LOCAL TO GLOBAL COMPATIBILITY ON THE EIGENCURVE

(l ≠ p)

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Abstract. We generalise Coleman’s construction of Hecke operators to define
an action of $GL_2(\mathbb{Q}_l)$ on the space of finite slope overconvergent $p$-adic modular
forms ($l \neq p$). In this way we associate to any $\mathbb{C}_p$-valued point on the tame level
$N$ Coleman-Mazur eigencurve an admissible smooth representation of $GL_2(\mathbb{Q}_l)$
extending the classical construction. Using the Galois theoretic interpretation
of the eigencurve we associate a 2-dimensional Weil-Deligne representation to
such points and show that away from a discrete set they agree under the Local
Langlands correspondence.

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1. INTRODUCTION

In the mid 1990s, Coleman wrote a series of papers (see [10], [9], [12]) which
rejuvenated the study of $p$-adic properties of modular forms as initiated by Katz
([22]) and Serre ([24]) almost thirty years earlier. Perhaps the most profound
output of this work was the construction of the eigencurve ([9]): a type of universal
parameter space for both overconvergent $p$-adic eigenforms and residually modular
Galois representations. In recent years much work has been done to understand how
these objects behave over the eigencurve. For example, work of Kisin \([23]\) applies Fontaine theory to \(p\)-adic Galois representations on the eigencurve to deduce certain cases of the Fontaine-Mazur conjecture. In this paper we study how automorphic representations attached to classical cuspidal forms (away from \(p\) and \(\infty\)) vary across the eigencurve.

More precisely, fix a prime \(p\) and let \(\mathcal{A}\) denote the adeles over \(\mathbb{Q}\). To \(f \in S_k(\Gamma_1(N), \chi)\), a normalised classical cuspidal eigenform, we can associate (non-canonically) a cuspidal algebraic automorphic representation, \(\pi_f\), of \(GL_2(\mathbb{A})\). The finite part of this representation in naturally the restricted tensor product of smooth irreducible representations \(\pi_{f,l}\) of \(GL_2(\mathbb{Q}_l)\) as \(l\) ranges over all rational primes.

Also associated to \(f\) is a compatible system of Galois representations:

\[
\rho_f : G_\mathbb{Q} \rightarrow GL_2(\overline{\mathbb{Q}}_p).
\]

The global Langlands philosophy links these two objects, but the correspondence can be seen at the local level. If we restrict \(\rho_f\) to a decomposition group at \(l \neq p\) then Grothendieck’s abstract monodromy theorem attatches to \(f\) a Weil-Deligne representation \((\rho_{f,l}, N_f)\). Work of Carayol \([7]\) shows that this matches with \(\pi_{f,l}\) under the (correctly normalised) local Langlands correspondence. This expresses the compatibility of the local and global Langlands correspondences. In this paper we generalise this result to overconvergent \(p\)-adic eigenforms.

Let \(\mathcal{E}\) denote the reduced tame level \(N\) cuspidal eigencurve (as constructed in \([9], [4]\)). To any \(\mathbb{C}_p\)-valued point (equivalent to a overconvergent eigenform of tame level \(N\) of some weight \(\kappa\)) we associate a admissible smooth (not necessarily irreducible) representation of \(GL_2(\mathbb{Q}_l)\) for \(l \neq p\). This is done by defining a weight zero action using the moduli problem of elliptic curves and then introducing a weight \(\kappa\) twist. As in Coleman’s construction of Hecke operators this twist factor is obtained by \(p\)-adically deforming the \(q\)-expansions of classical Eisenstein series. In the study of the \(p\)-adic properties of automorphic forms this is a common theme - embedding objects in large ambient \(p\)-adic Banach space which come equipped with some kind of weight zero action and then defining arbitrary weights by introducing certain twist factors (see \([8], [4]\)).

The Galois theoretic construction of \(\mathcal{E}\) ensures that there is an admissible open dense subspace \(\mathcal{E}^o \subset \mathcal{E}\) together with an admissible covering by open affinoids \(U\) such that for any \(Sp(A) \in U\) there exists a continuous Galois representation

\[
\rho : G_\mathbb{Q} \rightarrow GL_2(A).
\]

This representation is unique up to isomorphism over \(A\). \(\mathcal{E} \backslash \mathcal{E}^o\) is the discrete set corresponding to pseudocharacters which give rise to reducible representations. \(\mathcal{E}^o\) contains all the classical cuspidal locus, which forms a Zariski dense set. If \(Z \subset \mathcal{E}^o\) is an irreducible component (see \([13]\)) then there is naturally an admissible cover of \(Z\) with the above property. A mild generalisation of Grothendieck’s monodromy theorem allows us to construct a family of two dimensional Weil-Deligne representations across \(Z\), which agrees with the ordinary construction on the classical locus. Thus for any \(\kappa \in \mathcal{E}^o(\mathbb{C}_p)\) we have \(\pi_{f_{\kappa,l}}\), an admissible smooth representation of \(GL_2(\mathbb{Q}_l)\) and a two dimensional Weil-Deligne representation \((\rho_{(\kappa,l)}, N_\kappa)\). Let \(\pi_m\) be the modification of the Tate normalised local Langlands correspondence introduced in \(\S5.3\).

Our main result is the following:
THEOREM A. Away from a discrete set of points local to global compatibility holds on $\mathcal{E}^o$, i.e.

$$\pi_m(\rho_\kappa, N_\kappa) \cong \pi_{f_\kappa,l}$$

for all $\kappa \in \mathcal{E}^o(\mathbb{C}_p)$ away from some discrete set. More precisely if $Z \subset \mathcal{E}^o$ (taken over $\mathbb{C}_p$) is an irreducible component then either:

(i) $Z$ is Supercuspidal. In this case local to global compatibility holds on all $Z$.

(ii) $Z$ is Special. In this case local to global compatibility holds everywhere except at points where monodromy vanishes. For such $\kappa$, $\pi_{f_\kappa,l}$ is the unique special irreducible sub-representation of $\pi_m(\rho_\kappa, N_\kappa)$.

(iii) $Z$ is Principal Series. Here local to global compatibility holds except at points where the ratio of the Satake parameters becomes $l^{\pm 1}$. At such points all we know is that there is a smooth admissible representation $\pi$ and a $GL_2(\mathbb{Q}_l)$ equivariant surjection $\pi \rightarrow \pi_{f_\kappa,l}$ where the semi-simplification of $\pi$ is isomorphic to the semi-simplification of $\pi_m(\rho_\kappa, N_\kappa)$.

We briefly outline the proof: Fix a sufficiently small open compact subgroup $U < GL_2(\mathbb{Z}_l)$ which is bonne in the sense of [2] and such that $\pi_{f_\kappa,l}^U$ is non-zero for all $\kappa \in \mathcal{E}^o(\mathbb{C}_p)$. Also fix a Haar measure and let $\mathcal{H}_U$ denote the Hecke algebra of compactly supported $\mathbb{C}_p$-valued $U$ bi-invariant functions on $GL_2(\mathbb{Q}_l)$. One of the central results of [2] is that there is an equivalence of categories between admissible smooth representations of $GL_2(\mathbb{Q}_l)$ and finite dimensional $\mathcal{H}_U$ modules. Because we are working over $\mathbb{C}_p$ such a module is determined up to semisimplicity by traces. Let $\tilde{\mathcal{E}}^o$ be a normalisation of $\mathcal{E}^o$, considered as rigid spaces over $\mathbb{C}_p$. Using the automorphic interpretation of $\mathcal{E}$ we construct a function

$$tr_{aut}: \mathcal{H}_U \rightarrow O(\tilde{\mathcal{E}}^o),$$

which agrees with the trace map of $\mathcal{H}_U$ acting on $\pi_{f_\kappa,l}^U$ away from a discrete subset. Similarly, using the Galois map of $\mathcal{E}$ we construct a function

$$tr_{Lan}: \mathcal{H}_U \rightarrow O(\tilde{\mathcal{E}}^o),$$

which agrees with the trace map of $\mathcal{H}_U$ acting on $\pi_m(\rho_\kappa, N_\kappa)^U$ away from a discrete set. By Coleman’s classicality result and classical local to global compatibility we deduce that both functions must agree on a Zariski dense set. Hence we deduce that $tr_{aut} = tr_{Lan}$. This proves local to global compatibility on $\mathcal{E}$ away from a discrete set. The rest of Theorem A is deduced from a careful study of the finer properties of both these functions.

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2. THE CLASSICAL CASE

We briefly recall how to attach an automorphic representation to a cuspidal eigenform.

Let us fix the following notation:
\( \mathbb{A} \): the adeles over \( \mathbb{Q} \)

\( \mathbb{A}_f \): the finite adeles over \( \mathbb{Q} \)

\( \hat{\mathbb{Z}} := \prod_p \mathbb{Z}_p \subset \mathbb{A}_f \)

\( U \subset GL_2(\mathbb{A}_f) \): an open compact subgroup

\( U_0(N) := \{ g \in GL_2(\hat{\mathbb{Z}}) | g \equiv (\ast \ast \ast \ast \ast \ast) (N) \} \)

\( U_1(N) := \{ g \in GL_2(\hat{\mathbb{Z}}) | g \equiv (\ast \ast 0 1) (N) \} \)

\( U(N) := \{ g \in GL_2(\hat{\mathbb{Z}}) | g \equiv (1 0 0 1) (N) \} \)

We can decompose \( GL_2(\mathbb{A}) \) with respect to \( U \) using the following famous result of Borel:

**Lemma 1.** If \( U \subset GL_2(\mathbb{A}_f) \) is an open, compact subgroup then

\[
GL_2(\mathbb{A}) = \bigoplus_{j=1}^{r} GL_2(\mathbb{Q}) \cdot g_j U \cdot GL_2^+(\mathbb{R}) \text{ where } g_j \in GL_2(\mathbb{A}), \ r < \infty
\]

If \( \text{det}(U) = \hat{\mathbb{Z}}^* \) then \( r = 1 \) and \( g_1 \) can be trivial.

**Proof.** This is just an application of the strong approximation theorem. \qed

Let \( \Gamma \in GL_2^+(\mathbb{Q}) \) be an arithmetic subgroup such that \( U \cap GL_2^+(\mathbb{Q}) = \Gamma \). Let us further assume that \( \text{det}(U) = \hat{\mathbb{Z}}^* \). We make the following important definition:

**Definition.** For \( f \in \mathcal{M}_k(\Gamma) \) a classical modular form of weight \( k \in \mathbb{Z} \) and level \( \Gamma \) define

\[
\varphi_f : GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) / U \rightarrow \mathbb{C}
\]

\[
[g = \gamma u \delta] \rightarrow f(\delta i)j(\delta, i)^{-k \text{det}(\delta)}
\]

Where \( \gamma \in GL_2(\mathbb{Q}), u \in U \) and \( \delta \in GL_2^+(\mathbb{R}) \), as in the decomposition of lemma 1.

This is not a canonical procedure. There are different conventions about what power of determinant to choose. This choice ensures that classical and adelic Hecke operators (when correctly normalised) agree. We embed such functions in the ambient space of weight \( k \) modular forms:

\[
\mathcal{M}_k := \{ \varphi : GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) \rightarrow \mathbb{C} \mid (\ast) \}
\]

Where \( (\ast) \) is a set of conditions:

(i) \( \varphi \) must be left invariant under some open compact subgroup \( U \subset GL_2(\mathbb{A}_f) \).

(ii) \( \forall g \in GL_2(\mathbb{A}_f) \) we have a map \( f_g^\varphi : \mathcal{H} \rightarrow \mathbb{C} \) given by:

\[
f_g^\varphi(\delta i) = \varphi(g\delta)j(\delta, i)^{k \text{det}(\delta)}^{-1}
\]

where \( \delta \in GL_2^+(\mathbb{R}) \). This must be well defined and holomorphic.

(iii) \( \varphi \) is slowly increasing, see [17].
\[ M_k \subset A, \] where \( A \) is the space of automorphic forms for \( GL_2(\mathbb{Q}) \). It naturally comes equipped with a smooth admissible action \( GL_2(A_f) \). There is a very direct relation between this space and classical modular forms. If \( \varphi \in \mathcal{M}_k^U \) then \( f_\varphi^g \in \mathcal{M}_k(\Gamma_g) \) where \( \Gamma_g = gUg^{-1} \cap GL_2^+(\mathbb{Q}) \). Condition (ii) assures us that it satisfies the automorphic invariance property with respect to \( \Gamma_g \), and condition (iii) assures us that it is well behaved at cusps. If 

\[
GL_2(A_f) = \prod_{j=1}^r GL_2(\mathbb{Q}) \cdot g_j \cdot U \quad \text{where} \quad g_j \in GL_2(A_f), \ r < \infty
\]

and \( \Gamma_j = g_jU\bar{g}_j^{-1} \cap GL_2^+(\mathbb{Q}) \) then there is an isomorphism of complex vector spaces 

\[
\mathcal{M}_k^U \rightarrow \bigoplus_{j=1}^r \mathcal{M}_k(\Gamma_j)
\]

\[
\varphi \rightarrow (f_\varphi^g).
\]

In this way \( \mathcal{M}_k^{U_1(N)} \cong \mathcal{M}_k(\Gamma_1(N)) \). If we include a nebentypus character \( \chi \) then there is an isomorphism. \( \mathcal{M}_k^{U_1(N)}(\chi) \cong \mathcal{M}_k(\Gamma_1(N), \chi^{-1}) \) Our choice of conventions ensures that with is an isomorphism of Hecke modules away from \( N \).

**Definition.** For \( \varphi \in \mathcal{M}_k \) and \( g \in GL_2(A_f) \) we define the \( g \)-\( q \)-expansion of \( \varphi \) as being the \( q \)-expansion at \( \infty \) of \( f_\varphi^g \).

From the above it is clear that any \( \varphi \in \mathcal{M}_k \) is uniquely determined by a finite number of these \( q \)-expansions.

The space of cusp forms can also be expressed in this adelic language and we obtain the space \( S_k \subset M_k \) by demanding a cuspidal condition, see [17]. This property means that \( f_\varphi^g \in S_k(\Gamma_g) \). \( S_k \subset A^p \), the space of cuspidal automorphic forms. This latter space decomposes discretely into irreducible automorphic representations of \( GL_2(\mathbb{A}) \) each with multiplicity one. In the usual way, this is really a \( (gl_1, O_2(\mathbb{R})) \)-module at infinity but we suppress this from the notation. For \( k \geq 2 \) the span of \( S_k \) within \( A^p \) is precisely the direct sum of the automorphic representations whose infinite component is \( D_k \), the holomorphic weight \( k \) discrete series (with some fixed central character that we will suppress from the notation). In this way, \( S_k \) decomposes as the direct sum of the irreducible admissible representations of \( GL_2(A_f) \) appearing in \( A^p \) whose infinity component is \( D_k \). If \( f \in S_k(\Gamma_1(N), \chi) \) is a newform then \( \varphi_f \) lies in a unique such irreducible constituent. This sets us a bijection between newforms and irreducible subrepresentations of \( S_k \). More generally, any eigenform \( f \) lies in a unique irreducible sub-representation.

**Definition.** If \( f \in S_k(\Gamma_1(N), \chi) \) is an eigenform we define 

\[
\pi_f = C[GL_2(A_f)] \varphi_f = S_k
\]

as the irreducible smooth automorphic representation attached to \( f \).

By the general theory of such representations we know that it decomposes as the restricted tensor product of local representations: 

\[
\pi_f = \bigotimes_l \pi_{f,l},
\]

where the product is over all rational primes and the local factors are smooth irreducible representations of \( GL_2(\mathbb{Q}_l) \) in the sense of [21]. Clearly \( \pi_{f,l} \cong C[GL_2(\mathbb{Q}_l)] \varphi_f \).
The theory overconvergent modular forms is built on the $p$-adic properties of $q$-expansions. If $f$ is an overconvergent eigenform in the sense of \cite{9} then to attach a representation of $GL_2(\mathbb{Q}_l)$ we must make sense of this action at the level of $q$-expansions.

**Proposition 2.** Let $U \in GL_2(\mathbb{A}_f)$ an open compact subgroups, such that $\det(U) = \mathbb{Z}_l^*$. Let $g, h \in GL_2(\mathbb{A}_f)$ with $h g = \gamma \cdot u$ where $\gamma \in GL_2^+(\mathbb{Q})$ and $u \in U$. For any $\varphi \in \mathcal{M}_k$ the $h$-$q$-expansion at of $g(\varphi)$ is the $q$-expansion at infinity of $\det^{-k} \cdot f_{\gamma}^h$. 

**Proof.** The conventions we choose are that $f_{\gamma}^h := (\det \gamma)^{k-1} \cdot j(\gamma, \tau)^{-k} \cdot f(\gamma \tau)$. By definition $f_{\gamma}^h(\delta) = \varphi(h \delta g) j(\delta, i)^k \det(\delta)^{-1} = \varphi(\gamma \delta) j(\delta, i)^k \det(\delta)^{-1}$. $\varphi$ is left invariant by $GL_2(\mathbb{Q})$, so this simplifies to $\varphi(\gamma^{-1} \delta) j(\delta, i)^k \det(\delta)^{-1}$. By the elementary properties of the $j$ function this simplifies to the expression we are looking for.

Ultimately we will be interested on the effect of this action on the $q$-expansions of Eisenstein series as it is these which $p$-adically interpolate. We want therefore to be able to explicitly determine the effect of this action on $q$-expansion of classical forms of level $U_1(p)$ and character $\chi$ under the action of the Diamond operators at $p$. From now on fix a rational prime $l \neq p$.

**Lemma 3.** Let $g \in GL_2(\mathbb{Q}_l) \subset GL_2(\mathbb{A}_f)$. Then

$$g = \begin{pmatrix} l^m & a \cr 0 & l^n \end{pmatrix} \cdot u, \text{ with } a, r, m, n \in \mathbb{Z}, u \in U_0(p)$$

Where the first term is embedded diagonally in $GL_2(\mathbb{A}_f)$.

**Proof.** By the Iwasawa decomposition

$$GL_2(\mathbb{Q}_l) = B(\mathbb{Q})GL_2(\mathbb{Z}_l)$$

where $B(\mathbb{Q})$ is the Borel subgroup of upper triangular matrices. Therefore

$$g = \begin{pmatrix} A & B \\
0 & C \end{pmatrix} \cdot v \text{ where } A, B, C \in \mathbb{Q} \text{ and } v \in GL_2(\mathbb{Z}_l).$$

Post-multiplying the first term by an an appropriate diagonal matrix with coefficients in $\mathbb{Q}$ contained in $GL_2(\mathbb{Z}_l)$ we can get it into the form

$$\begin{pmatrix} l^m & a \\
0 & l^n \end{pmatrix} v \text{ where } m, n, r, N, a \in \mathbb{Z}, r \geq 0, (N, l) = 1$$

We want $N = 1$. Assume this is not the case and consider the following:

$$\begin{pmatrix} l^m & a \\
0 & l^n \end{pmatrix} \cdot \begin{pmatrix} c \\
0 \end{pmatrix} = \begin{pmatrix} l^m & a \\
0 & l^n \end{pmatrix} \cdot \begin{pmatrix} c \cr l^n \end{pmatrix}, c \in \mathbb{Z}$$

Note that the second factor is in $GL_2(\mathbb{Z}_l)$ for any such choice of $c$, and we may assume $r + m \geq 0$. We are reduced to finding $c \in \mathbb{Z}$ such that $N \parallel a + cl^{r+m}$. But $l$ and $N$ are coprime so Euclid’s algorithm assures us that there is such a $c$. Hence we are done.

**Proposition 4.** Let $\varphi \in \mathcal{M}_k^{U_1(p)}(\chi), g \in GL_2(\mathbb{Q}_l), h \in U_0(p), h_i$ its component at $l$, and $h_i^l$ the trivialisation of $h$ at $l$. Let

$$h_i g = \begin{pmatrix} l^m & a \\
0 & l^n \end{pmatrix} \cdot u, \text{ with } a, r, m, n \in \mathbb{Z}, u \in U_0(p)$$
If the 1-\(q\)-expansion at infinity of \(\varphi\) is \(f_1^{\varphi}(q)\) then the \(h\)-\(q\)-expansion at infinity of \(g(\varphi)\) is
\[
f_h^{g(\varphi)}(q) = \chi(uh^1)^{-1}(m+n) + nk f_1^{\varphi}(q^{n-m} \cdot e^{2\pi i (\frac{m+n}{m+n})})
\]

Proof. Let \(\begin{pmatrix} l_n & 0 \\ 0 & l_m \end{pmatrix} = \gamma\). By definition
\[
f_h^{g(\varphi)}(\delta i) = \varphi(h\delta g)j(\delta, i)^k \det(\delta)^{-1} = \varphi(h_1 g h_1^1)j(\delta, i)^k \det(\delta)^{-1}
\]

This final term is equal to
\[
\varphi(\gamma^{-1}uh^1)j(\delta, i)^k \det(\delta)^{-1} = \chi(\gamma^{-1}u) \varphi(\gamma^{-1}d)j(\delta, i)^k \det(\delta)^{-1}.
\]

The proof of proposition 2 tells us that this term is equal to
\[
\chi(\gamma^{-1}u) \det(\gamma)^{-k-2} \cdot f_1^{\varphi}(q^{n-m})
\]

If \(\tau\) is the parameter on the upper half plane then we know that
\[
f_1^{\varphi}(l_n^{-1}) = \chi(l_m^{-1}) \cdot f_1^{\varphi}(l_m^{-1})
\]

Recalling that \(q = e^{2\pi i \tau}\) we know that
\[
f_1^{\varphi}(l_n^{-1}) = l^{-m+1}(m+n) \cdot f_1^{\varphi}(l_m^{-1} + l_m^{-1})
\]

Multiplying through by the constant term in proposition 2 yields the result. \(\square\)

We wish to reinterpret these results geometrically. Let us fix the following notation:
\[
\mathcal{U} := \{ U \subset GL_2(\mathbf{A}_f) \mid \text{open, compact subgroup s.t. } U \cap GL_2^+(\mathbb{Q}) \text{ is torsion free} \}
\]
\[
\Sigma := \{ U \in \mathcal{U} | U \subset GL_2(\hat{\mathbb{Z}}) \}
\]

For \(U \in \mathcal{U}\) we define the complex Shimura variety:
\[
Y_U = GL_2(\mathbb{Q}) \backslash (\mathbb{C} \backslash \mathbb{R}) \times GL_2(\mathbf{A}_f) / U
\]

Here \(GL_2(\mathbb{Q})\) acts on \(\mathbb{C} \backslash \mathbb{R}\) by Mobius transformations and \(GL_2(\mathbf{A}_f)\) by left translation. \(U\) acts trivially on \(\mathbb{C} \backslash \mathbb{R}\) and by right translation on \(GL_2(\mathbf{A}_f)\). If
\[
GL_2(\mathbf{A}_f) = \prod_{j=1}^{r} GL_2(\mathbb{Q}) \cdot g_j U \text{ where } g_j \in GL_2(\mathbf{A}_f)
\]

and \(\Gamma_j = g_j U g_j^{-1} \cap GL_2^+(\mathbb{Q})\). Then there is a homeomorphism of topological spaces
\[
\prod_{j=1}^{r} \Gamma_j \backslash \mathcal{H} \rightarrow Y_U
\]
\[
\Gamma_j \backslash \mathcal{H} \rightarrow Y_U
\]
\[
[x] \rightarrow [x, g_j]
\]
Hence $Y_U$ inherits the structure of a riemann surface. Ordering $\mathcal{U}$ by inclusion gives the projective system

$$\varprojlim_{\mathcal{U}} Y_U$$

$GL_2(\mathbb{A}_f)$ naturally acts on this projective system. For $U \in \Sigma$, $Y_U$ is naturally the complex points of a non-singular algebraic curve $Y(U)$ over $\mathbb{C}$, with a (canonical, after fixing a Shimura datum) model over $\mathbb{Q}$, which is the moduli space of elliptic curve with level $U$ structure, see [15]. The push forward of the differential sheaf on the universal elliptic curve gives an invertible sheaf $\omega_U$. $Y(U)$ has a canonical compactification $X(U)$ and this sheaf uniquely extends to a sheaf on it which we also denote $\omega_U$. This naturally gives rise to the direct limit

$$\varinjlim_{\Sigma} \Gamma(X(U), \omega_U^{\otimes k})$$

for $k \in \mathbb{Z}$, which inherits an action of $GL_2(\mathbb{A}_f)$. There is a natural identification

$$\mathcal{M}_k^U \cong \Gamma(X(U), \omega_U^{\otimes k}).$$

This gives rise to an isomorphism of complex vector spaces

$$\mathcal{M}_k \cong \varprojlim_{\Sigma} \Gamma(X(U), \omega_U^{\otimes k}) \subset \varinjlim_{\Sigma} \Gamma(Y(U), \omega_{U}^{\otimes k})$$

**Proposition 5.** This is an isomorphism of $GL_2(\mathbb{A}_f)$ modules.

**Proof.** This is very well known, but for want for a reference we prove it. It is sufficient to prove that the inclusion of $\mathcal{M}_k$ in $\varprojlim_{\Sigma} \Gamma(Y(U), \omega_{U}^{\otimes k})$ respects the $GL_2(\mathbb{A}_f)$ action. We can work entirely in the complex analytic category and consider global section of $\omega_{U}^{\otimes k}$ as being global sections of a line bundle $L_{U}^{k}$. We can take the direct limit as being over all of $\mathcal{U}$.

We are able to decompose $GL_2(\mathbb{A}_f)$ with respect to $U$, getting double coset representatives $\{g_j\}$. Using these, we are deduce that $\{g_jg^{-1}\}$ are a set of double coset representatives with respect to $g_Ug^{-1}$. Notice that

$$\Gamma'_j = GL_2^+(\mathbb{Q}) \cap g_j g^{-1}(g_U g^{-1}) g g^{-1} = \Gamma_j$$

We have the isomorphism of riemann surfaces

$$\prod_{j=1}^{r} \Gamma_j \backslash \mathcal{H} \longrightarrow Y_{gUg^{-1}}$$

$$[x_j] \longrightarrow [x_j, g_j g^{-1}]$$

On connected components we get the maps

$$(Y_{gUg^{-1}})_j \xrightarrow{g} (Y_U)_j$$

$$[x_j, g_j g^{-1}] \xrightarrow{g} [x_j, g_j]$$

$$\Gamma_j \backslash \mathcal{H} \xrightarrow{g} \Gamma_j \backslash \mathcal{H}$$

$$[x_j] \xrightarrow{g} [x_j]$$

From this point of view the map induced by $g$ is trivial and the action on bundles is trivial. Hence for $\varphi \in \mathcal{M}_k^U$ the pullback of $f_{g_j}^{\varphi} \in M_k(\Gamma_j)$ by $g$ must be left the same and we deduce that
for any $h \in GL_2^+(\mathbb{R})$. Clearly $g(\varphi)$ satisfies this condition and we deduce that $g^* \varphi = g(\varphi)$. 

3. The $p$-adic Case

3.1. Overconvergent Modular Forms of Integer Weight. Fix once and for all an isomorphism $C \cong \mathbb{C}_p$. Let $A_f^p$ be the finite adeles away from $p$. For $U \in \Sigma$ let $U^p$ denote its image in $GL_2(A_f^p)$. For $n \in \mathbb{N}$, define $\Gamma_1(p^n), \Gamma_0(p^n) \subset GL_2(\mathbb{Z}_p)$ as the matrices which reduce to the mirabolic and borel respectively mod $p^n$. Let us modify the set $\Sigma$ such that

$$\Sigma := \{ U \subset GL_2(\hat{\mathbb{Z}}) \mid \text{open, compact subgroup s.t. } U^p \times GL_2(\mathbb{Z}_p) \cap GL_2^+(\mathbb{Q}) \text{ is torsion free} \}$$

As in the previous section, for any $n \in \mathbb{N}$ and $k \in \mathbb{Z}$,

$$\lim_{U \in \Sigma} \Gamma(X(U^p \times \Gamma_1(p^n)), \omega_{U^p \times \Gamma_1(p^n)}^\otimes)$$

comes with a left action of $GL_2(A_f^p)$. The complex smooth projective curve $X(U^p \times \Gamma_1(p^n))$ can be considered as an algebraic variety over $\mathbb{C}_p$ via the above isomorphism. It can be rigidly analytified to give the smooth projective rigid space $X(U^p \times \Gamma_1(p^n))^{an}$ over $\mathbb{C}_p$. The sheaf $\omega_{U^p \times \Gamma_1(p^n)}$ can also be analytified and we denote this by $\omega_{U^p \times \Gamma_1(p^n)}^{an}$. The functorial nature of analytification together with the rigid GAGA principle tells us that there is an isomorphism of $GL_2(A_f^p)$ modules

$$\lim_{U \in \Sigma} \Gamma(X(U^p \times \Gamma_1(p^n)), \omega_{U^p \times \Gamma_1(p^n)}^\otimes) \cong \lim_{U \in \Sigma} \Gamma(X(U^p \times \Gamma_1(p^n))^{an}, (\omega_{U^p \times \Gamma_1(p^n)}^{an})^\otimes).$$

$X(U^p \times \Gamma_1(p^n))$ is naturally the moduli space of elliptic curves with level $U^p \times \Gamma_1(p^n)$ structure and consequently has a canonical model over $\mathbb{Q}_p$. As such it can be viewed as a nonsingular projective curve over $\mathbb{Q}_p$. We can rigidly analytify this curve to give a smooth projective rigid curve which we denote $X(U^p \times \Gamma_1(p^n))^{an}/\mathbb{Q}_p$. Base change to $\mathbb{C}_p$ in the rigid analytic category recovers $X(U^p \times \Gamma_1(p^n))^{an}$. The points of $X(U^p \times \Gamma_1(p^n))^{an}/\mathbb{Q}_p$ are elliptic curves (in the rigid or algebraic category) over a finite extension of $\mathbb{Q}_p$ together with appropriate level structure.

For an elliptic curve $E$ over a finite extension of $\mathbb{Q}_p$ there is a rational number $V(E)$, defined using a lift of the Hasse invariant, which is a measure of the supersingularity of $E$. If $0 \leq V(E) < \frac{p^{2n-1}}{p-1}$ then $E$ possesses a canonical subgroup of order $p^n$. For any $r \in \mathbb{Q}$ in this range we can define $X(U^p \times \Gamma_1(p^n))_{\geq p^{-r}}/\mathbb{Q}_p$ as the admissible open affinoid subspace of $X(U^p \times \Gamma_1(p^n))^{an}/\mathbb{Q}_p$ whose non-cuspidal points correspond to elliptic curves $E$ with $V(E) \leq r$, a level $U^p$ structure and a point in the canonical subgroup of order $p^n$. Similarly we define $X(U^p \times \Gamma_0(p^n))_{\geq p^{-r}}/\mathbb{Q}_p$ as the admissible open affinoid subspace of $X(U^p \times \Gamma_0(p^n))^{an}/\mathbb{Q}_p$ whose non-cuspidal points correspond to elliptic curves $E$ with $V(E) \leq r$, a level $U^p$ structure and a subgroup of order $p^n$ which is the canonical subgroup. Geometrically, on any connected component of $X(U^p \times \Gamma_0(p))$ the supersingular locus is an annulus separating two ordinary components, one containing $\infty$ and the other $0$. The geometric connected components of $X(U^p \times \Gamma_1(p))_{\geq p^{-r}}/\mathbb{Q}_p$ are affinoid subspaces of these.
which strictly contain the ordinary locus with $\infty$ in some precise sense. We deduce that if $X(U^p \times \Gamma_1(p^n))_{an}/\mathbb{Q}_p$ has $m$ connected components then so too does $X(U^p \times \Gamma_1(p^n))_{\geq p^{-r}}/\mathbb{Q}_p$. These subspaces have been defined over $\mathbb{Q}_p$ and we denote $X(U^p \times \Gamma_1(p^n))_{\geq p^{-r}}/K$ as the base change to any complete extension $K$ of $\mathbb{Q}_p$. We denote the base change to $\mathbb{C}_p$ simply by $X(U^p \times \Gamma_1(p^n))_{\geq p^{-r}} \subset X(U^p \times \Gamma_1(p^n))_{an}$. If $U = GL_2(\hat{\mathbb{Z}})$ we simply write $X_1(p^n)_{\geq p^{-r}}$ or $X_0(p^n)_{\geq p^{-r}}$. These still make sense even if we are dealing with a coarse moduli problem. If $p$ is odd let $q = p$, otherwise $q = 4$.

**Definition.** For $K/\mathbb{Q}_p$ a complete extension, $U \in \Sigma$, $k \in \mathbb{Z}$ and non-zero $r \in \mathbb{Q}$ in the appropriate range we define

$$\mathcal{M}_k(r, U^p, K) = \Gamma(X(U^p \times \Gamma_1(q))_{\geq p^{-r}}/K, (\omega_{U^p \times \Gamma_1(q)}^{an})^{\otimes k}),$$

the space of $r$-overconvergent modular form over $K$ of tame level $U^p$ and weight $k$ with trivial action of the Diamond operators at $p$.

Note that this demand on the Diamond operators means that we actually of level $\Gamma_0(q)$. We have phrased it in this way because once we have redefined these spaces as functions using Eisenstein series in §3.3 this will introduce a non-trivial action.

It is clear that classical forms are overconvergent by restriction. If we let $r$ vary over the appropriate range then we get a direct system and we have

$$\mathcal{M}_k^1(U^p, K) := \lim_{r \to} \mathcal{M}_k(r, U^p, K),$$

the space of overconvergent modular forms of weight $k$ and tame level $U^p$.

For $V, U \in \Sigma$ open compact subgroups such that $V^p \subset U^p$ and $K/\mathbb{Q}_p$ a complete extension there is the functorial commutative diagram

$$\xymatrix{ X(V^p \times \Gamma_1(p^n))_{\geq p^{-r}}/K \ar[r]^{i} \ar[d] & X(U^p \times \Gamma_1(p^n))_{\geq p^{-r}}/K \ar[d] \\
X(V^p \times \Gamma_1(p^n))^{an}/K \ar[r] & X(U^p \times \Gamma_1(p^n))^{an}/K }$$

where the bottom horizontal arrow is just the usual inclusion functor. This tells us that pullback of $r$-overconvergent forms of level $U^p$ and weight $k$ by the map $i$ gives $r$-overconvergent forms of level $V^p$ and weight $k$. This gives the space of $r$-overconvergent modular forms over $K$ of integer weight a direct limit structure over $\Sigma$. We denote this space by

$$\mathcal{M}_k(r, K) := \lim_{U \in \Sigma} \mathcal{M}_k(r, U^p, K).$$

Similarly we have

$$\mathcal{M}_k^1(K) := \lim_{U \in \Sigma} \mathcal{M}_k^1(U^p, K).$$

We drop the $K$ from the notation if $K = \mathbb{C}_p$. There is clearly an embedding of $\mathbb{C}_p$ vector spaces

$$\mathcal{M}_k^{\Gamma_1(q)} = \lim_{U \in \Sigma} \Gamma(X(U^p \times \Gamma_1(q))^{an}, (\omega_{U^p \times \Gamma_1(q)}^{an})^{\otimes k}) \subset \mathcal{M}_k(r)$$
Proposition 6. There is an action of $GL_2(A_f)$ on $\mathcal{M}_k(r)$ such that this embedding is $GL_2(A_f)$-equivariant.

Proof. Let $g \in GL_2(A_f)$ and $V, U \in \Sigma$ such that $V^p \subset gU^p g^{-1}$. We need to show that the functorial map

$$X(V^p \times \Gamma_1(p^n))^{an} \xrightarrow{g} X(U^p \times \Gamma_1(p^n))^{an}$$

gives rise to the commutative diagram

$$
\begin{array}{ccc}
X(V^p \times \Gamma_1(p^n))_{\geq p^{-r}} & \xrightarrow{g} & X(U^p \times \Gamma_1(p^n))_{\geq p^{-r}} \\
\downarrow & & \downarrow \\
X(V^p \times \Gamma_1(p^n))^{an} & \xrightarrow{g} & X(U^p \times \Gamma_1(p^n))^{an}
\end{array}
$$

If we can show this over $\mathbb{Q}_p$ then the result follows after base change. Pullback by the above map will give the desired action. Let $K$ be a finite extension of $\mathbb{Q}_p$. If $E/K$ is an elliptic curve such that $0 \leq V(E) < \frac{p^{2-n}}{p+1}$ then it is naturally a complex curve using the isomorphism fixed at the beginning of §3. If $T(E)$ is the Tate module of $E$ then let us fix an isomorphism

$$\mu : T(E) \cong \hat{\mathbb{Z}}^2$$

to an action (on the right) of $V^p \times \Gamma_1(p^n) \subset GL_2(\hat{\mathbb{Z}})$ which fixes a point in the canonical subgroup of order $p^n$. If we tensor both sides with $\mathbb{Q}$ then $\mu$ extends to an isomorphism

$$\mu : T(E) \otimes \mathbb{Q} \cong A_f^2,$$

up to an action of $V^p \times \Gamma_1(p^n)$. This corresponds to a $K$-valued point of $X(V^p \times \Gamma_1(p^n))_{\geq p^{-r}}$. The action on the moduli problem induced by $g$ is to send $\mu$ to

$$\mu : T(E) \otimes \mathbb{Q} \cong A_f^2 \xrightarrow{g} A_f^2,$$

where the second arrow is the isomorphism induced by $g$ acting on the right. We will denote this map by $\mu_g$. This will correspond to an elliptic curve isogenous to $E$. If $E'$ is a second elliptic curve which possesses an isogeny onto $E$ which is prime to $p$ then $V(E) = V(E')$ and the image of the canonical subgroup of order $p^n$ in $E'$ is the canonical subgroup of order $p^n$ in $E$. Because we can decompose $g$ as the product of an element in $GL_2^\pm(\mathbb{Q})$ and $U_1(p^n)$, the problem is reduced to showing that for any $\gamma \in GL_2^\pm(\mathbb{Q}) \cap GL_2(\mathbb{Z}_p)$ and $\tau \in \mathcal{H}$, $E_\tau := C/\mathbb{Z} \tau \oplus \mathbb{Z}$ is isogenous to $E_{\gamma(\tau)}$ of order prime to $p$. Here

$$E_{\gamma(\tau)} \cong \mathbb{C}/(\mathbb{Z}(a\tau + b) \oplus \mathbb{Z}(c\tau + d)),$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \cap GL_2(\mathbb{Z}_p).$$

There is a natural isogeny

$$\phi : E_{\gamma(\tau)} \to E_\tau,$$

induced by the obvious inclusion of lattices. If $z = A(a\tau + b) + B(c\tau + d) \in ker(\phi)$ where $A, B \in \mathbb{R}$, then

$$(A \ B) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}_p^2 \cap \mathbb{Q}^2 \Rightarrow (A, B) \in \mathbb{Z}_p^2 \cap \mathbb{Q}^2$$
Hence if $z$ is non-trivial it cannot be in the $p$ torsion of $E_{\gamma(r)}$. Hence $\phi$ is of order prime to $p$, and we are done. \hfill \square

This action is defined on $\mathcal{M}_k(r, K)$ for any $K/\mathbb{Q}_p$ a complete extension. As for the space of classical modular forms,

$$\mathcal{M}_k(r, K)^{U^p} = \Gamma(X(U^p \times \Gamma_1(q)))_{\geq p^{-r}/K}, (\omega_{U^p \times \Gamma_1(q)})^k).$$

All of these spaces contain the as subspaces cuspidal forms which we always denote the space of classical modular forms, $(\omega_{U^p \times \Gamma_1(q)})^k)$. Now we make the following important definition:

**Definition.** If $f$ is a cuspidal overconvergent eigenform in the sense of $[9]$, then define

$$\pi_{f,l} := \mathbb{C}_p[GL_2(\mathbb{Q}_l)] f \subset \mathcal{M}_k(r) \subset S_k^1$$

Propositions 5 and the proof of proposition 6 immediately tell us that this agrees with the original definition if $f$ is classical. The remainder of this section will be devoted to generalising this to arbitrary weights.

3.2. **Rigid q-expansions.** Overconvergent modular forms of arbitrary weight are defined using the Eisenstein Family. To generalise the above constructions we need to understand the $q$-expansions of overconvergent modular forms of integer weight.

As before, let $K/\mathbb{Q}_p$ be a complete field extension. Because

$$\mathcal{M}_k(r, K) = \varinjlim_{U(N) \in \Sigma} \mathcal{M}_k(r, K)^{U(N)^p}$$

it will be sufficient to determine $q$-expansions for forms of full level $N$ structure where $(N, p) = 1$. As in the algebraic setting this can be done using the Tate curve.

Fix $K \subset \mathbb{C}_p$, a complete extension of $\mathbb{Q}_p$, such that it contains the primitive $N^{th}$ root of unity $\zeta = e^{2\pi i/N}$. Also fix $f \in \mathcal{M}_k(r, K)^{U(N)^p}$ where $(N, p) = 1$. Let $S$ denote the punctured disc of radius 1 with parameter $q^{1/N}$ in the category of rigid analytic spaces over $K$. As usual, $\mathbb{G}_m/(q)$ is the Tate curve over $S$. A full level $N$ structure on this curve is an isomorphism of group objects in the category of rigid spaces over $S$:

$$i_N : (\mathbb{Z}/N\mathbb{Z})^2 \cong \mathbb{G}_m/(q)[N]$$

Because $K$ contains a primitive $N^{th}$ root of unity this can be specified on points. For example:

$$i_N : (a, b) \longrightarrow \zeta^a \cdot q^{b/N}$$

The canonical subgroup of the Tate curve is $\mu_p$ which naturally embeds in $\mathbb{G}_m$. Now consider the triple $(\mathbb{G}_m/q^2, i_N, i_p)$, where $i_N$ is a full level $N$ structure as above and $i_p$ is an embedding of $\mu_p$ in $\mathbb{G}_m$. The non-cuspidal points of $X(U(N)^p \times \Gamma_1(q))_{\geq p^{-r}/K}$ have the usual functorial interpretation, so this triple induces the map:

$$S \longrightarrow X(U(N)^p \times \Gamma_1(q))_{\geq p^{-r}/K}$$

which we denote $(i_N, i_p)$. There is a canonical differential on the Tate curve, $dt/t$, where $t$ is the parameter on $\mathbb{G}_m$. The pullback of $f$ under this map gives a global differential on the Tate curve over $S$. All such global differentials are of the form $\mathcal{O}(S)(dt/t)^{\otimes k}$. Because the parameter on $S$ is $q^{1/N}$ we know that the global functions on it are finite-tailed laurent series in this variable with coefficients in $K$. The holomorphicity condition at cusps ensures that the pullback of $f$ is a differential
which extends to the whole disc. The \( q \)-expansion of \( f \) at the cusp defined by \((i_N, i_p)\) is \( f_{(i_N, i_p)}(q) \in K[[q^{1/N}]] \) given by

\[
(i_N, i_p)^* f = f_{(i_N, i_p)}(q)(dt/t)^{\otimes k}.
\]

Recall that classically automorphic forms have \( q \)-expansions for each \( g \in GL_2(A_f) \). In this setting we can almost recover this. More precisely, observe that \( U_1(q) \rightrightarrows U(N)^p \times \Gamma_1(q) \) and

\[
U_1(q)/(U(N)^p \times \Gamma_1(q)) \cong GL_2(\mathbb{Z}/N\mathbb{Z})
\]

This group naturally acts on our level structures and for \( g \in U_1(q) \) we denote the new level structure by \( g(i_N, i_p) \). There is a canonical choice of level structure \((\iota_N^{can}, \iota_p^{can})\) such that for \( f \) classical, the \( q \)-expansion given by this process agrees with the \( 1 \)-\( q \)-expansion as defined in \( \S 2 \). Similarly for \( g \in U_1(q) \) the \( g \)-\( q \)-expansion agrees with the \( q \)-expansion given by \( g(i_N, i_p) \).

Because \( \text{det}(U_1(q)) = \mathbb{Z}^* \) we are assured that a finite number of choices of \( g \) will give maps from \( S \) into each component of \( X(U(N)^p \times \Gamma_1(q))_{\geq p^{-\epsilon}}/K \). Any function on a connected rigid space which vanishes on an admissible affinoid subspace is identically zero everywhere. From this we deduce that \( f \) is uniquely determined by a finite number of these \( q \)-expansions.

3.3. Eisenstein Series of Integer Weight. We will adopt the notational conventions of [3] and [12]. Let \( W/\mathbb{Q}_p \) denote weight space; the moduli space (in the category of rigid spaces) of continuous morphisms \( B_k \rightarrow K \) naturally bijects with continuous homomorphisms \( \kappa : \mathbb{Z}_p^* \rightarrow C^*_k \) that extend to the whole disc. The new level structure by \((i_N, i_p)\) extends to the whole disc. The category of rigid spaces) of continuous morphisms \( B \rightarrow K \) naturally bijects with continuous homomorphisms \( \kappa : \mathbb{Z}_p^* \rightarrow C^*_k \) that extends to the whole disc. This group naturally acts on our level structures and for \( g \in U_1(q) \) we denote the new level structure by \( g(i_N, i_p) \). There is a canonical choice of level structure \((\iota_N^{can}, \iota_p^{can})\) such that for \( f \) classical, the \( q \)-expansion given by this process agrees with the \( 1 \)-\( q \)-expansion as defined in \( \S 2 \). Similarly for \( g \in U_1(q) \) the \( g \)-\( q \)-expansion agrees with the \( q \)-expansion given by \( g(i_N, i_p) \).

Because \( \text{det}(U_1(q)) = \mathbb{Z}^* \) we are assured that a finite number of choices of \( g \) will give maps from \( S \) into each component of \( X(U(N)^p \times \Gamma_1(q))_{\geq p^{-\epsilon}}/K \). Any function on a connected rigid space which vanishes on an admissible affinoid subspace is identically zero everywhere. From this we deduce that \( f \) is uniquely determined by a finite number of these \( q \)-expansions.

For \( s \) in this range and \( \chi \), any \( C_p \)-valued character of finite order on \( \mathbb{Z}_p^* \), we define \((s, \chi) \in W(C_p)\) by \((s, \chi)(d) = \langle d \rangle^s \chi(d)\). Such characters are called arithmetic. If \( \chi = \tau^n \) for some integer \( n \) we abbreviate this notation to \((s, \chi) = (s, n)\).

The Iwasawa \( p \)-adic \( L \)-function is non-vanishing on \( B \setminus \{0\} \). Hence for any \( \kappa \in \mathcal{B}(C_p) \setminus \{0\} \) there is a corresponding formal \( q \)-expansion \( E_{\kappa}(q) \) which \( p \)-adically interpolates classical Eisenstein series. More precisely, for \( k \) an even integer greater than 2 which is divisible by \((p - 1)\), \( E_{(k, k)} \) is the level \( \Gamma_0(q) \), \( p \)-deprived Eisenstein series, in the sense of [5]. If we set \( E_{(0, 0)} = 1 \) then we get the Eisenstein family \( \mathcal{E}(q) \in O(B)[[q]] \).

For \( k \in \mathbb{Z} \), \( E_{(k, 0)} \) is an overconvergent modular form of weight \( k \) and level \( \Gamma_1(q) \) over \( \mathbb{Q}_p \). It has character \( \tau^{-k} \) with respect to the Diamond operators at \( p \). If \( k \) is at least 1 then it is classical. Proposition 6.1 of [4] tells us that for some \( r \neq 0 \),
$E_{(k,0)}$ is non-vanishing on $X(U^p \times \Gamma_1(q))_{\geq p-r}$ for all $U \in \Sigma$. This $r$ is independent of $k$. Fix such an $r$. For
\[
f \in \Gamma(X(U^p \times \Gamma_1(q))_{\geq p-r}, (a_{U^p \times \Gamma_1(q)})^{\otimes k}) \text{ where } U \in \Sigma,
\]
f/$E_{(k,0)}$ is a rigid analytic function on $X(U^p \times \Gamma_1(q))_{\geq p-r}$. Hence we get an embedding
\[
\theta_k : M_k(r) \longrightarrow \varinjlim_{U \in \Sigma} O(X(U^p \times \Gamma_1(q))_{\geq p-r}) := M(r)
\]
f \longrightarrow f/E_{(k,0)}.

We do not refer to this space as the the space of weight zero overconvergent modular forms because we make no demands about the action of the diamond operators. The image of the $\theta_k$ is precisely the space of functions where the diamond operators act by $\tau^k$.

Let $g \in GL_2(A_f^p)$ and $V, U \in \Sigma$ such that $V^p \subset gU^pg^{-1}$. For $f \in M_k(r)^{U^p}$,
\[
g(f) = g^*(f)
= g^*(E_{(k,0)}f/E_{(k,0)})
= g^*(E_{(k,0)})g^*(f/E_{(k,0)})
= g(E_{(k,0)})g(f/E_{(k,0)}),
\]
where $g^*$ is the pullback under the functorial morphism induced by $g$ as in proposition 6. We deduce that $\theta_k$ is not an embedding of $GL_2(A_f^p)$ modules. To rectify this we make the following definition:

**Definition.** For $k \in \mathbb{Z}$, $g \in GL_2(A_f^p)$ and $f \in M(r)$ define
\[
g_k(f) = g(E_{(k,0)})/E_{(k,0)} \times g(f).
\]

**Proposition 7.** This gives rise to an action of $GL_2(A_f^p)$ on $M(r)$ which we refer to as the weight $k$ action. With respect to the weight $k$ action on $M(r)$, $\theta_k$ is an embedding of $GL_2(A_f^p)$ modules.

**Proof.** Immediate from the above. \qed

To extend this to arbitrary weights we generalise the twist factor $g(E_{(k,0)})/E_{(k,0)}$.

3.4. Overconvergent Modular forms of Arbitrary Weight. In §3.3 we observed that for $k \in \mathbb{Z}$ we could define a weight $k$ action on $M(r)$ such that $\theta_k$ became an embedding of $GL_2(A_f^p)$-modules. For $g \in GL_2(A_f^p)$ the standard action was twisted by the factor $g(E_{(k,0)})/E_{(k,0)}$. We now generalise this to arbitrary $\kappa \in \mathcal{B}(C_p)$.

**Theorem 8.** Let $g \in GL_2(\mathbb{Q}_l)$, $l \neq p$ and $N \in \mathbb{Z}$ such that $U(N) \subset gGL_2(\hat{\mathbb{Z}})^p g^{-1}$. There exists a unique rigid analytic function
\[
eps \in O(\mathcal{B} \times X(U(N)^p \times \Gamma_0(p))_{\geq 1}/\mathbb{Q}_p)
\]
such that on any affinoid subdomain of $\mathcal{B}$ it is strictly overconvergent for some $r \neq 0$, and such that for any $\kappa \in \mathcal{B}(C_p)$ where $\kappa = (k, 0)$ for $k \in \mathbb{Z}$,
\[
eps_{g, \kappa} = g(E_{(k,0)})/E_{(k,0)}.
Define the weight 1, level \( \Gamma_1(q) \) classical modular form over \( \mathbb{Z}/p \mathbb{Z} \),
\[
E := E_{(1,0)}
\]
with character \( \tau^{-1} \) for the action of the diamond operators at \( p \). Recall that using the conventions of \( \S 2 \), \( E \in \mathcal{M}_1^{U_1(q)}(\tau) \). At the cusp \( \infty \) on \( X_1(q) \) it has a \( q \)-expansion with the property
\[
E(q) \equiv 1 \mod q.
\]
In particular for any \( s \in B^*(\mathbb{C}/p) \), \( E(q)^s \) makes sense as a formal \( q \)-expansion. Theorem B4.1 of [12] and the comment immediately afterwards tell us that
\[
E_{(s,0)}(q)/E(q)^s
\]
is the \( q \)-expansion at the cusp \( \infty \) of an invertible rigid analytic function on \( B[0,t] \times X_0(q)_{\geq p^{-r}} \) for some \( t \) in the above range and \( r \) non-zero which we now fix. As explained in proposition 6, \( g \) induces the morphism:
\[
X(U(N)^p \times \Gamma_1(q))_{\geq p^{-r}} \overset{g}{\longrightarrow} X_0(q)_{\geq p^{-r}}
\]
This gives rise to the morphism:
\[
B[0,t] \times X(U(N)^p \times \Gamma_1(q))_{\geq p^{-r}} \overset{1 \times g}{\longrightarrow} B[0,t] \times X_0(q)_{\geq p^{-r}}.
\]
By pullback we may consider \( g^*(E_{(s,0)}/E^s) \) as a rigid function on \( B[0,t] \times X(U(N)^p \times \Gamma_1(q))_{\geq p^{-r}} \).

For \( A \in (\mathbb{Z}/l^d\mathbb{Z})^* \) define \( h^A \) to be the element of \( U_0(q) \) trivial everywhere except \( l \) where it is of form
\[
h^A_l = \begin{pmatrix} A' & 0 \\ 0 & 1 \end{pmatrix},
\]
where \( A' \) is a fixed integer lift of \( A \). Let \( c_A \) be the cusp in \( X(U(N)^p \times \Gamma_1(q))_{\geq p^{-r}} \) determined by \( h^A \) in the sense of \( \S 3.2 \). For any component of \( X(U(N)^p \times \Gamma_1(q))_{\geq p^{-r}} \).
there is an $A$ such that the cusp $c_A$ is on that component. Hence any function on $X(U(N)^p \times \Gamma_1(q))_{\geq p-r}$ is uniquely determined by its $q$-expansions at these cusps.

For positive $k \in \mathbb{Z}$, $E_{(k,0)}$ and $E^k$ are classical of weight $k$, level $U_1(q)$ and character $\tau^k$. Let us adopt the notational conventions of proposition 4. Hence at $c_A$, 

$$g^*E_{(k,0)}(q) = \tau^k(u;l^{-(n+m)+nk}E_{(k,0)}(\alpha q^{n-m}))$$

and 

$$g^*E^k(q) = \tau^k(u;l^{-(n+m)+nk}E^k(\alpha q^{n-m})),$$

Where $\alpha = e^{2\pi i\langle \frac{m}{\langle m+n \rangle} \rangle}$. We deduce that at $c_A$, 

$$g^*(E_{(k,0)}/E^k)(q) = g^*E_{(k,0)}(q)/g^*E^k(q) = \frac{E_{(k,0)}(\alpha q^{n-m})}{E(\alpha q^{n-m})^k}.$$ 

Within $\mathcal{B}[0,t]$ there is a zariski dense set of positive integers, hence we deduce that at $c_A$, 

$$g^*(E_{(s,0)}/E^s)(q) = E_{(s,0)}/E^s(q^{n-m}\cdot \zeta) = \frac{E_{(s,0)}(\alpha q^{n-m})}{E(\alpha q^{n-m})^s}.$$ 

In a similar way $g^*(E)/E$ may be considered as a rigid function on $X(U(N)^p \times \Gamma_1(q))_{\geq p-r}$. At $c_A$, 

$$(g^*(E)/E)(q) = \tau(u;l^{-(n+m)+n}E(\alpha q^{n-m})/E(q).$$ 

Note that $\tau(u) = \tau(l^{-n})$ and hence the constant term of this $q$-expansion is $l^{-(m+n)}\langle l^n \rangle$. We want to normalise this function so that on each component of $X(U(N)^p \times \Gamma_1(q))_{\geq p-r}$ there is a cusp with $q$-expansion congruent to 1 mod $q$. To do this we will make use of the following proposition:

**Lemma 9.** The $q$-expansions of $g^*(E)/E$ at the cusps $c_A$ all have the constant term $l^{-(m+n)}\langle l^n \rangle$.

**Proof.** It is enough to show that if 

$$h_i^aq = \begin{pmatrix} l^m & a \\ 0 & l^n \end{pmatrix} \cdot u, \text{ with } a, r, m, n \in \mathbb{Z}, u \in U_0(q),$$

then $n$ and $m$ and $\tau(u) = \tau(l^{-n})$ are independent of $A$. This can be checked by elementary means. \hfill $\square$

Hence we may normalise $g^*(E)/E$ to give $f$, such that for every component of $X(U(N)^p \times \Gamma_1(q))_{\geq p-r}$, $f$ has a cusp with $q$-expansion congruent to 1 modulo $q$. By the $q$-expansion principle such a function must reduce to 1 on the components of the special fibre of the Deligne-Rapaport/Katz-Mazur model of $X(U(N)^p \times \Gamma_1(q))$ containing the reduction of the cusps in $X(U(N)^p \times \Gamma_1(q))_{\geq 1}$. Hence 

$$|f - 1|_{X(U(N)^p \times \Gamma_1(q))_{\geq 1}} \leq |q|.$$ 

By continuity 

$$|f - 1|_{X(U(N)^p \times \Gamma_1(q))_{\geq 1}} = \lim_{r \to 0^+} |f - 1|_{X(U(N)^p \times \Gamma_1(q))_{\geq p-r}}.$$ 

We conclude that for any $\epsilon \in \mathbb{R}, \|q\| < \epsilon < 1$, there exists a non-zero $r$ such that 

$$|f - 1|_{X(U(N)^p \times \Gamma_1(q))_{\geq p-r}} \leq \epsilon.$$
Fix such an \( r \) and \( \epsilon \). Fix \( t \in \mathbb{Q}^* \) such that
\[
\left( \frac{s}{n} \right) T^n \to 0, \quad n \to \infty, \text{ for } |T| \leq \epsilon, |s| \leq t.
\]
If we view \( f \) as a rigid analytic function on \( X(U(N)^p \times \Gamma_1(q))_{\geq p^{-r}} \) then it makes sense to talk about \( f^* \) as a rigid analytic function on \( \mathcal{B}[0, t] \times X(U(N)^p \times \Gamma_1(q))_{\geq p^{-r}} \).

At \( c_A \) it has \( q \)-expansion
\[
f^*(q) = \frac{E(\alpha q^{n-m})^s}{E(q)^s}.
\]
We may multiply this function by \( E^*/E_{(s,0)}^* \) and \( g^*(E_{(s,0)}/E^*) \) to get \( F \). At \( c_A \) we have
\[
F(q) = \frac{E_{(s,0)}(\alpha q^{n-m})^s}{E(\alpha q^{n-m})^s} \cdot \frac{E(\alpha q^{n-m})^s}{E(q)^s} \cdot \frac{E(q)^s}{E_{(s,0)}(q)} = \frac{E_{(s,0)}(\alpha q^{n-m})^s}{E_{(s,0)}(q)}.
\]

Multiply this function by \( l^{-m+n}\langle l^n \rangle^s \), to give a function \( e_g \). At \( k \in \mathbb{Z}, k \in \mathcal{B}[0, t](\mathbb{C}_p) \) this function has the same \( q \)-expansion at \( c_A \) as \( g^*(E_{(k,0)}/E_{(k,0)}^*) \) for all \( A \in (\mathbb{Z}/l^d\mathbb{Z})^* \).

As observed on page 5 of [11] the Eisenstein family over \( \mathcal{B} \) has \( q \)-expansion in \( \Lambda[[q]] \), where \( \Lambda \) is the Iwasawa algebra. Let \( \Lambda_K \) denote the Iwasawa algebra over the ring of integers of \( K \). Viewing \( \kappa \) as a parameter on \( \mathcal{B}, l^{-(m+n)}\kappa(l^n) \) becomes an element of \( \Lambda_K \). We deduce that at any cusp \( e_g(q) \in \Lambda_K[[q]] \). Now we can apply the results of Thm 2.1 [11], which states that \( e_g \) must extend uniquely over \( \mathcal{B} \), replacing \( \Lambda \) with \( \Lambda_K \) and \( Z \) with \( X(U(N)^p \times \Gamma_1(q))_{\geq 1} \). The proof is identical to that given in [11]. Note that by construction \( e_g \) has trivial action of the Diamond operators at \( p \) so is of level \( U_0(q) \).

Finally we must prove that \( e_g \) descends to a function on \( \mathcal{B} \times X(U(N)^p \times \Gamma_0(q))_{\geq 1}/\mathbb{Q}_p \). To do this we will use a Galois descent argument.

We may assume that \( K = \mathbb{Q}_p(\zeta) \). This is a Galois extension of \( \mathbb{Q}_p \) with Galois group \( G = (\mathbb{Z}/l^d\mathbb{Z})^* \). Let us fix an affinoid subdomain \( Y = Sp(A) \subset \mathcal{B} \) defined over \( \mathbb{Q}_p \). Let \( Y_K = Sp(A \otimes K) \) be the base change to \( K \). When we restrict the function \( e_g \) to \( Y_K \) we know by compactness that there exists a non-zero \( r \) such that \( e_g \) is a function on \( Y_K \times X(U(N)^p \times \Gamma_0(q))_{\geq p^{-r}}/K \). \( X(U(N)^p \times \Gamma_0(q))_{\geq p^{-r}}/\mathbb{Q}_p \) is an affinoid space which we denote \( Sp(B) \). In this sense \( e_g \in (A \hat{\otimes} B) \otimes K \), where both products are over \( \mathbb{Q}_p \). We wish to show that \( e_g \in A \hat{\otimes} B \). There is an natural action of \( G \) on \( A \hat{\otimes} B \otimes K \) and any function which is invariant under \( G \) is in \( A \hat{\otimes} B \). Hence we must show that for any \( \sigma \in G, e_g \) and \( \sigma(e_g) \) are equal. It is enough to show that they have the same \( q \)-expansions.

Let \( c \) be a cusp in \( X(U(N)^p \times \Gamma_0(q))_{\geq p^{-r}} \), corresponding to \( h \in U_0(q) \). Let \( t \in K^*, |t| < 1 \) and \( \mu \in Y(K) \). Such points naturally have an action of \( G \). The data
\[
(\mathbb{G}_m/(t^N), x, y, \mu),
\]
where \( x \) and \( y \) are generators of the \( l^d \) torsion of \( \mathbb{G}_m/(t^N) \), gives a \( K \)-valued point of \( Y_K \times X(U(N)^p \times \Gamma_0(q))_{\geq p^{-r}}/K \) which we denote by \( \phi \). Note that we make no demands on level structure at \( p \) as it must be the canonical subgroup by construction.
We also demand that
\[
\begin{pmatrix} x \\ y \end{pmatrix} = \gamma \begin{pmatrix} \zeta \\ t \end{pmatrix}
\]
where \( \gamma \) is the reduction of \( h \) modulo \( U(N)^p \times \Gamma_0(\mathfrak{q}) \). Hence the \( q \)-expansion at \( c \) tells us what \( e_g \) evaluated at \( \phi \) is.

For any \( \sigma \in G \) we have the following commutative diagram:
\[
\begin{array}{ccc}
K & \xrightarrow{\sigma} & K \\
\downarrow & & \downarrow \\
(A \otimes B) \otimes K & \xrightarrow{\sigma} & (A \otimes B) \otimes K \\
\downarrow & & \downarrow \\
K & \xrightarrow{\sigma^{-1}} & K \\
\end{array}
\]

The composition of the bottom arrows in this diagram give a new \( K \)-valued point of \( Y_K \times X(U(N)^p \times \Gamma_0(\mathfrak{q}))_{\geq p^{-r}}/K \) which we will call \( \sigma^{-1}(\phi) \). Moduli theoretically this point corresponds to the data
\[
(G_m/\langle \sigma^{-1}(t)^N \rangle, \sigma^{-1}(x), \sigma^{-1}(y), \sigma^{-1}(\mu)),
\]
where
\[
\begin{pmatrix} \sigma^{-1}(x) \\ \sigma^{-1}(y) \end{pmatrix} = \gamma \begin{pmatrix} \sigma^{-1}(\zeta) \\ \sigma^{-1}(t) \end{pmatrix}.
\]

We deduce that \( \sigma(e_g) \) evaluated at \( \phi \) is equal to \( \sigma \) applied to the evaluation of \( e_g \) at \( \sigma^{-1}(\phi) \). If we can show that this is equal to \( e_g \) evaluated at \( \phi \) we are done.

If \( \sigma^{-1} = A \in (\mathbb{Z}/l^d\mathbb{Z})^* \) then the cusp corresponding to \( \sigma^{-1}(\phi) \) is determined by \( h^A \cdot h \). Let us call this cusp \( \sigma^{-1}(c) \). If
\[
h_l \cdot g = \begin{pmatrix} l^m & a \\ 0 & l^n \end{pmatrix} \cdot u, \text{ where } a, r, m, n \in \mathbb{Z}, u \in U_0(\mathfrak{q})
\]
then at \( c \),
\[
e_g(q) = l^{-(m+n)} \kappa(l^n) \frac{E_\kappa(aq^{n-m})}{E_\kappa(q)}.
\]
Here \( \alpha \) is some power of \( \zeta \). Note that
\[
h_l^A \cdot h_l \cdot g = \begin{pmatrix} l^m & a \cdot A' \\ 0 & l^n \end{pmatrix} \cdot u', \text{ where } a, r, m, n \in \mathbb{Z}, u' \in U_0(\mathfrak{q})
\]
Hence the \( q \)-expansion of \( e_g \) at \( \sigma^{-1}(c) \) is
\[
e_g(q) = l^{-(m+n)} \kappa(l^n) \frac{E_\kappa(a^\kappa(q^{n-m}))}{E_\kappa(q)} = l^{-(m+n)} \kappa(l^n) \frac{E_\kappa(\sigma^{-1}(\alpha)q^{n-m})}{E_\kappa(q)}.
\]
The coefficients of the Eisenstein family are elements of \( \Lambda \). The same is true of the function \( l^{-(m+n)} \kappa(l^n) \). Any such function evaluated at \( \mu \) is therefore the same as \( \sigma \) applied to it evaluated at \( \sigma^{-1}(\mu) \). We deduce that \( e_g \) and \( \sigma(e_g) \) have the same \( q \)-expansion at \( c \). Therefore \( e_g \) and \( \sigma(e_g) \) are equal and we are done.

\( \square \)

For the remainder of this section let all rigid spaces be over \( \mathbb{C}_p \). An arbitrary weight is a morphism of rigid spaces \( \phi : Y \to W \). For ease of exposition let us assume that \( Y = Sp(A) \) is affinoid and the image of \( \phi \) is contained in the component.
\( \mathcal{W}_i \) defined by the character \( \tau^i \). For \( U \in \Sigma \) we define the space of \( r \)-overconvergent modular forms of tame level \( U^P \) and weight \( \phi \) to be the subspace
\[
\mathcal{M}_\phi(r, U^P) \subset O(Y \times X(U^P \times \Gamma_1(q))_{\geq p^{-r}}),
\]
whose elements have character \( \tau^i \) with respect to the diamond operators at \( p \). Now let
\[
\mathcal{M}_\phi^1(U^P) = \lim_{\substack{r \to 0}} \mathcal{M}_\phi(r, U^P),
\]
and define
\[
\mathcal{M}_\phi(r) := \lim_{U \in \Sigma} \mathcal{M}_\phi(r, U^P),
\]
\[
\mathcal{M}_\phi^1 := \lim_{U \in \Sigma} \mathcal{M}_\phi^1(U^P)
\]
the space of \( r \)-overconvergent (resp. overconvergent) modular forms of weight \( \phi \). All these spaces are \( \Lambda \)-modules and the come equipped with a (weight zero) action of \( GL_2(\mathbb{A}_f^P) \). Similarly we may replace all these spaces by the subspaces of cusp forms \( S_\phi(r, U^P) \), \( S_\phi^1(U^P) \) and \( S_\phi^1 \). Both \( S_\phi(r) \) and \( S_\phi^1 \) are naturally (weight zero) \( GL_2(\mathbb{A}_f^P) \)-modules. Note that if \( \phi \) is classical (i.e. \( \phi = k \in \mathbb{Z} \)) then \( \mathcal{M}_\phi^1 \) is precisely the image of \( \theta_k \) in §3.3.

There is a natural isomorphism between \( \mathcal{W}_i \) and the identity component \( B \). For \( g \in GL_2(\mathbb{Q}_l) \) we can pullback \( e_g \) along this isomorphism and then along \( \phi \) to get
\[
e_{(g, \phi)} \in \lim_{U \in \Sigma} \lim_{r \to 0} O(Y \times X(U^P \times \Gamma_0(q))_{\geq p^{-r}}).
\]

If \( f \in \mathcal{M}_\phi^1 \) and \( g \in GL_2(\mathbb{Q}_l) \) define
\[
g(f) := e_{(g, \phi)} \cdot g^*(f) \in \mathcal{M}_\phi^1,
\]
where \( g^* \) is the weight 0 action.

**Proposition 10.** This defines an action (weight \( \phi \)) of \( GL_2(\mathbb{Q}_l) \) on \( \mathcal{M}_\phi^1 \) (and on \( \mathcal{M}_\phi(r) \) for some \( r \neq 0 \) dependent on \( \phi \)) which is functorial with respect to the weight.

**Proof.** Let \( h, g \in GL_2(\mathbb{Q}_l) \). Showing that this defines an action is equivalent to showing that \( e_h \cdot h^*(e_g) = e_{hg} \), where \( h^* \) is the weight zero action. This is true after restricting to any integer point of \( B \). These are Zariski dense in \( B \), hence they must have the same \( q \)-expansions so are equal.

If \( X = Sp(B) \) is an affinoid rigid space together with a morphism \( \pi : X \to Y \). Then pull back by morphisms of form
\[
X \times_{\mathbb{Q}_p} X(U^p \times \Gamma_1(p))_{\geq p^{-r}} \xrightarrow{\pi \times 1} Y \times_{\mathbb{Q}_p} X(U^p \times \Gamma_1(p))_{\geq p^{-r}}
\]
duces a map \( \Pi : \mathcal{M}_\phi^1 \to \mathcal{M}_\phi^1 \). This maps is actually a \( GL_2(\mathbb{Q}_l) \) module homomorphism. This follows from the fact that \( (\pi \times 1)^*(e_{(g, \phi)}) = e_{(g, \phi \cdot \pi)} \), where \( (\pi \times 1) \) is a morphism at the appropriate tame level.

This action preserves the subspace of overconvergent cusps \( S_\phi^1 \subset \mathcal{M}_\phi^1 \). It is also true that for any \( U \in \Sigma \),
\[
(\mathcal{M}_\phi^1)^U = \mathcal{M}_\phi^1(U^P)
\]
There is an analogous statement for $r$-overconvergent forms where the action is defined.

Note that if $\phi$ is a classical weight and $f \in \mathcal{M}_\phi^1$ is classical then the action of $GL_2(\mathbb{Q}_l)$ we have constructed agrees with the classical one.

4. $p$-adic analytic families of Admissible Representations of $GL_2(\mathbb{Q}_l)$

Fix $N$ a positive integer prime to $p$. Recall that for $f \in S_k^{U_1(N)}$, a cuspidal eigenform, we defined $\pi_{f,1} = \mathbb{C}[GL_2(\mathbb{Q}_l)]f \subset S_k$. We are now in a position to generalise this to overconvergent cuspidal eigenforms in the sense of [9]. Keeping the conventions of §3.4, the $A$-module $\mathcal{M}_\phi^1(U_1(N)^p)$ is equipped with a set of commuting $A$-linear endomorphisms, commonly known as the Hecke algebra $\mathcal{T}$. These preserve the space of cusp forms. The operator $U_p$ is contained in this algebra and for $r$ where it is defined it acts compactly on the Banach space $\mathcal{M}_\phi(r, U_1(N)^p)$. In fact $U_p$ acts on the whole of $\mathcal{M}_\phi^1$. Fix $f \in S_k^1(U_1(N)^p)$, a normalised cuspidal eigenform with respect to $\mathcal{T}$, such that the eigenvalue of $f$ with respect to $U_p$ is a unit in $A$. This last condition is commonly known as being finite slope. By the work of [9] and [12] this is equivalent to a commutative diagram of rigid spaces

$$
\begin{array}{c}
\mathcal{Y} \\
\phi \\
\downarrow \\
\mathcal{E} \\
\downarrow \\
\mathcal{W}
\end{array}
$$

where $\mathcal{E}$ is the reduced tame level $N$ cuspidal eigencurve as constructed in [4].

**Definition.** For $f \in S_k^1(U_1(N)^p)$, a finite slope, cuspidal eigenform, define

$$
\pi_{f,1} := A[GL_2(\mathbb{Q}_l)]f \subset S_k^1,
$$

where this is the weight $\phi$ action as defined in §3.4.

If $\phi$ and $f$ are classical then this recovers the previous definition. Note that this is a smooth representation in the obvious sense.

**Lemma 11.** Let $a_p \in A^*$, $U \in \Sigma$, then

$$
\mathcal{M}_\phi^1(U)^{U_p=a_p} = \mathcal{M}_\phi(r, U_p)^{U_p=a_p},
$$

for some non-zero $r$, which depends only on $\phi$.

**Proof.** This the statement that all overconvergent modular forms of fixed weight which are finite slope eigenforms for $U_p$ are $r$-overconvergent for some fixed $r$. Following [12], there is an overconvergent function $e$ on the space $\mathcal{B} \times X(GL_2(\hat{\mathbb{Z}})^p \times \Gamma_0(q))_{\geq 1}$ which is used to define $U_p$. The $q$-expansion at infinity is $e(q) = E_\phi(q) / E_\phi(q^p)$. Hence, given $U \in \Sigma$, $e$ is naturally an overconvergent function on $Y \times X(U^p \times \Gamma_0(q))_{\geq 1}$ which we also denote $e$. By compactness there exist a non-zero $r$ such that $e$ is a function on $Y \times X(U^p \times \Gamma_0(q))_{\geq r}$. Let $U_0$ be the weight 0 $U_p$ operator defined by a correspondence on the moduli problem. By proposition 3.5 of [3], the operator $U_0$ increases overconvergence. If $f \in \mathcal{M}_\phi^1(U^p)$ then by definition

$$
U_p(f) = U_0(e \cdot f).
$$
Hence, we deduce that if \( f \in \mathcal{M}_p(U^p)^{U_p=a_p} \), then
\[
f = a_p^{-1}U_0(e \cdot f),
\]
which means that \( f \) must be at least as overconvergent as \( e \). Hence we are done. \( \square \)

**Lemma 12.** Let \( f \in S^l_\phi(U_1(N)^p) \) be an eigenform whose eigenvalue at \( p \) is \( a_p \in A^* \). Then
\[
\pi_{f,l} \subset (S^l_\phi)^{U_p=a_p}.
\]

**Proof.** This statement will follow by showing that the action \( GL_2(\mathbb{Q}_l) \) on \( S^l_\phi \) commutes with the action of \( U_p \).

Fix \( g \in GL_2(\mathbb{Q}_l) \) and \( V, U \in \Sigma \) such that \( V^p \subset gU^pg^{-1} \) and both \( e \) and \( e_{(g, \phi)} \) are functions on \( Y \times X(V^p \times \Gamma_0(q))_{\geq p-r} \) for some non-zero \( r \). We will compare \( U_p(g(f)) \) and \( g(U_p(f)) \).
\[
U_p(g(h)) = U_p(e_{(g, \phi)}g^*(h)) = U_0(e \cdot e_{(g, \phi)} \cdot g^*(h)).
\]

Conversely,
\[
g(U_p(h)) = e_{(g, \phi)}g^*(U_0(e \cdot h)) = e_{(g, \phi)}U_0(g^*(e) \cdot g^*(h)).
\]

The second equality comes from the fact that \( U_0 \) and \( g^* \) are derived from commuting actions on the moduli problem. A standard result of Coleman tells us that
\[
e_{(g, \phi)}(q)U_0(g^*(e) \cdot g^*(h)) = U_0(e_{(g, \phi)}(q^p)g^*(e) \cdot g^*(h)).
\]

By considering \( q \)-expansions at any cusp we see that \( e(q) \cdot e_{(g, \phi)}(q) = e_{(g, \phi)}(q)g^*(e)(q) \).

We deduce that the action of \( GL_2(\mathbb{Q}_l) \) commutes with \( U_p \). \( \square \)

Fix \( U \triangleleft GL_2(\mathbb{Z}_l) \), an open compact subgroup such that \( U_1(N)^l \times U \subset U_1(N) \).

Note that \( \pi_{f,l}^U \neq 0 \).

**Proposition 13.** \( \pi_{f,l}^U \) is a finitely generated \( A \)-module.

**Proof.** By lemma 10 and lemma 11, we know that \( \pi_{f,l} \subset S_\phi(r)^{U_p=a_p} \) for some non-zero \( r \). In fact
\[
\pi_{f,l} \subset \lim_{V \subset GL_2(\mathbb{Z}_l)} \text{O}(Y \times X(U_1(N)^l)^p \times V \times \Gamma_1(q))_{\geq p-r}.
\]

where the limit is taken over appropriate open compact subgroups. This is because \( GL_2(\mathbb{Q}_l) \) only affects the moduli problem at \( l \). Hence for \( V \in \Sigma \) such that \( V^p \subset U_1(N)^l \times U \),
\[
\pi_{f,l}^U \subset S_\phi(r, V^p)^{U_p=a_p}.
\]

Because \( U_p \) acts compactly on this space, we know that the right hand side must be a finite \( A \)-module. Affinoid algebras are noetherian, hence \( \pi_{f,l}^U \) is a finite \( A \)-module. \( \square \)

We deduce that if \( A = \mathbb{C}_p \) then \( \pi_{f,l} \) is a smooth, admissible \( GL_2(\mathbb{Q}_l) \)-module in the classical sense.

Fix \( V \in \Sigma \) such that \( V^p \subset U_1(N)^l \times U \). Hence
\[
\pi_{f,l}^U \subset S_\phi(r, V^p)^{U_p=a_p}.
\]
The space $S_\phi(r, V^p)$ is a potentially ON-able Banach space (using the terminology of [4]) which compact operator $U_p$. If $P_\phi(T)$ is the characteristic power series of $U_p$ acting on $S_\phi(r, V^p)$ then it naturally has $a_p^{-1}$ as a root. This data determines a closed embedding $\theta$ of $Sp(A)$ into the spectral curve associated to $U_p$ acting on $M_\phi(r, V^p)$. As in [4], there is an admissible cover $\mathcal{U}$ of the spectral curve upon which we may construct an eigenvariety by appropriately gluing together Hecke algebras. Hence we may admissibly cover $Sp(A)$ by affinoids such that each affinoid is contained in some element of $\theta^{-1}(U)$.

Let us replace $A$ by an element of this cover. By definition of $\mathcal{U}$ we can find $Q(T) \in A[[T]]$ and $S(T) \in A[[T]]$ such that $Q(T)$ and $S(T)$ are coprime, $P_\phi(T) = Q(T)S(T)$ and $a_p^{-1}$ is a root of $Q(T)$. We can also guarantee that the leading term of $Q(T)$ is a unit. By theorem 3.3 of [4], $S_\phi(r, V^p)$ decomposes as the closed direct sum $S_\phi(r, V^p) = N \oplus F$, where $N$ is a finite, projective $A$-module. $Q^*(U_p)$ acts invertibly on $F$ and as zero on $N$. We deduce that

$$N = S_\phi(r, V^p)Q^*(U_p)=0.$$  

But $a_p$ is a root of $Q^*(T)$, hence

$$S_\phi(r, V^p)^{U_p=a_p} \subset N.$$  

Let $\tilde{E}$ be a normalisation of $E$ (over $\mathbb{C}_p$). By the above we may admissibly cover $\tilde{E}$ by open affinoids $\phi : Sp(A) \subset \tilde{E}$ such that $A$ is a Dedekind domain and if $f$ is the eigenform corresponding to $\phi$ then

$$\pi^U_{f,l} \subset S_\phi(r, V^p)^{U_p=a_p}$$

with

$$S_\phi(r, V^p) = N \oplus F,$$

where $N$ is a finite, projective $A$-module upon which $U_p - a_p$ acts as zero. Furthermore $U_p - a_p$ is never zero on $F$. Let us fix such a cover $\mathcal{U}$.

Now restrict to the case where $(Y, \phi) \in \mathcal{V}$. Note that $Y$ is an irreducible, smooth, connected, one dimensional rigid space over $\mathbb{C}_p$.

**Lemma 14.** $\pi^U_{f,l}$ is a finite, projective $A$-module.

**Proof.** $A$ being a Dedekind domain means that any finite $A$-module is projective if and only if it is torsion free. Hence any submodule of a finite, projective $A$-module is projective. By construction

$$\pi^U_{f,l} \subset S_\phi(r, V^p)^{U_p=a_p}.$$  

By definition of $\mathcal{V}$ we know that the right hand side is contained in $N$ which is finite and projective. Hence

$$\pi^U_{f,l} \subset N,$$

and we are done. Another way to see this is that the right hand side is a finitely generated $A$-module and each element in it is determined by its $q$-expansions. A function is zero if and only if its $q$-expansions (which have coefficients in an integral domain) are all zero. Hence, $S_\phi(r, V^p)$ is torsion free. $\square$
Let $(Y, φ) ∈ V$. Lemma 14 implies that any element of $End_A(π^U_{f,l})$ has a trace in $A$, which extends the usual definition for free modules. This trace map is functorial in the following sense:

Let $κ : A → B$ be a map of affinoid algebras over $ℂ_p$. Then $π^U_{f,l} ⊗ κ B$ is a projective $B$-module. $Ψ ∈ End_A(π^U_{f,l})$ then $Ψ ⊗ 1 ∈ End_B(π^U_{f,l} ⊗ κ B)$ and $tr(Ψ ⊗ 1) = κ(tr(Ψ))$.

Now specialise to the case where $B = ℂ_p$. In this situation $κ ∈ Sp(A)$ so is given by a maximal ideal $m_κ ⊂ A$. This will correspond to $κ ∈ E(ℂ_p)$. By the functoriality (see proposition 10) of our construction there is a $GL_2(ℚ_l)$-equivariant morphisms of $ℂ_p$ vector spaces

$$S^î_\phi \xrightarrow{λ} S^î_\phi_{œκ}.$$  

This map preserves $r$-overconvergence. We will be interested in the kernel of this map.

**Lemma 15.** $ker(λ) = m_κ · S^î_\phi$.

**Proof.** Let $U ∈ Σ$ and $f ∈ S^î_\phi(r, U^p)$. $X(U^p × Γ_1(q))_{≥ p−}$ is an affinoid over $ℂ_p$ whose associated affinoid algebra we denote as $B$. Hence $f ∈ A⊗ B$. Let us choose a presentation of $A$ and $B$ in terms of Tate algebras. By this we mean isomorphisms

$$A ≅ ℂ_p<T_1,...T_n>/I, \quad B ≅ ℂ_p<S_1,...S_m>/J,$$

for some $n, m ∈ ℤ$. Hence,

$$A⊗ B ≅ ℂ_p<T_1,...T_n,S_1,...S_m>/(I,J).$$

Let $f_κ$ denote the image of $f$ in $S^î_{œκ}(r, U^p)$. By definition $f_κ ∈ B$ and is formed by evaluating $T_1,...T_n$ at $κ$. Now assume that $f ∈ ker(λ)$. Let

$$f' ∈ ℂ_p<T_1,...T_n,S_1,...S_m>$$

be a lift of $f$, then evaluating at $κ$ gives

$$f'_κ ∈ ℂ_p<S_1,...S_m>,$$

which is a lift of $f_κ$. Now $f' − f'_κ$ is also a lift of $f$, whose evaluation at $κ$ is zero. Hence

$$f' − f'_κ ∈ m_κ · ℂ_p<T_1,...T_n,S_1,...S_m>.$$

From this we deduce that $f ∈ m_κ · S^î_{œκ}(r, U^p)$ and that

$$ker(λ) ⊂ m_κ · S^î_\phi.$$  

The reverse inclusion is obvious, hence we are done. ☐

Let $f_κ$ denote the image of $f$ in $S^î_{œκ}$. Clearly $f_κ ∈ S^î_{œκ}(U_1(N)^p)$ is a finite slope, cuspidal eigenform and so we naturally have the $GL_2(ℚ_l)$-module $π_{f_κ,l} ⊂ S^î_{œκ}$. Because $κ$ is surjective there is a surjection of $GL_2(ℚ_l)$-modules

$$π_{f,l} \xrightarrow{λ} π_{f_κ,l}.$$  

**Lemma 16.** The natural map

$$π^U_{f,l} \xrightarrow{λ} π^U_{f_κ,l}$$

is a surjection.
Proof. There is a short exact sequence of $\mathbb{C}_p[U]$-modules:

$$0 \longrightarrow \ker(\lambda) \longrightarrow \pi_{f,l} \longrightarrow \pi_{f_a,l} \longrightarrow 0.$$ 

$U$ is a profinite group so we can take continuous group cohomology to give the long exact sequence:

$$0 \longrightarrow \ker(\lambda)_U \longrightarrow \pi_{U,f,l}^U \longrightarrow \pi_{f_a,l}^U \longrightarrow H^1_{\text{con}}(U, \ker(\lambda)) \longrightarrow \cdots$$

By definition

$$H^1_{\text{con}}(U, \ker(\lambda)) = \lim_{\mathcal{V} \triangleleft U} H^1(U/V, \ker(\lambda)_V),$$

where the limit is taken over all open normal subgroups of $U$, which are necessarily of finite index. By Corollary 1 of [1] we know that each $H^1(U/V, \ker(\lambda)_V)$ is annihilated by $|U/V|$. However $H^1(U/V, \ker(\lambda)_V)$ naturally has the structure of a $\mathbb{C}_p$-vector space and we deduce that $H^1(U/V, \ker(\lambda)_V) = 0$ and therefore that $H^1_{\text{con}}(U, \ker(\lambda)) = 0$. □

**Proposition 17.** There is a surjection of $\mathbb{C}_p$-vector spaces:

$$\pi_{f,l}^U \otimes_\kappa \mathbb{C}_p \longrightarrow \pi_{f_a,l}^U,$$

which for all but finitely many $\kappa \in \text{Sp}(A)$ is an isomorphism.

Proof. By lemma 16 this statement is equivalent to $\ker(\lambda) = m_\kappa \cdot \pi_{f,l}^U$ for all but finitely many $\kappa$. By lemma 15

$$\ker(\lambda) = \pi_{f,l}^U \cap (m_\kappa \cdot S_\phi(r, V^p)).$$

This can be further simplified to give

$$\ker(\lambda) = \pi_{f,l}^U \cap (m_\kappa \cdot S_\phi(r, V^p))_{U_p = \mathfrak{a}_p}.$$

By definition there is a decomposition $S_\phi(r, V^p) = N \oplus F$ with $U_p - \mathfrak{a}_p$ acting by zero on $N$. $U_p - \mathfrak{a}_p$ never acts as zero on $F$. Hence we deduce

$$(m_\kappa \cdot S_\phi(r, V^p))_{U_p = \mathfrak{a}_p} \subset m_\kappa \cdot N.$$

By construction $\pi_{f,l}^U \subset N$, so the statement of the proposition is reduced to proving that

$$m_\kappa \cdot \pi_{f,l}^U = \pi_{f,l}^U \cap (m_\kappa \cdot N)$$

for all but finitely many $m_\kappa \in \text{Sp}(A)$. This statement is equivalent to proving that

the natural map

$$\pi_{f,l}^U \otimes_\kappa \mathbb{C}_p \longrightarrow N \otimes_\kappa \mathbb{C}_p$$

is injective for all but finitely many $\kappa$. Let $C$ be the cokernel of the natural inclusion $\pi_{f,l}^U \subset N$. Hence there is a short exact sequence of $A$-modules:

$$0 \longrightarrow \pi_{f,l}^U \longrightarrow N \longrightarrow C \longrightarrow 0$$

This gives rise to the long exact sequence:

$$\cdots \longrightarrow \text{Tor}^1_A(N, A/m_\kappa) \longrightarrow \text{Tor}^1_A(C, A/m_\kappa) \longrightarrow \pi_{f,l}^U \otimes_\kappa \mathbb{C}_p \longrightarrow \cdots$$

$$\cdots \longrightarrow N \otimes_\kappa \mathbb{C}_p \longrightarrow C \otimes_\kappa \mathbb{C}_p \longrightarrow 0.$$
Hence the injectivity of this map is reduced to the vanishing of \(\text{Tor}^1_A(C, A/m_\kappa)\). Computing \(\text{Tor}\) using either the left or right factor gives the same result. Hence the short exact sequence of \(A\)-modules:

\[
0 \rightarrow m_\kappa \rightarrow A \rightarrow A/m_\kappa \rightarrow 0
\]

gives rise to the long exact sequence:

\[
\cdots \rightarrow \text{Tor}^1_A(C, A) \rightarrow \text{Tor}^1_A(C, A/m_\kappa) \rightarrow C \otimes_A m_\kappa \rightarrow \cdots
\]

\[
\cdots \rightarrow C \rightarrow C \otimes_\kappa \mathbb{C}_\mu \rightarrow 0.
\]

\(A\) is a flat \(A\)-module hence \(\text{Tor}^1_A(C, A)\) is trivial. We deduce that \(\text{Tor}^1_A(C, A/m_\kappa) = 0 \Leftrightarrow C \otimes_A m_\kappa \rightarrow C\) is injective.

\(C\) is a finitely generated module over a Dedekind domain. The structure theory of such modules is well understood. If \(C_{\text{tor}} \subset C\) denotes the torsion submodule then \(C \cong C_{\text{tor}} \oplus C/C_{\text{tor}}\).

By construction the right hand summand is torsion free. Hence it is projective so flat. We deduce that \(\text{Tor}^1_A(C/C_{\text{tor}}, A/m_\kappa) = 0\). \(\text{Tor}\) respects direct summands hence

\[
\text{Tor}^1_A(C, A/m_\kappa) = \text{Tor}^1_A(C_{\text{tor}}, A/m_\kappa).
\]

The general structure theory for finitely generated torsion modules over a Dedekind domain implies that \(C_{\text{tor}}\) is the finite direct sum of modules of form \(A/I\) for some ideal \(I \subset A\). Thus we have further reduced the problem to determining \(\text{Tor}^1_A(A/I, A/m_\kappa)\). The same argument as above implies that

\[
\text{Tor}^1_A(A/I, A/m_\kappa) = 0 \Leftrightarrow A/I \otimes_A m_\kappa \rightarrow A/I\) is injective.
\]

\(A/I \otimes_A m_\kappa \cong m_\kappa/I \cdot m_\kappa\), so this map is injective if and only if \(m_\kappa \cap I = I \cdot m_\kappa\). This is if and only if \(m_\kappa\) and \(I\) are coprime. There are only finitely many maximal ideals diving \(I\), hence \(\text{Tor}^1_A(A/I, A/m_\kappa) = 0\) for all but finitely many \(m_\kappa \in \text{Sp}(A)\). Because \(\text{Tor}^1_A(C, A/m_\kappa)\) is the direct sum of finitely many such groups we deduce the result. Where \(\text{Tor}^1_A(C, A/m_\kappa) \neq 0\) there is a strict inclusion \(m_\kappa \cdot \pi_{f,1}^U \subset \ker(\lambda)\) and all we know is that the \(\lambda\) map in the statement of the proposition is a surjection.

□

Fix a Haar measure on \(GL_2(\mathbb{Q}_l)\). Let \(\mathcal{H}_U\) denote the Hecke algebra of compactly supported, \(U\) bi-invariant functions on \(GL_2(\mathbb{Q}_l)\) with coefficients in \(\mathbb{C}_p\). \(\pi_{f,1}^U\) and \(\pi_{f,1}^U\) are naturally \(\mathcal{H}_U\)-modules and and the map \(\lambda\) in lemma 16 is \(\mathcal{H}_U\)-equivariant.

**Theorem 18.** Let \(\tilde{E}\) be a normalisation of \(E\) (taken over \(\mathbb{C}_p\)). There is a discrete subset \(S \subset \tilde{E}\) and a unique map

\[
\text{tr}_{\text{aut}} : \mathcal{H}_U \rightarrow O(\tilde{E})
\]

such that for any \(h \in \mathcal{H}_U\) and \(\kappa \notin S\) the specialisation of \(\text{tr}_{\text{aut}}(h)\) to \(\kappa\) is equal to the trace of \(h\) acting on \(\pi_{f,1}^U\). In this sense traces vary analytically, at least away from a discrete set.
Proof. Fix for the moment an element of this cover \((Y, \phi) \in \mathcal{V}\). Let \(f_Y\) be the corresponding eigenform. By lemma 14 \(\pi_{f_Y,1}^U\) is a finite projective \(A\)-module. Fix \(h \in \mathcal{H}_U\). There is a natural inclusion \(h \in \text{End}_A(\pi_{f_Y,1}^U)\). Hence such an element possesses a trace in \(A\). The map \(\lambda\) in lemma 16 is a map of Hecke modules. By proposition 17 and the comments immediately preceding lemma 15 the restriction of this trace to all but finitely many \(\kappa \in \text{Sp} (A)\) gives the trace of \(h\) acting on \(\pi_{f_Y,1}^U\), where \(f_\kappa\) is the restriction of \(f_Y\) to \(\kappa\). Hence \(h\) gives rise to a function on \(\tilde{E}\) whose restriction to all but finitely many points (independent of \(h\)) is the trace of \(h\) acting on \(\pi_{f_Y,1}^U\).

\(\tilde{E}\) is a separated rigid space hence the intersection of any two element of \(\mathcal{V}\) is an affinoid curve. Any two functions on an affinoid curve which agree at infinitely many points must be the same. Hence we deduce that the trace of \(h\) lifts to a global function on \(\tilde{E}\). Letting \(S\) be the set of points for which the map in proposition 17 is not an isomorphism then we are done.

\[\square\]

5. Galois Side

5.1. Galois Representations on the Eigencurve. Let \(E\) denote the tame level \(N\), cuspidal eigencurve over \(\mathbb{Q}_p\), and \(G_{Q_l} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\). \(E\) naturally comes equipped with a universal continuous pseudocharacter:

\[T : G_{Q_l} \rightarrow O(E)\ .\]

If \(\kappa \in E(\mathbb{C}_p)\) we denote the specialisation to \(\kappa\) by \(T_\kappa\). A theorem of R. Taylor (\[29\]) ensures that this is the trace of a unique continuous semi-simple 2 dimensional representation \(\rho_\kappa\), over \(\mathbb{C}_p\). Let \(S\) denote the absolutely reducible locus in the sense that \(\rho_\kappa\) is absolutely reducible. This is a discrete subset and contains no classical cuspsforms. Let \(E^\circ = E\setminus S\). The classical locus of \(E\) is a Zariski dense subset. We deduce that the classical locus is still dense in \(E^\circ\). If \(Y = Sp(A) \subset E^\circ\) is an admissible open subset then let \(T_Y\) be the restriction of \(T\) to \(Y\). By lemma 7.2 of \[8\] this is the trace of a unique, continuous representation

\[\rho_Y : G_{Q_l} \rightarrow B^*\]

where \(B\) is a rank 4 Azumaya algebra over \(A\). By the definition of an Azumaya algebra, \(B\) is locally a matrix algebra. We deduce that \(Y\) has an admissible open affinoid cover, \(\mathcal{V}\), such that for any \(X \in \mathcal{V}\), \(T_X\) is the trace of a unique (over \(O(X)\)) continuous representation

\[\rho_X : G_{Q_l} \rightarrow GL_2(O(X))\ .\]

Hence we can find an admissible open affinoid cover of \(E^\circ\) with this property. Let us fix such a cover, \(\mathcal{U}\).

Let \(F/\mathbb{Q}_p\) be a finite extension and \(\kappa \in E^\circ(F)\). Let \(W_{Q_l}\) denote the Weil group at \(l\). By the above we can attach to \(\kappa\) a unique, continuous, 2 dimensional representation of \(W_{Q_l}\) over \(F\). Grothendieck’s Monodromy theorem (4.2.2 \[28\]) allows us to attach a 2 dimensional Frobenius semi-simple Weil-Deligne representation (4.1.2 \[28\]) to such a \(\kappa\). Applying the local Langlands correspondence (see \(\S 5.3\)) we get an irreducible smooth representation of \(GL_2(\mathbb{Q}_l)\). We will generalise this to arbitrary \(\kappa \in E^\circ(\mathbb{C}_p)\) and study how these objects vary across \(E^\circ\).
5.2. Weil-Deligne Representations in Families. Grothendieck’s Monodromy theorem (4.2.2 [28]) provides a dictionary between p-adic representations of $G_{Q_l} = \text{Gal}(\overline{Q}/Q_l)$ and Weil-Deligne representations of $W_{Q_l}$.

More precisely, fix a finite field extension $F/Q_p$, an inverse (geometric) Frobenius element $\Phi$ and a nonzero additive homomorphism $t_p : I_{Q_l} \rightarrow Q_p$. $(t_p$ is unique up to constant scalar multiple.) If $(\rho, N)$ is an $n$-dimensional Weil-Deligne representation defined over $F$ then we define a continuous (with respect to the $p$-adic topology on $F$) representation

$$\rho_l : W_{Q_l} \rightarrow GL_n(F)$$

according to $\rho_l(\Phi^n \cdot u) = \rho(\Phi^n \cdot u)\exp(t_p(u) \cdot N)$, for all $n \in \mathbb{Z}$ and $u \in I_{Q_l}$. Grothendieck’s theorem allows us to get backwards (4.2.1 [28]) giving a bijection between isomorphism classes of $n$-dimensional Weil-Deligne representations over $F$ and isomorphism classes of continuous $n$-dimensional representations of $W_{Q_l}$ over $F$.

Let $A/Q_p$ be a reduced, affinoid algebra and $\rho_l$ a continuous $A$-linear representation of $W_{Q_l}$ on a free $A$-module of rank $n$. Fixing a basis naturally gives a morphism

$$\rho_l : W_{Q_l} \rightarrow GL_n(A).$$

At any $F$-valued point of $Sp(A)$ we can invoke the above dictionary to get an $n$-dimensional Weil-Deligne representation. If we take an arbitrary $C_p$-valued point then this procedure breaks down because $\mathbb{C}_p$ is not even an algebraic extension of $Q_p$. Instead we generalise Grothendieck’s construction to work across all of $Sp(A)$ simultaneously.

**Proposition 19.** Let $\rho_l$ be a continuous $A$-linear representation of $G_{Q_l}$ on the free $A$-module $M$ of rank $n$. There exist a nilpotent endomorphism $N$ of $M$ such that $\rho_l(u) = \exp(t_p(u)N)$ for $u$ in an open subgroup of $I_{Q_l}$.

**Proof.** The proof is almost identical to Grothendieck’s original which can be found in the appendix of [27]. Let $|.|$ denote the spectral norm ($A$ is reduced) on $A$. Note that this gives rise to an ultrametric on $A$. Define $A^o = \{ a \in A : |a| \leq 1 \}$. Consider the open subgroup $H := 1 + p^2 M_n(A^o) \subset GL_n(A)$. After fixing a basis for $M$, the inverse image of $H$ under $\rho_l$ is an open subgroup of $G_{Q_l}$. Hence there is a finite field extension $K/Q_l$ such that if $G_K := \text{Gal}(\overline{Q}/K)$ then $\rho_l(G_K) \subset H$. Let $\kappa \in Sp(A)(F)$ for some $F/Q_p$ and $O_F$ its ring of integers. Specialising $\rho_l$ at $\kappa$ and then restricting to $G_K$ yields a continuous homomorphism

$$\rho_{l,\kappa} : G_K \rightarrow 1 + p^2 M_n(O_F)$$

If $K^t$ is the maximal tame extension of $K$ then the wild ramification group $P_K := \text{Gal}(K/K^t)$ is pro-$l$. Because the image of $G_K$ is necessarily pro-$p$ we deduce that $\rho_l$ is trivial on $P_K$. If we further restrict to $I_K$ we know that it factors through $I_K/P = \text{Gal}(K^t/K^{nr})$, where $K^{nr}$ is the maximal unramified extension of $K$. This group is (non-canonically) isomorphic to $\prod_{p \neq l} \mathbb{Z}_p$. The same argument as above shows that $\rho_{l,\kappa}$ is trivial on all components of this product away from $p$. This is true of any such $\kappa$ point, the set of which naturally forms a zariski dense set of $Sp(A)$. Hence we deduce that restricted of $\rho_l$ to $I_K$ is determined by a continuous homomorphism

$$\rho_l : \mathbb{Z}_p \rightarrow 1 + p^2 M_n(A^o).$$
$\mathbb{Z}$ is dense in $\mathbb{Z}_p$, hence this morphism is determined by $\alpha := \rho(t(1))$. There exists $c \in \mathbb{Q}_p$ such that $ct_p(I_F) = \mathbb{Z}_p$ and $\rho_t(u) = \alpha^{ct_p(u)}$. Note that raising $\alpha$ to something in $\mathbb{Z}_p$ makes sense because $\alpha \in 1 + p^2 M_n(A^o)$. The log map is a group homomorphism from $1 + p^2 M_n(A^o)$ to $M_n(A^o)$ defined by power series as follows:

$$
\log(1 + B) = B - \frac{B^2}{2} + \frac{B^3}{3} - \frac{B^4}{4} + \ldots
$$

If $1 + B \in 1 + p^2 M_n(A^o)$ then there is map $exp$, which is defined at the level of power series and has the property that $exp(\log(1+B)) = 1 + B$. Now define $N := c \cdot \log(\alpha)$, then $\rho_t(u) = exp(t_p(u)N)$. We can now do the usual trick of conjugating by $\Phi$ to show that $\rho_t(\Phi)N\rho_t(\Phi)^{-1} = p^{-1}N$ and hence that $N$ is nilpotent.

$\square$

Given any such $\rho_t$ we can find an $N \in \text{End}_A(M)$ as in the above proposition. Define the representation

$$
\rho : W_{\mathbb{Q}_p} \rightarrow GL_n(A)
$$

by $\rho(\Phi^n \cdot u) = \rho_t(\Phi^n \cdot u)exp(-t_p(u) \cdot N)$. By construction $\rho$ is continuous with respect to the discrete topology on $M$. An elementary check shows that $\rho(w)N\rho(w)^{-1} = |w|N$ for all $w \in W_{\mathbb{Q}_p}$, where $|.|$ is inherited from the local reciprocity isomorphism, sending $\Phi$ to a uniformiser (see §5.3). Hence $(\rho, N)$ is a Weil-Deligne representation on $M$, interpreted in the appropriate sense. Specialising to any $F$-valued of $Sp(A)$ recovers classical construction. One advantage of this approach is that specialising to any $\mathbb{C}_p$-valued point of $Sp(A)$ gives a Weil-Deligne representation over $\mathbb{C}_p$. Note that even though $N$ may be nonzero it may specialise to zero at some $\kappa \in Sp(A)(\mathbb{C}_p)$.

Ultimately we are interested in applying the local Langlands correspondence to such Weil-Deligne representations across $\mathcal{E}^o$. Let $Y = Sp(A) \in \mathcal{U}$. Then $\rho_Y$ restricted to $W_{\mathbb{Q}_p}$ is a representation to which we can apply proposition 18. Hence we get $(\rho_Y, N)$, a 2 dimensional Weil-Deligne representation over $A$. This gives a family of Weil-Deligne representations over all $\mathcal{E}^o$.

5.3. Local Langlands and Local-Global compatibility. The local Langlands correspondence for $GL_n$ over a $p$-adic field $F$ (proven in [20]) gives a natural bijection between isomorphism classes of irreducible smooth representations of $GL_n(F)$ ([21]) and $n$-dimensional $\Phi$-semisimple Weil-Deligne representations of $W_F$ ([28]). The base field is generally taken to be algebraically closed of characteristic 0. [14] provides a fairly comprehensive survey in the case $n = 2$. The bijection, though natural, can be normalised in different ways. The first choice to be made is how to normalise the local class field reciprocity isomorphism $W_{F}^{\text{ab}} \simeq F^*$. We will adopt the conventions of [14], where geometric Frobenius elements map to uniformisers. If $|.|$ is the absolute value on $F$ then the local reciprocity map induces a character on $W_{F}$ which we also denote $|.|$. The local reciprocity isomorphism induces a character $|.|$ on $GL_n(F)$ after composing with $\text{det}$. The unitary normalisation, $\pi_u$, is determined uniquely by stipulating that $L$ and $\epsilon$ factor agree. The Tate normalisation, $\pi_t$, is defined by twisting the unitary correspondence by $|.|^{\frac{1}{2}}$. More precisely, if $\sigma$ is an Weil-Deligne representation of the the above type over an algebraically closed field of characteristic 0 then $\pi_u(\sigma) \otimes |.|^{\frac{1}{2}} = \pi_t(\sigma)$. The Tate normalisation has the property that if $\sigma$ is defined over any field of characteristic 0 then so is $\pi_t(\sigma)$ (3.2.7).
1-dimensional representations factoring through the determinant. If \( GL \) smooth irreducible representations of \( \pi \), then define \( \pi_l \) other shows that

This is local-global compatibility away from \( p \).

Let \( f \) be a classical normalised cuspidal eigenform of weight \( k \geq 1 \) and level \( N \) and Nebentypus \( \chi \) defined over \( \bar{Q}_p \). Attached to such an \( f \) is a continuous two dimensional representation \( \rho_f \) of the absolute Galois group \( G_{\mathbb{Q}} \) over \( \bar{Q}_p \) characterised by the following conditions: it is unramified outside \( Np \) and for \( l \) not dividing \( Np \) the characteristic polynomial of a geometric Frobenius element is

where \( a_l \) is the eigenvalue of the Hecke operator \( T_l \) acting on \( f \). If we restrict to a decomposition group at \( l \neq p \) then by results of the previous section we can associate to \( f \) a 2-dimensional \( \Phi \)-semisimple Weil-Deligne representation of \( W_{\mathbb{Q}_l} \) which we denote \( \sigma_{f,l} \).

On the other hand, we can associate to \( f \) an automorphic representation of \( GL_2(A_f) \) over \( \bar{Q}_p \), albeit up to a choice of normalisation. Our choice of normalisation is determined by the fact that away from \( pN \) the eigenvalues of classical and adelic Hecke operators agree. Such a representation factors into the restricted tensor product of smooth irreducible representations of \( GL_2(Q_l) \) as \( l \) ranges over all finite places of \( Q \). In this way we can associate to \( f \) a smooth irreducible representation of \( GL_2(Q_l) \) over \( \bar{Q}_p \), which we denote \( \pi_{f,l} \). Work of Carayol \((11)\) and other shows that

This is local-global compatibility away from \( p \). Given the appropriate reformulation \((2.1.2 \ 18)\) there is a similar result for \( l = p \) as proven by Saito \((24)\).

In \((18)\) the classical local Langlands correspondence is modified at non-generic smooth irreducible representations of \( GL_2(Q_l) \). In the case of \( GL_2 \) this is just the 1-dimensional representations factoring through the determinant. If

then define \( \pi_m(\sigma) = B(\chi, |\cdot|^\frac{1}{2}, \chi, |\cdot|^{-\frac{1}{2}}) \), where \( B(\chi, |\cdot|^\frac{1}{2}, \chi, |\cdot|^{-\frac{1}{2}}) \) is the (unitary, see §11.2 \( 16) \) reducible principal series with 1-dimensional quotient \( \pi_1(\sigma) \). For all other \( \sigma \), \( \pi_m(\sigma) = \pi_1(\sigma) \).

5.4. Local Langlands in Families. We now study how the local Langlands correspondence behaves over a 1-dimensional, irreducible, connected, smooth affinoid rigid space \( Y = Sp(A) \) over \( \mathbb{C}_p \). This is equivalent to the affinoid algebra \( A \) being a Dedekind domain. Our ultimate goal is to apply these result to a normalisation of the eigencurve which can be admissibly covered by such spaces.

Let \( K \) be the field of fractions of \( A \). Fix \( \sigma = (\rho, N) \), a Weil-Deligne representation on a free \( A \)-module of rank 2. If \( \kappa \in Sp(A)(\mathbb{C}_p) \) then we can specialise \( (\rho, N) \) in two ways: locally at \( \kappa \) or generically at \( K \). If we \( \Phi \)-semisimplify we get two 2-dimensional Weil-Deligne representations: \( (\rho_{\kappa}, N_{\kappa}) := \sigma_{\kappa} \) and \( (\rho_K, N_K) := \sigma_K \). Invoking Local Langlands gives \( \pi_m(\sigma_{\kappa}) \) and \( \pi_m(\sigma_K) \), two smooth irreducible representations of \( GL_2(Q_l) \) over \( \mathbb{C}_p \) and \( K \) respectively. Let \( \overline{K} \) be an algebraic closure of \( K \) and
\(\sigma_K = (\rho_K, N)\) be the \(\Phi\)-semi-simplification of \(\sigma \otimes K\). By compatibility of \(\pi_m\) with extension of scalars we know that \(\pi_m(\sigma_K) \cong \pi_m(\sigma_K) \otimes K\). \(\pi_m(\sigma_K)\) can take one of three forms: Principal Series, Supercuspidal or Special. We deal with each in turn.

5.5. **Principal Series.** Fix a square root of \(l\) in \(\mathbb{C}_p\). This case corresponds \(\sigma_K = (\rho_K, 0)\), where \(\rho_K\) is reducible. More precisely:

\[
\rho_K \sim \begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix},
\]

where \(\chi_1\) and \(\chi_2\) are \(K\)-valued quasi-characters of \(W_{Q_l}\) such that \(\chi_1/\chi_2^{-1} \neq |.|^\pm 1\).

We ignore the degenerate case because this cannot occur on \(E\). By definition, \(\chi_1\) and \(\chi_2\) are continuous with respect to the discrete topology on \(K\). \(I_{Q_l}\) is profinite so compact, hence \(\chi_i(I_{Q_l})\) is a finite group for \(i = 1, 2\). We deduce that \(\chi_i(I_{Q_l}) \subset \mathbb{C} \subset A\) for \(i = 1, 2\). By construction we know that the characteristic polynomial, \(P_\Phi\), of \(\rho_K(\Phi)\) has coefficients in \(A\), hence

\[
\chi_1(\Phi)\chi_2(\Phi) \in A^*; \chi_1(\Phi) + \chi_2(\Phi) \in A.
\]

There is no reason for either \(\chi_1\) or \(\chi_2\) to be defined over \(A\) or even \(K\). We make use of the following proposition:

**Proposition 20.** Let \(A\) be a reduced integral affinoid algebra over \(\mathbb{C}_p\), \(K\) its field of fractions. Let \(L\) be a finite field extension of \(K\) and, \(B\) be the integral closure of \(A\) in \(L\). Then \(B\) is a finite \(A\)-module and in particular is affinoid.

**Proof.** Theorem 3.5.1 of \([19]\) tells us that the integral closure of \(A\) in \(K\) is a finite \(A\)-module which we denote \(R\). \(A\) is noetherian and hence we deduce that \(R\) is a noetherian, integrally closed domain. \(B\) is naturally the integral closure of \(R\) in \(L\). \(K\) is of characteristic zero hence \(L/K\) is separable and we can apply proposition 8 of \([20]\) to deduce that \(B\) is a finite \(R\)-module. Hence \(B\) is a finite \(A\)-module. \(\square\)

Define \(L\) to be the splitting field of \(P_\Phi\) over \(K\). This is a Galois extension of degree at most 2. Let \(B\) be the integral closure of \(A\) in \(L\). By the above proposition this is an affinoid algebra which is finite over \(A\). By construction \(\chi_1\) and \(\chi_2\) are defined over \(B\). The local Langlands correspondence in this case is correspondence

\[
\pi_m(\sigma_K) = \pi(\chi_1|.|^{\frac{1}{2}}, \chi_2|.|^{\frac{1}{2}}),
\]

where we are adopting the notation and conventions of §11.2 \([19]\). On the right hand side we are naturally viewing the \(\chi_i\) as characters of \(Q_l^*\) via the local reciprocity isomorphism.

Define

\[
\Pi_B = \pi_B(\chi_1|.|^{\frac{1}{2}}, \chi_2|.|^{\frac{1}{2}}) := \{f : GL_2(Q_l) \rightarrow B; f \in \pi(\chi_1|.|^{\frac{1}{2}}, \chi_2|.|^{\frac{1}{2}})\}.
\]

This makes sense because \(\chi_1(a)\chi_2(d) \in B\), for all \(a, d \in Q_l^*\).

Let \(P\) be the Borel subgroup consisting of upper triangular matrices in \(GL_2(Q_l)\), and \(P(Z_l) = GL_2(Z_l) \cap P\). Fix \(U \triangleleft GL_2(Z_l)\), an open compact subgroup such that \(\chi_1\chi_2\) is trivial on \(P(Z_l) \cap U\). This implies that \(\Pi_B \neq 0\).

**Lemma 21.** \(\Pi_B^U\) is a finite free \(B\)-module of whose rank \(d\) is equal to the order of \(P(Z_l) \backslash GL_2(Z_l)/U\). Moreover,

\[
\Pi_B^U \otimes_B K \cong \pi(\chi_1|.|^{\frac{1}{2}}, \chi_2|.|^{\frac{1}{2}})^U.
\]
Proof. The Iwasawa decomposition (4.5.2 [3]) tells us that $GL_2(\mathbb{Q}_l) = P \cdot GL_2(\mathbb{Z}_l)$. Hence $f \in \pi(\chi_1|.|^{\frac{1}{2}}, \chi_2|.|^{\frac{1}{2}})$ is uniquely determined by its behaviour on $GL_2(\mathbb{Z}_l)$. By the conditions imposed on $U$ we are free to choose the behaviour of $f$ on each element of the double coset space $P(\mathbb{Z}_l) \backslash GL_2(\mathbb{Z}_l)/U$. □

Let $\mathcal{H}_U$ be the Hecke algebra introduced in §4. It naturally acts in a compatible way on both $\pi(\chi_1|.|^{\frac{1}{2}}, \chi_2|.|^{\frac{1}{2}})$ and $\Pi_B^U$. Because $\pi(\chi_1|.|^{\frac{1}{2}}, \chi_2|.|^{\frac{1}{2}})$ is irreducible we know that $\pi(\chi_1|.|^{\frac{1}{2}}, \chi_2|.|^{\frac{1}{2}})^U$ is a simple finite dimensional $\mathcal{H}_U$-module (4.2.3 [3]). Such a module is determined up to isomorphism by its trace $tr : \mathcal{H}_U \rightarrow \overline{K}$. By the above lemma we know that this map factors through $B$.

**Theorem 22.** If $tr : \mathcal{H}_U \rightarrow B$ is the trace of $\mathcal{H}_U$ acting on $\Pi_B^U$, then it factors through $A$. Moreover for any $\kappa \in Sp(A)(\mathbb{C}_p)$, the specialisation of this map to $\kappa$ is the trace of $\mathcal{H}_U$ acting on $\pi_m(\sigma_\kappa)^U$.

*Proof.* Let $\tau \in Sp(B)(\mathbb{C}_p)$ be a point lying over $\kappa$, $m_\tau$ the defining maximal ideal of $B$. Let $\chi_{1,\tau}$ and $\chi_{2,\tau}$ be the $\mathbb{C}_p$-valued quasi-character induced by $\tau$. The Weil representation $\chi_{1,\tau} \otimes \chi_{2,\tau}$ has the same trace as $\rho_\kappa$, hence by the Brauer-Nesbitt theorem we deduce that they are isomorphic. If $B(\chi_{1,\tau}|.|^{\frac{1}{2}}, \chi_{2,\tau}|.|^{\frac{1}{2}})$ denotes the (potentially reducible) principal series representation then there is a surjective morphism of $GL_2(\mathbb{Q}_l)$-modules:

$$
\Pi_B^U \longrightarrow B(\chi_{1,\tau}|.|^{\frac{1}{2}}, \chi_{2,\tau}|.|^{\frac{1}{2}}).
$$

The usual group cohomology argument (see lemma 16) tells us that the induced map:

$$
\Pi_B^U \longrightarrow B(\chi_{1,\tau}|.|^{\frac{1}{2}}, \chi_{2,\tau}|.|^{\frac{1}{2}})^U
$$

is surjective. $\Pi_B^U$ is finite and free by lemma 20. We can choose a basis whose image in $B(\chi_{1,\tau}|.|^{\frac{1}{2}}, \chi_{2,\tau}|.|^{\frac{1}{2}})^U$ is a basis. We deduce that the kernel is equal to $m_\tau \cdot \Pi_B^U$. Hence there is an isomorphism

$$
\Pi_B^U \otimes \mathbb{C}_p \cong B(\chi_{1,\tau}|.|^{\frac{1}{2}}, \chi_{2,\tau}|.|^{\frac{1}{2}})^U.
$$

If $\chi_{1,\tau}/\chi_{2,\tau} \neq |.|^{-1}$, then $\pi_m(\sigma_\kappa) = B(\chi_{1,\tau}|.|^{\frac{1}{2}}, \chi_{2,\tau}|.|^{\frac{1}{2}})$. If $\chi_{1,\tau}/\chi_{2,\tau} = |.|^{-1}$ then $\pi_m(\sigma_\kappa) = B(\chi_{2,\tau}|.|^{\frac{1}{2}}, \chi_{1,\tau}|.|^{\frac{1}{2}})$. However, the semi-simplifications of $B(\chi_{2,\tau}|.|^{\frac{1}{2}}, \chi_{1,\tau}|.|^{\frac{1}{2}})^U$ and $B(\chi_{1,\tau}|.|^{\frac{1}{2}}, \chi_{2,\tau}|.|^{\frac{1}{2}})^U$ as $\mathcal{H}_U$-modules are isomorphic. Hence they the same trace. We deduce that the specialisation to $\tau$ of the trace of $\mathcal{H}_U$ acting on $\Pi_B^U$ must equal the trace of $\mathcal{H}_U$ acting on $\pi_m(\sigma_\kappa)^U$. This is true of any $\tau$ lying over $\kappa$, hence the trace is $Gal(L/K)$-equivariant so the image lies in $K \cap B = A$. □

5.6. Supercuspidal. If $\sigma_K = (\rho_K, 0)$, where $\rho_K$ is irreducible then $\pi_m(\sigma_K)$ is supercuspidal. By 2.2.1 [28] we know that $\rho_K$ is of the form $\nu \otimes \chi$ where $\nu$ is an irreducible Weil representation of Galois type and $\chi$ is an unramified character taking values in $K$. Any irreducible Weil representation of Galois type on an algebraically closed field of characteristic zero necessarily has finite image so is defined over $\mathbb{C}_p$. Hence there is a Weil-Deligne representation $\sigma_{\mathbb{C}_p}$, defined over $\mathbb{C}_p$ such that $\sigma_K \cong (\sigma_{\mathbb{C}_p} \otimes K) \otimes \chi$. Recall that the Tate normalisation of local Langlands is compatible with base change and twisting. Therefore,

$$
\pi_m(\sigma_K) = (\pi_m(\sigma_{\mathbb{C}_p} \otimes K)) \otimes \chi.
$$
Although $χ$ takes values in $K$ we know that $χ^2(Q^*_l) ∈ A^*$ by considering determinants. Let $L$ be the splitting field of $x^2 - χ(l)^2$ over $K$. Let $B$ be the integral closure of $A$ in $L$ and $R$ the integral closure of $A$ in $K$. $χ$ is defined over $B$ and hence we can define the $B$-module

$$Π_B := (π_m(σ_{C_p}) ⊗ B) ⊗ χ.$$ 

$Π_B$ is a free $B$-module because we are tensoring over a field.

**Lemma 23.** If $U ⊂ GL_2(ℤ_l)$ is an open compact subgroup then $Π_B^U$ is a finite free $B$-module.

**Proof.** $χ$ is unramified so is trivial on $det(U)$. Hence we need only consider $(π_m(σ_{C_p}) ⊗ B)^U$. Clearly we have the inclusion

$$π_m(σ_{C_p})^U ⊗ B ⊂ (π_m(σ_{C_p}) ⊗ B)^U.$$ 

The left hand side is free because we are taking the tensor product over a field. Let $v ∈ (π_m(σ_{C_p}) ⊗ B)^U$. It is the sum of finitely many components of form $x ⊗ b$, where $x ∈ π_m(σ_{C_p})$ and $b ∈ B$. Because $(π_m(σ_{C_p})$ is smooth we can choose $V ⊂ U$, an open subgroup of index $n$, which stabilises each component in $v$. If $g_1, g_2, ..., g_n$ are right coset representatives for $V$ in $U$ then by definition:

$$v = \frac{1}{n} \sum_{i=1}^n g_i(v).$$ 

The right hand side is the sum of components of form

$$\frac{1}{n} (\sum_{i=1}^n g_i(x)) ⊗ b.$$ 

By construction the left hand term in this product is invariant under $U$ and we deduce that

$$π_m(σ_{C_p})^U ⊗ B = (π_m(σ_{C_p}) ⊗ B)^U.$$ 

□

**Theorem 24.** If $tr : H_U → B$ is the trace of $H_U$ acting on $Π_B^U$, then it factors through $A$. Moreover for any $κ ∈ Sp(A)(C_p)$, the specialisation of this map to $κ$ is the trace of $H_U$ acting on $π_m(σ_κ)^U$.

**Proof.** Let $τ ∈ Sp(B)(C_p)$ be a point lying over $κ$ defined by the maximal ideal $m_τ$. By the Brauer-Nesbitt theorem $ρ_κ ≡ ν ⊗ χ_τ$. Hence,

$$π_m(σ_κ) = π_m(σ_{C_p}) ⊗ χ_τ.$$ 

There is a surjective morphism of $GL_2(ℚ_l)$-modules

$$Π_B → π_m(σ_{C_p}) ⊗ χ_τ,$$ 

induced by reducing by $m_τ$. We get the short exact sequence

$$0 → m_τΠ_B → Π_B → π_m(σ_κ) → 0.$$ 

The by now familiar group cohomology argument gives the short exact sequence of $H_U$-modules

$$0 → (m_τΠ_B)^U → Π_B^U → π_m(σ_κ)^U → 0.$$
Because everything is smooth the same argument as used in lemma 23 tells us that $(m_\tau \Pi_B^U)^U = m_\tau \Pi_B^U$. We deduce there is an isomorphism of $\mathcal{H}_U$-modules:

\[ \Pi_B^U \odot \tau \mathbb{C}_p \cong \pi_m(\sigma_\kappa)^U. \]

Hence the specialisation to $\tau$ of the trace of $\mathcal{H}_U$ acting on $\Pi_B^U$ must equal the trace of $\mathcal{H}_U$ acting on $\pi_m(\sigma_\kappa)^U$. This is true of any $\tau$ lying over $\kappa$, hence the trace is $Gal(L/K)$-equivariant so the image lies in $K \cap B = A$. $\square$

5.7. **Special.** The final case to consider is when $\sigma$ has non-trivial monodromy. This is the indecomposable case (4.1.5 [28]) and over $K$

\[ \sigma_K \cong \left( \begin{array}{ccc} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \]

for some $K$-valued quasicharacter $\chi$. By construction, the trace and determinant of $\rho_K$ are contained in $A$. Hence $\chi$ takes values in $A^*$. By definition $\sigma$ is defined over $A$. This means that $N \in \text{End}_A(M)$, where $M$ is free of rank 2. It is possible that $N$ will vanish after specialising to some $\kappa \in Sp(A)(\mathbb{C}_p)$. Over $K$ however we can always find a basis such that $N$ has the above form. What this means its that generic behaviour (over $K$) is too crude to tell us about local behaviour everywhere. However, if $A$ is 1-dimensional then then $N$ can only specialise to zero at finitely many $\mathbb{C}_p$-valued points of $Sp(A)$. Let us denote this set $S \subset Sp(A)$.

In this case,

\[ \pi_m(\sigma_K) = \pi(\chi^\frac{-1}{2}, \chi^\frac{1}{2}), \]

Where the right hand side is the unique irreducible quotient of the reducible principal series $B(\chi^\frac{-1}{2}, \chi^\frac{1}{2})$. Define the $A$-modules

\[ \Delta_A := \{ f \in B(\chi^\frac{-1}{2}, \chi^\frac{1}{2}) \mid f(g) \in A, \forall g \in GL_2(\mathbb{Q}_l) \}, \]

and

\[ \Omega_A := \{ f \in \Pi_A \mid f(g) = \chi(\text{det}(g))a \text{ for some } a \in A \} \]

Note that $\Omega_A \neq 0$. We are interested in the $A$-module $\Pi_A := \Delta_A/\Omega_A$. Note that $\Delta_A \odot_A K \cong \pi_m(\sigma_K)$. Fix $U < GL_2(\mathbb{Z}_l)$, an open compact subgroup such that $\text{det}(U) \subset \text{ker}(\chi)$. Hence $U$ acts trivially on $\Omega_A$.

**Lemma 25.** $\Pi_A^U$ is a finite projective $A$-module.

**Proof.** Let $f \in \Delta_A$ such that $f - u(f) \in \Omega_A$ for all $u \in U$. Hence the image of $f$ in $\Pi_A$ is contained in $\Delta_A^U$. This is equivalent to the statement that for all $u \in U$, there exists $a_u \in A$ such that $f(g) - f(gu) = \chi(\text{det}(g))a_u, \forall g \in GL_2(\mathbb{Q}_l)$. This gives a function from $\phi : U \rightarrow A, \phi(u) = a_u$. This is actually a group homomorphism (with the additive group structure on $A$). Recall that by definition $f$ is smooth, hence $\phi$ is trivial on some open subgroup of $U$. $U$ is compact so the image of $\phi$ is finite. But $A$ is a $\mathbb{C}_p$-vector space so $\phi$ is trivial. Therefore $f \in \Delta_A^U$. By the same argument as lemma 21 $\Delta_A^U$ is free of rank $|P(\mathbb{Z}_l)/GL_2(\mathbb{Z}_l)/U|$. Hence $\Pi_A^U$ is isomorphic to $\Delta_A^U/\Omega_A$, which is a finite, torsion free $A$-module, and is hence projective. In fact it is to be free of rank $|P(\mathbb{Z}_l)/GL_2(\mathbb{Z}_l)/U| - 1$. $\square$

**Theorem 26.** Let $\kappa \in Sp(A)$ and $U \in GL_2(\mathbb{Z}_l)$ an open compact subgroup such that $\text{det}(U) \subset \text{ker}(\chi)$. There is an embedding of $\mathcal{H}_U$-modules

\[ \Pi_A^U \odot \kappa \mathbb{C}_p \rightarrow \pi_m(\sigma_\kappa)^U. \]
Hence it has an admissible affinoid cover which we also denote $C$. Local-Global Compatibility. 5.8. Using the constructions of the previous paragraph we will prove an analogous result to Theorem 18 on the Galois side.

As indicated in §5.1 there is an admissible affinoid cover of $E^o$ (taken over $Q_p$) such that each element of this cover comes equipped with a global Galois representation. Restricting to $W_{Q_l}$ we may apply proposition 19 to each element of this cover to get a family of 2 dimensional Weil-Deligne representation across $E^o$. Base change to $C_p$ and then consider a normalisation which we denote $\tilde{E}^o$. By the above construction it has an admissible affinoid cover which we also denote $U$ (each member being 1-dimensional, connected, irreducible and smooth), which has the property that for $Y = Sp(A) \in U$ there is a Weil-Deligne representation over $A$, which agree on intersections. $Y \in U$ implies that $A$ is a Dedekind domain. Hence the results of §5.4 are applicable and we may apply the local Langlands correspondence to each of these families of Weil-Deligne representations. To any $\kappa \in \tilde{E}^o(C_p)$ we therefore have
a Weil-Deligne representation \((\rho_\kappa, N_\kappa)\) (over \(\mathbb{C}_p\)) and consequently an admissible, smooth representation of \(GL_2(\mathbb{Q}_l)\), \(\pi_m(\rho_\kappa, N_\kappa)\).

**Proposition 27.** If \(\tilde{Z} \subset \tilde{E}^o\) is a connected component then for \(\kappa \in \tilde{Z}(\mathbb{C}_p)\), \(\pi_m(\rho_\kappa, N_\kappa)\) is either generically (away from a discrete set) supercuspidal, principal series or special.

**Proof.** As in [13], for \(\kappa \in \tilde{E}^o(\mathbb{C}_p)\) the connected component containing \(\kappa\) is defined to the set of points \(z \in \tilde{E}^o(\mathbb{C}_p)\) such that there is a finite set \(Z_1, ..., Z_n\) of connected admissible affinoid opens in \(\tilde{E}^o\) with \(\kappa \in Z_1, z \in Z_n\) and \(Z_i \cap Z_{i+1}\) non empty for all \(i\).

Because any connected component \(\tilde{Z} \subset \tilde{E}^o\) is an admissible open there is a subset of \(U\) which admissibly covers \(\tilde{Z}\). On any element of this cover there is a family of admissible smooth representations of \(GL_2(\mathbb{Q}_l)\). By construction they must agree on intersections. \(\tilde{Z}\) is separated so the intersection of two admissible connected affinoid opens is again connected and affinoid. If two open affinoid subdomains have non-trivial intersection in \(\tilde{Z}\) then they must generically have the same type of \(GL_2(\mathbb{Q}_l)\) representation. Hence by the definition of a connected component \(\tilde{Z}\) must either be generically supercuspidal, principal series or special. \(\square\)

Let us follow to the conventions of §4 fixing a Haar measure on \(GL_2(\mathbb{Q}_l)\). Let \(\mathcal{H}_U\) denote the Hecke algebra of compactly supported, \(U\) bi-invariant functions on \(GL_2(\mathbb{Q}_l)\) with coefficients in \(\mathbb{C}_p\). Fix \(U < GL_2(\mathbb{Z}_l)\), an open compact subgroup such that \(U_1(\mathbb{N})^l \times U \subset U_1(\mathbb{N})\). This implies that \(\pi_m(\rho_\kappa, N_\kappa)^U\) is generically non-zero.

We will now prove the analogue of Theorem 18 on the Galois side.

**Theorem 28.** There is a map

\[
\text{tr}_{\text{Lan}} : \mathcal{H}_U \longrightarrow O(\tilde{E}^o),
\]

such that if \(\tilde{Z} \subset \tilde{E}^o\) a connected component then either

(i) \(\tilde{Z}\) is supercuspidal and for all \(\kappa \in \tilde{Z}(\mathbb{C}_p)\), \(\text{tr}_{\text{Lan}}\) restricted to \(\kappa\) is the trace of \(\mathcal{H}_U\) acting on \(\pi_m(\rho_\kappa, N_\kappa)^U\).

(ii) \(\tilde{Z}\) is special and for \(\kappa \in \tilde{Z}(\mathbb{C}_p)\) such that monodromy does not vanish \(\text{tr}_{\text{Lan}}\), restricted to \(\kappa\) is the trace of \(\mathcal{H}_U\) acting on \(\pi_m(\rho_\kappa, N_\kappa)^U\). When monodromy does vanish, \(\text{tr}_{\text{Lan}}\) restricted to \(\kappa\) is the trace of \(\mathcal{H}_U\) acting on the \(U\) fixed points of the special subrepresentation of \(\pi_m(\rho_\kappa, N_\kappa)\).

(iii) \(\tilde{Z}\) is Principal Series and and for all \(\kappa \in \tilde{Z}(\mathbb{C}_p)\), \(\text{tr}_{\text{Lan}}\) restricted to \(\kappa\) is the trace of \(\mathcal{H}_U\) acting on \(\pi_m(\rho_\kappa, N_\kappa)^U\).

**Proof.** Parts (i) and (iii) are direct from Theorems 22 and 24 applied to each the admissible cover of \(\tilde{Z}\) described in the proof of proposition 27. For part (ii) let \(Y = \text{Sp}(A) \in U\) be contained in \(\tilde{Z}\). Keeping the notation of §5.7 we may construct \(\text{tr}_{\text{Lan}}\) by gluing together the trace functions of \(\mathcal{H}_U\) acting on \(\Pi_A^U\). We know that they agree on intersections because they must do generically. For \(\kappa \in \tilde{Z}(\mathbb{C}_p)\) where monodromy doesn’t vanish we are done by theorem 26. The case where monodromy vanishes is due to the construction of \(\Pi_A\) and our choice of normalisation of the local Langlands correspondence. \(\square\)
There is a natural map from $\tilde{E}^o$ to $\tilde{E}$. Hence by pullback there is a map

$$tr_{aut} : \mathcal{H}_U \rightarrow O(\tilde{E}^o),$$

which satisfies the same properties as given in theorem 18.

**Theorem 29.** $tr_{aut} = tr_{Lan}$

**Proof.** It is enough to show that they agree on on connected components. Coleman’s classicality result tells us that there are a Zariski dense set of classical points in any connected component $\tilde{Z} \subset \tilde{E}^o$. Let $Y = Sp(A) \subset \tilde{Z}$ be a member of $\mathcal{U}$, which contains a classical point. Because classical points are never discrete it must contain infinitely many such points. By theorem 18 and theorem 28 there are a Zariski dense set of classical points, $\kappa \in Y(\mathbb{C}_p)$, such that $tr_{Lan}$ and $tr_{aut}$ restricted to $\kappa$ are the trace functions of $\mathcal{H}_U$ acting on $\pi_m(\rho_\kappa, N_\kappa)^U = \pi_\kappa(\rho_\kappa, N_\kappa)^U$ (classical forms always give generic representations) and $\pi_{f_\kappa,l}^U$ respectively. Here $f_\kappa$ is the cuspidal eigenform corresponding to $\kappa$. By classical local to global compatibility we know that these two $\mathcal{H}_U$ modules are isomorphic, hence traces must agree. We deduce that on $Y$, $tr_{Lan}$ and $tr_{aut}$ agree on a Zariski dense set, and hence are equal. By lemma 2.1.4 of [13] they must agree on all of $\tilde{Z}$. Hence we deduce that $tr_{Lan} = tr_{aut}$. \hfill \Box

Let $Z$ be the irreducible component of $E^o$ which is the image of $\tilde{Z}$ under the natural projection of $\tilde{E}^o$ to $E^o$. Let us also fix $U$ as above but take the property that it is bonne in the sense of [2]. This property means that there is an equivalence of categories between finite vector spaces over $\mathbb{C}_p$ with an action of $\mathcal{H}_U$ (such objects are determined up to semi-simplification by traces) and admissible smooth representations of $GL_2(\mathbb{Q}_l)$ over $\mathbb{C}_p$. The equivalence is given by taking $U$ invariants. Such subgroups form a basic for neighbourhoods of the identity so we may choose such a subgroup which also satisfies the condition $U_1(N)^l \times U \subset U_1(N)$. Fix such a $U$. This condition implies that $\pi_{f_\kappa,l}^U$ is non zero for all $\kappa \in E^o(\mathbb{C}_p)$. We now come to our central result.

**Theorem 30 (Theorem A).** Away from a discrete set of points local to global compatibility holds on $E^o$, i.e.

$$\pi_m(\rho_\kappa, N_\kappa) \cong \pi_{f_\kappa,l}$$

for all $\kappa \in E^o(\mathbb{C}_p)$ away from some discrete set. More precisely if $Z \subset E^o$ is an irreducible component then either:

(i) $Z$ is Supercuspidal (i.e. $\tilde{Z}$ is Supercuspidal) In this case local to global compatibility holds on all $Z$.

(ii) $Z$ is Special (i.e. $\tilde{Z}$ is Special). In this case local to global compatibility holds everywhere except at points where monodromy vanishes. For such $\kappa$, $\pi_{f_\kappa,l}$ is the unique special irreducible sub-representation of $\pi_m(\rho_\kappa, N_\kappa)$.

(iii) $Z$ is Principal Series (i.e. $\tilde{Z}$ is Principal Series). Here local to global compatibility holds except at points where the ratio of the Satake parameters becomes $l^{\pm 1}$. At such points all we know is that there is a smooth admissible representation $\pi$ and a $GL_2(\mathbb{Q}_l)$-module surjection $\pi \rightarrow \pi_{f_\kappa,l}$ where the semisimplification of $\pi$ if isomorphic to the semisimplification of $\pi_m(\rho_\kappa, N_\kappa)$. 


Proof. Let us fix an irreducible component \( Z \subset \tilde{E}^o \). In proving the result it will be enough to do it on \( \tilde{E}^o \), which surjects onto \( E^o \). We will go through each case in turn.

In case (i) for any \( \kappa \in \tilde{Z}(\mathbb{C}_p) \), we know by theorem 28 that \( \pi_m(\rho_\kappa, N_\kappa)^U \) is irreducible because \( U \) is bonne and \( \pi_m(\rho_\kappa, N_\kappa) \) is irreducible. By theorem 29 we know that the trace of \( \text{tr}_{\text{aut}} \) restricted to \( \kappa \) is the trace of this irreducible \( \mathcal{H}_U \) module. We know by proposition 17 that this irreducible module surjects (\( \mathcal{H}_U \) equivariantly) onto \( \pi_{f_\kappa,l}^U \). Hence they must be isomorphic as \( \pi_{f_\kappa,l}^U \neq 0 \). Therefore \( \pi_{f_\kappa,l}^U \) and \( \pi_m(\rho_\kappa, N_\kappa)^U \) are isomorphic irreducible \( \mathcal{H}_U \) modules. Because \( U \) is bonne we deduce that \( \pi_{f_\kappa,l} \cong \pi_m(\rho_\kappa, N_\kappa) \).

In case (ii) when monodromy does not vanish an identical argument to above yields the result. At the \( \kappa \) where monodromy vanishes a similar argument works taking the \( U \) fixed vectors of the special sub-representation of \( \pi_m(\rho_\kappa, N_\kappa) \).

In case (iii) the same argument as above works where \( \pi_m(\rho_\kappa, N_\kappa) \) is irreducible (i.e. when the ratio of the satake parameters is not \( l^{\pm 1} \)). When this does occur \( \pi_m(\rho_\kappa, N_\kappa)^U \) is reducible and by proposition 17 we know that there is a finite dimensional \( \mathcal{H}_U \) module which surjects onto \( \pi_{f_\kappa,l}^U \) whose semi-simplification is isomorphic the the semi-simplification of \( \pi_m(\rho_\kappa, N_\kappa)^U \). Because \( U \) is bonne this corresponds to an a smooth irreducible \( GL_2(\mathbb{Q}_l) \) representation \( \pi \) which surjects onto \( \pi_{f_\kappa,l} \). This completes the proof. \( \square \)

References


