spinor equations (4.47) for both of these solutions lead to conditions that $\epsilon(x)$ must be zero over half of the available basis. As a result, half of the possible supersymmetry has been lost. The solutions also possesses the property that the ADM mass density, or tension, is equal to the conserved charge $U$ or $V$.

4.3.1 The $M2$ and $M5$ solutions as group elements

Using the solution generating technique, let us consider the exceptional root $\beta = \alpha_{11}$. We have already seen that this root is the lowest weight vector of the three-form representation $E_{\alpha_{11}} = R^{91011}$. The Cartan element $H_{11}$ such that $[H_{11}, E_{\alpha_{11}}] = 2E_{\alpha_{11}}$ can be written in the $GL(11)$ basis of $K_{ij}$ as

$$H_{11} = -\frac{1}{3} (K^1 + K^2 + \ldots + K^8) + \frac{2}{3} (K^9 + K^{10} + K^{11})$$

and it is easily verified that $[H_{11}, E_{\alpha_i}] = (2\delta_{i,11} - \delta_{i,8})E_{\alpha_i}$ as required. The group element (4.41) is then

$$g_{M2} = \exp \left\{ \frac{1}{2} \log(N) \left( \frac{1}{3} (K^1 + \ldots + K^8) - \frac{2}{3} (K^9 + K^{10} + K^{11}) \right) \right\}$$

$$\times \exp \left\{ (1 - N) R^{91011} \right\}$$

$$= \exp \left\{ h_{a\ b} K_a^b \right\} \exp \left\{ A_{a_1a_2a_3} R^{a_1a_2a_3} \right\}$$

where the harmonic function $N(r)$ is given by equation (4.42):

$$N(r) = 1 + \frac{k}{r^6}.$$  

The index structure of the three-form transforms under the vielbein $(e^h)_\mu^a$, which defines the metric

$$ds^2 = N^{1/3} (dx_1^2 + \ldots + dx_8^2) + N^{-2/3} (dx_9^2 + dx_{10}^2 - dt^2)$$

\footnote{We omit the proof of this statement here, but a very clear review can be found in [68]. The $M2$ and $M5$ are both considered in this reference, as well as marginal bound states, in section 3.3.}

\footnote{Another statement which we will not prove for these solutions, but for which a calculation can be found in section 3.2 of [68].}
when we perform a Wick rotation on one of the worldvolume coordinates, $dx_1 \rightarrow idt$. The gauge field in (4.59) is given in locally flat coordinates so that
\[ A_{01011} = e_9^9 e_{10}^{10} e_{11}^{11} A_{01011} = N^{-1} - 1. \] (4.62)

By comparison with equation (4.54) of the previous subsection we recognise the $M2$-brane solution.

The Cartan elements for the simple roots $\alpha_i$, $i \leq 10$ are given in the GL(11) basis by
\[ H_i = K_{ii} - K_{i+1,i+1}. \] (4.63)

With the positive generator $E_\beta = R_{67891011}$ the solution group element
\[ g_{M5} = \exp \left\{ -\frac{1}{2} \Log(N) \times H_\beta \right\} \exp \left\{ (1 - N) R_{67891011} \right\} \] (4.65)
yields vielbein which generate
\[ ds^2 = N^{2/3} (dx_1^2 + \ldots + dx_5^2) + N^{-1/3} (dx_6^2 + \ldots + dx_9^2 - dt^2), \] (4.66)

after Wick rotation, and a gauge potential
\[ A_{01011} = e_6^6 e_7^7 e_8^8 e_9^9 e_{10}^{10} e_{11}^{11} A_{01011} = N^{-1} - 1. \] (4.67)

Again, by inspection, we find that when
\[ N(r) = 1 + \frac{k}{r^3} \] (4.68)

we recover the $M5$-brane solution of equation (4.55) from the previous subsection.

### 4.3.2 Type IIA/B Solutions

The ansatz which led to the $p$-brane solutions of eleven-dimensional supergravity is also valid for truncations of string theory supergravities. For example, the type IIA supergravity with Lagrangian (3.62) contains D0, D2 and D4-branes, as well as the dual D6-brane, fundamental string and NS5-brane. Each of these objects has an intuitive interpretation from the dimensional reduction of eleven-dimensional supergravity, as shown in table 4.3.2. The solutions can be constructed, as in equation (4.53), so that the $D_p$-branes are given by
\[ ds^2 = H^{-1/2} \left( -dt^2 + dx_1^2 + \ldots + dx_p^2 \right) + H^{1/2} \left( dx_{p+1}^2 + \ldots + dx_9^2 \right) \]
\[ e^{2\Phi} = H^{-(p-3)/2} \quad H = H(x_{p+1}, \ldots, x_9) = 1 + \frac{q}{r^{p-3}} \tag{4.69} \]
Table 4.3.1: Eleven-dimensional compactified solutions as type IIA solutions.

<table>
<thead>
<tr>
<th>Type IIA Solution</th>
<th>M-theory Solution on $S^1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>D0-brane</td>
<td>$KK$-excitation over $S^1$</td>
</tr>
<tr>
<td>String</td>
<td>$M2$ wrapped over $S^1$</td>
</tr>
<tr>
<td>D2-brane</td>
<td>$M2$ not wrapped</td>
</tr>
<tr>
<td>D4-brane</td>
<td>$M5$ wrapped over $S^1$</td>
</tr>
<tr>
<td>NS5-brane</td>
<td>$M5$ not wrapped</td>
</tr>
<tr>
<td>D6-brane</td>
<td>$KK6$ wrapped over $S^1$</td>
</tr>
</tbody>
</table>

Figure 4.3.1: Decomposition of $E_{11}$ which produces the type IIA spectrum.

the fundamental string by

$$ds^2 = H^{-1}(-dt^2 + dx_7^2) + dx_2^2 + \ldots + dx_9^2)$$
$$e^{2\phi} = H^{-1} \quad H = H(x_2, \ldots, x_9) = 1 + \frac{q}{r^6}$$

and the NS5-brane by

$$ds^2 = d\Omega^2_{(1,5)} + H (dx_6^2 + \cdots + dx_9^2)$$
$$e^{2\phi} = H \quad H = H(x_6, \ldots, x_9) = 1 + \frac{q}{r^2}.$$  \hspace{1cm} (4.71)

The IIA supergravity solutions have associated field strengths of the gauge fields which source them, with solutions also given by the $p$-brane ansatz.

The identification of type IIA [18] and IIB [69] solutions from $E_{11}$ has already been performed. This was done by taking $SL(10, \mathbb{R})$ decompositions where the choice of nodes which are levelled provides the distinction between the two. We obtain the type IIA field content by levelling all of the roots $\alpha = \sum_{i=1}^{11} a_i \alpha_i$ with respect to the nodes 10 and 11, as shown in figure 4.3.2. The fundamental weights $\lambda_8$ and $\lambda_9$ (which are contained within the roots $\alpha_{11}$ and $\alpha_{10}$, respectively) give the 55 two-form and 10 vector representation. The generators for the levelled algebra now include $E_{\alpha_{10}} = R^{10}$ and $E_{\alpha_{11}} = R^{11}$, as well as the usual $SL(10)$ generators $E_{\alpha_i} = K^{i}_{i+1}$. There is also a scalar $R$ which will generate the dilaton field. These can be obtained from the $E_{11}$ generators by isolating the compactified coordinate [18], so that the lowest level IIA generators $\tilde{E}$ are:

$$\tilde{K}^{i_1}_{i_2} = K^{i_1}_{i_2}, \quad \tilde{R}^i = K^{i}_{i+1}, \quad \tilde{R}^{i_1}_{i_2}_{i_3} = R^{i_1}_{i_2}_{i_3}, \quad \tilde{R}^{i_1}_{i_2} = R^{i_1}_{i_2}_{11}$$

$$\tilde{R} = \frac{1}{12} \left( 8K^{11}_{11} - \sum_{i=1}^{10} K^i_i \right) \hspace{1cm} (4.72)$$
where all of the \( i \in \{1, \ldots, 10\} \). The Cartan elements are then given by [20]

\[
H_i = \tilde{K}_i^i - \tilde{K}_{i+1}^{i+1} \quad H_{10} = -\frac{1}{8} \left( \tilde{K}_1^1 + \cdots + \tilde{K}_9^9 \right) + \frac{7}{8} \tilde{K}_{10}^{10} - \frac{3}{2} \tilde{R} \\
H_{11} = -\frac{1}{4} \left( \tilde{K}_1^1 + \cdots + \tilde{K}_8^8 \right) + \frac{3}{4} \left( \tilde{K}_9^9 + \tilde{K}_{10}^{10} \right) + \tilde{R},
\]

which will be essential in the construction of solutions. Of course there are infinitely many levels in the decomposition beyond the vector and two-form. The \( SL(10, \mathbb{R}) \) representation with compound level \((a_{10}, a_{11})\) will be a tensor product of \( a_{10} \) vectors and \( a_{11} \) two-forms and must also be contained within the level \( l = a_{11} \) representations of \( E_{11} \). The various representations and the associated IIA gauge fields are summarised in table 4.3.2.

We can generate solutions to the IIA theory using the same group element (4.41) with harmonic function \( N \) given by equation (4.42). For the \( p \)-brane solutions we find Cartan
generators \[20\]^7

\[
\beta_{p+1} \cdot H = \frac{7-p}{8} (K^1 + \ldots + K^{p+1}_{p+1}) - \frac{p + 1}{8} (K^{p+2}_{p+2} + \ldots + K^{10}_{10}) + \frac{3p-3}{2} R. \tag{4.74}
\]

With the vielbein ansatz we obtain the metric

\[
ds^2 = \frac{N(p-7)}{2} d\Omega_{1,p}^2 + \frac{N(p+1)}{6} d\Sigma_{9-p}^2 \tag{4.75}
\]
as well as the dilaton field

\[
e^A = N^{\eta=-3}. \tag{4.76}
\]

The gauge field is given by an additional ansatz where \(A_{1\ldots p+1} = 1 - N^{-1}\). With these we recover the standard brane solutions of type IIA supergravity presented above.

The type IIB theory obtained from the decomposition of \(E_{11}\) with respect to the nodes 9 and 10, as shown in figure 4.3.2. By levelling node 9 we produce two disconnected diagrams and the level \(a_9\) will not only have an \(SL(10, \mathbb{R})\) representation associated with the \(\lambda_8\) weight contained in \(\alpha_9\), but also the vector representation of \(SL(2, \mathbb{R})\). The Cartan elements are then

\[
H_i = K^i + K^{i+1}_{i+1} \quad \text{for } i = 1, \ldots, 8
\]
\[
H_9 = -\frac{1}{4} (K^1 + \ldots + K^8) + \frac{3}{4} (K^9_9 + K^{10}_{10}) + R
\]
\[
H_{10} = -2R \quad H_{11} = K^9_9 - K^{10}_{10}. \tag{4.77}
\]

In order to understand the field content it is easiest to produce a modified orthonormal basis of \(f_i, i = 1, \ldots, 10\) with two additional vectors \(g_1\) and \(g_2\), such that

\[
\alpha_i = f_i - f_{i+1} \quad \text{for } i = 1, \ldots, 8
\]
\[
\alpha_9 = f_9 + f_{10} + g_2 \quad \alpha_{10} = g_1 - g_2 \quad \alpha_{11} = f_9 - f_{10}. \tag{4.78}
\]

The inner product between two roots \(\alpha = \sum_i a_i\alpha_i = u \cdot f + v \cdot g\) and \(\beta = \sum_i b_i\alpha_i = w \cdot f + x \cdot g\), with Euclidean inner product \(\cdot\) is given by

\[
\langle \alpha, \beta \rangle = u \cdot w + v \cdot x - a_9 b_9. \tag{4.79}
\]
CHAPTER 4. CONJECTURE AND CONSEQUENCES

Taking each lowest weight root, we find the representations as two young tableaux: one of the tensor representation of $SL(10, \mathbb{R})$ with the form of figure 2.6.1 over the $f_i$ values and one of the tensor representation of $SL(2, \mathbb{R})$ with a young tableau using the $g_i$ values. These fill out the type IIB supergravity solutions [1].

4.3.3 Exotic Solutions

The ten-dimensional supergravity theories are only part of a string theory which contains infinitely many excited states. Duality symmetries of the theory allow us to construct new solutions from seeds. A standard example is the D8 brane of type IIA supergravity [70], produced by T-duality from other D-branes, which will be our first exotic solution. The D8 brane is not a solution for the theory truncated from eleven-dimensional supergravity, but requires an extension of the theory into the massive spectrum of IIA string theory. Known as Romans massive type IIA supergravity, this extension was originally formulated in [71] and understood within the Kac-Moody/Supergravity correspondence in [72].

The Romans massive action is obtained by introducing a mass term on the reduced gauge two-form so that $F_{\mu\nu}^{R-IIA} = F_{\mu\nu}^{IIA} + m A_{\mu\nu}^{IIA}$ and the new bosonic sector of the action is:

$$L^{[R-IIA]} = L^{[IIA]} - \frac{\sqrt{-g}}{2} m^2 e^{5\phi/2} \epsilon_{\mu_1\ldots\mu_9} \left( \frac{m}{12 \times 4!} \partial_{\mu_1} A_{\mu_2\mu_3\mu_4} A_{\mu_5\mu_6} A_{\mu_7\mu_8} A_{\mu_9} m \right) + \frac{m^2}{12 \times 2^7} A_{\mu_1\mu_2} A_{\mu_3\mu_4} A_{\mu_5\mu_6} A_{\mu_7\mu_8} A_{\mu_9\mu_{10}}.$$  (4.80)

As shown in [72], the level $(a_{10}, a_{11}) = (1, 4)$ roots, which are elements of the nine-form representation of $SL(10, \mathbb{R})$, generate the mass term with a field

$$A_{\mu_1\ldots\mu_9} = \frac{1}{2} N e^{5\phi/2} \epsilon_{\mu_1\ldots\mu_9} m.$$  (4.81)

This results in a solution

$$ds^2 = N^{-1/2} d\Omega_{1,8}^2 + N^{1/2} dy^2$$

$$e^{-4\phi} = N^5 \quad H = m|y - y_0|,$$  (4.82)

where we have chosen the form of the function $N$ as an ansatz in order to obtain the full D8-brane solution [73]. By introducing this nine-form field we also have a zero-form dual $F_0 = m$ which possesses a source term on the hyperplane $y = y_0$, known as a domain wall.

Just as the other $Dp$-brane solutions are the result of compactification of eleven-dimensional solutions, the D8 should have a corresponding eleven-dimensional solution. This has been discussed, as the $M9$-brane, in [74] and the associated $[10, 1, 1]$-form gauge field was found at level $l = 4$ in [18].

The $[10, 1, 1]$-form in $E_{11}$ and D8-brane are only the first pair in an infinite set of exotic objects proposed by the Kac-Moody/Supergravity conjecture. The diagonal vielbein for solutions given by (4.41) only require us to determine the Cartan element of some root $\alpha$ and so can be found for any root. While this could be a time-consuming task using the $\alpha_i$
basis it can be simplified with the $e_i$ basis. For each copy of the Cartan element $H_{11}$ we introduce a factor of $N^{1/6}$ on each $e_i$ for $i = 1, \ldots, 8$ and $N^{-1/3}$ on each $e_i$ for $i = 9, 10, 11$. Each $H_i$ then modifies by a factor of $N^{-1/2}$ on $e_i$ and $N^{1/2}$ on $e_i$ for $i \neq 1$. For some root, such as $\beta = (0, 0, 0, 1, 2, 3, 4, 5, 3, 1, 3)$ we find with a little work that

$$ds_3^2 = N (dx_1^2 + dx_2^2 + dx_3^2) + N^0 (dx_4^2 + \ldots + dx_{10}^2) + N^{-1} dx_{11}^2.$$  \hspace{1cm} (4.83)

For a root $\beta = \sum_{i=1}^{11} v_i e_i$ of level $l = a_{11}$ in the $e_i$ basis, each $v_i$ for $i \leq 8$ indicates the number of $H_i - H_{i-1}$ in the Cartan element, while for $i \geq 9$ this is increased by level. We therefore find that

$$e_i^i = N^{(l/6 - v_i/2)} \Rightarrow ds_3^2 = N^{l/3} \left( \sum_{i=1}^{11} N^{-v_i} dx_i^2 \right).$$  \hspace{1cm} (4.84)

This very easily reproduces the example of $\beta = (0, 0, 0, 1, 2, 3, 4, 5, 3, 1, 3) = 2e_{11} + \sum_{i=4}^{10} e_i$ above with metric (4.83) and provides a simple method of calculating the diagonal vielbein for any root. The task of interpreting the fields of the positive generators, as challenging as that might be, is all that remains.

### 4.4 Bound States

Besides the membrane and the fivebrane solutions there are two gravitational solutions, the $KK$-wave and $KK$-monopole (or $KK6$-brane), which when dimensionally reduced produce brane solutions of lower-dimensional supergravities. These solutions are naturally encoded by the $l = 0$ and $l = 3$ roots of $E_{11}$ using the solution generating method.

Other than these fundamental brane solutions and purely gravitational solutions alone, there are many more solutions associated with real roots of $E_{11}$ which are the bound states of these solutions. The intersections of the fundamental eleven-dimensional supergravity solutions and of lower-dimensional supergravity theories have been extensively studied [75]. One class of these types of bound states, known as marginal states are constructed by taking a brane solution and adding another static brane such that the gravitational background of the first brane exerts no force on the second [76]. This can be extended with $n$ branes, with $n$ independent harmonic functions, which preserve $2^{-n}$ of the possible supersymmetry. The number of intersecting dimensions for each possible bound state is a particular value which is given by the inner product of roots.

The marginal solutions have harmonic functions which are restricted to the mutually transverse space. The metrics of the intersecting solutions can be constructed using the harmonic superposition rule [76] and the gauge fields simply sum. For example, consider the $M2 \perp M2$ where the worldvolume coordinates are $x_9, x_{10}, x_{11}$ and $x_7, x_8, x_{11}$ with $x_{11} = t$. The superposed metric is

$$ds_{M2 \perp M2}^2 = N_1^{1/3} N_2^{1/3} (dx_1^2 + \ldots + dx_6^2) + N_1^{1/3} N_2^{-2/3} (dx_9^2 + dx_8^2) + N_1^{-2/3} N_2^{1/3} (dx_9^2 + dx_{10}^2) - N_1^{-2/3} N_2^{-2/3} dt^2,$$  \hspace{1cm} (4.85)
where the harmonic functions are now in the relative transverse space with \( r^2 = x_1^2 + \ldots + x_6^2 \):

\[
N_i = 1 + \frac{q_i}{r^4}.
\]  

(4.86)

The field strength is given by the sum of the two field strengths of the membranes:

\[
F_{[4]} = -dt \wedge (dN_1^{-1} \wedge dx^9 \wedge dx^{10} + dN_2^{-1} \wedge dx^7 \wedge dx^8).
\]  

(4.87)

These intersections were originally understood from the Kac-Moody algebra perspective as coming from a collection of \( n \) roots \( \beta_i \) with inner products \( \beta_i \cdot \beta_j = 2\delta_{ij} \) [24]. We give a few examples in table 4.4.1, where the value in parentheses indicates the number of space-like coordinates which are in both worldvolumes and the square brackets indicate that there are different relative configurations for the same collection of branes. While the \( M2 \perp M2 \perp M2 \) has one unique arrangement, the \( M5 \perp M5 \perp M5 \) has three nonequivalent configurations. To the roots

\[
\beta_1 = e_6 + \ldots + e_{11} \quad \beta_2 = e_4 + e_5 + e_8 + \ldots + e_{11}
\]  

(4.88)

we could add any of the three

\[
\begin{align*}
\beta_{3[1]} & = e_4 + \ldots + e_7 + e_{10} + e_{11} \\
\beta_{3[2]} & = e_3 + e_5 + e_7 + e_9 + e_{10} + e_{11} \\
\beta_{3[3]} & = e_2 + e_3 + e_8 + \ldots + e_{11}
\end{align*}
\]  

(4.89, 4.90, 4.91)

and obtain three orthogonal roots which generate different marginal intersections. The group element

\[
g = \exp \left\{ -\frac{1}{2} \sum_{i=1}^{n} \Log(N_i) H_{\beta_i} \right\} \exp \left\{ \sum_{i=1}^{n} (1 - N_i) E_{\beta_i} \right\}
\]  

(4.92)

generates the marginal bound state of \( n \) fundamental solutions. Taking \( n = 2 \) with \( \beta_1 = e_9 + e_{10} + e_{11} \) and \( \beta_2 = e_7 + e_8 + e_{11} \) we find exactly the \( M2 \perp M2 \) solution above.
CHAPTER 4. CONJECTURE AND CONSEQUENCES

Non-marginal bound states have also been studied using the solution generating technique, as originally developed in [25]. It was shown that for two roots $\beta_1$ and $\beta_2$, with inner product $(\beta_1, \beta_2) = -1$, a modified group element

$$
g = \exp \left[ -\frac{1}{2} \log(N_1) H_{\beta_1} - \frac{1}{2} \log(N_2) H_{\beta_2} \right]
\times \exp \left[ (1 - N_1)^{1/2} (1 - \frac{N_1}{N_2})^{1/2} E_{\beta_1} + (1 - N_2)^{1/2} \left( 1 - \frac{N_2}{N_1} \right)^{1/2} E_{\beta_2} \right]$$

$$
+ (1 - N_1)^{1/2} (1 - N_2)^{1/2} \left( \frac{N_1}{N_2} \right)^{1/2} E_{\beta_1+\beta_2} \right]$$

(4.93)
could be used to generate the non-marginal bound state of two fundamental solutions. While this was originally an ansatz, we will give justification for the fields which appear in the group element in the following chapters. This is based on the work of [26]. First we provide an example with the prototype non-marginal bound state.

4.4.1 The dyonic membrane

Consider the two $E_{11}$ roots $\beta_1 = e_6 + e_7 + e_8$ and $\beta_2 = e_9 + e_{10} + e_{11}$ which have inner product $\beta_1 \cdot \beta_2 = -1$ and, with $\beta_1 + \beta_2$, form the positive root space of $A_2$. The Cartan elements are

$$
H_{\beta_1} = H_6 + 2H_7 + 3H_8 + 2H_9 + H_{10} + H_{11}
= -\frac{1}{3} (K_{11}^1 + \ldots + K_{55}^5 + \ldots + K_{1010}^{10} + K_{1111}^{11}) + \frac{2}{3} (K_{66}^6 + \ldots + K_{88}^8) \quad (4.94)
$$

and the positive generators are

$$
E_{\beta_1} = R_{678}^{678} \quad E_{\beta_2} = R_{91011}^{91011} \quad E_{\beta_1+\beta_2} = R_{67891011}^{67891011}. \quad (4.95)
$$

The solution generating group element for these roots is then given by (4.93). By taking the vielbein as $e_\mu^a = (e^h)_\mu^a$ from the group element we find a metric

$$
ds^2 = (N_1 N_2)^{1/3} \left[ (dx_1^2 + \ldots + dx_5^2) + N_1^{-1} \left( dx_6^2 + \ldots + dx_8^2 \right) + N_2^{-1} \left( dx_9^2 + dx_{10}^2 - dt^2 \right) \right].$$

(4.96)

As with the marginal bound states, the harmonic functions will depend on the mutually transverse space. As a result we find that

$$
N_1 = 1 + \frac{q_1}{r^3} \quad N_2 = 1 + \frac{q_2}{r^3} \quad (4.97)
$$

In order to proceed from this point we will introduce a relation between the two charges

$$
q_2 = q_1 \cos^2(\theta) \Rightarrow \left( \frac{1 - N_2}{1 - N_1} \right) = \cos^2(\theta). \quad (4.98)
$$

*We will find in the next chapter that this equation comes from solving an $SL(3,\mathbb{R})/SO(1,2)$ coset model.*
Figure 4.5.1: The $E_8^{+++}$ Dynkin diagram used to define the $l_1$ representation of $E_{11}$.

This allows us to simplify the locally flat space gauge fields

$$A_{\beta_1}E_{\beta_1} + A_{\beta_2}E_{\beta_2} + A_{\beta_1+\beta_2}E_{\beta_1+\beta_2} = \left( (1 - N_1)N_2^{-1/2}\sin(\theta) \right) R^{678}$$

$$+ i \left( (1 - N_2)N_1^{-1/2}\tan(\theta) \right) R^{91011} + \left( (1 - N_1)(N_1N_2)^{-1/2}\cos(\theta) \right) R^{6\cdots11} \quad (4.99)$$

which after being transformed using the vielbein and dualising the seven-form yield a field strength

$$F_{[4]} = \sin(\theta)dN_1^{-1} \wedge dx_6 \wedge dx_7 \wedge dx_8 - \cos(\theta) \star_T dN_1$$

$$- \frac{i\sin(2\theta)}{2N_2^2} dN_1 \wedge dx_9 \wedge dx_{10} \wedge dx_{11}, \quad (4.100)$$

where the dualisation $\star_T$ occurs over the mutually transverse five-dimensional space. This is the eleven-dimensional lift of the dyonic membrane solution, originally found in [28].

### 4.5 The $l_1$ Representation and Charges

In the non-linear realisation of gravity we broke the diffeomorphism algebra into the inhomogeneous $IGL(n) = GL(n) \rtimes V_n$ with the vector representation $V_n$. In order to reproduce the results, the posited $G_{11}$ algebra contained $SL(11)$ with the three- and six-form representations as well as the vector $P_a$. This vector representation does not appear at any level in $E_{11}$ and the inclusion of the translation generators as elements of the ‘$l_1$’ representation of $E_{11}$ also introduces a number of additional fields which can be interpreted as the charges associated with eleven-dimensional supergravity and its $E_{11}$ extension.

As originally shown in [19], the $l_1$ representation of $E_{11}$ will contain the vector $P_a$, as well as an infinite set of other generators. The construction can be performed using the $E_8^{+++}$ extension of $E_{11}$ with another node, labeled zero, attached to node one as seen in figure 4.5. We introduce the additional node and then decompose the quadruple-extension with respect to it and isolate a particular level, for general $\alpha = \sum_{i=0}^{11} a_i\alpha_i$, of $a_0$. Just as in the $a_{11}$ decomposition of $E_{11}$, the root $\alpha_0 = x - l_1$ contains the fundamental weight $l_1$ of node one with some $x$ orthogonal to all of $E_{11}$.\(^9\) The roots can be defined as

$$\alpha_0 = y + \frac{3}{2}z - \lambda_1$$

$$\alpha_{11} = z - \lambda_8 \quad (4.101)$$

\(^9\)In the original paper [19] and the general literature the fundamental weight is given as $l_1$, giving the name of the representation. Note that this is the $E_{11}$ weight, rather than the $SL(11)$ weight $\lambda_1$.\)
where we have introduced a new $y$ which is orthogonal to all of the $E_{11}$ roots. Using the $e_i$ basis we can write these as \[77\]:
\[
    y = e_0 - \frac{1}{2}(e_1 + \ldots + e_{11})
\]
\[
    z = \frac{3}{11}(e_1 + \ldots + e_{11}).
\]

(4.102)

If we isolate only the $a_0 = 0$ roots we would recover the $E_{11}$ root space, where the generators form the adjoint. Taking $a_0 = 1$ we obtain the $l_1$ representation. This representation is still infinite-dimensional but we can decompose the algebra with respect to another node and find finite-dimensional representations at each level. When we decompose the $a_0 = 0$ $l_1$ representation with respect to $\alpha_{11}$ we find an infinite set of $SL(11, \mathbb{R})$ representations. This is analogous to the level decomposition of $E_{11}$ but now contains a fundamental $A_{10}$ weight $\lambda_1$. These representations include

\[
    K^i_j \overset{a_0=1}{\rightarrow} K^{0}_j = P_j
\]
\[
    R^{ijk} \overset{a_0=1}{\rightarrow} R^{0ijk} = Z^{jk}
\]
\[
    R^{i_1 \ldots i_6} \overset{a_0=1}{\rightarrow} R^{0c_2 \ldots c_6} = Z^{c_2 \ldots c_6}
\]

(4.103)

so that we find vector, two-form and five-form representations at the first $a_{11} = 0, 1, 2$ levels.

To summarise: in order to introduce the vector $P_a$ we surprisingly have to consider additional forms which correspond exactly with the charges of eleven-dimensional supergravity - and indeed at higher levels the representations are those of the charges for the infinite collection of proposed gauge fields \[78, 79\]. For convenience we show the $SL(10, \mathbb{R})$ representations of the first six $a_{11}$ levels in the $l_1$ representation in table 4.5.1. Note that at each level the Dynkin coefficients are only modified from those of table 2.6.2 by having one $\lambda_i \rightarrow \lambda_{i+1}$.

<table>
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<th>$a_{11}$</th>
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<th>Simple roots $\alpha_i$</th>
<th>$\alpha^2$</th>
<th>Dim</th>
<th>Mult</th>
<th>O. mult</th>
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Table 4.5.1: $A_{10}$ representations in $E_8^{+++}$ with $a_0 = 1$ and $a_{11} \leq 5$. 

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In this chapter we present the essential framework for the results of this thesis: one-dimensional $\sigma$-models which are embedded in a $G^{+++}/K(G^{+++})$ theory. The first section introduces the brane $\sigma$-model based on [22] and which was extended to apply to bound states of branes in [24] and [26]. The $A_{n(n)}$ models with $n \leq 3$, which had been solved in that body
CHAPTER 5. ONE-DIMENSIONAL SIGMA MODELS

of work, are reproduced and their application to supergravity solutions is discussed with examples. We then begin the results of the thesis proper. The $A_{n(n)}$ models are extended and solved for more general automorphic involution invariant subalgebras. By using these solutions with various involutions as limit points we construct and solve the $D_{4(4)}$ model which notably contains a root $\alpha = \sum a_i \alpha_i$ with some $a_i > 1$. This solution is then extended for general $D_{n(n)}$ with arbitrary involution. The $E_{6(6)}$ model with particular involution is constructed in all detail and solved explicitly before we present the general solution method for $E_{n(n)}$. Each model is accompanied by a discussion of the subgroup generated charge modifications which is central to the construction of supergravity solutions.

5.1 The Brane $\sigma$-model

The $G^{+++/K(G^{+++})}$ coset that was studied in [22] can be constructed using the same techniques as we discussed in section 2.4. From this point forward we will write roots in boldface $\alpha$ in order to distinguish them from constants which appear in the solutions. We consider a map from $\xi \in \mathbb{R}$ into the coset with a group element in Borel gauge

$$g(\xi) = \exp\left\{ h_a b(\xi) K^{a}_b \right\} \exp \left\{ \sum_{\alpha \in \Delta(G^{+++})} A_{\alpha}(\xi) E_{\alpha} \right\}.$$  \hspace{1cm} (5.1)

For a given involution we can isolate the anti-invariant subset $p \subset g_1$ and use $P(\xi)$, the Maurer-Cartan restricted to that subalgebra, to make the action

$$S_{G^{+++}/K(G^{+++})} = \int d\xi \frac{1}{n(\xi)} (P(\xi), P(\xi))$$ \hspace{1cm} (5.2)

where $n(\xi)$ is a lapse function. Reparametrisation invariance of the action results in the constraint that $(P(\xi), P(\xi)) = 0^2$, which is the equation of motion for the lapse constraint.

For the purposes of this thesis we will focus on the $E_8^{+++}/K(E_8^{+++})$ coset which we have already decomposed by levels of the exceptional node in section 2.6.1. The level $l = 0$ generators in this decomposition are those of the normal real form $\mathfrak{gl}(11, \mathbb{R})$. Of the many involutions we consider a temporal involution which leaves an invariant $K = SO(1,10)$ group. In order to make direct connection with supergravity we now choose to take the $(e^{-\phi}) \mu^a$ to be the vielbein of a spacetime manifold, each of the tensor representations as gauge fields over that spacetime and the fields which appear in $P$ will be general field strengths. This is sometimes referred to as the supergravity/Kac-Moody dictionary.

The various non-zero level fields which appear in $P$ are tensor representations of the local

---

1This is the complement of the invariant subalgebra $t$ which generates the subgroup of the coset. The fields of $p$ are then the fields of the coset algebra in a particular gauge.

2This can be seen as the Hamiltonian constraint for an arbitrary parameter invariant Lagrangian, which for a line element $ds$ requires null solutions.
Lorentz group and we label them by their flat space indicies as
\[ P(\xi) = \sum_{\alpha \in \Delta(g^{++})} E_{\alpha} - \Omega(E_{\alpha}) \frac{P_{\alpha}(\xi)}{2} \]
\[ = \frac{1}{3!} P_{a_1 a_2 a_3} s^{a_1 a_2 a_3} + \frac{1}{6!} P_{a_1 \cdots a_6} s^{a_1 \cdots a_6} + \frac{1}{9!} P_{a_1 \cdots a_8 | a_9} s^{a_1 \cdots a_8 | a_9} + \ldots \]
with an infinite number of fields and generators with \( l \geq 4 \). The equations of motion for the action (5.2) are not approachable due to the fact that, for any \( s_{\alpha} \), the equation of motion for this generator will involve products of all of the fields \( P_{\alpha + \beta} P_{\beta} \) such that both \( \beta, \beta + \alpha \in \Delta(g^{++}) \). Since this is an infinite dimensional algebra we instead choose to isolate a subset of roots which form the root space of some finite subalgebra. We can then use the machinery of section 2.4 to solve the equations of motion for the submodel and find a supergravity interpretation of the results.

5.1.1 Example: \( SL(2, \mathbb{R})/SO(1,1) \)

By isolating an individual root \( \alpha = \sum a_i \alpha_i \) from \( E_{11} \) we can construct the \( sl(2, \mathbb{R}) \) algebra with \( E_{\alpha}, F_{\alpha} \) and the Cartan element \( H = \sum a_i H_i \). If we select a root which satisfies \( \Omega(E_{\alpha}) = +F_{\alpha} \) then the involution invariant subalgebra will be \( so(1,1) \). The group element for this coset is
\[ g(\xi) = \exp \{ \phi H \} \exp \{ A_{\alpha} E_{\alpha} \} \] (5.5)
The Maurer-Cartan form is then
\[ \omega = (\partial_{\xi} g) g^{-1} = H \partial_{\xi} \phi + E_{\alpha} e^{2\phi} \partial_{\xi} A_{\alpha} \] (5.6)
and by defining \( s_{\alpha}(E_{\alpha} + F_{\alpha})/2 \) it is clear that \( P_{\alpha} = e^{2\phi} \partial_{\xi} A_{\alpha} \). There are three equations of motion from equation (2.43)
\[ 0 = \partial_{\xi}^2 \phi + \frac{1}{2} e^{4\phi} (\partial_{\xi} A_{\alpha})^2 \]
\[ 0 = \partial^2 A_{\alpha} + 4 \partial_{\xi} A_{\alpha} \partial_{\xi} \phi \]
\[ 0 = (\partial_{\xi} \phi)^2 - \frac{1}{4} e^{4\phi} (\partial_{\xi} A_{\alpha})^2 \] (5.7)
which come from the \( H, s_{\alpha} \) and the lapse constraint. The first and last equations yield an equation for \( \phi \),
\[ 0 = \partial_{\xi}^2 \phi + (\partial_{\xi} \phi)^2, \] (5.8)
which is solved by
\[ \phi(\xi) = \frac{1}{2} \log(a + b\xi) \equiv \frac{1}{2} \log N(\xi), \] (5.9)
where \( N \) is a (trivially, due to the dimension) harmonic function. Subsequently, we find that
\[ A_{\alpha} = \pm \frac{1}{N} + c \quad P_{\alpha} = \pm N \partial_{\xi} N^{-1}. \] (5.10)
The signs of \( A_\alpha \) and \( P_\alpha \) are not determined from the equations of motion but by considering the associated brane solution we shall choose appropriately. The constants \( a \), \( b \) and \( c \) can be set using group transformations. They affect the asymptotics of the solution, charge and gauge symmetry of the \( A_\alpha \) field.

Using these solutions to describe the fields and the Cartan elements \( (e^{-\phi})_\mu^a \) to describe the vielbein it was found that \( l = 0, 1, 2, 3 \) levels of \( E_{11} \) naturally describe the four fundamental solutions of eleven-dimensional supergravity, as we will discuss in section 6.1.1 and as was originally shown in [22].

5.1.2 Example: \( SL(2, \mathbb{R})/SO(2) \)

This model is constructed in the same manner as the \( SL(2, \mathbb{R})/SO(1, 1) \) but the opposite involution results in sign changes in the equations of motion:

\[
0 = \partial_\xi^2 \phi - \frac{1}{2} e^{4\phi} (\partial_\xi A_\alpha)^2 \\
0 = \partial_\alpha^2 A_\alpha - 4\partial_\xi A_\alpha \partial_\xi \phi \\
0 = (\partial_\xi \phi)^2 + \frac{1}{4} e^{4\phi} (\partial_\xi A_\alpha)^2. \tag{5.11}
\]

The first and last equations still yield the constraint of equation (5.8) which requires \( \phi = \frac{1}{2} \log(a + b\xi) \). The sign change in the second equation now forces the solution to be complex:

\[
A_\alpha = \pm i \frac{1}{N} + c \quad P_\alpha = \pm i N \partial_\xi N^{-1}. \tag{5.12}
\]

This is due to the fact that we are mapping from a one-dimensional space into the coset. If we considered a larger space the equations of motion would not be so restrictive. Throughout the rest of the chapter we will find real solutions for a variety of cosets. While this complex solution will appear at various locations within a ‘charge-space’ of solutions, we will consider subspaces which are wholly real. In the next chapter we will make some comments on the connection between complex solutions and ‘spacelike’ brane solutions.

5.1.3 Example: \( SL(3, \mathbb{R})/SO(1, 2) \)

In order to create an \( SL(3, \mathbb{R}) \) model we only need two roots \( \beta_1, \beta_2 \in E_{11} \) with inner product \( \langle \beta_1, \beta_2 \rangle = -1 \). If each root is expanded in the simple root basis as \( \beta_i = \sum_{j=1}^r a_{ij} \alpha_j \) we take two Cartan elements \( H_i = \sum_{j=1}^r a_{ij} H_j \). We will now start labeling the roots in the finite subalgebra by the simple root basis of the subalgebra so that \( \beta_1 = (1, 0) \), \( \beta_2 = (0, 1) \) and \( \beta_1 + \beta_2 = (1, 1) \). The group element for the coset in Borel gauge is

\[
g(\xi) = \exp \{ \phi_1 H_1 + \phi_2 H_2 \} \exp \{ A_{(1,0)} E_{(1,0)} + A_{(0,1)} E_{(0,1)} + A_{(1,1)} E_{(1,1)} \} \tag{5.13}
\]

and the Maurer-Cartan form is

\[
\omega = H_1 \partial_\xi \phi_1 + H_2 \partial_\xi \phi_2 + E_{(1,0)} e^{2\phi_1 - \phi_2} \partial_\xi A_{(1,0)} + E_{(0,1)} e^{2\phi_2 - \phi_1} \partial_\xi A_{(0,1)} + E_{(1,1)} e^{\phi_1 + \phi_2} \left( \partial_\xi A_{(1,1)} + \frac{1}{2} (A_{(1,0)} \partial_\xi A_{(0,1)} - A_{(0,1)} \partial_\xi A_{(1,0)}) \right). \tag{5.14}
\]
CHAPTER 5. ONE-DIMENSIONAL SIGMA MODELS

There are 5+1 equations of motion for the dimension of \( p \) plus the lapse constraint. In order to calculate them we will have to stipulate an involution and we will consider first the \( t = so(1, 2) \) which is generated by the involution where \( \Omega(E(1, 0)) = +F(1, 0) \) and \( \Omega(E(0, 1)) = -F(0, 1) \). After calculating all of the commutators for the resulting \( k_{\alpha} \) and \( s_{\alpha} \) we find

\[
H_1 : \quad 0 = \partial_\xi^2 \phi_1 + \frac{1}{2} (P_{(1, 0)}^2 + P_{(1, 1)}^2)
\]

\[
H_2 : \quad 0 = \partial_\xi^2 \phi_2 + \frac{1}{2} (-P_{(0, 1)}^2 + P_{(1, 1)}^2)
\]

\[
s_{(1, 0)} : \quad 0 = \partial_\xi P_{(1, 0)} + P_{(1, 0)} (2\partial_\xi \phi_1 - \partial_\xi \phi_2) - P_{(1, 1)}P_{(0, 1)}
\]

\[
s_{(0, 1)} : \quad 0 = \partial_\xi P_{(0, 1)} + P_{(0, 1)} (2\partial_\xi \phi_2 - \partial_\xi \phi_1) - P_{(0, 0)}P_{(1, 1)}
\]

\[
s_{(1, 1)} : \quad 0 = \partial_\xi P_{(1, 1)} + P_{(1, 1)} (\partial_\xi \phi_1 + \partial_\xi \phi_2)
\]

\[
n(\xi) : \quad 0 = (\partial_\xi \phi_1)^2 - \partial_\xi \phi_1 \partial_\xi \phi_2 + (\partial_\xi \phi_2)^2 + \frac{1}{4} (-P_{(0, 0)}^2 + P_{(0, 1)}^2 - P_{(1, 1)}^2)
\]

where the \( P_{\alpha} \) fields are those given in the Maurer-Cartan form expansion of equation (5.14) and \( \omega = \sum_i H_i \partial_\xi \phi_i + \sum_{\alpha} P_{\alpha} E_{\alpha} \). The solution to these equations of motion is given by [26]:

\[
\phi_i = 1/2 \log(a_i + b_i) \equiv 1/2 \log N_i
\]

\[
P_{(1, 0)}^2 = \frac{\partial_\xi N_1 (N_2 \partial_\xi N_1 - N_1 \partial_\xi N_2)}{N_1^2 N_2}
\]

\[
P_{(0, 1)}^2 = \frac{\partial_\xi N_2 (N_2 \partial_\xi N_1 - N_1 \partial_\xi N_2)}{N_1^2 N_2^2}
\]

\[
P_{(1, 1)}^2 = \frac{\partial_\xi N_1 \partial_\xi N_2}{N_1 N_2}
\]

The signs are not uniquely determined from the equations of motion, just as we observed in the \( SL(2, \mathbb{R}) \) model. However, the relative signs between the \( P \)-fields do have some restrictions. The numerators of each \( P_{\alpha} \) are constants which we will label \( c_{\alpha} \), so that

\[
P_{\alpha} = \frac{c_{\alpha}}{f(N_i)}.
\]

The \( s_{(1, 0)} \) and \( s_{(0, 1)} \) equations of motion then indicate that

\[
c_{(1, 1)} = \frac{-c_{(1, 0)} \partial_\xi N_2}{c_{(0, 1)} \partial_\xi N_1} = \frac{-c_{(0, 1)} \partial_\xi N_2}{c_{(1, 0)}}.
\]

The solutions that we have presented satisfy these with all \( P_{\alpha} \) being negative or exactly one being negative.

As we found in section 2.4, the symmetries of the models generate charge-modifying, as well as scaling and gauge, transformations. The latter will allow us to set the \( a_i \) and \( c_i \) as we see fit. In order to understand the subgroup symmetry we set the \( a_i = 1 \), take the provided solution and evaluate the Noether current at \( \xi = 0 \)

\[
J(\xi = 0) = g^{-1} P g = \begin{pmatrix}
  b_1 & -b_1 \sqrt{b_1 - b_2} & -b_2 \\
  & b_1 \sqrt{b_1 - b_2} & -b_2 \sqrt{b_1 - b_2} \\
  & -b_2 \sqrt{b_1 - b_2} & b_1 \sqrt{b_1 - b_2}
\end{pmatrix}.
\]

\[\text{Note: This is effectively the } r \to \infty \text{ limit when unsmeared for a (super)gravity solution.}\]
Note that if we set \( b_1 = 0 \) we would recover the \( SL(2, \mathbb{R})/SO(2) \) model with complex solutions. In order to avoid complex solutions we must have \( b_1 > b_2 \) and in the larger models which follow there will be many similar constraints on the charges in order to find real solutions. The Noether current transforms under the subgroup elements as

\[
J \rightarrow K_\alpha(\theta) J (K_\alpha(\theta))^{-1} \quad K_\alpha(\theta) = e^{b_\alpha \theta}.
\]  

(5.28)

If we set \( b_1 = q \) and \( b_2 = q \cos^2(\psi) \) the action of the compact symmetry \( K_{(0,1)}(\theta) \) takes the form [26]

\[
J(\psi) \rightarrow K_{(0,1)}(\theta) J(\psi) (K_{(0,1)}(\theta))^{-1} = J(\psi + \theta).
\]  

(5.29)

In this construction we have isolated one orbit which is generated by the \( K_{(0,1)} \) symmetry. This is precisely how the \( \cos \theta \) of the coset symmetry is introduced in the dyonic membrane, as was claimed in section 4.4.1. While the first \( M_2 \) charge is left invariant, the symmetry acts to introduce the \( S_2 \) charge and interpolate between the \( M_2 \) and \( M_5 \) solutions.

5.2 General Model Construction

In the previous section we presented an action which contains infinitely many fields. The equations of motion contain infinitely many fields as well, so we chose to isolate an individual root in order to create an action whose equations of motion we could solve. When we extend this work by including a collection of roots which form some finite subalgebra the thinking is reversed: we build the finite model, solve this model and then find ways of embedding the model within the action (5.2). Considerable progress has already been made on the last front [1] with the identification of simply laced algebras constructed out of low-level roots.

The models that we are building are based on section 2.4 with a base manifold \( \mathcal{M} = \mathbb{R} \) so that the group element for some finite algebra \( g \) in Borel gauge is

\[
g(\xi) = \exp \left\{ \sum_{i=1}^r \phi_i(\xi) H_i \right\} \exp \left\{ \sum_{\alpha \in \Delta(g)} A_\alpha(\xi) E_\alpha \right\}.
\]  

(5.30)

Independent of the involution used in the coset construction, we can expand the Maurer-Cartan form

\[
\omega = e^{\phi H} \partial_\xi e^{-\phi H} + e^{\phi H} \left( e^{\sum_{\alpha} A_\alpha E_\alpha} \partial_\xi e^{-\sum_{\alpha} A_\alpha E_\alpha} \right) e^{-\phi H} = \sum_{i=1}^r \partial_\xi \phi_i H_i + e^{\phi H} Ze^{-\phi H}
\]  

(5.31)

where the series

\[
W = \sum_{\alpha \in \Delta(g)} \partial_\xi A_\alpha E_\alpha + \frac{1}{2} \left[ \sum_{\alpha \in \Delta(g)} \partial_\xi A_\alpha E_\alpha, \sum_{\alpha \in \Delta(g)} A_\alpha E_\alpha \right] + \ldots
\]  

(5.32)

\[\footnote{Besides the content of this paper we also refer the reader to the tables of root configurations which complement the work. These can be found (strangely only in version 2) at http://arxiv.org/abs/1109.6595v2} \]
must terminate since the algebra is finite. We then find that
\[
\omega = \sum_{i=1}^{r} \partial_{\xi} \phi_{i} H_{i} + W - \left[ W, \sum_{i=1}^{r} \phi_{i} H_{i} \right] + \frac{1}{2} \left[ \left[ W, \sum_{i=1}^{r} \phi_{i} H_{i} \right], \sum_{i=1}^{r} \phi_{i} H_{i} \right] + \ldots
\]
\[
= \sum_{i=1}^{r} \partial_{\xi} \phi_{i} H_{i} + \sum_{\alpha \in \Delta(g)} P_{\alpha} E_{\alpha}.
\]
(5.33)
The series in (5.33) does not terminate but if we expand \(W\) over the basis of positive generators
\[
W = \sum_{\alpha} D_{\alpha} E_{\alpha}
\]
(5.34)
the \(P\)-fields can be calculated as
\[
P_{\alpha} E_{\alpha} = D_{\alpha} \sum_{j=0}^{\infty} \frac{1}{j!} \left( \text{ad}(\sum_{i=1}^{r} \phi_{i} H_{i}) \right)^{j} \left( E_{\alpha} \right)
\]
\[
= D_{\alpha} \exp \left\{ \sum_{i=1}^{r} \alpha \cdot \phi_{i} H_{i} \right\} E_{\alpha}.
\]
(5.35)
Since the \(s_{\alpha} = \frac{1}{2} (E_{\alpha} - \Omega(E_{\alpha}))\) and \(k_{\alpha} = \frac{1}{2} (E_{\alpha} + \Omega(E_{\alpha}))\) sum as \(s_{\alpha} + k_{\alpha} = E_{\alpha}\), the involution invariant subalgebra \(Q\)-fields are identical:
\[
Q = \sum_{\alpha \in \Delta(g)} P_{\alpha} k_{\alpha}.
\]
(5.36)
Now that we have defined and calculated the \(P\)-fields we can construct the now-familiar action \(L = n(\xi)^{-1} (P, P)\), which requires the choice of a particular involution for the coset.
The involutions that we will consider can be defined by their action on the simple root generators \(E_{\alpha}\),
\[
\Omega(E_{\alpha}) = -\epsilon_{\alpha} F_{\alpha}.
\]
(5.37)
The \(\epsilon_{i}\) form an \(r\)-dimensional vector which we will use to define an involution. The generator \(E_{\alpha}\) for an arbitrary root \(\alpha = \sum_{i=1}^{r} a_{i} \alpha_{i}\) will be mapped under the involution to
\[
\Omega(E_{\alpha}) = -\left( \prod_{i=1}^{r} (\epsilon_{i})^{a_{i}} \right) F_{\alpha} \equiv -\epsilon_{\alpha} F_{\alpha}.
\]
(5.38)
The equation of motion for the lapse function will be
\[
0 = (P, P) = \sum_{i=1}^{r} (\partial_{\xi} \phi_{i})^{2} + \sum_{i \neq j} A_{ij} \partial_{\xi} \phi_{i} \partial_{\xi} \phi_{j} + \frac{1}{4} \sum_{\alpha \in \Delta(g)} \epsilon_{\alpha} P_{\alpha}^{2}
\]
(5.39)
where we have introduced the Cartan Matrix to give terms when the nodes for \(\phi_{i}\) and \(\phi_{j}\) are connected. The equations of motion from equation (2.43) are
\[
\partial_{\xi} P - [Q, P] = 0
\]
(5.40)
and give us \( \dim(p) \)-many individual equations. Of these, \( r \) of them are Cartan equations of motion, one for each \( H_i \), which contain contributions from the associated \( \partial_\xi \phi_i \) and each \( P_\alpha \) where \( \alpha = \sum_{j=1}^{r} a_j \alpha_j \) has \( a_i \neq 0 \):

\[
H_i : \quad 0 = \partial_\xi^2 \phi_i - \frac{1}{2} \sum_{\alpha \in \Delta(\mathfrak{g})} \epsilon_\alpha a_i P_\alpha^2.
\]  

(5.41)

The remaining equations of motion come from each \( s_\alpha \) and the terms in the equations of motion can be produced in three ways:

1. One \( \partial_\xi P_\alpha \) term from \( \partial_\xi \mathcal{P} \)
2. One \( (\sum c_i \phi_i) P_\alpha \) term from \([H_i, k_\alpha]\) in \([\mathcal{Q}, \mathcal{P}]\)
3. Various \( P_{\beta_1} P_{\beta_2} \) terms which come from \([s_{\beta_1}, k_{\beta_2}]\) in \([\mathcal{Q}, \mathcal{P}]\), where \( \beta_1 - \beta_2 = \pm \alpha \).

The first two items are easy to determine and do not depend on the involution while the last item includes (potentially) many terms which depend both on the involution and the structure constants of the algebra. Because of this last observation we have to include an additional term which depends on the constants used in the realisation of the generators of the algebra, as we show below. In order to find out what \( P_{\beta_1} P_{\beta_2} \) terms appear we must explore all of the possible \([k_{\beta_1}, s_{\beta_2}]\) possibilities. Our construction of the \( k \) and \( s \) generators

\[
k_\alpha = \frac{E_\alpha + \Omega(E_\alpha)}{2} \quad s_\alpha = \frac{E_\alpha - \Omega(E_\alpha)}{2}
\]

(5.42)

allows us to consider the only pairs of commutators which can introduce \( P_{\beta_1} P_{\beta_2} \) terms:

\[
4 [k_{\beta_1}, s_{\beta_2}] = [E_{\beta_1}, E_{\beta_2}] + [E_{\beta_1}, -\Omega(E_{\beta_2})] + [\Omega(E_{\beta_1}), E_{\beta_2}] + [\Omega(E_{\beta_1}), -\Omega(E_{\beta_2})]
\]

\[
4 [k_{\beta_2}, s_{\beta_1}] = [E_{\beta_2}, E_{\beta_1}] + [E_{\beta_2}, -\Omega(E_{\beta_1})] + [\Omega(E_{\beta_2}), E_{\beta_1}] + [\Omega(E_{\beta_2}), -\Omega(E_{\beta_1})].
\]

When \( \beta_1 + \beta_2 \) is a root the sum of these two cancel each other, as the first and last terms have \( \beta_1 \leftrightarrow \beta_2 \). Therefore if both \( \beta_1 - \beta_2 \) and \( \beta_2 - \beta_1 \) are not roots in \( \mathfrak{g} \) these commutators must vanish, so in order to obtain some \( s_\alpha \) from these commutators we must have some \( \alpha + \beta \) and \( \beta \) which are both roots. Let us define the Cartan basis structure constants for our generators to be

\[
[E_\alpha, E_\beta] = c_{\alpha, \beta} E_{\alpha + \beta}.
\]

(5.43)

We can then find the sum of the two commutators with roots \( \alpha, \beta, \alpha + \beta \in \Delta(\mathfrak{g}) \) to be

\[
4 ([k_{\alpha + \beta}, s_\beta] + [k_\beta, s_{\alpha + \beta}]) = 2 ([E_{\alpha + \beta}, \epsilon_\beta F_\beta] - [\epsilon_{\alpha + \beta} F_{\alpha + \beta}, E_\beta])
\]

\[
= 2 \epsilon_\beta ([E_{\alpha + \beta}, F_\beta] - \epsilon_\beta [F_{\alpha + \beta}, E_\beta])
\]

\[
= 4 \epsilon_\alpha \epsilon_\beta c_{\alpha, \beta} s_\alpha
\]

(5.44)

since \( \epsilon_\alpha \epsilon_\beta = \epsilon_{\alpha + \beta} \) and \( [E_{\alpha + \beta}, F_\beta] = c_{\alpha, \beta} E_\alpha \). The equations of motion for the \( s_\alpha \) are therefore

\[
s_\alpha : \quad 0 = \partial_\xi P_\alpha + P_\alpha \left( \sum_{i=1}^{r} \langle \alpha, \alpha_i \rangle \partial_\xi \phi_i \right) + \sum_{\{\beta | \beta + \beta \in \Delta(\mathfrak{g})\}} c_{\alpha, \beta} \epsilon_\beta P_{\alpha + \beta} P_\beta.
\]

(5.45)
In order to give a more concrete picture of the equations of motion we will now provide more explicit calculations for several models of interest.

5.2.1 \( SL(n, \mathbb{R})/SO(1, n-1) \) with involution \( \epsilon = (-, +, \ldots, +) \)

These models are the natural extension of the \( SL(2, \mathbb{R})/SO(1, 1) \) and \( SL(3, \mathbb{R})/SO(1, 2) \) models that we started the chapter with. We use the usual \( SL(n, \mathbb{R}) \) generators and their commutators:

\[
H_i = K^i_i - K^{i+1}_{i+1}, \quad E_{\alpha_i} = K^i_i, \quad F_{\alpha_i} = K^{i+1}_i
\]

\[
[K^i_j, K^{m}_l] = \delta^i_j K^m_l - \delta^m_l K^i_j. \tag{5.46}
\]

With an involution \( \Omega(E_{\alpha_i}) = -\epsilon_i F_{\alpha_i} \) given by \( \epsilon = (-, +, \ldots, +) \), the involution invariant subalgebra \( \mathfrak{t} \) includes all of the elements

\[
k_{\alpha_1 + \beta} = \frac{E_{\alpha_1 + \beta} + F_{\alpha_1 + \beta}}{2}, \quad k_{\beta} = \frac{E_{\beta} - F_{\beta}}{2} \tag{5.47}
\]

where \( \beta \) is any root of the form \( (0, 1, \ldots) \). These generate the algebra \( \mathfrak{so}(1, n-1) \). The Cartan equations of motion are

\[
H_1 : \quad 0 = \partial_\xi^2 \phi_1 + \frac{1}{2} \left( P_{(1,0,\ldots,0)}^2 + P_{(1,1,0,\ldots,0)}^2 + \cdots + P_{(1,\ldots,1)}^2 \right) \tag{5.48}
\]

\[
H_i : \quad 0 = \partial_\xi^2 \phi_i + \frac{1}{2} \left( P_{(1,\ldots,1,0,\ldots,0)}^2 + P_{(1,\ldots,1,1,0,\ldots,0)}^2 + \cdots + P_{(1,\ldots,1)}^2 \right) - \frac{1}{2} \left( P_{0,1,\ldots,1,0,\ldots,0}^2 + \cdots + P_{(0,1,\ldots,1)}^2 \right) - \cdots
\]

\[
- \frac{1}{2} \left( P_{0,0,1,\ldots,1,0,\ldots,0}^2 + \cdots + P_{(0,0,1,\ldots,1)}^2 \right) \tag{5.49}
\]

for \( 1 < i \leq n-1 \)

For some \( \alpha = (0, \cdots, 0, 1_a, \cdots, 1_b, 0, \cdots, 0) \) the equation of motion from (5.45) for the associated generator is

\[
s_{\alpha} : \quad 0 = \partial_\xi P_{\alpha} + P_{\alpha} \left( \partial_\xi \phi_0 + \partial_\xi \phi_b - \partial_\xi \phi_{a-1} - \partial_\xi \phi_{b+1} \right)
\]

\[
+ \frac{1}{2} \left( -P_{(1,\ldots,1,0,\ldots,0)} P_{(1,\ldots,1,0,\ldots,0)} + \cdots \right.
\]

\[
+ \left. P_{(0,0,1,\ldots,1,0,\ldots,0)} P_{(0,0,1,\ldots,1,0,\ldots,0)} \right) \tag{5.50}
\]

\[
- \frac{1}{2} \left( P_{0,0,1,\ldots,1,0,\ldots,0} P_{0,0,1,\ldots,1,0,\ldots,0} + \cdots \right.
\]

\[
+ \left. P_{(0,\ldots,0,1,\ldots,1)} P_{(0,\ldots,0,1,\ldots,1)} \right).
\]

Only the first \( P_{\beta_1} P_{\beta_2} \) term contains an \( \epsilon \) modification from our involution and we have used the \( c_{\alpha, \beta} \) which come from our commutators (5.46). If we wish to consider some other involution the only modification to the equations of motion will occur in the \( P_{\beta_1} P_{\beta_2} \) terms, so it is straightforward to obtain any of the \( SL(n, \mathbb{R})/SO(m, n-m) \) null geodesic equations by substituting another \( \epsilon \).
5.2.2 \( SO(4,4)/(SO(2,2) \times SO(2,2)) \) with involution \( \epsilon = (+,-,+,+,-) \)

The algebra \( SO(4,4) \) is the normal real form of \( D_4 \) which has the Dynkin diagram of figure 5.2.2. \( D_4 \) has twelve positive roots with highest root \( \Theta = (1,2,1,1) \). The commutators of the 28 generators of \( SO(4,4) \) can be given by the matrix

\[
\begin{pmatrix}
-H_2 + H_3 + H_4 & -E_{0,0,0}^1 & -F_{1,1,0}^0 & F_{0,1,0}^0 & 0 & -E_{0,0,1}^0 & E_{1,1,1}^1 & E_{0,1,1}^1 \\
-F_{0,0,0}^1 & H_3 - H_4 & -F_{1,1,0}^0 & F_{0,1,0}^0 & E_{0,0,1}^0 & 0 & E_{1,1,1}^1 & -E_{0,1,1}^1 \\
-E_{0,1,0}^0 & -E_{1,1,0}^1 & H_1 & -E_{1,0,0}^1 & -E_{1,1,1}^1 & -E_{0,1,1}^0 & 0 & -E_{1,2,1}^1 \\
E_{0,1,0}^0 & E_{1,1,0}^1 & -F_{1,0,0}^0 & -H_1 + H_2 & -E_{0,1,1}^1 & E_{0,1,1}^0 & E_{1,2,1}^1 & 0 \\
0 & F_{0,0,1}^0 & -F_{1,1,1}^1 & -F_{0,1,1}^0 & H_2 - H_3 - H_4 & -E_{0,1,0}^1 & E_{0,1,0}^0 & -E_{0,1,0}^0 \\
-F_{0,0,1}^0 & 0 & -F_{1,1,1}^1 & F_{0,1,1}^0 & E_{0,0,1}^0 & -H_3 + H_4 & E_{1,1,0}^1 & -E_{0,1,0}^1 \\
F_{1,1,1}^1 & F_{1,1,1}^0 & 0 & F_{1,2,1}^1 & F_{1,1,0}^0 & F_{1,1,0}^1 & -H_1 & F_{1,0,0}^0 \\
F_{0,1,1}^1 & -F_{0,1,1}^0 & -F_{1,2,1}^1 & 0 & -F_{0,1,0}^0 & -F_{0,1,0}^0 & E_{1,0,0}^0 & H_1 - H_2
\end{pmatrix}
\]

where we have temporarily used the notation \( E_{(a_1,a_2,a_3,a_4)} = E_{a_1 a_2 a_3 a_4} \). The commutators are then given by taking the commutators of the matrix basis elements. For example, \([E_{(1,0,0,0)}, E_{(0,1,0,0)}] = +E_{(1,1,0,0)} \) based on the matrix. The set of \( c_{\alpha,\beta} \) presented here are precisely those given by the level decomposition of \( \alpha_4 \) to give \( SL(4,\mathbb{R}) \) and its two-form representation.

Finding what matrix Lie algebra the involution leaves invariant is cumbersome in this notation because we have isolated the Cartan elements in the basis. The compact real form \( D_4(-28) \) is isomorphic to \( so(8) \), which is given by the anti-symmetric \( 8 \times 8 \) matrices. The only nontrivial automorphic involution is given by conjugation with \( \Omega \) (defined in (2.29)) so that

\[
\begin{pmatrix}
1_4 & 0 \\
0 & -1_4
\end{pmatrix}
\begin{pmatrix}
A & B \\
-B^T & C
\end{pmatrix}
\begin{pmatrix}
1_4 & 0 \\
0 & -1_4
\end{pmatrix}
= \begin{pmatrix}
A & -B \\
B^T & C
\end{pmatrix}
\]

where \( A \) and \( C \) are anti-symmetric \( 4 \times 4 \) matrices. The invariant subalgebra is therefore

\[
t = \begin{pmatrix}
A & 0 \\
0 & C
\end{pmatrix}
\]

By using the Weyl unitary trick \( p \rightarrow ip \) we obtain the normal real form for which the maximal compact subalgebra is \( t = so(4) \oplus so(4) \). This is equivalent to taking \( \epsilon = (+,+,+,-) \) and by modifying this vector we will obtain some \( t = so(p,q) \oplus so(p,q) \). An easy method for determining which of the three possible \( t \) is left invariant for a given \( \epsilon \) is to count the (non-
Table 5.2.1: Involution invariant subalgebras of $D_{4(4)}$ with all nonequivalent $\epsilon$.

<table>
<thead>
<tr>
<th>$\epsilon$</th>
<th>Invariant t</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(+,+,+,+)$</td>
<td>$so(4) \oplus so(4)$</td>
</tr>
<tr>
<td>$(-,+,+,+)$</td>
<td>$so(1,3) \oplus so(1,3)$</td>
</tr>
<tr>
<td>$(+,-,+,+)$</td>
<td>$so(2,2) \oplus so(2,2)$</td>
</tr>
<tr>
<td>$(-,-,+,+)$</td>
<td>$so(1,3) \oplus so(1,3)$</td>
</tr>
<tr>
<td>$(-,+,+,+)$</td>
<td>$so(1,3) \oplus so(1,3)$</td>
</tr>
<tr>
<td>$(-,-,-,+)$</td>
<td>$so(1,3) \oplus so(1,3)$</td>
</tr>
<tr>
<td>$(-,+,+,-)$</td>
<td>$so(2,2) \oplus so(2,2)$</td>
</tr>
<tr>
<td>$(-,-,+-)$</td>
<td>$so(2,2) \oplus so(2,2)$</td>
</tr>
</tbody>
</table>

Figure 5.2.2: Dynkin diagram of $E_6$ with Coxeter labels.

By choosing the involution $\epsilon = (+,-,+,+)$ we obtain the last required information, (5.53), needed to compute the equations of motion. This is done in the appendix B.1.

5.2.3 $E_{6(6)}/Sp(8,\mathbb{R})$ with involution $\epsilon = (+,+,-,+,+)$

The $E_{6(6)}$ algebra with Dynkin diagram 5.2.3 has 36 positive generators and a dimension of 72 but the equations of motion are straightforward, if tedious, to write out. The only challenging part of this task is to find all of the many $c_{\alpha,\beta}$ which will affect hundreds of signs throughout the equations. As we have done several times before, we will level the algebra with respect to the exceptional node and obtain $SL(6,\mathbb{R})$ with the 20-dimensional three-form...
and one-dimensional six-form. The commutators that define our structure constants are

\[
H_i = K^i_i - K^{i+1}_{i+1}, \quad E_{\alpha} = K^i_{i+1}, \quad F_{\alpha} = K^{i+1}_{i}
\]

\[
H_6 = -\frac{1}{3} (K^1_1 + K^2_2 + K^3_3) + \frac{2}{3} (K^4_4 + K^5_5 + K^6_6)
\]

\[
E_{\alpha \beta} = R^{456}, \quad F_{\alpha \beta}, \quad E_{(1,2,3,2,1,2)} = R^{123456}, \quad F_{(1,2,3,2,1,2)} = R_{123456}
\]

\[
[K^i_j, R_{a_1a_2a_3}] = \delta^a_1 R^{a_2a_3} + \text{sym}, \quad [K^i_j, R_{a_1a_2a_3}] = -\delta^i_1 R_{j a_2a_3} + \text{sym}
\]

\[
[R_{a_1a_2a_3}, R_{b_1b_2b_3}] = R^{a_1a_2a_3} b_1 b_2 b_3, \quad [R_{a_1a_2a_3}, R_{b_1b_2b_3}] = R_{a_1a_2a_3 b_1 b_2 b_3}
\]

\[
[R_{a_1a_2a_3}, R_{b_1b_2b_3}] = 18 \delta^{a_1a_2}_{b_1b_2} K^{a_3}_{b_3} - 2 \delta^{a_1a_2}_{b_1b_2} \sum_{i=1}^{6} K^i_i
\]

\[
[K^i_j, K^l_m] = \delta^j_l K^i_m - \delta^l_m K^i_j \quad [R_{123456}, R_{123456}] = \frac{1}{3} \sum_{i=1}^{6} K^i_i.
\]

The involution invariant subalgebras are most conveniently obtained by again finding the number of (non-)compact generators. For example, with the involution \( \epsilon = (+, +, - , +, +) \) the only roots for which \( \epsilon_\alpha = -1 \) are those with \( a_3 = 1, 3 \). There are 20 roots with either of those values and the only non-compact generators are these \( (E_\alpha + F_\alpha)/2 \). By looking up a table of the Riemannian symmetric spaces\(^5\) we find that the invariant subalgebra is \( \mathfrak{t} = \mathfrak{sp}(8, \mathbb{R}) \). The involution invariant subalgebras with one \( \epsilon_i = -1 \) are given in table 5.2.3. For our example we will consider \( \epsilon = (+, +, - , +, +) \). The equations of motion with our structure constants are shown in appendix B.2.

### 5.3 Solution Generating Technique

It was shown in [26] that the \( SL(4, \mathbb{R})/SO(p, q) \) model could be solved by using the solution from an \( SL(3, \mathbb{R})/SO(p, q) \) model and adding another \( SL(2) \) which could be ‘turned on’ using the subgroup transformations to generate the full solution. We will use a general form of the converse to construct solutions of simply laced cosets: the larger solution contains limits under supergroup motion which are the solutions of any subalgebra.

\(^5\)For example, see table 9.7 of [40].
5.3.1 \( SL(n, \mathbb{R})/SO(1, n - 1) \) with involution \( \epsilon = (-, +, \ldots, +) \)

In order to solve the arbitrary rank \( A_{n-1(n-1)} \) model we will start with the smallest models and build an inductive argument.

**Solutions of \( SL(2, \mathbb{R})/SO(p, q) \)**

The first two models in this set, \( n = 2, 3 \) have already been solved in the first section. We found that the solution for \( SL(2, \mathbb{R})/SO(1, 1) \) has two fields

\[
\phi(\xi) = 1/2\log(a + b\xi) = 1/2\log N
\]

\[
P_\alpha = \pm N\partial_\xi N^{-1} \quad \Rightarrow \quad A_\alpha = \pm N^{-1} + c
\]  

(5.61)

with three constants which are related to the asymptotics of the solution, charge and \( A \)-gauge symmetry. The sign of \( P_\alpha \) is undetermined by the equations of motion. We can evaluate the Noether current at \( \xi = 0 \) as we did in equation (5.27). Here the subgroup is the noncompact \( SO(1, 1) \) and the transformation is

\[
\mathcal{J}(\xi = 0) \rightarrow \left( \begin{array}{cc} \cosh(\theta) & \sinh(\theta) \\ \sinh(\theta) & \cosh(\theta) \end{array} \right) \left( \begin{array}{cc} b & -b \\ b & -b \end{array} \right) \left( \begin{array}{cc} \cosh(\alpha) & -\sinh(\alpha) \\ -\sinh(\alpha) & \cosh(\alpha) \end{array} \right) = e^{2\theta} \mathcal{J}. \]  

(5.62)

The other involution that we could consider leaves \( SO(2) \) invariant and the signs of the \( P_\alpha^2 \) terms will be opposite that of (5.11). The solution for the \( SL(2, \mathbb{R})/SO(2) \) model is therefore

\[
\phi(\xi) = 1/2\log(a + b\xi) = 1/2\log N
\]

\[
P_\alpha = \pm iN\partial_\xi N^{-1} \quad \Rightarrow \quad A_\alpha = \pm iN^{-1} + c.
\]  

(5.63)

The subgroup motion is now

\[
\mathcal{J}(\xi = 0) \rightarrow e^{\theta(E - F)} \left( \begin{array}{cc} b & ib \\ ib & -b \end{array} \right) e^{-\theta(E - F)} = e^{2\theta} \mathcal{J}. \]  

(5.64)

**Solutions of \( SL(3, \mathbb{R})/SO(p, q) \)**

The \( H = SO(1, 2) \) coset model was solved above with

\[
\phi_i = 1/2\log(a_i + b_i) \equiv 1/2\log N_i \quad P_{(1,0)} = \pm \sqrt{\partial_\xi N_1 \alpha_{12}} \quad N_1 \sqrt{N_2}
\]  

(5.65)

\[
P_{(0,1)} = \pm \sqrt{\partial_\xi N_2 \alpha_{12}} \quad N_2 \sqrt{N_1} \quad P_{(1,1)} = \pm \sqrt{\partial_\xi N_1 \partial_\xi N_2} \quad N_1 N_2.
\]  

(5.66)

where we have defined the constant \( \alpha_{12} = N_2\partial_\xi N_1 - N_1\partial_\xi N_2 \). An essential quality of this solution is that it limits to \( SL(2, \mathbb{R})/SO(p, q) \) solutions when we set the \( b_i \) to natural limit points:

\[
(b_1, b_2) = (q, 0) \quad \Rightarrow \quad P_{(0,1)} = P_{(1,1)} = 0, \quad P_{(1,0)} = \pm N\partial_\xi N^{-1} \quad [SL(2\mathbb{R})/SO(1, 1)]
\]

\[
(b_1, b_2) = (0, q) \quad \Rightarrow \quad P_{(1,0)} = P_{(1,1)} = 0, \quad P_{(0,1)} = \pm iN\partial_\xi N^{-1} \quad [SL(2\mathbb{R})/SO(2)]
\]

\[
(b_1, b_2) = (q, q) \quad \Rightarrow \quad P_{(1,0)} = P_{(0,1)} = 0, \quad P_{(1,1)} = \pm N\partial_\xi N^{-1} \quad [SL(2\mathbb{R})/SO(1, 1)]
\]
Figure 5.3.1: $SL(3, \mathbb{R})/SO(1, 2)$ subgroup motion on the space of charges through the point $(b_1, b_2) = (q, 0)$.

These solutions should be identifiable in all of the subsequent models whenever we isolate individual $b_i$ charges.

We have already reviewed the result from [26] in section 4.4.1 where if $b_1 = q$ and $b_2 = q \cos^2(\theta)$ the compact subgroup symmetry with parameter $\psi$ acts by mapping $\theta \to \theta + \psi$. It is not difficult to see that the $K_{(1,0)}$ symmetry only modifies $b_1$ with an exponential and $K_{(1,1)}$ should give some combination of these transformations. The $K_{(1,1)}$ transformation produces various paths through the real $b_1, b_2$-space which is bounded by $b_1 = b_2$ and $b_2 = 0$. We show the paths which include the point $(b_1, b_2) = (q, 0)$ with the action of only one subgroup symmetry in figure 5.3.1. The action of the compact symmetry generated by $K_{(0,1)}$ moves $b_2$ between 0 and $b_1$ while the noncompact symmetries generate scalings on $b_1$ and along $b_1 - b_2 = \text{constant}$ lines. When constructing larger models with $SL(3, \mathbb{R})/SO(1, 2)$ submodels we should be able to identify this $b$-space motion within a larger space.

If we set the $a_i = 1$ and isolate the compact symmetry we find that $b_1 \equiv q$ remains constant while $b_2 = q \sin^2(\theta)$. As a result $\alpha_{12} = q \cos^2(\theta)$ and the constants $c_\alpha$ defined in equation 5.25 are

$$c_{(1,0)} = \pm q \cos(\theta) \quad c_{(0,1)} = \pm q \sin(\theta) \cos(\theta) \quad c_{(1,1)} = \pm q \sin(\theta). \quad (5.67)$$

These $\xi$-constants parametrise the compact symmetry and also allow us to more easily verify the equations of motion.

The other two nonequivalent involutions that we could consider are given by $\epsilon = (+, +)$ and $\epsilon = (-, -)$, which leave invariant $SO(3)$ and $SO(1, 2)$, respectively. While the subgroup of the latter may be the same as what we have just explored, the $P_\alpha$ fields, charges and brane models will be different. This model does not appear in any of the $SL(n, \mathbb{R})/SO(1, n - 1)$ models but we will solve it for use in the next subsection.
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The solutions for these involutions only differ by signs in the $P_{\alpha}P_{\alpha+\beta}$ terms. The equations of motion for the $\epsilon = (+, +)$ involution are

\[
\begin{align*}
H_1 : 0 &= \partial_\xi^2 \phi_1 - \frac{1}{2} (P_{(1,0)}^2 + P_{(1,1)}^2) \\
H_2 : 0 &= \partial_\xi^2 \phi_2 - \frac{1}{2} (P_{(0,1)}^2 + P_{(1,1)}^2) \\
s_{(1,0)} : 0 &= \partial_\xi P_{(1,0)} + P_{(1,0)} (2\partial_\xi \phi_1 - \partial_\xi \phi_2) - P_{(1,1)} P_{(0,1)} \\
s_{(0,1)} : 0 &= \partial_\xi P_{(0,1)} + P_{(0,1)} (2\partial_\xi \phi_2 - \partial_\xi \phi_1) + P_{(1,0)} P_{(1,1)} \\
s_{(1,1)} : 0 &= \partial_\xi P_{(1,1)} + P_{(1,1)} (\partial_\xi \phi_1 + \partial_\xi \phi_2) \\
n(\xi) : 0 &= (\partial_\xi \phi_1)^2 - \partial_\xi \phi_1 \partial_\xi \phi_2 + (\partial_\xi \phi_2)^2 + \frac{1}{4} \left( P_{(1,0)}^2 + P_{(0,1)}^2 + P_{(1,1)}^2 \right). 
\end{align*}
\]

These are satisfied by the same $\phi_i$ definitions and the following $P_{\alpha}$:

\[
\begin{align*}
P_{(1,0)}^2 &= \frac{-\partial_\xi N_1 (N_2 \partial_\xi N_1 - N_1 \partial_\xi N_2)}{N_1^2 N_2} \\
P_{(0,1)}^2 &= \frac{\partial_\xi N_2 (N_2 \partial_\xi N_1 - N_1 \partial_\xi N_2)}{N_1 N_2^2} \\
P_{(1,1)}^2 &= \frac{-\partial_\xi N_1 \partial_\xi N_2}{N_1 N_2}. 
\end{align*}
\]

As a quick check we can verify that if $b_1 = 0$, $b_2 = 0$ or $b_1 = b_2$ we obtain the $SL(2, \mathbb{R})/SO(2)$ solution from $P_{(1,0)}$, $P_{(0,1)}$ or $P_{(1,1)}$, respectively, while the others vanish.

The equations of motion for the $\epsilon = (-, -)$ involution are

\[
\begin{align*}
H_1 : 0 &= \partial_\xi^2 \phi_1 + \frac{1}{2} (P_{(1,0)}^2 - P_{(1,1)}^2) \\
H_2 : 0 &= \partial_\xi^2 \phi_2 + \frac{1}{2} (P_{(0,1)}^2 - P_{(1,1)}^2) \\
s_{(1,0)} : 0 &= \partial_\xi P_{(1,0)} + P_{(1,0)} (2\partial_\xi \phi_1 - \partial_\xi \phi_2) + P_{(1,1)} P_{(0,1)} \\
s_{(0,1)} : 0 &= \partial_\xi P_{(0,1)} + P_{(0,1)} (2\partial_\xi \phi_2 - \partial_\xi \phi_1) - P_{(1,0)} P_{(1,1)} \\
s_{(1,1)} : 0 &= \partial_\xi P_{(1,1)} + P_{(1,1)} (\partial_\xi \phi_1 + \partial_\xi \phi_2) \\
n(\xi) : 0 &= (\partial_\xi \phi_1)^2 - \partial_\xi \phi_1 \partial_\xi \phi_2 + (\partial_\xi \phi_2)^2 + \frac{1}{4} \left( -P_{(1,0)}^2 - P_{(0,1)}^2 + P_{(1,1)}^2 \right)
\end{align*}
\]

and the solutions are

\[
\begin{align*}
P_{(1,0)}^2 &= \frac{\partial_\xi N_1 (N_2 \partial_\xi N_1 - N_1 \partial_\xi N_2)}{N_1^2 N_2} \\
P_{(0,1)}^2 &= \frac{-\partial_\xi N_2 (N_2 \partial_\xi N_1 - N_1 \partial_\xi N_2)}{N_1 N_2^2} \\
P_{(1,1)}^2 &= \frac{-\partial_\xi N_1 \partial_\xi N_2}{N_1 N_2}.
\end{align*}
\]

This is again an $SL(3, \mathbb{R})/SO(1, 2)$ model but now the compact symmetry $K(1,1)$ performs the function of moving between the two $SL(2, \mathbb{R})/SO(1, 1)$ solutions which are associated with $P_{(1,0)}$ and $P_{(0,1)}$, rather than $P_{(1,1)}$. When taking the $a_i = 1$ and isolating the compact
symmetry we find two disjoint orbits which are symmetric under \( \alpha_1 \leftrightarrow \alpha_2 \) (which does not conflict with the involution). Taking the \( b_1 = q \sin^2(\theta) \) and \( b_2 = -q \cos^2(\theta) \) orbit we find that
\[
\begin{align*}
\alpha_{12} &= N_2 \partial_\xi N_1 - N_1 \partial_\xi N_2 = 1 \\
 c_{(1,0)} &= \pm q \cos(\theta) \\
 c_{(0,1)} &= \pm q \sin(\theta) \\
 c_{(1,1)} &= \pm q \cos(\theta) \sin(\theta). 
\end{align*}
\]
Another notable difference in this model is that the charges must be of opposite signs if we start at some \( b_i = q, b_j = 0 \) point in the charge space. We present the diagram for the \( b \)-space motion in figure 5.3.2, where it is understood that the orbits of the non-compact symmetry transformations do not pass through the origin.

**Solutions of \( SL(4, \mathbb{R})/SO(p, q) \)**

While it is not necessary to give the solutions for \( n = 4 \) in order to continue with our argument for general \( n \), these computations will be explicitly used to solve the \( SO(4, 4)/(SO(2, 2) \times SO(2, 2)) \) model, as it contains multiple \( SL(4, \mathbb{R})/SO(p, q) \) submodels with several different involutions. The \( \epsilon = (-, +, +) \) and \( \epsilon = (+, -, +) \) models were solved already in [26], so we will only present the results.

The equations of motion for the \( \epsilon = (-, +, +) \) are
\[
\begin{align*}
H_1 : \quad 0 &= \partial_\xi^2 \phi_1 + 1/2 \left( P_{(1,0,0)}^2 + P_{(1,1,0)}^2 + P_{(1,1,1)}^2 \right) \\
H_2 : \quad 0 &= \partial_\xi^2 \phi_2 + 1/2 \left( -P_{(0,1,0)}^2 + P_{(1,1,0)}^2 - P_{(0,1,1)}^2 + P_{(1,1,1)}^2 \right) \\
H_3 : \quad 0 &= \partial_\xi^2 \phi_3 - 1/2 \left( P_{(0,0,1)}^2 + P_{(0,1,1)}^2 - P_{(1,1,1)}^2 \right) \\
s_{(1,0,0)} : \quad 0 &= \partial_\xi P_{(1,0,0)} + P_{(1,0,0)} \left( 2 \partial_\xi \phi_1 - \partial_\xi \phi_2 \right) - P_{(1,1,0)} P_{(0,1,0)} - P_{(1,1,1)} P_{(0,1,1)}
\end{align*}
\]
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\[ s_{(0,1,0)} : \quad 0 = \partial_\xi P_{(0,1,0)} + P_{(0,1,0)} (2\partial_\xi \phi_2 - \partial_\xi \phi_1 - \partial_\xi \phi_3) \tag{5.91} \]

\[ s_{(0,0,1)} : \quad 0 = \partial_\xi P_{(0,0,1)} + P_{(0,0,1)} (2\partial_\xi \phi_3 - \partial_\xi \phi_2) \tag{5.92} \]

\[ s_{(1,0,0)} : \quad 0 = \partial_\xi P_{(1,0,0)} + P_{(1,0,0)} (\partial_\xi \phi_1 + \partial_\xi \phi_2 - \partial_\xi \phi_3) - P_{(1,1,1)} P_{(0,0,1)} \tag{5.93} \]

\[ s_{(0,1,1)} : \quad 0 = \partial_\xi P_{(0,1,1)} + P_{(0,1,1)} (\partial_\xi \phi_2 \partial_\xi \phi_3 - \partial_\xi \phi_1) - P_{(1,1,1)} P_{(1,0,0)} \tag{5.94} \]

\[ s_{(1,1,0)} : \quad 0 = \partial_\xi P_{(1,1,0)} + P_{(1,1,0)} (\partial_\xi \phi_1 + \partial_\xi \phi_3 - \partial_\xi \phi_2) \tag{5.95} \]

\[ n(\xi) : \quad 0 = (\partial_\xi \phi_1)^2 - \partial_\xi \phi_1 \partial_\xi \phi_2 + (\partial_\xi \phi_2)^2 - \partial_\xi \phi_2 \partial_\xi \phi_3 + (\partial_\xi \phi_3)^2 \tag{5.96} \]

\[ + 1/4(-P_{(1,0,0)}^2 + P_{(0,0,1)}^2 + P_{(0,1,0)}^2 - P_{(1,1,0)}^2 + P_{(0,1,1)}^2 - P_{(1,1,1)}^2). \]

We can solve the equations with

\[ \phi_i = 1/2\log(a_i + b_i) \equiv 1/2\log N_i \tag{5.97} \]

\[ P_{(1,0,0)}^2 = \frac{\partial_\xi N_1 (N_2 \partial_\xi N_1 - N_1 \partial_\xi N_2)}{N_1^2 N_2^2 N_3} \tag{5.98} \]

\[ P_{(0,0,1)}^2 = \frac{\partial_\xi N_3 (N_3 \partial_\xi N_2 - N_2 \partial_\xi N_3)}{N_2^2 N_3^2} \tag{5.99} \]

\[ P_{(1,1,0)}^2 = \frac{\partial_\xi N_1 (N_3 \partial_\xi N_2 - N_2 \partial_\xi N_3)}{N_1 N_2 N_3} \tag{5.100} \]

\[ P_{(0,1,1)}^2 = \frac{\partial_\xi N_3 (N_2 \partial_\xi N_1 - N_1 \partial_\xi N_2)}{N_1 N_3^2} \tag{5.101} \]

\[ P_{(1,1,1)}^2 = \frac{\partial_\xi N_1 \partial_\xi N_3}{N_1 N_3}. \tag{5.102} \]

Again we notice that the \( b \) limits

\[ b = (q, 0, 0) \rightarrow [SL(2, \mathbb{R})/SO(1, 1)] \quad b = (0, q, 0) \rightarrow [SL(2, \mathbb{R})/SO(2)] \]

\[ b = (0, q, 0) \rightarrow [SL(2, \mathbb{R})/SO(2)] \quad b = (q, q, 0) \rightarrow [SL(3, \mathbb{R})/SO(1, 2)] \]

\[ b = (0, q, q) \rightarrow [SL(3, \mathbb{R})/SO(1, 2)] \quad b = (q, q, q) \rightarrow [SL(2, \mathbb{R})/SO(1, 1)] \]

all produce submodel solutions which we recognise from the previous subsections. In the diagram of the \( b \) space motion under the symmetry transformations, figure 5.3.3, we recognise the \( SL(3, \mathbb{R})/SO(1, 2) \) with \( e = (-, +) \) diagram of figure 5.3.1 within the \( b_3 = 0, b_1 = b_2 \)

and \( b_2 = b_3 \) planes, which each correspond with limits producing that submodel.

If we take all of the \( a_i = 1 \) and isolate the compact symmetries we limit ourselves to the triangle with vertices \((q, 0, 0), (q, q, 0)\) and \((q, q, q)\), which has only a two-dimensional orbit. Starting at the first of these points, it is natural to take the \( K_{(0,1,0)} \) and \( K_{(0,1,1)} \) symmetries.
In particular, there are

The equations of motion have only minor sign changes and it is clear by now that we can

Construct a solution simply by identifying the subalgebra models. In particular, there are

Figure 5.3.3: \( SL(4, \mathbb{R})/SO(1, 3) \) with involution \( \epsilon = (−, +, +) \) subgroup motion on the space

of charges through the point \((b_1, b_2, b_3) = (q, 0, 0)\).

With these charges the \( P \)-field constants are then

With these expressions the task of verifying the equations of motion and determining possible signs for each \( P \)-field is much simpler. The \( b \) limits which produce \( SL(3, \mathbb{R})/SO(1, 2) \) submodels can also be interpreted as taking particular values for the compact symmetry transformations: \( \theta_{(0, 1, 0)} = 0, \theta_{(0, 1, 0)} = \pi/2 \) and \( \theta_{(0, 1, 0)} = \theta_{(0, 1, 1)} \).

Of the other possible involutions we will only briefly discuss two which will play a central

role in the \( SO(4, 4) \) solution. The first is the \( SL(4, \mathbb{R})/SO(2, 2) \) with involution \( \epsilon = (+, −, +) \). The equations of motion have only minor sign changes and it is clear by now that we can construct a solution simply by identifying the subalgebra models. In particular, there are four \( SL(3, \mathbb{R})/SO(1, 2) \) submodels which are obtained by setting \( b_1 = 0, b_1 = b_2, b_3 = 0 \) or \( b_2 = b_3 \). By requiring that all of the submodel solutions are recovered we can easily construct
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Figure 5.3.4: $SL(4,\mathbb{R})/SO(2,2)$ with involution $\epsilon = (+, -, +)$ subgroup motion on the space of charges through the point $(b_1, b_2, b_3) = (0, q, 0)$.

The full solution with those limits and, importantly, the correct signs:

$$P_{(1,0,0)} = \pm \sqrt{\frac{\partial_3 N_1 \alpha_{12}}{N_1 \sqrt{N_2}}} \quad P_{(0,1,0)} = \pm \sqrt{\frac{\alpha_{12} \alpha_{32}}{N_2 \sqrt{N_1 N_3}}}$$

$$P_{(0,0,1)} = \pm \sqrt{\frac{\partial_3 N_3 \alpha_{23}}{N_3 \sqrt{N_2}}} \quad P_{(1,1,0)} = \pm \sqrt{\frac{\alpha_{23} \alpha_{32}}{N_1 \sqrt{N_2}}}$$

$$P_{(0,1,1)} = \pm \sqrt{\frac{\partial_3 N_1 \alpha_{12}}{N_1 \sqrt{N_2}}} \quad P_{(1,1,1)} = \pm \sqrt{\frac{\alpha_{12} \alpha_{32}}{N_1 \sqrt{N_2}}}$$

where $\alpha_{12} = N_1 \partial_3 N_2 - N_2 \partial_3 N_1$ and $\alpha_{23} = N_3 \partial_3 N_2 - N_2 \partial_3 N_3$. Notice that there is a sign difference between the definitions of $\alpha_{12}$ in this model and the last. Taking the $a_i = 1$ the charge motion under $K$-symmetries are depicted in figure 5.3.4. The $b_1 = 0, b_1 = b_2, b_3 = 0$ or $b_2 = b_3$ are then surfaces within the $b$ space which are recognisable as the $SL(3,\mathbb{R})/SO(1,2)$ models of figure 5.3.1. Isolating the two compact symmetries $K_{(1,0,0)}$ and $K_{(0,0,1)}$ restricts us to the square with vertices $(q, 0, 0), (q, q, 0), (q, 0, q)$ and $(q, q, q)$ which is parametrised by the symmetry parameters as

$$b_1 = q \sin^2(\theta_{(1,0,0)}), \quad b_2 = q, \quad b_3 = q \sin^2(\theta_{(0,0,1)}),$$

$$\alpha_{12} \equiv N_1 \partial_3 N_2 - N_2 \partial_3 N_1 = q \cos^2(\theta_{(1,0,0)}),$$

$$\alpha_{23} \equiv N_3 \partial_3 N_2 - N_2 \partial_3 N_3 = q \cos^2(\theta_{(0,0,1)}).$$

The $P$-field $\xi$-constants are

$$c_{(1,0,0)} = \pm q \sin(\theta_{(1,0,0)}) \cos(\theta_{(1,0,0)}) \quad \quad c_{(0,1,0)} = \pm q \sin(\theta_{(1,0,0)}) \sin(\theta_{(0,0,1)}),$$

$$c_{(0,0,1)} = \pm q \sin(\theta_{(0,0,1)}) \cos(\theta_{(0,0,1)}) \quad \quad c_{(1,1,0)} = \pm q \sin(\theta_{(1,0,0)}) \cos(\theta_{(0,0,1)}),$$

$$c_{(0,1,1)} = \pm q \cos(\theta_{(1,0,0)}) \sin(\theta_{(0,0,1)}), \quad \quad c_{(1,1,1)} = \pm q \sin(\theta_{(1,0,0)}) \sin(\theta_{(0,0,1)}).$$
Figure 5.3.5: $SL(4, \mathbb{R})/SO(2, 2)$ with involution $\epsilon = (-, +, -)$ subgroup motion on the space of charges through the points $(b_1, b_2, b_3) = (\pm q, 0, 0)$ and $(0, 0, \pm q)$.

The $SL(4, \mathbb{R})/SO(2, 2)$ with $\epsilon = (-, +, -)$ is another model which will appear in the $D_4$ model, which also has two $\epsilon = (-, +)$ $SL(3, \mathbb{R})$ submodels when $b_1 = 0$ or $b_3 = 0$ and two $\epsilon = (-, -)$ models when $b_1 = b_2$ or $b_3 = b_2$. Requiring these limits allows us to produce the solution

$$P_{(0,1,0)} = \pm \sqrt{\frac{\partial_N N_1 \partial_0 \alpha_{12}}{N_1 \sqrt{N_2}}}$$
$$P_{(0,1,0)} = \pm \sqrt{-\frac{\partial_N N_1 \partial_0 \alpha_{23}}{N_2 \sqrt{N_1 N_3}}}$$

(5.120)

$$P_{(1,0,0)} = \pm \sqrt{\frac{\partial_N N_2 \partial_1 \alpha_{12}}{N_2 \sqrt{N_1}}}$$
$$P_{(1,0,0)} = \pm \sqrt{-\frac{\partial_N N_2 \partial_1 \alpha_{23}}{N_1 \sqrt{N_2 N_3}}}$$

(5.121)

$$P_{(1,1,0)} = \pm \sqrt{-\frac{\partial_N N_3 \partial_1 \alpha_{12}}{N_3 \sqrt{N_2}}}$$
$$P_{(1,1,0)} = \pm \sqrt{-\frac{\partial_N N_3 \partial_1 \alpha_{23}}{N_2 \sqrt{N_1 N_3}}}$$

(5.122)

where now we have $\alpha_{12} = N_2 \partial_N N_1 - N_1 \partial_N N_2$ and $\alpha_{23} = N_2 \partial_N N_3 - N_3 \partial_N N_2$ defined with new signs once more. The $b$-space motion under the $K_\alpha$ symmetries is shown in figure 5.3.5. Notice that the $b_1 = b_2$ and $b_2 = b_3$ limits reproduce the $\epsilon = (-, -)$ figure 5.3.2 while the $b_1 = 0$ and $b_3 = 0$ limits give us the $\epsilon = (-, +)$ model of figure 5.3.1 with two disjoint orbits. This observation reminds us that we could have also produced solutions in the $b_2 < b_1 < 0$ sector of that model. Isolating the compact symmetries restricts us the patches of the $b_1 + b_3 = q$ surface bounded in green and teal. We choose the orbit containing $b = (+q, 0, 0)$ to find

$$b_1 = q \cos^2(\theta(1,1,1))$$
$$b_2 = q \left(\sin^2(\theta(0,1,0)) - \sin^2(\theta(1,1,1))\right)$$
$$b_3 = -q \sin^2(\theta(1,1,1))$$

(5.123)

$$\alpha_{12} \equiv N_2 \partial_N N_1 - N_1 \partial_N N_2 = \cos^2(\theta(0,1,0))$$

(5.124)

$$\alpha_{23} \equiv N_2 \partial_N N_3 - N_3 \partial_N N_2 = -\sin^2(\theta(0,1,0))$$

(5.125)
and subsequently the $P$-field constants are

\begin{align}
    c_{(1,0,0)} &= \pm q\cos(\theta_{(0,1,0)})\cos(\theta_{(1,1,1)}) \\
    c_{(0,1,0)} &= \pm q\cos(\theta_{(0,1,0)})\sin(\theta_{(1,1,1)}) \\
    c_{(0,0,1)} &= \pm q\sin(\theta_{(0,1,0)})\sin(\theta_{(1,1,1)}) \\
    c_{(1,1,0)} &= \pm q\sin(\theta_{(0,1,0)})\cos(\theta_{(1,1,1)}) \\
    c_{(1,1,1)} &= \pm q\sin(\theta_{(1,1,1)})\cos(\theta_{(1,1,1)})
\end{align}

(5.126)

(5.127)

(5.128)

The $\epsilon = (-, -, +)$, $\epsilon = (-, -, -)$ and $\epsilon = (+, +, +)$ models can be easily solved using the same method. The first two possess a variety of $SL(3, \mathbb{R})/SO(1, 2)$ submodels while the $SO(4)$ model is an intuitive extension of the $SO(3)$, as well will see in the next section.

**Solutions for general $SL(n, \mathbb{R})/SO(1, n - 1)$ with $\epsilon = (-, +, \ldots, +)$**

We have already given the general formula for the equations of motion in equations (5.41, 5.45, 5.39) and we have written out the particular equations of motion for an $SL(n, \mathbb{R})$ with $\epsilon = (-, +, \ldots, +)$ in section 5.2.1. Our approach to solving the equations of motion for $n \leq 4$ was to build a solution which limits to the submodel solutions which have already been constructed. We will now extend to arbitrary $n$ inductively. First we consider the simplest example of $SL(n, \mathbb{R})/SO(n)$.

We already have proven that the solution for $n = 2$ is given by

$$\phi = 1/2\text{Log}N \quad P = \pm iN\partial_N N^{-1}$$

(5.129)

and for $n = 3$ we found

\begin{align}
    P_{(1,0)} &= \pm \sqrt{-\partial_N N_1 (N_3\partial_N N_1 - N_1\partial_N N_3)} \quad \frac{N_1\sqrt{N_2}}{N_1} \\
    P_{(0,1)} &= \pm \sqrt{-\partial_N N_2 (N_2\partial_N N_1 - N_1\partial_N N_2)} \quad \frac{N_2\sqrt{N_1}}{N_2} \\
    P_{(1,1)} &= \pm \sqrt{-\partial_N N_1 \partial_N N_2} \quad \frac{\sqrt{N_1 N_2}}{\sqrt{N_1 N_2}}
\end{align}

(5.130)

(5.131)

(5.132)

We can find the solution for $n = 4$ by requiring that in each appropriate $b$ limit we recover an $n < 4$ solution. For example, when $b \equiv (b_1, b_2, b_3) = (q, 0, 0)$ we find $P_{(1,0,0)} = \pm iN\partial_N N^{-1}$ and all else zero. In general whenever the entries $b = \alpha$ for some root $\alpha$ we find $P_{\alpha} = \pm iN\partial_N N^{-1}$. When $b_1 = 0$ we must find the $n = 3$ solution above with the $P_{(0,1,0)}$, $P_{(0,0,1)}$ and $P_{(0,1,1)}$. In general we obtain this submodel whenever $b_1 = 0$, $b_3 = 0$, $b_1 = b_2$ or $b_2 = b_3$.

Using these constraints we find that all of the $n = 4$ fields are given by

$$P_{\alpha} = \pm \frac{c_{\alpha}}{\prod_{i=1}^{n-1} N_i^{(\alpha, \alpha^i)/2}}$$

(5.133)

with the constants

$$c_{(0,\ldots,0,1,\ldots,1,0,\ldots,0)} = \pm \sqrt{(N_{a}\partial_N N_{a-1} - N_{a-1}\partial_N N_{a}) (N_{b+1}\partial_N N_{b} - N_{b}\partial_N N_{b+1})}$$

(5.134)

where $N_i = 1$ if $i \notin 1, \ldots, n - 1$. 

Let us now assume that we have a solution with fields given by equations (5.133,5.134) with some general \( n \). To the collection of roots \( \alpha_i \), \( i = 1, \ldots, n - 1 \) we add a new root \( \alpha_n \) such that \( (\alpha_i, \alpha_n) = -\delta_{i,n-1} \). The solution to the \( SO(n+1) \) model has this \( SL(n, \mathbb{R})/SO(n) \) solution as a sub-solution when \( b_n = 0 \). All of the roots \( (0, \cdots, 0, 1, \cdots, 1, 0, 0) \) cannot make any connected Dynkin diagram with \( \alpha_n \) and as a result none of the \( P_{(0, \cdots, 0, 1, \cdots, 1, 0, 0)} \) will have \( N_n \) dependence. The solutions provided by the \( SL(n, \mathbb{R})/SO(n) \) submodel with \( b_n = 0 \) are therefore the solutions for those fields in the \( SL(n + 1, \mathbb{R})/SO(n + 1) \) model. Each of the roots \( (0, \cdots, 0, 1, \cdots, 1, 0) \), however, forms an \( SL(3, \mathbb{R})/SO(3) \) with \( \alpha_n \) and from the solution that we have just constructed we know that each \( P_{(0, \cdots, 0, 1, \cdots, 1, 0)} \) must be modified as

\[
P_{(0, \cdots, 0, 1, \cdots, 1, 0)}(b_n = 0) \rightarrow P_{(0, \cdots, 0, 1, \cdots, 1, 0)} \times \frac{\sqrt{N_n \partial \xi N_{n-1} - N_{n-1} \partial \xi N_n}}{\sqrt{N_n \partial \xi N_{n-1}}} \tag{5.135}
\]

while the new simple root \( P \)-field is

\[
P_{\alpha_n} = \pm \frac{\sqrt{\partial \xi N_n(N_n \partial \xi N_{n-1} - N_{n-1} \partial \xi N_n)}}{N_n \sqrt{N_{n-1}}} \tag{5.136}
\]

and the third field in each \( SL(3, \mathbb{R})/SO(3) \) is

\[
P_{(0, \cdots, 0, 1, \cdots, 1, 0)} = P_{(0, \cdots, 0, 1, \cdots, 1, 0)} \times \frac{\sqrt{N_{n-1} \partial \xi N_n}}{\sqrt{N_p \partial \xi N_{n-1} - N_{n-1} \partial \xi N_n}}. \tag{5.137}
\]

These solutions are exactly those of equations (5.133,5.134) for the \( n + 1 \) model, so they are solutions for arbitrary \( n \).

The argument for \( SL(n, \mathbb{R})/SO(1, n - 1) \) with \( \epsilon = (-, +, \ldots, +) \) is similar. As we have already noticed, the only difference in the \( P_{\alpha} \) between different involutions is in the constant \( c_{\alpha} \) while the functional dependence is the same. As a result the functional dependence of equation (5.133) is still valid while we need a slight modification to the constants. The \( SL(n, \mathbb{R})/SO(1, n - 1) \) model has a \( SL(n - 1, \mathbb{R})/SO(n - 1) \) submodel when \( b_1 = 0 \). Isolating all of the fields \( P_{(0,0, \cdots, 0,1, \cdots, 1,0, \cdots, 0)} \) with \( a_1 = a_2 = 0 \) we find that these fields have no \( N_1 \) dependence as their roots are orthogonal to \( \alpha_1 \). The \( P_{(0,1, \cdots, 1,0, \cdots, 0)} \) fields do limit into \( SL(3, \mathbb{R})/SO(1, 2) \) models with \( P_{\alpha_1} \) so that they are modified as

\[
P_{(0,1, \cdots, 1,0, \cdots, 0)} \rightarrow P_{(0,1, \cdots, 1,0, \cdots, 0)} \times \frac{\sqrt{N_2 \partial \xi N_1 - N_1 \partial \xi N_2}}{\sqrt{N_2 \partial \xi N_1}}, \tag{5.138}
\]

and the new fields are

\[
P_{\alpha_1} = \pm \frac{\sqrt{\partial \xi N_1(N_2 \partial \xi N_1 - N_1 \partial \xi N_2)}}{N_1 \sqrt{N_2}} \tag{5.139}
\]

\[
P_{(1, \cdots, 1,0, \cdots, 0)} = P_{(0,1, \cdots, 1,0, \cdots, 0)} \times \frac{\sqrt{N_2 \partial \xi N_1}}{\sqrt{N_2 \partial \xi N_1 - N_1 \partial \xi N_2}}. \tag{5.140}
\]
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The solution for $SL(n, \mathbb{R})/SO(1, n-1)$ with general $n$ is therefore provided by fields of equation (5.133) with constants

$$c_{(0,\ldots,0,\ldots,1,\ldots,0)} = \begin{cases} \pm \sqrt{\partial_k N_1 (N_{b+1} \partial_k N_b - N_b \partial_k N_{b+1})} & \text{if } \alpha = 1 \\ \pm \sqrt{(N_a \partial_k N_{a-1} - N_{a-1} \partial_k N_a)} & \text{if } \alpha > 1 \end{cases}$$

(5.141)

5.3.2 $SO(4,4)/(SO(2,2) \times SO(2,2))$ with involution $\epsilon = (+, -, +, +)$

In the sequence of $D_n$ algebras, $D_4$ is the first with a triply-connected node. It also has a highest root which contains more than one copy of a single simple root: $\Theta = \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$. The normal real form with involution $\epsilon = (+, -, +, +)$ has the following $SL(4, \mathbb{R})/SO(p, q)$ submodels:

$$[\alpha_1, \alpha_2, \alpha_3] + \text{perm. } \epsilon = (+, -, +)$$

(5.142)

$$[(1,1,0,0), \alpha_3, \alpha_4] + \text{perm. } \epsilon = (-, +, +)$$

(5.143)

$$[(1,1,1,0), \alpha_4, \alpha_2] + \text{perm. } \epsilon = (-, +, -)$$

(5.144)

$$[(1,1,0,0), \alpha_3, (0,1,0,1)] + \text{perm. } \epsilon = (-, +, -)$$

(5.145)

where $\epsilon$ is the involution which acts on those submodels and there exist other submodels given by permutation of the indices. We will continue to demand that each submodel solution exists as some limit, so the first assumption is that the Cartan fields are given by the familiar expression

$$\phi_i = 1/2 \log(a_i + b_i \xi) \equiv 1/2 \log N_i.$$  

(5.146)

Using the results from the previous section we know that each of the first (5.142) models provide us limits where the fields become the solutions (5.120) when one of the $b_i = 0$ $i = 1, 3, 4$. While the $P_{\alpha_1 \neq 2}$ are the same in each submodel, the $P$-fields which contain $\alpha_2$ limit differently in each model. For example

$$P_{\alpha_2} \rightarrow \begin{cases} \pm \sqrt{\partial_k N_1 \partial_k N_2} \over N_2 \sqrt{N_1 N_4} & \text{if } b_2 = 0 \\ \pm \sqrt{\partial_k N_1 \partial_k N_3} \over N_3 \sqrt{N_1 N_4} & \text{if } b_3 = 0 \\ \pm \sqrt{\partial_k N_1 \partial_k N_4} \over N_4 \sqrt{N_1 N_3} & \text{if } b_4 = 0 \end{cases}$$

(5.147)

$$P_{(1,1,0,0)} \rightarrow \begin{cases} 0 & \text{if } b_1 = 0 \\ \pm \sqrt{\partial_k N_1 \partial_k N_2} \over \sqrt{N_1 N_2 N_3} & \text{if } b_3 = 0 \\ \pm \sqrt{\partial_k N_1 \partial_k N_3} \over \sqrt{N_1 N_2 N_3} & \text{if } b_4 = 0 \end{cases}$$

(5.148)

$$P_{(1,1,1,0)} \rightarrow \begin{cases} 0 & \text{if } b_1 = 0 \text{ or } b_3 = 0 \\ \pm \sqrt{\partial_k N_1 \partial_k N_2} \over \sqrt{N_1 N_3} & \text{if } b_4 = 0 \end{cases}$$

(5.149)

where we have defined the $\alpha_{12} = N_1 \partial_k N_2 - N_2 \partial_k N_1$, $\alpha_{23} = N_2 \partial_k N_3 - N_3 \partial_k N_2$ and $\alpha_{24} = N_3 \partial_k N_4 - N_4 \partial_k N_3$. The simplest fields which limit to these contain an extra $\alpha_{12} \frac{b_2=0}{\partial_k N_2}$
so that the fields are

\[
P_{\alpha_2} = \pm \frac{\sqrt{\alpha_{12} \alpha_{23} \alpha_{24}}}{N_2 \sqrt{N_1 N_3 N_4} \partial_\xi N_2}
\]

(5.150)

\[
P_{(1,1,0,0)} = \pm \frac{\sqrt{\partial_\xi N_1 \alpha_{23} \alpha_{24}}}{\sqrt{N_1 N_2 N_3 N_4} \partial_\xi N_2}
\]

(5.151)

\[
P_{(1,1,1,0)} = \pm \frac{\sqrt{\alpha_{24} \partial_\xi N_1 \partial_\xi N_3}}{\sqrt{N_1 N_3 N_4} \partial_\xi N_2}
\]

(5.152)

These clearly limit as required and the four other fields related by symmetry can be constructed in the same way. These submodels do not contain the \(P_{(1,1,1,1)}\) and \(P_{(1,2,1,1)}\) fields, however. The next models (5.143) contain the field \(P_{(1,1,1,1)}\), which becomes

\[
P_{(1,1,1,1)} \xrightarrow{b_i = b_2} \sqrt{\partial_\xi N_j \partial_\xi N_k} \sqrt{N_j N_k}
\]

(5.153)

for each choice of \(i\) and \(i \neq j \neq k\). The expression which limits in each of these ways is

\[
P_{(1,1,1,1)} = \sqrt{N_2 \partial_\xi N_1 \partial_\xi N_3 \partial_\xi N_4} \sqrt{N_1 N_3 N_4} \partial_\xi N_2
\]

(5.154)

The collection of these general fields which limit correctly to all of the submodels (5.142,5.143) form what we will refer to as a ‘Boleyn’ solution\(^6\) to the equations of motion (which are written out in appendix B.1) when we set the field \(P_{(1,2,1,1)} = 0\). The highest root \(\Theta\) field in all of our models has the \(s_\Theta\) equation of motion

\[
\partial_\xi P_\Theta + P_\Theta \left( \sum_{i=1}^{r} (\Theta, \alpha_i) \partial_\xi \phi_i \right) = 0
\]

(5.155)

which for \(D_4\) is

\[
\partial_\xi P_{(1,2,1,1)} - P_{(1,2,1,1)} \partial_\xi \phi_2 = 0 \Rightarrow P_{(1,2,1,1)} = \frac{c_{(1,2,1,1)}}{\sqrt{N_2}}.
\]

(5.156)

One of our requirements is that each \(P\)-field should limit to \(\pm (i) N \partial_\xi N^{-1}\) for some choice of the \(b\) charges. In order to maintain this with equation (5.156) we must break our harmonic \(N_i\) ansatz so that \(N_2 \to N^2\) for the \(SL(2, \mathbb{R})\) model with only the highest root. We can in general write the now quadratic \(N_2\) as the product of two harmonic functions\(^7\)

\[
N_2 = (a_1 + b_1 \xi)(a_2 + b_2 \xi) \equiv N_{21} N_{22}.
\]

(5.157)

---

\(^6\)This name references the fact that the field of the highest root generator, or head, has been removed.

\(^7\)Another way to understand this comes from the brane sigma model interpretation, where each root in \(E_{11}\) is associated to an individual brane. The combined highest root in some finite-dimensional bound state will have \(n\) copies of each brane which is associated to a simple root, where \(n = (\Theta, \alpha_i)\), and there are \(n\) harmonic functions associated with those branes. We will later find that the number of harmonic functions will generally be equal to the height of the highest root of the algebra.
With this new product of harmonic functions we might be concerned about the validity of the Boleyn solution, but this is still valid in the limit where $N_2$ becomes harmonic (which coincides with $P_{(1,2,1,1)} = 0$). Let’s now bring the head back into the full $D_4$ model with the third submodel (5.144). Before treating the whole $SL(4, \mathbb{R})$ first consider the $SL(3, \mathbb{R})$ submodel with $\epsilon = (-,-)$ submodel with

$$\beta_1 = (1, 1, 1, 1) \quad \beta_2 = \alpha_2 \quad \beta_1 + \beta_2 = (1, 2, 1, 1).$$

(5.158)

The solution to this model is given by

$$P_{(1,1,1,1)} \rightarrow \frac{\sqrt{\partial \xi M_1 (M_2 \partial \xi M_1 - M_1 \partial \xi M_2)}}{M_1 \sqrt{M_2}}$$

(5.159)

$$P_{\alpha_2} \rightarrow \frac{\sqrt{-\partial \xi M_2 (M_2 \partial \xi M_1 - M_1 \partial \xi M_2)}}{M_2 \sqrt{M_1}}$$

(5.160)

$$P_{(1,2,1,1)} \rightarrow \frac{\sqrt{-\partial \xi M_1 \partial \xi M_2}}{\sqrt{M_1 M_2}}$$

(5.161)

where the submodel harmonic functions are related to the $D_4$ harmonic functions by the $b$ limit required to obtain the model:

$$M_1 = N_1 = N_3 = N_4 = N_2, \quad M_2 = N_2.$$  

(5.162)

We can immediately make two essential observations: (1) we have given no reason to prefer setting $(i,j) = (1,2)$ or $(2,1)$ - indeed there should be a symmetry $N_2_i \leftrightarrow N_2_j$ and (2) the Boleyn submodel solutions do not limit to these $SL(3, \mathbb{R})$ solutions! The fact that $N_2$ was assumed to be harmonic in our earlier work is responsible for the latter point. If we introduce the two harmonic functions $N_{2_i}$ we can find fields with appropriate functional dependence

$$P_{(1,1,1,1)} = c_{(1,1,1,1)} \left[ \frac{N_{2_1}}{N_1 N_2 N_3 N_4} \right] \quad P_{\alpha_2} = c_{\alpha_2} \left[ \frac{N_{2_1}}{N_{2_2} \sqrt{N_1 N_3 N_4}} \right].$$

(5.163)

The pair of these fields limit correctly for the $SL(3, \mathbb{R})$ solution but when we set $b_{2_1} = 0$ or $b_{2_2} = 0$ only one of the fields limits to the correct functional dependence of the Boleyn solution. If we use the other observation, that there should be a $N_{2_1} \leftrightarrow N_{2_2}$ symmetry, the fields

$$P_{(1,1,1,1)} = c_{(1,1,1,1)}[1] \left[ \frac{N_{2_1}}{N_1 N_2 N_3 N_4} \right] + c_{(1,1,1,1)}[2] \left[ \frac{N_{2_2}}{N_1 N_2 N_3 N_4} \right]$$

$$P_{\alpha_2} = c_{\alpha_2}[1] \left[ \frac{N_{2_2}}{N_{2_1} \sqrt{N_1 N_3 N_4}} \right] + c_{\alpha_2}[2] \left[ \frac{N_{2_1}}{N_{2_2} \sqrt{N_1 N_3 N_4}} \right].$$

(5.164)

limit to the correct functional dependence of the $SL(3, \mathbb{R})$ when

$$M_1 = N_{2_1} \quad \text{and} \quad c_{(1,1,1,1)}[2] = c_{\alpha_2}[1] = 0 \quad \text{or} \quad M_1 = N_{2_2} \quad \text{and} \quad c_{(1,1,1,1)}[1] = c_{\alpha_2}[2] = 0$$

and to the Boleyn solution when either

$$b_{2_1} = c_{(1,1,1,1)}[1] = c_{\alpha_2}[1] = 0 \quad \text{or} \quad b_{2_2} = c_{(1,1,1,1)}[2] = c_{\alpha_2}[2] = 0.$$
In order for the fields to possess the $N_2 \leftrightarrow N_2$ invariance we must stipulate that the as-yet-unknown $\xi$-constants swap under that transformation: $c_\alpha[1] \leftrightarrow c_\alpha[2]$. This replacement of the individual constants $c_\alpha$ with a set of constants $c_\alpha[i]$, labeled by some index or indices (in this case simply $i = 1, 2$), will be a standard feature whenever the highest root of the algebra satisfies $\text{height} \Theta > r$. To determine these constants we focus on the $c_{(1,1,1,1)}[i]$ first and collect all of the constraints:

\[
\begin{align*}
&c_{(1,1,1,1)}[1] \quad N_2 \leftrightarrow N_2 \\
&c_{(1,1,1,1)}[i] \quad b_i = 0 \\
&c_{(1,1,1,1)}[i] \quad N_1 = N_3 = N_2 \\
&c_{(1,1,1,1)}[i] \quad b_{2i} = 0 \\
&c_{(1,1,1,1)}[i] \quad N_1 = N_3 = N_2, \\
&c_{(1,1,1,1)}[i] \quad N_1 = N_3 = N_2, \\
&\xi_{(1,1,1,1)}[i] \quad \sqrt{\frac{\partial \xi N_1 \partial \xi N_3 \partial \xi N_4}{\partial \xi N_2}} \\
&\xi_{(1,1,1,1)}[i] \quad \sqrt{\partial \xi N_2, (N_2, \partial \xi N_2, -N_2, \partial \xi N_2)}
\end{align*}
\]

The constants should also be symmetric in $N_1, N_3, N_4$. From these constraints we find that the constants

\[
c_{(1,1,1,1)}[i] = \sqrt{\frac{(N_2, \partial \xi N_1 - N_2, \partial \xi N_1) (N_2, \partial \xi N_3 - N_2, \partial \xi N_3) (N_2, \partial \xi N_4 - N_2, \partial \xi N_4) \partial \xi N_2}{(N_2, \partial \xi N_2, -N_2, \partial \xi N_2)^2}}
\]

(5.170)
satisfy all of the conditions. Generalising our notation so that $\alpha_{12} = N_2, \partial \xi N_1 - N_2, \partial \xi N_1$, $\alpha_{23} = N_2, \partial \xi N_3 - N_2, \partial \xi N_3$, $\alpha_{24} = N_2, \partial \xi N_4 - N_2, \partial \xi N_4$ and $\alpha_{212} = N_2, \partial \xi N_2, -N_2, \partial \xi N_2$, the full solution for $P_{(1,1,1,1)}$ is

\[
P_{(1,1,1,1)} = \pm \left( \frac{-\alpha_{12} \alpha_{23} \alpha_{24} \partial \xi N_2}{\alpha_{212}^2} \sqrt{\frac{N_2}{N_2, N_1, N_3 N_4}} + [N_2 \leftrightarrow N_2] \right)
\]

(5.171)

Using exactly the same technique to solve for the $P_{\alpha_2}$ constants we find the solution

\[
P_{\alpha_2} = \pm \left( \sum_{i \neq j} \frac{-\alpha_{12} \alpha_{23} \alpha_{24} \partial \xi N_{2i}}{\alpha_{212}^2} \sqrt{\frac{N_{2j}}{N_2, N_1, N_3 N_4}} \right)
\]

(5.172)

and then placing both of these into the $s_{(1,1,1,1)}$ equation of motion allows us to solve for $c_{(1,2,1,1)}$, giving us

\[
P_{(1,2,1,1)} = \pm \sqrt{\frac{-\partial \xi N_2, \partial \xi N_2}{N_2, N_2}} = \pm \sqrt{\frac{-\partial \xi^2 N_2}{2 N_2}}
\]

(5.173)

The process of calculating the other $P$-fields proceeds uneventfully by using our known
submodel solutions. The full set of fields is given by:

\[ P_{\alpha_1} = \pm \sqrt{\frac{-\alpha_{12}}{N_1}} \sqrt{\frac{1}{N_2}} = \pm \sqrt{\frac{\partial_\xi N_1 \alpha_{12} - 1/2N_1^2\partial_\xi^2 N_2}{N_1N_2}} \]  

\[ P_{\alpha_2} = \pm \left( \sum_{i \neq j} \sqrt{\frac{-\alpha_{12,3} \alpha_{2,4} \partial_\xi N_2}{\alpha_{2,2}}^2} \right) \]  

\[ P_{(1,1,0,0)} = \pm \left( \sum_{i \neq j} \sqrt{\frac{-\alpha_{12,3} \alpha_{2,4} \partial_\xi N_2}{\alpha_{2,2}}^2} \right) \]  

\[ P_{(1,1,1,0)} = \pm \left( \sum_{i \neq j} \sqrt{\frac{-\alpha_{12,3} \alpha_{2,4} \partial_\xi N_2}{\alpha_{2,2}}^2} \right) \]  

\[ P_{(1,1,1,1)} = \pm \left( \sum_{i \neq j} \sqrt{\frac{-\alpha_{12,3} \alpha_{2,4} \partial_\xi N_2}{\alpha_{2,2}}^2} \right) \]  

\[ P_{(1,2,1,1)} = \pm \left( \sum_{i \neq j} \sqrt{\frac{-\alpha_{12,3} \alpha_{2,4} \partial_\xi N_2}{\alpha_{2,2}}^2} \right) \]  

with symmetric permutations for the remaining fields.

The subgroup transformations act on the charges just as in the $SL(n, \mathbb{R})$ models. In particular, the compact symmetries will allow us to move between any points $b$ and $b'$ where both correspond to some $SL(2, \mathbb{R})/SO(1, 1)$ submodel. Taking all $\alpha_i = 1$ we calculate

\[ J(\xi = 0) \to K_\alpha(\theta_\alpha)J(K_\alpha(\theta_\alpha))^{-1} \]

to find how $b = (b_1, b_2, b_3, b_4)$ transforms. If we start at some location where only one $b_i = q$ is non-zero, the action of the compact $K_\alpha, i = 1, 3, 4$ generate transformations which are each the same as the compact symmetry of $SL(3, \mathbb{R})$ with $\epsilon = (-, +)$, as shown in figure 5.3.1. In the three-dimensional $b_i = q, b_j = 0$ subspace the simple compact transformations map the origin to any point in the cube with opposite vertex $(q, q, q)$. The five-dimensional charge space is difficult to visualise but we use a coordinate-less depiction for the transformations of $K_\alpha, i = 1, 3, 4$ on the point $b = (0, q, 0, 0, 0)$ in figure 5.3.6. The compact $K(1,2,1,1)$ symmetry performs the same role as the $K(1,1)$ in $SL(3, \mathbb{R})$ with $\epsilon = (-, -)$: it takes a point in the charge space where the coefficients match a root and maps it to a point with charge coefficients equal to the difference with the symmetry root\(^8\). Schematically we write

\[ b_{(a_1, a_2, a_3, a_4)} = q(a_1, a_2, a_3, a_4) K(1,2,1,1) = q(a_1 - 1, a_2 - 1, a_3 - 1, a_4 - 1) \]

where we decompose $a_1 + a_2 = a_2$ with each $a_i = 0, \pm 1$. Acting on each of the vertices

---

\(^8\)The original charge root $(a_1, a_2, a_3, a_4)$ cannot be orthogonal to the symmetry root $(1, 2, 1, 1)$ as otherwise the Noether current will commute with $K(1,2,1,1)$.\]
CHAPTER 5. ONE-DIMENSIONAL SIGMA MODELS

Figure 5.3.6: Simple $SO(2, 2) \times SO(2, 2)$ compact symmetry transformations from the point $b = (0, q, 0, 0, 0)$. The transformations are color coded: $K_{\alpha_1}$ is blue, $K_{\alpha_3}$ is violet and $K_{\alpha_4}$ is green.

Figure 5.3.7: Full set of compact transformations on $b$-space component including $b_1 = (0, q, 0, 0, 0)$ and $b_2 = (0, 0, -q, 0, 0)$. The compact transformations of $K_{(1,2,1,1)}$ are in orange.

of the cube of figure (5.3.6) maps them to vertices of another cube with these new charges, as shown in in figure (5.3.7). In total the compact transformations map around the volume of this four-dimensional hypercube. This is one of two disjoint components in the $b$-space (just as in the $SL(3, \mathbb{R})$ $e = (-,-)$ model), the other being generated from the points $(0, -q, 0, 0, 0)$ and $(0, 0, q, 0, 0)$. The concrete construction which is useful in verifying the equations of motion is obtained by mapping the Noether current at $\xi = 0$ and a point in $b$ under the $K_\alpha$ with parameters $\theta_\alpha$. We find that the component containing $b = (0, q, 0, 0, 0)$ has charges and $\alpha_{ij}$:

$$
\begin{align*}
b_1 &= q(\sin^2(\theta_{\alpha_1}) - \sin^2(\theta_{(1,2,1,1)})) \\
b_2 &= q\cos^2(\theta_{(1,2,1,1)}) \\
b_3 &= q(\sin^2(\theta_{\alpha_3}) - \sin^2(\theta_{(1,2,1,1)})) \\
b_4 &= q(\sin^2(\theta_{\alpha_4}) - \sin^2(\theta_{(1,2,1,1)}))
\end{align*}
$$

(5.181)

$$
\begin{align*}
\alpha_{i2_1} &= N_i \partial_{\xi} N_{2_1} - N_{2_1} \partial_{\xi} N_i = q\cos^2(\theta_{\alpha_1}) \\
\alpha_{i2_2} &= N_i \partial_{\xi} N_{2_2} - N_{2_2} \partial_{\xi} N_i = -q\sin^2(\theta_{\alpha_1}) \\
\alpha_{2_1,2_2}^2 &= (N_{2_1} \partial_{\xi} N_{2_2} - N_{2_2} \partial_{\xi} N_{2_1})^2 = q^2
\end{align*}
$$

(5.184)

(5.185)
for $i = 1, 3, 4$. With these charges we can then compute the $P$-field constants

$$c_{\alpha \neq 2} = \pm q \sin(\theta_{\alpha}) \cos(\theta_{1,2,1,1}) \quad c_{1,2,1,1} = \pm q \sin(\theta_{1,2,1,1}) \cos(\theta_{1,2,1,1})$$

(5.186)

$$c_{\alpha 2}[1] = q \cos(\theta_{\alpha 1}) \cos(\theta_{\alpha 3}) \cos(\theta_{\alpha 4}) \cos(\theta_{1,2,1,1})$$

(5.187)

$$c_{\alpha 2}[2] = q \sin(\theta_{\alpha 1}) \sin(\theta_{\alpha 3}) \sin(\theta_{\alpha 4}) \sin(\theta_{1,2,1,1})$$

(5.188)

$$c_{1,1,0,0}[1] = q \sin(\theta_{\alpha 1}) \cos(\theta_{\alpha 3}) \cos(\theta_{\alpha 4}) \cos(\theta_{1,2,1,1})$$

(5.189)

$$c_{1,1,0,0}[2] = q \cos(\theta_{\alpha 1}) \sin(\theta_{\alpha 3}) \sin(\theta_{\alpha 4}) \sin(\theta_{1,2,1,1})$$

(5.190)

$$c_{1,1,1,0}[1] = q \sin(\theta_{\alpha 1}) \sin(\theta_{\alpha 3}) \cos(\theta_{\alpha 4}) \cos(\theta_{1,2,1,1})$$

(5.191)

$$c_{1,1,1,0}[2] = q \cos(\theta_{\alpha 1}) \cos(\theta_{\alpha 3}) \sin(\theta_{\alpha 4}) \sin(\theta_{1,2,1,1})$$

(5.192)

$$c_{1,1,1,1}[1] = q \sin(\theta_{\alpha 1}) \sin(\theta_{\alpha 3}) \sin(\theta_{\alpha 4}) \cos(\theta_{1,2,1,1})$$

(5.193)

$$c_{1,1,1,1}[2] = q \cos(\theta_{\alpha 1}) \cos(\theta_{\alpha 3}) \cos(\theta_{\alpha 4}) \sin(\theta_{1,2,1,1})$$

(5.194)

with the remaining constants filled in by symmetry.

### 5.3.3 $SO(5, 5)/(SO(2, 3) \times SO(2, 3))$ with involution $\epsilon = (+, +, -, +, +)$

We have already solved a $D_{4(4)}$ model with highest root $\Theta$ which has height greater than the rank of the algebra by one. The equation of motion for $s_{\Theta}$ and submodel solutions required us to introduce another harmonic function so that $N_2 = N_2', N_2''$. The algebra $D_5$ with Dynkin diagram (5.3.3) has highest root $(1, 2, 2, 1, 1)$. From the brane $\sigma$-model perspective we expect the solution to be the superposition of seven solutions which correspond to the 5 roots embedded in $E_{11}$, with two appearing twice in the full bound state. The $D_{4(4)}$ submodel requires us to set $N_3 = N_3', N_3''$ while the equation of motion for $s_{\Theta}$

$$s_{1,2,2,1,1} : \quad 0 = \partial_\xi P_{1,2,2,1,1} + P_{1,2,2,1,1} \left( \partial_\xi \phi_2 \right)$$

(5.195)

is subject to the same set of arguments that constrained the form of $P_{1,2,1,1}$ in the previous section, so that indeed $N_2 = N_2', N_2''$, and we have seven charges $b_i, i = 1, 2, 1, 2, 3, 1, 3, 2, 4, 5$.

To solve the model we identify subalgebras and use the submodel solutions to constrain the full $P$-fields. There are three $D_4$ subalgebras which are formed from

$$\beta_1 = \alpha_1 + \alpha_2 \quad \beta_2 = \alpha_3 \quad \beta_3 = \alpha_4 \quad \beta_4 = \alpha_5$$

(5.196)
To these we add the Boleyn solution with $P_{(1,2,2,1,1)} = P_{(1,1,2,1,1)} = P_{(0,1,2,1,1)} = 0$ and several SL$(4, \mathbb{R})$ models which contain a non-zero $P_\Theta$, including:

\[
\begin{align*}
\beta_1 &= \alpha_2 \quad \beta_2 = \alpha_3 \quad \beta_3 = \alpha_5 \quad \beta_4 = (1, 1, 1, 1, 0) \quad (5.197) \\
\beta_1 &= \alpha_2 \quad \beta_2 = \alpha_3 + \alpha_4 \quad \beta_3 = \alpha_5 \quad \beta_4 = (1, 1, 1, 0, 0) \quad (5.198) \\
\beta_1 &= \alpha_1 + \alpha_2 \quad \beta_2 = \alpha_3 + \alpha_4 \quad \beta_3 = \alpha_5 \quad \beta_4 = \alpha_2 + \alpha_3 \quad (5.199) \\
\beta_1 &= (0, 1, 1, 1, 0) \quad \beta_2 = \alpha_5 \quad \beta_3 = \alpha_3 \quad \beta_4 = \alpha_1 + \alpha_2. \quad (5.200)
\end{align*}
\]

The process is similar to what has been done before and we will discuss general $D_{n(\alpha)}$ solutions later in this chapter, so we simply present the solution:

\[
\begin{align*}
P_{(1,0,0,0,0)} &= \frac{\sqrt{-\alpha_{12} \alpha_{23}}}{N_1 \sqrt{N_2}} \quad P_{(0,0,0,1,0)} = \frac{\sqrt{-\alpha_{34} \alpha_{45}}}{N_4 \sqrt{N_5}} \quad (5.201) \\
P_{(0,1,0,0,0)} &= \sum_{i \neq j} \sqrt{\frac{-\alpha_{12} \alpha_{23} \alpha_{45}}{\alpha_{21}^2 \alpha_{32}^2}} \frac{N_2}{N_2 \sqrt{N_1 N_3}} \quad (5.202) \\
P_{(0,0,1,0,0)} &= \sum_{i \neq j} \sqrt{\frac{\alpha_{23} \alpha_{45}}{\alpha_{32}^2}} \frac{N_3}{N_2 \sqrt{N_1 N_4 N_5}} \quad (5.203) \\
P_{(1,1,0,0,0)} &= \sum_{i \neq j} \sqrt{\frac{-\alpha_{12} \alpha_{23} \alpha_{45}}{\alpha_{21}^2 \alpha_{32}^2}} \frac{N_2}{N_1 N_2 \sqrt{N_3 N_4 N_5}} \quad (5.204) \\
P_{(0,1,1,0,0)} &= \sum_{i \neq j \neq k \neq l} \sqrt{\frac{-\alpha_{12} \alpha_{23} \alpha_{45}}{\alpha_{21}^2 \alpha_{32}^2}} \frac{N_2}{N_1 N_2 \sqrt{N_3 N_4 N_5}} \quad (5.205) \\
P_{(0,0,1,1,0)} &= \sum_{i \neq j} \sqrt{\frac{-\alpha_{23} \alpha_{45}}{\alpha_{32}^2}} \frac{N_3}{N_2 N_3 N_4 N_5} \quad (5.206) \\
P_{(1,1,1,0,0)} &= \sum_{i \neq j \neq k \neq l} \sqrt{\frac{-\alpha_{12} \alpha_{23} \alpha_{45}}{\alpha_{21}^2 \alpha_{32}^2}} \frac{N_2}{N_1 N_2 N_3 N_4 N_5} \quad (5.207) \\
P_{(0,1,1,1,0)} &= \sum_{i \neq j \neq k \neq l} \sqrt{\frac{\alpha_{23} \alpha_{45}}{\alpha_{32}^2}} \frac{N_2}{N_1 N_2 N_3 N_4 N_5} \quad (5.208) \\
P_{(0,0,1,1,1)} &= \sum_{i \neq j \neq k \neq l} \sqrt{\frac{\alpha_{23} \alpha_{45}}{\alpha_{32}^2}} \frac{1}{\sqrt{N_2 N_3 N_4 N_5}} \quad (5.209) \\
P_{(1,1,1,1,0)} &= \sum_{i \neq j \neq k \neq l} \sqrt{\frac{-\alpha_{12} \alpha_{23} \alpha_{45}}{\alpha_{21}^2 \alpha_{32}^2}} \frac{1}{\sqrt{N_1 N_4 N_5}} \quad (5.210) \\
P_{(0,1,1,1,1)} &= \sum_{i \neq j \neq k \neq l} \sqrt{\frac{-\alpha_{12} \alpha_{23} \alpha_{45}}{\alpha_{21}^2 \alpha_{32}^2}} \frac{N_2}{N_1 N_2 N_3 N_4 N_5} \quad (5.211) \\
P_{(1,1,1,1,1)} &= \sum_{i \neq j \neq k \neq l} \sqrt{\frac{\alpha_{23} \alpha_{45}}{\alpha_{32}^2}} \frac{N_3}{N_1 N_3 N_4 N_5} \quad (5.212)
\end{align*}
\]
\[ P_{(0,1,2,1,1)} = \sum_{i \neq j}^{\alpha_{12}, \alpha_{23}, \alpha_{23}, \alpha_{23}} \sqrt{-\frac{\alpha_{12}, \alpha_{23}, \alpha_{23}, \alpha_{23}}{\alpha_{23}, \alpha_{23}}} \frac{1}{\sqrt{N_1 N_3}} \]  
\[ P_{(1,1,2,1,1)} = \sum_{i \neq j}^{\alpha_{12}, \alpha_{23}, \alpha_{23}, \alpha_{23}} \sqrt{\frac{\alpha_{12}, \alpha_{23}, \alpha_{23}, \alpha_{23}}{\alpha_{23}, \alpha_{23}}} \frac{N_2}{\sqrt{N_1 N_2 N_3}} \]  
\[ P_{(1,2,2,1,1)} = -\frac{\partial_{N_2} N_2}{\sqrt{N_2}} \]  

where every \( \alpha_{ij} = N_i \partial_{\xi} N_j - N_j \partial_{\xi} N_i \). We only determine each \( P^2 \) so each field should have an additional \( \pm \) factor. We omit any fields which are related by symmetries of the Dynkin diagram, for example \( P_{\alpha_4} \) appears while \( P_{\alpha_2} \) does not.

### 5.3.4 \( E_{6(6)}/Sp(8, \mathbb{R}) \) with involution \( \epsilon = (+, +, - , + , + , +) \)

The \( E_6 \) equations of motion, which appear in appendix section B.2, are significantly more complicated than the \( D_{4(4)} \) but the same methods for solution generation will give us positive results. The \( D_{5(5)} \) submodel solutions associated with \( b_1 = 0 \) or \( b_5 = 0 \) each indicate that \( N_4 = N_4, N_5 \) and \( N_2 = N_2, N_2 \). The highest root equation of motion \( s_{(1,2,3,2,1,2)} \), combined with the requirement of \( SL(2, \mathbb{R}) \) submodel solutions, indicates that \( N_6 = N_6, N_6 \). From the brane-sigma model perspective we expect three charges associated with \( \alpha_3 \). A useful submodel to demand the last charge is the \( D_4 \) with simple roots:

\[ \beta_1 = \alpha_1 + \alpha_2 \quad \beta_2 = \alpha_3 \quad \beta_3 = \alpha_4 + \alpha_5 \quad \beta_4 = (0, 1, 1, 1, 0, 1) \]  

This submodel is built out of a limit where \( b_{3_a} = b_{2_1} = b_{4_m} = b_{6_v} \), for one value of each of the latin indices, and the \( D_{4(4)} \) solution indicates that there are two other \( b_3 \) charges. It should be noted that, just as in the examples above, the number of harmonic functions needed to construct the solution is equal to the height of the highest root of the algebra (in this case eleven).

The \( P_{\alpha_1} \) and \( P_{\alpha_5} \) will be unchanged from the \( D_5 \) subsolution while an \( A_5 \) submodel

\[ \beta_1 = (0, 1, 1, 0, 0, 1) \quad \beta_2 = \alpha_4 \quad \beta_3 = \alpha_3 \quad \beta_4 = \alpha_1 + \alpha_2 \quad \beta_5 = (0, 0, 1, 1, 1, 1) \]  

will assist us. The \( P_{(1,1,0,0,0,0,0)} \) must now interact with each \( N_{3_n} \) so that the constant in equation (5.204) must have \( \partial_{\xi} N_2 \rightarrow \alpha_{23}, \alpha_3 \) for the \( E_6 \) solution:

\[ P_{(1,1,0,0,0,0,0)} = \sum_{i \neq j}^{\alpha_{12}, \alpha_{23}, \alpha_{23}, \alpha_{23}} \sqrt{\frac{\alpha_{12}, \alpha_{23}, \alpha_{23}, \alpha_{23}}{\alpha_{23}, \alpha_{23}}} \frac{N_2}{\sqrt{N_1 N_2 N_3}} \]  

In order to satisfy these submodels, as well as the \( D_3 \) made from simple roots, the \( P_{(1,1,1,0,0,0)} \) must then have the functional dependence

\[ P_{(1,1,1,0,0,0,0,0)} = \frac{1}{\sqrt{N_1 N_2 N_6}} \sum_{a \neq b \neq c} c_{(1,1,1,0,0,0)}[a, b, c] \frac{N_3}{{\sqrt{N_3}}_n} \]
where we have labeled each of the constants \(c[a, b, c]\) with indices in the summation. The constant must contain the usual \(\alpha_{12}\) terms from the \(D_5\) sub-solution, an analog of the triple \(\alpha_{23}\) in equation (5.218) and a sum over the three \(N_{3a}\) with \(\alpha_{34}\) and \(\alpha_{36}\) terms. By considering all of the submodels we can fix the denominator as well, obtaining

\[
c(1,1,1,0,0,0)[a, b, c] = \sum_{i \neq f} \sqrt{\frac{\alpha_{12}\alpha_{23}\alpha_{23}\alpha_{34}\alpha_{34}\alpha_{36}\alpha_{41}\alpha_{41}\alpha_{61}}{\alpha_{21}\alpha_{22}\alpha_{33}\alpha_{33}\alpha_{33}\alpha_{33}}}. \tag{5.220}
\]

The rest of the \(P\)-fields can be determined from numerous submodels and will contain up to four summations with a total of twenty-four terms. The full solution is

\[
P_{\alpha_1} = \frac{\sqrt{-\alpha_{12}\alpha_{12}}}{N_1\sqrt{N_2}} \tag{5.221}
\]

\[
P_{\alpha_2} = \frac{\alpha_{12}\alpha_{23}\alpha_{23}\alpha_{34}\alpha_{34}\alpha_{36}\alpha_{36}}{\alpha_{21}\alpha_{22}\alpha_{33}\alpha_{33}\alpha_{33}\alpha_{33}} \frac{N_2}{\sqrt{N_1N_3}} \tag{5.222}
\]

\[
P_{\alpha_3} = \frac{\alpha_{21}\alpha_{22}\alpha_{23}\alpha_{34}\alpha_{34}\alpha_{36}\alpha_{36}}{\alpha_{21}\alpha_{22}\alpha_{33}\alpha_{33}\alpha_{33}\alpha_{33}} \frac{N_3\sqrt{N_2N_1N_4}}{N_3a\sqrt{N_2N_1N_4}N_6} \tag{5.223}
\]

\[
P_{\alpha_6} = \frac{-\alpha_{23}\alpha_{23}\alpha_{34}\alpha_{34}\alpha_{36}\alpha_{36}}{\alpha_{21}\alpha_{22}\alpha_{33}\alpha_{33}\alpha_{33}\alpha_{33}} \frac{N_6}{\sqrt{N_3}} \tag{5.224}
\]

\[
P_{(1,1,1,0,0,0)} = \sqrt{\frac{\alpha_{12}\alpha_{23}\alpha_{23}\alpha_{34}\alpha_{34}\alpha_{36}\alpha_{36}}{\alpha_{21}\alpha_{22}\alpha_{33}\alpha_{33}\alpha_{33}\alpha_{33}}} \frac{N_2}{\sqrt{N_1N_2N_3}} \tag{5.225}
\]

\[
P_{(0,1,1,0,0,0)} = \sqrt{\frac{\alpha_{21}\alpha_{22}\alpha_{23}\alpha_{34}\alpha_{34}\alpha_{36}\alpha_{36}}{\alpha_{21}\alpha_{22}\alpha_{33}\alpha_{33}\alpha_{33}\alpha_{33}}} \frac{N_3\sqrt{N_2N_1N_4}}{N_3a\sqrt{N_2N_1N_4}N_6} \tag{5.226}
\]

\[
P_{(0,1,1,0,1,0)} = \sqrt{\frac{\alpha_{21}\alpha_{22}\alpha_{23}\alpha_{34}\alpha_{34}\alpha_{36}\alpha_{36}}{\alpha_{21}\alpha_{22}\alpha_{33}\alpha_{33}\alpha_{33}\alpha_{33}}} \frac{N_3\sqrt{N_2N_1N_4}N_6}{N_2N_3a\sqrt{N_1N_4}N_6} \tag{5.227}
\]

\[
P_{(1,1,1,0,0,0)} = \sqrt{\frac{\alpha_{12}\alpha_{23}\alpha_{23}\alpha_{34}\alpha_{34}\alpha_{36}\alpha_{36}}{\alpha_{21}\alpha_{22}\alpha_{33}\alpha_{33}\alpha_{33}\alpha_{33}}} \frac{N_2\sqrt{N_1N_2N_3}}{N_1N_2N_3a\sqrt{N_1N_4}N_6} \tag{5.228}
\]

\[
P_{(0,1,1,1,0,0)} = \sqrt{\frac{\alpha_{12}\alpha_{23}\alpha_{23}\alpha_{34}\alpha_{34}\alpha_{36}\alpha_{36}}{\alpha_{21}\alpha_{22}\alpha_{33}\alpha_{33}\alpha_{33}\alpha_{33}}} \frac{N_2\sqrt{N_1N_2N_3}}{N_1N_2N_3a\sqrt{N_1N_4}N_6} \tag{5.229}
\]

\[
P_{(0,1,1,0,1,0)} = \sqrt{\frac{\alpha_{12}\alpha_{23}\alpha_{23}\alpha_{34}\alpha_{34}\alpha_{36}\alpha_{36}}{\alpha_{21}\alpha_{22}\alpha_{33}\alpha_{33}\alpha_{33}\alpha_{33}}} \frac{N_2\sqrt{N_1N_2N_3}}{N_1N_2N_3a\sqrt{N_1N_4}N_6} \tag{5.230}
\]

\[
P_{(1,1,1,1,0,0)} = \sqrt{\frac{\alpha_{12}\alpha_{23}\alpha_{23}\alpha_{34}\alpha_{34}\alpha_{36}\alpha_{36}}{\alpha_{21}\alpha_{22}\alpha_{33}\alpha_{33}\alpha_{33}\alpha_{33}}} \frac{N_2\sqrt{N_1N_2N_3}}{N_1N_2N_3a\sqrt{N_1N_4}N_6} \tag{5.231}
\]
\[ P_{(0,1,1,1,0,1)} = \sqrt{\frac{-\alpha_{12} \alpha_{23} \alpha_{34} \alpha_{45} \alpha_{56} \alpha_{67} \alpha_{78} \alpha_{89} \alpha_{910}}{\alpha_{21} \alpha_{32} \alpha_{43} \alpha_{54} \alpha_{65} \alpha_{76} \alpha_{87} \alpha_{98} \alpha_{109}} \frac{N_3 \ N_6}{N_1 N_4 N_9}} \]  

(5.232)

\[ P_{(1,1,1,0,1,0)} = \sqrt{\frac{-\alpha_{12} \alpha_{23} \alpha_{34} \alpha_{45} \alpha_{56} \alpha_{67} \alpha_{78} \alpha_{89} \alpha_{910}}{\alpha_{21} \alpha_{32} \alpha_{43} \alpha_{54} \alpha_{65} \alpha_{76} \alpha_{87} \alpha_{98} \alpha_{109}} \frac{N_3 \ N_6}{N_1 N_4 N_9}} \]  

(5.233)

\[ P_{(1,1,1,1,1,0)} = \sqrt{\frac{-\alpha_{12} \alpha_{23} \alpha_{34} \alpha_{45} \alpha_{56} \alpha_{67} \alpha_{78} \alpha_{89} \alpha_{910}}{\alpha_{21} \alpha_{32} \alpha_{43} \alpha_{54} \alpha_{65} \alpha_{76} \alpha_{87} \alpha_{98} \alpha_{109}} \frac{N_3 \ N_6}{N_1 N_4 N_9}} \]  

(5.234)

\[ P_{(1,1,1,1,1,1)} = \sqrt{\frac{-\alpha_{12} \alpha_{23} \alpha_{34} \alpha_{45} \alpha_{56} \alpha_{67} \alpha_{78} \alpha_{89} \alpha_{910}}{\alpha_{21} \alpha_{32} \alpha_{43} \alpha_{54} \alpha_{65} \alpha_{76} \alpha_{87} \alpha_{98} \alpha_{109}} \frac{N_3 \ N_6}{N_1 N_4 N_9}} \]  

(5.235)

\[ P_{0(1,2,1,0,1)} = \sqrt{\frac{-\alpha_{12} \alpha_{23} \alpha_{34} \alpha_{45} \alpha_{56} \alpha_{67} \alpha_{78} \alpha_{89} \alpha_{910}}{\alpha_{21} \alpha_{32} \alpha_{43} \alpha_{54} \alpha_{65} \alpha_{76} \alpha_{87} \alpha_{98} \alpha_{109}} \frac{N_3 \ N_6}{N_1 N_4 N_9}} \]  

(5.236)

\[ P_{(1,2,1,0,1,1)} = \sqrt{\frac{-\alpha_{12} \alpha_{23} \alpha_{34} \alpha_{45} \alpha_{56} \alpha_{67} \alpha_{78} \alpha_{89} \alpha_{910}}{\alpha_{21} \alpha_{32} \alpha_{43} \alpha_{54} \alpha_{65} \alpha_{76} \alpha_{87} \alpha_{98} \alpha_{109}} \frac{N_3 \ N_6}{N_1 N_4 N_9}} \]  

(5.237)

\[ P_{(1,2,1,1,1,0)} = \sqrt{\frac{-\alpha_{12} \alpha_{23} \alpha_{34} \alpha_{45} \alpha_{56} \alpha_{67} \alpha_{78} \alpha_{89} \alpha_{910}}{\alpha_{21} \alpha_{32} \alpha_{43} \alpha_{54} \alpha_{65} \alpha_{76} \alpha_{87} \alpha_{98} \alpha_{109}} \frac{N_3 \ N_6}{N_1 N_4 N_9}} \]  

(5.238)

\[ P_{(1,2,2,1,0,1)} = \sqrt{\frac{-\alpha_{12} \alpha_{23} \alpha_{34} \alpha_{45} \alpha_{56} \alpha_{67} \alpha_{78} \alpha_{89} \alpha_{910}}{\alpha_{21} \alpha_{32} \alpha_{43} \alpha_{54} \alpha_{65} \alpha_{76} \alpha_{87} \alpha_{98} \alpha_{109}} \frac{N_3 \ N_6}{N_1 N_4 N_9}} \]  

(5.239)

\[ P_{(1,2,2,1,1,1)} = \sqrt{\frac{-\alpha_{12} \alpha_{23} \alpha_{34} \alpha_{45} \alpha_{56} \alpha_{67} \alpha_{78} \alpha_{89} \alpha_{910}}{\alpha_{21} \alpha_{32} \alpha_{43} \alpha_{54} \alpha_{65} \alpha_{76} \alpha_{87} \alpha_{98} \alpha_{109}} \frac{N_3 \ N_6}{N_1 N_4 N_9}} \]  

(5.240)

\[ P_{(1,2,2,1,1,0)} = \sqrt{\frac{-\alpha_{12} \alpha_{23} \alpha_{34} \alpha_{45} \alpha_{56} \alpha_{67} \alpha_{78} \alpha_{89} \alpha_{910}}{\alpha_{21} \alpha_{32} \alpha_{43} \alpha_{54} \alpha_{65} \alpha_{76} \alpha_{87} \alpha_{98} \alpha_{109}} \frac{N_3 \ N_6}{N_1 N_4 N_9}} \]  

(5.241)
for each $c \epsilon$ while in the model with involution.

The only differences between the solutions of models with different involutions are in the $\xi$-constants. Of course, we only determine $P^2$ uniquely so each term should contain an additional $\pm$ factor. We omit any fields which are related by symmetries of the Dynkin diagram, for example $P_{\alpha_1}$ appears while $P_{\alpha_5}$ does not.

5.4 Solutions for Simply-Laced Algebras

Having presented solutions for a variety of $SL(n, \mathbb{R}), SO(n, n)$ and $E_{6(6)}$ coset models we have observed several important qualities and patterns which can be used to generate solutions for the coset models of normal real forms of simply-laced algebras over involution invariant subalgebras defined by a $r$-dimensional vector $\epsilon$. While determining the charge dependence of the parameters of the subgroup is a task for each individual model and involution, it can be simplified with the brane $\sigma$-model perspective as well as knowledge of submodel motion.

5.4.1 $A_{n(n)}$ solutions

We have already found the solutions for $SL(n, \mathbb{R})$ with $\epsilon = (+, \ldots, +)$ and $\epsilon = (-, +, \ldots, +)$ in section 5.3.1. Independent of the involution, the number of harmonic functions and charges is equal to the rank $r = n - 1$, while the $(n^2 - n)/2$ $P$-fields are labeled by the positive roots $\alpha$:

\[
\phi_i = 1/2 \log N_i = 1/2 \log(a_i + b_i \xi) \quad i = 1, \ldots, r \quad (5.246)
\]
\[
P_\alpha = \pm \frac{c_\alpha}{\prod_{i=1}^r N_i^{||\alpha, \alpha_i||/2}}. \quad (5.247)
\]

The only differences between the solutions of models with different involutions are in the constants $c_\alpha$. For $t = so(n)$ we found that the constants (5.134) are

\[
c_{(0, \ldots, 0, a, a, \ldots, 0, 0, \ldots, 0)} = \pm \sqrt{(N_a \partial_\xi N_{a-1} - N_{a-1} \partial_\xi N_a) (N_{b+1} \partial_\xi N_b - N_b \partial_\xi N_{b+1})} \quad (5.248)
\]

while in the model with involution $\epsilon = (-, +, \ldots, +)$ the sign inside each radical is opposite for each $c_\alpha$ where $a = 1$. It is easy to see that this generalises for any model with involution.
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\[ \epsilon = (+, \ldots, +, -a, +, \ldots, +) \] with one \( \Omega(E_{\alpha_a}) = F_{\alpha_a} \). The field \( P_{\alpha_a} \) must limit to the \( SL(2, \mathbb{R})/SO(1, 1) \) solution when \( b_i = \delta_{i,a} \) and

\[ c_{\alpha_a} \xrightarrow{b_i = \delta_{i,a}} \pm \sqrt{+ (\partial \xi N_a)^2}. \quad (5.249) \]

Any \( P_{\alpha} \), where \( \alpha_a \in \alpha \) will also limit in this way since \( \epsilon_{\alpha} = -1 \). For a general \( \epsilon \) the constants \( c_{\alpha} \) depend only on the value of \( \epsilon_{\alpha} \) and the generalisation of the constants in equation (5.134) can be given rather simply by

\[ c_{\alpha} = c_{(0, \ldots, 1_a, \ldots, 1_b, 0, \ldots, 0)} = \pm \sqrt{\epsilon_{\alpha} (N_a \partial \xi N_{a-1} - N_{a-1} \partial \xi N_a) (N_{b+1} \partial \xi N_b - N_b \partial \xi N_{b+1})}. \quad (5.250) \]

5.4.2 \( D_{n(n)} \) solutions

In the \( D_{4(4)} \) model solution, presented in section 5.3.2, we found that the simple root which appears twice in the highest root must have two harmonic functions and charges within the Cartan field \( \phi \). In the extension to \( D_{5(5)} \) the same argument applies to the new root which appears twice in the highest roots, as well. Assuming that the solution to \( D_{n(n)} \) with the number of charges equal to \( \text{height}(\Theta) = 2n - 3 \), the \( D_{n+1(n+1)} \) solution will have one extra charge for the new node as well as one from the root which has been promoted to having two copies in the highest root. The second new charge comes from the same highest root argument: the field \( P_{\Theta} \) must limit to some \( SL(2, \mathbb{R}) \) solution and also must satisfy the equation of motion (5.155). The field associated to the newly created simple root will be

\[ P_{\alpha_1}[D_{n+1(n+1)}] = \frac{\sqrt{\alpha_{12} \alpha_{12}^2}}{N_1 \sqrt{N_2}} \quad (5.251) \]

while the \( \alpha_1 \) field from the \( D_n \) solution will be modified as

\[ P_{\alpha_1}[D_{n(n)}] = \frac{\sqrt{\alpha_{23} \alpha_{23}^2}}{N_2 \sqrt{N_3}} \quad (5.252) \]

\[ \rightarrow P_{\alpha_2}[D_{n+1(n+1)}] = \sum_{i \neq j} \frac{\sqrt{\alpha_{12} \alpha_{2,3,1} \alpha_{2,3,2} \partial \xi N_{2i} \partial \xi N_{2j}}}{\alpha_{21}^2 \cdot N_2 \sqrt{N_1 N_3}} \cdot N_2_j \quad (5.253) \]
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exactly as in the $D_{4(4)} \to D_{5(5)}$ solution. The next simple root field must now be modified as

$$P_{\alpha_2}[D_{n(n)}] = \sum_{i \neq j} \sqrt{\frac{\alpha_{2i3} \alpha_{314} \alpha_{342} \partial_i N_{3i}}{\alpha_{3i3j}^2}} \cdot \frac{N_{3j}}{N_{3i} \sqrt{N_{2j} N_4}}$$

(5.254)

$$\rightarrow P_{\alpha_3}[D_{n+1(n+1)}] = \sum_{i \neq j} \sqrt{\frac{\alpha_{2i3} \alpha_{2i4} \alpha_{342} \alpha_{3i42} \partial_i N_{3i}}{\alpha_{3i3j}^2}} \cdot \frac{N_{3j}}{N_{3i} \sqrt{N_{2j} N_4}}.$$  (5.255)

We can determine the rest of the $P$-fields using submodels and other patterns which are apparent when we solve the equations of motion. For example, the constants of the fields associated to two roots whose sum is another root have the general property that

$$c_{\beta_1} c_{\beta_2} = \pm c_{\beta_1 + \beta_2} \times \sqrt{X^2}$$  (5.256)

with $X$ the terms which are shared by both $c_{\beta_1}$ and $c_{\beta_2}$. This observation comes from the equations of motion, where $c_\alpha$ contains all of the possible $P_\beta P_{\alpha + \beta}$. In order to find a solution to the equations of motion, the constants for each of these products must contain $c_\alpha \sqrt{X^2}$ in order to cancel the radical (which will appear in each product $P_\beta P_{\beta_j}$) and find a solution to the equation, which implies equation (5.256). One particular consequence of this is that for any two roots $\beta_1 + \beta_2 = \Theta$ the difference between the two constants $c_{\beta_1}$ and $c_{\beta_2}$ is only the relabeling $\partial_i N_{2i} \leftrightarrow \partial_i N_{2j}$. The $P_{(1,1,\ldots)}$, $P_{(0,1,\ldots)}$ and $P_{(0,0,1,\ldots)}$ fields will all be modified as one would expect from the simple root fields above. In particular, the highest root field from $D_{n(n)}$ becomes

$$P_{\Theta}[D_{n(n)}] = \sqrt{\frac{\partial_i N_{31} \partial_i N_{32}}{N_{3i}^3}}$$

$$\rightarrow P_{\Theta}[D_{n+1(n+1)}] = \sum_{i \neq j} \sqrt{\frac{\alpha_{12i} \alpha_{2i3} \alpha_{2i4} \partial_i N_{3i}}{\alpha_{3i3j}^2}} \cdot \frac{1}{\sqrt{N_{1i} N_{3j}}}.$$  (5.257)

At this point we can give the general solution for $D_{n(n)}$ with involution $\epsilon$ starting with the fields

$$P_{(0,0,\ldots,0,1,\ldots,1,0,\ldots,0)}^2 = \pm \sum_{i \neq j} \sum_{k \neq l} \cdots \sum_{u \neq v} \sum_{w \neq x} \left( \frac{\alpha_{(a-1)1} \alpha_{(a-1)2} \alpha_{a+1} \alpha_{a+1}}{\alpha_{a1a2}^2} \times \cdots \times \alpha_{b1b2}^2 \right) \times \cdots \times \frac{\alpha_{(b-1)1} \alpha_{(b-1)2} \alpha_{b+1} \alpha_{b+1}}{N_{(a-1)} N_{a1} N_{b1} N_{b+1}}.$$  (5.258)

where $3 \leq a < b \leq n-3$. The sign of the square is determined first by finding which ordering produces the $SL(2,\mathbb{R})/SO(2)$ solution of $-(\partial N)^2/N^2$ when the only non-zero charges are $b_{a1} = \ldots = b_{a3} = 1$, and then multiplying by $\epsilon_\alpha$, where $\alpha$ is the root associated with the
field. The set of fields with this form (5.258) can be extended on the left by

\[
P_{(0,1,\ldots,1,0,\ldots,0)^2} = \pm \sum_{i \neq j} \sum_{k \neq l} \sum_{u \neq v} \sum_{w \neq x} \left( \frac{\alpha_{a_1} \alpha_{a_2} \alpha_{a_3} \partial_{\xi} N_{a_2}}{\alpha_{a_2} \alpha_{a_3}^2 \times \cdots \times \alpha_{a_1} \alpha_{a_2}^2} \times \ldots \times \alpha_{a_1} \alpha_{a_2}^2 \times \ldots \times \alpha_{a_1} \alpha_{a_2}^2 \right) \frac{\alpha_{a_1} \alpha_{a_2} \alpha_{a_3} \partial_{\xi} N_{a_2}}{N_1 N_{a_3} N_{(b+1)}}
\]

(5.259)

\[
P_{(1,\ldots,1,0,\ldots,0)^2} = \pm \sum_{i \neq j} \sum_{k \neq l} \sum_{u \neq v} \sum_{w \neq x} \left( \frac{\alpha_{a_1} \alpha_{a_2} \alpha_{a_3} \partial_{\xi} N_{a_2}}{\alpha_{a_2} \alpha_{a_3}^2 \times \cdots \times \alpha_{a_1} \alpha_{a_2}^2} \times \ldots \times \alpha_{a_1} \alpha_{a_2}^2 \times \ldots \times \alpha_{a_1} \alpha_{a_2}^2 \right) \frac{\alpha_{a_1} \alpha_{a_2} \alpha_{a_3} \partial_{\xi} N_{a_2}}{N_1 N_{a_2} N_{(b+1)}}
\]

(5.260)

where \( b \leq n - 3 \) and the signs are determined in the same fashion. On the right side we find the extensions

\[
P_{(0,0,\ldots,0,1,\ldots,1,0)^2} = \pm \sum_{i \neq j} \sum_{k \neq l} \sum_{u \neq v} \sum_{w \neq x} \left( \frac{\alpha_{a_1} \alpha_{a_2} \alpha_{a_3} \alpha_{a_4} \alpha_{a_5} \partial_{\xi} N_{a_2}}{\alpha_{a_2} \alpha_{a_3}^2 \times \cdots \times \alpha_{a_1} \alpha_{a_2}^2} \times \ldots \times \alpha_{a_1} \alpha_{a_2}^2 \times \cdots \times \alpha_{a_1} \alpha_{a_2}^2 \right) \frac{\alpha_{a_1} \alpha_{a_2} \alpha_{a_3} \partial_{\xi} N_{a_2}}{N_{(a-1)} N_{a_1} N_{(a-2)} N_{(a-1)}^n}
\]

(5.261)

\[
P_{(0,0,\ldots,0,1,\ldots,1,1)^2} = \pm \sum_{i \neq j} \sum_{k \neq l} \sum_{u \neq v} \sum_{w \neq x} \left( \frac{\alpha_{a_1} \alpha_{a_2} \alpha_{a_3} \alpha_{a_4} \alpha_{a_5} \partial_{\xi} N_{a_2}}{\alpha_{a_2} \alpha_{a_3}^2 \times \cdots \times \alpha_{a_1} \alpha_{a_2}^2} \times \ldots \times \alpha_{a_1} \alpha_{a_2}^2 \times \cdots \times \alpha_{a_1} \alpha_{a_2}^2 \right) \frac{\alpha_{a_1} \alpha_{a_2} \alpha_{a_3} \partial_{\xi} N_{a_2}}{N_{(a-1)} N_{a_1} N_{(a-2)} N_{(a-1)}^n}
\]

(5.262)

\[
P_{(0,0,\ldots,0,1,\ldots,1,1,0)^2} = \pm \sum_{i \neq j} \sum_{k \neq l} \sum_{u \neq v} \sum_{w \neq x} \left( \frac{\alpha_{a_1} \alpha_{a_2} \alpha_{a_3} \alpha_{a_4} \alpha_{a_5} \partial_{\xi} N_{a_2}}{\alpha_{a_2} \alpha_{a_3}^2 \times \cdots \times \alpha_{a_1} \alpha_{a_2}^2} \times \ldots \times \alpha_{a_1} \alpha_{a_2}^2 \times \cdots \times \alpha_{a_1} \alpha_{a_2}^2 \right) \frac{\alpha_{a_1} \alpha_{a_2} \alpha_{a_3} \partial_{\xi} N_{a_2}}{N_{(a-1)} N_{a_1} N_{(a-2)} N_{(a-1)}^n}
\]

(5.263)

where \( 3 \leq a \). Both of these extensions, to the left and right, can be performed to obtain all of the fields with at most one copy of \( \alpha_{a-2} \). Of the roots with two copies of \( \alpha_{a-2} \), for all of the roots \( \alpha = (0, 1, 2, 2, 2, 1, 1) \) and \( \alpha = (1, 2, 2, 2, 1, 1) \) there exists some root \( \beta \) such that \( \alpha + \beta = \Theta \). A consequence of the observation (5.256) is that these ‘\( \Theta \)-dual’ roots have constants which are identical except for the exchange of \( \partial_{\xi} N_{2_j} \leftrightarrow \partial_{\xi} N_{2_j} \) and the field with lower height has an additional factor of \( N_{2_j}/N_{2_j} \). The only remaining \( P \)-fields are those with root \( (0, 1, 2, 2, 2, 1, 1) \), with \( 3 \leq a < b \leq n - 2^3 \), for which the field

\[\text{When } a = 2 \text{ the root } \beta = (1, 1, 0, 0, 0, 0) \text{ satisfies } \alpha + \beta = \Theta \text{ and when } a = 1 \text{ the root is } \beta = (0, 1, 2, 2, 1, 1). \text{ When } a \geq 3 \text{ there is no such } \beta.\]
is
\[ P_{(0,\cdots,0,1_n,\cdots,1_h,2,\cdots,2,1,1)^2} = \pm \sum_{i \neq j} \sum_{k \neq l} \sum_{w \neq \overline{w}} \sum_{x \neq \overline{x}} \left( \frac{\alpha(a-1)_a \alpha(a-1)_a \alpha_{a_1(a+1)} \alpha_{a_1(a+1)}}{\alpha_{a_1(a+1)}^2} \right) \times \cdots \times \alpha_{b_1b_2}^2 \times \frac{\alpha(b-1)_b \alpha(b-1)_b \alpha_{b_1(b+1)} \alpha_{b_1(b+1)}}{N_{(a-1)} N_{a_b} N_{b_1(b+1)}}. \]  

Note that the field (5.264) is very similar to (5.258) with a difference in the \(\alpha\)-indices. For these fields we expect to obtain an \(SL(2,\mathbb{R})\) field solution when

\[ q = (q_1, q_2, \cdots, q_{(n-2)_1}, q_{(n-2)_2}, q_{(n-1)_1}, q_n) \]

or whenever any combination of \(q_i \leftrightarrow q_j\) are switched. In order to understand the trick in index ordering which is implicit in our construction, we consider the constants of the solutions for two fields with roots \(\beta_1 = (0, 1, 1, 0, \cdots, 0)\) and \(\beta_2 = (0, 1, 1, 2 \cdots, 2, 1, 1)\). The numerators of \(c_\beta q^2\) are given by

\[ \beta_1 : \sum_{i \neq j} \sum_{k \neq l} \frac{\partial \xi N_2 \alpha_{12} \alpha_{2,3} \alpha_{2,3} \alpha_{3,4} \alpha_{3,4}^2}{\partial \xi N_2} \]

\[ = \frac{\partial \xi N_2 \alpha_{12} \alpha_{2,3} \alpha_{2,3} \alpha_{3,4} \alpha_{3,4}^2}{\partial \xi N_2} + \frac{\partial \xi N_2 \alpha_{12} \alpha_{2,3} \alpha_{2,3} \alpha_{3,4} \alpha_{3,4}^2}{\partial \xi N_2} \cdots \]

\[ \beta_2 : \sum_{i \neq j} \sum_{k \neq l} \frac{\partial \xi N_2 \alpha_{12} \alpha_{2,3} \alpha_{2,3} \alpha_{3,4} \alpha_{3,4}^2}{\partial \xi N_2} \]

\[ = \frac{\partial \xi N_2 \alpha_{12} \alpha_{2,3} \alpha_{2,3} \alpha_{3,4} \alpha_{3,4}^2}{\partial \xi N_2} + \frac{\partial \xi N_2 \alpha_{12} \alpha_{2,3} \alpha_{2,3} \alpha_{3,4} \alpha_{3,4}^2}{\partial \xi N_2} \cdots \]

It is straightforward to verify that the charge values \(q\) above that must give the \(SL(2,\mathbb{R})\) submodels do indeed return \(\pm q\) for the constants. Moreover, any allowable\(^{10}\) configuration of 0 and 1 entries which is not the one given (up to the mentioned permutations) yields a constant \(c_\beta = 0\).

The fact that there are only two choices for each sign makes the challenge of determining the appropriate signs for each \(c_\alpha^q\) manageable. The simplest method (which was used in the \(r = 4, 5\) examples) of fixing signs is to first identify the charge \(q\) which limits the full model to an individual root \(\alpha\). One can then choose the appropriate sign for the constant \(c_\alpha\) so that in this limit we recover the \(SL(2,\mathbb{R})/SO(2)\) constant \(c_\alpha \rightarrow \sqrt{-q^2}\). It only remains to multiply the constant by \(\sqrt{-q^2}\) so that any fields associated with non-compact elements of \(t\) will instead limit to the \(SL(2,\mathbb{R})/SO(1,1)\) solution. A quick example is the field \(P_{\alpha_1}\) in the \(D_4\) solution (5.174) with \(\epsilon = (+, -, +, +)\). Since the \(k_{\alpha_1}\) generates a compact group element this field should limit as

\[ P_{\alpha_1}^{b_1=0, b_2=0} = \pm iN^{-1} \partial N. \]  

\(^{10}\) Allowable in this context means that there exists a root \(\alpha = \sum a_i \alpha_i\) with \(a_i = q_i\).
The constant should be arranged so that the $N_2(\partial N_1)^2$ term in the radical is negative: $c_{\alpha_1} = \pm \sqrt{-a_{12}}a_{12}$. For a more challenging example consider the $P_{1(1,1,2,1,1,1)}$ of $E_6$ with the involution $\epsilon = (+, +, -, +, +, +)$ found in equation (5.239). In any of the limits where $N = N_1 = N_2 = N_4 = N_5 = N_6$ and $N_3 = N^2$ we should find the $SL(2, \mathbb{R})/SO(2)$ field solution. By considering one of those limits (say $b_1 = b_2 = b_3 = b_4 = b_5 = c = q$ and others zero) we can fix the sign as found in that expression.

### 5.4.3 $E_{n(n)}$ solutions

Having solved the $E_{6(6)}$ coset model with a particular involution, the solutions for arbitrary involutions can be obtained in exactly the method proposed above. The 70 and 128-dimensional cosets for $E_{7(7)}$ and $E_{8(8)}$ possess as many equations of motion and our solutions contain height($\Theta$) = 17 and 29 harmonic functions/charges, respectively. We will not write out the equations of motion or explicit solution for any involution but provide the method for generating an $E_{8(8)}$ solution with arbitrary involution. The solution can be obtained by extensive exploration of the possible simply-laced submodels which we have already solved.

The functional dependence for some $P_{\beta}$ field is in general given by a sum of functions $N_{i_j}$ over the index $j$. For example, we wrote

$$
\sum_{a \neq b \neq c} \frac{N_{a_b}N_{a_c}}{N_{a_a}} = \frac{N_{a_2}N_{a_3}}{N_{a_1}} + \frac{N_{a_3}N_{a_3}}{N_{a_2}} + \frac{N_{a_1}N_{a_2}}{N_{a_3}}
$$

in the $P_{\alpha_3}$ of $E_{6(6)}/Sp(8, \mathbb{R})$. The general functional dependence of the field $P_{\beta}$ in $E_{8(8)}$ is given, except for the special cases where the root $\beta$ has zero copies of $\alpha_i$ or the maximal (identical to the number in $\Theta$), by

$$
\prod_{i=1}^{8} \left( \frac{N_{a_i}}{\prod_{b \neq a} N_{b_b}} \right)^{-1} \langle \beta, \alpha_i \rangle^{1/2}
$$

where $x_i$ is the number of $\alpha_i$ in $\beta$. In those cases there is no sum over $a$ and instead we find simply $(N_i)^{-1/2}$ in the product. This rule for the functional dependence can be found by observing the $E_{6(6)}$ submodel solutions and then requiring $P$-fields to limit correctly in $SL(n, \mathbb{R})$ submodels which contain $P_{\beta}$.

Consider the simple root $\alpha_a$ which has $-1$ inner product with the roots $\alpha_b$, $\alpha_c$ and $\alpha_d$. If the highest root is expanded as $\Theta = w_1\alpha_a + x_1\alpha_b + y_1\alpha_c + z_1\alpha_d + \ldots$, then the simple root field $P_{\alpha_a}$ is given by

$$
P_{\alpha_a}^2 = \pm \sum_{i=1}^{w_1} \left( \prod_{j=1}^{x_1} \alpha_{a_b} \right) \left( \prod_{k=1}^{y_1} \alpha_{a_c} \right) \left( \prod_{l=1}^{z_1} \alpha_{a_d} \right) \left( \prod_{m \neq i} N_{a_m} \right)^2 \frac{N_{a_1}^2 N_{b_1} N_{c_1} N_{d_1}}{N_{a_1} N_{b_1} N_{c_1} N_{d_1}}.
$$

Roots $\alpha_i$ with fewer connections, such as $\alpha_1$ which only connect to one other node, will only contain $\alpha_{ij}$ constants and factors of $N_{j}$ when $(\alpha_i, \alpha_j) \neq 0$. An extremely useful observation,
which we have used in equation (5.256), is that for any two roots $\beta_1$ and $\beta_2$ which sum to form another root, the constants for the associated $P$-fields are related by

$$c_{\beta_1}c_{\beta_2} = \pm c_{\beta_1+\beta_2}\sqrt{X},$$

(5.270)

where $X$ is the collection of terms shared by both $c_{\beta_1}$ and $c_{\beta_2}$. Using this rule it is straightforward, if tedious, to calculate all of the constants which, combined with the rule for functional dependence, yields the full solution.

We mention one helpful trick which can be generalised from the previous work to make this task easier. The highest root $\Theta = (2,3,4,5,6,4,2,3)$ has $(\Theta, \alpha_i) = \delta_{i,1}$, so the equation of motion for $s_\Theta$ is solved by

$$P_{\Theta}^2 = \pm \frac{\partial_\xi N_{11} \partial_\xi N_{12}}{N_1}.$$  

(5.271)

All of the pairs $P_{\beta_1} P_{\beta_2}$ with $\beta_1 + \beta_2 = \Theta$ will have identical constant terms with an exchange of $\partial_\xi N_{11} \leftrightarrow \partial_\xi N_{12}$, just as in the $E_6(6)$ solution. In order to find all of the constants we need only get to $(1,1,1,2,3,2,1,1)$. The remaining fields, including those which are not $\Theta$-dual to those built up to this root, can be found by relatively simple additions of the simple root constants $c_{\alpha_i}$.

The correct signs for each $P_{\alpha}^2$ can be deduced by using the same method as described in the $D_n(n)$ models: arrange the terms in the numerator of $c_{\alpha}^2$ so that at the appropriate $SL(2,\mathbb{R})$ limit we find $c_{\alpha} = \sqrt{-q^2}$ and then multiply by $\sqrt{e_{\alpha}}$. 


Having found solutions for the one-dimensional cosets in the previous chapter we will now use them to find bound states of solutions in M-theory and type IIA/B string theory by embedding the algebras within \( E_{11}/K(E_{11}) \). This is equivalent to finding consistent finite truncations in the \( E_{11} \) \( \sigma \)-model discussed in [22]. In section 1 we review the \( SL(n, \mathbb{R}) \) \( n \leq 4 \) embeddings into \( l \leq 4 \) subalgebras of \( E_{11} \) and the bound state solutions that can be obtained [26]. In the next sections we consider a variety of embeddings of \( D_4 \) and \( E_6 \) which were identified, but not solved, in [1]. We conclude with a discussion of higher rank models and exotic solutions with \( l \geq 5 \) in \( E_{11} \).
6.1 Review of $A_n(n)$ Solutions

In the brane $\sigma$-model of $E_{11}$ all of the tensor representations at $l \geq 1$ transform under the temporal involution invariant level zero generators, which exponentiate to form the group $SO(1, 10)$. While the indices of the group are not a priori related, we choose to identify them with spacetime. Taking the Borel-gauged group element

$$g = \exp \left\{ \sum_{i=1}^{11} \phi_i(\xi) H_i \right\} \exp \left\{ \sum_{\alpha \in \Delta(g)} A_\alpha(\xi) E_\alpha \right\}$$  \hspace{1cm} (6.1)

where all of the Cartan fields $\phi_i$ transform as, and therefore can be identified with, the vielbein of spacetime, while all of the $A_\alpha$ transform as tensors. In this process we find that the vielbein are given by

$$g_{\mu\nu} = \left( e^{-\phi} \right)_\mu^a \left( e^{-\phi} \right)_\nu^b \eta_{ab}$$  \hspace{1cm} (6.2)

and that each of the gauge fields $A_\alpha$ and field strengths $P_\alpha$ are given in the locally orthonormal frame.

6.1.1 Single brane $SL(2, \mathbb{R})/SO(1, 1)$ solutions

The single root model has a two-dimensional coset with fields $\phi$ and $P$ which solve the equations of motion when

$$\phi = 1/2\log(a + b\xi) = 1/2\log N$$

$$P = e^{2\phi} \partial_\xi A = N \partial_\xi N^{-1}.$$  \hspace{1cm} (6.3)

When we embed the single root $\alpha$ within $E_{11}$ the Cartan generator $H$ will take on the $gl(11)$ indicies of the corresponding $E_{11}$ Cartan generator $H_\alpha$ such that $[H_\alpha, E_\alpha] = 2E_\alpha$. These indices then define the relationship between the spacetime metric and the field $\phi(\xi)$. The gauge field $A$ then assumes the tensor representation associated with the level of $\alpha$. The field strength $P$ contains a derivative of the geodesic parameter $\xi$ which we identify with one of the transverse coordiantes to the brane solution. The resulting field strength and metric describes a supergravity solution for all of the roots with levels $l \leq 3$.

The $l = 0$ $KK$-wave

At level zero the generators are simply elements of $GL(11)$ which generate additional vielbein. We have chosen a basis where the Cartan fields give a diagonal metric and the new $A$ produces off-diagonal modifications. Consider the root $\alpha_{10}$ which has

$$H_{\alpha_{10}} = K_{10}^{10} - K_{11}^{11} \hspace{1cm} E_{\alpha_{10}} = K_{10}^{10}.$$  \hspace{1cm} (6.4)

In order to have the $SL(2, \mathbb{R})/SO(1, 1)$ model we need an involution which takes $\Omega(E_{\alpha_{10}}) = +E_{\alpha_{10}}$, which is equivalent to taking $dx_{10} = t$. The solution from the coset model indicates
that the Cartan field \( \phi = 1/2 \log N \) while the gauge field \( A = N^{-1} + c \) for arbitrary constant \( c \). The vielbein generated by identifying these fields are

\[
e_{10}^{10} = N^{-1/2} \quad e_{10}^{11} = A \quad e_{11}^{11} = N^{1/2}
\]

so that the metric is

\[
ds^2 = d\Omega_9^2 - \frac{1}{N} dt^2 + N (dx_{11} + Adt)^2
\]

(6.5)

where \( d\Omega_9^2 \) is the \( n \)-dimensional Euclidean line element. In the one-dimensional result we have only identified \( N \) as a function of one coordinate in the nine-dimensional transverse space. We can unsmear the solution by increasing the dimension of the space that \( N \) is a function of while maintaining the dependence on the radial coordinate \( r = \sum_{i=1}^{d} x_i^2 \) and taking the harmonic function to be \( N = 1 + kr^{-d+2} \) for \( d \geq 3 \). When we unsmear to the full transverse space we recover the \( KK \)-wave [80].

**The \( l = 1 \) M2-brane**

For a level one solution take the root \( \beta = \alpha_{11} = e_9 + e_{10} + e_{11} \) and a temporal involution which identifies \( x_{11} \) as time, so that

\[
\Omega(E\beta) = +F_\beta.
\]

The Cartan elements with \((e^{-\phi})_\mu^a\) generate vielbein for this solution exactly as we found in section 4.3.1

\[
ds^2 = N^{1/3} \left( dx_1^2 + \ldots dx_8^2 \right) + N^{-2/3} \left( dx_9^2 + dx_{10}^2 + dt^2 \right),
\]

but the justification for the field strength no longer requires an ansatz for the gauge field. Having solved the equations of motion we found that

\[
A_{91011} = N^{-1} + c \quad F_{91011} = N \partial_t N^{-1}.
\]

(6.8)

After identifying \( \xi \) with any one of the transverse coordinates \( \hat{i} \) and transforming to curved-space coordinates we find the field strength

\[
F_{\hat{i}91011} = \partial_{\hat{i}} N^{-1} \quad \Rightarrow \quad F_{[4]} = dN^{-1} \wedge dx_9 \wedge dx_{10} \wedge dx_{11}.
\]

(6.9)

After unsmearing the solution to the full transverse space with harmonic function \( N = 1 + k/r^6 \) we recover the \( M2 \)-brane solution.

**The \( l = 2 \) M5-brane**

The level two root \( \beta = (0, 0, 0, 0, 0, 1, 2, 3, 2, 1, 2) \) has a Cartan element

\[
H_\beta = \frac{2}{3} \left( K_1^1 + \ldots + K_5^5 \right) + \frac{1}{3} \left( K_6^6 + \ldots + K_{11}^{11} \right)
\]

(6.10)
CHAPTER 6. BOUND STATES OF BRANES

which generates the metric
\[ ds^2 = N^{1/3} (dx_1^2 + \ldots + dx_5^2) + N^{-2/3} (dx_6^2 + \ldots + dx_{10}^2 - dt^2), \]  
again exactly as in section 4.3.1. The gauge field \( A \) and field strength \( P \) are now promoted to six- and seven-forms in the locally flat coordinates
\[ A_{67891011} = N^{-1} + c \quad F_{\xi67891011} = N\partial_\xi N^{-1} \]  
so that the curved-space field strength is given by
\[ F_{\xi1\bar{6}\cdots11} = \partial_{\xi1} N^{-1}. \]  
The dualised field strength is then
\[ F_{i_1\bar{i}_2\cdots\bar{i}_5} = \partial_{i_1} N, \]  
where every \( i \) coordinate is within the space transverse to the brane coordinates. After unsmeasuring the solution to the full transverse space with harmonic function \( N = 1 + k/r^3 \) we recover the \( M5 \)-brane solution.

The \( l = 3 \) \( KK6 \)-brane

The level three root \( \beta = (0, 0, 0, 1, 2, 3, 4, 5, 3, 1, 3) \) forms the lowest root of the mixed symmetry \([8, 1]\)-form tensorm, which can be dualised into a vielbein. The Cartan element
\[ H_\beta = H_4 + 2H_5 + 3H_6 + 4H_7 + 5H_8 + 3H_9 + H_{10} + 3H_{11} \]  
\[ = - (K^1_1 + K^2_2 + K^3_3) + K^{11}_{11} \]  
generates the diagonal metric
\[ ds^2 = N (dx_1^2 + dx_2^2 + dx_3^2) + N^{-1} dx_{11}^2 + d\Omega^2_{1,6}. \]  
The field strength from the coset solution gives
\[ F_{\xi4567891011} = N\partial_\xi N^{-1} \quad \Rightarrow \quad F_{i\bar{4}\cdots11} = \partial_{i1} N^{-1} \]  
so that the dualised field strength is
\[ F_{j\bar{k}11} = \partial_{j1} A_{k\bar{1}}^{11} = \partial_{j1} N \]  
where we have introduced the new dual gauge field \( A \). With this new off-diagonal vielbein the metric is generally
\[ ds^2 = N (dx_1^2 + dx_2^2 + dx_3^2) + N^{-1} (dx_{11}^2 + A \cdot dx) + d\Omega^2_{1,6}. \]  
When the gauge field is unsmeared over the full three-dimensional transverse space the equation defining the dual field is
\[ dA = \ast dN \]  
which completes the description of the \( KK6 \)-brane solution. Note that this is a purely gravitational solution and that the four-dimensional submanifold with coordinates \( x_1, x_2, x_3, x_{11} \) is the Euclidean Taub-NUT space [81].
6.1.2 $SL(2, \mathbb{R})/SO(2)$ and space-like solutions

In section 5.1.2 we found that the solutions for null geodesics mapped from a one-dimensional space possessed complex positive generator fields $A_\alpha$. These solutions correspond to supergravity solutions, known as space-like solutions or $S$-branes, without a time coordinate within their worldvolume. These were originally found in string theory in [82] and the eleven-dimensional solutions were first shown in [83], where they are referred to as $SM2/5$-branes. This was later extended with multiple brane intersection rules in [84].

It was observed in [85] that space-like brane solutions can be obtained from a truncated supergravity solution with opposite sign on the $F^2$ term in the Lagrangian. In this case the field strength is explicitly imaginary. This was discussed in a Kac-Moody/supergravity context in [86] where solutions are considered for a variety of space-time signatures. It was pointed out by Cook and West that signature change can be achieved by Wick rotating coordinates while leaving a metric unaltered, and that this may introduce factors of $i$ on the Lie algebra fields of a coset group element. For example, taking the $KK$-wave solution above (6.6) and Wick rotating the coordinates $dx_1$ and $dx_{10}$, we would obtain the $SKK$-wave solution

$$ds^2 = d\Omega^2_{1,8} + (1 - K)dx^2_{10} - 2iKdx_{10}dx_{11} + (1 + K)dx^2_{11},$$

(6.21)

where the time coordinate is contained within the transverse coordinates of $d\Omega^2_{1,8}$ and $K$ is a harmonic function of the transverse space. From the group theory perspective, the change in involution from $H = SO(1, 1) \rightarrow SO(2)$ yields a coset Lagrangian with opposite sign for the field strength $F^2_\alpha$ term and if we proceed with our complex solutions we will find precisely the $SKK$-wave solution at level 0. While the bound states described in this chapter contain space-like solutions for particular charge choices, we will describe motion through the space of real solutions as developed in the previous chapter.

6.1.3 $n$-brane $SL(n + 1, \mathbb{R})$ bound state solutions

Constructing $SL(n, \mathbb{R})$ bound brane models requires us to find which configurations of root in $E_{11}$ provide the correct inner products. From an algebraic perspective we are constrained in our constructions of subalgebras by the rank of the algebra and possible combinations of roots, while from a physical perspective we will also want to maintain as many transverse dimensions as possible and potentially stay within levels $l \leq 3$ in order to apply the Kac-Moody/Supergravity dictionary. For example, a perfectly reasonable algebra may contain roots of table 6.1.1, in the $e_i$ basis. This collection of roots forms the algebra $A_8$ and has two transverse dimensions but various limits of the (extended-)supergravity solution have fields with unusual mixed-symmetry tensors, such as the $[9, 3], [9, 6]$ and $[9, 8, 1]$. We will discuss these constructions in the last section.

If we limit ourselves to eleven-dimensional supergravity solutions which correspond to roots whose sum is at most level $l = 3$ we find relatively few possible configurations. If we restrict this further by eliminating $l = 0$ solutions the largest remaining algebra is the
CHAPTER 6. BOUND STATES OF BRANES

Table 6.1.1: Embedding $A_8$ using level $l = 1$ roots in $E_{11}$

<table>
<thead>
<tr>
<th>Root</th>
<th>1</th>
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<tbody>
<tr>
<td>$\beta_1$</td>
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<td>$\beta_4$</td>
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<td>$\beta_7$</td>
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<td>$\beta_8$</td>
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$SL(4, \mathbb{R})$ which was described in [26]. In ten-dimensional supergravity we can also include a level $l = 4$ solution and build the $SL(5, \mathbb{R})$ which was described in [1].

**Dyonic Membrane within a KK6-brane**

We can construct an $A_3$ of simple roots, as shown in table 6.1.3, which are all level $l = 1$ in $E_{11}$ and with a choice of $x_{10} = t$ so that the involution $\epsilon = (-, +, +)$ generates the coset $SL(4, \mathbb{R})/SO(1, 3)$. The equations of motion and solution are discussed in section 5.3.1. The

Table 6.1.2: Roots of $E_{11}$ generating the dyonic membrane within a KK6-brane.

<table>
<thead>
<tr>
<th>Branes</th>
<th>1</th>
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<th>8</th>
<th>9</th>
<th>10(t)</th>
<th>11</th>
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<tr>
<td>M2</td>
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</table>

Cartan fields are $\phi_i = 1/2\text{Log}N_i = 1/2\text{Log}(a_i + q_i\xi)$ and we choose to set $a_i = 1$ in order to obtain asymptotic flatness. The charges, as mapped under the compact symmetries, are

$$q_1 \equiv q, \quad q_2 = q(1 - \cos^2(\theta_{(0,1,0)}))\cos^2(\theta_{(0,1,1)})), \quad q_3 = q\sin^2(\theta_{(0,1,1))}) \quad (6.22)$$
and the $P$-fields are therefore

$$P_{(1,0,0)} = P_{ξ91011} = -\cos(θ_{(0,1,0)})cos(θ_{(0,1,1)}) \frac{∂_k N_1}{N_1 \sqrt{N_2}}$$

$$P_{(0,1,0)} = P_{ξ678} = -\cos(θ_{(0,1,0)})\sin(θ_{(0,1,0)})\cos^2(θ_{(0,1,1)}) \frac{∂_k N_2}{1 - \cos^2(θ_{(0,1,0)})\cos^2(θ_{(0,1,1)})} N_2 \sqrt{N_1 N_3}$$

$$P_{(0,0,1)} = P_{ξ4511} = -\sin(θ_{(0,1,0)})\cot(θ_{(0,1,1)}) \frac{∂_k N_3}{N_3 \sqrt{N_2}}$$

$$P_{(1,1,0)} = P_{ξ67891011} = -\sin(θ_{(0,1,0)})\cos(θ_{(0,1,1)}) \frac{∂_k N_1}{\sqrt{N_1 N_2 N_3}}$$

$$P_{(0,1,1)} = P_{ξ4567811} = -\cos(θ_{(0,1,0)})\cot(θ_{(0,1,1)}) \frac{∂_k N_3}{\sqrt{N_1 N_2 N_3}}$$

$$P_{(1,1,1)} = P_{ξ456789101111} = -\sin(θ_{(0,1,1)}) \frac{∂_k N_1}{\sqrt{N_1 N_3}}.$$  \hspace{1cm} (6.23)

The last of these fields introduces an off-diagonal vielbein $A_{11}$ so that the metric is

$$ds^2 = Λ(\sum dx_i^2 + dx_9^2 + dx_{10}^2 + dx_{11}^2 + N_3^{-1}(dx_1^2 + dx_2^2) + N_2^{-1}(dx_6^2 + dx_7^2 + dx_8^2))$$

$$+ N_1^{-1}(dx_3^2 - dt^2) + N_1^{-1}N_3^{-1}(dx_{11} + A \cdot dx)^2.$$  \hspace{1cm} (6.24)

where $Λ = (N_1 N_2 N_3)^{1/3}$. Each of the other $P$-fields contributes to the four-form field strength directly or through dualisation. After identifying $ξ$ with a transverse direction, transforming to curved-space coordinates and unsmeasuring, the field strength is

$$F_{[4]} = \cos(θ_{(0,1,0)})\cos(θ_{(0,1,1)})dN_1^{-1} \wedge dx_9 \wedge dx_{10} \wedge dx_{11}$$

$$- \frac{\cos(θ_{(0,1,0)})\sin(θ_{(0,1,0)})\cos^2(θ_{(0,1,1)})}{1 - \cos^2(θ_{(0,1,0)})\cos^2(θ_{(0,1,1)})}dN_2^{-1} \wedge dx_6 \wedge dx_7 \wedge dx_8$$

$$+ \sin(θ_{(0,1,0)})\cot(θ_{(0,1,1)})dN_3^{-1} \wedge dx_4 \wedge dx_5 \wedge dx'_{11}$$

$$+ \sin(θ_{(0,1,0)})\cos(θ_{(0,1,1)})dN_1^{-1}(⋆dN_3) \wedge dx_9 \wedge dx_{10}$$

$$+ \cos(θ_{(0,1,0)})\cot(θ_{(0,1,1)})dN_3^{-1}(⋆dN_1) \wedge dx_4 \wedge dx_5.$$  \hspace{1cm} (6.25)

where $dx'_{11} = dx_{11} + \sum_{i=1}^{3} A_i^{11}dx_i$. This solution, which interpolates between the $M$-brane ($θ_{(0,1,0)} = θ_{(0,1,1)} = 0$), the $M5$-brane ($θ_{(0,1,0)} = π/2$, $θ_{(0,1,1)} = 0$) and the $KK6$-brane ($θ_{(0,1,1)} = π/2$), was originally shown in [26].

### 6.2 Branes with $D_4$ Configuration

As mentioned in section 6.1.3, we cannot embed subalgebras of rank greater than three in $E_{11}$ unless we: (1) allow $l = 0$ roots, (2) consider ‘exotic’ objects with mixed-symmetry tensor fields or (3) take the ten-dimensional string theory configurations of [1]. We consider in this section examples which include $KK$-waves and string theory bound states.

---

1Note that there is a difference in symmetry parameters and $P_{(0,1,0)}$ constant in our presentation.
6.2.1 Example: $KK$-Wave $\subset M2^3 \subset M5^3 \subset KK6$

As our first case we consider, as the simple roots, a collection of eleven-dimensional branes with the following orientations: These branes are identified with the $E_{11}$ roots:

<table>
<thead>
<tr>
<th>Branes</th>
<th>1</th>
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<th>4(t)</th>
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Table 6.2.1: Configuration of $E_{11}$ roots in $e_i$ basis and simple roots for $KK$-Wave $\subset M2^3 \subset M5^3 \subset KK6$ $D_4$ model.

$$\beta_1 = e_9 + e_{10} + e_{11} \quad \beta_2 = e_4 - e_{11} \quad \beta_3 = e_7 + e_8 + e_{11} \quad \beta_4 = e_5 + e_6 + e_{11},$$

which form a $D_4$ algebra. The $E_{11}/\text{supergravity}$ dictionary allows us to identify the four Cartan fields $\phi$ with vielbein of the solution. The vielbein are given by

$$e_i^j = \exp \left( \sum_k \phi_k (\beta_k \cdot H) \right)^j_i = \exp \left( f(N_1, N_2, N_3, N_4)K^{j,i} \right)$$

where we are now considering the Cartan generators of $E_{11(11)}$ which can be decomposed as representations of $SL(11, \mathbb{R})$. For this example we find that

$$\sum_k \phi_k (\beta_k \cdot H) = \frac{1}{3} \left( K^1 + K^2 + K^3 \right) (N_1 + N_3 + N_4) + K^4 \left( \frac{1}{3} (N_1 + N_3 + N_4) - N_2 \right)$$

$$+ (K^5 + K^6) \left( \frac{1}{3} (N_1 + N_3) - \frac{2}{3} N_4 \right) + (K^7 + K^8) \left( \frac{1}{3} (N_1 + N_4) - \frac{2}{3} N_3 \right)$$

$$+ (K^9 + K^{10}) \left( \frac{1}{3} (N_3 + N_4) - \frac{2}{3} N_1 \right) + K^{11} \left( N_2 - \frac{2}{3} (N_1 + N_3 + N_4) \right)$$

and the diagonal metric is given by

$$ds^2 = \Lambda (dx_1^2 + dx_2^2 + dx_3^2) + \Lambda (-N_2^{-1} dx_1^2 + N_4^{-1} (dx_5^2 + dx_6^2))$$

$$+ \Lambda (N_3^{-1} (dx_2^2 + dx_3^2) + N_1^{-1} (dx_5^2 + dx_6^2)) + \Lambda^{-2} N_2 dx_{11}^2$$

where $\Lambda = (N_1 N_3 N_4)^{1/3}$. The charges are mapped under the compact symmetries as

$$q_{21} \equiv q \quad q_{22} = -q \sin^2(\theta_2)$$

$$q_i = q (\sin^2(\theta_i) - \sin^2(\theta_2)) \quad i = 1, 3, 4$$

where the parameters have been relabeled for convenience as $\theta_{\alpha_i} = \theta_i$ and $\theta_{(1,2,1,1)} = \theta_2$. The $\alpha_{ij}$ terms are functions the parameters of the four compact generators:

$$\alpha_{121} = N_2 \partial N_1 - N_1 \partial N_2 = -\sin^2(\beta_1)$$

$$\alpha_{122} = N_2 \partial N_1 - N_1 \partial N_2 = \cos^2(\beta_1)$$

$$\alpha_{212} = N_2 \partial N_1 - N_2 \partial N_2 = 1$$

$$\alpha_{212} = N_2 \partial N_1 - N_2 \partial N_2 = 1$$

$$\alpha_{212} = N_2 \partial N_1 - N_2 \partial N_2 = 1$$
CHAPTER 6. BOUND STATES OF BRANES

with symmetric remaining terms. Each of the $P_\alpha$ fields are identified with a field strength in flat-space coordinates according to the $E_{11}$ dictionary and the $R$ element $\xi$ is identified with a coordinate of the mutually transverse space. One of the field strengths will be

$$F_{\xi 91011} = P_{(1,0,0,0)} = \frac{\sqrt{\alpha_{121} \alpha_{122}}}{N_1 \sqrt{N_2}}$$

which we can then transform using the vielbein (6.29)

$$F_{\xi 91011} = e_9^g e_{10}^i e_{11}^j P_{(1,0,0,0)} = \frac{\sin(\beta_1) \cos(\beta_1)}{N_1^2}$$

$$= - \frac{\sin(\beta_1) \cos(\beta_1)}{\cos^2(\beta_2) - \sin^2(\beta_1)} \partial_N N_1^{-1}.$$ (6.34)

We have arrived at the one-dimensional description of the field strength for the $S^2$-brane which can be unsmeared over the three mutually transverse coordinates. We are led to the form of (6.34) by group theoretic definition of the $P_\alpha$. In this simple case the gauge field is

$$A_{(1,0,0,0)} = - \sqrt{\alpha_{121} \alpha_{122}} \frac{1}{N_1}.$$ (6.35)

Each of gauge fields for simple root generators are solved directly and the solutions for generators with root $\beta = a_i \alpha_i$, where $a = \sum a_i$ can be inductively solved with the solutions from $a - 1$. The gauge field solutions for this involution are, without repetition for symmetric solutions:

$$A_{(1,0,0,0)} = - \sqrt{\alpha_{121} \alpha_{122}} \frac{1}{N_1} A_{(0,1,0,0)} = - \sum_{i \neq j} \sqrt{\alpha_{121} \alpha_{23} \alpha_{24} \alpha_{12}} \frac{1}{N_1}$$

(6.36)

$$A_{(1,1,0,0)} = \sum_{i \neq j} \sqrt{\alpha_{121} \alpha_{23} \alpha_{24} \alpha_{12}} \frac{1}{N_1} \left( \frac{1}{2N_1 \partial N_2} + \frac{1}{2N_2 \partial N_1} \right)$$

(6.37)

$$A_{(1,1,1,0)} = \sum_{i \neq j} \sqrt{\alpha_{121} \alpha_{23} \alpha_{24} \alpha_{12}} \frac{1}{N_1} \left( \frac{1}{3N_1 \partial N_3} + \frac{1}{3N_3 \partial N_1} \right)$$

(6.38)

$$+ \frac{\partial N_2}{6N_2 \partial N_1 \partial N_3} + \frac{N_2}{6N_1 \partial N_3 \partial N_2}$$

(6.39)

$$A_{(1,1,1,1)} = \frac{1}{12} \sum_{i \neq j} \sqrt{\alpha_{121} \alpha_{23} \alpha_{24} \alpha_{12}} \frac{3N_2^2}{N_1 N_3 N_4 \partial N_2 j}$$

$$+ \frac{N_2}{2N_1 N_3 N_4 \partial N_1 \partial N_3 \partial N_4} + \frac{3(\partial N_2)^2}{N_2 \partial N_1 \partial N_3 \partial N_4}.$$ (6.40)
The rest of the field strengths are

\[
F_{\xi^8\xi^1} = -\frac{\sin(\beta_3)\cos(\beta_3)}{\cos^2(\beta_2) - \sin^2(\beta_3)} \partial_\xi N_3^{-1} \\
F_{\xi^9\xi^0} = \frac{\sin(\beta_1)\cos(\beta_2)\cos(\beta_3)\sin(\beta_2)}{N_1 N_2} + \frac{\cos(\beta_1)\sin(\beta_3)\sin(\beta_4)\cos(\beta_2)}{N_1 N_2} \\
F_{\xi^7\xi^8} = \frac{\cos(\beta_1)\sin(\beta_3)\cos(\beta_4)\sin(\beta_2)}{N_2 N_3} + \frac{\sin(\beta_1)\cos(\beta_3)\sin(\beta_4)\cos(\beta_2)}{N_2 N_3} \\
F_{\xi^6\xi^5} = \frac{\cos(\beta_1)\cos(\beta_2)\sin(\beta_4)\sin(\beta_2)}{N_2 N_4} + \frac{\sin(\beta_1)\sin(\beta_3)\cos(\beta_4)\cos(\beta_2)}{N_2 N_4}
\]

and the remaining three fields are related to purely gravitational terms. The \( A_{(0,1,0,0)} \) gauge field is directly associated with the vielbein while the \( P_{(1,1,1,1)} \) fields must be dualised over the 11-dimensional space before the vielbein can be identified. This ansatz is exactly what has been used to construct the KK-wave and KK6-monopole solutions, respectively. We find the remaining (dual-)field strengths:

\[
F_{\xi^4^i} = -\frac{\sin(\beta_1)\sin(\beta_3)\sin(\beta_4)}{\cos(\beta_2)} \partial_\xi N_2^{-1} - \frac{\cos(\beta_1)\cos(\beta_2)\cos(\beta_4)}{\sin(\beta_2)} \partial_\xi N_2^{-1} \\
F_{\xi^i^j} = \epsilon_{ijk} \left( \frac{\sin(\beta_1)\sin(\beta_3)\sin(\beta_4)}{\cos(\beta_2)} \partial_\xi N_2^{-1} + \frac{\cos(\beta_1)\cos(\beta_2)\cos(\beta_4)}{\sin(\beta_2)} \partial_\xi N_2^{-1} \right)
\]

The full unsmeared solution then requires us to solve these differential equations in their 3-dimensional setting. We will require that the unsmeared functions possess spherical symmetry. If in three dimensions the functions are

\[
N_i = 1 + g_i(\beta_1, \beta_2, \beta_3, \beta_4) f(r),
\]

where \( \Delta f(r) = 0 \), then the dual gravity field (6.50) is solved by

\[
B_i^\lambda = \sin(\beta_2)\cos(\beta_2) B_i
\]

where \( B_i \) is the conjugate of \( f(r) \), \( dB = *df(r) \). We can find an explicit solution and the equations (6.49) can be easily solved if we consider polar coordinates and set \( B_\theta = 0 \). The conjugate to \( f(r) = 1/r \) is then \( B_\theta = \cos(\theta) \).

The solution which we have presented contains several branes as limits of the compact sub-group parameter space. For each root \( \beta \) such that \( \Omega(E_\beta) = +F_\beta \) we obtain a brane which should be reproduced in some limit. For this involution we find 8 such objects which include three distinct \( M2 \)-branes, three \( M5 \)-branes, one \( KK \)-wave and one \( KK6 \)-brane. Sub-group
Figure 6.2.1: Simple $SO(2, 2) \times SO(2, 2)$ compact symmetry transformations from the point $KK$-wave point in the solution space. The parameters are color coded so that $\theta_{\alpha_1}$ is blue, $\theta_{\alpha_3}$ is violet and $\theta_{\alpha_4}$ is green.

motion through these branes is depicted in a particular diagram of figure 5.3.6 shown in figure 6.2.1. Each of the vertices corresponds to a choice of the $\beta_i$ parameters and motion along the black lines is generated by the $k_{(1,0,0,0)}$, $k_{(0,0,1,0)}$ and $k_{(0,0,0,1)}$ symmetries. The $k_{1,2,1,1}$ symmetry acts to take us between branes whose associated roots $\beta_1 + \beta_2 = (1, 2, 1, 1)$ and is depicted with green curves. Note that the $k_{(1,2,1,1)}$ symmetry, whose action is not depicted here, switches the sign of the charges in the harmonic functions when acting on a brane limit.

6.2.2 Example: $M^2 \subset M^5 \subset KK^6^2$

If we consider the same root embedded in $E_{11}$ with time chosen as $x_{10}$, we find the $SO(4, 4)/(SO(1, 3) \times (SO(1, 3))$ model with involution $\epsilon = (-, +, +, +)$. In this case the equations of motion will

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Table 6.2.2: Configuration of $E_{11}$ roots in $e_i$ basis and simple roots for $M^2^3 \subset M^5^3 \subset KK^6^2$ $D_4$ model.

differ from those in appendix B.1 only by signs. The solutions constructed in the previous section will be valid for this model with appropriate sign modifications. Unfortunately, the details of the model are entirely within the signs and we must treat them with care. However, after calculating the equations of motion for one involution it is straightforward
to modify them for another. The Cartan equations of motion contain $\epsilon_\beta P_\beta^2$ terms where $\Omega(E_\beta) = \epsilon_\beta F_\beta$. These terms will switch sign if the $\epsilon_\beta$ switches with the new involution. In the $S_\alpha$ equations the sign of each $P_\alpha P_\beta$ term only changes if $\epsilon_\beta$ is switched. For example, when we calculated the $SL(3, \mathbb{R})$ equations of motion with $\epsilon = (-, +)$ we found the $H_1$ equation

$$0 = \partial_\xi^2 \phi_1 + \frac{1}{2} P_{(1,0)}^2 + \frac{1}{2} P_{(1,1)}$$

(6.51)

taking each of the other possible $\epsilon$ we would find

$$\epsilon = (+, +) : 0 = \partial_\xi^2 \phi_1 - \frac{1}{2} P_{(1,0)}^2 - \frac{1}{2} P_{(1,1)}$$

(6.52)

$$\epsilon = (+, -) : 0 = \partial_\xi^2 \phi_1 - \frac{1}{2} P_{(1,0)}^2 + \frac{1}{2} P_{(1,1)}$$

(6.53)

$$\epsilon = (-, -) : 0 = \partial_\xi^2 \phi_1 + \frac{1}{2} P_{(1,0)}^2 - \frac{1}{2} P_{(1,1)}.$$  

(6.54)

For this particular embedding (6.2.2) in $E_{11}$ we find a metric and field strength which are both similar to the previous section, but the sub-group parameters modify the harmonic functions differently. While there are six compact generators for the sub-group, two of them are redundant. We only possess six distinct brane solutions which we can map between. These include two $M2$-branes, two $M5$-branes and two $KK6$-branes.

### 6.2.3 Type IIA Bound States

As discussed in section 4.3.2 we can also identify solutions to Type IIA/B supergravity theories using a decomposition of $E_{11}$ with respect to two nodes. Since there are more fundamental solutions in these theories the possible spatial orientations for bound states are more numerous and we can construct more interesting non-exotic subalgebras with higher rank than in eleven-dimensional supergravity. The reduction of table 6.2.1 to type IIA supergravity results in a combination of D-branes, shown in table 6.2.3, which result from the $KK$-wave, $M2^3$, $M5^3$ and $KK6$, respectively. Another bound state which involves the

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Table 6.2.3: Roots of $E_{11}$ in $e_i$ basis associated with the $D0 \subset D2^3 \subset D4^3 \subset D6$ bound state.

extension to the massive Romans type IIA theory is the similar model which contains $l_{11} =$
1, 2, 3, 4 roots associated with D2, D4, D6 and D8 branes. One configuration is presented in table 6.2.3. Both of these $D_4$ models have involution $\epsilon = (+, -, +, +)$ so that the coset

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Table 6.2.4: Roots of $E_{11}$ in $e_i$ basis associated with the $D_2 \subset D_4^3 \subset D_6^3 \subset D_8$ bound state.

is $SO(4, 4)/(SO(2, 2) \times SO(2, 2))$ and we can again apply the solution described in section 5.3.2.

### 6.3 Bound States of Branes with $E_n$ Configurations

The exceptional algebras contain at highest roots $\Theta$ of height 11, 17 and 29, so each of the solutions presented in section 5.4.3 will limit to some solution associated with at least $h(\Theta)$-many copies of the simple root solutions. Unless most of the simple roots are level $l_{11} = 0$ in $E_{11}$, the bound state will clearly limit to exotic solutions. We are therefore limited to considering bound states which mostly consist of $KK$-waves mixed with a few low level branes.

#### 6.3.1 Non-Exotic $E_6$ Bound States

Consider the set of $E_{11}$ roots $\alpha_i$ with $i \geq 6$. These form the most obvious $E_6$ subalgebra, although we cannot use the $\epsilon = (+, +, -, +, +, +)$ involution solution. Taking the six $E_6$ simple roots $\beta$ to be the $E_{11}$ roots $\beta_i = \alpha_{i+5}$ we could set time $t = x_{11}$ and find the $\epsilon = (+, +, -, +, +, +)$, for example, which generates the coset $E_{6(6)}/Sp(2, 2)$. The number of non-compact generators in $t$ is equal to the number of $P$-fields which limit to $SL(2, \mathbb{R})/SO(1, 1)$ solutions for a certain location in the charge $q$-space. These correspond with all of the time-like supergravity solutions:

$KKW : [6, 11], [7, 11], [8, 11], [9, 11], [10, 11]$

$M2 : [6711], [6811], [6911], [61011], [7811],$

$\quad [7911], [71011], [8911], [81011], [91011]$  \quad (6.55)

$M5 : [67891011].$
If we had chosen to take $t = x_6$ we would have found the same involution invariant subalgebra and the same number/type of time-like solutions which appear in limits, only with different coordinates. Of course these both have the maximum number of transverse dimensions. The only other involution invariant subalgebra, with some non-compact generators, that can be produced with $\epsilon$ involutions is $\mathfrak{sp}(8, \mathbb{R})$. There is no choice of $t = x_i$ for the $\beta_i = \alpha_{i+5}$ configuration of roots which produces this subalgebra, but instead consider the subalgebra

$$
\beta_1 = \alpha_4, \quad \beta_2 = \alpha_5, \quad \beta_3 = \alpha_6, \quad \beta_4 = \alpha_7 \tag{6.56}
$$

$$
\beta_5 = \alpha_8 + \alpha_{11}, \quad \beta_6 = (0, 0, 0, 0, 0, 1, 2, 2, 1, 1) \tag{6.57}
$$

for which the $E_{11}$ generators are

$$
K_i^{i+1} R_{81011}^{8i} R_{789}^{7i} \quad i = 4, 5, 6, 7. \tag{6.58}
$$

Taking $t = x_9$ gives an involution $\epsilon = (+, +, +, +, +, -)$ and generates the $\mathfrak{sp}(8, \mathbb{R})$ invariant subalgebra with twenty non-compact generators. These twenty generators are associated with the twenty time-like brane limits, of which ten are $M2$-branes and ten are $M5$-branes:

$$
M2 : [459], [469], [479], [489], [569], [579], [589], [679], [689], [789]
$$

$$
M5 : [45691011], [45791011], [45891011], [46791011], [46891011], [47891011], [56791011], [56891011], [57891011], [67891011]. \tag{6.59}
$$

If we limit ourselves to constructing non-exotic $E_6$ submodels there are no possible configurations with $l = 2, 3$ roots\(^2\), so any other embedded $E_6$ would have to include exotic objects.

### 6.3.2 The Non-Exotic $E_8$ Bound State

The only non-exotic solution which is an $E_8$ bound state includes one level one root together with seven level zero roots. Since no $D_4$ can be embedded into level zero roots in $E_{11}$ the only example is the trivial case where the $E_8$ roots $\beta_i$ are given by the $E_{11}$ roots $\alpha_i$ by

$$
\beta_i = \alpha_{i+3}, \quad i = 1, \ldots, 8.
$$

Our only choice now is between $t = x_i$, and subsequently the involution invariant subalgebra. Besides the Cartan involution, with one of $t = x_1, x_2, x_3$, invariant $\mathfrak{so}(16)$ the only invariant subalgebras are $\mathfrak{so}(8,8)$ and $\mathfrak{so}^*(16)$.\(^3\) While the $SO(8,8)$ invariant subalgebra can be obtained from a variety of $\epsilon$, such as $(+, +, -, +, +, +, +, +)$ or $(+, +, +, +, +, +, +, -)$, any choice of one time coordinate generates the coset $E_{6(6)}/SO^*(16)$. Taking, for example

\(^2\)To convince yourself of this note that no combination of $l = 0$ roots can form the inner products for the triply-connected node, i.e. $D_4$ cannot be embedded in level zero. Any $l = 2, 3$ root would have to be included as $\beta_1$ or $\beta_3$. The combined observations prohibit any $E_6$ embedded into $E_{11}$ with $l = 2, 3$ simple roots.

\(^3\)The group $SO^*(2n) \cong O(n, \mathbb{H})$ is the subgroup of $SO(2n, \mathbb{C})$ which leaves invariant the sesquilinear antisymmetric metric. For more details see, for example, chapter 6 of [40].
CHAPTER 6. BOUND STATES OF BRANES

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t = x_4 and \( \epsilon = (-, +, +, +, +, +, +, +) \), there are 56 non-compact generators in the invariant subalgebra and consequently 56 \( SL(2, \mathbb{R})/SO(1, 1) \) limits which include:

\[
\begin{align*}
K KW \ [7] & : [4, i] \quad i = 5, \ldots, 11 \\
M2 \ [21] & : [4 ij] \quad i < j = 5, \ldots, 11 \\
M5 \ [21] & : [4 (567891011) \setminus ij] \quad i < j = 5, \ldots, 11 \\
KK6 \ [7] & : [4567891011, i] \quad i = 5, \ldots, 11.
\end{align*}
\]

(6.60)

Any other choice \( t = x_i \) will contain the same number/types of time-like solutions within the bound state.

6.4 Exotic Bound States

The infinitely many roots of \( l \geq 4 \) which are conjectured to describe exotic solutions using the Kac-Moody/Supergravity dictionary can also be placed in bound states. Bound states can interpolate between well-understood solutions and exotic objects which have less developed interpretation. Many algebras which describe bound states, such as \( E_8 \), can hardly be embedded in \( E_{11} \) without involving higher-level roots and exotic branes. The eleven-dimensional \( D_4 \) bound state solution which limits to the bound state described in table 6.2.3 can be constructed with the \( E_{11} \) roots

\[
\begin{align*}
\beta_1 &= (0, 1, 2, 2, 2, 2, 1, 0, 1) \\
\beta_2 &= (0, 0, 0, 0, 0, 0, 1, 1, 1, 1) \\
\beta_3 &= (0, 0, 0, 1, 2, 2, 2, 1, 0, 1) \\
\beta_4 &= (0, 0, 0, 0, 0, 1, 2, 2, 1, 0, 1).
\end{align*}
\]

(6.61)

This model contains mostly canonical eleven-dimensional supergravity solutions, as summarised in table 6.4, but also contains the root

\[
(1, 2, 1, 1)_{D_4} = (0, 1, 2, 3, 4, 5, 6, 7, 4, 1, 4)_{E_{11}}.
\]

(6.62)

This is precisely the lowest weight of the \([10\bar{b}b, 1, 1]-\)form which appears as the 715-dimensional representation at level 4 in \( E_{11} \), as shown in table 2.6.2. The work of [1] gives a classification of some low level exotic solutions but interpretations of the higher-level branes are required in order to construct bound states containing them.

6.4.1 Bound States within the \( E_9 \) Multiplet

The set of \( E_{11} \) roots which, when the algebra is levelled with respect to \( \alpha_{11} \), are elements of tensor representations with Young Tableau symmetries containing no more than nine boxes in any column form a special \( E_9 \) subalgebra, as discussed in section 2.6.1 [47]. By taking Weyl reflections in the affine group each of these roots were given interpretations as codimension two analogues of the four canonical solutions which appear at levels \( l = 0, 1, 2, 3 \) [31]. We should therefore be able to use the solutions from the simply-laced cosets and the
Kac-Moody/Supergravity dictionary to create complex bound states with these towers of solutions. Consider, for example, the $D_4$ model with $\epsilon = (+, -, +, +)$ and simple roots giving the configuration of table 6.4.1. This model is quite familiar, but the $D_4$ roots

\[
(1, 1, 1, 0) = (0, 0, 1, 2, 3, 4, 5, 6, 4, 2, 4) \quad \text{and} \quad (6.63)
\]

\[
(1, 1, 1, 1) = (0, 0, 1, 3, 5, 7, 9, 11, 7, 3, 6) \quad (6.64)
\]

are elements of the $[9,3]$-form and $[9,8,1]$-form representations which appear in table 2.6.2 as the 8470 and 57200 at levels four and six, respectively. While not canonical, these have an interpretation and so does the bound state. We can therefore consider any bound state with codimension $\geq 2$ and provide a solution for the metric and gauge fields. Exploring these bound states could potentially provide information about the extension of supergravity theories which are predicted by the $E_{11}/M$-theory conjecture, as we will explore in the next chapter.
Our main aim in this work is to use the non-linear realisation to investigate continuous solution generating sub-algebras within $A_{D-3}^{++-}$. Although we are ultimately motivated to understand the non-linear realisation of the affine sub-groups we will in this chapter identify finite sub-algebras which are embedded within $A_{D-3}^{++-}$ and which may be combined to give a set of continuous transformations which cover a set of solutions generated by an affine group. We will identify truncations of the algebra in the first instance to $\mathfrak{sl}(2, \mathbb{R})$ sub-algebras encoding gravity solutions and in the second instance to $\mathfrak{sl}(3, \mathbb{R})$ sub-algebras which we will interpret as bound states of pp-waves, KK-branes and other exotic gravitational objects. An infinite set of sequential $\mathfrak{sl}(3, \mathbb{R})$ sub-algebras will be presented which interpolate between...
the solutions discretely mapped to by the Geroch group. For the case when \( D = 11 \) we find that the null geodesic motion on the cosets of \( SL(3, \mathbb{R}) \) reproduce the gravitational tower of solutions identified in [31], but now the continuous interpolation between these solutions is identified as a solution to an extension of the Einstein-Hilbert action. A corollary of the work presented here is that the M2-M5 infinite tower of solutions found in [31] may be understood to originate from the (gravitational) Geroch group acting within \( A_9^{+++} \).

The chapter is organised as follows in section 2 we define the algebra \( A_D^{+++} \) and interpret each of the roots as Young tableaux carrying the symmetries of their associated generator. In section 3 we set up our notation by briefly reviewing the construction of the sigma model for bound states found in [26]. In section 4 the main body of our work is presented. In section 4.1 the null geodesic motion on cosets of \( SL(2, \mathbb{R}) \) and \( SL(3, \mathbb{R}) \) embedded within \( A_D^{+++} \) are studied. We give particular emphasis to understanding the appearance of the fundamental gravitational solutions, the pp-wave and the KK(\( D - 5 \)) brane, before we investigate more exotic solutions including the interpolating \( SL(3, \mathbb{R}) \) gravitational solutions which continuously map between the Geroch solutions. These one dimensional coset-model solutions do not lift to solutions of the Einstein-Hilbert action, so in section 5 we investigate the supergravity dictionary used for mixed-symmetry fields and identify gravity and exotic matter actions which do admit the full interpolation as a solution to their \( D \)-dimensional equations of motion. In section 6 we focus on the problem of dualising these exotic actions to the Einstein-Hilbert action and understanding why the full interpolating solution is lost. Section 7 is devoted to a discussion of the results.

### 7.1 The \( A_D^{+++} \) algebra.

The collection of very-extended Kac-Moody algebras \( A_D^{+++} \) where \( D \geq 4 \) has a root space spanned by the simple roots \( \{\alpha_1, \alpha_2, \ldots, \alpha_D\} \) which may be conveniently embedded in \( \mathbb{R}^D \). Let \( \{e_1, e_2, \ldots, e_D\} \) be an orthonormal basis for \( \mathbb{R}^D \) and an embedding of the basis of the root space is

\[
\alpha_i = e_i - e_{i+1} \quad \text{where} \quad i < D
\]

\[
\alpha_D = e_D + \sum_{i=3}^D e_i
\]  

(7.1)

For root vectors \( \alpha \equiv \sum_{i=1}^D a_i e_i \) and \( \beta \equiv \sum_{i=1}^D b_i e_i \) the inner product on the root space is given by

\[
\langle \alpha, \beta \rangle = \sum_{i=1}^D a_i b_i - \frac{1}{D-2} \sum_{j=1}^D \sum_{k=1}^D a_j b_k
\]  

(7.2)

One can confirm that this inner product acting on the positive simple roots embedded in \( \mathbb{R}^D \) reproduces the Cartan matrix of \( A_D^{+++} \) when \( D \geq 4 \).

An indefinite Kac-Moody algebra may be decomposed into an infinite set of highest weight representations of a classical Lie algebra each labelled by the level at which they occur in
the decomposition. $A^{+++}_{D-3}$ may be decomposed into a set of highest weight representations of $SL(D, \mathbb{R})$ corresponding to nodes 1 to $D - 1$ in figure 1.1 and the level specified by a single integer. If a generic root of the algebra is given by $\beta = \sum_{i \leq D} m_i \alpha_i$ in the simple root basis then the decomposition amounts to partitioning the root coefficients $(m_1, m_2, \ldots, m_D)$ into two sets of labels $(m_1, m_2, \ldots, m_{D-1})$ and $(m_D)$. The first set labels a highest weight representation of $SL(D, \mathbb{R})$; the $D - 1$ numbers are the coefficients of the root $\hat{\beta} = \sum_{i \leq D-1} m_i \alpha_i$ associated to the highest weight in the representation. The remaining integer $m_D$ is called the level in the decomposition and labels where each representation of $SL(D, \mathbb{R})$ occurs in the decomposition of $A^{+++}_{D-3}$.

The example of $A^{+++}_8$, occurring when we choose $D = 11$, is relevant to M-theory and we will emphasise this particular example. Its field content at low levels was first found in [87]. At levels 0 and 1 of the decomposition of $A^{+++}_8$ into representations of $SL(11, \mathbb{R})$ $K_{ab}$ and $R^a_1 \cdots R^a_8$ are associated with the KK-wave and KK6 brane solutions of M-theory. Indeed when $D = 11$ the inner product (7.2) coincides with the inner product of $E_{11}$ embedded in $\mathbb{R}^{11}$ and the algebra $A^{+++}_8$ is a sub-algebra of $E_{11}$. The results in this chapter will be readily adapted to $A^{+++}_8$. For the case of $A^{+++}_8$ the low level roots are shown in table 2.6.3 up to level $m_{11} = 3$ which was produced using [88]. These same representations at levels 0, 1, 2 and 3 occur within the decomposition of $E_{11}$ but at levels 0, 3, 6 and 9 and the pattern continues for all higher levels i.e. an $SL(11, \mathbb{R})$ representation at level $m_{11}$ in the decomposition of $A^{+++}_8$ is also always found at level $3m_{11}$ in the decomposition of $E_{11}$.

Reproducing all the information of table 2.6.3 is computationally challenging. A very efficient way to reproduce the highest weight representations appearing in the decomposition relies on the embedding of the root space within $\mathbb{R}^D$ advocated in equation (7.1). The choice of the embedding is such that the covariant and contravariant index structure of the $SL(D, \mathbb{R})$ highest weight representations associated to each of the generators is encoded in the coefficients of the $e_i$ in a manner that we will now describe. Each highest weight $SL(D, \mathbb{R})$ representation may be represented by a Young tableau whose columns are antisymmetrised and whose rows have widths $w_i$ where $1 \leq i \leq D$ and $w_i \leq w_{i+1}$. For a root of $A^{+++}_{D-3}$ associated with a highest weight representation, the Young tableau carrying the symmetries of the associated generator has rows of width $w_i$, running from the bottom row of width $w_1$ up to the top row of width $w_D$, which may be read from the root by virtue of the embedding in $\mathbb{R}^D$

$$\beta \equiv \sum_{i \leq D} m_i \alpha_i = \sum_{i \leq D} w_i e_i. \quad (7.3)$$

The roots at level zero which make up the algebra of $SL(D, \mathbb{R})$ provide an immediate, but isolated, puzzle for the preceding definition. Namely these roots have the form $e_i - e_j$ which we interpret as a Young tableau having an $i$'th row of width one but a $j$'th row of width negative one. The interpretation is best stated in terms of the generator index structure and a negative coefficient indicates covariant indices on the tensor component while a positive index indicates contravariant indices. The root $e_i - e_j$ is associated to the generator $K^i_{-j}$.
whose commutator relations are those of the positive generators of the \( \mathfrak{sl}(D, \mathbb{R}) \) algebra. The simple root at level one \( \alpha_D \) as given in equation (7.1) has a mixed-symmetry \( \mathfrak{sl}(D, \mathbb{R}) \) generator \( R^{456...D,D} \). The gauge field associated to this generator has the index structure of a dual graviton, which one may see by dualising the first set of \((D−3)\) indices to find a tensor field related to the vielbein. The highest weight Young table has two columns of \( D−3 \) and \( D \) boxes:

\[
\begin{array}{c|c}
\hline
D & D \\
\hline
D-1 & \\
\vdots & \\
5 & 4 \\
\hline
\end{array}
\] (7.4)

The Young table above is associated to the particular generator \( R^{456...D,D} \), in what follows we will frequently drop the labels in the Young table where we refer to the unique highest weight. The lower weights arise following commutation with the \( K_{ij} \) generators of \( \mathfrak{sl}(D, \mathbb{R}) \).

For \( D = 4 \) the level one Young tableau has the same structure as the metric representing a symmetric two tensor, while for \( D > 4 \) the level one Young tableaux are hooked. The associated gauge field component \( A_{456...D,D} \) is dualised on its antisymmetric \( D−3 \) indices to \( A_{i,D} \) where \( i \in \{1, 2, 3\} \) which, as we will see, may be associated to the vielbein. The field appearing at level one is therefore referred to as the dual graviton.

The embedding of equation (7.1) into \( \mathbb{R}^D \) furnishes a simple method to uncover the non-trivial commutators of the low-level generators which we will now outline. The commutators of the Kac-Moody algebra are controlled by the Serre relations, these relations may be recast as an algebraic condition on the root length squared \([25]\). The commutator of two generators associated to the distinct roots \( \alpha \) and \( \beta \) is either trivial or gives rise to a new generator in the algebra depending upon the length squared of the sum of the roots \( (\alpha + \beta)^2 \):

\[
[E_\alpha, E_\beta] = \begin{cases} 
0 & \text{if } (\alpha + \beta)^2 > 2 \text{ and} \\
E_{\alpha+\beta} & \text{if } (\alpha + \beta)^2 \leq 2.
\end{cases} (7.5)
\]

The generator structure for \( E_{\alpha+\beta} \) may be read directly from the root \( \alpha + \beta \) using equation (7.3). For example we asserted earlier that the generators \( K^i_j \) associated to the roots \( e_i - e_j \) gave rise to the algebra of \( \mathfrak{sl}(D, \mathbb{R}) \) - we can now confirm this assertion using the inner product (7.2). To find the general commutator let \( \alpha = e_i - e_j \) and \( \beta = e_k - e_l \). As \((\alpha + \beta)^2 = (e_i - e_j + e_k - e_l)^2 = 4 - 2\delta_{jk} - 2\delta_{il}\) the commutator is only non-trivial if \( i = l \) or \( j = k \). Note that if both \( i = l \) and \( j = k \) then the roots are non-distinct as \( \alpha = \beta \) and the commutator is trivially zero. Consequently we may write the commutator relation for the level zero roots as:

\[
[K^i_j, K^k_l] = \delta^i_k K^j_l - \delta^j_k K^i_l. (7.6)
\]

The minus sign difference between the terms follows from the antisymmetry of the Lie bracket. The commutators above are recognisable as the commutators of the generators associated to the positive roots of \( \mathfrak{sl}(D, \mathbb{R}) \).
A necessary criterion for existence of roots (up to root multiplicity or outer multiplicity) may be summarised as: if \( \beta^2 = 2, 0, -2, -4, \ldots \) then \( \beta \) is a root in the root lattice of \( A_{D-3}^{++} \). One may find the \( SL(D, \mathbb{R}) \) Young tableaux at level \( L \) by noting that the nested commutator of \( L \) copies of the level one generator whose Young tableau has \( D - 2 \) boxes will consist of Young tableaux with \( L(D - 2) \) boxes. By drawing all the tableaux formed of \( L(D - 2) \) boxes and projecting out all those whose associated root length squared is greater than two one arrives at a close approximation of the algebraic content of \( A_{D-3}^{++} \). It is, perhaps, simpler to find one Young tableau at each level whose length squared is two and then construct the other Young tableaux at level \( L \) by moving the Young tableau boxes between columns - a transformation which has a simple impact on the root length squared. The movement of a Young tableau box by one column to the left has the effect of lowering the associated root length squared by two, as can be quickly verified, using (7.2). Consequently the reverse manoeuvre of transferring a box one column to the right raises the root length squared by two. Using these rules on a Young tableau, whose associated root length squared is known, one can quickly construct all Young tableaux in the decomposition at a particular level together with their associated root length squared but without computation. Additionally there always exists a root at any level \( L \) whose generator has the symmetries of the highest weight Young table

\[
\begin{array}{ccccccc}
\text{D} & \text{D} & \ldots & \text{D} & \text{D} & \\
\text{D-1} & \text{D-1} & \ldots & \text{D-1} & \text{D-1} & \\
\vdots & \vdots & \ldots & \vdots & \vdots & \\
4 & 4 & \ldots & 4 & 4 & \\
3 & 3 & \ldots & 3 & \\
\end{array}
\]  

(7.7)

where there are \( (L - 1) \) columns of height \( (D - 2) \), one column of height \( (D - 3) \) and a single column of height one, whose length squared for any dimension \( D \geq 4 \) is always two. The Young tableaux represented in (7.7) exist within the affine subalgebra \( A_{D-3}^+ \subset A_{D-3}^{++} \). The construction, in this way, of the Young tableaux in the decomposition of \( A_{D-3}^{++} \) may be confirmed at low levels by comparison with table 2.6.3. As one can see this construction is almost sufficient to reproduce the decomposed algebra. However the reader will notice that the information in the column headed \( \mu \) which gives the outer multiplicity of the generators (the number of copies of a particular generator) has not been reproduced. The outer multiplicity is particularly crucial when it is zero, as then, contrary to our expectations, the generator does not appear within the algebra, however the calculation of outer multiplicity is time-consuming. It would be very useful to find a quick computation to determine whether the outer multiplicity of a generator is zero.

The highest weight generator corresponding to the deleted node defining the level has a Young tableau containing \( (D - 2) \) boxes. Consequently a generator appearing at level \( L \) has a Young tableau formed of \( L(D - 2) \) boxes, as the generator is defined by \( L \) nested

\[\sqrt{2} \] here, consequently all roots have an even length-squared which is less than or equal to two.
commutators involving level one generators. Note that the inner product of two roots at levels \( L_1 \) and \( L_2 \) in the decomposition is given by:

\[
\langle \alpha^{L_1}, \beta^{L_2} \rangle = \sum_i w(\alpha)_i w(\beta)_i - L_1 L_2 (D - 2) \tag{7.8}
\]

where \( \alpha^{L_1} = \sum_i w(\alpha)_i e_i \) and \( \beta^{L_2} = \sum_i w(\beta)_i e_i \).

7.2 The coset model for gravitational solutions.

Throughout this chapter we will be using the coset model approach developed in [26] to construct gravitational solutions from the \( A_{++}^{D-3} \) algebras. We will restrict our interest to cosets of \( A_1 \) and \( A_2 \) sub-groups embedded in \( A_{++}^{D-3} \). The work of [26] describes a method that associates a one-dimensional solution of M-theory and string theory to a null geodesic motion on cosets \( SL(n, \mathbb{R})/H \) where the algebra of \( H \) is the fixed point set under some generalised involution.

7.2.1 The generalised involution \( \Omega \).

The Chevalley-Cartan involution \( \Omega_C \) is defined on a semisimple Lie algebra \( \mathfrak{g} \) of the group \( G \) by

\[
\Omega_C(H_i) = -H_i, \quad \Omega_C(E_i) = -F_i \quad \text{and} \quad \Omega_C(F_i) = -E_i \tag{7.9}
\]

where we have used the Chevalley basis to present the algebra so that \( H_i \) indicates the Cartan sub-algebra, \( E_i \) are the generators associated to the positive simple roots, \( F_i \) the generators associated to the negative simple roots and the index \( i \in \{1, 2, 3, \ldots, R\} \) where \( R \) is the rank of the Lie algebra. The Chevalley-Cartan involution action on the remainder of the algebra is derived from its action on the generators associated to the simple roots as

\[
\Omega_C([E_i, E_j]) = [\Omega_C(E_i), \Omega_C(E_j)] \equiv \Omega_C(E_{i+j}) \tag{7.10}
\]

\[
\Omega_C([F_i, F_j]) = [\Omega_C(F_i), \Omega_C(F_j)] \equiv \Omega_C(-F_{i+j})
\]

where we have defined \( E_{i+j} \equiv [E_i, E_j] \) and \( -F_{i+j} \equiv [F_i, F_j] \). The algebra \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \) is split into a part which is fixed under the involution \( \mathfrak{k} \) and the complement \( \mathfrak{p} \). The basis generators of \( \mathfrak{k} \) are (with normalisation) \( k_i \equiv \frac{1}{2}(E_i - F_i) \) while the algebra \( \mathfrak{p} \) has basis elements \( P_i \equiv \frac{1}{2}(E_i + F_i) \) and \( H_i \). The group \( \mathcal{K}(\mathcal{G}) \), found by exponentiating the \( \Omega_C \)-invariant \( \mathfrak{k} \), is the maximal compact sub-group of \( \mathcal{G} \).

A more general involution\(^2\) \( \Omega \) can be defined by its action on the Cartan and simple root generators [22]

\[
\Omega(H_i) = -H_i, \quad \Omega(E_i) = -\epsilon_i F_i \quad \text{and} \quad \Omega(F_i) = -\epsilon_i E_i \tag{7.11}
\]

\(^2\)The involutions and invariant algebras depend on whether the time dimension is contained within the brane and we will assume a \((1, D-1)\)-signature metric in this chapter. For the group theoretic description of alternative signature M-theories of Hull [89] we refer the reader to [90].
where $\epsilon_i$ is either $-1$ or $+1$. Collecting the $R$ values $\epsilon_i$ as a vector $\epsilon$ we can write the Chevalley-Cartan involution as $\epsilon = (+, +, \ldots, +)$. In the case of $g = sl(3, \mathbb{R})$ the Chevalley-Cartan involution generates the coset $SL(3, \mathbb{R})/SO(3)$, while the involutions with $\epsilon = (-, +)$ and $(-, -)$ both generate $SL(3, \mathbb{R})/SO(1, 2)$. For the general normal real form $A_n(n) \cong sl(n + 1, \mathbb{R})$ each of the possible $K(G)$ constructed in this way, which are $SO(p, q)$ with $p + q = n + 1$, can be obtained by taking the involution with $\epsilon = (+, \ldots, +, -p, +, \ldots, +)$.

### 7.2.2 Solutions as null-geodesics on cosets.

We define a map from the real line parameterised by $\xi$ into the coset $G_{K(G)}$ in the Borel gauge

$$g = \exp\left(\sum_{i=1}^{n} \phi_i H_i\right) \exp\left(\sum_{E_\alpha \in \Delta^+} C_\alpha E_\alpha\right)$$

(7.12)

where $\phi_i \equiv \phi_i(\xi)$, $C_\alpha = C_\alpha(\xi)$, $H_i$ are the Cartan sub-algebra generators and the second summation runs over the generators of the numerator algebra $g$ associated with the set of postive roots $\Delta^+$. We can decompose the Maurer-Cartan form

$$\partial_\xi gg^{-1} = Q_\xi + P_\xi$$

(7.13)

into components of the fixed point algebra $Q_\xi$ and the complement in $g$ denoted $P_\xi$. The Lagrangian for this model

$$L = \eta^{-1} (P_\xi | P_\xi)$$

(7.14)

where $(M|N) = Tr(MN)$ is the Killing form for $G$ and $\eta$ is the lapse function encoding reparameterisation invariance. The Lagrangian is invariant under global $G$ transformations and local $K(G)$ transformations. Its equations of motion are

$$\partial_\xi P_\xi - [Q_\xi, P_\xi] = 0$$

(7.15)

$$\partial_\xi P_\xi = 0$$

(7.16)

where the first set comes from variation of the coset representative $g$ and the second is due to variation of the lapse function $\eta$. The solution to these equations is a null geodesic on a coset. The null geodesics encode all the $\frac{1}{2}$-BPS solutions [20–22] as well as the bound state solutions in supergravity, string theory and M-theory [1, 25–27]. The bound state solutions possess a manifest global $G$ symmetry which permutes the branes in the bound state and a local $K(G)$ whose compact symmetries interpolate continuously between brane solutions [26], the group element associated to the remaining symmetries in $K(G)$ are responsible for shifts in the gauge field, Ehlers transformations on the gauge fields and the Cartan sub-algebra acts as a conformal transformation on the metric, for a review of these ideas see [57] and the references therein.
7.3 Null geodesics on cosets of $SL(2, \mathbb{R})$ and $SL(3, \mathbb{R})$.

Our aim in this work is to understand the continuous symmetries within $A_{D-3}^{++}$ Kac-Moody algebras when truncated to $\mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{sl}(3, \mathbb{R})$ sub-algebras. The relevant cosets are $\frac{SL(2, \mathbb{R})}{SO(1,1)}$ and $\frac{SL(3, \mathbb{R})}{SO(1,2)}$ and solutions encoding the null geodesic motion on the coset are presented in [26]. The major technical challenge is the extension of the dictionary that relates null geodesics on cosets to gravitational solutions. We commence this section with the mapping to the well-known pp-wave and KK($D-5$) brane.

7.3.1 $SL(2, \mathbb{R})/SO(1,1)$: the pp-wave and KK($D-5$) brane.

The algebra of $\mathfrak{sl}(2, \mathbb{R})$ consists of a one-dimensional Cartan sub-algebra spanned by $H$ and a single generator associated to a positive root $E$ and its negative root counterpart $F$. The coset representative in the Borel gauge of $SL(2, \mathbb{R})_K$, where $K$ is either $SO(1,1)$ or $SO(2)$, is

$$g = \exp(\phi H)\exp(CE).$$

(7.17)

The null geodesic solution [22] is given by

$$\phi = \frac{1}{2} \ln N \quad \text{and} \quad C = \pm N^{-1} + K$$

(7.18)

where $K$ is a constant and $N \equiv a + b\xi$ with $a$ and $b$ real constants. The solution has

$$P_\xi = e^{2\phi} \partial_\xi C = \pm N \partial_\xi N^{-1}.$$  

(7.19)

where we choose to work with the positive sign in the following. The field $\phi$, as it premultiplies an element of the Cartan sub-algebra $H$ embedded in $A_{D-3}^{++}$, encodes the vielbein components. More precisely the vielbein $e_{i}^{j} = (e^{-h})_{i}^{k}(e^{-h})_{k}^{j}$ where $h_{ij}$ is the coefficient of the generator $K_{ij}$ in the coset representative $g$ and $i < k \leq j$. We will discuss the construction of the vielbein in detail below but we also refer the reader to section 2 of [91] for a detailed argument. The dependence of $h$ on $\xi$ will depend crucially upon the embedding of the Cartan sub-algebra $H$ of $\mathfrak{sl}(2, \mathbb{R})$ into the Cartan sub-algebra of $A_{D-3}^{++}$.

By establishing a dictionary one identifies $P_\xi$ with a field strength whose index structure will depend upon the embedding of the positive generator $E$ in $A_{D-3}^{++}$. Let us fix the prescription for identifying one-dimensional solutions in this way by considering the examples of the pp-wave and the KK($D-5$) brane.

The pp-wave. To identify an $SL(2, \mathbb{R})$ root system we need find only a single real root in the root system of $A_{D-3}^{++}$. Consider a positive root $\alpha$ of the $SL(D, \mathbb{R})$ sub-group singled out under the decomposition of $A_{D-3}^{++}$ carried out in section 7.1. In the $e_i$ basis we have

$$\alpha = e_i - e_j \quad \text{for} \quad 1 \leq i < j \leq D$$

(7.20)
and it is associated to the generator $K^i_j$ and has associated Cartan sub-algebra element $H = K^i_i - K^j_j$. Let us commence with the case where the index $i$ is timelike but all other indices are spacelike. This defines the involution on the $\mathfrak{sl}(D,\mathbb{R})$ algebra to be

$$\Omega(K_a^{a+1}) = \begin{cases} -K_a^{a+1} & \text{for } 1 \leq a < i \\ K_a^{a+1} & \text{for } i \leq a < D. \end{cases}$$

(7.21)

The sub-algebra fixed by $\Omega$ is $\mathfrak{so}(1,1)$. We can now use the solution for the null geodesic given in equations (7.18,7.19), and originally found in [22], to read off the line element.

It would however be helpful to illuminate why one may “read off” the vielbein components. Consider the introduction of a translation generator

$$3 \hat{P}_i$$

whose commutator with $\mathfrak{sl}(D,\mathbb{R})$ is [19]

$$[\hat{P}_i, K^j_k] = \delta_i^j \delta_k^k P_k.$$

(7.22)

Conjugation of the translation generator by a representative element of the coset $SL(D,\mathbb{R})$ having diagonal $h^i_i$ and off-diagonal $A^j_k$ fields non-zero gives

$$gP_m g^{-1} = \exp(h^i_i K^i_i) \exp(A^j_k K^j_k) P_m \exp(-A^j_k K^j_k) \exp(-h^i_i K^i_i)$$

(7.23)

$$= \exp(h^i_i K^i_i) [P_m - A^j_k \delta^j_m \delta^k_k P_k + \frac{1}{2!} A^j_k \delta^j_m A^l_k \delta^l_{kn} P_n - \ldots] \exp(-h^i_i K^i_i)$$

(7.24)

$$= \exp(h^i_i K^i_i) [(e^A)^m_k P_k] \exp(-h^i_i K^i_i)$$

(7.25)

$$= (e^A)^m_k (P_k - \delta^j_k \delta^i_j P_j + \frac{1}{2!} \delta^j_k \delta^i_j \delta^j_i \delta^j_i P_j - \ldots)$$

(7.26)

$$= (e^A)^m_k (e^{-h})^k_k P_k$$

(7.27)

Now we see that the combined exponentials act on $P_k$ as a vielbein:

$$e^j_i \equiv (e^{-A})^k_i (e^{-h})^j_k$$

(7.28)

where $h$ is diagonal and $k > i$. If we had repeated the conjugation of $P_m$ without any off-diagonal contributions to the vielbein (i.e. $A^j_k = 0$) then we would have found

$$e^j_i \equiv (e^{-h})^j_i.$$

(7.29)

Returning to the example we may now read off the non-trivial components of the vielbein for the solution

$$e^i_i = N^{-\frac{1}{2}}, \quad e^j_j = N^{\frac{1}{2}} \quad \text{and} \quad e^j_i = (e^{-A})^j_i = -A^j_i$$

(7.30)

---

3 We have adopted a hatted index to indicate a curved space-time coordinate while an unhatted index to indicate a flat tangent space coordinate as in [26].

4 We remain in the Borel gauge for the group element so that $k > j$. 
CHAPTER 7. DUAL GRAVITY SOLUTIONS

where \( i \) and \( j \) now take fixed values given by the choice of root and \( i < j \). The field \( A^i{}_j \) is determined from the null geodesic motion on the coset. \( P_\xi = N\partial_\xi N^{-1} \) is identified with the components of a field strength for \( A^i{}_j \) as follows

\[
F^i{}_j = \partial_\xi A^j{}_i = N\partial_\xi N^{-1}. \tag{7.31}
\]

With this definition we have differentiated between the sets of (antisymmetrised) coordinates, after all one may wonder why we have assumed that the exterior derivative hits the first set of indices on \( A \) and not the second. It is also worth emphasising that with this definition we are treating \( A^i{}_j \) as a scalar object under the covariant derivative as the dictionary identifies components of \( F \) with the components of a one-form. We only have a vector field strength, which we identified with a component of the Maurer-Cartan form. We do not construct a full \([2,1]\) field strength tensor and for the moment the extra indices on \( A \) play no role. We may put all sets of indices on an equal footing by forming the \([2,2]\) field strength

\[
F^i{}_j = D_\xi \left( N\partial_\xi N^{-1} \right) \equiv D^\xi(\partial_\xi A^j{}_i),
\]

where there is no summation over the repeated \( \xi \) indices and the derivative \( D \) is covariant with respect to the vielbein encoded in the Cartan sub-algebra of the coset. However in order to find the off-diagonal vielbein components we will immediately remove the second derivative and the covariant derivative plays no role in this solution. It will however be important in later mixed symmetry solutions that we will develop. We may return to equation (7.31). As \( A^i{}_j = A^j{}_i(\xi) \) then \( \partial_\xi A^j{}_i = \partial_\xi A^i{}_j \). It is useful to embed the field strength in space-time using the vielbein so that

\[
F^\hat{i}\hat{j} = e^\hat{i}\hat{k} e^\hat{j}\hat{l} F^\hat{k}\hat{l} = \partial_\xi N^{-1}. \tag{7.32}
\]

Hence \( A^\hat{i}\hat{j} = N^{-1} + c \), with some constant \( c \). Consequently \( A^i{}_j = e^k{}_j A^i{}_k = N^{-\frac{1}{2}}(1 + cN) \) and so the off-diagonal component of the vielbein is

\[
e^i{}_j = -N^{-\frac{1}{2}}(1 + cN). \tag{7.33}
\]

Imposing that the solution is asymptotically flat fixes \( c = -1 \) and writing \( N = 1 + K \) we have:

\[
e^i{}_i = \frac{1}{\sqrt{1 + K}}, \quad e^j{}_j = \sqrt{1 + K} \quad \text{and} \quad e^i{}_j = \frac{K}{\sqrt{1 + K}}. \tag{7.34}
\]

This gives non-trivial metric components

\[
g_{\hat{i}\hat{i}} = -\frac{1}{1 + K} + \frac{K^2}{1 + K} = -(1 - K) \tag{7.35}
\]

\[
g_{\hat{i}\hat{j}} = K \tag{7.36}
\]

\[
g_{\hat{j}\hat{j}} = (1 + K) \tag{7.37}
\]

and the line element

\[
ds^2 = -(1 - K)(dt^i)^2 + 2K dt^i dx^j + (1 + K)(dx^j)^2 + dy^k dy^l \eta^{(D-2)}_{kl} \tag{7.38}
\]

\[
= 2K du^2 - 2dudv + dy^k dy^l \eta^{(D-2)}_{kl} \tag{7.39}
\]
where \( u = \frac{1}{\sqrt{2}}(t + x) \) and \( u = \frac{1}{\sqrt{2}}(t - x) \) are light cone coordinates. The linear function \( N(\xi) \) once embedded in space-time becomes a linear function of one of the transverse \( y^k \) coordinates. A crucial step in deriving the one-dimensional solution was the assumption that \( N \) was a harmonic function of the single variable. Once the fields of the solution have been embedded in space time it is clear that all \( D - 2 \) transverse coordinates are equivalent. Hence it is possible to promote the linear function \( N \) to be a harmonic function in the transverse \( D - 2 \)-dimensional sub-space. Choosing \( N \) to have the form

\[
N = 1 + \frac{M}{r^{D-4}},
\]

(7.40)

where \( r^2 = \sum_{k=1}^{D-2}(y^k)^2 \), gives a single centre pp-wave solution of eleven dimensional supergravity found in [80] and found in the case of general relativity in \( D = 4 \) in [92].

The KK monopole. Let us consider a second example of a solution derived from a single root. We will derive the KK(\( D - 5 \)) monopole from a real root at level one in the decomposition of \( A_{D-3}^{++} \). This example will be the prototype for associating solutions to roots in the remainder of this chapter. Let us consider the root whose generator has the Young table shown in equation (7.4). The \( SL(2, \mathbb{R}) \) coset representative is again of the form shown in equation (7.17) but now we take the level one root to be the single positive root of an \( SL(2, \mathbb{R}) \) embedded in \( A_{D-3}^{++} \). The generators are

\[
H = -(K^1 + K^2 + K^3) + K^D_D \quad \text{and} \quad E = R^{456...D,D}.
\]

(7.41)

We will work with the involution with \( \epsilon = (-) \), where the sub-group is \( SO(1, 1) \), by choosing \( x^4 \) to be the time dimension. We note that we could have picked any one of \( x^4, x^5, \ldots x^{(D-1)} \) to be the temporal coordinate while ensuring that \( \Omega(E) = F \), however if we had picked \( x^D \) to be temporal as the index appears twice in the generator \( E \) we would have found \( \Omega(E) = -F \) and the local sub-group would have been \( SO(2) \).

The field \( A_{456...D,D} \) is related by Hodge duality to a vielbein field \( A_{i}^{D} \) where \( i \in \{1, 2, 3\} \). Once again we will use the null geodesic motion to find an expression for the off-diagonal components of the vielbein in terms of \( N \), the linear function encoding the solution. As we suggested earlier we may simply take covariant derivatives, indicated here by \( \mathcal{D} \), on all sets of indices

\[
F_{\xi456...D,D} = \mathcal{D}_\xi (N \mathcal{D}_\xi N^{-1}) \equiv \mathcal{D}_\xi \mathcal{D}_\xi A_{456...D,D}.
\]

(7.42)

The non-trivial diagonal elements of the vielbein are

\[
e_i^i = N^{\frac{1}{2}} \quad \text{for} \quad i \in \{1, 2, 3\} \quad \text{and} \quad e_D^D = N^{-\frac{1}{2}}.
\]

(7.43)

Upon using the diagonal vielbein to embed the field strength in space-time we have

\[
F_{\xi456...D,D} = \mathcal{D}_\xi (\mathcal{D}_\xi N^{-1}) \equiv \mathcal{D}_\xi \mathcal{D}_\xi A_{456...D,D}.
\]

(7.44)
where we have made use of the identity $D_{i}(e_{i}^{J}) = 0$. Before carrying out the Hodge dualisation it will be useful to pick an embedding of $\xi$ identifying the parameter with one of the three transverse directions labelled by $\{\hat{1}, \hat{2}, \hat{3}\}$ - for this example we will pick $\hat{\xi} = \hat{1}$. Next we Hodge dualise the first set of indices to get

$$\star F_{1456...D,\hat{\xi}D} \equiv F_{23,1D} = -D_{1}(N\partial_{1}N^{-1}) \equiv D_{1}\partial_{[2}A_{3]}D \quad (7.45)$$

$$\Rightarrow \partial_{1}N = \partial_{[2}A_{3]}D. \quad (7.46)$$

We have implicitly used $D_{i}(g_{\hat{\mu}\hat{\nu}}) = 0$ to derive the expressions above. Had we embedded the solution with $\hat{\xi} = \hat{2}$ or $\hat{\xi} = \hat{3}$ we would have found

$$-\partial_{2}N = \partial_{[1}A_{3]}D \quad \text{or} \quad (7.47)$$

$$\partial_{3}N = \partial_{[1}A_{2]}D, \quad (7.48)$$

respectively. The three individual one-dimensional scalars $A_{i}D$ may be collected together to form a three-dimensional vector $A_{D}$ and simultaneously $N$ may be made a harmonic function both of which are now dependent on three coordinates. We emphasise that this enhancement of the fields is an application of the symmetry of the background metric in the three transverse directions labelled by $\{\hat{1}, \hat{2}, \hat{3}\}$. We are left with a single equation in the three dimensional subspace

$$\nabla N = \nabla \wedge A_{D} \quad (7.49)$$

which, in the Euclidean signature, is the equation defining a Taub-NUT [81, 93] or the generalised Gibbons-Hawking instanton metric [94, 95] on the four-dimensional Euclidean subspace spanned by $x_{\hat{1}}, x_{\hat{2}}, x_{\hat{3}}$, and $x_{\hat{D}}$. Explicitly we suppose $N$ takes the form $N = 1 + \frac{2K}{r}$ and then upon changing to spherical coordinates we find $A_{\phi} = 2K \cos \theta$ up to a function of $\phi$ and the non-trivial vielbein components are

$$e_{r}^{r} = N^{\frac{1}{2}}, \quad e_{\theta}^{\theta} = N^{\frac{1}{2}}, \quad e_{\phi}^{\phi} = N^{\frac{1}{2}}, \quad e_{\phi}^{D} = 2KN^{-\frac{1}{2}} \cos \theta \quad \text{and} \quad e_{D}^{D} = N^{-\frac{1}{2}}. \quad (7.50)$$

The metric is the Euclidean Taub-NUT metric embedded in a $D$-dimensional Minkowski space-time discovered in [96, 97]:

$$ds^{2} = N(dx^{2} + r^{2}d\theta^{2} + r^{2}\sin^{2}\theta d\phi^{2}) + N^{-1}(dx^{D} + 2K \cos \theta d\phi)^{2} + d\Sigma_{(1,D-5)}. \quad (7.51)$$

To avoid a conical singularity $dx^{D}$ is periodically identified. When $D = 5$ this is the KK-monopole metric, for $D > 5$ we refer to this as the KK($D-5$) brane which upon dimensional reduction along $dx^{D}$ gives a ($D-5$) brane metric.

The Young tableaux appearing at level $L$ and associated to real roots shown in equation (7.7) all have a two-dimensional transverse space. It will benefit us to return to equation (7.49) and investigate the solution when it is smeared down to two dimensions. Let us suppose we smear the solution along $x^{3}$ so that $N$ and $A_{i}D$ depend only on $x^{1}$ and $x^{2}$. The components of equation (7.49) become:

$$\partial_{2}A_{3}D = \partial_{1}N, \quad \partial_{1}A_{3}D = -\partial_{2}N \quad \text{and} \quad \partial_{1}A_{2}D = \partial_{2}A_{1}D. \quad (7.52)$$
The last equation trivialises the field strength component \( F_{\hat{1}\hat{2}}^D \), while the first pair are the Cauchy-Riemann equations for an analytic function

\[
f = N + iA_3^D.
\]  

(7.53)

Smearing the harmonic function \( N \) to two dimensions we have

\[
N = 1 + K \ln (r^2) = 1 + K \ln (z \bar{z}) = 1 + K \ln (z) + K \ln (\bar{z})
\]  

(7.54)

where \( z = x^1 + ix^2 \) and

\[
f = 1 + K \ln (z) + K \ln (\bar{z}) + iA_3^D
\]  

(7.55)

will be holomorphic if \( A_3^D = -iK \ln (\bar{z}/z) = 2K \theta \), where \( \theta \) is the argument of \( z \). We note that \( |f|^2 = N^2 + 4K^2 \theta^2 \). Given the distinguished transverse space the appearance of holomorphic functions to describe co-dimension two solutions is not surprising - they will be a feature of the higher level solutions as well.

7.3.2 \( SL(2, \mathbb{R})/SO(1, 1) \): higher level solutions

In this section we will truncate \( A^{+++}_{D-3} \) to \( sl(2, \mathbb{R}) \) sub-algebras using roots of higher level. The null geodesic motion on these cosets will encode the solutions generated by the Geroch group - in particular we will reproduce the infinite tower of gravitational solutions found in [31] and in so doing we shall understand the appearance of holomorphic functions that describe the solution.

The \( A^{+++}_{D-3} \) algebra contains infinitely many positive real roots associated with generators having the symmetries of the Young table in equation (7.7). There are a set of roots of this type occurring at each level \( L \) which can be expressed in the \( e_i \) basis using only \( e_3, e_4, \ldots e_D \) as

\[
\alpha = e_i - e_j + L \left( \sum_{k=3}^{D} e_k \right)
\]  

(7.56)

where \( i, j \in \{3, \ldots, D\} \) and \( i \neq j \). Let the specific \( \alpha \) with \( i = D \) and \( j = 3 \) be the positive simple root of an \( SL(2, \mathbb{R}) \) embedded within \( A^{+++}_{D-3} \). The generators of \( SL(2, \mathbb{R}) \) in terms of the generators of \( A^{+++}_{D-3} \) are

\[
H = -L(K_1^1 + K_2^2) - K_3^3 + K_3^D D,
\]  

(7.57)

\[
E = R_{345 \ldots D} | 345 \ldots D | 456 \ldots D | D | D
\]  

(7.58)

\[
F = R_{345 \ldots D} | 345 \ldots D | 456 \ldots D | D | D.
\]  

(7.59)

In the previous section the level one root of the above type was shown to be associated with the KK\((D - 5)\) brane. In that case the solution was constructed from the level one root

\[
\alpha = e_4 + e_5 + \ldots + e_{D-1} + 2e_D
\]  

(7.60)
and one of the set of coordinates \( \{x^4, x^5, \ldots, x^{D-1}\} \) was chosen to be timelike. The involution \( \Omega \) is chosen so that it acts on the generator \( R^{45\ldots(D-1)D} \) as \( \Omega(R^{45\ldots(D-1)D}) = R_{45\ldots(D-1)D} \) and the involution invariant sub-algebra is \( SO(1, 1) \). For the higher level roots

\[
\alpha = (L - 1)e_3 + L(e_4 + e_5 + \ldots + e_{D-1}) + (L + 1)e_D
\]  

(7.61)

choosing one of the coordinates \( \{x^4, x^5, \ldots, x^{D-1}\} \) to be timelike implies that the involution acts on the associated element of the algebra as

\[
\Omega(E) = (-1)^{L+1}F \quad \text{where} \quad t \in \{x^4, x^5, \ldots, x^{D-1}\}
\]

(7.62)

where we have used the notation of equations (7.58) and (7.59) to indicate the positive and negative generators. The sub-algebra left invariant under the involution will consequently be \( SO(1, 1) \) for odd \( L \) and \( SO(2) \) for even \( L \). Of course by choosing \( x^3 \) or \( x^D \) to be the sole timelike coordinate the situation is reversed as the involution then acts as

\[
\Omega(E) = (-1)^L F \quad \text{where} \quad t \in \{x^3, x^D\}.
\]

(7.63)

We will focus our attention on null geodesics on \( \frac{SL(2, \mathbb{R})}{SO(1, 1)} \) and will differentiate in the following between odd and even level roots where needed.

To construct the solutions in the previous section we made use of a coset model dictionary which identified the field multiplying the level one generator in the Maurer-Cartan form with the \([D - 2]_1\)-form field strength \( F_{\xi 34\ldots D} \) which could then be dualised. We therefore propose an extension of the dictionary which includes higher level objects appearing at arbitrary level \( L \) and identifies them with a democratic field strength. These would be associated with a \([D - 1]_1\ldots[D - 2]_2\)-form field strength which we can convert into a \([1]_1\ldots[1]_2\)-form field strength through by dualisation of the first \( L \) sets of indices. The procedure for this is simply the application of covariant derivatives which are antisymmetrised with each set of indices giving the dictionary defining components of tensors on the geodesic

\[
F_{\xi 3\ldots D} = \mathcal{P}_{\xi 3\ldots D} \equiv \mathcal{P}_{\xi 3\ldots D} \equiv D^\xi_L \mathcal{P}_\xi.
\]

(7.64)

In order to Hodge dualise this field strength we identify the coset coordinate \( \xi \) with a dimension in the transverse space \( (x^1, x^2) \) and embed the field strength in space-time using the diagonalised vielbein (see equation (7.29)) which is derived from the Cartan element in equation (7.57). The non-trivial vielbein components are

\[
e_1^1 = N^\frac{1}{2}, e_2^2 = N^\frac{1}{2}, e_3^3 = N^\frac{1}{2} \quad \text{and} \quad e_D^D = N^{-\frac{1}{2}}.
\]

(7.65)

The field strength is then dualised over its first \( L \) sets of indices, and the remaining \( D \) index is raised. The dualisation is sensitive to the choice of temporal coordinate in the background space-time. All the solutions associated to the positive root \( \alpha \) give product space-time manifolds of the form \( M_4 \otimes N_{D-4} \), where \( M_4 \) is a four-dimensional manifold and \( N_{D-4} \) is
a \((D-4)\)-dimensional manifold which is not warped in the solution. The coordinates of \(N_{D-4}\) are \(\{x^4, x^5, \ldots, x^{D-1}\}\) and the dualised field strength depends upon whether \(N_{D-4}\) has Euclidean or Minkowski signature. When \(N\) is Minkowski the dualised field strength is given by

\[
F_{\hat{D}i\hat{j}\hat{k}\hat{l}} = (-1)^{(D-1)(L-1)} D^L_i (NP_j) \quad \text{for } \hat{\xi} = \hat{1} \quad \text{and} \quad F_{\hat{D}i\hat{j}\hat{k}\hat{l}} = (-1)^{D(L-1)} D^L_i (NP_j) \quad \text{for } \hat{\xi} = \hat{2}.
\]

(7.66)

(7.67)

While if \(N\) is Euclidean (so that either \(x^3\) or \(x^D\) is the sole temporal coordinate) then the dual field strength is

\[
F_{\hat{D}i\hat{j}\hat{k}\hat{l}} = (-1)^{D(L-1)} D^L_i (NP_j) \quad \text{for } \hat{\xi} = \hat{1} \quad \text{and} \quad F_{\hat{D}i\hat{j}\hat{k}\hat{l}} = (-1)^{(D-1)(L-1)} D^L_i (NP_j) \quad \text{for } \hat{\xi} = \hat{2}.
\]

(7.68)

(7.69)

The signs of the pair of equations in each case have switched if \(N\) is chosen to have Euclidean rather than Minkowski signature. From the null geodesic motion on \(SL(2,\mathbb{R})/SO(1,1)\) we have \(P_{\xi} = N\partial_{\xi} N^{-1}\) and the set of equations above may be summarised by

\[
F_{\hat{D}i\hat{j}\hat{k}\hat{l}} = \kappa (\epsilon_{ij})^L D^L_i A_{\hat{D}j} \quad \text{where} \quad \begin{cases} 
\kappa = -(-1)^{(D-1)} & \text{if } N \text{ is Euclidean} \\
\kappa = -(-1)^{(D-1)(L-1)} & \text{if } N \text{ is Minkowski}
\end{cases}
\]

(7.70)

where \(i, j \in \{\hat{1}, \hat{2}\}\) and \(\epsilon_{ij}\) is the Levi-Civita symbol in the two-dimensional sub-space with coordinates \((x^1, x^2)\), normalised such that \(\epsilon_{\hat{1}\hat{2}} = 1\). The dual field strengths are derivatives of the off-diagonal components of the vielbein \(A_{\hat{g}}\)

\[
F_{\hat{D}i\hat{j}\hat{k}\hat{l}} = D_i D_j D^L_k A_{\hat{g}} \quad \text{for} \quad \hat{D} \equiv \hat{1}\hat{D} = D_i D_j A_{\hat{g}} \hat{D}_j = D_i D_j A_{\hat{g}} \quad \text{for} \quad \hat{D} \equiv \hat{2}\hat{D}
\]

(7.71)

where we have assumed that \(A_{g\hat{D}}\) is dependent only on the transverse coordinates \(x^1\) and \(x^2\). The combination of equations (7.70) and (7.71) give a monopole-like partial differential equation of order \(L + 1\) to solve for \(A_{\hat{g}\hat{D}}\) which may be trivially solved for a one-dimensional harmonic function \(N\).

We may unsmear the one-dimensional equation to two dimensions\(^5\) by taking advantage of the symmetry between the \(x^1\) and \(x^2\) coordinates in the metric (see equation (7.65)) for the objects associated to arbitrary level \(L\) generators of \(A^{++}_{D-3}\). The constraint, coming from the null geodesic motion on the coset, that \(N\) is a harmonic function is maintained so that \(N\) is a harmonic function in the two transverse dimensions \((x^1, x^2)\) and takes the form \(N = a + b \ln(r)\) where \(r^2 \equiv (x^1)^2 + (x^2)^2\). The consistent two-dimensional version of equations (7.70) and (7.71) gives

\[
D_i D_{j_1} D_{j_2} \cdots D_{j_L} A_{\hat{g}\hat{D}} = \kappa (\epsilon_{i_1 j_1})(\epsilon_{i_2 j_2}) \cdots (\epsilon_{i_L j_L}) D_i D_{j_1} D_{j_2} \cdots D_{j_L} N
\]

(7.72)

\(^5\)In section 7.3.1 we were able to unsmear the one-dimensional KK-monopole solution to three dimensions due to the isometries of the metric in \(\{x^1, x^2, x^3\}\).
where \( \hat{n}, \hat{r} \in \{1, \hat{2}\} \) for \( 0 < n \leq L \) and \( n \in \mathbb{Z} \). For guidance in determining solutions to this equation we isolate the terms with only partial derivatives, having integrated both sides with respect to \( x^l \) and setting the constant to zero, we have

\[
\partial_{j_1} \partial_{j_2} \ldots \partial_{j_L} A^\beta_3 = \kappa (e_{i_1 j_1})(e_{i_2 j_2}) \ldots (e_{i_L j_L}) \partial_{i_1} \partial_{i_2} \ldots \partial_{i_L} N \quad (7.73)
\]

which have a convenient set of solutions that can be summarised for the odd and even levels as

Odd \( L \) : \( A^\beta_3 = \kappa (-1)^{\frac{L+1}{2}} B \) \quad (7.74)

Even \( L \) : \( A^\beta_3 = \kappa (-1)^{\frac{L}{2}} N \) \quad (7.75)

where \( B \) is the harmonic conjugate of \( N \) such that \( \partial_iB = \epsilon_{ij} \partial_jN \). Equation (7.73) gives \( 2^L \) equations to solve each of which is identical to one of the \( L + 1 \) equations of the form

\[
\partial^n_1 \partial^n_2 \ldots \partial^n_L A^\beta_3 = \kappa (-1)^{L-n} \partial^n_1 \partial^n_2 \ldots \partial^n_L N \quad (7.76)
\]

where \( 0 \leq n \leq L \) for \( n \in \mathbb{Z} \). Consider first the case when \( L \) is even: upon substitution of \( A^\beta_3 = \kappa (-1)^{\frac{L-1}{2}} N \), by virtue of the commutativity of the partial derivative, there remain only \( \frac{L}{2} \) equations to solve (those for which \( 0 \leq n < \frac{L}{2} \)). As \( N \) is harmonic in \( (x^1, x^2) \) we have \( \partial^2_1 N = -\partial^2_2 N \) and by applying this identity \( m = \frac{L}{2} - n \) times we see the \( \frac{L}{2} \) equations are all solved identically:

\[
\partial^n_1 \partial^n_2 \ldots \partial^n_L A^\beta_3 = \kappa (-1)^{\frac{L+1}{2} + m} \partial^n_1 \partial^n_2 \ldots \partial^n_L N = \kappa (-1)^{L-n} \partial^n_1 \partial^n_2 \ldots \partial^n_L N. \quad (7.77)
\]

For odd values of \( L \) we substitute equation (7.74) into (7.73) to obtain

\[
\kappa (e_{j_1 j_L})(e_{j_2 j_{L-1}}) \ldots \partial_{j_1} \partial_{j_2} \ldots \partial_{j_L-1} N = \kappa (e_{i_1 j_1})(e_{i_2 j_2}) \ldots (e_{i_L j_L}) \partial_{i_1} \partial_{i_2} \ldots \partial_{i_L} N \quad (7.78)
\]

which simplifies to (dropping constant terms)

\[
(-1)^{\frac{L+1}{2}} \partial_{j_1} \partial_{j_2} \ldots \partial_{j_{L-1}} N = (e_{i_1 j_1})(e_{i_2 j_2}) \ldots (e_{i_{L-1} j_{L-1}}) \partial_{i_1} \partial_{i_2} \ldots \partial_{i_{L-1}} N \quad (7.79)
\]

where, as \( L - 1 \) is even, these equations are identical to the set obtained for even \( L \) and shown to be identities in equation (7.77).

The proof that equations (7.74) and (7.75) are solutions of equation (7.73) relied solely upon the fact that partial derivatives commute. The covariant derivatives of the full equation (7.72) do not in general commute. However if the harmonic function \( N \) is further constrained to be a holomorphic or anti-holomorphic function in the complex variables \( z = x^1 + ix^2 \) or \( \bar{z} = x^1 - ix^2 \) then the component of the curvature tensor \( \hat{R}_{12}^{12} \) in the transverse space vanishes for arbitrary level (see Appendix D.1) and the covariant derivatives in these coordinates do commute. This observation guarantees that the equations for the dual field strength may be rearranged so that they take the form

\[
\mathcal{D}_{1}^{n} \mathcal{D}_{2}^{L-n} A^\beta_3 = \kappa (-1)^{L-n} \mathcal{D}_{1}^{n} \mathcal{D}_{2}^{L-n} N \quad (7.80)
\]
where $0 \leq n \leq L$ for $n \in \mathbb{Z}$. We note that $D_1 D_1 N = -D_2 D_2 N$ and so the arguments for the partial derivatives acting on the harmonic function carry across to the covariant derivatives acting on the (anti-)holomorphic function and the vielbein component $A_3 \hat{D}$ is given by equations (7.74) and (7.75) for the odd and even level fields.

The form of the full metric depends on the whether the dual gravition field appears at an even or odd level in the decomposition of $A_4^{1+1}$. 

**SL(2, \mathbb{R})/SO(1, 1): arbitrary even levels**

The even level root given in equation (7.56) is associated with a coset $SL(2, \mathbb{R})/SO(1, 1)$ if the temporal coordinate is $x^i$ or $x^j$ and consequently the flat $D - 4$-dimensional space $\mathcal{N}$ is Euclidean. Hence from equation (7.70) we have $\kappa = (-1)^{(1 - D)}$ and from equation (7.75) $A_3 \hat{D} = (-1)^{(1 - D + \frac{L}{2})} N$. For the example with $i = D$ and $j = 3$ we have

$$ds^2 = NL \left( (dx^1)^2 + (dx^2)^2 \right) + N(dx^3)^2 - N^{-1}(dx^D - A_3 \hat{D} dx^3)^2 + d\Sigma^2_{(D-4)}$$  

(7.81)

when $x^D$ is the temporal coordinate, and

$$ds^2 = NL \left( (dx^1)^2 + (dx^2)^2 \right) - N(dx^3)^2 + N^{-1}(dx^D - A_3 \hat{D} dx^3)^2 + d\Sigma^2_{(D-4)}$$  

(7.82)

when $x^3$ is the temporal coordinate. The only non-zero components of the Einstein tensor for this metric are proportional to $\left((\partial_1 N)^2 + (\partial_2 N)^2\right)$ when $N$ is harmonic. For all levels, except the level 0 solution which trivially satisfies the Einstein equations, this becomes a vacuum solution when $N$ is (anti-)holomorphic.

**SL(2, \mathbb{R})/SO(1, 1): arbitrary odd levels**

The odd level root given in equation (7.56) is associated with a coset $SL(2, \mathbb{R})/SO(1, 1)$ if the temporal coordinate is neither $x^i$ or $x^j$ but one of the set $\{x^4, x^5, \ldots, x^{D-1}\}$ and the flat $D - 4$-dimensional space-time $\mathcal{N}$ is Minkowski. Hence from equation (7.70) we have $\kappa = 1$ and from equation (7.74) $A_3 \hat{D} = (-1)^{\frac{L+1}{2}} B$ where $\partial_1 B = \partial_2 N$ and $\partial_2 B = -\partial_1 N$. For the example with $i = D$ and $j = 3$ we have

$$ds^2 = NL \left( (dx^1)^2 + (dx^2)^2 \right) + N(dx^3)^2 + N^{-1} \left( dx^D - A_3 \hat{D} dx^3 \right)^2 + d\Sigma^2_{(1, D-5)}.$$  

(7.83)

The only non-zero components of the Einstein tensor for this metric are again proportional to $\left((\partial_1 N)^2 + (\partial_2 N)^2\right)$. Taking $N$ to be (anti-)holomorphic ensures this is a solution to the vacuum Einstein equations.

### 7.3.3 $SL(3, \mathbb{R})/SO(1, 2)$: composite gravitational solutions

In this section we will construct what we will refer to as bound states of KK-monopoles consisting of pairs of the solutions described in section 4.2. To do this we will truncate $A_4^{1+1}$ to $\mathfrak{sl}(3, \mathbb{R})$ sub-algebras whose simple positive roots consist of two real roots which
individually are the simple positive roots of two of the $\mathfrak{sl}(2, \mathbb{R})$ sub-algebras discussed in section 7.3.1 and 7.3.2.

Bound states of two KK-monopoles, just as for the dyonic membrane - the bound state of the membrane and fivebrane in supergravity [28] discussed in the context of $E_{11}$ in [25, 26], correspond to null geodesics on the coset of $\frac{SL(3, \mathbb{R})}{SO(1, 2)}$. The algebra $\mathfrak{sl}(3, \mathbb{R})$ has non-trivial commutators

$$\begin{align*}
[H_1, E_1] &= 2E_1, \quad [H_1, E_2] = -E_2, \quad [H_1, E_{12}] = E_{12} \\
[H_2, E_1] &= -E_1, \quad [H_2, E_2] = 2E_2, \quad [H_1, F_{12}] = F_{12} \\
[E_1, E_2] &= E_{12} \\
[F_1, F_2] &= -F_{12}
\end{align*}$$

(7.84)

and the sub-algebra $\mathfrak{so}(1, 2)$ is invariant under the involution $\Omega$ which acts as $\Omega(E_1) = F_1$, $\Omega(E_2) = -F_2$, $\Omega(E_{12}) = F_{12}$, $\Omega(H_1) = -H_1$ and $\Omega(H_2) = -H_2$. There are three canonical $\mathfrak{sl}(2, \mathbb{R})$ sub-algebras within $\mathfrak{sl}(3, \mathbb{R})$ and truncation to any of these three sub-algebras leaves one of two cosets either $\frac{SL(2, \mathbb{R})}{SO(1, 1)}$ or $\frac{SL(2, \mathbb{R})}{SO(2)}$. For the involution given above the subsets of generators $\{E_1, H_1, F_1\}$ and $\{E_{12}, H_1 + H_2, F_{12}\}$ defines the coset $\frac{SL(2, \mathbb{R})}{SO(1, 1)}$ and correspond to KK-monopole solutions and other solutions as described in section 7.3.1 and 7.3.2, while the generators $\{E_2, H_2, F_2\}$ and the involution define an $\frac{SL(2, \mathbb{R})}{SO(2)}$ coset. In this way the full solution related to the null geodesic on the full coset $\frac{SL(3, \mathbb{R})}{SO(1, 2)}$ is understood to correspond to a bound state of a pair of KK-monopole and similar objects. There are two harmonic functions $N_1$ and $N_2$ which are used to define the solution and related to each other by

$$N_2 = \sin^2(\beta) + \cos^2(\beta)N_1$$

(7.85)

where $\beta \in \mathbb{R}$, each of which may be thought of as the harmonic function defining the $SL(2, \mathbb{R})$ coset solutions of the previous sections. The parameter $\beta$ encodes the action of the generator of the compact symmetry in $\mathfrak{so}(1, 2)$ which transforms the charges of the harmonic functions.\(^6\)

**Bound states consisting of two KK-monopoles** We will construct the bound state solutions which possess a common two-dimensional transverse space whose simple positive roots both appear at level one in the decomposition of $A_1^{++}$. Since $\mathfrak{sl}(3, \mathbb{R})$ cannot be constructed from level one roots in $A_1^{++}$ this is only applicable for algebras with $D \geq 5$ and the solutions we find will be five-dimensional solutions embedded in a $D$-dimensional background, that is space-time manifolds found are product manifolds of the form $M_5 \otimes N_{D-5}$, where the dimension of the manifold is indicated by the subscript label and $N_{D-5}$ is either a Euclidean or a Minkowski space. Consider the pair of real level one roots of $A_1^{++} \(^6\)See section 3.6 of [26] for a discussion of the action of the compact transformations of the sub-group $SO(1, 2)$.
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given by

\[ \alpha_1 = e_D - e_3 + \sum_{i=3}^{D} e_i \]  

(7.86)

\[ \alpha_2 = e_{D-1} - e_D + \sum_{i=3}^{D} e_i \]  

(7.87)

and note that these satisfy

\[ \langle \alpha_1, \alpha_2 \rangle = -1 \]  

(7.88)

and so are the simple positive roots of an \( \mathfrak{sl}(3, \mathbb{R}) \) sub-algebra whose third positive root is

\[ \alpha_1 + \alpha_2 = e_{D-1} - e_3 + 2 \sum_{i=3}^{D} e_i. \]  

(7.89)

They have the Cartan elements

\[ H_1 = - (K^1_1 + K^2_2 + K^3_3) + K^D_D \]  

(7.90)

\[ H_2 = - (K^1_1 + K^2_2 + K^D_D) + K^{(D-1)}_{(D-1)} \]  

(7.91)

and the three positive generators for this example are

\[ E_1 = R^{3\ldots D|D}, \ E_2 = R^{3\ldots (D-1)|(D-1)} \quad \text{and} \quad E_{12} = [E_1, E_2] = - R^{3\ldots D|4\ldots D|(D-1)}. \]  

(7.92)

Imposing the involution to act as \( \Omega(E_1) = F_1 \) and \( \Omega(E_2) = -F_2 \) is equivalent to setting \( x^{D-1} \) to be the temporal coordinate. Bound state solutions are identified with null-geodesics on cosets of \( \frac{SO(3, \mathbb{R})}{SO(1,2)} \) where the representative coset group element is expressed as

\[ g = \exp(\phi_1 H_1 + \phi_2 H_2) \exp(C_1 E_1 + C_2 E_2 + C_{12} E_{12}) \]

where \( \phi_1, \phi_2, C_1, C_2 \) and \( C_{12} \) are functions of the null geodesic parameter \( \xi \). These brane coset models were first solved for bound state solutions in [26] where the ansatz \( \phi_1 = \frac{1}{2} \ln N_1 \) and \( \phi_2 = \frac{1}{2} \ln N_2 \) is used. The diagonal portion of the metric for our bound state as

\[ ds_{\text{diagonal}}^2 = N_1 N_2 \left( (dx^1)^2 + (dx^2)^2 \right) + N_1 (dx^3)^2 - N_2^{-1} (dx^{D-1})^2 + \frac{N_2}{N_1} (dx^D)^2 + d\Omega^2_{(D-5)}, \]

(7.93)

where \( N_1 = a + b \xi, \ N_2 = c + d \xi \) and \( \xi \) is the parameter labelling translations along the null geodesic. As for the \( \mathfrak{sl}(2, \mathbb{R}) \) case described in detail earlier the Maurer-Cartan form is split as \( \partial_\xi g^{-1} = \mathcal{P}_\xi + Q_\xi \) where \( Q_\xi \) indicates generators of the \( \mathfrak{so}(2, 1) \) algebra of the isometry group of the coset. The remainder of the coset is spanned by the Cartan elements \( H_1, H_2 \) and \( H_{12} \) and \( S_1 = \frac{1}{2} (E_1 - F_1), \ S_2 = \frac{1}{2} (E_2 + F_2) \) and \( S_{12} = \frac{1}{2} (E_{12} - F_{12}) \), where \( F_1, F_2 \) and \( F_{12} \) are the generators associated to the negative roots \( -\alpha_1, -\alpha_2 \) and \( -\alpha_{12} \). The coefficients of the \( S_1, S_2 \) and \( S_{12} \) generators in the Maurer-Cartan form are found [26] to be

\[ \mathcal{P}_{\xi,1} = - \sqrt{\frac{\alpha}{b}} \frac{\partial_\xi N_1}{N_1 \sqrt{N_2}}, \ \mathcal{P}_{\xi,2} = - \sqrt{\frac{\alpha}{d}} \frac{\partial_\xi N_2}{N_2 \sqrt{N_1}} \quad \text{and} \quad \mathcal{P}_{\xi,12} = - \sqrt{\frac{d}{b}} \frac{\partial_\xi N_1}{\sqrt{N_1 N_2}}, \]

(7.94)
where $\alpha = N_2 \partial_\xi N_1 - N_1 \partial_\xi N_2 = bc - ad$. The form of these $P_{\xi,i}$ are chosen so that terms in the square root correspond to functions of the parameter $\beta$ which is mapped between $[0, \pi/2]$ by the compact local symmetry. Specifically the harmonic functions are $N_1 = 1 + q \xi$ and $N_2 = 1 + q \xi \cos^2 \beta$ so that $\alpha = q \sin^2 \beta$. Each $P_{\xi}$ corresponds to a dual-gravity field whose construction is given by the same dictionary used in the previous sections:

$$F_{\xi 4...D}\mid_{D} = P_{\xi,1},$$
$$F_{\xi 3...D-1}\mid_{D-1} = P_{\xi,2} \quad \text{and} \quad (7.95)$$
$$F_{\xi 4...[3...D-1} = D\xi P_{\xi,12}.\quad (7.96)$$

We first identify $\xi$ with one of the transverse coordinates $x^1$ or $x^2$. After dualisations and transformations, which are identical to those performed in previous sections, the dual field strengths are, when $\xi$ is identified with $x^1$,

$$F_{23}\mid_{D} = \sin(\beta) \frac{\partial_1 N_1}{N_2} = D_2 (A_1)_3 \mid_{D},\quad (7.97)$$
$$F_{2\mid3-1} = (-1)^D \tan(\beta) \frac{\partial_1 N_2}{N_1} = D_2 (A_2)_\mid_{D-1} \quad \text{and} \quad (7.100)$$
$$F_{2\mid23-1} = (-1)^D \cos(\beta) D_1 D_1 N_1 = D_2 D_2 (A_{12})_3 \mid_{D-1}.\quad (7.101)$$

As with the previous examples, the one-dimensional solutions may be unsmeared using the symmetry of the transverse directions to two-dimensional fields which are functions of $x^1$ and $x^2$. The full equations are then

$$F_{i3}\mid_{D} = \sin(\beta) \epsilon_{ij} \frac{\partial_j N_1}{N_2} = D_i (A_1)_3 \mid_{D},\quad (7.99)$$
$$F_{i\mid3-1} = (-1)^D \tan(\beta) \epsilon_{ij} \frac{\partial_j N_2}{N_1} = D_i (A_2)_\mid_{D-1} \quad \text{and} \quad (7.100)$$
$$F_{i3}\mid_{D-1} = (-1)^D \cos(\beta) \epsilon_{ij} \epsilon_{i1j1} D_1 D_1 N_1 = D_i D_i (A_{12})_3 \mid_{D-1}.\quad (7.101)$$

An integrability condition on these equations is provided, in the two dimensional space, by

$$\text{d} A_1 = \sin(\beta) \frac{\text{d} N_1}{N_2} \quad \Rightarrow \quad 0 = \text{d} d A_1 = \sin(\beta) \text{d} \left( \frac{\text{d} N_1}{N_2} \right)\quad (7.102)$$

or, in components,

$$0 = \epsilon^{ki} \partial_k \partial_i (A_1)_3 \mid_D = \sin(\beta) \epsilon^{ki} \epsilon_{ij} \partial_k \left( \frac{\partial_j N_1}{N_2} \right) \quad (7.103)$$

so that, in order for this system to be integrable we need:

$$\partial_i \left( \frac{\partial^i N_1}{N_2} \right) = 0 \quad (7.104)$$
with summation over \( i \) implied. If we take \( N_1 \) to be harmonic and \( N_2 \) to limit to \( N_1 \) as described by (7.85) the integrability condition reduces to:

\[
(\partial_1 N_1)^2 + (\partial_2 N_1)^2 = 0. \tag{7.106}
\]

This is solved by taking \( N_1 \), and subsequently \( N_2 \), to be (anti-)holomorphic, which was the same condition required for the component KK-monopole solutions to be vacuum Einstein solutions. The holomorphic functions \( N_1 = 1 + f(z) \) and \( N_2 = 1 + \cos^2(\beta)f(z) \) have dual gravity fields given by

\[
\begin{align*}
(A_1)_{3}^D &= B_1 \sin \beta \tag{7.107} \\
(A_2)_{D}^{D-1} &= (-1)^D B_2 \tan \beta \tag{7.108} \\
(A_{12})_{3}^{D-1} &= (-1)^{(D+1)} N_1 \cos \beta \tag{7.109}
\end{align*}
\]

where

\[
\begin{align*}
\partial_i B_1 &= \epsilon_{ij} \frac{\partial_j N_1}{N_2} \tag{7.110} \\
\partial_i B_2 &= \epsilon_{ij} \frac{\partial_j N_2}{N_1} \tag{7.111}
\end{align*}
\]

When \( \beta = \frac{\pi}{2} \) this reduces to a single KK-monopole solution, while when \( \beta = 0 \) we find a level two object as described section 7.3.1. One can solve equations (7.107) and (7.108) to find

\[
\begin{align*}
(A_1)_{3}^D &= i \tan(\beta) \sec(\beta) \ln(N_2) \quad \text{and} \\
(A_2)_{D}^{D-1} &= i(-1)^D \sin(\beta) \cos(\beta) \ln(N_1). \tag{7.112}
\end{align*}
\]

In the case of anti-holomorphic functions where \( f = f(\bar{z}) \) the \( A_1 \) and \( A_2 \) solutions are simply the opposite sign of those above. Including the off-diagonal dual gravity fields we find the metric

\[
ds^2 = N_1 N_2 \left( (dx^1)^2 + (dx^2)^2 \right) + N_1 (dx^3)^2 + d\Omega^2_{[D-5]} - N_2^{-1} \left( dx^D - (A_2)_{D}^{D-1} dx^D - (A_{12})_{3}^{D-1} dx^3 \right)^2 + \frac{N_2}{N_1} \left( dx^D - (A_1)_{3}^D dx^3 \right)^2. \tag{7.114}
\]

We recall the construction of the harmonic functions in equation (7.85) of the model and find that when \( \beta = \pi/2 \) the harmonic function \( N_2 = 1 \) and we recover the individual root solution corresponding to \( \alpha_1 = (D,3)_1 \). When \( \beta = 0 \) only the \( A_{12} \) gauge field is present and \( N_1 = N_2 \equiv N \) leaves us with precisely the \( \mathfrak{sl}(3,\mathbb{R}) \) model solution with the level 2 root \( \alpha_{12} = \alpha_1 + \alpha_2 \). The metric in equation (7.114) with (anti-)holomorphic \( N_1 \) and \( N_2 \) has a

\[
\text{This integrability condition can also be satisfied for arbitrary harmonic functions} \ N_1 \text{ and } N_2 \text{ when they are harmonic conjugates. However, the only pair of harmonic functions which would limit to each other as described by equation (7.85) are constant functions.}.
\]
vanishing Ricci scalar for all $\beta$ values (see Appendix B) and has a vanishing curvature tensor at the endpoints which correspond with the level 1 and level 2 solutions. However the bound state is not a solution to the vacuum Einstein equations for $\beta$ taking values in the open set $(0, \frac{\pi}{2})$. In sections 5 and 6, by considering the lift of the sigma-model to $D$ dimensions, we will investigate this obstruction to the full interpolating bound state being a solution to the Einstein-Hilbert action.

### 7.3.4 $\text{SL}(3, \mathbb{R})/\text{SO}(1, 2)$: composite gravitational solutions with arbitrary levels

The above construction can be generalised to include arbitrary level roots and we present here the $\mathfrak{sl}(3, \mathbb{R})$ model which has simple real roots found at arbitrary level $L_1$ and $L_2 = 1$ which will possess an $\mathfrak{so}(1, 2)$ invariant sub-algebra and whose associated solutions have a common two dimensional transverse space. Taking

$$\alpha_1 = e_D - e_3 + L_1 \sum_{i=3}^D e_i \quad \text{and} \quad \alpha_2 = e_{D-1} - e_D + \sum_{i=3}^D e_i,$$

(7.156)

whose inner product is $-1$, the algebra associated with these roots

$$H_{\alpha_1} = -(K^1_1 + K^2_2) + K^3_3 - K^D_D, \quad H_{\alpha_2} = -(K^1_1 + K^2_2 + K^D_D) - K^{D-1}_{D-1},$$

(7.17)

$$E_1 = R^{D[4...D][3...D]_1[3...D]_{L_1-1}}, \quad E_2 = R^{D-1[3...D]-1}, \quad E_{12} = R^{D-1[4...D][3...D]_1[3...D]_{L_1}}$$

leads us to identify the $\mathcal{P}_i$ fields from the coset model equations of motion with fields

$$F_{D}[\xi...D][\xi3...D]_{1[3...D]_{L_1-1}} = D^{L_1-1}_\xi \mathcal{P}_1, \quad F_{D-1[3...D]-1} = \mathcal{P}_{\xi, 2}, \quad F_{D-1[4...D][3...D]_1[3...D]_{L_1}} = D^{L_1}_\xi \mathcal{P}_{\xi, 12}.$$

(7.118) - (7.120)

We may now employ the same techniques from previous sections to obtain the differential equations which describe our dual gravity fields and unsmear these to find

$$F_{i3[\hat{(i)}_1]...[\hat{(i)}_{L_1-1}] D} = \sin(\beta) D_{i1}^{L_1-1} \left( \frac{\partial_j N_1}{N_2} \right) = (\epsilon_{ij})^{L_1} D_{j1}^{L_1}(A_1)^D_{\hat{3}},$$

(7.121)

$$F_{iD}^{D-1} = \tan(\beta) \frac{\partial_j N_2}{N_1} = \epsilon_{ij} D_j^{D-1}(A_2)^D_{\hat{3}},$$

(7.122)

$$F_{i3[\hat{(i)}_1]...[\hat{(i)}_{L_1}] D}^{D-1} = \cos(\beta) D_{i1}^{L_1+1} N_1 = (\epsilon_{ij})^{L_1+1} D_{j1}^{L_{1+1}}(A_2)^D_{\hat{3}}.$$

(7.123)

As we had found in the $\mathfrak{sl}(3, \mathbb{R})$ model above, these equations only have complex solutions where $N_1$ and $N_2$ are (anti-)holomorphic functions. The methods used to find arbitrary level
solutions in previous sections are still valid since $D_1D_1N = -D_2D_2N$ for harmonic functions
$N$ and $R_1212 = 0$. We therefore find that

\[ A_1 = (i)^{L_1}\tan(\beta)\sec(\beta) \log N_2 \]  \hspace{1cm} (7.124)
\[ A_2 = i\sin(\beta)\cos(\beta) \log N_1 \]  \hspace{1cm} (7.125)
\[ A_{12} = (i)^{L_1+1}\cos(\beta)N_1 \]  \hspace{1cm} (7.126)

when $N_1$ and $N_2$ are holomorphic. When they are anti-holomorphic every $i$ is replaced by $-i$.
We note that these solutions are valid for odd and even $L_1$ when the involution is correctly
chosen so that $\Omega(E_1) = F_1$ and $\Omega(E_2) = -F_2$.

We must now specify the level $L_1$ and find the appropriate involution in order to build
the full solutions. This requires us to consider the odd and even $L_1$ separately. For odd $L_1$
the involution required has $t = x_{D-1}$ and for even $L_1$ $t = x_D$ so that the full set of solutions
of this form are given by

\[
\begin{aligned}
    ds^2 &= N_1^{L_1} N_2 \left( dx_1^2 + dx_2^2 \right) + N_1 dx_3^2 + (-1)^{L_1} N_2^{-1} (dx_{D-1} + A_2 dx_D + A_{12} dx_3)^2 \\
    &\quad + (-1)^{L_1+1} N_2 \frac{N_2}{N_1} \left( dx_D + A_1 dx_3 \right)^2 + d\Omega^2_{(D-5)}.
\end{aligned}
\]  \hspace{1cm} (7.127)

### 7.4 The supergravity dictionary and multiforms.

The dimensional reduction to three-dimensions of a $D$-dimensional theory allows the remaining
part of the theory to be expressed in terms of scalars that parameterise a coset. The
scalars of a general theory arise from both the gravity and the matter sectors of the theory,
however upon dimensional reduction some information and structure of the $D$-dimensional
theory is lost. In particular, without this $D$-dimensional information, there is no informa-
tion in the three-dimensional theory concerning the index-structure of the $D$-dimensional
field strength which sources any scalar field. In the present section we face the problem of
lifting the one-dimensional coset invariant Lagrangian to $D$-dimensions where the fields are
mixed-symmetry tensors of $GL(D, \mathbb{R})$. For the coset $\frac{SL(3, \mathbb{R})}{SO(1,2)}$ the invariant Lagrangian (7.14)
is

\[
\mathcal{L} = -2(\partial_\xi \phi_1)^2 - 2(\partial_\xi \phi_2)^2 + 2(\partial_\xi \phi_1)(\partial_\xi \phi_2) + \frac{1}{2}(P_{\xi,1})^2 - \frac{1}{2}(P_{\xi,2})^2 + \frac{1}{2}(P_{\xi,12})^2
\]  \hspace{1cm} (7.128)

where we have set the lapse function $\eta$ to minus one and $\xi$ denotes the single spatial coordi-
nate. The $\phi_i$ encode the diagonal components of the vielbein from which the diagonal entries
of the $D$-dimensional metric may be reconstructed. The $\phi_i$ terms lift to the Ricci scalar con-
structed from the diagonal entries of the $D$-dimensional metric as shown in appendix D.2.
The remaining $P_{\xi,i}$ terms correspond to kinetic terms for the three mixed symmetry fields
associated with the Borel sub-algebra of $\mathfrak{sl}(3, \mathbb{R})$ being considered. As commented upon in the
earlier sections there is an ambiguity in identifying $P_{\xi,i}$ with $D$-dimensional field strengths:
the supergravity dictionary for mixed symmetry fields remains to be written. Consider the
example embedded at levels zero, one and two of the decomposition of the $A^{++}_{D-3}$ algebra where the mixed-symmetry gauge fields are $A_{\mu_1...\mu_{D-3}[^\nu}$ and $A_{\mu_1...\mu_{D-2}[\rho|\nu_1...\nu_{D-3}]|^\rho$ (indices labelled with the same letter are implicitly antisymmetrised). Let the notation $\Omega[^{[a_1|a_2|...|a_n]}_\mu_1...\mu_{D-3}[^\nu] \in \Omega^{[D-3][1]}$ and $A_{\mu_1...\mu_{D-2}|a_1...|a_\nu_{D-3}[^\rho} \in \Omega^{[D-2][D-3][1]}$. The exterior derivative acts on the space of mixed-symmetry tensors as

$$\mathcal{d} : \Omega^{[a_1|a_2|...|a_n]} \rightarrow \Omega^{[a_1+1|a_2|...|a_n]} \oplus \Omega^{[a_1|a_2+1|...|a_n]} \oplus ... \oplus \Omega^{[a_1|a_2|...|a_n+1]} \oplus \Omega^{[a_1|a_2|...|a_n][1]}$$

(7.129)

by introducing a partial derivative which is projected with the symmetries of each of the multi-form spaces [98–102]. We include the possibility indicated by the last multi-form in the sequence above that the derivative is not-antisymmetrised with respect to any of the indices of the multi-form that it acts on. For example the exterior derivative acts on level one multiform components to define a set of field strength components within three different spaces of mixed symmetry forms

$$\mathcal{d} : (A_{\mu_2...\mu_{D-2}[\nu_2} \rightarrow \begin{cases} \partial_{\mu_1}A_{\mu_2...\mu_{D-2}[\nu_2} \in \Omega^{[D-3][1]} \\ \partial_{\nu_1}A_{\mu_2...\mu_{D-2}[\nu_2} \in \Omega^{[D-2][2]} \\ \partial_{\rho}A_{\mu_2...\mu_{D-2}[\rho} \in \Omega^{[D-2][1][1]} \end{cases}.$$ (7.130)

Unlike form fields, where the exterior derivative takes $p$-form gauge fields to $p+1$-form fields strengths, a multi-form gauge field is mapped to multiple multi-form field strengths. It is therefore ambiguous which higher-dimensional field strength components should be preferred in the supergravity dictionary and equated with $\mathcal{P}_\xi,1$ and $\mathcal{P}_\xi,2$. We will argue that there is a minimal consistent way to identify $\mathcal{P}_\xi,\mathcal{d}$ which is indicated by the embedding of the $\mathfrak{sl}(3,\mathbb{R})$ into the algebra of $A^{++}_{D-3}$ and our guiding principle will be to ensure that the equations for $\mathcal{P}_\xi$ which reflect the $\mathfrak{sl}(3,\mathbb{R})$ structure are maintained by the dictionary.

For a representative coset element

$$g = \exp(\phi_1 H_1 + \phi_2 H_2) \exp(C_1 E_1 + C_2 E_2 + C_{12} E_{12})$$

(7.131)

we compute

$$\mathcal{P}_\xi,1 = \exp(2\phi_1 - \phi_2)\partial_\xi C_1,$$

$$\mathcal{P}_\xi,2 = \exp(2\phi_2 - \phi_1)\partial_\xi C_2$$

and

$$\mathcal{P}_\xi,12 = \exp(\phi_1 + \phi_2)(\partial_\xi C_{12} - \frac{1}{2}\partial_\xi C_1 C_2 + \frac{1}{2}\partial_\xi C_2 C_1)$$

$$= \exp(\phi_1 + \phi_2)\partial_\xi C_{12} - \frac{1}{2}\mathcal{P}_\xi,1 C_2 + \frac{1}{2}\mathcal{P}_\xi,2 C_1.$$

(7.132)

Our proposal will be most simply motivated by first considering a simple alternative dictionary. Suppose that, contrary to our proposition, the multiforms were treated as form fields by declaring that a set of their antisymmetric indices are in the privileged position
of being space-time form indices while the remaining indices are treated as internal indices. For example suppose that we treat \( A_{\mu_1...\mu_{D-3}|\nu} \) as a \( D-3 \) form (carrying an internal vector index) and \( A_{\mu_1...\mu_{D-2}|\nu\nu_D...\nu} \) as a \( D-2 \) form with corresponding field strengths given by

\[
F_{[D-2]|1} = (D-2)\partial_{\mu_1}A_{\mu_2...\mu_{D-2}|\nu} \quad \text{and} \quad G_{[D-1]|D-3|1} = (D-1)\partial_{\nu_1}A_{\nu_2...\nu_{D-3}|\rho} - \frac{(D-2)}{2} \partial_{\nu_1}A_{\nu_2...\nu_{D-3}|\rho D_{\nu_1}\nu_D...\nu_{D-3}|\mu_{D-2}} + \frac{(D-2)}{2} \partial_{\nu_1}A_{\nu_2...\nu_{D-3}|\rho}A_{\nu_1...\nu_{D-3}|\mu_{D-2}}.
\]

The definition of \( G_{[D-1],D-3,1} \) is found using

\[
[R^{a_1a_2...a_{D-3}|b}, R^{c_1c_2...c_{D-3}|d}] = R^{a_1...a_{D-3}|b|c_1...c_{D-3}|d} - R^{c_1...c_{D-3}|b|a_1...a_{D-3}|d} + \ldots
\]

where the ellipsis indicates level two generators in the full \( \mathcal{a}_{D-3} \) algebra beyond the truncation to the \( \mathfrak{s}(3, \mathbb{R}) \) algebra encoding the bound state we are considering here. The gauge transformations are:

\[
\delta A_{\mu_1...\mu_{D-3}|\nu} = (D-3)\partial_{\mu_1}A_{\mu_2...\mu_{D-3}|\nu} \quad \text{(7.136)}
\]

\[
\delta A_{\mu_1...\mu_{D-2}|\nu\nu_D...\nu} = (D-2)\partial_{\nu_1}A_{\nu_2...\nu_{D-3}|\rho} - \frac{(D-4)}{2}(D-3)\partial_{\nu_1}A_{\nu_2...\nu_{D-3}|\rho}A_{\nu_1...\nu_{D-3}|\mu_{D-2}} + A_{\mu_1...\mu_{D-3}|\nu_1}A_{\nu_2...\nu_{D-3}|\mu_{D-2}}.
\]

By comparison with the expression for \( \mathcal{P}_{\xi,2} \) in equation (7.132) we see that while \( \mathcal{P}_{\xi,2} \) is identified with a component of \( F_{[D-2]|1} \), \( \mathcal{P}_{\xi,1} \) would be identified with \( 2\partial_{\mu_1}A_{\nu_1...\nu_{D-3}|\rho_2} \equiv F_{[D-3]|2} \). The dictionary is not well defined at level one as the two level one fields are treated differently. Consider instead the proposition that the exterior derivative acts on level two multiform fields in the following minimal manner

\[
\mathcal{d} : \Omega^{[D-2]|D-3|1} \to \Omega^{[D-1]|D-3|1} \oplus \Omega^{[D-2]|D-2|1} \oplus \Omega^{[D-2]|D-3|2} \quad \text{(7.138)}
\]

we refer to this as a minimal action when we neglect the mapping into the multiforms with the symmetries of four column wide Young tableaux. At level one we consider the full mapping:

\[
\mathcal{d} : \Omega^{[D-3]|1} \to \Omega^{[D-2]|1} \oplus \Omega^{[D-3]|2} \oplus \Omega^{[D-3]|1|1} \quad \text{(7.139)}
\]

so that in both cases the derivative is distributed across three sets of indices. Consider a five-dimensional example\(^8\) constructed using the Borel sub-algebra

\[
H_1 = -(K_{1}^{1} + K_{2}^{2}) - K_{3}^{3} + K_{5}^{5},
\]

\[
H_2 = -(K_{1}^{1} + K_{2}^{2}) - K_{5}^{5} + K_{4}^{4},
\]

\[
E_1 = R^{45|5}, \quad E_2 = R^{34|4} \quad \text{and} \quad E_{12} = R^{345|45}|4
\]

\(^8\)As the bound states are product spaces in which only a five-dimensional sub-manifold has a non-trivial metric, we will not lose any generality by focussing on a five-dimensional example.
where the involution $\Omega$ is chosen to be consistent with taking $x^4$ as the single temporal coordinate. The field strengths are exterior derivatives of multiform tensors $A_{\mu_1\mu_2|\nu}$ and $A_{\mu_1\mu_2\mu_3|\nu_1\nu_2\nu_3}$ and the dictionary identifies $P_\xi$ with components across different multiform spaces which reduce to a sum of vectors as

$$
P_{\xi, 1} = F_{\xi 45|5} + F_{45|5} + F_{45|5} = \partial_\xi A_{45|5}, \tag{7.141}
$$

$$
P_{\xi, 2} = F_{\xi 34|4} + F_{34|4} + F_{34|4} = \partial_\xi A_{34|4} \quad \text{and} \quad \tag{7.142}
$$

$$
P_{\xi, 12} = G_{\xi 345|45|4} + G_{345|45|4} + G_{345|45|4} = \partial_\xi A_{345|45|4}. \tag{7.143}
$$

Now, as vectors,

$$
F_{\xi 45|5} = F_{45|5} = F_{45|5} = \partial_\xi (A_{45|5}) \tag{7.144}
$$

whereas upon the lift to five dimensions these components, while all equal, arise from three different multiform field strengths. While at level two we have

$$
G_{\xi 345|45|4} = \partial_\xi A_{345|45|4} - \frac{1}{2} F_{45|5} A_{34|4} + \frac{1}{2} F_{34|4} A_{45|5} \tag{7.145}
$$

$$
= G_{345|45|4} = \partial_\xi A_{345|45|4} - \frac{1}{2} F_{45|5} A_{34|4} + \frac{1}{2} F_{34|4} A_{45|5} \tag{7.146}
$$

$$
= G_{345|45|4} = \partial_\xi A_{345|45|4} - \frac{1}{2} F_{45|5} A_{34|4} + \frac{1}{2} F_{34|4} A_{45|5} \tag{7.147}
$$

consequently

$$
P_{\xi, 12} = \partial_\xi A_{345|45|4}
$$

$$
- \frac{1}{2} (F_{\xi 45|5} + F_{45|5} + F_{45|5}) A_{34|4} + \frac{1}{2} (F_{34|4} + F_{34|4} + F_{34|4}) A_{45|5} \tag{7.148}
$$

$$
= \partial_\xi C_{12} - \frac{1}{2} P_{\xi, 1} C_2 + \frac{1}{2} P_{\xi, 2} C_1
$$

where the partial derivative is understood to distribute as the component of an exterior derivative across the multiform fields. While the dictionary definition contains a redundancy in the three-fold generation of field strengths from a single gauge field it has the advantage that it reproduces the non-trivial structure equation (7.148) of the $\mathfrak{sl}(3, \mathbb{R})$ sub-algebra.

The redundancy in the dictionary permits us to prefer a set of field strengths, that is as the components of each field strength are equal, for example $F_{\xi 45|5} = F_{45|5} = F_{45|5} = -\frac{1}{2} \sin \beta \partial_\xi N^{-1}_1$, we may eliminate field strengths algebraically in the action. In practice we may return to treating a set of indices in a privileged manner, at least for identifying a $D$-dimensional action whose equations of motion are satisfied by the null geodesic on $SO(1, 12)$. For example the case where the derivative is antisymmetrised with the leading column of each Young tableau has the action

$$
S_1 = \int R \star 1 - \frac{1}{2} F_{[D-2|1]} \wedge \star F_{[D-2|1]} - \frac{1}{2} G_{[D-1|D-3|1]} \wedge \star G_{[D-1|D-3|1]} \tag{7.149}
$$

where $F_{\xi 4...D|D} = P_{\xi, 1}$, $F_{\xi 3...(D-1)|(D-1)} = P_{\xi, 2}$, $G_{\xi 3...(D-1)|(D-1)} = P_{\xi, 12}$, $\star$ denotes the Hodge dual on the form indices, while the remaining internal indices on the kinetic terms are
contracted with the metric. The equations of motion for the metric, $A_{[D-3]}$ and $A_{[D-2][D-3]}$ from (7.149) are satisfied for all points on the interpolating bound state described by a null-geodesic on $SL(3,\mathbb{R})/SO(1,2)$, that is by the non-zero field strength components

$$
F_{\xi 4...(D-1)}(D-1) = -\sin \beta \partial_{\xi} N_1^{-1}, \\
F_{\xi 3...(D-2)(D-1)}(D-1) = -\tan \beta \partial_{\xi} N_2^{-1}
$$

and

$$G_{\xi 3...D|4...(D-1)} = -\cos \beta \frac{\partial_{\xi} N_1}{N_1 N_2} (7.150)
$$

where $N_1 = 1 + Q \xi$, $N_2 = 1 + Q \xi \cos^2 \beta$ and $\xi$ labels a single transverse direction either $\xi = 1$ or $\xi = 2$, see appendix D.3 to see the equations of motions satisfied in the five-dimensional case. Furthermore the field strength components may be unsmeread to two-dimensions and still satisfy the equations of motion of the above action. The unsmearing takes advantage of the spherical symmetry in the two transverse directions, in this case the solution is described as above in equation (7.150) but allowing $\xi \in \{1, 2\}$ and redefining $N_1 = 1 + Q \ln (r)$ and $N_2 = 1 + Q \ln (r) \cos^2 \beta$ where $r \equiv \sqrt{(x^1)^2 + (x^2)^2}$.

Similarly one could have considered the actions

$$S_2 = \int R \star_2 \mathbb{I} - \frac{1}{2} F_{[D-3][2]} \wedge \star_2 F_{[D-3][2]} - \frac{1}{2} G_{[D-2][D-2][1]} \wedge \star_2 G_{[D-2][D-2][1]}.$$

where $\star_2$ indicates the Hodge dual on the second set of indices of the field strength, or

$$S_3 = \int R \star_3 \mathbb{I} - \frac{1}{2} F_{[D-3][1][1]} \wedge \star_3 F_{[D-3][1][1]} - \frac{1}{2} G_{[D-2][D-3][2]} \wedge \star_3 G_{[D-2][D-3][2]}.$$

and $\star_3$ indicates the Hodge dual on the third set of indices of the field strength. Each action has equations of motion solved by fields encoded in the null geodesic on $SL(3,\mathbb{R})/SO(1,2)$.

### 7.5 An obstruction to an $A_{D-3}^{+++}$ symmetry of Einstein-Hilbert action.

The algebra $A_{D-3}^{+++} \subset E_{11}$ was first identified with an extended symmetry of gravity in [32]. An action for the $D$-dimensional dual graviton was given in [18] and investigated further in [33]. The reason $A_{D-3}^{+++}$ is relevant to gravity is apparent: the fields associated with the fundamental generators of the algebra are of the correct index type to be associated with the vielbein at level zero and with the dual graviton at level one, all other generators in the algebra are constructed by taking commutators of this pair. It is also evident that a traceless, massless field associated with $R^{\mu_1...\mu_{D-3}[\nu}$ carries $\frac{D}{2}(D-3)$ degrees of freedom as does the $D$-dimensional graviton. But one may wonder whether a theory containing both fields is a theory of a single graviton or a pair of gravitons. In the first case one would expect to identify components of $g_{\mu \nu}$ and $A_{\mu_1...\mu_{D-3}[\nu}$ by duality relations, and solutions of exotic gravity and matter actions would be mapped to solutions of the Einstein-Hilbert action. Consequently we face the puzzle of how to dualise the exotic actions of the previous section, which admit
the full interpolating bound state of dual gravitons as a solution, to the Einstein-Hilbert action. We are aware from the discussion in the first half of the chapter that the dualisation of the bound state solution is not a solution of the Einstein-Hilbert action apart from at the end points of the interpolation. We therefore expect to find that the action (7.149) is only equivalent to the Einstein-Hilbert action when $\beta = 0, \frac{\pi}{2}$ that is at the end points of the interpolation. In this section we will show that this is the case and show that the full interpolating solution is preserved when we treat the mixed-symmetry fields as multiforms.

The prototype bound state solution encoded as a null geodesic on $SL(3,\mathbb{R})/SO(1,\mathbb{R})$ is the dyonic membrane of supergravity [28]. The equation of motion for $A_{\mu_1\mu_2\mu_3}$ has contributions from the Chern-Simons term and it is precisely the interpolating parts of the bound state where the Chern-Simons term has an active role. In the present context, where we have shown that the end-points of the interpolating gravitational bound state are solutions to Einstein-Hilbert gravity but the interpolating points are not solutions, we anticipate that the full interpolating bound state will be a solution of the Einstein-Hilbert action with an additional Chern-Simons-like term.

Commencing with the $D$-dimensional action:

$$S_1 = \int R \ast \ast - \frac{1}{2} F_{D-2[1]} \wedge \ast F_{D-2[1]} - \frac{1}{2} G_{[D-1]D-3[1]} \wedge \ast G_{[D-1]D-3[1]}$$

(7.153)

where $R$ is the Ricci curvature formed from the fields associated with the Cartan subalgebra, i.e. from the diagonal part of the metric, $\ast$ indicates the Hodge dual and $\wedge$ the exterior product on the form indices (all other indices are contracted using the metric). To make the connection with the Einstein-Hilbert action one must dualise the higher rank mixed symmetry field strengths at the level of the action to find an action for the vielbein. The dualisation would be carried out in two steps with the first step eliminating $A_{\mu_1\mu_2\mu_3}$ symmetry field strengths at the level of the action to find an action for the vielbein. The equation of motion for $\chi$ term

$$\partial_\mu G_{\mu \rho_1 \rho_2 ... \rho_{D-3}} dx^{\rho_1} \wedge dx^{\rho_2} \wedge ... dx^{\rho_D} \otimes dx'^{\rho_1} \wedge ... dx'^{\rho_{D-3}} \otimes dx'^\rho$$

and $Y$ denotes the young projector that projects into $\Omega^{[D-3][1]}$. The Lagrange multiplier $\chi$ is introduced to dualise only one set of indices: in terms of components the first term above is $-\chi^{[D-3][1]} \partial_\mu G_{\rho_2 ... \rho_D | \rho_1 ... \rho_{D-3} | \mu} dx^{\rho_1} \wedge dx^{\rho_2} \wedge ... dx^{\rho_D} \otimes dx'^{\rho_1} \wedge ... dx'^{\rho_{D-3}} \otimes dx'^\rho$ i.e. the terms above each have $D$ antisymmetric ‘$\mu$ indices’ which are dual to $\chi$ in the ‘$\mu$ indices’. After carrying out the dualisation $\chi^{[D-3][1]}$ will be identified, by assumption, with additional components of $A^{[D-3][1]}$, the level one field. Varying the action with respect to $\chi$ gives the term in brackets

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above which is identically zero (when treating the fields as forms). Varying the action with respect to $G$ gives the algebraic identity:

$$F_{[1]}^{D-3[1]} = \frac{d\chi^{[D-3]}{1}}{\star} = \star G_{[D-1]}^{D-3[1]}.$$  \hfill (7.155)

This is a component of a new one-form field strength and not a component of $F_{[D-2][1]}$ and is related to $A_{[D-3][1]}$ by

$$F_{\mu_1[\nu_1...\nu_{D-3}]\rho} \equiv \partial_{\mu_1}A_{\nu_1...\nu_{D-3}\rho}.$$  \hfill (7.156)

This observation is already enough to motivate treating the fields as multiforms. However we understand from the previous section that components of $F_{\mu_1[\nu_1...\nu_{D-3}]\rho}$ and $F_{\mu_1\nu_1...\nu_{D-3}\rho}$ are equal and one may take advantage of this to identify a new non-zero component of $F_{[D-2][1]}$ in the action. Substituting our dualisation and identity into the action we find

$$S_1 = \int R \star I - \frac{1}{2} F_{[D-2][1]} \wedge \star F_{[D-2][1]} + A^{D-3[1]}Y \cdot (F_{[2]D-3}F_{[D-2][1]}).$$  \hfill (7.157)

After dualising $G$, the bound state solution has non-zero field strength components given by (see equation (7.150))

$$F_{\xi4...(D-1)D}[D] = -\sin \beta \partial_\xi N_1^{-1},$$

$$F_{\xi3...(D-2)(D-1)[D-1]} = -\tan \beta \partial_\xi N_2^{-1} \quad \text{and}$$

$$F_{\xi4...D[D-1]} = \cos \beta \frac{\partial_\xi N_1}{N_1 N_2}$$  \hfill (7.158)

where $\xi' \neq \xi$ and $\xi, \xi' \in \{1, 2\}$. These components now fail to solve the metric’s equation of motion for the full interpolation, but do solve it at the end points. We have erred in our dualisation. The source of our mistake is the elimination of $A_{[D-3][D-2][1]}$ in the Bianchi identity for $G$. While we have attempted to treat it as a $(D-3)$-form so that $d^2A_{[D-3][D-2][1]} = 0$ the structure of the algebra necessitates that it is a multi-form field such that $\delta^n A_{[D-3][D-2][1]} = 0$ only for $n \geq 4$. Modifying the Lagrange multiplier term in (7.154) to include the second derivatives on $A_{[D-2][D-3][1]}$ and integrating by parts gives

$$+F_{[1]}^{D-3[1]} \left( G_{[D-1][D-3][1]} - Y \cdot (dA_{[D-2][D-3][1]} - \frac{1}{2} F_{[D-3][2]A_{[D-3][1]} + \frac{1}{2} F_{[D-2][1]A_{[D-3][1]}})). \right.$$  \hfill (7.159)

Repeating the dualisation fails to eliminate $A_{[D-2][D-3][1]}$ from the action and we are left with

$$S'_1 = \int R \star I - \frac{1}{2} F_{[D-2][1]} \wedge \star F_{[D-2][1]} - F_{[1]}^{D-3[1]}dA_{[D-2][D-3][1]} - A_{[D-3[1]}Y \cdot (F_{[2]D-3}F_{[D-2][1]}$$  \hfill (7.160)

where for the bound state we have

$$A_{[D-2][D-3][1]} = \frac{1}{2} \cos \beta \left( \frac{1}{N_2} + \frac{1}{N_1 \cos^2 \beta} \right).$$  \hfill (7.161)

This action does admit the full interpolating bound state as a solution to its equations of motion, see appendix D.3 for the five-dimensional equations of motion. We note that due
to the contraction of indices the Chern-Simons term contributes to the metric equation of motion.

Following the proposal that the mixed-symmetry fields must be treated as multiforms to preserve the solutions under dualisation, there is no possibility to remove $A_{[D−2][D−3]}$ without considering a higher derivative action. The required Bianchi identity is fourth order in derivatives:

$$\mathfrak{d}^3(G_{[D−2][D−3]}) = \frac{1}{2} \mathfrak{d}(Y \cdot (−F_{[D−2][2]} F_{[D−3][2]} + F_{[D−2][2]} F_{[D−2][1]})). \quad (7.162)$$

The Bianchi identity is trivially zero for functions of one variable but contributes for functions of two variables. Generically as $\mathfrak{d}^2G \in \Omega^{[D−1][D−2]}$ then $\mathfrak{d}^3G \in \Omega^{[D−1][D−2]} \oplus \Omega^{[D−1][D−1]} \oplus \Omega^{[D−1][D−2][3]}$. A Lagrange multiplier would exist in the (thrice) dual space $\Omega^{[0][2][D−2]} \oplus \Omega^{[1][1][D−2]} \oplus \Omega^{[1][2][D−3]}$ which is the same space in which $\mathfrak{d}^2\Omega^{[D−3][1][0]}$ exists. The appropriate Lagrange multiplier term is

$$\mathfrak{d}^2\chi_{[D−3][1][0]} \left( \mathfrak{d}^3(G_{[D−1][D−2][2]}) - \frac{1}{2} Y \cdot (F_{[D−2][2]} F_{[D−3][2]} - F_{[D−2][2]} F_{[D−2][1]}) \right) \quad (7.163)$$

where $Y$ acts so that the appropriate Young tableau symmetries are projected onto the product of multiforms. Now the Lagrange multiplier term is a six-derivative term and the non-trivial fields for the bound state will only solve the equations of motion of an action in which all terms are also six-derivative terms, that is the level one field strength would be $F_{[D−2][2][1]}$ with components $\partial_\mu \partial_\nu \partial_\kappa A_{(\mu_2...\mu_{D−2})\nu}$ and the level zero field associated to the diagonal part of the vielbein would have an equivalent six-derivative term. The precise form of the term can be reconstructed from a six-derivative sigma-model\(^9\) where the Lagrangian is

$$\mathcal{L}' = -okus\mathfrak{d}^2\mathcal{P}_{\xi} \mathfrak{d}^2\mathcal{P}_{\xi} \quad (7.164)$$

we find

$$R_6 \equiv -2 \partial_\mu \partial_\nu \partial_\sigma \partial_\kappa \partial_\lambda h^{\sigma \lambda} + 2 \partial_\mu \partial_\nu \partial_\sigma \partial_\kappa \partial_\rho \partial_\lambda h^{\rho \lambda} + 4 \partial_\mu \partial_\nu \partial_\rho \partial_\sigma h^{\lambda \rho \sigma \lambda} - 4 \partial_\mu \partial_\nu \partial_\rho \partial_\kappa h^{\lambda \rho \sigma \lambda} + 2 \partial_\mu \partial_\nu \partial_\rho \partial_\kappa h^{\lambda \rho \sigma \lambda} - \partial_\mu \partial_\nu \partial_\rho \partial_\kappa h^{\lambda \rho \sigma \lambda} \quad (7.165)$$

where indices have been raised and lowered with the diagonal metric (7.93) encoded in the coefficients Cartan sub-algebra. It is not clear to the authors that this term has a simple geometrical expression. For example the equivalent four-derivative set of terms are not related to the Gauss-Bonnet gravity terms.

The 3D-dimensional six-derivative action after dualisation is

$$S = \int \int \int R_6 \star_2 \star_3 \mathbb{I} - \frac{1}{2} F_{[D−2][2]} \wedge^3 \star_1 \star_2 \star_3 F_{[D−2][2]} \quad (7.166)$$

$$+ \frac{1}{2} F_{[D−2][2]} \wedge^3 (Y \cdot (F_{[D−2][2]} F_{[D−3][2]} F_{[D−2][2]} F_{[D−2][1]}))$$

\(^9\)In the same way that the Ricci scalar is found from the two-derivative sigma-model in appendix D.2.
where \( Y \) is the appropriate Young tableau projector, \( \wedge^3 \) denotes the triple wedge product applied to each of the three sets of indices and \( \star_i \) denotes the Hodge dual on the \( i \)th set of antisymmetric indices. One might hope to continue the dualisation in the same manner by constructing the field strength with derivatives on each set of indices of \( A_{[D-3][1]} \), i.e. \( F_{[D-2][2]} \) and construct a four derivative term, nested within a further two derivatives, in the action with a Lagrange multiplier and the Bianchi identity for \( F_{[D-2][2]} \) generically

\[
\int \mathcal{O}^2 \int \mathcal{O} \chi_{[1][D-3]} \mathcal{O}(F_{[D-2][2]})
\]

(7.167)

The field is self-dual when dualisation is carried out over all sets of indices. Restricting to a dualisation over a single index requires the term

\[
\int \mathcal{O}^4 \int \mathcal{O} \chi_{[1][1]} \mathcal{O}(F_{[D-2][1]} - F_{[D-3][2]})
\]

(7.168)

so that \( d\chi_{[1]} = \star_1 F_{[D-2][1]} \) but the field \( A_{[D-3][1]} \) remains present in the action.

### 7.6 Discussion

In this chapter we have investigated the gravitational solutions associated to real roots of \( A_{D-3}^{++} \) algebras. For the case when \( D = 11 \) the solutions we have presented here are also present in the non-linear realisation of \( \epsilon_{11} \) which is conjectured to be the extension of supergravity relevant to M-theory [18]. Such affine classes of solutions have been studied before \([30, 31]\) and it has been shown that the Geroch group of solutions, defined using the Weyl reflections of \( A_1^+ \), are not only relevant to the gravitational sector of M-theory but also to the M2-M5 branes as well \([31]\). This work established solutions for every co-dimension two object which is predicted to exist in M-theory from \( E_{11} \). In the present work we have understood each solution within the gravity tower sub-sector of \([31]\) in terms of a null geodesic motion on coset \( SL(2,\mathbb{R})/SO(1,1) \). At low levels of \( A_{D-3}^{++} \) the solutions include the pp-wave and the KK\((D-5)\)-brane. For a different choice of real form of \( A_{D-3}^{++} \) the KK\((D-5)\)-brane solution is the Euclidean Taub-NUT solution, which is a solution of four dimensional gravity theory trivially embedded in a \( D \)-dimensional Minkowski spacetime. Similarly the co-dimension two solutions investigated in this chapter all possess a large transverse isometry and we may regard them as five-dimensional gravity solutions trivially embedded in a \( D \)-dimensional background.

The observation of \([25]\) that bound-state solutions, such as the dyonic membrane, could be described by an \( E_{11} \) group element led to the investigation of Lagrangians on cosets of groups of rank two and greater \([26]\). It was shown that the bound state solutions could also be understood as encoding a null geodesic motion on a coset. In both \([25, 26]\) the solutions were characterised by a continuous interpolating parameter that moved from one \( \frac{1}{2} \)-BPS solution to another. By considering the \( \mathfrak{sl}(3,\mathbb{R}) \) sub-algebras embedded within the algebra of \( A_{D-3}^{++} \) model we have constructed interpolating solutions which move between
any two gravitational solutions which appear at adjacent levels in the level decomposition of $A_{D-3}^{++}$ into representations of $A_{D-1}$. Such a construction allows the possibility to, with a combination of several models in succession, interpolate between the solutions occurring at any positive levels by means of a smooth interpolating parameter. Consider the combination of roots

$$\alpha_i = \begin{cases} 
e D - e_3 + \sum_{i=3}^{D} e_i & \text{if } i = 1 \\ e_{D-1} - e_D + \sum_{i=3}^{D} e_i & \text{if } i > 1 \text{ and } i \text{ even} \\ e_D - e_{D-1} + \sum_{i=3}^{D} e_i & \text{if } i > 1 \text{ and } i \text{ odd} \end{cases} \quad (7.169)$$

with $t = x_{D-1}$. Any root $\alpha_{12...n} = \alpha_1 + \alpha_2 + \ldots + \alpha_n$ will have inner product of $-1$ with $\alpha_{n+1}$. Let the involution be chosen such that the generator associated with the first root is involution invariant while the second is not, so that the involution invariant sub-algebra is $\mathfrak{so}(1,2)$. If we begin at any level $m$ we may create a coset model which will limit to the level $m+1$ solution from which we can construct a new model. This shows us that by advancing the parameters in each $\mathfrak{sl}(3,\mathbb{R})$ to produce the next level object we may telescopically reach an arbitrary positive level from a series of $\mathfrak{sl}(3,\mathbb{R})$ models. However the bound states of $A_{D-3}^{++}$ constructed as null geodesics on $\mathbb{SL}(3,\mathbb{R})/\mathbb{SO}(1,2)$ do not lift to interpolating solutions of the Einstein-Hilbert action. As shown in [31], they are however solutions of Einstein-Hilbert gravity at the end points of the bound states. As shown explicitly in appendix D.2 the bound state is a solution to a gravity and matter theory, which may be simply constructed using the index structure of the mixed-symmetry fields and the one-dimensional sigma-model Lagrangian. This is puzzling as one would expect to be able to dualise the action on fields of $A_{D-3}^{++}$ to and action written in terms of the vielbein, while preserving the full interpolating solution. The loss of the full interpolating solution is a consequence of the vanishing of terms such as $d^2 A_{[D-2]...[D-2][D-3]}^{[1]}$, while if the field is retained in the action the full interpolating solution persists. This observation, discussed in section 7.5, motivates the consideration of an extension of the exterior derivative to a derivative which distributes over all the indices of multiform fields [98–102]. We argued in section 7.4 that such a treatment of multiform fields maintained the structure of the equations for the null geodesic on the coset and gave a simple extension of the supergravity dictionary to multiform field-strengths. The brane coset model remains to be extended to include a generalised vielbein as appears in the non-linear realisation of $l_1 \times E_{11}$ [103] and such an extension may suggest an alternative interpretation for the mixed-symmetry field strengths in the supergravity dictionary.

It was argued in section 7.5 that the exotic gravity and matter action could not be dualised to an action of just the Einstein-Hilbert term alone, instead Chern-Simons-like terms will remain. It is interesting to wonder, in the context of recent observations in massive gravity [104, 105], whether the Chern-Simons term retained from the $\mathfrak{sl}(3,\mathbb{R})$ sigma-model action might be consistently identified with a product of vielbein components as seen in the actions of [106] - at first glance this would seem unlikely due to the presence of derivatives in the sigma-model terms. If such a link were made the dual graviton would be reinterpreted...
as a second graviton.

The initial motivation for studying $A_{D-3}^{++}$ algebras was that $E_{11}$ is the dimensional reduction of $A_{9}^{++}$ and contains $A_{8}^{++}$. Consequently the M-theory bound states of branes encoded as null geodesics on cosets of sub-groups of $E_{11}$ should lift to bound states described within the gravitational algebra $A_{9}^{++}$. As argued in this chapter there are no interpolating solutions of the Einstein-Hilbert term alone, suggesting that $A_{D-3}^{++}$ is a continuous symmetry of an extension of the Einstein-Hilbert action. We have given examples in section 7.5 of both matter terms and Chern-Simons terms which can be added to the Einstein-Hilbert term so that the extended action admits full interpolating bound state solutions. The class of solutions that we have considered has not included the dimensional lift of the dyonic membrane. The embedding of dyonic membrane into the twelve-dimensional theory merits a closer examination as it provides a link between bound states involving mixed-symmetry fields in twelve dimensions and form fields in eleven dimensions and one expects to recover a full interpolating solution in twelve dimensions. The level one and level two generators of \( e_{11} \) are lifted to the $A_{9}^{++}$ generators

\[
R_{\mu_0\mu_1\mu_1\mu_1} = \frac{1}{8!} \epsilon_{\mu_1...\mu_11} R^{\mu_1...\mu_8} \rightarrow R^{\mu_1...\mu_9}\mu_9
\]

(7.170)

\[
R_{\mu_0\mu_7\mu_9\mu_9\mu_9\mu_11} = \frac{1}{5!11!} \epsilon_{\mu_1...\mu_11} \epsilon_{\nu_1...\nu_11} R^{\nu_1...\nu_11\mu_1...\mu_5} \nu_1...\nu_12\mu_1...\mu_6|\rho|\sigma.
\]

(7.171)

The M2-M5 tower of solutions [31] relevant to M-theory may also be re-interpreted in the context of $A_{D-3}^{++}$ algebras. Via the dimensional reduction of $A_{12}^{++}$ it is possible to understand the affine multiplet of states including the M2 and M5 brane uncovered in [31] as part of the Geroch group associated to the twelve-dimensional gravitational theory. The dimensional reduction of the solutions related by the Geroch group in twelve dimensions includes all the states recognised in both the gravity tower and the M2-M5 tower of [31].

It is to be hoped that the analysis of brane solutions as null geodesics on cosets of finite sub-groups of Kac-Moody algebras will be extended to null geodesics on cosets of affine sub-algebras. The difference between an affine algebra and the series of $\mathfrak{sl}(3,\mathbb{R})$ sub-algebras we have investigated in the present work is seemingly small. After all the affine coset model would be expected to describe solutions with continuous parameters that move directly between any pair of generators appearing at any level in the algebraic decomposition, while the sequence of $\mathfrak{sl}(3,\mathbb{R})$ sub-algebras we have investigated only interpolate directly between generators appearing at adjacent levels. However suitable combinations of interpolating solutions encoded in $\mathfrak{sl}(3,\mathbb{R})$ sub-algebras may be found that interpolate between any two levels in the algebra. The algebra $A_{2}^{+}$ formed by the roots $\alpha_1$, $\alpha_2$ and $\alpha_0 = e_3 - e_{10}$ contains all of the roots $\alpha_i$ listed above and would include them in one model forming a substantial subsector of M-theory.
Appendices
A.1 Extended Affine Algebra $G_{11}$

The algebra $G_{11}$, which contains a truncation of $E_{11}$ of the $SL(11,\mathbb{R})$ generators $K^i{}_j$ as well as the three-form and six-form representations $R^{i_1i_2i_3}$ and $R^{i_1\cdots i_6}$ of $SL(11,\mathbb{R})$ with the translation generating vector representation $P_i$. The commutators are given by the standard $IGL(11)$

$$[K^i{}_j, K^m{}_n] = \delta^i{}_m K^j{}_n - \delta^j{}_m K^i{}_n \quad [K^i{}_j, P_m] = -\delta^i{}_m P_j \quad (A.1)$$

along with

$$[K^i{}_j, R^{m_1m_2m_3}] = 3\delta^i{}_{[m_1} R^{j]m_2m_3} = \delta^i{}_{[m_1} R^{j]m_2m_3} + \text{sym} \quad [K^i{}_j, R^{m_1\cdots m_6}] = 6\delta^i{}_{[m_1} R^{j]m_2\cdots m_6} \quad [R^{i_1i_2i_3}, R^{j_1j_2j_3}] = 2R^{i_1i_2i_3j_1j_2j_3} \quad (A.2)$$

and all others zero. In order to evaluate the Maurer-Cartan form of the group element

$$g = \exp \{ x^\mu P_\mu \} \exp \left\{ h_a^b K^a{}_b \right\} \exp \left\{ \frac{1}{3!} A_{a_1a_2a_3} R^{a_1a_2a_3} \right\} \exp \left\{ \frac{1}{3!} A_{a_1\cdots a_6} R^{a_1\cdots a_6} \right\} \quad (A.3)$$

we make repeated use of one of our BCH lemmas

$$A e^B = e^B \left( A + [A, B] + \frac{1}{2!} [[A, B], B] + \cdots \right) \quad (A.4)$$
by commuting over the derivative. The form \(dg\) can then be rewritten as

\[
\partial \mu g = e^{x^a P_a} \left[ \partial \mu + (\partial \mu x^b) P_b \right] e^{h^i K^i} \exp \left( \frac{1}{3!} A_{c_1 c_2 c_3} R^{c_1 c_2 c_3} + \frac{1}{6!} A_{c_1 \cdots c_6} R^{c_1 \cdots c_6} \right) (A.5)
\]

\[
= e^{x^a P_a} e^{h^i K^i} \left[ \partial \mu + (e^h)^\mu_b P_b + (\partial \mu h^a_b) K^a \right] \exp \left( \frac{1}{3!} A_{c_1 c_2 c_3} R^{c_1 c_2 c_3} \right.
\]

\[
+ \frac{1}{6!} \left( A_{c_1 \cdots c_6} R^{c_1 \cdots c_6} \right)
\] (A.6)

\[
= g \times \left( e^h \right)_\mu ^a P_a + (\partial \mu h^a_b) K^a_b + \frac{1}{3!} \left( \partial \mu A_{c_1 c_2 c_3} + 3 \partial \mu h_{c}^a A_{a|c_2 c_3} \right) R^{c_1 c_2 c_3}
\]

\[
+ \frac{1}{6!} \left( \partial \mu A_{c_1 \cdots c_6} + 6 \partial \mu h_{c}^a A_{a|c_2 \cdots c_6} + 20 \partial \mu A_{c_4 c_5 c_6} \right).
\] (A.7)

When we redefine the fields \(e^a, \Omega^b_{\mu a}, \tilde{D}_\mu A_{a_1 a_2 a_3}\) and \(\tilde{D}_\mu A_{a_1 \cdots a_6}\) as in section 4.2 we find that the Maurer-Cartan form is given simply as

\[
\omega = dx^\mu \left( g^{-1} \partial \mu g - \lambda_\mu \right)
\]

\[
= dx^\mu \left( e^a P_a + \Omega^b_{\mu a} K^a_b + \frac{1}{3!} R^{a_1 a_2 a_3} \tilde{D}_\mu A_{a_1 a_2 a_3} + \frac{1}{6!} R^{a_1 \cdots a_6} \tilde{D}_\mu A_{a_1 \cdots a_6} \right). \quad (A.8)
\]

### A.2 Conformal Algebra

With the group element

\[
g = \exp \{ x^a P_a \} \exp \{ \phi^b C_b \} \exp \{ \sigma D \} \quad (A.9)
\]

we again use BCH to rearrange the derivatives on the group element

\[
\partial \mu g = e^{-x^a P_a} \left[ \partial \mu - (\partial \mu x^b) P_b \right] e^{\phi^b C_b} e^{\sigma D} \quad (A.10)
\]

\[
= e^{-x^a P_a} e^{\phi^b C_b} \left[ \partial \mu - (\partial \mu x^b) P_b - 2 \phi_\mu D + 2 \delta^a_\mu \phi^b J_{ab} \right.
\]

\[
-2 \phi_\mu \phi^a C_a + \phi^a \phi_\mu \delta^b_\mu C_b \right] e^{\sigma D}
\] (A.11)

\[
= g \times \left( - e^{-\sigma \delta^a_\mu P_a} + e^{-\sigma} \left( \partial \mu \phi^a + \phi_\sigma \phi_\mu \delta^a_\mu - 2 \phi_\mu \phi^a \right) C_a \right.
\]

\[
+ \left( \partial \mu \sigma - 2 \phi_\mu \right) + \left( \delta^a_\mu \phi^b - \delta^b_\mu \phi^a \right) J_{ab} \right) \quad (A.12)
\]
Sigma Model Equations of Motion

There is one equation of motion for the $\sigma$-model of (2.43) for each of the elements of the involution anti-invariant generators. These can in general be labeled by the basis of the involution anti-invariant subset $p$ with $H_i$ and $s_\alpha$, while the involution invariant generators are $k_\alpha$. The equations of motion are given by equations (5.41,5.45):

\[ H_i : 0 = \partial^2 \phi_i - \frac{1}{2} \sum_{\alpha \in g} \epsilon_\alpha a_i P_{\alpha}^2 \quad (B.1) \]

\[ s_\alpha : 0 = \partial^2 P_\alpha + P_\alpha \left( \sum_{i=1}^{r} \langle \alpha, \alpha_i \rangle \partial \phi_i \right) + \sum_{\beta | \alpha + \beta \in g} c_{\alpha, \beta} \epsilon_\beta P_{\alpha + \beta} P_{\beta} \quad (B.2) \]

The signs for each term are determined by the involution and definition of the structure constants through the constants $\epsilon_\alpha$ and $c_{\alpha, \beta}$.

B.1 $D_4(4)$ Equations of Motion

With the choice of $\epsilon = (+, -, +, +)$ involution and commutators which are defined by the matrix in section 5.2.2 we find the equations of motion to be:

\[ H_1 : 0 = 2 \partial^2 \phi_1 - P_{(1,0,0,0)} + P_{(1,1,0,0)} + P_{(1,1,1,0)} + P_{(1,1,0,1)} + P_{(1,1,1,1)} - P_{(1,2,1,1)} \]
\[ H_2 : 0 = 2 \partial^2 \phi_2 + P_{(0,1,0,0)} + P_{(1,1,0,0)} + P_{(0,1,1,0)} + P_{(0,1,0,1)} + P_{(1,1,0,1)} + P_{(1,1,1,0)} + P_{(1,1,1,1)} - 2 P_{(1,2,1,1)} \]
\[ H_3 : 0 = 2 \partial^2 \phi_3 - P_{(0,0,1,0)} + P_{(0,1,1,0)} + P_{(1,1,1,0)} + P_{(0,1,1,1)} + P_{(1,1,1,1)} + P_{(1,2,1,1)} \]
\[ H_4 : 0 = 2 \partial^2 \phi_4 - P_{(0,0,0,1)} + P_{(0,1,0,1)} + P_{(1,0,0,1)} + P_{(1,0,1,1)} + P_{(1,1,1,1)} - P_{(1,2,1,1)} \]
APPENDIX B. SIGMA MODEL EQUATIONS OF MOTION

\[ S_{(1,0,0,0)} : \quad 0 = \partial P_{(1,0,0,0)} + P_{(1,0,0,0)}(2\partial \phi_1 - \partial \phi_2) + P_{(1,1,0,0)}P_{(0,1,1,0)} + P_{(1,1,1,0)}P_{(0,1,0,1)} + P_{(1,1,1,1)}P_{(0,1,1,1)} \]

\[ S_{(0,1,0,0)} : \quad 0 = \partial P_{(0,1,0,0)} + P_{(0,1,0,0)}(2\partial \phi_2 - \partial \phi_1 - \partial \phi_3 - \partial \phi_4) + P_{(1,1,0,0)}P_{(1,0,0,0)} - P_{(0,1,1,0)}P_{(0,0,1,0)} - P_{(0,1,0,1)}P_{(0,0,0,1)} + P_{(1,2,1,1)}P_{(1,1,1,1)} \]

\[ S_{(0,0,1,0)} : \quad 0 = \partial P_{(0,0,1,0)} + P_{(0,0,1,0)}(2\partial \phi_3 - \partial \phi_2) - P_{(0,1,1,0)}P_{(0,1,0,0)} - P_{(1,1,0,0)}P_{(1,1,1,0)} + P_{(0,1,1,1)}P_{(0,1,0,1)} + P_{(1,1,1,1)}P_{(1,0,1,0)} \]

\[ S_{(0,0,0,1)} : \quad 0 = \partial P_{(0,0,0,1)} + P_{(0,0,0,1)}(2\partial \phi_4 - \partial \phi_2) - P_{(1,0,1,0)}P_{(1,0,0,0)} - P_{(1,1,0,0)}P_{(1,1,0,0)} + P_{(0,1,1,1)}P_{(0,1,1,0)} + P_{(1,1,1,1)}P_{(1,1,1,0)} \]

\[ S_{(1,1,0,0)} : \quad 0 = \partial P_{(1,1,0,0)} + P_{(1,1,0,0)}(\partial \phi_1 + \partial \phi_2 - \partial \phi_3 - \partial \phi_4) - P_{(1,1,1,0)}P_{(0,0,1,0)} - P_{(1,1,0,1)}P_{(0,0,0,1)} - P_{(1,2,1,1)}P_{(0,1,1,1)} \]

\[ S_{(0,1,1,0)} : \quad 0 = \partial P_{(0,1,1,0)} + P_{(0,1,1,0)}(\partial \phi_2 + \partial \phi_3 - \partial \phi_1 - \partial \phi_4) + P_{(1,1,1,0)}P_{(1,0,0,0)} + P_{(0,1,1,1)}P_{(0,0,0,1)} + P_{(1,2,1,1)}P_{(1,1,0,1)} \]

\[ S_{(0,1,0,1)} : \quad 0 = \partial P_{(0,1,0,1)} + P_{(0,1,0,1)}(\partial \phi_2 + \partial \phi_4 - \partial \phi_1 - \partial \phi_3) + P_{(1,1,0,1)}P_{(1,0,0,0)} + P_{(0,1,1,1)}P_{(0,0,1,0)} + P_{(1,2,1,1)}P_{(1,1,1,0)} \]

\[ S_{(1,1,1,0)} : \quad 0 = \partial P_{(1,1,1,0)} + P_{(1,1,1,0)}(\partial \phi_1 + \partial \phi_3 - \partial \phi_4) + P_{(1,1,1,1)}P_{(0,0,0,1)} - P_{(1,2,1,1)}P_{(0,1,0,1)} \]

\[ S_{(1,1,0,1)} : \quad 0 = \partial P_{(1,1,0,1)} + P_{(1,1,0,1)}(\partial \phi_1 + \partial \phi_4 - \partial \phi_3) + P_{(1,1,1,1)}P_{(0,0,1,0)} - P_{(1,2,1,1)}P_{(0,1,1,0)} \]

\[ S_{(0,1,1,1)} : \quad 0 = \partial P_{(0,1,1,1)} + P_{(0,1,1,1)}(\partial \phi_3 + \partial \phi_4 - \partial \phi_1) + P_{(1,1,1,1)}P_{(1,0,0,0)} + P_{(1,2,1,1)}P_{(1,1,0,0)} \]

\[ S_{(1,1,1,1)} : \quad 0 = \partial P_{(1,1,1,1)} + P_{(1,1,1,1)}(\partial \phi_1 + \partial \phi_2 + \partial \phi_3 + \partial \phi_4) - P_{(1,2,1,1)}P_{(0,1,0,0)} \]

\[ S_{(1,2,1,1)} : \quad 0 = \partial P_{(1,2,1,1)} + P_{(1,2,1,1)}\partial \phi_2 \]

B.2 \( E_{6(6)} \) Equations of Motion

Here we present the equations of motion for the \( E_{6(6)}/Sp(8,\mathbb{R}) \) coset with involution \( \epsilon = (+,+,−,+,+,+), \) with structure constants given by the commutators in section 5.2.3 which is solved in section 5.3.4. The equations of motion are organised by the \( \alpha_6 \) level in the following subsections.
B.2.1 Cartan Equations of Motion

\[ H_1 : \quad 0 = 2\xi\phi_1 - P^2_{(1,0,0,0,0)} - P^2_{(1,1,0,0,0)} + P^2_{(1,1,1,0,0)} + P^2_{(1,1,1,1,0)} + P^2_{(1,1,1,1,1,0)} \\
+ P^2_{(1,1,1,0,0,1)} + P^2_{(1,1,1,1,0,1)} + P^2_{(1,1,1,1,1)} - P^2_{(1,1,2,1,0,1)} - P^2_{(1,2,1,2,1,1)} \]  
(B.3)

\[ H_2 : \quad 0 = 2\xi\phi_2 - P^2_{(0,1,0,0,0)} - P^2_{(1,1,0,0,0)} + P^2_{(0,1,1,0,0)} + P^2_{(1,1,1,0,0)} + P^2_{(0,1,1,1,0,0)} \\
+ P^2_{(1,1,1,1,0)} + P^2_{(1,1,1,1,0,0)} + P^2_{(0,1,1,1,1,0)} + P^2_{(1,1,1,1,0,0)} + P^2_{(1,1,1,1,1,0)} \]  
(B.4)

\[ H_3 : \quad 0 = 2\xi\phi_3 + P^2_{(0,0,1,0,0)} + P^2_{(0,1,1,0,0)} + P^2_{(0,0,1,1,0,0)} + P^2_{(0,0,1,1,1,0)} \\
+ P^2_{(1,1,0,0,0)} + P^2_{(1,1,1,0,0)} + P^2_{(0,0,1,1,1)} + P^2_{(0,1,1,1,0)} + P^2_{(1,1,1,1,0,0)} \]  
(B.5)

\[ H_4 : \quad 0 = 2\xi\phi_4 - P^2_{(0,0,0,1,0,0)} + P^2_{(0,0,1,1,0,0)} - P^2_{(0,0,0,1,1,0)} + P^2_{(0,0,1,1,1,0)} + P^2_{(0,1,1,1,0,0)} \\
+ P^2_{(1,1,1,1,0,0)} + P^2_{(1,1,1,1,1,0)} + P^2_{(0,0,1,1,1,0)} + P^2_{(0,1,1,1,0,0)} \]  
(B.6)

\[ H_5 : \quad 0 = 2\xi\phi_5 - P^2_{(0,0,0,0,1,0,0)} - P^2_{(0,0,0,1,1,0,0)} + P^2_{(0,0,0,1,1,1,0)} + P^2_{(0,0,1,1,1,1,0,0)} \\
+ P^2_{(0,1,1,1,1,0)} + P^2_{(1,1,1,1,1,0)} + P^2_{(0,1,1,1,1,1,0)} + P^2_{(0,1,1,1,1,1,1,0)} \]  
(B.7)

\[ H_6 : \quad 0 = 2\xi\phi_6 - P^2_{(0,0,0,0,0,1,0,0)} + P^2_{(0,0,1,0,0,0,1,0)} + P^2_{(0,1,1,0,0,0,1,0)} + P^2_{(0,0,1,0,0,1,0,1)} \\
+ P^2_{(1,1,1,0,0,0,1,0)} + P^2_{(1,1,1,0,0,1,0,1)} + P^2_{(0,1,1,0,0,1,0,1)} + P^2_{(1,1,1,0,0,1,1,0)} \]  
(B.8)
B.2.2 Level 0 Equations of Motion

\[ S_{(1,0,0,0,0,0)} : 0 = \partial_\xi P_{(1,0,0,0,0,0)} + P_{(1,0,0,0,0,0)} (2\partial_\xi \phi_1 - \partial_\xi \phi_2) - P_{(1,1,0,0,0,0)} P_{(1,0,0,0,0,0)} + P_{(1,1,1,0,0,0)} P_{(1,1,0,0,0,0)} + P_{(1,1,1,1,0,0)} P_{(1,1,1,0,0,0)} - P_{(1,1,2,2,1,1)} P_{(1,1,2,2,1,1)} + P_{(1,1,1,1,1,1)} P_{(1,1,1,1,1,1)} + P_{(1,1,1,1,0,1)} P_{(1,1,1,1,0,1)} \]  

\[ \text{Eq. (B.9)} \]

\[ S_{(0,1,0,0,0,0)} : 0 = \partial_\xi P_{(0,1,0,0,0,0)} + P_{(0,1,0,0,0,0)} P_{(1,1,0,0,0,0)} + P_{(0,1,1,0,0,0)} P_{(0,1,1,0,0,0)} + P_{(0,1,1,1,0,0)} P_{(0,1,1,1,0,0)} - P_{(1,2,2,2,1,1)} P_{(1,2,2,2,1,1)} + P_{(1,1,1,1,1,1)} P_{(1,1,1,1,1,1)} + P_{(0,1,1,1,0,1)} P_{(0,1,1,1,0,1)} - P_{(1,2,2,1,0,1)} P_{(1,2,2,1,0,1)} \]  

\[ \text{Eq. (B.10)} \]

\[ S_{(0,0,1,0,0,0)} : 0 = \partial_\xi P_{(0,0,0,1,0,0)} - P_{(0,0,0,1,0,0)} P_{(0,0,0,1,0,0)} - P_{(0,0,1,0,0,0)} P_{(0,0,1,0,0,0)} - P_{(0,1,1,1,0,0)} P_{(0,1,1,1,0,0)} + P_{(0,1,1,0,0,1)} P_{(0,1,1,0,0,1)} - P_{(1,2,2,2,1,1)} P_{(1,2,2,2,1,1)} - P_{(1,1,2,1,1,1)} P_{(1,1,2,1,1,1)} + P_{(0,1,2,1,0,1)} P_{(0,1,2,1,0,1)} + P_{(0,0,0,1,0,0)} P_{(0,0,0,1,0,0)} - P_{(1,1,2,1,0,1)} P_{(1,1,2,1,0,1)} - P_{(0,0,1,0,0,1)} P_{(0,0,1,0,0,1)} \]  

\[ \text{Eq. (B.11)} \]

\[ S_{(0,0,0,1,0,0)} : 0 = \partial_\xi P_{(0,0,0,0,1,0)} - P_{(0,0,0,0,1,0)} P_{(0,0,0,0,1,0)} - P_{(0,0,0,1,0,0)} P_{(0,0,0,1,0,0)} - P_{(0,0,1,1,0,0)} P_{(0,0,1,1,0,0)} + P_{(0,0,0,0,1,0)} P_{(0,0,0,0,1,0)} - P_{(1,2,2,2,1,1)} P_{(1,2,2,2,1,1)} - P_{(1,1,2,1,1,1)} P_{(1,1,2,1,1,1)} + P_{(0,1,1,1,0,1)} P_{(0,1,1,1,0,1)} + P_{(0,0,1,0,0,0)} P_{(0,0,1,0,0,0)} - P_{(1,1,1,1,0,0)} P_{(1,1,1,1,0,0)} + P_{(1,1,1,1,0,1)} P_{(1,1,1,1,0,1)} + P_{(0,0,1,0,0,0)} P_{(0,0,1,0,0,0)} \]  

\[ \text{Eq. (B.12)} \]

\[ S_{(0,0,0,0,1,0)} : 0 = \partial_\xi P_{(0,0,0,0,1,0)} + P_{(0,0,0,0,1,0)} P_{(0,0,0,0,1,0)} - P_{(0,0,0,0,1,0)} P_{(0,0,0,0,1,0)} - P_{(0,0,0,0,1,0)} P_{(0,0,0,0,1,0)} + P_{(0,1,1,1,1,0)} P_{(0,1,1,1,1,0)} - P_{(1,2,2,1,1,1)} P_{(1,2,2,1,1,1)} - P_{(1,1,1,1,1,1)} P_{(1,1,1,1,1,1)} + P_{(0,0,1,0,0,1)} P_{(0,0,1,0,0,1)} - P_{(0,1,1,1,0,0)} P_{(0,1,1,1,0,0)} + P_{(0,1,1,1,0,1)} P_{(0,1,1,1,0,1)} + P_{(0,0,1,1,1,1)} P_{(0,0,1,1,1,1)} + P_{(0,0,1,1,1,1)} P_{(0,0,1,1,1,1)} \]  

\[ \text{Eq. (B.13)} \]
\begin{align}
S_{(1,1,0,0,0,0)} : \quad 0 &= \partial_\xi P_{(1,1,0,0,0,0)} - P_{(1,1,1,0,0,0)} P_{(0,0,1,0,0,0)} - P_{(1,1,1,1,0,0)} P_{(0,0,1,1,0,0)} \\
&\quad + P_{(1,1,0,0,0,0)} (\partial_\xi \phi_1 + \partial_\xi \phi_2 - \partial_\xi \phi_3) \\
&\quad + P_{(1,2,2,2,1,1)} P_{(0,1,2,2,1,1)} + P_{(1,2,2,1,1,1)} P_{(0,1,2,1,1,1)} \\
&\quad + P_{(1,1,1,1,1,1)} P_{(0,0,1,1,1,1)} + P_{(1,1,1,1,0,1)} P_{(0,0,1,0,1,0)} \\
&\quad - P_{(1,1,1,1,1,0)} P_{(0,0,1,1,1,0)} + P_{(1,2,2,1,0,1)} P_{(0,1,2,1,0,1)} + P_{(1,1,1,0,0,1)} P_{(0,0,1,0,0,1)} \\
&= (B.14)
\end{align}

\begin{align}
S_{(0,1,1,0,0,0)} : \quad 0 &= \partial_\xi P_{(0,1,1,0,0,0)} + P_{(1,0,0,0,0,0)} P_{(1,1,0,0,0,0)} - P_{(0,1,1,0,0,0)} P_{(0,0,1,0,0,0)} \\
&\quad + P_{(0,1,1,0,0,0)} (\partial_\xi \phi_2 + \partial_\xi \phi_3 - \partial_\xi \phi_1 - \partial_\xi \phi_4 - \partial_\xi \phi_6) \\
&\quad + P_{(1,2,3,2,1,1)} P_{(1,1,2,2,1,1)} + P_{(1,2,2,1,1,1)} P_{(1,1,1,1,1,1)} \\
&\quad - P_{(0,1,2,1,1,1)} P_{(0,0,1,1,1,1)} - P_{(0,1,2,1,0,1)} P_{(0,0,1,0,1,0)} \\
&\quad - P_{(0,1,1,1,1,0)} P_{(0,0,0,1,1,0)} + P_{(1,2,2,1,0,1)} P_{(1,1,1,1,0,1)} - P_{(0,1,1,0,0,1)} P_{(0,0,0,0,0,1)} \\
&= (B.15)
\end{align}

\begin{align}
S_{(0,0,1,1,0,0)} : \quad 0 &= \partial_\xi P_{(0,0,1,1,0,0)} + P_{(0,1,0,0,0,0)} P_{(0,1,1,1,0,0)} + P_{(1,1,0,0,0,0)} P_{(1,1,1,1,0,0)} \\
&\quad + P_{(0,0,1,1,0,0)} (\partial_\xi \phi_3 + \partial_\xi \phi_4 - \partial_\xi \phi_2 - \partial_\xi \phi_5 - \partial_\xi \phi_6) \\
&\quad - P_{(1,2,3,2,1,1)} P_{(1,2,2,1,1,1)} - P_{(1,1,2,2,1,1)} P_{(1,1,1,1,1,1)} \\
&\quad - P_{(0,1,2,2,1,1)} P_{(0,1,1,1,1,1)} + P_{(0,1,2,1,0,1)} P_{(0,1,1,0,0,1)} \\
&\quad - P_{(0,0,1,1,1,0)} P_{(0,0,0,0,1,0)} + P_{(1,2,1,2,0,1)} P_{(1,1,1,0,0,1)} + P_{(0,0,1,1,0)} P_{(0,0,0,0,0,1)} \\
&= (B.16)
\end{align}

\begin{align}
S_{(0,0,0,1,1,0)} : \quad 0 &= \partial_\xi P_{(0,0,0,1,1,0)} - P_{(0,0,1,0,0,0)} P_{(0,0,1,1,1,0)} - P_{(0,1,1,0,0,0)} P_{(0,1,1,1,1,0)} \\
&\quad + P_{(0,0,1,1,1,0)} (\partial_\xi \phi_4 + \partial_\xi \phi_5 - \partial_\xi \phi_3) \\
&\quad - P_{(1,2,2,2,1,1)} P_{(1,2,2,1,0,1)} - P_{(1,1,2,2,1,1)} P_{(1,1,2,1,0,1)} \\
&\quad - P_{(0,1,2,2,1,1)} P_{(0,1,2,1,0,1)} - P_{(0,1,1,1,1,1)} P_{(0,1,1,0,0,1)} \\
&\quad - P_{(1,1,1,0,0,0)} P_{(1,1,1,1,1,0)} - P_{(1,1,1,1,1,0)} P_{(1,1,1,0,0,1)} - P_{(0,0,1,1,1,1)} P_{(0,0,1,0,0,1)} \\
&= (B.17)
\end{align}

\begin{align}
S_{(1,1,0,0,0,0)} : \quad 0 &= \partial_\xi P_{(1,1,0,0,0,0)} - P_{(1,1,1,1,0,0)} P_{(0,0,0,1,0,0)} - P_{(1,1,1,1,1,0)} P_{(0,0,0,1,1,0)} \\
&\quad + P_{(1,1,0,0,0,0)} (\partial_\xi \phi_1 + \partial_\xi \phi_3 - \partial_\xi \phi_4 - \partial_\xi \phi_6) \\
&\quad - P_{(1,2,3,2,1,1)} P_{(0,1,2,2,1,1)} - P_{(1,2,2,1,1,1)} P_{(0,1,1,1,1,1)} - P_{(1,2,2,1,0,1)} P_{(0,1,1,1,0,1)} \\
&\quad - P_{(1,1,2,1,1,1)} P_{(0,0,1,1,1,1)} - P_{(1,1,2,1,0,1)} P_{(0,0,1,1,0)} - P_{(1,1,1,0,0,1)} P_{(0,0,0,0,1)} \\
&= (B.18)
\end{align}
APPENDIX B. SIGMA MODEL EQUATIONS OF MOTION

B.2.3 Level 1 Equations of Motion

\( S_{(0,1,1,1,0)} : \quad 0 = \partial_{\xi} P_{(0,1,1,1,0)} + P_{(1,1,1,1,0)} P_{(0,0,0,0,0)} - P_{(0,1,1,1,0)} P_{(0,0,0,0,1)} + P_{(0,1,1,1,0)} \left( \partial_{\xi} \phi_2 + \partial_{\xi} \phi_4 - \partial_{\xi} \phi_1 + \partial_{\xi} \phi_5 - \partial_{\xi} \phi_6 \right) \)  
\( + P_{(1,2,3,2,1,1)} P_{(1,1,2,1,0,1)} - P_{(1,2,2,2,1,1)} P_{(1,1,1,1,1,1)} + P_{(1,2,2,1,1,0,1)} P_{(1,1,1,1,1,0,1)} + P_{(1,2,2,1,1,0)} P_{(0,0,1,0,0,1,1)} + P_{(0,1,1,1,1,0,1)} P_{(0,0,0,0,0,1)} \)  
\( \)  
\( S_{(0,0,1,1,1,0)} : \quad 0 = \partial_{\xi} P_{(0,1,0,1,1)} + P_{(0,1,1,1,1)} P_{(0,0,0,0,0)} + P_{(1,1,1,0,0)} P_{(0,1,1,1,0)} + P_{(0,0,1,1,1,0)} \left( \partial_{\xi} \phi_3 + \partial_{\xi} \phi_4 - \partial_{\xi} \phi_1 + \partial_{\xi} \phi_5 - \partial_{\xi} \phi_6 \right) \)  
\( - P_{(1,2,3,2,1,1)} P_{(1,2,1,0,1)} - P_{(1,2,2,2,1,1)} P_{(1,1,1,1,0,1)} + P_{(1,2,2,2,1,1)} P_{(1,1,1,0,0,1)} - P_{(0,1,1,1,0,0,1)} P_{(0,0,1,1,1,1,0,1)} + P_{(0,1,2,2,1,0)} P_{(1,0,1,0,1,0,1)} + P_{(1,1,2,1,0,1)} P_{(0,0,1,0,0,1,1)} \)  
\( S_{(1,1,1,1,0)} : \quad 0 = \partial_{\xi} P_{(1,1,1,1,0)} + P_{(1,0,1,0,0,0)} P_{(1,1,1,1,0)} + P_{(0,1,1,1,1,1,0)} \left( \partial_{\xi} \phi_2 + \partial_{\xi} \phi_4 - \partial_{\xi} \phi_1 + \partial_{\xi} \phi_5 - \partial_{\xi} \phi_6 \right) \)  
\( + P_{(1,2,3,2,1,1)} P_{(1,1,2,1,0,1)} - P_{(1,2,2,2,1,1)} P_{(1,1,1,1,0,1)} + P_{(1,2,2,2,1,1)} P_{(1,1,1,0,0,1)} + P_{(0,1,2,2,1,0)} P_{(0,1,0,1,0,1,0)} + P_{(1,1,2,1,0,1)} P_{(0,0,1,0,1,0,1)} + P_{(0,1,1,1,1,1,0)} P_{(0,0,0,0,0,1)} \)  
\( S_{(1,1,1,1,1,0)} : \quad 0 = \partial_{\xi} P_{(1,1,1,1,1,0)} \left( \partial_{\xi} \phi_1 + \partial_{\xi} \phi_3 - \partial_{\xi} \phi_2 + \partial_{\xi} \phi_5 - \partial_{\xi} \phi_6 \right) \)  
\( + P_{(1,2,3,2,1,1)} P_{(0,1,2,1,0,1)} + P_{(1,2,2,2,1,1)} P_{(0,1,1,1,0,1,1)} + P_{(1,2,2,1,1,0,1)} P_{(0,1,1,0,0,1,1)} + P_{(1,1,2,2,1,1,0)} P_{(0,0,1,1,1,0,1)} + P_{(1,1,2,2,1,1,0,1)} P_{(0,0,0,0,1,0,1)} - P_{(1,1,1,1,1,0)} P_{(0,1,1,1,1,1,0,1)} \)  

B.2.3 Level 1 Equations of Motion

\( S_{(0,0,0,0,0,1)} : \quad 0 = \partial_{\xi} P_{(0,0,0,0,0,1)} + P_{(0,0,0,0,0,1)} \left( 2 \partial_{\xi} \phi_6 - \partial_{\xi} \phi_3 \right) \)  
\( - P_{(0,0,1,1,0,0)} P_{(0,0,1,1,0,0,1)} + P_{(0,0,1,1,0,0,1)} P_{(0,0,1,1,1,0)} - P_{(0,0,1,1,1,0)} P_{(0,0,1,1,1,1,0)} + P_{(0,1,1,1,1,0,0)} P_{(0,1,1,1,1,0,0,1)} + P_{(0,1,1,1,1,0,0,1)} P_{(0,1,1,1,1,1,0)} + P_{(0,1,1,1,1,1,0,0)} P_{(0,1,1,1,1,1,1,0)} + P_{(1,1,1,1,0,0,0)} P_{(1,1,1,1,0,0,1)} + P_{(1,1,1,1,0,0,1)} P_{(1,1,1,1,1,0)} + P_{(1,1,1,1,1,1,0)} P_{(1,1,1,1,1,1,1,0)} + P_{(1,2,3,2,1,1,2)} P_{(1,2,3,2,1,1)} \)
APPENDIX B. SIGMA MODEL EQUATIONS OF MOTION

\[ S_{(0,0,1,0,0,1)} : 0 = \partial_\xi P_{(0,0,1,0,0,1)} + P_{(0,0,1,0,0,1)} (\partial_\xi \phi_0 + \partial_\xi \phi_3 - \partial_\xi \phi_2 - \partial_\xi \phi_4) \]
\[ + P_{(0,0,1,0,0,1)} P_{(0,0,1,1,0)} - P_{(0,0,0,1,1,0)} P_{(0,1,1,1,1,1)} + P_{(0,1,0,0,0,0)} P_{(0,1,1,1,0,1)} \]
\[ + P_{(0,1,1,1,0,0)} P_{(0,1,2,1,0,1)} - P_{(0,1,1,1,1,0)} P_{(0,1,2,1,1,1)} + P_{(1,1,1,0,0,0)} P_{(1,1,1,1,0,1)} \]
\[ + P_{(1,1,1,1,0,0)} P_{(1,1,2,1,1,1)} - P_{(1,1,1,1,1,0)} P_{(1,1,2,2,1,1)} \]
\[ - P_{(1,2,3,2,1,2)} P_{(1,2,2,1,1,1)} \]  
(B.25)

\[ S_{(0,0,1,1,0,1)} : 0 = \partial_\xi P_{(0,0,1,1,0,1)} + P_{(0,0,1,1,0,1)} (\partial_\xi \phi_0 + \partial_\xi \phi_4 - \partial_\xi \phi_2 - \partial_\xi \phi_5) \]
\[ + P_{(0,0,0,1,1,0)} P_{(0,0,1,1,1,1)} + P_{(0,1,0,0,0,0)} P_{(0,1,1,1,0,1)} + P_{(0,1,0,0,0,0)} P_{(0,1,1,1,0,1)} \]
\[ - P_{(0,1,1,1,1,0)} P_{(0,1,2,1,1,1)} + P_{(1,1,1,0,0,0)} P_{(1,1,1,1,0,1)} + P_{(1,1,1,0,0,0)} P_{(1,1,2,1,0,1)} \]
\[ - P_{(1,1,1,1,0,0)} P_{(1,1,2,1,1,1)} + P_{(1,2,3,2,1,2)} P_{(1,2,2,1,1,1)} \]  
(B.26)

\[ S_{(0,0,1,1,1,1)} : 0 = \partial_\xi P_{(0,0,1,1,1,1)} + P_{(0,0,1,1,0,1)} (\partial_\xi \phi_0 + \partial_\xi \phi_5 - \partial_\xi \phi_2) \]
\[ + P_{(0,1,0,0,0,0)} P_{(0,1,1,1,1,1)} + P_{(0,1,1,0,0,0)} P_{(0,1,2,1,1,1)} - P_{(0,1,1,1,1,0)} P_{(0,1,2,2,1,1)} \]
\[ + P_{(1,1,0,0,0,0)} P_{(1,1,1,1,1,1)} + P_{(1,1,1,0,0,0)} P_{(1,1,2,1,1,1)} - P_{(1,1,1,1,0,0)} P_{(1,1,2,2,1,1)} \]
\[ - P_{(1,2,3,2,1,2)} P_{(1,2,2,1,0,1)} \]  
(B.27)

\[ S_{(0,1,1,0,0,1)} : 0 = \partial_\xi P_{(0,1,1,0,0,1)} + P_{(0,0,1,0,0,1)} (\partial_\xi \phi_0 + \partial_\xi \phi_2 - \partial_\xi \phi_1 - \partial_\xi \phi_4) \]
\[ + P_{(0,0,0,1,1,0)} P_{(0,1,1,1,1,0)} - P_{(0,0,0,1,1,0)} P_{(0,1,1,1,1,1)} - P_{(0,0,1,1,1,0)} P_{(0,1,2,1,0,1)} \]
\[ + P_{(0,0,1,1,1,0)} P_{(0,1,2,1,1,1)} + P_{(1,0,0,0,0,0)} P_{(1,1,1,0,0,1)} + P_{(1,1,1,0,0,0)} P_{(1,2,2,1,0,1)} \]
\[ - P_{(1,1,1,1,1,0)} P_{(1,2,2,1,1,1)} + P_{(1,2,3,2,1,2)} P_{(1,1,2,2,1,1)} \]  
(B.28)

\[ S_{(0,1,1,1,0,1)} : 0 = \partial_\xi P_{(0,1,1,1,0,1)} + P_{(0,1,1,1,0,1)} (\partial_\xi \phi_0 + \partial_\xi \phi_2 + \partial_\xi \phi_4 - \partial_\xi \phi_1 - \partial_\xi \phi_3 - \partial_\xi \phi_5) \]
\[ + P_{(0,0,0,1,0,0)} P_{(0,1,1,1,1,1)} - P_{(0,0,0,1,0,0)} P_{(0,1,2,1,0,1)} + P_{(0,0,1,1,1,0)} P_{(0,1,2,1,1,1)} \]
\[ + P_{(1,0,0,0,0,0)} P_{(1,1,1,1,1,0)} + P_{(1,1,1,0,0,0)} P_{(1,2,2,1,0,1)} - P_{(1,1,1,1,1,0)} P_{(1,2,2,2,1,1)} \]
\[ - P_{(1,2,3,2,1,2)} P_{(1,1,2,1,1,1)} \]  
(B.29)

\[ S_{(0,1,1,1,1,1)} : 0 = \partial_\xi P_{(0,1,1,1,1,1)} + P_{(0,1,1,1,1,1)} (\partial_\xi \phi_0 + \partial_\xi \phi_2 + \partial_\xi \phi_3 - \partial_\xi \phi_1 - \partial_\xi \phi_3) \]
\[ - P_{(0,0,1,0,0,0)} P_{(0,1,2,1,1,1)} + P_{(0,0,1,1,0,0)} P_{(0,1,2,2,1,1)} + P_{(1,0,0,0,0,0)} P_{(1,1,1,1,1,1)} \]
\[ + P_{(1,1,1,0,0,0)} P_{(1,2,2,1,1,1)} - P_{(1,1,1,1,1,0)} P_{(1,2,2,2,1,1)} \]
\[ + P_{(1,2,3,2,1,2)} P_{(1,1,2,1,0,1)} \]  
(B.30)
\[ S_{(0,1,2,1,0,1)} : 
\begin{aligned} 
0 &= \partial_\xi P_{(0,1,2,1,0,1)} + P_{(0,1,2,1,0,1)} (\partial_\xi \phi_3 - \partial_\xi \phi_1 - \partial_\xi \phi_5) \\
&\quad + P_{(0,0,0,1,0,0)} P_{(0,1,2,1,1,1)} + P_{(0,0,0,1,1,0)} P_{(0,1,2,2,1,1)} + P_{(1,0,0,0,0,0)} P_{(1,1,2,1,0,1)} \\
&\quad - P_{(1,1,0,0,0,0)} P_{(1,2,2,1,0,1)} - P_{(1,1,1,1,1,0)} P_{(1,2,3,2,1,1)} \\
&\quad + P_{(1,2,3,2,1,2)} P_{(1,1,1,1,1,1)} 
\end{aligned} \] (B.31)

\[ S_{(0,1,2,1,1,1)} : 
\begin{aligned} 
0 &= \partial_\xi P_{(0,1,2,1,1,1)} + P_{(0,1,2,1,1,1)} (\partial_\xi \phi_3 + \partial_\xi \phi_5 - \partial_\xi \phi_1 - \partial_\xi \phi_4) \\
&\quad + P_{(0,0,0,1,0,0)} P_{(0,1,2,2,1,1)} + P_{(1,0,0,0,0,0)} P_{(1,1,2,1,1,1)} - P_{(1,1,0,0,0,0)} P_{(1,2,2,1,1,1)} \\
&\quad - P_{(1,1,1,1,0,0)} P_{(1,2,3,2,1,1)} \\
&\quad - P_{(1,2,3,2,1,2)} P_{(1,1,1,1,0,1)} 
\end{aligned} \] (B.32)

\[ S_{(0,1,2,2,1,1)} : 
\begin{aligned} 
0 &= \partial_\xi P_{(0,1,2,2,1,1)} + P_{(0,1,2,2,1,1)} (\partial_\xi \phi_4 - \partial_\xi \phi_1) \\
&\quad + P_{(1,0,0,0,0,0)} P_{(1,1,2,2,1,1)} - P_{(1,1,0,0,0,0)} P_{(1,2,2,2,1,1)} - P_{(1,1,1,0,0,0)} P_{(1,2,3,2,1,1)} \\
&\quad + P_{(1,2,3,2,1,2)} P_{(1,1,1,0,0,1)} 
\end{aligned} \] (B.33)

\[ S_{(1,1,1,0,0,1)} : 
\begin{aligned} 
0 &= \partial_\xi P_{(1,1,1,0,0,1)} + P_{(1,1,1,0,0,1)} (\partial_\xi \phi_0 + \partial_\xi \phi_1 - \partial_\xi \phi_4) \\
&\quad + P_{(0,0,0,1,0,0)} P_{(1,1,1,1,0,1)} - P_{(0,0,0,1,1,0)} P_{(1,1,1,1,1,1)} - P_{(0,0,1,1,0,0)} P_{(1,1,2,1,0,1)} \\
&\quad + P_{(0,0,1,1,1,0)} P_{(1,1,2,1,1,1)} - P_{(0,1,1,1,0,0)} P_{(1,2,2,1,0,1)} + P_{(0,1,1,1,1,0)} P_{(1,2,2,1,1,1)} \\
&\quad - P_{(1,2,3,2,1,2)} P_{(0,1,2,2,1,1)} 
\end{aligned} \] (B.34)

\[ S_{(1,1,1,1,0,1)} : 
\begin{aligned} 
0 &= \partial_\xi P_{(1,1,1,1,0,1)} + P_{(1,1,1,1,0,1)} (\partial_\xi \phi_0 + \partial_\xi \phi_1 + \partial_\xi \phi_4 - \partial_\xi \phi_3 - \partial_\xi \phi_5) \\
&\quad + P_{(0,0,0,1,0,0)} P_{(1,1,1,1,1,1)} - P_{(0,0,1,0,0,0)} P_{(1,1,2,1,0,1)} + P_{(0,0,1,1,1,0)} P_{(1,1,2,2,1,1)} \\
&\quad - P_{(0,1,1,0,0,0)} P_{(1,2,2,1,0,1)} + P_{(0,1,1,1,1,0)} P_{(1,2,2,2,1,1)} \\
&\quad + P_{(1,2,3,2,1,2)} P_{(0,1,2,1,1,1)} 
\end{aligned} \] (B.35)

\[ S_{(1,1,1,1,1,1)} : 
\begin{aligned} 
0 &= \partial_\xi P_{(1,1,1,1,1,1)} + P_{(1,1,1,1,1,1)} (\partial_\xi \phi_0 + \partial_\xi \phi_1 + \partial_\xi \phi_3 - \partial_\xi \phi_5) \\
&\quad - P_{(0,0,1,0,0,0)} P_{(1,1,2,1,1,1)} + P_{(0,0,1,1,0,0)} P_{(1,1,2,2,1,1)} - P_{(0,1,1,0,0,0)} P_{(1,2,2,1,1,1)} \\
&\quad + P_{(0,1,1,1,0,0)} P_{(1,2,2,2,1,1)} \\
&\quad - P_{(1,2,3,2,1,2)} P_{(0,1,2,1,0,1)} 
\end{aligned} \] (B.36)
\[ S_{(1,1,2,1,0,1)} : 0 = \partial_\xi P_{(1,1,2,1,0,1)} + P_{(1,1,2,1,0,1)} \left( \partial_\xi \phi_1 + \partial_\xi \phi_3 - \partial_\xi \phi_2 - \partial_\xi \phi_5 \right) + P_{(0,0,0,1,0)} P_{(1,1,2,1,1,1)} + P_{(0,0,0,1,1,0)} P_{(1,1,2,2,1,1)} + P_{(0,1,0,0,0,0)} P_{(1,2,2,1,0,1)} + P_{(0,1,1,1,1,0)} P_{(1,2,3,2,1,1)} - P_{(1,2,3,2,1,2)} P_{(0,1,1,1,1,1)} \] (B.37)

\[ S_{(1,1,2,1,1,1)} : 0 = \partial_\xi P_{(1,1,2,1,1,1)} + P_{(1,1,2,1,1,1)} \left( \partial_\xi \phi_1 + \partial_\xi \phi_3 + \partial_\xi \phi_5 - \partial_\xi \phi_2 - \partial_\xi \phi_4 \right) + P_{(0,0,0,1,0,0)} P_{(1,1,2,2,1,1)} + P_{(0,1,0,0,0,0)} P_{(1,2,2,1,1,1)} + P_{(0,1,1,1,1,0)} P_{(1,2,3,2,1,1)} + P_{(1,2,3,2,1,2)} P_{(0,1,1,1,0,1)} \] (B.38)

\[ S_{(1,1,2,2,1,1)} : 0 = \partial_\xi P_{(1,1,2,2,1,1)} + P_{(1,1,2,2,1,1)} \left( \partial_\xi \phi_1 + \partial_\xi \phi_4 - \partial_\xi \phi_2 \right) + P_{(0,1,0,0,0,0)} P_{(1,2,2,2,1,1)} + P_{(0,1,1,0,0,0)} P_{(1,2,3,2,1,1)} - P_{(1,2,3,2,1,2)} P_{(0,1,1,0,0,1)} \] (B.39)

\[ S_{(1,2,2,1,0,1)} : 0 = \partial_\xi P_{(1,2,2,1,0,1)} + P_{(1,2,2,1,0,1)} \left( \partial_\xi \phi_2 - \partial_\xi \phi_5 \right) + P_{(0,0,0,1,0,0)} P_{(1,2,2,1,1,1)} + P_{(0,0,0,1,1,0)} P_{(1,2,2,2,1,1)} - P_{(0,0,1,1,1,0)} P_{(1,2,3,2,1,1)} + P_{(1,2,3,2,1,2)} P_{(0,0,1,1,1,1)} \] (B.40)

\[ S_{(1,2,2,1,1,1)} : 0 = \partial_\xi P_{(1,2,2,1,1,1)} + P_{(1,2,2,1,1,1)} \left( \partial_\xi \phi_2 + \partial_\xi \phi_5 - \partial_\xi \phi_4 \right) + P_{(0,0,0,1,1,0)} P_{(1,2,2,2,1,1)} - P_{(0,0,1,1,0,0)} P_{(1,2,3,2,1,1)} - P_{(1,2,3,2,1,2)} P_{(0,0,1,1,1,0)} \] (B.41)

\[ S_{(1,2,2,2,1,1)} : 0 = \partial_\xi P_{(1,2,2,2,1,1)} + P_{(1,2,2,2,1,1)} \left( \partial_\xi \phi_2 + \partial_\xi \phi_4 - \partial_\xi \phi_3 \right) - P_{(0,0,1,0,0,0)} P_{(1,2,3,2,1,1)} + P_{(1,2,3,2,1,2)} P_{(0,0,1,0,0,1)} \] (B.42)

\[ S_{(1,2,3,2,1,1)} : 0 = \partial_\xi P_{(1,2,3,2,1,1)} + P_{(1,2,3,2,1,1)} \left( \partial_\xi \phi_3 - \partial_\xi \phi_6 \right) - P_{(1,2,3,2,1,2)} P_{(0,0,0,0,0,1)} \] (B.43)

**B.2.4 Level 2 Equation of Motion**

\[ S_{(1,2,3,2,1,2)} : 0 = \partial_\xi P_{(1,2,3,2,1,2)} + \partial_\xi \phi_6 P_{(1,2,3,2,1,2)} \] (B.44)
APPENDIX C

Verification of Solutions

The equations of motion and solutions provided in chapter 5 are complicated enough that we provide this section on verifying their validity. The $P$ fields of the $G = SL(n, \mathbb{R})$ solutions do not contain summation, so we present them first before making comments on the $D_{n(n)}$ and $E_{n(n)}$ models where $\text{height(}\Theta\text{)} > r$ (and therefore the solutions contain sums over the indices of the harmonic functions).

C.1 The $A_{n(n)}$ solutions

The solution for $SL(n, \mathbb{R})$ models with arbitrary $\epsilon$ was given by equations (5.246, 5.247, 5.250) as

\[
\phi_i = \frac{1}{2} \log(N_i) \quad \text{(C.1)}
\]

\[
P_\alpha = \pm \sqrt{\epsilon_\alpha} \frac{(N_a \partial_\xi N_{a-1} - N_{a-1} \partial_\xi N_a) (N_{b+1} \partial_\xi N_{b} - N_b \partial_\xi N_{b+1})}{\prod_{i=1}^{n} N_i^{||\alpha_i||/2}}. \quad \text{(C.2)}
\]

The Cartan equations of motion

\[0 = \partial_\xi^2 \phi_i - \frac{1}{2} \sum_{\alpha \in \mathfrak{g}} \epsilon_\alpha a_i P_\alpha^2, \quad \text{(C.3)}\]

where $a_i = 1$ for all of the roots in $A_n$, then does not depend on the involution $\epsilon$. For our solutions these can be written out quite simply - take the $H_1$ equation of $SL(n+1, \mathbb{R})$ for example:

\[0 = 2\partial_\xi^2 \phi_1 + \frac{\partial_\xi N_1 \alpha_{12}}{N_1^2 N_2} + \frac{\partial_\xi N_1 \alpha_{23}}{N_1 N_2 N_3} + \ldots + \frac{\partial_\xi N_1 \alpha_{n-1,n}}{N_1 N_{n-1} N_n} + \frac{\partial_\xi N_1 \partial_\xi N_n}{N_1 N_n}, \quad \text{(C.4)}\]
where \( \alpha_{ij} = N_j \partial \xi N_i - N_i \partial \xi N_j \). Pulling out the common \( N_1^{-1} \partial \xi N_1 \) we find

\[
0 = \frac{-\partial \xi N_1}{N_1} + \frac{N_2 \partial \xi N_1 - N_1 \partial \xi N_2}{N_1 N_2} + \frac{N_3 \partial \xi N_2 - N_2 \partial \xi N_3}{N_2 N_3} + \ldots + \frac{N_n \partial \xi N_{n-1} - N_{n-1} \partial \xi N_n}{N_{n-1} N_n} + \frac{\partial \xi N_n}{N_n}.
\]

(C.5)

This pattern is a feature of all of the Cartan equations. The \( H_2 \) equation of motion requires less effort to verify. Let us first consider the \( 0 = \partial \phi_2 + \partial \xi N_1 \left[ \frac{\alpha_{23}}{N_2 N_3} + \ldots + \frac{\alpha_{n-1,n}}{N_{n-1} N_n} + \frac{\partial \xi N_n}{N_n} \right] \]

\[
- \frac{\alpha_{12}}{N_1 N_2} \left[ \frac{\alpha_{23}}{N_2 N_3} + \ldots + \frac{\alpha_{n-1,n}}{N_{n-1} N_n} + \frac{\partial \xi N_n}{N_n} \right]
\]

(C.6)

\[
= - \frac{\partial \xi N_2}{N_2^2} + \frac{\partial \xi N_1 \partial \xi N_2}{N_1 N_2} - \frac{\alpha_{12} \partial \xi N_2}{N_1 N_2^2}.
\]

The general Cartan equation for \( H_i \) is then satisfied since

\[
0 = \partial^2 \xi \phi_i + \left( \frac{\partial \xi N_1}{N_1} - \frac{\alpha_{12}}{N_1 N_2} \ldots - \frac{\alpha_{i-1,i}}{N_{i-1} N_i} \right) \left[ \frac{\partial \xi N_n}{N_n} + \sum_{k=1}^{n-1} \frac{\alpha_{k,k+1}}{N_k N_{k+1}} \right]
\]

\[
= - \frac{(\partial \xi N_i)^2}{N_i^2} + \left( \frac{\partial \xi N_1}{N_1} - \sum_{j=1}^{i-1} \frac{\alpha_{j,j+1}}{N_j N_{j+1}} \right) \left[ \frac{\partial \xi N_i}{N_i} \right]
\]

\[
= \frac{\partial \xi N_i}{N_i} \left( \frac{\partial \xi N_1}{N_1} - \frac{\partial \xi N_i}{N_i} - \sum_{j=1}^{i-1} \frac{\alpha_{j,j+1}}{N_j N_{j+1}} \right).
\]

(C.7)

The \( S_\alpha \) equations of motion require less effort to verify. Let us first consider the \( S_{(1,0,\ldots,0)} \) equation of motion for \( SL(n, \mathbb{R})/SO(n) \) in order to avoid sign complications:

\[
0 = \partial \phi_1 + \partial \xi N_1 \left[ 2 \partial \phi_1 - \partial \xi \phi_2 \right] - \frac{\partial \phi_2}{(1,0,\ldots,0)} \frac{\partial \xi N_2}{(1,0,\ldots,0)} - \frac{\partial \phi_2}{(1,1,1,0,\ldots,0)} \frac{\partial \xi N_2}{(1,1,1,0,\ldots,0)} - \ldots - \frac{\partial \phi_2}{(1,\ldots,1)} \frac{\partial \xi N_2}{(1,\ldots,1)}.
\]

(C.8)

The first line simplifies to be \( -2 \partial \xi \phi_2 P_{(1,0,\ldots,0)} \) while in the second line each of the pairs of constants have the property that

\[
c_{(0,1,\ldots,1,0,\ldots,0)c_{(1,1,\ldots,1,0,\ldots,0)} = c_{(1,0,\ldots,0)} \sqrt{\alpha_{i+1}^2 + \alpha_{i+1}^2}.
\]

(C.9)

The RHS of (C.8) can then be rewritten as

\[
- \frac{\partial \xi N_2}{N_2 N_1^{3/2}} + \frac{\sum_{i=2}^{n-1} c_{(1,0,\ldots,0)} \alpha_{i,i+1}}{N_1 N_1^{3/2} N_i N_{i+1}} + \frac{c_{(1,0,\ldots,0)} \partial \xi N_n}{N_1 N_2^{1/2} N_n}
\]

(C.10)

so that when we remove the common factor we find our usual null series

\[
\frac{\partial \xi N_2}{N_2} - \frac{\sum_{i=2}^{n-1} \alpha_{i,i+1}}{N_i N_{i+1}} - \frac{\partial \xi N_n}{N_n} = 0.
\]

(C.11)
APPENDIX C. VERIFICATION OF SOLUTIONS

For any root $\alpha = (0, \ldots, 0, 1_a, \ldots, 1_b, 0, \ldots, 0)$, which has inner product $-1$ with the two simple roots $\alpha_{a-1}$ and $\alpha_{b+1}$, the equation of motion $S_\alpha$ will simplify into two series

\[
0 = \frac{c_\alpha}{N_a^{1/2}N_a} \left[ \frac{\partial_\xi N_1}{N_1} - \left( \sum_{i=1}^{n-2} \frac{\alpha_{i,i+1}}{N_i N_{i+1}} \right) - \frac{\partial_\xi N_{a-1}}{N_{a-1}} \right] + \frac{c_\alpha}{N_b N_{b+1}^{1/2}} \left[ \frac{\partial_\xi N_{b+1}}{N_{b+1}} - \left( \sum_{i=b+1}^{n-1} \frac{\alpha_{i,i+1}}{N_i N_{i+1}} \right) - \frac{\partial_\xi N_n}{N_n} \right],
\]

(C.12)

where both bracketed expressions are individually zero.

C.2 The $D_{n(n)}$ and $E_{n(n)}$ solutions

The first difference that makes these solutions more complicated is that the fields $P_\alpha$ generally include a sum over the indices of any $N_i$ which is the product of $m$ harmonic functions $N_i = \prod_{j=1}^m N_{i_j}$, where $m$ is the coefficient of $\alpha_i$ in $\Theta$. The Cartan equations of motion contain squares of $P$-fields which in the $SL(n, \mathbb{R})$ case eliminated the radicals. This also occurs in the $D_{n(n)}$ and $E_{n(n)}$ solutions, and leads to the same patterns of null series, but also introduces cross products where radicals remain. To see how these terms vanish we take the fully-worked example of $D_{4(4)}$ (with solutions given in (5.174-5.179) and equations of motion in appendix B.1) and consider the cross terms in the $H_1$ equation of motion. There are four fields with a sum. The first is

\[
P_{(1,1,0,0)} = \pm \sqrt{-\frac{N_2 \alpha_{12} \alpha_{23} \alpha_{24} \partial_\xi N_2}{N_2 N_1 N_3 N_4 \alpha_{2;2}^2}} \pm \sqrt{-\frac{N_2 \alpha_{12} \alpha_{23} \alpha_{24} \partial_\xi N_2}{N_2 N_1 N_3 N_4 \alpha_{2;2}^2}}
\]

(C.13)

which introduces a cross term in the square

\[
\pm 2 \frac{\alpha_{12} \alpha_{12} \alpha_{23} \alpha_{23} \alpha_{24} \alpha_{24} \partial_\xi N_2 \partial_\xi N_2}{\alpha_{2;2}^4} \frac{1}{N_1 N_3 N_4}.
\]

(C.14)

This cross term is identical to the cross terms which appear in the equation of motion from $P_{(1,1,1,0)}^2$, $P_{(1,1,0,1)}^2$ and $P_{(1,1,1,1)}^2$, so by taking the relative signs in the summations appropriately these terms cancel. This observation has analogues in the other $D_{n(n)}$ and $E_{n(n)}$ solutions, where the (multiple) cross terms are identical to others in every Cartan equation of motion.

The squared terms in the Cartan equations of motion form series which are identically zero and so verification proceeds in the same fashion as for $SL(n, \mathbb{R})$ solutions. This is also the case for the $S_\alpha$ equations of motion. While the number of series increases rather rapidly with $n$, the summation does not increase the difficulty of verifying the solutions. To show this we take the least complicated (while still allowing us to make the pertinent comments) $S_{(1,2,2,1,1,1)}$ equation of motion from $E_{6(6)}$ (B.41):

\[
0 = (\partial_\xi P_{(1,2,2,1,1,1)} + P_{(1,2,2,1,1,1)}(\partial_\xi \phi_2 + \partial_\xi \phi_5 - \partial_\xi \phi_6)) + P_{(0,0,0,1,0,0)} P_{(1,2,2,1,1)} - P_{(0,0,1,1,0,0)} P_{(1,2,3,2,1,1)} - P_{(0,0,1,1,0,1)} P_{(1,2,3,2,1,2)}.
\]

(C.15)
APPENDIX C. VERIFICATION OF SOLUTIONS

We take the solutions from (5.221-5.245) and simplify the first line of (C.15)

\[
\partial_\zeta P_{1(2,2,1,1,1)} + P_{1(2,2,1,1,1)}(\partial_\zeta \phi_2 + \partial_\zeta \phi_5 - \partial_\zeta \phi_6) = \sum_{m=1}^{2} c_{(1,2,2,1,1,1)}[a, m, x] \frac{-\partial_\zeta N_{4m} N_{1/2}^{1/2}}{N_{4m}^{1/2} N_{2}^{1/2} N_{5}^{1/2}},
\]

where

\[
c_{(1,2,2,1,1,1)}[a, m, x] = \sqrt{\frac{\alpha_{213a} \alpha_{234a} \alpha_{3a4m} \alpha_{3a4m} \alpha_{3a4m} \alpha_{4m5a} \alpha_{3a6x} \alpha_{3a6x} \alpha_{3a6x} \partial_\zeta N_{6x}}{\alpha_{212} \alpha_{233} \alpha_{3a4} \alpha_{3a4} \alpha_{3a4} \alpha_{3a4} \alpha_{3a4} \alpha_{3a4} \alpha_{3a4}}} \]

and summation is implied over all of the indices in \(c_\alpha\). When we consider the product \(P_{(0,0,0,1,0,0)}P_{1(2,2,2,1,1)}\) the observation that

\[
\alpha_{3a4m} \alpha_{3a4m} \alpha_{3a4m} \alpha_{3a4m} \alpha_{3a4m} = \alpha_{3a4m}^{2} \alpha_{3a4m} \alpha_{3a4m} \alpha_{3a4m}
\]

allows us to write

\[
P_{(0,0,0,1,0,0)}P_{1(2,2,2,1,1)} = c_{(1,2,2,1,1,1)}[a, m, x] \frac{\alpha_{3a4m} N_{1/2}^{1/2}}{N_{3a} N_{4m}^{1/2} N_{2}^{1/2} N_{5}}
\]

while the other two products are

\[
P_{(0,0,1,1,0,0)}P_{1(2,3,2,1,1)} = c_{(1,2,2,1,1,1)}[a, m, x] \frac{\alpha_{3a6x} N_{1/2}^{1/2}}{N_{3a} N_{4m} N_{6x}^{1/2} N_{2}^{1/2} N_{5}^{1/2}}
\]

\[
P_{(0,0,1,1,0,1)}P_{1(2,3,2,1,2)} = c_{(1,2,2,1,1,1)}[a, m, x] \frac{-\partial_\zeta N_{6x} N_{1/2}^{1/2}}{N_{4m} N_{6x}^{1/2} N_{2}^{1/2} N_{5}^{1/2}}.
\]

The equation of motion containing all of these expressions is then

\[
\sum_{a=1}^{3} \sum_{m=1}^{2} \sum_{x=1}^{2} c_{(1,2,2,1,1,1)}[a, m, x] \left( \frac{N_{1/2}^{1/2}}{N_{4m}^{1/2} N_{2}^{1/2} N_{5}} \right) \left[ \frac{-\partial_\zeta N_{4m}}{N_{4m}} - \frac{\alpha_{3a4m}}{N_{3a} N_{4m}} - \frac{\alpha_{4m6x}}{N_{4m} N_{6x}} - \frac{-\partial_\zeta N_{6x}}{N_{6x}} \right] = 0.
\]

While there is a summation over all of the indices of the harmonic functions, the bracketed term is zero for each element of the sum. This is a general feature in all of the \(S_\alpha\) equations of motion: the product of fields \(P_\beta P_\beta^\alpha \) will have the same summation (over the harmonic function indices) as \(P_\alpha\). For each element of the index sum the equation of motion becomes a series, as in the \(SL(n, \mathbb{R})\) model, which vanishes independently of the other terms of the index sum.
D.1 \( SL(3, \mathbb{R})/SO(1, 2) \) gravitational solutions Ricci scalar

In chapter 7 we produce several solutions from the coset model which describe a 5 dimensional metric of the form

\[ ds^2 = f(x_1 + ix_2)(dx_1^2 + dx_2^2) + g_{\mu\nu}(x_1 + ix_2)dx^\mu dx^\nu \]  

(D.1)

where Greek indices run over coordinates \( x_3 \) to \( x_5 \) and we let Latin indices represent the coordinates \( x_1 \) and \( x_2 \). It can be easily shown that particular components of the Riemann tensor vanish namely

\[ R_{a\mu\nu\rho} = 0 \quad \text{and} \quad R_{\mu\nu\rho\sigma} = 0. \]  

(D.2)  

(D.3)

The first equality is true for metrics where \( f \) and the components of \( g_{\mu\nu} \) are functions of \( x_1 \) and \( x_2 \), while the second equality requires \( f \) and \( g_{\mu\nu} \) to be (anti-)holomorphic functions. It is also easy to verify that, due to the fact that the components of the metric are holomorphic,

\[ R_{\mu_1\nu_1} = -R_{\mu_2\nu_2}. \]  

(D.4)

Therefore the only non-zero components of the Ricci tensor are \( R_{ab} \). Another direct result of equation (D.4) is that \( R_{11} = -R_{22} \) so that all of these solutions have zero Ricci scalar.

For the solution presented in the first \( \mathfrak{sl}(3, \mathbb{R}) \) model (7.114) it is easiest to transform to \( z = x_1 + ix_2 \) and \( \bar{z} = x_1 - ix_2 \) where the only non-zero Ricci tensor component is

\[ R_{zz} = -N_2^2 A'_1 - N_2^2 (N'_1)^2 + N_1^2 (N_1 A'_2 - (N'_2)^2) + N_1 N_2 ((A'_{12})^2 - 2A_1 A'_{12} A'_2 + A_1^2 (A'_2)^2 + N'_1 N'_2) \]  

(D.5)
where the prime indicates a derivative with respect to \( z \). For the dual gravity fields we found in section 7.3.3 each term cancels except for those which contain factors of \( A_1 \). While these terms vanish for the limits where \( \beta = 0 \) or \( \pi/2 \) they are non-zero for the interpolating metrics.

### D.2 The Ricci scalar in the sigma model

The metric is related to the fields \( h_{a \, b} \) which appear at level zero in the decomposition of the \( A_{D=3}^{+++} \) algebra by:

\[
g_{\mu \nu} = (e^{-h})^a_{\, \mu} (e^{-h})^b_{\, \nu} \eta_{ab} \quad (D.6)
\]

the Christoffel symbols are

\[
\Gamma^\rho_{\mu \nu} = -\partial_\mu h_{\nu \rho} - \partial_\nu h_{\rho \mu} + \partial_\rho h_{\mu \nu} \quad (D.7)
\]

and the Ricci scalar is

\[
R = -2\partial_\kappa \partial^\lambda h_{\lambda \kappa} + 2\partial_\kappa h_{\lambda \kappa} \partial^\lambda h_{\sigma} - 4\partial_\kappa h_{\lambda \kappa} \partial^\lambda h_{\sigma} - 4\partial_\kappa h_{\lambda \kappa} \partial^\lambda h_{\sigma} \quad (D.8)
\]

The bound state of dual gravitons that we have studied in chapter 7 have representative coset group elements

\[
g = \exp(\phi_1 H_1 + \phi_2 H_2) \exp(C_1 E_1 + C_2 E_2 + C_{12} E_{12}) \quad (D.9)
\]

The bound state constructed from real roots at levels one and two in the algebra decomposition has

\[
H_1 = -(K_1^1 + K_2^2 + K_3^3) + K_{DD} \\
H_2 = -(K_1^1 + K_2^2 + K_{DD}) + K_{D-1 D-1} \quad (D.10, D.11)
\]

and hence the non-zero components of \( h_{\mu \nu} \) are

\[
h_1^1 = h_2^2 = (\phi_1 + \phi_2), \quad h_3^3 = -\phi_1, \quad h_{D-1 D-1} = \phi_2 \quad \text{and} \quad h_D^D = \phi_1 - \phi_2 \quad (D.12)
\]

and consequently, for the resulting diagonal metrics,

\[
R = -2\partial_\xi \phi_1 \partial^\xi \phi_1 + 2\partial_\xi \phi_2 \partial^\xi \phi_2 - 2\partial_\xi \phi_1 \partial_\xi \phi_2 - 2\partial_\xi \partial^\xi \phi_1 - 2\partial_\xi \partial^\xi \phi_2 \quad (D.13)
\]

where \( \phi_1 \) and \( \phi_2 \) are functions \( \xi \) which is identified with \( x^1 \) or \( x^2 \) (and \( \xi \) is not summed over). The Ricci scalar is identical to the set of terms in \( \phi_1 \) and \( \phi_2 \) that appear in the brane sigma-model Lagrangian for the \( SL(3, \mathbb{R})/SO(1, 2) \) coset when the lapse function \( \eta \) is set to minus one and the total derivative terms in the above are discarded.
D.3 The Einstein Equations for the \( D = 5 \) bound state.

The equations of motion for the action in equation (7.149) when \( D = 5 \) are

\[
0 = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \frac{1}{24} F_{\mu_1\mu_2\mu_3} \nu F_{\mu_1\mu_2\mu_3} \nu + g_{\mu\nu} \frac{1}{192} G_{\mu_1\mu_2\mu_3\mu_4} |_{\nu_1\nu_2} |_{\rho} G_{\mu_1\mu_2\mu_3\mu_4} |_{\nu_1\nu_2} |_{\rho}
\]

\( (D.14) \)

\[
- \frac{1}{4} F_{\mu_1\mu_2\mu_3} |_{\nu_1} F_{\nu_1} \mu_2 \mu_3 |_{\nu_1} - \frac{1}{12} F_{\mu_1\mu_2\mu_3} |_{\nu} F_{\mu_1\mu_2\mu_3} |_{\nu}
\]

\[
- \frac{1}{24} G_{\mu_1\mu_2\mu_3\mu_4} |_{\nu_1\nu_2} |_{\rho} G_{\nu_2} \mu_2 \mu_3 |_{\nu_1} |_{\rho} - \frac{1}{48} G_{\mu_1\mu_2\mu_3\mu_4} |_{\nu_2} |_{\rho} G_{\mu_1\mu_2\mu_3\mu_4} |_{\nu_1} |_{\rho}
\]

\[
- \frac{1}{96} G_{\mu_1\mu_2\mu_3\mu_4} |_{\nu_1\nu_2} |_{\rho} G_{\mu_1\mu_2\mu_3\mu_4} |_{\nu_1\nu_2} |_{\rho}
\]

\( 0 = \partial_{\nu_3} (\sqrt{-g} F_{\mu_1\mu_2\mu_3} |_{\nu_1} |_{\rho}) - \frac{1}{2} \partial_{\nu_1} (\sqrt{-g} G_{\nu_4} \nu_2 \nu_3 |_{\nu_1} |_{\rho}) A_{\nu_1\nu_2} |_{\rho}
\]

\( (D.15) \)

\[
+ \frac{1}{2} \partial_{\nu_1} (\sqrt{-g} G_{\mu_1\mu_2\mu_3} |_{\nu_2} |_{\nu_1} |_{\rho}) A_{\nu_1\nu_2} |_{\rho} + \frac{\sqrt{-g}}{4} G_{\mu_1\mu_2\mu_3} |_{\nu_2} |_{\nu_1} |_{\rho} F_{\nu_1\nu_2} |_{\mu_3} |_{\rho}
\]

\( 0 = \partial_{\nu_1} (\sqrt{-g} G_{\mu_1\mu_2\mu_3} |_{\nu_2} |_{\rho})
\]

\( (D.16) \)

The null-geodesic motion on the coset \( \frac{SL(3,\mathbb{R})}{SO(1,2)} \) parameterised by \( \xi = x^1 \) is given by the line element

\[
ds^2 = N_1 N_2 \left( (dx^1)^2 + (dx^2)^2 + \frac{1}{N_2} (dx^3)^2 - \frac{1}{N_1 N_2} (dx^4)^2 + \frac{1}{N_1^2} (dx^5)^2 \right)
\]

\( (D.17) \)

where \( N_1 = 1 + Q x^1 \) and \( N_2 = 1 + Q x^1 \cos^2 \beta \) and the non-zero field strength components

\[
F_{145|5} = - \sin \beta \partial_1 N_1^{-1},
\]

\[
F_{134|4} = - \tan \beta \partial_1 N_1^{-1}
\]

and

\[
G_{1345|45|4} = - \cos \beta \frac{\partial_1 N_1}{N_1 N_2}
\]

implying the non-zero gauge-field components are

\[
A_{45|5} = - \sin \beta N_1^{-1},
\]

\[
A_{34|4} = - \tan \beta N_2^{-1}
\]

and

\[
A_{345|45|4} = \frac{1}{2} \cos \beta \left( \frac{1}{N_2} + \frac{1}{N_1 \cos^2 \beta} \right).
\]

We will now show that the diagonal metric and gauge fields given satisfy the equations of motion (D.14-D.16). Equation (D.16) is satisfied as \( N_1 \) is a harmonic function:

\[
0 = \partial_1 (\sqrt{-g} G_{1345|45|4}) = \cos \beta \partial_1 \partial_1 N_1 = 0.
\]

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APPENDIX D. COMPOSITE GRAVITATIONAL SOLUTIONS

Splitting equation (D.15) into the two non-trivial equations gives as the coefficient of the variations $\delta A_{15|5}$

$$\partial_1(\sqrt{-g}F^{145|5}) - \sqrt{-g}G^{1345|4}F_{134|1} = -\sin \beta \partial_1(\frac{\partial_1 N_1}{N_2}) - \cos \beta \partial_1 N_1 (-\tan \beta \partial_1 N_2^{-1})$$

$$= 0$$

and for $\delta A_{34|4}$ we have

$$\partial_1(\sqrt{-g}F^{134|4}) + \sqrt{-g}G^{1345|4}F_{45|15} = \tan \beta \partial_1(\frac{\partial_1 N_2}{N_1}) + \cos \beta \partial_1 N_1 (-\sin \beta \partial_1 N_1^{-1})$$

$$= 0$$

where in the final equation we note that $\partial_1 N_2 = \cos^2 \beta \partial_1 N_1$. For the Einstein equations (D.14) it is notationally useful to write $F_1^2 \equiv 6F_{145|5}F^{145|5}$, $F_2^2 \equiv 6F_{134|4}F^{134|4}$, $G^2 \equiv 48G_{1345|4}G^{1345|4}$ and $\tilde{G}_{\mu
u} = R_{\mu
u} - \frac{1}{2}g_{\mu
u}R$, so that the five non-trivial Einstein equations are

$$\tilde{G}_{11} = \frac{1}{2}(F_1^2 + F_2^2 + G^2) + g_{11} \frac{1}{2}(F_1^2 + F_2^2 + G^2)$$

$$= \frac{1}{4N_1^2 N_2^2}(N_2^2(\partial_1 N_1)^2 - N_1 N_2 \partial_1 N_1 \partial_1 N_2 + N_1^2(\partial_1 N_2)^2)$$

$$\tilde{G}_{22} = \frac{1}{4N_1^2 N_2^2}(N_2^2(\partial_1 N_1)^2 - N_1 N_2 \partial_1 N_1 \partial_1 N_2 + N_1^2(\partial_1 N_2)^2)$$

$$\tilde{G}_{33} = \frac{1}{4N_1^2 N_2^2}(N_2^2(\partial_1 N_1)^2 - N_1 N_2 \partial_1 N_1 \partial_1 N_2 + N_1^2(\partial_1 N_2)^2)$$

$$\tilde{G}_{44} = \frac{1}{4N_1^2 N_2^2}(N_2^2(\partial_1 N_1)^2 - N_1 N_2 \partial_1 N_1 \partial_1 N_2 + N_1^2(\partial_1 N_2)^2)$$

$$\tilde{G}_{55} = \frac{1}{4N_1^2 N_2^2}(N_2^2(\partial_1 N_1)^2 - N_1 N_2 \partial_1 N_1 \partial_1 N_2 + N_1^2(\partial_1 N_2)^2)$$

which gives the components of the Einstein tensor corresponding to the metric of equation (D.17) and completes the proof.

The dualised action, where $A_{D-2[D-3|1]}$ has not been eliminated is

$$S'_{1} = \int R \ast 1 - \frac{1}{2} F_{D-2[D-2]1} \wedge \ast F_{D-2[D-2]1} - F_{1[D-3|1]} dA_{D-2[D-3|1]} - A_{D-3|1} F_{2[D-3]} \wedge F_{2[D-2]}$$

(D.28)
where the metric is unchanged from equation (D.17). The equations of motion for the action in equation (D.28) when \(D = 5\) are

\[
0 = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \frac{1}{24} F_{\mu1\nu2\rho3\sigma} F^{\mu1\nu2\rho3\sigma} + \frac{1}{4} F_{\mu1\nu2\rho3\sigma} F_{\nu2\rho3\sigma}^\rho - \frac{1}{12} F_{\mu1\nu2\rho3\sigma} F_{\nu2\rho3\sigma}^\rho \\
+ \frac{1}{\sqrt{g}} F_{1|\mu|\nu2|\rho} \left( \partial_{\nu2} A_{\mu345|\rho\sigma} - A_{\mu2|\nu2} F_{\mu345|\rho\sigma} \right) \\
+ \frac{1}{\sqrt{g}} F_{1|\mu|\nu2|\rho} \left( \partial_{\nu2} A_{\mu345|\rho\sigma} - A_{\mu2|\nu2} F_{\mu345|\rho\sigma} \right)
\]

\[
0 = \partial_{\nu2} (\sqrt{-g} F_{1|\mu|\nu2|\rho}) - \frac{1}{2} \partial_{\nu1} (F_{\nu2|\mu1\rho\sigma} A_{\mu345|\rho\sigma}) - \frac{1}{2} F_{\nu2|\mu1\rho\sigma} F_{\nu1\nu3|\rho}\]

The dualised bound state has non-trivial field strength and gauge field components, the non-trivial field strength components are

\[
F_{1|5|5} = -\sin \beta \partial_{1} N_{1}^{-1}, \\
F_{1|3|4} = -\tan \beta \partial_{1} N_{2}^{-1} \\
F_{2|4|5} = F_{2|5|4} = \cos \beta \partial_{1} N_{1} N_{2}
\]

implying the non-zero gauge-field components are

\[
A_{1|5|5} = -\sin \beta N_{1}^{-1}, \\
A_{3|4|4} = -\tan \beta N_{2}^{-1} \\
A_{3|4|5|5} = \frac{1}{2} \cos \beta \left( \frac{1}{N_{2}} + \frac{1}{N_{1} \cos^{2} \beta} \right).
\]

We now demonstrate that the same diagonal metric (D.17) with these gauge fields satisfy the equations of motion (D.29-D.31). Equation (D.31) is trivially satisfied for all field strength components, the least trivial equation being

\[
0 = \partial_{1} (F_{2|5|4}) = \partial_{1} (\cos \beta \partial_{1} N_{1}) = 0
\]

which holds as \(N_{1}\) is a harmonic function in \(x^{1}\). Equation (D.30) splits into two non-trivial equations, the coefficient of the variations \(\delta A_{1|5|5}\) gives

\[
\partial_{1} \sqrt{-g} F_{1|5|5} - \frac{1}{2} \partial_{1} (F_{2|5|4} A_{3|4|4}) - \frac{1}{2} F_{2|5|4} F_{1|3|4} = 0
\]

and the coefficient of \(\delta A_{3|4|4}\)

\[
\partial_{1} \sqrt{-g} F_{3|4|4} + \frac{1}{2} F_{2|5|4} F_{1|5|5} + \frac{1}{2} \partial_{1} (F_{2|5|4} A_{4|5|5}) = 0
\]
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For the Einstein equations (D.29) it is notationally useful to write \( F_{12}^2 = 6F_{245\ldots}F^{245\ldots} \) so that the five non-trivial Einstein equations are\(^1\)

\[
\begin{align*}
\hat{G}_{11} &= -g_{11}\frac{1}{4}(F_1^2 + F_2^2 + F_{12}^2) + g_{11}\frac{1}{2}(F_1^2 + F_2^2) \\
&= -\frac{1}{4N_1^2N_2^2}(N_2^2(\partial_1 N_1)^2 - N_1 N_2 \partial_1 N_1 \partial_1 N_2 + N_2^2(\partial_1 N_2)^2) \\
\hat{G}_{22} &= -g_{22}\frac{1}{4}(F_1^2 + F_2^2 + F_{12}^2) + g_{22}\frac{1}{2}(F_{12}^2) \\
&= \frac{1}{4N_1^2N_2^2}(N_2^2(\partial_1 N_1)^2 - N_1 N_2 \partial_1 N_1 \partial_1 N_2 + N_2^2(\partial_1 N_2)^2) \\
\hat{G}_{33} &= -g_{33}\frac{1}{4}(F_1^2 + F_2^2 + F_{12}^2) + g_{33}\frac{1}{2}(F_{12}^2) \\
&= \frac{1}{4N_1^2N_2^2}(N_2^2(\partial_1 N_1)^2 - N_1 N_2 \partial_1 N_1 \partial_1 N_2 - N_1 N_2 \partial_1 N_1 \partial_1 N_2) \\
\hat{G}_{44} &= -g_{44}\frac{1}{4}(F_1^2 + F_2^2 + F_{12}^2) + g_{44}\frac{1}{2}(F_1^2 + 2F_2^2 + 2F_{12}^2) \\
&\quad - g_{44}\frac{2}{9g}F_{2145\ldots}(\partial_1 A_{434\ldots}^{45\ldots} + \frac{1}{2}F_{134\ldots}^{45\ldots}A_{45\ldots}^{43\ldots} - \frac{1}{2}F_{145\ldots}^{43\ldots}A_{34\ldots}^{41\ldots}) \\
&= -g_{44}\frac{1}{4}(F_1^2 + F_2^2 + F_{12}^2) + g_{44}\frac{1}{2}(F_1^2 + 2F_2^2 + 2F_{12}^2) - 2g_{44}(F_{12}^2) \\
&= \frac{1}{4N_1^2N_2^2}(N_2^2(\partial_1 N_1)^2 + 2N_2^2(\partial_1 N_2)^2 + N_1 N_2 \partial_1 N_1 \partial_1 N_2) \\
\hat{G}_{55} &= -g_{55}\frac{1}{4}(F_1^2 + F_2^2 + F_{12}^2) + g_{55}\frac{1}{2}(2F_1^2 + 2F_{12}^2) \\
&\quad - g_{55}\frac{1}{9g}F_{2145\ldots}(\partial_1 A_{434\ldots}^{45\ldots} + \frac{1}{2}F_{134\ldots}^{45\ldots}A_{45\ldots}^{43\ldots} - \frac{1}{2}F_{145\ldots}^{43\ldots}A_{34\ldots}^{41\ldots}) \\
&= -g_{55}\frac{1}{4}(F_1^2 + F_2^2 + F_{12}^2) + g_{55}\frac{1}{2}(2F_1^2 + 2F_{12}^2) - g_{55}(F_{12}^2) \\
&= \frac{1}{4N_1^2N_2^2}(-3N_2^2(\partial_1 N_1)^2 + N_1 N_2 \partial_1 N_1 \partial_1 N_2 - N_1 N_2 \partial_1 N_1 \partial_1 N_2)
\end{align*}
\]

which gives the non-zero components of the Einstein tensor corresponding to the metric of equation (D.17) and completes the proof.

\(^1\)For comparison with the previous set of Einstein equations it is useful to note that \( F_{12}^2 = -G^2 \).


http://books.google.co.uk/books?id=G_8AAAIAAAJ.


http://dx.doi.org/10.1007/BFb0104623.


BIBLIOGRAPHY


