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Fuzzy Guaranteed Cost Output Tracking Control for Fuzzy Discrete-Time Systems with Different Premise Variables

Di Liu, Chengwei Wu, Hak-Keung Lam and Qi Zhou

Abstract

This paper investigates the problem of output tracking control for a class of discrete-time interval type-2 (IT2) fuzzy systems subject to mismatched premise variables. Based on the IT2 T-S fuzzy model, the criterion to design the desired controller is obtained, which guarantees the closed-loop system to be asymptotically stable and satisfies the predefined cost function. Moreover, the controller to be designed does not need to share the same premise variables of the system, which enhances the flexibility of controller design and reduces the conservativeness. Finally, a numerical example is provided to demonstrate the effectiveness of the method proposed in this paper.

Keywords: Interval type-2; T-S fuzzy model; Output tracking control; Guaranteed cost control

I. INTRODUCTION

It is well known that the Takagi-Sugeno (T-S) fuzzy-model-based (FMB) model [1] can deal with the complicated nonlinear systems, in which nonlinear systems are obtained via a “blending” of every linear

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sub-system with membership functions [2]–[9]. With the T-S FMB approach, considerable yet remarkable results on stability analysis and controller synthesis for nonlinear systems have been reported, see for example, [6], [10]–[20] and the references therein. To mention a few, in [20], the authors investigated the sampled-data H_∞ control problem for uncertain active suspension systems through T-S fuzzy control approach; the H_∞ model approximation for discrete-time T-S fuzzy time-delay systems was considered in [19]; the stability and stabilization problems for switched systems with delays were studied in [18].

The output tracking control is an active research field due to it has a variety of applications in practical systems. And the main idea of output tracking control is that the outputs of the given reference model and the plant are as close as possible, that is, the output of the plant can track that of the reference model. Some results have been published in open literatures [8], [21]–[25]. The authors in [26] developed an observer-based fuzzy controller to guarantee the reference tracking performance through the T-S fuzzy model; output tracking control for nonlinear time-delay systems was researched in [8]; a new output-feedback polynomial fuzzy controller was designed to satisfy the tracking performance in [27].

However, it should be mentioned that the above output tracking control results are on the basis of the type-1 T-S fuzzy model approach. There exist not only the nonlinearity but also parameter uncertainties due to [the complex environment often changes in industrial process](#). The type-1 T-S fuzzy model can handle the nonlinearity effectively rather than the parameter uncertainties. However, parameter uncertainties may become the resource of instability, which may degrade the system performance. To cope with the parameter uncertainties, the interval type-2 (IT2) fuzzy set [28], [29] was proposed on the basis of the type-2 fuzzy set theory. [On the strength of IT2 fuzzy set theory](#), the IT2 fuzzy model has been developed and some significant results have been reported in [30]–[36]. In [32], the authors proposed an IT2 T-S fuzzy model and achieved sufficient conditions to stabilize the system subject to parameter uncertainties. In [37], [the footprint of uncertainty was taken into consideration in more information of the uncertain systems](#). Moreover, the mismatched premise variables were considered when the IT2 controller was designed for complex systems in [36], which facilitates less conservativeness and more flexibility for controller design. However, to the authors' best knowledge, there are few results on the output tracking control results for the complex nonlinear systems with uncertainties describing by the IT2 FMB systems. Therefore, it is a very challenge to design a novel output tracking controller for IT2 fuzzy systems with the guaranteed cost performance index.

In this study, the guaranteed cost output tracking control problem for the discrete-time IT2 fuzzy system under imperfect premise matching is considered for the first time. [The main contributions of this paper can be summarized as follows: 1. The nonlinear systems subject to parameter uncertainties are modeled](#)

by the IT2 T-S fuzzy model approach, in which the lower and upper membership functions are introduced to represent and capture the uncertainties. 2. The premise variables of the controller to be designed do not share the same those of the system model, which makes the controller design more flexible. 3. The guaranteed cost and tracking control are first considered simultaneously. Finally, the availability of the proposed method is illustrated through a numerical example.

The paper is organized as follows. The IT2 T-S fuzzy output tracking control system and the IT2 state-feedback tracking controller are described in Section II. In Section III, an IT2 FMB state feedback output tracking controller is to be designed which can ensure the output of the given reference model is tracked by the one of the closed-system. Section IV shows a numerical example to illustrate the effectiveness of the proposed approach. Finally, Section V draws a conclusion for this paper.

Notation: The notation used throughout the paper is fairly standard. “ I ” and “ $0_{m \times n}$ ” stand for an identity matrix with appropriate dimension and $m \times n$ zero matrix, respectively. “ T ” represents the transpose. A real symmetric matrix $P > 0$ means that P is positive definite. $\mathbb{R}^n$ represents n -dimension Euclidean space. $\text{diag}\{\dots\}$ is used to stand for a block diagonal matrix. “ $*$ ” represents symmetric terms in a block matrix. Matrices, if they are not explicitly stated, are assumed to be compatible dimensions.

II. PROBLEM FORMULATION

Consider the following IT2 T-S fuzzy system with σ -rule :

Rule i : IF $f_1(k)$ is M_1^i , $\dots$, and $f_\alpha(k)$ is M_α^i , $\dots$, and $f_\delta(k)$ is M_δ^i , THEN

$$\begin{cases} x(k+1) = A_i x(k) + B_i u(k) + L_i w(k), \\ y(k) = C_i x(k) + D_i u(k). \end{cases} \quad (1)$$

where M_α^i represents an IT2 fuzzy set of i th rule according to the known function $f_\alpha(k)$ for $i = 1, 2, \dots, \sigma$ and $\alpha = 1, 2, \dots, \delta$; $x(k) \in \mathbb{R}^n$ represents the system state variable; $u(k) \in \mathbb{R}^q$ is the control input; $w(k) \in \mathbb{R}^l$ is assumed to be a disturbance input; $y(k) \in \mathbb{R}^m$ stands for the measured output. A_i, B_i, C_i, D_i and L_i are known real constant matrices with appropriate dimensions. The following interval sets stand for the emission intensity of the i -th rule:

$$\Phi_i(x(k)) = [\underline{\varphi}_i(x(k)), \overline{\varphi}_i(x(k))], i = 1, 2, \dots, \sigma; \quad (2)$$

where

$$\begin{aligned}
\underline{\varphi}_i(x(k)) &= \underline{\mu}_{F_1^i}(f_1(k)) \times \underline{\mu}_{F_2^i}(f_2(k)) \cdots \times \underline{\mu}_{F_\delta^i}(f_\delta(k)), \\
\overline{\varphi}_i(x(k)) &= \overline{\mu}_{F_1^i}(f_1(k)) \times \overline{\mu}_{F_2^i}(f_2(k)) \cdots \times \overline{\mu}_{F_\delta^i}(f_\delta(k)), \\
\overline{\mu}_{F_\alpha^i}(f_\alpha(k)) &\geq \underline{\mu}_{F_\alpha^i}(f_\alpha(k)) \geq 0, \quad \overline{\varphi}_i(x(k)) \geq \underline{\varphi}_i(x(k)) \geq 0, \\
1 &\geq \overline{\mu}_{F_\alpha^i}(f_\alpha(k)) \geq 0, \quad 1 \geq \underline{\mu}_{F_\alpha^i}(f_\alpha(k)) \geq 0,
\end{aligned}$$

$\underline{\varphi}_i(x(k))$ and $\overline{\varphi}_i(x(k))$ represent the lower grade of membership and the upper grade of membership, respectively. $\underline{\mu}_{F_\alpha^i}(f_\alpha(k))$ and $\overline{\mu}_{F_\alpha^i}(f_\alpha(k))$ represent the lower membership function and the upper membership function, respectively. Then the IT2 T-S fuzzy model from (1) can be written as:

$$\begin{cases} x(k+1) = \sum_{i=1}^{\sigma} \varphi_i(x(k)) (A_i x(k) + B_i u(k) + L_i w(k)), \\ y(k) = \sum_{i=1}^{\sigma} \varphi_i(x(k)) (C_i x(k) + D_i u(k)), \end{cases} \quad (3)$$

where for $i = 1, 2, \dots, \sigma$,

$$\begin{aligned}
\varphi_i(x(k)) &= \underline{\alpha}_i(x(k)) \underline{\varphi}_i(x(k)) + \overline{\alpha}_i(x(k)) \overline{\varphi}_i(x(k)) \geq 0, \quad \sum_{i=1}^{\sigma} \varphi_i(x(k)) = 1, \\
\underline{\alpha}_i(x(k)) &\in [0, 1], \quad \overline{\alpha}_i(x(k)) \in [0, 1], \quad 1 = \underline{\alpha}_i(x(k)) + \overline{\alpha}_i(x(k)),
\end{aligned}$$

where $\underline{\alpha}_i(x(k))$ and $\overline{\alpha}_i(x(k))$ rely on parameter uncertainties and not indispensable to know in the paper. $\varphi_i(x(k))$ is the grade of membership of the embedded membership functions.

In order to track the control system, the reference model is proposed as follows:

$$\begin{cases} x_r(k+1) = E x_r(k) + r(k), \\ y_r(k) = F x_r(k). \end{cases} \quad (4)$$

Define the tracking error as follow:

$$e(k) = y(k) - y_r(k). \quad (5)$$

Then, consider the fuzzy guaranteed cost state-feedback controller as follow:

Rule j : IF $g_1(k)$ is N_1^j , $\dots$, and $g_\beta(k)$ is N_β^j , $\dots$, and $g_z(k)$ is N_z^j , THEN

$$u(k) = K_j x(k) + K_{rj} x_r(k), \quad (6)$$

where N_z^j represents an IT2 fuzzy set of j th rule in the light of the known $g_z(k)$ for $j = 1, 2, \dots, \lambda$ and $\beta = 1, 2, \dots, \delta$, and K_j and K_{rj} are the controller gains. The following interval sets stand for the emission intensity of the j -th rule:

$$\Psi_j(x(k)) = [\underline{\psi}_j(x(k)), \overline{\psi}_j(x(k))], \quad j = 1, 2, \dots, \lambda, \quad (7)$$

where

$$\begin{aligned}
\underline{\psi}_j(x(k)) &= \underline{\mu}_{N_1^j}(g_1(k)) \times \underline{\mu}_{N_2^j}(g_2(k)) \cdots \times \underline{\mu}_{N_z^j}(g_z(k)), \\
\overline{\psi}_j(x(k)) &= \overline{\mu}_{N_1^j}(g_1(k)) \times \overline{\mu}_{N_2^j}(g_2(k)) \cdots \times \overline{\mu}_{N_z^j}(g_z(k)), \\
\overline{\mu}_{N_\beta^j}(g_\beta(k)) &\geq \underline{\mu}_{N_\beta^j}(g_\beta(k)) \geq 0, \quad \overline{\psi}_j(x(k)) \geq \underline{\psi}_j(x(k)) \geq 0, \\
1 &\geq \underline{\mu}_{N_\beta^j}(g_\beta(k)) \geq 0, \quad 1 \geq \overline{\mu}_{N_\beta^j}(g_\beta(k)) \geq 0,
\end{aligned}$$

in which $\underline{\mu}_{N_\beta^j}(g_\beta(k))$ stands for lower membership function and $\overline{\mu}_{N_\beta^j}(g_\beta(k))$ stands for the upper membership function. $\underline{\psi}_j(x(k))$ and $\overline{\psi}_j(x(k))$ represent the lower and the upper grade of membership, respectively. Then the IT2 T-S fuzzy controller in (6) can be written as:

$$u(k) = \sum_{j=1}^{\lambda} \psi_j(x(k)) [K_j x(k) + K_{rj} x_r(k)]. \quad (8)$$

where for $j = 1, 2, \dots, \lambda$,

$$\begin{aligned}
\psi_j(x(k)) &= \frac{\underline{\beta}_j(x(k)) \underline{\psi}_j(x(k)) + \overline{\beta}_j(x(k)) \overline{\psi}_j(x(k))}{\sum_{k=1}^{\lambda} (\underline{\beta}_k(x(k)) \underline{\psi}_k(x(k)) + \overline{\beta}_k(x(k)) \overline{\psi}_k(x(k)))} \geq 0, \quad \sum_{j=1}^{\lambda} \psi_j(x(k)) = 1, \\
\underline{\beta}_j(x(k)) &\in [0, 1], \quad \overline{\beta}_j(x(k)) \in [0, 1], \quad 1 = \underline{\beta}_j(x(k)) + \overline{\beta}_j(x(k)),
\end{aligned}$$

in which $\underline{\beta}_j(x(k))$ and $\overline{\beta}_j(x(k))$ are predefined nonlinear functions. $\psi_j(x(k))$ represents the grade of membership of the embedded membership functions.

The guaranteed cost function [38] can be defined as follows:

$$J = \sum_{k=0}^{\infty} [e^T(k) Q e(k) + u^T(k) R u(k)], \quad (9)$$

where $Q > 0$ and $R > 0$.

III. MAIN RESULTS

In this section, consider the equations in (3) and (8), the closed-loop system can be described as follows:

$$\begin{cases} x(k+1) = \sum_{i=1}^{\sigma} \sum_{j=1}^{\lambda} \varphi_i(x(k)) \psi_j(x(k)) [(A_i + B_i K_j) x(k) + B_i K_{rj} x_r(k) + L_i w(k)], \\ y(k) = \sum_{i=1}^{\sigma} \sum_{j=1}^{\lambda} \varphi_i(x(k)) \psi_j(x(k)) [(C_i + D_i K_j) x(k) + D_i K_{rj} x_r(k)]. \end{cases} \quad (10)$$

This paper studies the stability of IT2 fuzzy-model-based system. For the sake of analyzing the stability, Ψ_k represents the sub-state spaces, $k = 1, 2, \dots, \varrho$ and $\Psi = \cup_{k=1}^{\varrho} \Psi_k$. Then, on account of lower

membership function and upper membership function within the footprint is uncertain. We can consider the lower membership function and upper membership function as follows:

$$\begin{aligned}
\underline{h}_{ij}(x(k)) &= \sum_{k=1}^{\varrho} \sum_{i_1=1}^2 \sum_{i_2=1}^2 \cdots \sum_{i_n=1}^2 \prod_{r=1}^n v_{ri_r k}(x_r(k)) \underline{\epsilon}_{ij i_1 i_2 \cdots i_n k}, \\
\bar{h}_{ij}(x(k)) &= \sum_{k=1}^{\varrho} \sum_{i_1=1}^2 \sum_{i_2=1}^2 \cdots \sum_{i_n=1}^2 \prod_{r=1}^n v_{ri_r k}(x_r(k)) \bar{\epsilon}_{ij i_1 i_2 \cdots i_n k}, \\
1 &= \sum_{k=1}^{\varrho} \sum_{i_1=1}^2 \sum_{i_2=1}^2 \cdots \sum_{i_n=1}^2 \prod_{r=1}^n v_{ri_r k}(x_r(k)), \\
0 &\leq v_{ri_r k}(x_r(k)) \leq 1, \quad 0 \leq \underline{\epsilon}_{ij i_1 i_2 \cdots i_n k} \leq \bar{\epsilon}_{ij i_1 i_2 \cdots i_n k} \leq 1,
\end{aligned}$$

where $v_{r1k}(x_r(k)) + v_{r2k}(x_r(k)) = 1$, in which $r, \delta = 1, \dots, n$ and $k = 1, \dots, q$ and $\underline{\epsilon}_{ij i_1 i_2 \cdots i_n k}$ and $\bar{\epsilon}_{ij i_1 i_2 \cdots i_n k}$ are constant scalars; for $i_r, i_s = 1, 2, k = 1, 2, \dots, q$, and $x(k) \in \Psi_k$.

From (4) to (10), we define the augmented state vector $\varsigma(k) = [x^T(k) \ x_r^T(k)]^T$, the IT2 fuzzy closed-loop system can be rewritten as:

$$\begin{cases} \varsigma(k+1) = \sum_{i=1}^{\sigma} \sum_{j=1}^{\lambda} h_{ij}(x(k)) [\bar{A}_{ij} \varsigma(k) + \bar{r}(k)], \\ e(k) = \sum_{i=1}^{\sigma} \sum_{j=1}^{\lambda} h_{ij}(x(k)) H_i \varsigma(k), \end{cases} \quad (11)$$

where

$$\begin{aligned}
h_{ij}(x(k)) &\equiv \varphi_i(x(k)) \psi_j(x(k)) \\
&= \underline{\gamma}_{ij}(x(k)) \underline{h}_{ij}(x(k)) + \bar{\gamma}_{ij}(x(k)) \bar{h}_{ij}(x(k)), \forall i, j. \\
1 &= \sum_{i=1}^{\sigma} \sum_{j=1}^{\lambda} h_{ij}(x(k)), \quad 1 = \underline{\gamma}_{ij}(x(k)) + \bar{\gamma}_{ij}(x(k)), \quad (12)
\end{aligned}$$

$$A_{ij} = A_i + B_i K_j, \quad H_i = [C_i + D_i K_j \quad -F + D_i K_{rj}], \quad U_j = [K_j \quad K_{rj}], \quad (13)$$

$$0 \leq \underline{\gamma}_{ij}(x(k)) \leq \bar{\gamma}_{ij}(x(k)) \leq 1, \quad \forall i, j.$$

$$\bar{A}_{ij} = \begin{bmatrix} A_{ij} & B_i K_{rj} \\ 0 & E \end{bmatrix}, \quad \bar{r}(k) = \begin{bmatrix} L_i & 0 \\ 0 & I \end{bmatrix} [w^T(k) \quad r^T(k)]^T, \quad (14)$$

in which the $\underline{\gamma}_{ij}(x(k))$ and $\bar{\gamma}_{ij}(x(k))$ are two unknown functions.

For brevity, $\underline{\varphi}_i(x(k))$, $\bar{\varphi}_i(x(k))$, $\varphi_i(x(k))$, $\underline{\psi}_j(x(k))$, $\bar{\psi}_j(x(k))$, $\psi_j(x(k))$, $\underline{h}_{ij}(x(k))$, $\bar{h}_{ij}(x(k))$, $h_{ij}(x(k))$, $\underline{\gamma}_{ij}(x(k))$, $\bar{\gamma}_{ij}(x(k))$, $\epsilon_{ij i_1 i_2 \cdots i_n k}$, $\bar{\epsilon}_{ij i_1 i_2 \cdots i_n k}$ are denoted as $\underline{\varphi}_i$, $\bar{\varphi}_i$, φ_i , $\underline{\psi}_j$, $\bar{\psi}_j$, ψ_j , $\underline{h}_{ij}$, $\bar{h}_{ij}$,

$h_{ij}, \underline{\gamma}_{ij}, \bar{\gamma}_{ij}, \underline{\epsilon}, \bar{\epsilon}$, respectively. Moreover, from the details in the above content, we have $\sum_{i=1}^{\sigma} \varphi_i = \sum_{j=1}^{\lambda} \psi_j = \sum_{i=1}^{\sigma} \sum_{j=1}^{\lambda} \varphi_i \psi_j = \sum_{i=1}^{\sigma} \sum_{j=1}^{\lambda} h_{ij} = 1$.

Theorem 1: Consider the discrete-time IT2 fuzzy system (11) under imperfect premise matching and cost function (9). If there exist matrices $P > 0$, $Q > 0$, $R > 0$, $V_{ij} > 0$, $G_{ij} > 0$, $N_{ij} > 0$, $W_{ij} > 0$ and M with appropriate dimensions, for $i = 1, 2, \dots, \sigma; j = 1, 2, \dots, \lambda$ satisfying the following conditions:

$$\Theta_{ij} - \Xi_{ij} + \Pi_{ij} - \bar{A}_{ij}^T V_{ij} \bar{A}_{ij} + W_{ij} + M > 0, \forall i, j, \quad (15)$$

$$\sum_{i=1}^{\sigma} \sum_{j=1}^{\lambda} [\bar{\epsilon}(\Theta_{ij} + \Pi_{ij}) + (\bar{\epsilon} - \underline{\epsilon})W_{ij} + (\bar{\epsilon} - \frac{1}{\sigma\lambda})M] < 0, \forall i, j, \quad (16)$$

where

$$\begin{aligned} \Theta_{ij} &= \bar{A}_{ij}^T P \bar{A}_{ij} - P, \quad \Xi_{ij} = H_i^T G_{ij} H_i + U_j^T N_{ij} U_j, \\ \Pi_{ij} &= H_i^T Q H_i + U_j^T R U_j, \quad V_{ij} = \text{diag}\{V_{1ij}, V_{2ij}\}. \end{aligned} \quad (17)$$

Then, the closed-loop system is asymptotically stable and the cost function (9) satisfies the bound

$$J \leq J_0,$$

where $J_0 = \zeta^T(0)P\zeta(0) + \sum_{k=0}^{\infty} \bar{r}^T(k)P\bar{r}(k)$.

Proof: Firstly, consider a Lyapunov function for system (11) as follows:

$$V(k) = \zeta^T(k) P \zeta(k). \quad (18)$$

By using the well-known upper bound, we can obtain as follows:

$$\begin{aligned} \Delta V(k) &= \zeta^T(k+1) P \zeta(k+1) - \zeta^T(k) P \zeta(k) \\ &= \sum_{i=1}^{\sigma} \sum_{j=1}^{\lambda} \sum_{i=1}^{\sigma} \sum_{j=1}^{\lambda} h_{ij} h_{ij} \zeta^T(k) \bar{A}_{ij}^T P \bar{A}_{ij} \zeta(k) - \zeta^T(k) P \zeta(k) + \bar{r}^T(k) P \bar{r}(k) \\ &\leq \sum_{i=1}^{\sigma} \sum_{j=1}^{\lambda} h_{ij} \zeta^T(k) \bar{A}_{ij}^T P \bar{A}_{ij} \zeta(k) - \zeta^T(k) P \zeta(k) + \bar{r}^T(k) P \bar{r}(k). \end{aligned} \quad (19)$$

$$\begin{aligned} e^T(k) Q e(k) + u^T(k) R u(k) &= \sum_{i=1}^{\sigma} \sum_{j=1}^{\lambda} \sum_{i=1}^{\sigma} \sum_{j=1}^{\lambda} h_{ij} h_{ij} [\zeta^T(k) (H_i^T Q H_i + U_j^T R U_j) \zeta(k)] \\ &\leq \sum_{i=1}^{\sigma} \sum_{j=1}^{\lambda} h_{ij} \zeta^T(k) (H_i^T Q H_i + U_j^T R U_j) \zeta(k). \end{aligned} \quad (20)$$

Due to the conditions in Theorem 1, V_{ij} , G_{ij} , N_{ij} , W_{ij} are positive definite matrices, we have

$$\begin{aligned}
\Delta V(k) &\leq \sum_{i=1}^{\sigma} \sum_{j=1}^{\lambda} \left(\gamma_{ij} \underline{h}_{ij} + (1 - \gamma_{ij}) \bar{h}_{ij} \right) \varsigma^T(k) \Theta_{ij} \varsigma(k) + \bar{r}^T(k) P \bar{r}(k) \\
&\quad - \sum_{i=1}^{\sigma} \sum_{j=1}^{\lambda} (1 - \gamma_{ij}) (\underline{h}_{ij} - \bar{h}_{ij}) \varsigma^T(k) W_{ij} \varsigma(k) \\
&\quad + \left[\sum_{i=1}^{\sigma} \sum_{j=1}^{\lambda} \left(\gamma_{ij} \underline{h}_{ij} + (1 - \gamma_{ij}) \bar{h}_{ij} \right) - 1 \right] \varsigma^T(k) M \varsigma(k) \\
&\quad - \sum_{i=1}^{\sigma} \sum_{j=1}^{\lambda} \gamma_{ij} (\underline{h}_{ij} - \bar{h}_{ij}) \varsigma^T(k) \bar{A}_{ij}^T V_{ij} \bar{A}_{ij} \varsigma(k) \\
&\quad - \sum_{i=1}^{\sigma} \sum_{j=1}^{\lambda} \gamma_{ij} (\underline{h}_{ij} - \bar{h}_{ij}) \varsigma^T(k) H_i^T G_{ij} H_i \varsigma(k) \\
&\quad - \sum_{i=1}^{\sigma} \sum_{j=1}^{\lambda} \gamma_{ij} (\underline{h}_{ij} - \bar{h}_{ij}) \varsigma^T(k) U_j^T N_{ij} U_j \varsigma(k), \\
&= \varsigma^T(k) \left[\sum_{i=1}^{\sigma} \sum_{j=1}^{\lambda} \gamma_{ij} (\underline{h}_{ij} - \bar{h}_{ij}) (\Theta_{ij} + W_{ij} + M - \Xi_{ij} - \bar{A}_{ij}^T V_{ij} \bar{A}_{ij}) \right] \varsigma(k) \\
&\quad + \varsigma^T(k) \left[\sum_{i=1}^{\sigma} \sum_{j=1}^{\lambda} (\bar{h}_{ij} \Theta_{ij} - (\underline{h}_{ij} - \bar{h}_{ij}) W_{ij} + \bar{h}_{ij} M) - M \right] \varsigma(k) \\
&\quad + \bar{r}^T(k) P \bar{r}(k).
\end{aligned}$$

Then, by using the conditions (15), (16) and (20) in Theorem 1, we can get

$$\Delta V(k) \leq -[e^T(k) Q e(k) + u^T(k) R u(k)] + \bar{r}^T(k) P \bar{r}(k).$$

In the above inequality, Q and R are positive definite matrices. Then, we can know $\Delta V(k) \leq \bar{r}^T(k) P \bar{r}(k)$ holds. The “input to state stability” condition (ISSC) [39] is satisfied for this inequality and the system is asymptotically stable.

Then, the cost function

$$\begin{aligned}
J &= \sum_{k=0}^{\infty} [e^T(k)Qe(k) + u^T(k)Ru(k)] \\
&\leq -\sum_{k=0}^{\infty} \Delta V(k) + \sum_{k=0}^{\infty} \bar{r}^T(k)P\bar{r}(k) \\
&= V(0) - V(\infty) + \sum_{k=0}^{\infty} \bar{r}^T(k)P\bar{r}(k) \\
&\leq V(0) + \sum_{k=0}^{\infty} \bar{r}^T(k)P\bar{r}(k) = J_0,
\end{aligned}$$

where $J_0 = \varsigma^T(0)P\varsigma(0) + \sum_{k=0}^{\infty} \bar{r}^T(k)P\bar{r}(k)$. The proof is completed. $\blacksquare$

Theorem 2: Given the system (11) and the cost function (9), if there exist matrix $P > 0$, $X > 0$, $Q > 0$, $R > 0$, $\bar{V}_{ij} > 0$, $G_{ij} > 0$, $N_{ij} > 0$, $\bar{W}_{ij} > 0$ and $\bar{M}$ with appropriate dimensions, for $i = 1, 2, \dots, \sigma; j = 1, 2, \dots, \lambda$ such that the following optimization problem:

$$\min \quad J = \alpha + \sum_{k=0}^{\infty} \beta_k I, \quad (21)$$

$$S.t \left\{ \begin{array}{l} \Omega_{ij} > 0, \forall i, j, \\ \Delta_{ij} < 0, \forall i, j, \\ G_{ij} - Q > 0, V_{ij} - P > 0, N_{ij} - R > 0, \forall i, j, \\ \begin{bmatrix} -\alpha & \varsigma^T(0) \\ * & -X \end{bmatrix} < 0, \\ \begin{bmatrix} -\beta_k I & \bar{r}^T(k) \\ * & -X \end{bmatrix} < 0, \end{array} \right. , \quad (22)$$

has a solution $(\alpha_{\min}, X_{\min}, J_{\min})$. Then the guaranteed cost function has a minimum upper bound and the optimal guaranteed cost output tracking controller is given as follows:

$$\begin{aligned}
u(k) &= \sum_{j=1}^{\lambda} \psi_j [K_j x(k) + K_{rj} x_r(k)] \\
&= \sum_{j=1}^{\lambda} \psi_j \bar{U}_j X_{\min}^{-1} \varsigma(k),
\end{aligned}$$

where

$$\Omega_{ij} = \begin{bmatrix} \Omega_{1ij} & \Omega_{2ij} & P_1 A_i^T + \bar{K}_j^T B_i^T & 0 & P_1 C_i^T + \bar{K}_j^T D_i^T & \bar{K}_j^T \\ * & \Omega_{3ij} & \bar{K}_{rj}^T B_i^T & P_2 E^T & -P_2 F^T + \bar{K}_{rj}^T D_i^T & \bar{K}_{rj}^T \\ * & * & 3P_1 - \bar{V}_{1ij} & 0 & 0 & 0 \\ * & * & * & 3P_2 - \bar{V}_{2ij} & 0 & 0 \\ * & * & * & * & 2I + Q - G_{ij} & 0 \\ * & * & * & * & * & 2I + R - N_{ij} \end{bmatrix},$$

$$\Delta_{ij} = \begin{bmatrix} \Delta_{1ij} & \Delta_{2ij} & \sqrt{\bar{\epsilon}} P_1 A_i^T + \sqrt{\bar{\epsilon}} \bar{K}_j^T B_i^T & 0 & \sqrt{\bar{\epsilon}} P_1 C_i^T Q^T + \sqrt{\bar{\epsilon}} \bar{K}_j^T D_i^T Q^T & \sqrt{\bar{\epsilon}} \bar{K}_j^T R^T \\ * & \Delta_{3ij} & \sqrt{\bar{\epsilon}} \bar{K}_{rj}^T B_i^T & \sqrt{\bar{\epsilon}} P_2 E^T & -\sqrt{\bar{\epsilon}} P_2 F^T Q^T + \sqrt{\bar{\epsilon}} \bar{K}_{rj}^T D_i^T Q^T & \sqrt{\bar{\epsilon}} \bar{K}_{rj}^T R^T \\ * & * & -P_1 & 0 & 0 & 0 \\ * & * & * & -P_2 & 0 & 0 \\ * & * & * & * & -Q & 0 \\ * & * & * & * & * & -R \end{bmatrix},$$

$$\begin{aligned} \Omega_{1ij} &= -P_1 + \bar{W}_{1ij} + \bar{M}_1, \quad \Omega_{2ij} = \bar{W}_{2ij} + \bar{M}_2, \quad \Omega_{3ij} = -P_2 + \bar{W}_{3ij} + \bar{M}_3, \\ \Delta_{1ij} &= -\bar{\epsilon} P_1 + (\bar{\epsilon} - \underline{\epsilon}) \bar{W}_{1ij} + (\bar{\epsilon} - \frac{1}{\sigma\lambda}) \bar{M}_1, \quad \Delta_{2ij} = (\bar{\epsilon} - \underline{\epsilon}) \bar{W}_{2ij} + (\bar{\epsilon} - \frac{1}{\sigma\lambda}) \bar{M}_2, \\ \Delta_{3ij} &= -\bar{\epsilon} P_2 + (\bar{\epsilon} - \underline{\epsilon}) \bar{W}_{3ij} + (\bar{\epsilon} - \frac{1}{\sigma\lambda}) \bar{M}_3, \quad P = \text{diag}\{P_1^{-1}, P_2^{-1}\}, \\ \bar{U}_j &= [\bar{K}_j \quad \bar{K}_{rj}], \quad X = P^{-1}, \quad K_j = \bar{K}_j P_1^{-1}, \quad K_{rj} = \bar{K}_{rj} P_2^{-1}. \end{aligned} \quad (23)$$

Proof: Firstly, for the condition $\Omega_{ij} > 0$ in (22), by using $2I + Q - G_{ij} < (G_{ij} - Q)^{-1} = -(Q - G_{ij})^{-1}$ and $2I + R - N_{ij} < (N_{ij} - R)^{-1} = -(R - N_{ij})^{-1}$, the following LMI hold

$$\bar{\Omega}_{ij} > \Omega_{ij} > 0,$$

where

$$\bar{\Omega}_{ij} = \begin{bmatrix} \Omega_{1ij} & \Omega_{2ij} & P_1 A_i^T + \bar{K}_j^T B_i^T & 0 & P_1 C_i^T + \bar{K}_j^T D_i^T & \bar{K}_j^T \\ * & \Omega_{3ij} & \bar{K}_{rj}^T B_i^T & P_2 E^T & -P_2 F^T + \bar{K}_{rj}^T D_i^T & \bar{K}_{rj}^T \\ * & * & 3P_1 - \bar{V}_{1ij} & 0 & 0 & 0 \\ * & * & * & 3P_2 - \bar{V}_{2ij} & 0 & 0 \\ * & * & * & * & -(Q - G_{ij})^{-1} & 0 \\ * & * & * & * & * & -(R - N_{ij})^{-1} \end{bmatrix}. \quad (24)$$

According to Schur complement [40], (13) and (23), the (24) is rewritten as

$$\bar{\Lambda} = \begin{bmatrix} \bar{\Lambda}_1 & \bar{\Lambda}_2 \\ * & \bar{\Lambda}_3 \end{bmatrix} > 0, \quad (25)$$

where

$$\begin{aligned} \bar{\Lambda}_1 &= -P^{-1} + P^{-T}(-\Xi_{ij} + \Pi_{ij} + W_{ij} + M)P^{-1} \\ &= \begin{bmatrix} -P_1 + \bar{W}_{1ij} + \bar{M}_1 & \bar{W}_{2ij} + \bar{M}_2 \\ * & -P_2 + \bar{W}_{3ij} + \bar{M}_3 \end{bmatrix} \\ &\quad + \begin{bmatrix} P_1 C_i^T + \bar{K}_j^T D_i^T \\ -P_2 F^T + \bar{K}_{rj}^T D_i^T \end{bmatrix} (Q - G_{ij}) \begin{bmatrix} C_i P_1 + D_i \bar{K}_j & -F P_2 + D_i \bar{K}_{rj} \end{bmatrix} \\ &\quad + \begin{bmatrix} \bar{K}_j^T \\ \bar{K}_{rj}^T \end{bmatrix} (R - N_{ij}) \begin{bmatrix} \bar{K}_j & \bar{K}_{rj} \end{bmatrix}, \\ \bar{\Lambda}_2 &= P^{-T} \bar{A}_{ij}^T = \begin{bmatrix} P_1 A_i^T + \bar{K}_j^T B_i^T & 0 \\ \bar{K}_{rj}^T B_i^T & P_2 E^T \end{bmatrix}, \\ \bar{\Lambda}_3 &= 3P^{-T} - P^{-T} V_{ij} P^{-1} = \begin{bmatrix} 3P_1 - \bar{V}_{1ij} & 0 \\ * & 3P_2 - \bar{V}_{2ij} \end{bmatrix}. \end{aligned}$$

Define the following nonsingular matrices:

$$W_{ij} = \begin{bmatrix} W_{1ij} & W_{2ij} \\ * & W_{3ij} \end{bmatrix}, \quad M = \begin{bmatrix} M_1 & M_2 \\ * & M_3 \end{bmatrix},$$

$$\bar{W}_{1ij} = P_1 W_{1ij} P_1, \quad \bar{W}_{2ij} = P_1 W_{2ij} P_2, \quad \bar{W}_{3ij} = P_2 W_{3ij} P_2,$$

$$\bar{M}_1 = P_1 M_1 P_1, \quad \bar{M}_2 = P_1 M_2 P_2, \quad \bar{M}_3 = P_2 M_3 P_2,$$

$$\bar{V}_{1ij} = P_1 V_{1ij} P_1, \quad \bar{V}_{2ij} = P_2 V_{2ij} P_2, \quad \bar{K}_j = K_j P_1, \quad \bar{K}_{rj} = K_{rj} P_2.$$

Now, per-and post-multiplying (25) by $\text{diag}\{P, P\}$ and its transpose, one can obtain

$$\Lambda = \begin{bmatrix} \Lambda_1 & \Lambda_2 \\ * & \Lambda_3 \end{bmatrix} > 0,$$

where

$$\Lambda_1 = -P - \Xi_{ij} + \Pi_{ij} + W_{ij} + M,$$

$$\Lambda_2 = \bar{A}_{ij}^T P, \quad \Lambda_3 = 3P - V_{ij}.$$

By using $P(V_{ij} - P)^{-1}P > 2P - (V_{ij} - P) = 3P - V_{ij}$, we have

$$\Lambda = \begin{bmatrix} \Lambda_1 & \Lambda_2 \\ * & P(V_{ij} - P)^{-1}P \end{bmatrix} > 0.$$

According to Schur complement, the inequality (15) in Theorem 1 is satisfied.

Similarly, the condition $\Delta_{ij} < 0$ in (22) is rewritten as

$$\bar{\Gamma} = \begin{bmatrix} \bar{\Gamma}_1 & \bar{\Gamma}_2 \\ * & \bar{\Gamma}_3 \end{bmatrix} < 0, \quad (26)$$

where

$$\begin{aligned} \bar{\Gamma}_1 &= -\bar{\epsilon}P^{-1} + \bar{\epsilon}P^{-T}\Pi_{ij}P^{-1} + (\bar{\epsilon} - \underline{\epsilon})\bar{W}_{ij} + (\bar{\epsilon} - \frac{1}{\sigma\lambda})\bar{M} \\ &= \begin{bmatrix} -\bar{\epsilon}P_1 + (\bar{\epsilon} - \underline{\epsilon})\bar{W}_{1ij} + (\bar{\epsilon} - \frac{1}{\sigma\lambda})\bar{M}_1 & (\bar{\epsilon} - \underline{\epsilon})\bar{W}_{2ij} + (\bar{\epsilon} - \frac{1}{\sigma\lambda})\bar{M}_2 \\ * & -\bar{\epsilon}P_2 + (\bar{\epsilon} - \underline{\epsilon})\bar{W}_{3ij} + (\bar{\epsilon} - \frac{1}{\sigma\lambda})\bar{M}_3 \end{bmatrix} \\ &\quad + \begin{bmatrix} \sqrt{\bar{\epsilon}}P_1C_i^TQ^T + \sqrt{\bar{\epsilon}}\bar{K}_j^TD_i^TQ^T \\ -\sqrt{\bar{\epsilon}}P_2F^TQ^T + \sqrt{\bar{\epsilon}}\bar{K}_{rj}^TD_i^TQ^T \end{bmatrix} Q^{-1} \begin{bmatrix} \sqrt{\bar{\epsilon}}QC_iP_1 + \sqrt{\bar{\epsilon}}QD_i\bar{K}_j & -\sqrt{\bar{\epsilon}}QFP_2 + \sqrt{\bar{\epsilon}}QD_i\bar{K}_{rj} \end{bmatrix} \\ &\quad + \begin{bmatrix} \sqrt{\bar{\epsilon}}\bar{K}_j^TR^T \\ \sqrt{\bar{\epsilon}}\bar{K}_{rj}^TR^T \end{bmatrix} R^{-1} \begin{bmatrix} \sqrt{\bar{\epsilon}}R\bar{K}_j & \sqrt{\bar{\epsilon}}R\bar{K}_{rj} \end{bmatrix}, \\ \bar{\Gamma}_2 &= \sqrt{\bar{\epsilon}}P^{-T}\bar{A}_{ij}^T = \begin{bmatrix} \sqrt{\bar{\epsilon}}P_1A_i^T + \sqrt{\bar{\epsilon}}\bar{K}_j^TB_i^T & 0 \\ \sqrt{\bar{\epsilon}}\bar{K}_{rj}^TB_i^T & \sqrt{\bar{\epsilon}}P_2E^T \end{bmatrix}, \\ \bar{\Gamma}_3 &= -P^{-1} = \begin{bmatrix} -P_1 & 0 \\ * & -P_2 \end{bmatrix}. \end{aligned}$$

Per-and post-multiply (26) by $\text{diag}\{P, P\}$ and its transpose, which yields

$$\Gamma = \begin{bmatrix} \Gamma_1 & \Gamma_2 \\ * & \Gamma_3 \end{bmatrix} < 0,$$

where

$$\begin{aligned} \Gamma_1 &= -\bar{\epsilon}P + \bar{\epsilon}\Pi_{ij} + (\bar{\epsilon} - \underline{\epsilon})W_{ij} + (\bar{\epsilon} - \frac{1}{\sigma\lambda})M, \\ \Gamma_2 &= \sqrt{\bar{\epsilon}}\bar{A}_{ij}^TP, \quad \Gamma_3 = -P. \end{aligned}$$

By means of Schur complement, the inequality (16) in Theorem 1 holds. Furthermore, the condition $\begin{bmatrix} -\beta_k I & \bar{r}^T(k) \\ * & -X \end{bmatrix} < 0$ is equivalent to $\bar{r}^T(k)X^{-1}\bar{r}(k) < \beta_k I$, and the condition $\begin{bmatrix} -\alpha & \varsigma^T(0) \\ * & -X \end{bmatrix} < 0$ is

equivalent to $\varsigma^T(0)X^{-1}\varsigma(0) < \alpha$. Therefore, $J = \alpha + \sum_{k=0}^{\infty} \beta_k I$ implies the minimum value of optimal guaranteed cost. This completes the proof. $\blacksquare$

IV. SIMULATION EXAMPLES

Example 1: In this section, a numerical example is used to illustrate the effectiveness of the control design method. Consider the following IT2 fuzzy model:

Rule i : If $x_1(k)$ is M_1^i ,

Then

$$\begin{cases} x(k+1) = A_i x(k) + B_i u(k) + L_i w(k), \\ y(k) = C_i x(k) + D_i u(k), i = 1, 2, 3, 4. \end{cases}$$

The reference model and reference input for the tracking control system are given as:

$$\begin{cases} x_r(k+1) = E x_r(k) + r(k), \\ y_r(k) = F x_r(k), \end{cases}$$

where

$$\begin{aligned} A_1 &= \begin{bmatrix} -6.5019 & 4.9999 \\ 0.3144 & -2.5095 \end{bmatrix}, A_2 = \begin{bmatrix} 0.0102 & 1.1097 \\ 0.10971 & 1.0109 \end{bmatrix}, A_3 = \begin{bmatrix} -3.0200 & 4.0201 \\ 3.9159 & -1.9101 \end{bmatrix}, \\ B_1 &= \begin{bmatrix} 5.0261 & 5.0443 \end{bmatrix}^T, B_2 = \begin{bmatrix} 5.0392 & 5.1251 \end{bmatrix}^T, B_3 = \begin{bmatrix} 5.0948 & 5.0741 \end{bmatrix}^T, \\ C_1 &= \begin{bmatrix} 22.5000 & -0.1094 \end{bmatrix}, C_2 = \begin{bmatrix} -0.6000 & -0.9836 \end{bmatrix}, C_3 = \begin{bmatrix} 6.0000 & -1.5050 \end{bmatrix}, \\ L_1 &= \begin{bmatrix} 3.0261 & 3.0443 \end{bmatrix}^T, L_2 = \begin{bmatrix} 3.0392 & 3.1251 \end{bmatrix}^T, L_3 = \begin{bmatrix} 3.0948 & 3.0741 \end{bmatrix}^T, \\ D_1 &= -15.0261, D_2 = -15.0392, D_3 = 15.0948, Q = 0.1, R = 0.1, \\ E &= -0.990, F = 3.2, r(k) = 0.04 \cos(3.24k - 3.24), w(k) = 0.04 \cos(3.24k - 3.24). \end{aligned}$$

Table. I and Table. II show the lower and upper membership functions of the plant and the controller. Let $\underline{\beta}_i = \bar{\beta}_i = 0.5$, $x_1 \in [-81, 81]$, $\bar{\alpha}_i = 1 - \underline{\alpha}_i$ and $\bar{\varphi}_i$ and $\underline{\varphi}_i$ are defined in Preliminaries. The state x_1 is divided into 19 equal-size sub-states (i.e., $k = 1, 2, \dots, 19$).

Define

$$\begin{aligned} \underline{\alpha}_1 &= \frac{1}{2} \sin^2 x_1, \quad \underline{\alpha}_3 = \frac{1}{2} \sin^2 x_1, \\ \underline{\alpha}_2 &= \frac{1}{\bar{\varphi}_2 - \underline{\varphi}_2} (-1 + (\bar{\varphi}_1 - \underline{\alpha}_1(\bar{\varphi}_1 - \underline{\varphi}_1) + \bar{\varphi}_3 - \underline{\alpha}_3(\bar{\varphi}_3 - \underline{\varphi}_3) + \bar{\varphi}_2)), \end{aligned}$$

$$v_{11k}(x_1) = 1 - \frac{x_1 - \underline{x}_{1,k}}{\bar{x}_{1,k} - \underline{x}_{1,k}}, \quad v_{12k}(x_1) = 1 - v_{11k}(x_1),$$

$$\underline{x}_{1,k} = \frac{162}{19}(k - 10), \quad \bar{x}_{1,k} = \frac{162}{19}(k - 9), \quad k = 1, 2, \dots, 19.$$

$$\underline{\epsilon}_{ij1k} = \underline{\varphi}_i(\underline{x}_{1,k})\underline{\psi}_j(\underline{x}_{1,k}), \quad \underline{\epsilon}_{ij2k} = \underline{\varphi}_i(\bar{x}_{1,k})\underline{\psi}_j(\bar{x}_{1,k}),$$

$$\bar{\epsilon}_{ij1k} = \bar{\varphi}_i(\underline{x}_{1,k})\bar{\psi}_j(\underline{x}_{1,k}), \quad \bar{\epsilon}_{ij2k} = \bar{\varphi}_i(\bar{x}_{1,k})\bar{\psi}_j(\bar{x}_{1,k}), \quad \text{for all } k.$$

TABLE I
LOWER AND UPPER MEMBERSHIP FUNCTIONS FOR THE PLANT

Lower membership functions for the plant	Upper membership functions for the plant
$\underline{\mu}_{F_1^1}(x_1) = 0.8 - (0.8/(1 + \exp(-(x_1 + 81)/14)))$	$\bar{\mu}_{F_1^1}(x_1) = 1 - (1/(1 + \exp(-(x_1 + 81)/14)))$
$\underline{\mu}_{F_1^3}(x_1) = 0.8/(1 + \exp(-(x_1 - 81)/14))$	$\bar{\mu}_{F_1^3}(x_1) = 1/(1 + \exp(-(x_1 - 81)/14))$
$\underline{\mu}_{F_1^2}(x_1) = 1 - \bar{\mu}_{F_1^1}(x_1) - \bar{\mu}_{F_1^3}(x_1)$	$\bar{\mu}_{F_1^2}(x_1) = 1 - \underline{\mu}_{F_1^1}(x_1) - \underline{\mu}_{F_1^3}(x_1)$

TABLE II
LOWER AND UPPER MEMBERSHIP FUNCTIONS FOR THE CONTROLLER

Lower membership functions for the controller	Upper membership functions for the controller
$\underline{\mu}_{N_1^1}(x_1) = \exp(-x_1^2/5000)$	$\bar{\mu}_{N_1^1}(x_1) = \exp(-x_1^2/0.5)$
$\underline{\mu}_{N_1^2}(x_1) = 1 - \bar{\mu}_{N_1^1}(x_1)$	$\bar{\mu}_{N_1^2}(x_1) = 1 - \underline{\mu}_{N_1^1}(x_1)$

For demonstration, the initial state $x_0 = [0.2; -0.2]$, $x_{r0} = 0.2$. The optimization problem in Theorem 2 is solved by means of MATLAB Control Toolbox when minimizing the cost function J , and we can obtain the upper bound of the minimum cost $J_0 = 13.100452$.

Applying Theorem 2, we can obtain the optimal reliable guaranteed cost controller $K_1 = [0.0001 \ -0.2031]$, $K_{r1} = -0.0076$ and $K_2 = [0.0579 \ -0.1804]$, $K_{r2} = -0.0062$. The state trajectories of the open-loop and closed-loop system are shown in Fig.1-2. It can be observed that the instable open-loop system becomes stable after the controller is designed for the system. Fig.3 and Fig.4 show the tracking error of the open-loop and closed-loop system, respectively. Fig. 5 plots the outputs of the closed-loop system and the reference model. It can be seen that the tracking performance of the designed closed-loop system performs well. Additionally, Fig.6 plots the control input.

Fig. 1. State response $x(k)$ of the open-loop system. Fig. 2. State response $x(k)$ of the closed-loop system. Fig. 3. $e(k)$ of the open-loop system. Fig. 4. $e(k)$ of the closed-loop system. Fig. 5. Outputs $y(k)$ and $y_r(k)$ of the closed-loop system. Fig. 6. The controlled input $u(k)$.

TABLE III
MEMBERSHIP FUNCTIONS WITH LOWER AND UPPER BOUNDS OF THE CONTROLLER

Lower bounds	Upper bounds
$\underline{u}_{N_{11}}(x_1) = \exp\left(-\frac{x_1^2}{0.5}\right)$	$\bar{u}_{N_{11}}(x_1) = \underline{u}_{N_{11}}(x_1)$
$\underline{u}_{N_{12}}(x_1) = \underline{u}_{N_{11}}(x_1)$	$\bar{u}_{N_{12}}(x_1) = \underline{u}_{N_{12}}(x_1)$

Example 2: Taking into account the mass-spring-damping system shown in Fig. 12 and on the basis of Newton's law, we can obtain:

$$m\ddot{x} + F_f + F_s = u(t),$$

where m represents the mass; F_f represents the friction force; F_s represents the restoring force of the spring and u represents the external control input. The friction force $F_f = c\dot{x}$ with $c > 0$ and the hardening spring force $F_s = k(1 + a^2x^2)x$ with constants k and a . Thus, the dynamic equation is as follows:

$$m\ddot{x} + c\dot{x} + kx + ka^2x^3 = u(t),$$

in which x represents the displacement from a reference point. Define $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$ and $\tilde{f} = \frac{-k-ka^2x_1^2(t)}{m}$. Let $x_1(t) \in [-3, 3]$, $m = 1$ kg, $c = 5$ N·m/s, $k_{\min} = 5$ N/m, $k_{\max} = 10$ N/m and $a = 2$ m⁻¹. Then, $\tilde{f}_{\max} = -5$ (i.e., the maximum value of $\tilde{f}$) with $k = 5$ and $x_1(t) = 0$. $\tilde{f}_{\min} = -370$ (i.e., the minimum value of $\tilde{f}$) with $k = 10$ and $x_1^2(t) = 9$. According to the membership function property $m_1(x_1(t)) + m_2(x_1(t)) = 1$, $\tilde{f}$ can be represented as

$$\tilde{f} = m_1(x_1(t))\tilde{f}_{\min} + m_2(x_1(t))\tilde{f}_{\max}.$$

Then, it can be found that

$$m_1(x_1(t)) = \frac{-\tilde{f} + \tilde{f}_{\max}}{\tilde{f}_{\max} - \tilde{f}_{\min}}, \quad m_2(x_1(t)) = \frac{\tilde{f} - \tilde{f}_{\min}}{\tilde{f}_{\max} - \tilde{f}_{\min}}.$$

According to the uncertain parameter k , the membership functions for IT2 fuzzy system can be obtained as follows:

$$\begin{aligned} \underline{m}_1(x_1(t)) &= \frac{-\tilde{f} + \tilde{f}_{\max}}{\tilde{f}_{\max} - \tilde{f}_{\min}}, \quad \bar{m}_2(x_1(t)) = \frac{\tilde{f} - \tilde{f}_{\min}}{\tilde{f}_{\max} - \tilde{f}_{\min}} \quad \text{with } k = 5, \\ \bar{m}_1(x_1(t)) &= \frac{-\tilde{f} + \tilde{f}_{\max}}{\tilde{f}_{\max} - \tilde{f}_{\min}}, \quad \underline{m}_2(x_1(t)) = \frac{\tilde{f} - \tilde{f}_{\min}}{\tilde{f}_{\max} - \tilde{f}_{\min}} \quad \text{with } k = 10. \end{aligned}$$

Membership functions of the controller are those in Table III.

Then, we can get the following IT2 T-S fuzzy model for the mass-spring-damping system:

$$x(k+1) = \sum_{i=1}^2 \varphi_i(x(k)) [A_i x(k) + B_i u(k)],$$

where

$$\begin{aligned} A_1 &= \begin{bmatrix} 0 & 1 \\ \tilde{f}_{\min} & -\frac{c}{m} \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}, \\ A_2 &= \begin{bmatrix} 0 & 1 \\ \tilde{f}_{\max} & -\frac{c}{m} \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}. \end{aligned}$$

Under sampling time $T = 1$ s, we can get

$$\begin{aligned} A_1 &= \begin{bmatrix} 0.0824 & 0.0010 \\ -0.3517 & 0.0777 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.0025 \\ 0.0010 \end{bmatrix}, \\ A_2 &= \begin{bmatrix} 0.3897 & 0.1003 \\ -0.5014 & -0.1118 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.1221 \\ 0.1003 \end{bmatrix}. \end{aligned}$$

The reference model is defined the same as that in Example 1. For demonstration, the external disturbance will be add into the system. Other relevant matrices are given as

$$\begin{aligned} L_1 &= \begin{bmatrix} 0.0055 \\ 0.01399 \end{bmatrix}, \quad L_2 = \begin{bmatrix} -0.01776 \\ 0.0330 \end{bmatrix}, \quad F_1 = 0.0354, \\ C_1 &= \begin{bmatrix} -01.0474 & -01.1704 \end{bmatrix}, \quad D_1 = 01.10794, \quad E = -0.3466, \quad R = 1 \\ C_2 &= \begin{bmatrix} -01.0054 & -01.0881 \end{bmatrix}, \quad D_2 = 01.10463, \quad F = 0.3466, \quad Q = 1. \end{aligned}$$

In this example, the number of sub-state is 20. The initial state $x_0 = [0.0; 0.0]$, $x_{r0} = 13.5$. Applying Theorem 2, we can obtain the optimal reliable guaranteed cost controller $K_1 = [0.4792 \quad 0.4991]$, $K_{r1} = 0.1532$ and $K_2 = [0.4883 \quad 0.5410]$, $K_{r2} = 0.1632$. The state trajectories of the open-loop and closed-loop system are shown in Fig.7-8. It can be observed that the instable open-loop system becomes stable after the controller is designed for the system. Fig. 9 plots the outputs of the closed-loop system and the reference model. Fig.10 show the tracking error of the closed-loop system. It can be seen that the tracking performance of the designed closed-loop system performs well. Additionally, Fig.11 plots the control input.

Fig. 7. State response $x(k)$ of the open-loop system. Fig. 8. State response $x(k)$ of the closed-loop system. Fig. 9. Outputs $y(k)$ and $y_r(k)$ of the closed-loop system. Fig. 10. $e(k)$ of the closed-loop system. Fig. 11. The controlled input $u(k)$.

Fig. 12. The mass-spring-damping system.

V. CONCLUSION

In this paper, the problem of guaranteed cost output tracking control has been studied for the discrete-time IT2 fuzzy system under imperfect premise matching. For the involved system, sufficient conditions have been given for the existence of fuzzy guaranteed cost control law and cost upper bound. And the cost function minimization problem has been solved by the sufficient conditions on the basis of some LMIs. Furthermore, there is a good track between the output of controlled system and the output of a given reference model by the designed output tracking state-feedback controller. A numerical example has been used to verify the availability of the presented results. *In future work, the case of input with delays and constraints will be taken into consideration in the framework of the IT2 T-S fuzzy model.*

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