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An Indefinite Convection-Diffusion Operator

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Abstract

We give a mathematically rigorous analysis which confirms the surprising results in a recent paper [2] of Benilov, O’Brien and Sazonov about the spectrum of a highly singular non-self-adjoint operator that arises in a problem in fluid mechanics.

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1 Introduction

In a recent paper [2] Benilov, O’Brien and Sazonov have shown that the equation

$$\frac{\partial f}{\partial t} = \varepsilon \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial f}{\partial \theta} \right) + \frac{\partial f}{\partial \theta}$$

approximates the evolution of a liquid film inside a rotating horizontal cylinder. The variable $\theta$ is taken to lie in $[-\pi, \pi]$ and one assumes that the solutions $f$ are sufficiently smooth and satisfy periodic boundary conditions.

The operator $H$ is highly non-self-adjoint (NSA) and it is not amenable to standard elliptic techniques because the second order coefficient is indefinite. For $\theta \in (0, \pi)$ the second order term has a diffusive effect on the evolution but for $\theta \in (-\pi, 0)$ its effect is anti-diffusive. Many of the calculations in [2] are based on an asymptotic or WKB analysis for small $\varepsilon > 0$, but this has dangers because infinite order approximate eigenvalues of NSA operators need not be close to true eigenvalues. Nor need eigenvalues computed by truncations of a highly non-self-adjoint operator to large finite-dimensional subspaces by standard methods be close to the eigenvalues of the original operator; see [3, 4, 6] for examples and discussions of their relationship to pseudospectra. Our goal in this paper is to rederive some of the results in [2] for a fixed positive value of $\varepsilon$ by a rigorous and non-asymptotic technique. We also provide strong numerical evidence that the eigenvectors do not form a basis. In our numerical calculations we take $\varepsilon = 0.1$. 
2 A Reformulation of the Problem

We focus attention on the spectral properties of the operator

\[(Hf)(\theta) := \varepsilon \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial f}{\partial \theta} \right) + \frac{\partial f}{\partial \theta} \]

defined on all \(C^2\) periodic functions \(f \in L^2(-\pi, \pi)\). We normally assume that \(0 < \varepsilon < 2\) for reasons explained in Corollary 2. According to the WKB analysis of [2] the eigenvalue equation \(-iHf = \lambda f\) has a sequence of real eigenvalues which converge to the integers as \(\varepsilon \to 0\). This suggests that the evolution equation is neutrally stable, but Benilov et al. show that it exhibits explosive disturbances. This is closely related to the pseudospectra of the operator. Our goal is to prove that there are indeed real eigenvalues \(\lambda\) without depending on WKB analysis, and to provide a simple and rigorous method for computing them.

By expanding \(f \in L^2(-\pi, \pi)\) in the form

\[f(\theta) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} v_n e^{in\theta},\]

one may rewrite the eigenvalue problem in the form \(A v = \lambda v\), where \(A = -iH\) is given by

\[(Av)_n = \varepsilon n(n-1)v_{n-1} - \varepsilon n(n+1)v_{n+1} + nv_n.\]

The (unbounded) tridiagonal matrix \(A\) is of the form

\[
A = \begin{pmatrix}
A_- & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & A_+ \\
\end{pmatrix}
\]

where \(A_-\) acts in \(l^2(\mathbb{Z}_-)\), the central 0 acts in \(\mathbb{C}\) and \(A_+\) acts in \(l^2(\mathbb{Z}_+)\). The coefficient map \(n \to -n\) induces a unitary equivalence between \(A_-\) and \(-A_+\), so we only need study the spectrum of \(A_+\). Since \(A_+^* = DA_+D^{-1}\) where \(D_{r,s} = \delta_{r,s}(-1)^r\), \(A_+\) and \(A_+^*\) have the same spectrum. We assume that \(A_+\) has its natural maximal domain, and observe that it is a closed operator. We will see that its eigenvectors decrease more rapidly as \(n \to +\infty\) the smaller \(\varepsilon > 0\) is. We prove that the spectrum is discrete, i.e. that it consists only of isolated eigenvalues of finite multiplicity, in Section 3.

Benilov et al. correctly state in [2] that one obtains very poor numerical results if one simply truncates \(A\) to produce a finite matrix whose eigenvalues are then computed. We study the matrix \(A\) in a completely different manner.
The eigenvalue equation for $A_+$ may be written in the form

$$n(n - 1)v_{n-1} - n(n + 1)v_{n+1} + 2\frac{n - \lambda}{\varepsilon}v_n = 0. \tag{1}$$

The reality of the coefficients of (1) implies that if $\lambda$ is an eigenvalue then so is $\bar{\lambda}$. It does not, however, imply that the eigenvalues are real.

We confine attention to the solutions of (1) with support in $\mathbb{Z}^+$, and regard the $n = 1$ case, namely $\varepsilon v_2 = (1 - \lambda)v_1$, as an initial condition. Since it is a second order recurrence equation, the solution space of (1) is two-dimensional. We will see that one solution, often called the subordinate solution, lies in $l^2(\mathbb{Z}^+)$, but no others do so if $0 < \varepsilon < 2$. We say that $\lambda > 0$ is an eigenvalue of $A_+$ if the subordinate solution of the recurrence equation satisfies the initial condition.

If one assumes that (1) has a solution of the form $v_n = n^a(1 + b/n + O(1/n^2))$, then one finds that $a = -1 + 1/\varepsilon$ and $b = \lambda/\varepsilon$. This motivates our next two lemmas. In the following calculations we introduce constants $N_{\lambda,\varepsilon}$, and will use the fact that they can always be increased without affecting the results.

**Lemma 1** If $\lambda \geq 0$, there exists $N = N_{\lambda,\varepsilon}^{(1)}$ such that if $v_n$ is a solution of (1) satisfying $0 < v_{N-i} \leq (N - i)^a$ for $i = 1, 2$ where $a = -1 + 1/\varepsilon$, then $0 < v_n \leq n^a$ for all $n \geq N$.

**Proof** Suppose that $n \geq \lambda + 3$ and $0 < v_{n-i} \leq (n - i)^a$ for $i = 1, 2$. Then

$$0 < n^{-a}v_n = n^{-a} \left( \frac{n-2}{n}v_{n-2} + 2\frac{n-1-\lambda}{\varepsilon n(n-1)}v_{n-1} \right) \leq \left( 1 - \frac{2}{n} \right)^{a+1} + 2\frac{n-1-\lambda}{\varepsilon n(n-1)} \left( 1 - \frac{1}{n} \right)^a \leq 1$$

for all $n \geq N = N_{\lambda,\varepsilon}^{(1)}$. It follows inductively that $0 < v_n \leq n^a$ for all $n \geq N$.

**Corollary 2** If $\lambda \geq 0$ and $\varepsilon > 2$ then every solution of (1) lies in $l^2(\mathbb{Z}^+)$. In particular every such $\lambda$ is an eigenvalue of $A_+$.

**Proof** Let $N = N_{\lambda,\varepsilon}^{(1)}$. Let $u$ be the solution of (1) such that $u_{N-2} = 0$ and $u_{N-1} = (N - 1)^a$, and let $v$ be the solution such that $v_{N-2} = (N - 2)^a$ and $v_{N-1} = 0$. Since $a < -1/2$, Lemma 1 implies that both lie in $l^2(\mathbb{Z}^+)$. The space of all solutions is two-dimensional, so every solution lies in $l^2(\mathbb{Z}^+)$, and this applies in particular to the solution that satisfies the initial condition.

It is highly probably that one could avoid the above conclusion by imposing boundary conditions at $+\infty$ if $\varepsilon > 2$, i.e. by reducing the domain of $A_+$. We do not pursue this possibility.
Lemma 3  For every $\lambda \in \mathbb{R}$ there exists $N = N_{\lambda, \varepsilon}^{(2)}$ such that if $v_n$ is a solution of (1) satisfying

$$v_n \geq \left(1 + \frac{k}{n}\right) n^a$$

for $n = N - 1$ and $n = N - 2$, where $a = -1 + 1/\varepsilon$ and $k = 1 + \lambda/\varepsilon$, then the same inequality holds for all $n \geq N$.

Proof  Suppose that $n \geq \lambda + 3$ and that

$$v_{n-i} \geq \left(1 + \frac{k}{n-i}\right) (n-i)^a$$

for $i = 1, 2$. Then

$$n^{-a} v_n = n^{-a} \left(\frac{n-2}{n} v_{n-2} + \frac{2n-1-\lambda}{\varepsilon n(n-1)} v_{n-1}\right)$$

$$\geq \left(1 + \frac{k}{n-2}\right) \left(1 - \frac{2}{n}\right)^{a+1} + 2 \frac{n-1-\lambda}{\varepsilon n(n-1)} \left(1 + \frac{k}{n-1}\right) \left(1 - \frac{1}{n}\right)^a$$

$$= 1 + \frac{k}{n} + \frac{2}{n^2} + O(n^{-3})$$

$$\geq 1 + \frac{k}{n}$$

provided $n \geq N = N_{\lambda, \varepsilon}^{(2)}$. It follows inductively that (2) holds for all $n \geq N$.

Theorem 4  If $0 < \varepsilon < 2$ and $\lambda$ is a real eigenvalue of $A_+$ then $\lambda > 1$.

Proof  Suppose that $A_+ v = \lambda v$ where $\lambda \leq 1$ and $v_1 = 1$. The initial condition $\varepsilon v_2 = (1-\lambda) v_1$ implies that $v_2 \geq 0$, and it then follows from the signs of the coefficients in (1) that $v_n > 0$ for all $n \geq 3$. Lemma 1 implies that there exists a constant $c > 0$ such that

$$v_n \geq c \left(1 + \frac{k}{n}\right) n^a$$

for all $n \geq N_{\lambda, \varepsilon}^{(2)}$. The lower bound $a > -1/2$ implies that $v \notin l^2(\mathbb{Z}_+)$, and hence that $\lambda$ is not an eigenvalue of $A_+$.

Hypothesis  From this point onwards we assume that $0 < \varepsilon < 2$ and $\lambda \geq 0$.

Theorem 5  For every $\delta > 0$ there exists $N = N_{\lambda, \varepsilon, \delta}$ and a solution $v$ of (1) such that

$$n^a \leq v_n \leq (1 + \delta)n^a$$

for all $n \geq N$, where $a = -1 + 1/\varepsilon$. 

4
We put \( N = N_{\lambda,\varepsilon,\delta} = \max\{N_{\lambda,\varepsilon,1}^{(1)}, N_{\lambda,\varepsilon,1}^{(2)} + k/\delta\} \) where \( k = 1 + \lambda/\varepsilon \) and let \( v \) be the solution of (1) such that \( v_{N-i} = (1+\delta)(N-i)^a \) for \( i = 1, 2 \). Lemma 1 implies that \( 0 < v_n \leq (1+\delta)n^a \) for all \( n \geq N \). Since
\[
v_n \geq \left(1 + \frac{k}{n}\right)n^a
\]
for \( n = N - 1 \) and \( n = N - 2 \), we deduce by Lemma 3 that \( v_n \geq (1+k/n)n^a \geq n^a \) for all \( n \geq N \). This completes the proof.

We will show that, up to a multiplicative constant, there is exactly one ‘subordinate’ solution \( v \) of (1) such that \( \lim_{n \to +\infty} v_n = 0 \). We identify this solution by solving the recurrence relation backwards from \( n = M \) and then letting \( M \to +\infty \).

**Lemma 6** There exists \( N = N_{\lambda,\varepsilon}^{(3)} \) such that if \( M > N \) and \( v_n = (-1)^nw_n \) is a solution of (1) satisfying \( 0 < w_{M+i} \leq (M+i)^{-c} \) for \( i = 1, 2 \) where \( c = 1 + 1/\varepsilon \), then \( 0 < w_n \leq n^{-c} \) for all \( n \) satisfying \( N \leq n \leq M \).

**Proof** The sequence \( w_n \) satisfies the recurrence relation
\[
w_n = \frac{n+2}{n}w_{n+2} + \frac{2(n+1-\lambda)}{\varepsilon n(n+1)}w_{n+1}.
\]
This has positive coefficients for \( n \geq \lambda \) so the solution is positive if \( \lambda < n \leq M \). Suppose inductively that \( 0 < w_{n+2} \leq (n+2)^{-c} \) and \( 0 < w_{n+1} \leq (n+1)^{-c} \) for such an \( n \). Then
\[
n^c w_n \leq \left(1 + \frac{2}{n}\right)^{1-c} + \frac{2(n+1-\lambda)}{\varepsilon n(n+1)} \left(1 + \frac{1}{n}\right)^{-c}
\]
\[
= 1 - \frac{2\lambda}{\varepsilon n^2} + O(n^{-3})
\]
\[
\leq 1
\]
for all large enough \( n \). By induction there exists \( N = N_{\lambda,\varepsilon} \) such that \( 0 < w_n \leq n^{-c} \) provided \( N \leq n \leq M \).

**Lemma 7** There exists \( N = N_{\lambda,\varepsilon}^{(4)} \) such that if \( M > N \) and \( v_n = (-1)^nw_n \) is a solution of (1) such that
\[
w_n \geq \left(1 - \frac{h}{n}\right)n^{-c}
\]
for \( n = M + 1 \) and \( n = M + 2 \), where \( c = 1 + 1/\varepsilon \) and \( h = 1 + \lambda/\varepsilon \), then (4) holds for all \( n \) satisfying \( N \leq n \leq M \).

**Proof** Suppose that \( \max\{h, \lambda\} \leq n \leq M \) and (4) holds when \( n \) is replaced by \( n+1 \) or \( n+2 \). Then
\[
n^c w_n \geq \left(1 - \frac{h}{n+2}\right) \left(1 + \frac{2}{n}\right)^{1-c} + \frac{2n+1-\lambda}{\varepsilon n(n+1)} \left(1 - \frac{h}{n+1}\right) \left(1 + \frac{1}{n}\right)^{-c}
\]
\begin{align*}
&= 1 - \frac{h}{n} + \frac{2}{n^2} + O(n^{-3}) \\
&\geq 1 - \frac{h}{n}
\end{align*}

provided \( n \) is large enough. An induction now implies that there exists \( N = N^{(4)}_{\lambda, \varepsilon} \) such that (4) holds for all \( n \) such that \( N \leq n \leq M \).

**Theorem 8** There exists \( N = N^{(5)}_{\lambda, \varepsilon} \) and a unique solution \( v_n = (-1)^n w_n \) of (7) such that

\[
\left( 1 - \frac{h}{n} \right) n^{-c} \leq w_n \leq n^{-c}
\]

for all \( n \geq N \), where \( c = 1 + 1/\varepsilon \) and \( h = 1 + \lambda/\varepsilon \). Hence

\[
\lim_{n \to +\infty} w_n n^c = 1.
\] (5)

**Proof** Let \( M > N = N^{(5)}_{\lambda, \varepsilon} = \max\{N^{(3)}_{\lambda, \varepsilon}, N^{(4)}_{\lambda, \varepsilon}\} \) and let \( w^{(M)} \) denote the solution of (3) such that \( w^{(M)}_n = n^{-c} \) for \( n = M + 1 \) and \( n = M + 2 \). Lemmas 6 and 7 imply that

\[
\left( 1 - \frac{h}{n} \right) n^{-c} \leq w^{(M)}_n \leq n^{-c}
\]

for all \( n \) such that \( N \leq n \leq M \). By choosing a sequence \( M_r \to +\infty \) such that \( w^{(M_r)}_N \) and \( w^{(M_r)}_{N+1} \) converge as \( r \to +\infty \) we see using (3) that \( w^{(M_r)}_n \) converge for all \( n \geq 1 \). Denoting the limit by \( w_n^{(\infty)} \) we deduce that

\[
\left( 1 - \frac{h}{n} \right) n^{-c} \leq w^{(\infty)}_n \leq n^{-c}
\]

for all \( n \geq N \). Putting \( v_n^{(\infty)} = (-1)^n w^{(\infty)}_n \), the uniqueness of the solution \( v^{(\infty)} \) subject to the normalization condition (5) follows from the fact that the solution space of (1) is two-dimensional and it contains a divergent sequence by Theorem 5.

Numerical examples suggest that the following lemma is not the best possible and that \( w \) takes its maximum value very close to \( n = \lambda \). Figure 1 plots the eigenfunction \( v \) of the operator \( A_+ \) for the eigenvalue \( \lambda \sim 14.94784 \) with \( \varepsilon = 0.1 \).
Lemma 9 If $\lambda \geq 0$ then the unique subordinate solution $w$ of (3) satisfies $0 < w_{n+1} < w_n$ for all $n \geq 2\lambda$.

Proof Let $w^{(M)}$ denote the solution of (3) constructed in the proof of Theorem 8. Then

$$w^{(M)}_M = \frac{M + 2}{M} (M + 2)^{-c} + \frac{2(M + 1 - \lambda)}{\varepsilon M(M + 1)} (M + 1)^{-c}$$

$$= (M + 1)^{-c} \left( (1 + 2/M)^{1-c}(1 + 1/M)^c + \frac{2}{\varepsilon M}(1 - \lambda/(M + 1)) \right)$$

$$\geq (M + 1)^{-c}$$

provided $M$ is large enough. Therefore

$$w^{(M)}_M \geq w^{(M)}_{M+1} \geq w^{(M)}_{M+2}.$$ 

We prove inductively that $w^{(M)}_n \geq w^{(M)}_{n+1}$ for all $n$ such that $2\lambda \leq n \leq M$. If this holds with $n$ replaced by $n + 1$ or by $n + 2$ then

$$w^{(M)}_n - w^{(M)}_{n+1} = \frac{n + 2}{n} w^{(M)}_{n+2} + \frac{2(n + 1 - \lambda)}{\varepsilon n(n + 1)} w^{(M)}_{n+1}$$

$$- \frac{n + 3}{n + 1} w^{(M)}_{n+3} - \frac{2(n + 2 - \lambda)}{\varepsilon (n + 1)(n + 2)} w^{(M)}_{n+2}$$

$$= \frac{n + 2}{n} w^{(M)}_{n+2} - \frac{n + 3}{n + 1} w^{(M)}_{n+3}$$

$$+ \frac{2(n + 1 - \lambda)}{\varepsilon n(n + 1)} w^{(M)}_{n+1} - \frac{2(n + 2 - \lambda)}{\varepsilon (n + 1)(n + 2)} w^{(M)}_{n+2}$$

$$\geq \left( \frac{n + 2}{n} - \frac{n + 3}{n + 1} \right) w^{(M)}_{n+3}$$

Figure 1. Eigenvector $v$ for $\varepsilon = 0.1$ and $\lambda \sim 14.94784$
provided \( n \geq 2\lambda \). This completes the induction.

Finally we take the same sequence \( M(r) \) as in the proof of Theorem 8 to obtain \( 0 < w^{(\infty)}_{n+1} \leq w^{(\infty)}_n \) for all \( n \geq 2\lambda \).

### 3 Compactness of the Resolvent

In this section we prove that \( 0 \notin \text{Spec}(A_+) \) and that \( A^{-1}_+ \) is a Hilbert-Schmidt operator, and hence compact. This implies that the spectrum of \( A_+ \) is discrete and coincides with its set of eigenvalues. We cannot, however, prove that the spectrum is real. We define the Hilbert-Schmidt operator \( R \) on \( l^2(\mathbb{Z}_+) \) by

\[
(Rf)_m = \sum_{n=1}^{\infty} \rho_{m,n} f_n
\]

where \( \rho \in l^2(\mathbb{Z}_+ \times \mathbb{Z}_+) \) is given explicitly. We then show directly that \( R \) is the inverse of \( A_+ \).

Let \( \phi \) be the solution of

\[
\phi_n = \frac{n-2}{n} \phi_{n-2} + \frac{2}{\varepsilon n} \phi_{n-1}
\]

that satisfies the initial conditions \( \phi_1 = 1 \) and \( \phi_2 = \varepsilon^{-1} \). One sees immediately that \( \phi_n > 0 \) for all \( n \geq 1 \). Theorem 5 implies that there exists a constant \( c_1 > 0 \) such that

\[
c_1^{-1} n^a \leq \phi_n \leq c_1 n^a
\]

for all \( n \geq 1 \).

Let \( \psi_n = (-1)^n w_n \) be the unique subordinate solution of

\[
\psi_n = \frac{n-2}{n} \psi_{n-2} + \frac{2}{\varepsilon n} \psi_{n-1}
\]

such that \( w \) satisfies the asymptotic condition \( \lim_{n \to +\infty} n^c w_n = 1 \). Since

\[
w_n = \frac{n+2}{n} w_{n+2} + \frac{2}{\varepsilon n} w_{n+1}
\]

we see that \( w_n > 0 \) for all \( n \geq 1 \), and indeed that there exists a constant \( c_2 > 0 \) such that

\[
c_2^{-1} n^{-c} \leq w_n \leq c_2 n^{-c}
\]

for all \( n \geq 1 \).
We finally put
\[ \sigma_n = \frac{\varepsilon}{2} n(n-1) \phi_{n-1} w_n + \frac{\varepsilon}{2} n(n+1) \phi_n w_{n+1} + n \phi_n w_n \]
and observe that \( \sigma_n > 0 \) for all \( n \geq 1 \). The upper and lower bounds on \( \phi \) and \( w \) imply that there exists a constant \( c_3 > 0 \) such that
\[ c_3^{-1} n^{a-c+2} \leq \sigma_n \leq c_3 n^{a-c+2} \]
for all \( n \geq 1 \).

Theorem 10 If \( 0 < \varepsilon < 2 \) and
\[ \rho_{m,n} = \begin{cases} (-1)^n \phi_m \psi_n / \sigma_n & \text{if } m \leq n, \\ (-1)^n \psi_m \phi_n / \sigma_n & \text{if } m > n. \end{cases} \]
then \( \rho \in l^2(\mathbb{Z}_+ \times \mathbb{Z}_+) \). The Hilbert-Schmidt operator \( R \) defined by (6) satisfies \( A_+ R f = f \) for all \( f \in l^2(\mathbb{Z}_+) \). Indeed \( 0 \notin \text{Spec}(A_+) \) and \( R = A_+^{-1} \).

Proof The above bounds on \( \phi, \psi, \sigma \) imply that
\[ |\rho_{m,n}| \leq \begin{cases} c_4 m a^{n-a-2} & \text{if } m \leq n, \\ c_4 m^{-c} n^{c-2} & \text{if } m > n. \end{cases} \]
It follows that
\[ \sum_{m=1}^{\infty} |\rho_{m,n}|^2 \leq c_5 n^{-3} \]
and then that
\[ \sum_{m,n=1}^{\infty} |\rho_{m,n}|^2 < \infty. \]
We conclude that \( R \) is a compact operator. If \( \{e_n\}_{n=1}^{\infty} \) is the standard basis in \( l^2(\mathbb{Z}_+) \) then a direct calculation shows that \( A_+ R e_n = e_n \) for all \( n \). By using the fact that \( A_+ \) is closed one deduces that \( \text{Ran}(R) \subseteq \text{Dom}(A_+) \) and that \( A_+ R f = f \) for all \( f \in l^2(\mathbb{Z}_+) \). We conclude from this that \( \text{Ran}(A_+) = l^2(\mathbb{Z}_+) \). The bound \( 0 < \varepsilon < 2 \) implies that \( \text{Ker}(A_+) = \{0\} \) by Theorem [4] so we finally see that \( 0 \notin \text{Spec}(A_+) \) and that \( R = A_+^{-1} \).

4 \( \lambda \)-Dependence

In this section we prove that the unique normalized subordinate solution \( v_{\lambda,n} = (-1)^n w_{\lambda,n} \) of (1) provided by Theorem [8] depends continuously on \( \lambda \).

We first observe that for any \( \Lambda \geq 1 \) the various constants \( N^{(i)}_{\lambda, \varepsilon} \) are uniformly bounded with respect to \( \lambda \) provided \( 0 \leq \lambda \leq \Lambda \). We (incorrectly) use the notation \( N^{(i)}_{\lambda, \varepsilon} \) to refer to the relevant upper bounds.
Lemma 11 If \(0 \leq \lambda \leq \mu \leq \Lambda\) then
\[
0 < w_{\Lambda,n} \leq w_{\mu,n} \leq w_{\lambda,n} \leq w_{0,n} < \infty
\]
for all \(n \geq \Lambda\).

Proof The positivity of \(w_{\lambda,n}\) for \(n \geq \Lambda\) follows from the positivity of the coefficients of (3) for \(n \geq \Lambda\) and the positivity of \(w_{\lambda,n}\) for all \(n \geq N = N^{(5)}_{\Lambda,\varepsilon}\). We only need only prove the central inequality above since the other two are special cases of it.

Theorem 8 implies that if \(\delta > 0\) then
\[
w_{\mu,n} \leq (1 + \delta)w_{\lambda,n}
\]
for all \(n \geq N = N^{(6)}_{\Lambda,\varepsilon,\delta}\). This inequality persists for all \(n \in [\Lambda, N]\) by the monotonicity of the coefficients of (3). Since (7) holds for all \(\delta > 0\) and all \(n \geq \Lambda\), the required inequality follows by letting \(\delta \to 0\).

Lemma 12 If \(0 \leq \lambda \leq \mu \leq \Lambda\) and \(|\mu - \lambda| \leq \delta\) then
\[
0 < w_{\lambda,n} \leq p_{\Lambda,\varepsilon,n,\delta}w_{\mu,n}
\]
for all \(n \geq 2\Lambda\), where
\[
p_{\Lambda,\varepsilon,n,\delta} = (1 + \delta)\exp \left\{ 2\delta^{-1} \sum_{r=n}^{\infty} r^{-2} \right\}.
\]

Proof Since \(1 + \delta \leq p_{\Lambda,\varepsilon,n,\delta}\), Theorem 8 implies that (8) holds for all \(n \geq N = N^{(6)}_{\Lambda,\varepsilon,\delta}\). We prove inductively that the same inequality persists for \(n \in [2\Lambda, N]\). If (8) holds with \(n\) replaced by \(n + 1\) and by \(n + 2\), then, using Lemma 9, we obtain
\[
w_{\lambda,n} = \frac{n + 2}{n} w_{\lambda,n+2} + \frac{2(n + 1 - \lambda)}{\varepsilon n(n + 1)} w_{\lambda,n+1}
\]
\[
\leq \frac{n + 2}{n} p_{\Lambda,\varepsilon,n+2,\delta} w_{\mu,n+2} + \frac{2(n + 1 - \lambda)}{\varepsilon n(n + 1)} p_{\Lambda,\varepsilon,n+1,\delta} w_{\mu,n+1}
\]
\[
\leq p_{\Lambda,\varepsilon,n+1,\delta} \left( \frac{n + 2}{n} w_{\mu,n+2} + \frac{2(n + 1 - \lambda)}{\varepsilon n(n + 1)} w_{\mu,n+1} \right)
\]
\[
\leq p_{\Lambda,\varepsilon,n+1,\delta} \left( \frac{n + 2}{n} w_{\mu,n+2} + \frac{2(n + 1 - \mu)}{\varepsilon n(n + 1)} w_{\mu,n+1} + \frac{2\delta}{\varepsilon n^2} w_{\mu,n+1} \right)
\]
\[
\leq p_{\Lambda,\varepsilon,n+1,\delta} \left( \frac{n + 2}{n} w_{\mu,n} + \frac{2\delta}{\varepsilon n^2} w_{\mu,n} \right)
\]
\[
\leq p_{\Lambda,\varepsilon,n,\delta} w_{\mu,n}.
\]
This completes the induction.
Theorem 13. The subordinate solution \( v_\lambda \) depends continuously on \( \lambda \) for \( 0 \leq \lambda < \infty \). Hence the function
\[
f(\lambda) := \varepsilon v_{\lambda,2} - (1 - \lambda)v_{\lambda,1}
\]
is continuous on \([0, \infty)\).

Proof. It is sufficient to prove that \( f \) is continuous on \([0, \Lambda]\) for every positive integer \( \Lambda \). It follows directly from the estimates in Lemmas 11 and 12 that the map \( \lambda \in [0, \Lambda] \rightarrow (w_{\lambda,2\Lambda}, w_{\lambda,2\Lambda+1}) \) is continuous. Composing this with the linear (and therefore continuous) map \((w_{\lambda,2\Lambda}, w_{\lambda,2\Lambda+1}) \rightarrow \varepsilon v_{\lambda,2} - (1 - \lambda)v_{\lambda,1}\) yields the second statement of the theorem.

5 Numerical Calculations

Let \( v_\lambda \) denote the solution of (1) such that
\[
\lim_{n \to +\infty} v_{\lambda,n}(-1)^n n^c = 1.
\]

Then \( \lambda > 0 \) is an eigenvalue if and only if
\[
f(\lambda) := \varepsilon v_{\lambda,2} - (1 - \lambda)v_{\lambda,1}
\]
vanishes. Since this function is continuous, one can compute the roots of \( f(\lambda) = 0 \) by evaluating \( f(\lambda) \) numerically for a range of values of \( \lambda \). We determined the subordinate solution by solving (3), starting from \( M = 4000 \) (and also \( M = 8000 \) to check consistency) with \( w_{M+i} = (M+i)^{-c} \) for \( i = 1, 2 \). Figure 2 plots \( f(\lambda) \) for \( \varepsilon = 0.1 \) and \( 0 \leq \lambda \leq 4 \). The eigenvalues listed in Table 1 were obtained by solving \( f(\lambda) = 0 \) numerically, and are quite close to those obtained in [2].

\[
\begin{array}{ccc}
n & \lambda_n & \|P_n\| \\
1 & 1.00968 & 1.0189 \\
2 & 2.07334 & 1.1848 \\
3 & 3.22978 & 1.8868 \\
4 & 4.50134 & 4.3409 \\
5 & 5.89993 & 13.341 \\
6 & 7.43194 & 50.638 \\
7 & 9.10097 & 226.20 \\
8 & 10.9092 & 1152.9 \\
9 & 12.8578 & 6561.3 \\
10 & 14.9478 & 41018 \\
15 & 27.5331 & - \\
20 & 43.74 & - \\
\end{array}
\]

Table 1. Eigenvalues of \( A_+ \) for \( \varepsilon = 0.1 \).
The computation is very stable and one can confidently evaluate the first ten eigenvalues to much higher accuracy. The list of eigenvalues found is compatible with the asymptotic formula \( \lambda_n \sim \alpha n^\gamma \) where \( \alpha \sim 0.53 \) and \( \gamma \sim 1.44 \).

However, for \( \varepsilon = 1 \), the Fourier coefficients decrease much more slowly, and the eigenvalue calculation is correspondingly more onerous. We computed the first five eigenvalues for \( \varepsilon = 1 \), determining the subordinate solution as before with \( M \) between 1000 and 32000. The apparent numbers of eigenvalues increased from 7 to 11 as \( M \) increased in this range. For \( M = 4000 \) it appeared that the computation of the first five eigenvalues presented in Table 2 was reliable.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \lambda_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.4485</td>
</tr>
<tr>
<td>2</td>
<td>4.3159</td>
</tr>
<tr>
<td>3</td>
<td>8.6219</td>
</tr>
<tr>
<td>4</td>
<td>14.3638</td>
</tr>
<tr>
<td>5</td>
<td>21.5414</td>
</tr>
</tbody>
</table>

Table 2. Eigenvalues of \( A_+ \) for \( \varepsilon = 1 \)

![Figure 2. \( f(\lambda) \) for \( \varepsilon = 0.1 \) and \( 0 \leq \lambda \leq 4 \)](image)

We conclude with some comments about the conjecture in [2] that the eigenvectors form a basis. It seems quite plausible that they form a complete set in the sense that their linear span is dense. However, if they form a basis then the spectral projections

\[
P_n f = \frac{\langle f, \phi_n^* \rangle}{\langle \phi_n, \phi_n^* \rangle} \phi_n
\]
of $H$ must be uniformly bounded in norm, where $\phi_n$ are the eigenfunctions of $H$ and $\phi^*_n$ the corresponding eigenfunctions of $H^*$; see [4, Lemma 3.3.3]. However, it appears from [2, Figure 4] that the eigenfunctions $\phi_n$ concentrate more and more strongly around $\theta = \pi$ as $n$ increases; the eigenfunctions $\phi^*_n$ should concentrate around $\theta = 0$ as $n \to \infty$ for similar reasons. If this is indeed the case then the norms of the spectral projections

$$\|P_n\| = \|\phi_n\| \|\phi^*_n\|/|\langle \phi_n, \phi^*_n \rangle|$$

must diverge as $n \to \infty$ and the eigenfunctions do not form a basis. The norms of the first 10 spectral projections are presented in Table 1 and confirm the conjecture that they diverge as $n$ increases. See [5] for another highly non-self-adjoint operator arising in physics for which an apparently well-behaved sequence of eigenfunctions do not form a basis.

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**References**


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