TOTAL POSITIVITY, SCHUBERT POSITIVITY, AND GEOMETRIC SATAKE

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Abstract. Let G be a simple, simply-connected complex algebraic group, and let $X \subset G^\vee$ be the centraliser of a principal nilpotent. Ginzburg and Peterson independently related the ring of functions on X with the homology ring of the affine Grassmannian Gr_G. Peterson furthermore connected X to the quantum cohomology rings of partial flag varieties G/P.

In this paper we study three notions of positivity for X: (1) *Schubert positivity* arising via Peterson's work, (2) Lusztig's *total positivity* and (3) *Mirković-Vilonen positivity* obtained from the MV-cycles in Gr_G. The first main theorem establishes that these three notions of positivity coincide. Our second main theorem proves a parametrisation of the totally nonnegative part of X, confirming a conjecture of the second author.

In type A the parametrization and relationship with Schubert positivity were proved earlier by the second author. Here we tackle the general type case and also introduce a crucial new connection with the affine Grassmannian and geometric Satake correspondence.

1. Introduction

Let G be a simply connected, semisimple complex linear algebraic group, split over $\mathbb{R}$, and let $G^\vee$ be its Langlands dual group (over $\mathbb{C}$). The Peterson variety $\mathcal{Y}$ may be viewed as the compactification of the stabilizer $X := G^\vee_F$ of a standard principal nilpotent F in $(\mathfrak{g}^\vee)^*$ (with respect to the coadjoint representation of $G^\vee$), which one obtains by embedding X into the Langlands dual flag variety $G^\vee/B^\vee_-$ and taking the closure there.

Ginzburg [13] and Peterson [32] independently showed that the coordinate ring $\mathcal{O}(X)$ of the variety X was isomorphic to the homology $H_*(\mathrm{Gr}_G)$ of the affine Grassmannian Gr_G of G, and Peterson discovered moreover that the compactification $\mathcal{Y}$ encodes the quantum cohomology rings of all of the flag varieties G/P. Peterson's remarkable work in particular exhibited explicit homomorphisms between localizations of $qH^*(G/P, \mathbb{C})$ and $H_*(\mathrm{Gr}_G, \mathbb{C})$ taking quantum Schubert classes σ^P_w to affine homology Schubert classes ξ_x. These homomorphisms were verified in [25].

The first aim of this paper is to compare different notions of positivity for the real points of X: (i) the *affine Schubert positive* part $X^{\mathrm{af}}_{>0}$ where affine Schubert classes ξ_x take positive values via Ginzburg and Peterson's isomorphism $H_*(\mathrm{Gr}_G) \simeq \mathcal{O}(X)$; (ii) the *totally positive* part $X_{>0} := X \cap U^\vee_{-,>0}$ in the sense of Lusztig [27]; and (iii) the *Mirković-Vilonen positive* part $X^{\mathrm{MV}}_{>0}$ where the classes of the Mirković-Vilonen cycles from the geometric Satake correspondence [30] take positive values.

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Our first main theorem (Theorem 7.1) states that these three notions of positivity coincide. For G of type A the coincidence $X^{\mathrm{af}}_{>0} = X_{>0}$ was already established in [36], where instead of $X^{\mathrm{af}}_{>0}$, the notion of *quantum Schubert positivity* was used. In general quantum Schubert positivity is possibly weaker than affine Schubert positivity. It follows from [36] that the notions coincide in type A, and we verify that they coincide in type C in Appendix A.

The notion of Mirković-Vilonen positivity does not appear to have been studied in the literature before. We note that the MV-basis is expected to coincide with Lusztig's semicanonical basis which is distinct from the canonical basis used in Lusztig's approach to total positivity. So the coincidence $X^{MV}_{>0} = X_{>0}$ might not be immediately expected. As for the comparison $X^{MV}_{>0} = X^{\mathrm{af}}_{>0}$ we note that the classes of MV-cycles span $H_*(\mathrm{Gr}_G)$ over $\mathbb{C}$, but the $\mathbb{Z}$-lattice spanned by MV-cycles is known to be strictly contained in the lattice spanned by the Schubert basis.

Our second main theorem (Theorem 7.3) is a parametrization of the totally positive $X_{>0}$ and totally nonnegative $X_{\geq 0}$ parts of X. We show that they are homeomorphic to $\mathbb{R}^n_{>0}$ and $\mathbb{R}^n_{\geq 0}$ respectively. This was conjectured by the second author in [36] where it was established in type A. In type A_n we have that $X = G^\vee_F$ is the n-dimensional subgroup of lower-triangular unipotent Toeplitz matrices, and thus the parametrization $X_{\geq 0} \simeq \mathbb{R}^n_{\geq 0}$ is a "finite-dimensional" analogue of the Edrei-Thoma theorem [9] parametrizing *infinite* totally nonnegative Toeplitz matrices, appearing in the classification of the characters of the infinite symmetric group. The results of this article give an arbitrary type generalization.

The strategy of our proof is as follows: to show that $X^{\mathrm{af}}_{>0} \subseteq X^{\mathrm{MV}}_{>0}$ we use a result of Kumar and Nori [24] stating that effective classes in $H_*(\mathrm{Gr}_G)$ are Schubert-positive. We then use the geometric Satake correspondence [13, 30, 26] to describe $X^{\mathrm{MV}}_{>0}$ via matrix coefficients, and a result of Berenstein-Zelevinsky [2] to connect to the totally positive part $X_{>0}$.

Finally, to connect $X_{>0}$ back to $X^{\mathrm{af}}_{>0}$, we parametrize the latter directly by combining the positivity of the 3-point Gromov-Witten invariants of $qH^*(G/B)$ with the Perron-Frobenius theorem. This argument follows the strategy of [36].

There is a general phenomenon [27, 2] that totally positive parts have "nice parametrizations". This phenomenon is closely related to the relation between total positivity and the canonical bases [28], and also the cluster algebra structures on related stratifications [10]. Indeed our work suggests that the coordinate ring $\mathcal{O}(X)$ has the affine homology Schubert basis $\{\xi_w\}$ as a "dual canonical basis", and that the Hopf-dual universal enveloping algebra $U(\mathfrak{g}^\vee_F)$ has the cohomology affine Schubert basis $\{\xi^w\}$ as a "canonical basis". Certainly the affine Schubert bases have the positivity properties expected of canonical bases.

In [37] the type A parametrization result for the totally positive part $X_{>0}$ of the Toeplitz matrices X is proved in a completely different way, using a mirror symmetric construction of X. This approach does not however prove the interesting positivity properties of the bases we study in this paper. The mirror symmetric approach was partly generalized to other types in [39], where the existence of a totally positive point in X for any choice of positive quantum parameters is proved (but not its uniqueness). Since the first posting of the present paper two other related papers have appeared. Namely the uniqueness property of the totally positive critical point in the G/B case has been verified by Reda Chhaibi [6], completing the alternative (mirror symmetric) proof of [39] for the parametrization result in that case. Moreover, the construction from [37] of a unique positive critical point for a mirror Laurent polynomial was generalized and applied in the context of toric Fano varieties by Sergey Galkin in [12].

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2. Preliminaries and notation

Let G be a simple linear algebraic group over $\mathbb{C}$ split over $\mathbb{R}$. Usually G will be simply connected. Denote by $\mathrm{Ad} : G \to GL(\mathfrak{g})$ the adjoint representation of G on its Lie algebra $\mathfrak{g}$. We fix opposite Borel subgroups B^+ and B^- defined over $\mathbb{R}$ and intersecting in a split torus T. Their Lie algebras are denoted by $\mathfrak{b}^+$ and $\mathfrak{b}^-$ respectively. We will also consider their unipotent radicals U^+ and U^- with their Lie algebras $\mathfrak{u}^+$ and $\mathfrak{u}^-$. We use the notations $B^+_{\mathbb{R}}, B^-_{\mathbb{R}}, U^+_{\mathbb{R}}, U^-_{\mathbb{R}}$ to denote real points.

Let $X^*(T)$ be the character group of T and $X_*(T)$ the group of cocharacters together with the usual perfect pairing $\langle\, ,\, \rangle : X^*(T) \times X_*(T) \to \mathbb{Z}$. We may identify $X^*(T)$ with a lattice inside $\mathfrak{h}^*$, and $X_*(T)$ with the dual lattice inside $\mathfrak{h}$. These span the real forms $\mathfrak{h}^*_{\mathbb{R}}$ and $\mathfrak{h}_{\mathbb{R}}$, respectively.

Let $\Delta_+ \subset X^*(T)$ be the set of positive roots corresponding to $\mathfrak{b}^+$, and Δ_- the set of negative roots. There is a unique highest root in Δ_+ which is denoted by θ. Let $I = \{1, \dots, n\}$ be an indexing set for the set $\Pi := \{\alpha_i \mid i \in I\}$ of positive simple roots. The α_i-root space $\mathfrak{g}_{\alpha_i} \subset \mathfrak{g}$ is spanned by Chevalley generator e_i and $\mathfrak{g}_{-\alpha_i}$ is spanned by f_i. The split real form of $\mathfrak{g}$, denoted $\mathfrak{g}_{\mathbb{R}}$ is generated by the Chevalley generators e_i, f_i.

Let $Q := \langle \alpha_1, \dots, \alpha_n \rangle_{\mathbb{Z}}$ be the root lattice. We also have the fundamental weights $\omega_1, \dots, \omega_n$, and the weight lattice $L := \langle \omega_1, \dots, \omega_n \rangle_{\mathbb{Z}}$ associated to G. If G is simply connected, we have the relations

$$Q \subset X^*(T) = L \subset \mathfrak{h}^*.$$

Let $Q^\vee$ denote the lattice spanned by the simple coroots, $\alpha_1^\vee, \dots, \alpha_n^\vee$, and $L^\vee$ the lattice spanned by the fundamental coweights $\omega_1^\vee, \dots, \omega_n^\vee$. Then $Q^\vee$ is the dual lattice to L and $L^\vee$ the dual lattice to Q, giving

$$Q^\vee = X_*(T) \subset L^\vee \subset \mathfrak{h},$$

in the case where G is simply connected. We set $\rho = \sum_{i \in I} \omega_i$ and write $\mathrm{ht}(\lambda^\vee) = \langle \rho, \lambda^\vee \rangle$ for the height of $\lambda^\vee \in Q^\vee$.

For any Chevalley generator e_i, f_i of $\mathfrak{g}$ we may define a 'simple root subgroup' by

$$x_i(t) = \exp(te_i), \qquad y_i(t) = \exp(tf_i), \qquad\qquad \text{for } t \in \mathbb{C}.$$

Let $W = N_G(T)/T$ be the Weyl group of G. It is generated by simple reflections $s_1, \dots, s_n$. The length function $\ell : W \to \mathbb{N}$ gives the length of a reduced expression of $w \in W$ in the simple reflections. The unique longest element is denoted w_0, and for a root α, we let r_α denote the corresponding reflection. For any simple reflection s_i we choose a representative $\dot{s}_i$ in G defined by

$$\dot{s}_i := x_i(-1)y_i(1)x_i(-1).$$

If $w = s_{i_1} \dots s_{i_m}$ is a reduced expression, then $\dot{w} := \dot{s}_{i_1} \dots \dot{s}_{i_m}$ is a well-defined representative for w, independent of the reduced expression chosen. W is a poset under the Bruhat order $\leq$.

We denote the Langlands dual group of G by $G^\vee$, or $G^\vee_{\mathbb{C}}$ to emphasize that we mean the algebraic group over $\mathbb{C}$. The notations for $G^\vee$ are the same as those for G but with added ${}^\vee$ and any other superscripts moved down, for example $B^\vee_+$ for the analogue of B^+.

2.1. **Parabolic subgroups.** Let P denote a parabolic subgroup of G containing B^+, and let $\mathfrak{p}$ be the Lie algebra of P. Let I_P be the subset of I associated to P consisting of all the $i \in I$ with $\dot{s}_i \in P$ and consider its complement $I^P := I \setminus I_P$.

Associated to P we have the parabolic subgroup $W_P = \langle s_i \mid i \in I_P \rangle$ of W. We let $W^P \subset W$ denote the set of minimal coset representatives for W/W_P. An element w lies in W^P precisely if for all reduced expressions $w = s_{i_1} \cdots s_{i_m}$ the last index i_m always lies in I^P. We write w^P or w_0^P for the longest element in W^P, while the longest element in W_P is denoted w_P. For example $w_0^B = w_0$ and $w_B = 1$. Finally P gives rise to a decomposition

$$\Delta_+ = \Delta_{P,+} \sqcup \Delta_+^P.$$

Here $\Delta_{P,+} = \{\alpha \in \Delta_+ \mid \langle \alpha, \omega_i^\vee \rangle = 0 \text{ all } i \in I^P\}$, so that

$$\mathfrak{p} = \mathfrak{b}^+ \oplus \bigoplus_{\alpha \in \Delta_{P,+}} \mathfrak{g}_{-\alpha},$$

and Δ_+^P is the complement of $\Delta_{P,+}$ in Δ_+. For example $\Delta_{B,+} = \emptyset$ and $\Delta_+^B = \Delta_+$.

3. Total Positivity

3.1. **Total positivity.** A matrix A in $GL_n(\mathbb{R})$ is called *totally positive* (or *totally nonnegative*) if all the minors of A are positive (respectively nonnegative). In other words A acts by positive or nonnegative matrices in all of the fundamental representations $\bigwedge^k \mathbb{R}^n$ (with respect to their standard bases). In the 1990's Lusztig [27] extended this theory dating back to the 1930's to all reductive algebraic groups. This work followed his construction of canonical bases and utilized their deep positivity properties in types ADE.

Let G be a simple algebraic group, split over $\mathbb{R}$, not assumed to be simply connected for the purposes for this section. The definitions and results here are formulated for a general such group G, but will later on be applied to the group we have denoted by $G^\vee$.

Definition 3.1. The totally nonnegative part $U^+_{\geq 0}$ of $U^+_{\mathbb{R}}$ is defined to be the semigroup generated by $\{x_i(t) \mid i \in I \text{ and } t \in \mathbb{R}_{\geq 0}\}$. Similarly the totally nonnegative part $U^-_{\geq 0}$ of $U^-_{\mathbb{R}}$ is the semigroup generated by $\{y_i(t) \mid i \in I \text{ and } t \in \mathbb{R}_{\geq 0}\}$. The totally positive parts are given by $U^+_{>0} = U^+_{\geq 0} \cap B^- \dot{w}_0 B^-$ and $U^-_{>0} = U^-_{\geq 0} \cap B^+ \dot{w}_0 B^+$.

3.2. **Matrix coefficients.** Suppose $\lambda \in X^*(T)$ is dominant. Then we have a highest weight irreducible representation V_λ for G. The Lie algebra $\mathfrak{g}$ also acts on V_λ as does its universal enveloping algebra $U(\mathfrak{g})$. We fix a highest weight vector v_λ^+ in V_λ. The vector space V_λ has a real form given by $V_{\lambda,\mathbb{R}} = U(\mathfrak{g}_{\mathbb{R}}) \cdot v_\lambda^+$.

Let $(\ )^T : U(\mathfrak{g}) \to U(\mathfrak{g})$ be the unique involutive anti-automorphism satisfying $e_i^T = f_i$. We let $\langle .,. \rangle : V_\lambda \times V_\lambda \to \mathbb{C}$ denote the unique symmetric, non-degenerate bilinear form (Shapovalov form)[23, II, 2.3] satisfying

$$\langle u \cdot v, v' \rangle \;=\; \langle v, u^T \cdot v' \rangle \qquad \text{for all } u \in U(\mathfrak{g}), v, v' \in V_\lambda, \tag{3.1}$$

normalized so that $\langle v_\lambda^+, v_\lambda^+ \rangle = 1$. The Shapovalov form is real positive definite on $V_{\lambda,\mathbb{R}}$, see [23, Theorem 2.3.13].

We will be studying total positivity in the Langlands dual group of a simply-connected group. Thus in our application the group will be adjoint. Let G^{sc} be the simply-connected cover of the general complex simple algebraic group G. Then the unipotent subgroups of G^{sc} and G can be identified, and so can their totally positive (resp. negative) parts. The purpose of this observation is to allow the evaluation of matrix coefficients of fundamental representations on the unipotent subgroups of G. (As opposed to G^{sc}, the group G itself may not act on these representations.)

Thus for a fundamental weight ω_i (not necessarily a character of G!) and a vector v in the fundamental representation V_{ω_i} of $\mathfrak{g}$ we have a matrix coefficient

$$y \longmapsto \langle v, y \cdot v_{\omega_i}^+ \rangle$$

defined for $y \in U^-$. The following result follows from a theorem ([2, Theorem 1.5]) of Berenstein and Zelevinsky (note that every chamber weight is a w_0-chamber weight in the terminology of [2]).

Proposition 3.2. *Let $y \in U^-_{\mathbb{R}}$. Then y is totally positive if and only if for all $i \in I$ and $w \in W$ we have*

$$\langle \dot{w} \cdot v_{\omega_i}^+, y \cdot v_{\omega_i}^+ \rangle > 0,$$

where the matrix coefficient is as defined above.

We will need the following variation on the above Proposition.

Proposition 3.3. *Let $y \in U^-_{\mathbb{R}}$. Suppose that for all irreducible representations V_λ of G it holds that*

$$\langle v, y \cdot v_\lambda^+ \rangle > 0, \tag{3.2}$$

for some choice of highest weight vector v_λ^+ and any weight vector v satisfying the following two conditions.

- *The vector v lies in a one-dimensional weight space of the real form $V_{\lambda,\mathbb{R}}$.*
- *$\langle v, x \cdot v_\lambda^+ \rangle > 0$ for all totally positive $x \in U^-_{>0}$.*

Then y is totally positive.

The proof of Proposition 3.3 is delayed until Section 13. If G is simply connected then this Proposition 3.3 follows from Proposition 3.2. The difference arises if G is not simply connected, in which case the fundamental weights will not all be characters of the maximal torus of G, so that we may not be able to take λ to be ω_i.

Remark 3.4. Suppose G is simply-laced. Then the matrix coefficients of $x \in U^-_{>0}$ in the canonical basis of any irreducible representation V_λ are positive. It follows that for any $v \neq 0$ lying in a one-dimensional weight space of $V_{\lambda,\mathbb{R}}$, either v or $-v$ has the property that $\langle v, x \cdot v_\lambda^+ \rangle > 0$ for all $x \in U^-_{>0}$.

4. The affine Grassmannian and geometric Satake

In this section, G is a simple simply-connected linear algebraic group over $\mathbb{C}$. Let $\mathcal{O} = \mathbb{C}[[t]]$ denote the ring of formal power series and $\mathcal{K} = \mathbb{C}((t))$ the field of formal Laurent series. Let $\mathrm{Gr}_G = G(\mathcal{K})/G(\mathcal{O})$ denote the affine Grassmannian of G.

4.1. **Affine Weyl group.** Let $W_{\mathrm{af}} = W \ltimes X_*(T)$ be the affine Weyl group of G. For a cocharacter $\lambda \in X_*(T)$ we write $t_\lambda \in W_{\mathrm{af}}$ for the translation element of the affine Weyl group. We then have the commutation formula $wt_\lambda w^{-1} = t_{w\cdot\lambda}$. The affine Weyl group is also a Coxeter group, generated by simple reflections $s_0, s_1, \ldots, s_n$, where $s_0 = r_\theta t_{-\theta^\vee}$. It is a graded poset with its usual length function $\ell : W_{\mathrm{af}} \to \mathbb{Z}_{\geq 0}$, and Bruhat order $\geq$.

Let W_{af}^- denote the minimal length coset representatives of W_{af}/W. Thus we have canonical bijections

$$X_*(T) \longleftrightarrow W_{\mathrm{af}}/W \longleftrightarrow W_{\mathrm{af}}^-. \tag{4.1}$$

The intersection $X_*(T) \cap W_{\mathrm{af}}^-$ is given by the anti-dominant translations, that is t_λ where $\langle \alpha_i, \lambda\rangle \leq 0$ for each $i \in I$.

Note that an element λ of $X_*(T)$ viewed as a map from $\mathbb{C}^*$ to T can also be reinterpreted as an element of $T(\mathcal{K})$. We denote this element by t^λ. The two should not be confused since the isomorphism $W_{\mathrm{af}} \to N_{G(\mathcal{K})}(T)/T$ sends t_λ to $t^{-\lambda}$.

4.2. **Geometric Satake and Mirković-Vilonen cycles.** The affine Grassmannian is an ind-scheme [26], see also [23]. The $G(\mathcal{O})$-orbits Gr_λ on Gr_G are parametrized by the dominant cocharacters $\lambda \in X_*^+(T)$. Namely,

$$\mathrm{Gr}_\lambda := G(\mathcal{O})t^\lambda G(\mathcal{O})/G(\mathcal{O}).$$

The geometric Satake correspondence [13, 26, 30] states that the tensor category $\mathrm{Perv}(\mathrm{Gr}_G)$ of $G(\mathcal{O})$-equivariant perverse sheaves on Gr_G with $\mathbb{C}$-coefficients is equivalent to the tensor category $\mathrm{Rep}(G^\vee_{\mathbb{C}})$ of finite-dimensional representations of the Langlands dual group $G^\vee_{\mathbb{C}}$. (For our purposes the tensor structure will be unimportant.) The simple objects of $\mathrm{Perv}(\mathrm{Gr}_G)$ are the intersection cohomology complexes IC_λ of the $G(\mathcal{O})$-orbit closures $\overline{\mathrm{Gr}}_\lambda$. They correspond under the geometric Satake correspondence to the highest weight representations V_λ of $G^\vee$. Furthermore, we have a canonical isomorphism

$$IH^*(\overline{\mathrm{Gr}}_\lambda) = H^*(\mathrm{Gr}_G, IC_\lambda) \simeq V_\lambda. \tag{4.2}$$

Mirković and Vilonen found explicit cycles in Gr_G whose intersection homology classes give rise to a weight-basis of V_λ under the isomorphism (4.2). We denote by $\mathrm{MV}_{\lambda,v}$ the MV-cycle with corresponding basis vector $v \in V_\lambda$. For $w \in W$, the weight-space $V_\lambda(w\lambda)$ is one-dimensional. We denote by $\mathrm{MV}_{\lambda,w\lambda}$ the corresponding MV-cycle. Thus $[\mathrm{MV}_{\lambda,w\lambda}]_{IH} \in IH^*(\overline{\mathrm{Gr}}_\lambda) \simeq V_\lambda$ has weight $w\lambda$. All the statements of this section hold with $\mathbb{R}$-coefficients: we take perverse sheaves with $\mathbb{R}$-coefficients, and consider the representations of a split real form $G^\vee_{\mathbb{R}}$ of the Langlands dual group.

4.3. **Schubert varieties in** Gr_G**.** Let $\mathcal{I} \subset G(\mathcal{O})$ denote the Iwahori subgroup of elements $g(t)$ which evaluate to $g \in B^+$ at $t = 0$. The $\mathcal{I}$-orbits Ω_μ on Gr_G, called *Schubert cells*, are labeled by all (not necessarily dominant) cocharacters $\mu \in X_*(T)$. Explicitly,

$$\Omega_\mu = \mathcal{I}\, t^\mu G(\mathcal{O})/G(\mathcal{O}).$$

Alternatively, we may label Schubert cells by cosets $xW \in W_{\mathrm{af}}/W$ or minimal coset representatives $x \in W_{\mathrm{af}}^-$, using the bijection (4.1). Choosing a representative $\dot{x}$ of x we have

$$\Omega_x = \mathcal{I}\,\dot{x}G(\mathcal{O})/G(\mathcal{O}).$$

The Schubert cell $\Omega_\mu = \Omega_x$ is isomorphic to $\mathbb{C}^{\ell(x)}$ whenever $x \in W_{\mathrm{af}}^-$. We note that $\Omega_\mu = \Omega_{t_{-\mu}}$ if μ is dominant, compare Section 4.1. The Schubert varieties $X_x = \overline{\Omega_x}$, alternatively denoted $X_\mu = \overline{\Omega_\mu}$, are themselves unions of Schubert cells: $X_x = \sqcup_{v \leq x} \Omega_v$. The $G(\mathcal{O})$ orbits are also unions of Schubert cells:

$$\mathrm{Gr}_\lambda = \bigsqcup_{w \in W} \Omega_{w \cdot \lambda}.$$

In particular the largest one of these, $\Omega_\lambda \cong \mathbb{C}^{\ell(t_{-\lambda})}$, is open dense in Gr_λ (where we assumed λ dominant), and so

$$\overline{\mathrm{Gr}_\lambda} = \overline{\Omega_\lambda} = X_\lambda. \tag{4.3}$$

Thus every $G(\mathcal{O})$-orbit closure is a Schubert variety, but not conversely. Moreover $\overline{\mathrm{Gr}_\lambda}$ has dimension $\ell(t_{-\lambda})$, which equals $2\,\mathrm{ht}(\lambda)$.

We note that the MV-cycle $MV_{\lambda,v}$ is an irreducible subvariety of $\overline{\mathrm{Gr}_\lambda}$ of dimension $\mathrm{ht}(\lambda) + \mathrm{ht}(\nu)$ if v lies in the ν-weight space of V_λ, see [30, Theorem 3.2]. In particular $MV_{\lambda,\lambda} = \overline{\mathrm{Gr}_\lambda}$ and $MV_{\lambda,w_0\lambda}$ is just a point.

4.4. **The (co)homology of Gr_G.** The space Gr_G is homotopy equivalent to the based loop group ΩK of continuous maps of S^1 into the compact form $K \subset G$ [34, 33] (see [31] for a discussion of this). Thus the homology $H_*(\mathrm{Gr}_G;\mathbb{C})$ and cohomology $H^*(\mathrm{Gr}_G;\mathbb{C})$ are commutative and co-commutative graded dual Hopf algebras over $\mathbb{C}$.

Ginzburg [13] (see also [3]) and Dale Peterson [32] described $H_*(\mathrm{Gr}_G,\mathbb{C})$ as the coordinate ring of the stabilizer subgroup of a principal nilpotent in $(g^\vee)^*$. Namely, in our conventions, let $F \in (\mathfrak{g}^\vee)^*$ be the principal nilpotent element defined by

$$F = \sum_{i \in I} (e_i^\vee)^*,$$

where $(e_i^\vee)^*(\zeta) = 0$ if $\zeta \in \mathfrak{g}_\alpha^\vee$ for $\alpha \neq \alpha_i$, and $(e_i^\vee)^*(e_i^\vee) = 1$. Let $X = (G^\vee)_F$ denote the stabilizer of F inside $G^\vee$, under the coadjoint action. It is an abelian subgroup of $U_-^\vee$ of dimension equal to the rank of G. Then the result from [13, 32] says that $H_*(\mathrm{Gr}_G)$ is Hopf-isomorphic to the ring of regular functions on X. Moreover, the cohomology, $H^*(\mathrm{Gr}_G,\mathbb{C})$ is Hopf-isomorphic to the universal enveloping algebra $U(\mathfrak{g}_F^\vee)$ of the centralizer of F, as graded dual.

We note that Ginzburg [13] works over $\mathbb{C}$ while Peterson [32] works over $\mathbb{Z}$, but the details of Peterson's work are so far unpublished. Another approach is given in work of Yun and Zhu [41].

Our choice of principal nilpotent F is compatible via Peterson's isomorphism (6.3), see [25], with the conventions in [21, 22, 38], and is related to the choice in [13, 32] by switching the roles of B^+ and B^-.

In terms of the above presentation of $H_*(\mathrm{Gr}_G)$, the fundamental class of an MV-cycle can be described as follows. Let $\langle .,. \rangle : H^*(\mathrm{Gr}_G) \times H_*(\mathrm{Gr}_G) \to \mathbb{C}$ be the pairing obtained from cap product composed with pushing forward to a point.

Proposition 4.1. *Suppose* $\mathrm{MV}_{\lambda,v}$ *is the MV-cycle with corresponding weight vector* $v \in V_\lambda$ *under* (4.2). *Let* $u \in U(\mathfrak{g}_F^\vee) \simeq H^*(\mathrm{Gr}_G)$. *Then the fundamental class* $[\mathrm{MV}_{\lambda,v}] \in H_*(\mathrm{Gr}_G)$ *satisfies*

$$\langle u, [\mathrm{MV}_{\lambda,v}]\rangle = \langle u \cdot v, v_\lambda^- \rangle,$$

where v_λ^- *is the lowest weight vector of* V_λ *(in the MV-basis).*

Proof. The argument is essentially the same as [13, Proposition 1.9] (see also [14, 15]); the main difference is that in our conventions u is lower unipotent, rather than upper unipotent, however accordingly $\overline{\mathrm{Gr}_\lambda}$ is in our conventions the MV-cycle representing the highest weight vector, whereas it is the lowest weight vector in [13]. So the difference is that everywhere the roles of B^+ and B^- are interchanged. By [13, Theorem 1.7.6], the action of $u \in U(\mathfrak{g}_F^\vee)$ on V_λ is compatible with the action of the corresponding element in $H^*(\mathrm{Gr}_G)$ on $IH^*(\overline{\mathrm{Gr}}_\lambda)$. Under (4.2), the vector v is sent to $[\mathrm{MV}_{\lambda,v}]_{IH}$ which maps to the fundamental class $[\mathrm{MV}_{\lambda,v}]$ under the natural map from the intersection cohomology $IH^*(\overline{\mathrm{Gr}}_\lambda)$ to the homology $H_*(\overline{\mathrm{Gr}}_\lambda)$. Also, under the fundamental class map the action of $H^*(\mathrm{Gr}_G)$ on $IH^*(\overline{\mathrm{Gr}}_\lambda)$ is sent to the cap product of $H^*(\mathrm{Gr}_G)$ on $H_*(\overline{\mathrm{Gr}}_\lambda)$. Finally, pushing forward to a point is the same as pairing with v_λ^- (in our conventions). So we get the identity

$$\langle u, [\mathrm{MV}_{\lambda,v}]\rangle = \pi_*(u \cap [\mathrm{MV}_{\lambda,v}]) = \langle v_\lambda^-, u \cdot v\rangle,$$

where $\pi : X \to \{pt\}$. □

4.5. **Schubert basis.** We have

$$H_*(\mathrm{Gr}_G) = \bigoplus_{x \in W_{\mathrm{af}}^-} \mathbb{C} \cdot \xi_x, \qquad H^*(\mathrm{Gr}_G) = \bigoplus_{x \in W_{\mathrm{af}}^-} \mathbb{C} \cdot \xi^x,$$

where the ξ_w are the fundamental classes $[X_w]$ of the Schubert varieties, and $\{\xi^w\}$ is the cohomology basis (dual under the cap product). Suppose λ is dominant, then we also have

$$H_*(\overline{\mathrm{Gr}}_\lambda) = \bigoplus_{\substack{x \in W_{\mathrm{af}}^- \\ x \le t_{-\lambda}}} \mathbb{C} \cdot \xi_x,$$

because of (4.3) and the decomposition of X_λ into Schubert cells.

By Ginzburg/Peterson's isomorphism, we will often think of a Schubert basis element ξ_w as a function on X. The Schubert basis of $H_*(\mathrm{Gr}_G)$ has the following factorization property:

Proposition 4.2 ([32][25, Proposition 9.1])**.** *Suppose* $wt_\nu, t_\mu \in W_{\mathrm{af}}^-$. *Then* $\xi_{wt_\nu}\xi_{t_\mu} = \xi_{wt_{\nu+\mu}}$.

We remark that if $wt_\nu \in W_{\mathrm{af}}^-$, then necessarily ν is anti-dominant.

5. The quantum cohomology ring of G/P

5.1. **The usual cohomology of G/P and its Schubert basis.** For our purposes it will suffice to take homology or cohomology with complex coefficients, so $H^*(G/P)$ will stand for $H^*(G/P, \mathbb{C})$. By the well-known result of C. Ehresmann, the singular homology of G/P has a basis indexed by the elements $w \in W^P$ made up of the fundamental classes of the Schubert varieties,

$$X_w^P := \overline{(B^+ wP/P)} \subseteq G/P.$$

Here the bar stands for (Zariski) closure. Let $\sigma_w^P \in H^*(G/P)$ be the Poincaré dual class to $[X_w^P]$. Note that X_w^P has complex codimension $\ell(w)$ in G/P and hence σ_w^P lies in $H^{2\ell(w)}(G/P)$. The set $\{\sigma_w^P \mid w \in W^P\}$ forms a basis of $H^*(G/P)$ called the Schubert basis. The top degree cohomology of G/P is spanned by $\sigma_{w_0^P}^P$ and we have the Poincaré duality pairing

$$H^*(G/P) \times H^*(G/P) \longrightarrow \mathbb{C}, \qquad (\sigma, \mu) \mapsto \langle \sigma \cup \mu \rangle$$

which may be interpreted as taking (σ, μ) to the coefficient of $\sigma_{w_0^P}^P$ in the basis expansion of the product $\sigma \cup \mu$. For $w \in W^P$ let $PD(w) \in W^P$ be the minimal length coset representative in $w_0 w W_P$. Then this pairing is characterized by

$$\left\langle \sigma_w^P \cup \sigma_v^P \right\rangle = \delta_{w,PD(v)}.$$

5.2. **The quantum cohomology ring $\mathbf{qH^*(G/P)}$.** The (small) quantum cohomology ring $qH^*(G/P)$ is a deformation of the usual cohomology ring by $\mathbb{C}[q_1^P, \dots, q_k^P]$, where $k = \dim H^2(G/P)$, with structure constants defined by 3-point genus 0 Gromov-Witten invariants. For more background on quantum cohomology, see [11].

We have

$$qH^*(G/P) = \oplus_{w \in W^P} \mathbb{C}[q_1^P, \dots, q_k^P] \cdot \sigma_w^P$$

where σ_w^P now (and in the rest of the paper) denotes the quantum Schubert class. The quantum cup product is defined by

$$\sigma_v^P \cdot \sigma_w^P = \sum_{\substack{u \in W^P \\ \mathbf{d} \in \mathbb{N}^k}} \left\langle \sigma_u^P, \sigma_v^P, \sigma_w^P \right\rangle_{\mathbf{d}} \, \mathbf{q}^{\mathbf{d}} \sigma_{PD(u)}^P,$$

where $\mathbf{q}^{\mathbf{d}}$ is multi-index notation for $\prod_{i=1}^k q_i^{d_i}$, and the $\left\langle \sigma_u^P, \sigma_v^P, \sigma_w^P \right\rangle_{\mathbf{d}}$ are genus 0, 3-point Gromov-Witten invariants. These enumerate rational curves in G/P, with a fixed degree determined by $\mathbf{d}$, which pass through general translates of three Schubert varieties. In particular, $\left\langle \sigma_u^P, \sigma_v^P, \sigma_w^P \right\rangle_{\mathbf{d}}$ is a nonnegative integer.

The quantum cohomology ring $qH^*(G/P)$ has an analogue of the Poincaré duality pairing which may be defined as the symmetric $\mathbb{C}[q_1^P, \dots, q_k^P]$-bilinear pairing

$$qH^*(G/P) \times qH^*(G/P) \longrightarrow \mathbb{C}[q_1^P, \dots, q_k^P], \qquad (\sigma, \mu) \mapsto \langle \sigma \cdot \mu \rangle_{\mathbf{q}}$$

where $\langle \sigma \cdot \mu \rangle_{\mathbf{q}}$ denotes the coefficient of $\sigma_{w_0^P}^P$ in the Schubert basis expansion of the product $\sigma \cdot \mu$. In terms of the Schubert basis the quantum Poincaré duality pairing on $qH^*(G/P)$ is given by

$$\left\langle \sigma_w^P \cdot \sigma_v^P \right\rangle_{\mathbf{q}} = \left\langle \sigma_w^P \cup \sigma_v^P \right\rangle = \delta_{w,PD(v)}, \tag{5.1}$$

where $v, w \in W^P$, and $PD : W^P \to W^P$ is the involution defined in Section 5.1. Equation (5.1) is a consequence of properties of the small quantum cup product, see [8, Proposition 8.1.6].

6. Peterson's theory

In this section we summarize Peterson's results concerning his geometric realizations of $qH^*(G/P)$ and their relationship with $H_*(\mathrm{Gr}_G)$.

6.1. **Definition of the Peterson variety.** Each $\mathrm{Spec}(qH^*(G/P))$ turns out to be most naturally viewed as lying inside the Langlands dual flag variety $G^\vee/B^\vee$, where it appears as a stratum (non-reduced intersection with a Bruhat cell) of one n-dimensional projective variety called the *Peterson variety.* This remarkable fact was discovered and shown by Dale Peterson [32].

Recall the definition of $F \in (\mathfrak{g}^\vee)^*$ from Section 4.4. The condition

$$(\mathrm{Ad}(g^{-1}) \cdot F)(X) = 0 \quad \text{for all } X \in [\mathfrak{u}_-^\vee, \mathfrak{u}_-^\vee],$$

defines a closed subvariety of $G^\vee$ invariant under right multiplication by $B_-^\vee$. Thus it defines a closed subvariety of $G^\vee/B_-^\vee$. This subvariety $\mathcal{Y}$ is the *Peterson variety* for G. Explicitly we have

$$\mathcal{Y} = \left\{ gB_-^\vee \in G^\vee/B_-^\vee \;\middle|\; \mathrm{Ad}(g^{-1}) \cdot F \in [\mathfrak{u}_-^\vee, \mathfrak{u}_-^\vee]^\perp \right\}.$$

For any parabolic subgroup $W_P \subset W$ with longest element w_P define $\mathcal{Y}_P$ as non-reduced intersection,

$$\mathcal{Y}_P := \mathcal{Y} \times_{G^\vee/B_-^\vee} \left(B_+^\vee w_P B_-^\vee / B_-^\vee \right).$$

Remark 6.1. For $P = B$ we have a map

$$\mathcal{Y}_B \to \mathcal{A}_G \;:\; \quad uB_-^\vee \mapsto u^{-1} \cdot F, \tag{6.1}$$

where $\mathcal{A}_G \subset (\mathfrak{g}^\vee)^*$ is the degenerate leaf of the Toda lattice. This map is an isomorphism as follows from classical work of Kostant [20]. Kostant also showed that $\mathcal{Y}_B$ is irreducible [21].

The isomorphism between $qH^*(G/B)$ and the functions on the degenerate leaf of the Toda lattice was established by B. Kim [18] building on [16].

6.2. **Irreducibility of $\mathcal{Y}$.** It is not immediately obvious from the above definition that the Peterson variety $\mathcal{Y}$ is irreducible. In other words apart from the closure of $\mathcal{Y}_B$ it could a priori contain some other irreducible components coming from intersections with other Bruhat cells. We include a sketch of proof (put together from [32]) that this doesn't happen, and that therefore $\mathcal{Y}$ is irreducible, n-dimensional and equal to the closure of $\mathcal{Y}_B$. Namely we have the following proposition.

Proposition 6.2 (Dale Peterson)**.** *If $w = w_P$, the longest element in W_P for some parabolic subgroup P, then $\mathcal{Y} \cap B_+^\vee w B_-^\vee / B_-^\vee$ is nonempty and of dimension $|I^P|$. Otherwise $w^{-1} \cdot (-\Pi^\vee) \not\subset \Delta_-^\vee \cup \Pi^\vee$ and $\mathcal{Y} \cap B_+^\vee w B_-^\vee / B_-^\vee = \emptyset$.*

Sketch of proof. Clearly $\dot{w}^{-1} \cdot F$ needs to lie in $b_\Pi := [\mathfrak{u}_-^\vee, \mathfrak{u}_-^\vee]^\perp$ for $\mathcal{Y} \cap B_+^\vee w B_-^\vee / B_-^\vee$ to be non-empty. So $w^{-1} \cdot (-\Pi^\vee) \subset \Delta_-^\vee \cup \Pi^\vee$. This is the case if and only if $w = w_P$ for some parabolic P, by a lemma from [32] reproduced in [35, Lemma 2.2].

Consider the map

$$\begin{array}{rcl} \psi : (U^+)^\vee & \to & (\mathfrak{u}_-^\vee)^*, \\ u & \mapsto & (u^{-1} \cdot F)|_{\mathfrak{u}_-^\vee}. \end{array}$$

The coordinate rings of $(U^+)^\vee$ and $(\mathfrak{u}_-^\vee)^*$ are polynomial rings. On $(U^+)^\vee$ consider the $\mathbb{C}^*$-action coming from conjugation by the one-parameter subgroup of $T^\vee$ corresponding to $\rho \in X_*(T^\vee)$. On $(\mathfrak{u}_-^\vee)^*$ let $\mathbb{C}^*$ act by $z \cdot X_\alpha = z^{<\alpha,\rho>+1} X_\alpha$ for X_α in the α-weight space of $(\mathfrak{u}_-^\vee)^*$ and $\alpha \in \Delta_+^\vee$. Then ψ^* is a homomorphism of (positively) graded rings, namely it is straightforward to check that ψ is $\mathbb{C}^*$-equivariant.

Also $\psi^{-1}(0) = \{0\}$, in terms of $\mathbb{C}$-valued points or indeed over any algebraically closed field. Peterson proves this in [32] by considering the $B_-^\vee$-Bruhat decomposition intersected wtih $U_+^\vee$. Namely, the only way $\psi(u)$ can be 0 for $u \in U_+^\vee \cap B_-^\vee \dot{w} B_-^\vee$ is if $w = e$, wherefore u must be the identity element in $U_+^\vee$.

It follows from these two properties that ψ is finite. For example by page 660 in Griffiths-Harris and using the $\mathbb{C}^*$-action to go from the statement locally around zero, to a global statement, or by another proposition in Peterson's lectures [32].

Let

$$U_P := \psi^{-1}((w_P \cdot b_\Pi)|_{\mathfrak{u}_-^\vee}) = \{u \in U_+^\vee \mid (u^{-1} \cdot F)|_{\mathfrak{u}_-^\vee} \in (w_P \cdot b_\Pi)|_{\mathfrak{u}_-^\vee}\}.$$

Since $u^{-1} \cdot F \in F + \mathfrak{h} + (\mathfrak{u}_-^\vee)^*$ for $u \in U_+^\vee$ and $F + \mathfrak{h} \subset w_P \cdot b_\Pi$, we can drop the restriction to $\mathfrak{u}_-^\vee$ on both sides of the condition above, and we have a projection map

$$U_P = \{u \in U_+^\vee \mid u^{-1} \cdot F \in w_P \cdot b_\Pi\} = \{u \in U_+^\vee \mid w_P^{-1} u^{-1} \cdot F \in b_\Pi\} \to \mathcal{Y} \cap B_+^\vee w_P B_-^\vee / B_-^\vee$$

taking $u \in U_P$ to $u w_P B_-^\vee$, which is a fiber bundle with fiber $\cong \mathbb{C}^{\ell(w_P)}$.

Since ψ is finite the dimension of U_P is equal to the dimension of the subspace $(w_P \cdot b_\Pi)|_{\mathfrak{u}_-^\vee}$ inside $(\mathfrak{u}_-^\vee)^*$. This dimension is just $|I^P| + \ell(w_P)$, by looking at the weight space decomposition. So $\mathcal{Y} \cap B_+^\vee w_P B_-^\vee / B_-^\vee$ has dimension $|I_P|$. □

6.3. **Geometric realization of $qH^*(G/P)$.** Recall the stabilizer X of the principal nilpotent F, which is an n-dimensional abelian subgroup of $(U^-)^\vee$. Using an idea of Kostant's [19, page 304], the Peterson variety may also be understood as a compactification of X. Namely,

$$\mathcal{Y} = \overline{X \dot{w}_0 B_-^\vee / B_-^\vee} \subset G^\vee / B_-^\vee.$$

For the parabolic P let

$$\mathcal{Y}_P^* := \mathcal{Y}_P \times_{G^\vee/B_-^\vee} X \dot{w}_0 B_-^\vee / B_-^\vee = (X \times_{G^\vee} B_+^\vee \dot{w}_P \dot{w}_0 B_+^\vee) \dot{w}_0 / B_-^\vee \cong X \times_{G^\vee} B_+^\vee \dot{w}_P \dot{w}_0 B_+^\vee,$$

or equivalently,

$$\mathcal{Y}_P^* = \mathcal{Y}_P \times_{G^\vee/B_-^\vee} B_-^\vee \dot{w}_0 B_-^\vee / B_-^\vee.$$

We define

$$X_P := X \times_{G^\vee} B_+^\vee \dot{w}_P \dot{w}_0 B_+^\vee,$$

so that the above is an isomorphism $\mathcal{Y}_P^* \cong X_P$.

Theorem 6.3 (Dale Peterson)**.**

(1) *The $\mathcal{Y}_P$ give rise to a decomposition*

$$\mathcal{Y}(\mathbb{C}) = \bigsqcup_P \mathcal{Y}_P(\mathbb{C}).$$

where $\mathcal{Y}_B(\mathbb{C})$ is open dense in $\mathcal{Y}(\mathbb{C})$.

(2) *For $P = B$ we have an isomorphism*

$$\mathcal{P} : qH^*(G/B) \xrightarrow{\sim} \mathcal{O}(\mathcal{Y}_B), \tag{6.2}$$

via the isomorphism (6.1) *of $\mathcal{Y}_B$ with the degenerate leaf of the Toda lattice of $G^\vee$.*

(3) *For any Schubert class σ_w, let S^w denote regular function on $\mathcal{Y}_B$ which is the image $\mathcal{P}(\sigma_w)$ of σ_w under the isomorphism $\mathcal{P}$ from* (6.2)*. Suppose that $w \in W^P$. Then S^w extends regularly to $\mathcal{Y}_P$. In this case, let S_P^w denote the regular function in $\mathcal{O}(\mathcal{Y}_P)$ obtained from the extension of S^w by restriction to $\mathcal{Y}_P$. There is a (uniquely determined) isomorphism*

$$qH^*(G/P) \xrightarrow{\sim} \mathcal{O}(\mathcal{Y}_P)$$

which takes σ_w^P to S_P^w.

(4) *The isomorphisms above restrict, to give isomorphisms*

$$qH^*(G/P)[q_1^{-1}, \ldots, q_k^{-1}\ ] \xrightarrow{\sim} \mathcal{O}(\mathcal{Y}_P^*).$$

In particular, Theorem 6.3(1) gives

$$X(\mathbb{C}) = \bigsqcup_P X_P(\mathbb{C}).$$

6.4. **Quantum cohomology and homology of the affine Grassmannian.**

Theorem 6.4 (Dale Peterson)**.**

(1) *The composition of isomorphisms*

$$H_*(\mathrm{Gr}_G)[\xi_{(t_\lambda)}^{-1}] \cong \mathcal{O}(X_B) \cong \mathcal{O}(\mathcal{Y}_B^*) \cong qH^*(G/B)[q_1^{-1}, \ldots, q_n^{-1}\ ] \tag{6.3}$$

is given by

$$\xi_{wt_\lambda}\xi_{t_\mu}^{-1} \longmapsto q_{\lambda-\mu}\sigma_w^B \tag{6.4}$$

where $q_\nu = q_1^{a_1}q_2^{a_2}\cdots q_n^{a_n}$ if $\nu = a_1\alpha_1^\vee + \cdots + a_n\alpha_n^\vee$.

(2) *More generally, for an arbitrary parabolic P the composition*

$$(H_*(\mathrm{Gr}_G)/J_P)[(\xi_{(\pi_P(t_\lambda))}^{-1}] \cong \mathcal{O}(X_P) \cong \mathcal{O}(\mathcal{Y}_P^*) \cong qH^*(G/P)[q_1^{-1}, \ldots, q_k^{-1}\ ] \tag{6.5}$$

is given by

$$\xi_{w\pi_P(t_\lambda)}\xi_{\pi_P(t_\mu)}^{-1} \longmapsto q_{\eta_P(\lambda-\mu)}\sigma_w^P \tag{6.6}$$

where $J_P \subset H_*(\mathrm{Gr}_G)$ is an ideal, π_P maps W_{af} to a subset $(W^P)_{\mathrm{af}}$, and η_P is the natural projection $Q^\vee \mapsto Q^\vee/Q_P^\vee$ where $Q_P^\vee$ is the root lattice of W_P.

The first author and Shimozono [25] verified that the maps (6.4) (resp. (6.6)) are isomorphisms from $H_*(\mathrm{Gr}_G)[\xi_{(t_\lambda)}^{-1}]$ to $qH^*(G/B)[q_1^{-1}, \ldots, q_n^{-1}\ ]$ (respectively from, in the parabolic case, $(H_*(\mathrm{Gr}_G)/J_P)[(\xi_{(\pi_P(t_\lambda))}^{-1})]$ to $qH^*(G/P)[q_1^{-1}, \ldots, q_k^{-1}\ ]$). We do not review the definitions of J_P and π_P here, but refer the reader to [25].

Remark 6.5. In [25] it is not shown that the isomorphism (6.4) is the one induced by the geometry of X. We sketch how this can be achieved.

First, Kostant [21, Section 5] expresses the quantum parameters as certain ratios of 'chamber minors' on X. Ginzburg's [13, Proposition 1.9] also expresses the translation affine Schubert classes $\xi_{t_{-\lambda}}$ as matrix coefficients, since $\xi_{t_{-\lambda}} = [X_\lambda] = [\overline{\mathrm{Gr}_\lambda}] = [MV_{\lambda,\lambda}]$. This allows one to compare $\xi_{t_{-\lambda}}$ with q_λ as functions on X_B, and see that they agree. Namely both are equal to $x \mapsto \langle x \cdot v_\lambda^+, v_\lambda^- \rangle$.

Let $\lambda = m\omega_i^\vee$ be a positive multiple of $\omega_i^\vee$ contained in $Q^\vee$. We now compare the functions $\xi_{s_i t_{-\lambda}}$ and $q_{-\lambda}\sigma_{s_i}$ on X_B. For the function σ_{s_i}, Kostant gives a formula in [21, (119)] as a ratio of matrix coefficients on X. For the function $\xi_{s_i t_{-\lambda}}$, one notes that since λ is a multiple of $\omega_i^\vee$, then $t_{-\lambda} \in W_{\mathrm{af}}^-$ covers only $s_i t_{-\lambda}$ in the Bruhat order of W_{af}^-. It follows that $H_{2\ell(t_{-\lambda})-2}(\overline{\mathrm{Gr}}_\lambda)$ is one-dimensional, spanned by $\xi_{s_i t_{-\lambda}}$. Similarly the weight space $V_\lambda(\lambda - \alpha_i^\vee)$ is one-dimensional. If we let $[\mathrm{MV}_{\lambda,v}]$ be the unique MV-cycle with v of weight $\nu = \lambda - \alpha_i^\vee$, then this gives a cycle in homology of degree $2(\mathrm{ht}(\lambda) + \mathrm{ht}(\nu)) = 4\,\mathrm{ht}(\lambda) - 2 = 2\ell(t_{-\lambda}) - 2$. So we have that the homology class $[\mathrm{MV}_{\lambda,v}]$ is a positive integer multiple of the Schubert class $\xi_{s_i t_{-\lambda}}$. Proposition 4.1 allows one to write $[\mathrm{MV}_{\lambda,v}]$ as a matrix coefficient on X and compare it with Kostant's formula for $q_\lambda \sigma_{s_i}$. Finally we see in this way that $q_\lambda \sigma_{s_i}$ is a positive integral multiple of $\xi_{s_i t_{-\lambda}}$ as function on X_B.

Now we can compose this isomorphism $qH^*(G/B)[q_1^{-1}, \dots, q_n^{-1}] \cong \mathcal{O}(X_B) \cong H_*(Gr_G)[\xi_{t_\mu}^{-1}]$ which we have just seen takes σ_{s_i} to a positive integral multiple of $\xi_{s_i t_{-\lambda}} \xi_{t_{-\lambda}}^{-1}$, with the isomorphism $H_*(Gr_G)[\xi_{t_\mu}^{-1}] \to qH^*(G/B)[q_1^{-1}, \dots, q_n^{-1}]$ defined by (6.4) going the other way. This way we get a map $qH^*(G/B)[q_1^{-1}, \dots, q_n^{-1}] \to qH^*(G/B)[q_1^{-1}, \dots, q_n^{-1}]$ which takes every q_i to q_i but σ_{s_i} to a positive integral multiple of σ_{s_i}. However the images of the σ_{s_i} still need to obey the quadratic relation in the quantum cohomology ring, which arises from the Hamiltonian of the Toda lattice [18]. This quadratic relation identifies a quadratic form in the σ_{s_i}'s with a linear form in the q_i's. The quadratic form arises from the invariant symmetric bilinear form on $\mathfrak{h}^*$, with the fundamental weights ω_i identified with σ_{s_i}. Since this form is non-degenerate, the σ_{s_i}'s cannot be rescaled by positive integer multiples and this relation still hold, unless all of the integer factors are 1, which means that σ_{s_i} must go to σ_{s_i}.

Since $qH^*(G/B)[q_1^{-1}, \dots, q_n^{-1}]$ is generated by the σ_{s_i} as ring over $\mathbb{C}[q_1^{\pm 1}, \dots q_n^{\pm 1}]$ the map $qH^*(G/B)[q_1^{-1}, \dots, q_n^{-1}] \to qH^*(G/B)[q_1^{-1}, \dots, q_n^{-1}]$ considered above is the identity. Therefore (6.4) is the inverse to the map $qH^*(G/B)[q_1^{-1}, \dots, q_n^{-1}] \cong \mathcal{O}(X_B) \cong H_*(Gr_G)[\xi_{t_\mu}^{-1}]$, and we are done.

7. Main results

In the rest of the paper we will be working with the $\mathbb{R}$-structures on all our main objects. Since $X, \mathcal{Y}, H_*(\mathrm{Gr}_G), qH^*(G/P)$ are in fact all defined over $\mathbb{Z}$ there is no problem with this. All our notations for positivity and nonnegativity refer to $\mathbb{R}$-points. We now define

(AP) The subset
$$X_{>0}^{\mathrm{af}} = \{x \in X_{\mathbb{R}} \mid \xi_w(x) > 0 \,\text{for all}\, w \in W_{\mathrm{af}}^-\}$$
of *affine Schubert positive* elements.

(MVP) The subset
$$X_{>0}^{\mathrm{MV}} = \{x \in X_{\mathbb{R}} \mid [\mathrm{MV}_{\lambda,v}](x) > 0 \,\text{for all MV-cycles}\}$$
of *MV-positive* elements.

(TP) The *totally positive subset* defined by Lusztig's theory,
$$X_{>0} := X_{\mathbb{R}} \cap U_{-,>0}^\vee.$$

(QP) The subset
$$X_{>0}^{\mathrm{qSchubert}} := \{x \in (X_B)_{\mathbb{R}} \mid \sigma_B^w(x) > 0 \text{ all } w \in W\}$$
of *quantum Schubert positive* elements defined in terms of the quantum Schubert basis and Peterson's isomorphism $qH^*(G/B)[q_1^{-1}, \dots, q_n^{-1}] \cong \mathcal{O}(X_B)$.

We also define the totally *nonnegative* part $X_{\geq 0} = X_{\mathbb{R}} \cap U^{\vee}_{-,\geq 0}$ of X, and the affine Schubert *nonnegative* part $X^{\mathrm{af}}_{\geq 0}$ of X as the set of points $x \in X_{\mathbb{R}}$ such that $\xi_w(x) \geq 0$ for every affine Schubert class ξ_w.

We note that by (6.3) affine Schubert positivity is equivalent to quantum Schubert positivity with additional positivity of the quantum parameters. Therefore we have an immediate inclusion $X^{\mathrm{af}}_{>0} \subset X^{\mathrm{qSchubert}}_{>0}$. We can now state our first main theorem.

Theorem 7.1. *The first three notions of positivity in X agree: we have $X_{>0} = X^{\mathrm{af}}_{>0} = X^{\mathrm{MV}}_{>0}$. For the fourth notion we have the inclusion $X_{>0} \subset X^{\mathrm{qSchubert}}_{>0}$. Furthermore, $X_{\geq 0} = \overline{X_{>0}} = \overline{X^{\mathrm{af}}_{>0}} = X^{\mathrm{af}}_{\geq 0}$.*

By this Theorem the first three notions of positivity are equivalent, and the fourth is at worst weaker. We in fact conjecture all four of the notions to be equivalent.

Conjecture 7.2. *We have $X_{>0} = X^{\mathrm{qSchubert}}_{>0}$.*

Conjecture 7.2 was shown to hold for type A in [36]. We will verify it in Appendix A for type C.

Our second main result is about a description of the totally nonnegative part of X and its stratification. By the isomorphism

$$\mathbb{C}[\mathcal{Y}^*_P] \cong qH^*(G/P)[(q_1^P)^{-1}, \dots, (q_k^P)^{-1}]$$

from Theorem 6.3 combined with $X_P \cong \mathcal{Y}^*_P$ we have a morphism

$$\pi^P = (q_1^P, \dots, q_k^P) : X_P \to (\mathbb{C}^*)^k. \tag{7.1}$$

Let $X_{P,>0} := X_P \cap X_{\geq 0}$. In particular $X_{B,>0} = X_{>0}$.

Theorem 7.3.

(1) *π^P restricts to a bijection*

$$\pi^P_{>0} : X_{P,>0} \to \mathbb{R}^k_{>0}.$$

(2) *$X_{P,>0}$ lies in the smooth locus of X_P, and the map π^P is étale on $X_{P,>0}$.*

(3) *The maps $\pi^P_{>0}$ glue to give a homeomorphism*

$$\Delta_{\geq 0} : X_{\geq 0} \longrightarrow \mathbb{R}^n_{\geq 0}.$$

8. One direction of Theorem 7.1

The main goal of this section is to show that

$$X^{\mathrm{af}}_{>0} \subseteq X^{\mathrm{MV}}_{>0} \subseteq X_{>0}.$$

Lemma 8.1. *If $x \in X$ is affine Schubert positive, then it is MV-positive.*

Proof. The main result of Kumar and Nori [24], applied to Gr_G, shows that every effective cycle in Gr_G is homologous to a positive sum of Schubert cycles. It follows that the fundamental class $[\mathrm{MV}_{\lambda,v}]$ of an MV-cycle is a positive linear combination of the Schubert classes ξ_w. □

Lemma 8.2. *Suppose $X^{\mathrm{MV}}_{>0} \cap X_{>0} \neq \emptyset$. Then $X^{\mathrm{MV}}_{>0} \subseteq X_{>0}$.*

Proof. Suppose V_λ is an irreducible representation of $G^\vee$ with highest weight λ and μ is a weight of V_λ with one-dimensional weight space.

We work with the geometric Satake equivalence with $\mathbb{R}$-coefficients. Let $[\mathrm{MV}_{\lambda,\mu}]$ be the MV-cycle representing a weight vector v with weight μ in $V_{\lambda,\mathbb{R}}$ under the isomorphism (4.2). Then for $x \in X_{\mathbb{R}}$, we have by Proposition 4.1, $[\mathrm{MV}_{\lambda,\mu}](x) = \langle x, [\mathrm{MV}_{\lambda,\mu}]\rangle = \langle x \cdot v, v_\lambda^- \rangle = \langle v, x^T \cdot v_\lambda^-\rangle$. Note that we are really thinking of $V_{\lambda,\mathbb{R}}$ as a lowest weight representation by fixing the lowest weight vector v_λ^- (of weight $w_0\lambda$).

Now suppose that that there is a vector v' in the weight space $V_{\lambda,\mathbb{R}}(\mu)$ satisfying $\langle v', y \cdot v_\lambda^- \rangle > 0$ for all $y \in U^+_{>0}$. Since the weight space $V_{\lambda,\mathbb{R}}(\mu)$ is one-dimensional it follows that the MV-basis element $v = c_{\lambda,\mu}\, v'$ for a scalar $c_{\lambda,\mu}$. Choose $x_0 \in X_{>0}^{\mathrm{MV}} \cap X_{>0}$. Such an x_0 exists by our assumption. Then we see that the scalar is positive,

$$c_{\lambda,\mu} = \frac{\langle x_0 \cdot v, v_\lambda^- \rangle}{\langle x_0 \cdot v', v_\lambda^- \rangle} = \frac{[\mathrm{MV}_{\lambda,\mu}](x_0)}{\langle v', x_0^T \cdot v_\lambda^- \rangle} > 0.$$

(Here, if $x_0 = y_{i_1}(a_1) y_{i_2}(a_2) \cdots y_{i_\ell}(a_\ell) \in U^-_{>0}$, then $x_0^T = x_{i_\ell}(a_\ell) \cdots x_{i_2}(a_2) x_{i_1}(a_1) \in U^+_{>0}$.)

Now suppose $x \in X^{\mathrm{MV}}_{>0}$ is an arbitrary element. Then

$$\langle v', x^T \cdot v_\lambda^- \rangle = \frac{1}{c_{\lambda,\mu}} \langle v, x^T \cdot v_\lambda^- \rangle = \frac{1}{c_{\lambda,\mu}} [\mathrm{MV}_{\lambda,\mu}](x) > 0.$$

By Proposition 3.3 (applied with 'positive Borel' B^+ taken to be $B_-^\vee$) , this implies that x^T is totally positive. Clearly then x is totally positive. □

Lemma 8.3. *The principal nilpotent $f^\vee = \sum f_i^\vee$ goes to a positive multiple of the affine Schubert class $\xi^{s_0} \in H^2(\mathrm{Gr}_G)$ under the isomorphism $\mathcal{U}(\mathfrak{g}_F^\vee) \cong H^*(\mathrm{Gr}_G)$.*

Proof. Since $H^2(\mathrm{Gr}_G)$ is 1-dimensional we know that $f^\vee = c\xi^{s_0}$ under the identification $\mathcal{U}(\mathfrak{g}_F^\vee) \cong H^*(\mathrm{Gr}_G)$. We want to show that c is positive.

Consider the exponential $\exp(f^\vee) = \exp(c\xi^{s_0})$ as an element of the completion $\widehat{H^*(\mathrm{Gr}_G)}$, and choose λ such that $s_i t_\lambda \in W^-_{\mathrm{af}}$. If we evaluate the (localized) homology class $\xi_{s_i t_\lambda} \xi_{t_\lambda}^{-1}$ on this element we obtain

$$\begin{aligned} \xi_{s_i t_\lambda} \xi_{t_\lambda}^{-1}(\exp(f^\vee)) &= \xi_{s_i t_\lambda} \xi_{t_\lambda}^{-1}(\exp(c\xi^{s_0})) \\ &= \xi_{s_i t_\lambda} \xi_{t_\lambda}^{-1}(1 + c\xi^{s_0} + \frac{c^2}{2!}(a_2) + \frac{c^3}{3!}(a_3) + \cdots) \\ &= \frac{c^{k-1}}{(k-1)!} \frac{k!}{c^k} \xi_{s_i t_\lambda}(a_{k-1}) \xi_{t_\lambda}(a_k)^{-1} \\ &= \frac{k}{c}\ \xi_{s_i t_\lambda}(a_{k-1}) \xi_{t_\lambda}(a_k)^{-1}, \end{aligned}$$

where $\ell(t_\lambda) = k = \ell(s_i t_\lambda) + 1$ and the $a_i \in H^{2i}(\mathrm{Gr}_G)$ are positive linear combinations of Schubert classes, since all cup product Schubert structure constants of $H^*(\mathrm{Gr}_G)$ are positive [24]. Therefore c is positive if and only if $\xi_{s_i t_\lambda} \xi_{t_\lambda}^{-1}(\exp(f^\vee))$ is positive.

We now compute $\xi_{s_i t_\lambda} \xi_{t_\lambda}^{-1}(\exp(f^\vee))$ in a different way. Under Peterson's isomorphism (6.3)

$$\xi_{s_i t_\lambda} \xi_{t_\lambda}^{-1} \mapsto \sigma_{s_i}. \tag{8.1}$$

Consider the element of $X_{>0}$, the totally positive part of X_B, given by $\exp(f^\vee)$. Using (8.1) and identifying both $H_*(\mathrm{Gr}_G)$ and $qH^*(G/B)[q_1^{-1},\dots,q_n^{-1}]$ with $\mathbb{C}[X_B]$ as in (6.3), we can evaluate

$$\xi_{s_i t_\lambda}\xi_{t_\lambda}^{-1}(\exp(f^\vee)) = \sigma_{s_i}(\exp(f^\vee)). \tag{8.2}$$

The right hand side here is a quotient of two 'chamber minors' of $\exp(f^\vee)$ by Kostant's formula [21, Proposition 33]. Therefore the total positivity of $\exp(f^\vee)$ implies

$$\xi_{s_i t_\lambda}\xi_{t_\lambda}^{-1}(\exp(f^\vee)) > 0. \tag{8.3}$$

□

Lemma 8.4. *The element $x = \exp(f^\vee)$ lies in $X^{\mathrm{af}}_{>0}$.*

Proof. A Schubert class ξ_w in $H_*(\mathrm{Gr}_G)$ can be evaluated against $\exp(f^\vee)$ by viewing $f^\vee$ as element of $H^2(\mathrm{Gr}_G)$, expanding $\exp(f^\vee)$ as a power series and pairing ξ_w with each summand. But Lemma 8.3 implies that $\exp(f^\vee)$ expands as a positive linear combination of Schubert classes. This implies that $\xi_w(\exp(f^\vee)) > 0$ for all $w \in W_{\mathrm{af}}^-$. □

Lemma 8.5. *We have $X^{\mathrm{MV}}_{>0} \subseteq X_{>0}$ and $X^{\mathrm{MV}}_{\geq 0} \subseteq X_{\geq 0}$.*

Proof. In Lemma 8.4 we found a totally positive point $x \in X_{>0}$, namely $x = \exp(f^\vee)$, which is also affine Schubert positive. Since affine Schubert positive implies MV-positive, by Lemma 8.1, this means that $x \in X^{\mathrm{MV}}_{>0} \cap X_{>0}$. Now Lemma 8.2 implies that $X^{\mathrm{MV}}_{>0} \subseteq X_{>0}$. The second inclusion is an immediate consequence. □

Corollary 8.6. *We have $X^{\mathrm{af}}_{>0} \subseteq X^{\mathrm{MV}}_{>0} \subseteq X_{>0}$ and $X^{\mathrm{af}}_{\geq 0} \subseteq X^{\mathrm{MV}}_{\geq 0} \subseteq X_{\geq 0}$.*

9. Parametrizing the affine Schubert-positive part of X_P

Let $X^{\mathrm{af}}_{P,>0}$ denote the set of points $x \in (X_P)_{\mathbb{R}}$ such that $\xi_w(x) \geq 0$ for every affine Schubert class ξ_w, so

$$X^{\mathrm{af}}_{P,>0} := X^{\mathrm{af}}_{\geq 0} \cap X_P.$$

Note that while some of the functions ξ_w will vanish identically on X_P, the ones that don't will be strictly positive. Hence the notation of $X^{\mathrm{af}}_{P,>0}$. The goal of this section is to provide $X^{\mathrm{af}}_{P,>0}$ with a natural bijection to $\mathbb{R}^k_{>0}$.

Recall the map $\pi^P : X_P \to (\mathbb{C}^*)^k$ from (7.1) defined by $\pi^P(x) = (q_i^P(x))_{i=1}^k$. We begin by observing that π^P when restricted to $X^{\mathrm{af}}_{P,>0}$ takes values in $\mathbb{R}^k_{>0}$. We saw already that $q_i^P(x) \neq 0$ for $x \in X_P$, by identification of X_P with $\mathcal{Y}_P^*$. Expressing the quantum parameters q_i^P in terms of affine Schubert classes ξ_{t_λ} using Theorem 6.4(2), it follows that $x \in X^{\mathrm{af}}_{P,>0}$ has $q_i^P(x) > 0$ for all $i \in I^P$. We also have strict positivity for the quantum Schubert classes, as in the next Lemma.

Lemma 9.1. *Suppose $x \in X^{\mathrm{af}}_{P,>0}$. Then $\sigma_w^P(x) > 0$ for all $w \in W^P$.*

Proof. It follows from Theorem 6.4(2) and the definitions that $\sigma_w^P(x) \geq 0$. Suppose $\sigma_w^P(x) = 0$. Let r_θ denote the reflection in the longest root, and $\pi_P(r_\theta) \in W^P$ be the corresponding minimal length parabolic coset representative. Proposition 11.2 of [25] states that

$$\sigma^P_{\pi_P(r_\theta)}\,\sigma^P_w = q_{\eta_P(\theta^\vee - w^{-1}\theta^\vee)}\sigma^P_{\pi_P(r_\theta w)} + q_{\eta_P(\theta^\vee)}\sum_{s_i w < w} a_i^\vee\,\sigma^P_{s_i w}.$$

We refer the reader to Appendix A and [25] for the notation used here. Applying this repeatedly, we see that for large ℓ, the product $(\sigma^P_{\pi_P(r_\theta)})^\ell\,\sigma^P_w$ is a (positive) combination of quantum Schubert classes which includes a monomial in the q_i^P. This contradicts $q_i^P(x) > 0$ for each i. □

Remark 9.2. We note that the Lemma 9.1 together with the positivity of the quantum parameters implies that we have $X^{\mathrm{af}}_{B,>0} = X^{\mathrm{af}}_{>0}$.

The remainder of this section will be devoted to the proof of the following proposition.

Proposition 9.3. *The map $\pi^P_{>0} : X^{\mathrm{af}}_{P,>0} \longrightarrow \mathbb{R}^k_{>0}$ is bijective.*

We follow the proof in type A given in [36], shortening somewhat the proof of our Lemma 9.5 below (Lemma 9.3 in [36]), by using a result of Fulton and Woodward [11]: the quantum product of Schubert classes is always nonzero.

Fix a point $Q \in (\mathbb{R}_{>0})^k$ and consider its fiber under $\pi = \pi^P$. Let us define

$$R_Q := qH^*(G/P)/(q_1^P - Q_1, \dots, q_k^P - Q_k).$$

This is the (possibly non-reduced) coordinate ring of $\pi^{-1}(Q)$. Note that R_Q is a $|W^P|$-dimensional algebra with basis given by the (image of the) Schubert basis. We will use the same notation σ^P_w for the image of a Schubert basis element from $qH^*(G/P)$ in the quotient R_Q. The proof of the following result from [36] holds in our situation verbatim.

Lemma 9.4 ([36, Lemma 9.2])**.** *Suppose $\mu \in R_Q$ is a nonzero simultaneous eigenvector for all linear operators $R_Q \to R_Q$ which are defined by multiplication by elements in R_Q. Then there exists a point $p \in \pi^{-1}(Q)$ such that (up to a scalar factor)*

$$\mu = \sum_{w \in W^P} \sigma^P_w(p)\, \sigma^P_{PD(w)}.$$

□

Set

$$\sigma := \sum_{w \in W^P} \sigma^P_w \in R_Q.$$

Suppose the multiplication operator on R_Q defined by multiplication by σ is given by the matrix $M_\sigma = (m_{v,w})_{v,w \in W^P}$ with respect to the Schubert basis. That is,

$$\sigma \cdot \sigma^P_v = \sum_{w \in W^P} m_{v,w} \sigma^P_w.$$

Then since $Q \in \mathbb{R}^k_{>0}$ and by positivity of the structure constants it follows that M_σ is a nonnegative matrix.

Lemma 9.5 ([36, Lemma 9.3])**.** *There is no subset V of W^P such that the linear span of the set $\{\sigma^P_w \,|\, w \in V\}$ is invariant under the action of M_σ. In other words, M_σ is an indecomposable matrix.*

Proof. Suppose indirectly that the matrix M_σ is *reducible*. Then there exists a nonempty, proper subset $V \subset W^P$ such that the span of $\{\sigma_v \mid v \in V\}$ in R_Q is invariant under M_σ. We will derive a contradiction to this statement.

First let us show that the smallest element, 1, in W^P lies in V. Suppose not. Since $V \neq \emptyset$ we have a $v \neq 1$ in V. Since $1 \notin V$, the coefficient of σ_1 in $\sigma_w \cdot \sigma_v$ must be zero for all $w \in W^P$, or equivalently

$$\left\langle \sigma_w \cdot \sigma_v \cdot \sigma_{w_0^P} \right\rangle_Q = 0 \tag{9.1}$$

for all $w \in W^P$. Here by the bracket $\langle \quad \rangle_Q$ we mean $\langle \quad \rangle_{\mathbf{q}}$ evaluated at Q. But this relation (9.1) implies $\langle \sigma_w \cdot \sigma_v \cdot \sigma_{w_0^P} \rangle_{\mathbf{q}} = 0$, since the latter is a nonnegative polynomial in the q_i^P's which evaluated at $Q \in \mathbb{R}^k_{>0}$ equals 0. Therefore it follows that $\sigma_v \cdot \sigma_{w_0^P} = 0$ in $qH^*(G/P)$, by quantum Poincaré duality. This leads to a contradiction, since by work of W. Fulton and C. Woodward [11] no two Schubert classes in $qH^*(G/P)$ ever multiply to zero.

So V must contain 1. Since V is a proper subset of W^P we can find some $w \notin V$. In particular, $w \neq 1$. It is a straightforward exercise that given $1 \neq w \in W^P$ there exists $\alpha \in \Delta_+^P$ and $v \in W^P$ such that

$$w = v r_\alpha, \quad \text{and} \quad \ell(w) = \ell(v) + 1.$$

Now $\alpha \in \Delta_+^P$ means there exists $i \in I^P$ such that $\langle \alpha, \omega_i^\vee \rangle \neq 0$. And hence by the (classical) Chevalley Formula we have that $\sigma_{s_i} \cdot \sigma_v$ has σ_w as a summand. But if $w \notin V$ this implies that also $v \notin V$, since $\sigma \cdot \sigma_v$ would have summand $\sigma_{s_i} \cdot \sigma_v$ which has summand σ_w. Note that there are no cancellations with other terms by positivity of the structure constants.

By this process we can find ever smaller elements of W^P which do not lie in V until we end up with the identity element, so a contradiction. □

Given the indecomposable nonnegative matrix M_σ, then by Perron-Frobenius theory (see e.g. [29] Section 1.4) we know the following.

> The matrix M_σ has a positive eigenvector μ which is unique up to scalar (positive meaning it has positive coefficients with respect to the standard basis). Its eigenvalue, called the Perron-Frobenius eigenvalue, is positive, has maximal absolute value among all eigenvalues of M_σ, and has algebraic multiplicity 1. The eigenvector μ is unique even in the stronger sense that any nonnegative eigenvector of M_σ is a multiple of μ.

Suppose μ is this eigenvector chosen normalized such that $\langle \mu \rangle_Q = 1$. Then since the eigenspace containing μ is 1–dimensional, it follows that μ is joint eigenvector for all multiplication operators of R_Q. Therefore by Lemma 9.4 there exists a $p_0 \in \pi^{-1}(Q)$ such that

$$\mu = \sum_{w \in W^P} \sigma_w^P(p_0)\, \sigma_{PD(w)}^P.$$

Positivity of μ implies that $\sigma_w^P(p_0) \in \mathbb{R}_{>0}$ for all $w \in W^P$. Of course all of the $q_i(p_0) = Q_i$ are positive too. Hence $p_0 \in X_{P,>0}^{\mathrm{af}}$. Also the point p_0 in the fiber over Q with the property that all $\sigma_w^P(p_0)$ are positive is unique. Therefore

$$X_{P,>0}^{\mathrm{af}} \longrightarrow \mathbb{R}^k_{>0} \tag{9.2}$$

is a bijection and Proposition 9.3 is proved.

10. Proof of Theorem 7.3.(2)

We establish Theorem 7.3(2) for $X^{\mathrm{af}}_{P,>0}$ instead of $X_{P,>0}$. In Proposition 11.3, we will we establish the equality $X^{\mathrm{af}}_{P,>0} = X_{P,>0}$.

Since $qH^*(G/P)$ is free over $\mathbb{C}[q_1^P, \ldots, q_k^P]$, it follows that π^P is flat. Let $Q = \pi^P(p_0)$. Let $R = qH^*(G/P)$ and $I \subset R$ the ideal $(q_1 - Q_1, \ldots, q_k - Q_k)$. The Artinian ring $R_Q = R/I$ is isomorphic to the sum of local rings $R_Q \cong \bigoplus_{x \in (\pi^P)^{-1}(Q)} R_x/IR_x$. And for $x = p_0$ the local ring R_{p_0}/IR_{p_0} corresponds in R_Q to the Perron–Frobenius eigenspace of the multiplication operator M_σ from the above proof. Since this is a one-dimensional eigenspace (with algebraic multiplicity one) we have that $\dim(R_{p_0}/IR_{p_0}) = 1$. It follows that the map π^P is unramified at the point p_0. Thus, for example by [17, Ex.III.10.3], π^P is étale at p_0. Since $(\mathbb{C}^*)^k$ is smooth, it follows that X_P is smooth at p_0.

11. Proofs of Theorem 7.1 and Theorem 7.3(1)

Lemma 11.1. *$X_{\geq 0}$ and $X^{\mathrm{af}}_{\geq 0}$ are closed subsemigroups of X.*

Proof. For $X_{\geq 0}$ this follows from the fact that $(U_-^\vee)_{\geq 0}$ is a closed subsemigroup of $U_-^\vee$, and $X \subset U_-^\vee$ is a closed subgroup. For $X^{\mathrm{af}}_{\geq 0}$ being a closed subset follows from the definition. For the semigroup structure, suppose $x, y \in X^{\mathrm{af}}_{\geq 0}$. Then for any affine Schubert class ξ_w, we have

$$\xi_w(xy) = \Delta(\xi_w)(x \otimes y) = \sum_{v,u} c^w_{v,u} \xi_v(x) \otimes \xi_u(y) \geq 0$$

where Δ denotes the coproduct of $H_*(\mathrm{Gr}_G)$, and $c^w_{v,u} \geq 0$ are nonnegative integers [24]. Thus $xy \in X^{\mathrm{af}}_{\geq 0}$. ☐

The first statement of Theorem 7.1 follows from Corollary 8.6 and the following proposition.

Proposition 11.2. *The totally positive part and the affine Schubert positive part of X agree,*

$$X^{\mathrm{af}}_{>0} = X_{>0}.$$

Proof. Our proof is identical to the proof of Proposition 12.2 from [36].

By [21, Section 5] or combining Theorem 6.3 with [13, Proposition 1.9], we see that each q_i is a ratio of 'chamber minors' and so π^B takes positive values on $X_{>0}$. By Corollary 8.6 we have the following commutative diagram

$$\begin{array}{ccc} X^{\mathrm{af}}_{>0} & \hookrightarrow & X_{>0} \\ & \searrow \quad \swarrow & \\ & \mathbb{R}^n_{>0} & \end{array}$$

where the top row is clearly an open inclusion and the maps going down are restrictions of π^B. By (9.2) and Section 10, the left hand map to $\mathbb{R}^n_{>0}$ is a homeomorphism. It follows from this and elementary point set topology that $X^{\mathrm{af}}_{>0}$ must be closed inside $X_{>0}$. So it suffices to show that $X_{>0}$ is connected.

For an arbitrary element $u \in X$ and $t \in \mathbb{R}$, let

$$u_t := t^{-\rho} u t^{\rho}, \tag{11.1}$$

where $t \mapsto t^\rho$ is the one-parameter subgroup of $T^\vee$ corresponding to the coroot ρ (a coroot relative to $G^\vee$). Then $u_0 = e$ and $u_1 = u$, and if $u \in X_{>0}$, then also $u_t \in X_{>0}$ for all positive t.

Let $u, u' \in X_{>0}$ be two arbitrary points. Consider the paths

$$\begin{aligned} \gamma\ &: [0,1] \to X_{>0}\ , \quad \gamma(t) = uu'_t \\ \gamma' &: [0,1] \to X_{>0}\ , \quad \gamma'(t) = u_t u'. \end{aligned}$$

Note that these paths lie entirely in $X_{>0}$ since $X_{>0}$ is a semigroup (Lemma 11.1). Since γ and γ' connect u and u', respectively, to uu', it follows that u and u' lie in the same connected component of $X_{>0}$, and we are done. □

The second statement of Theorem 7.1 and Theorem 7.3(1) follow from:

Proposition 11.3. *We have* $\overline{X_{>0}} = X_{\geq 0}$ *and* $\overline{X^{\mathrm{af}}_{>0}} = X^{\mathrm{af}}_{\geq 0}$. *Moreover* $X_{\geq 0} = X^{\mathrm{af}}_{\geq 0}$ *and for each* P

$$X_{P,>0} = X^{\mathrm{af}}_{P,>0}.$$

Proof. Since $X_{\geq 0}$ is closed and contains $X_{>0}$ we clearly have that $\overline{X_{>0}} \subset X_{\geq 0}$. Suppose $x \in X_{\geq 0}$. Then for any $u \in X_{>0}$, we have $u_t \in X_{>0}$ for all positive t, where u_t is defined in (11.1). The curve $t \mapsto x(t) = xu_t$ starts at $x(0) = x$ and lies in $X_{>0}$ for all $t > 0$, compare [27]. Therefore $x \in \overline{X_{>0}}$ as desired and we have proved the first equality, $\overline{X_{>0}} = X_{\geq 0}$.

Now $X^{\mathrm{af}}_{\geq 0}$ is a closed semigroup in $X_{\geq 0}$ by a combination of Lemma 11.1 and Corollary 8.6. On the other hand since $X^{\mathrm{af}}_{>0} = X_{>0}$ by Proposition 11.2, we have $X^{\mathrm{af}}_{\geq 0} \supseteq \overline{X^{\mathrm{af}}_{>0}} = \overline{X_{>0}} = X_{\geq 0}$. If follows that $X^{\mathrm{af}}_{\geq 0} = \overline{X^{\mathrm{af}}_{>0}} = X_{\geq 0}$. □

12. Proof of Theorem 7.3(3)

To define $\Delta_{\geq 0}$, we set $\Delta_i = \xi_{t_{m_i \omega_i^\vee}}$, where m_i is chosen so that $m_i \omega_i^\vee \in Q^\vee$. Then $\Delta_{\geq 0} = (\Delta_1, \ldots, \Delta_n)$.

It follows from the explicit description [25] of $\pi_P(t_\lambda)$ and $\eta_P(\lambda)$ of Theorem 6.4 that for each i, some power of q_i^P is equal to $\xi_\lambda \xi_\mu^{-1}$ on X_P, for certain $\lambda, \mu \in Q^\vee$. Furthermore, the map

$$\pi^P_{>0} = (q_1^P, \ldots, q_k^P) : X_{P,>0} \to \mathbb{R}^k_{>0}$$

is related to the map

$$\Delta^P_{>0} = (\Delta_{i_1}, \Delta_{i_2}, \ldots, \Delta_{i_k}) : X_{P,>0} \to \mathbb{R}^k_{>0}$$

by a homeomorphism of $\mathbb{R}^k_{>0}$, where $I^P = \{i_1, i_2, \ldots, i_k\}$. But $X_{\geq 0} = \bigsqcup X_{P,>0}$, so we have that

$$\Delta_{\geq 0} : X_{\geq 0} \longrightarrow \mathbb{R}^n_{\geq 0}$$

is bijective. So $\Delta_{\geq 0}$ is continuous and bijective. Since Δ is finite it follows that it is closed, that is, takes closed sets to closed sets. (This holds true also in the Euclidean topology, since the preimage of a bounded set under a finite map must be bounded, compare [40, Section 5.3]). Since $X_{\geq 0}$ is closed in X the restriction $\Delta_{\geq 0}$ of Δ to $X_{\geq 0}$ is also closed. Therefore $\Delta^{-1}_{\geq 0}$ is continuous.

13. Proof of Proposition 3.3

It suffices to prove the Proposition for G of adjoint type. Call a dominant weight λ *allowable* if it is a character of the maximal torus of adjoint type G.

We note that the tensor product $V = V_\lambda \otimes V_\mu$ of two irreducible representations inherits a *tensor Shapovalov form* $\langle \cdot, \cdot \rangle$ defined by $\langle v \otimes w, v' \otimes w' \rangle = \langle v, v' \rangle \langle w, w' \rangle$. This is again a positive-definite non-degenerate symmetric form on $V_{\lambda,\mathbb{R}} \otimes V_{\mu,\mathbb{R}}$ satisfying (3.1). Note that for weight vectors $v_1, v_2 \in V$, we have $\langle v_1, v_2 \rangle = 0$ unless v_1 and v_2 have the same weight. It follows that if $V_\nu, V_\rho \subset V$ are irreducible subrepresentations, and $\nu \neq \rho$ then $\langle v, v' \rangle = 0$ for $v \in V_\nu$ and $v' \in V_\rho$. Thus if the highest-weight representation V_ν occurs in V with multiplicity one, the restriction of $\langle \cdot, \cdot \rangle$ from V to V_ν must be a positive-definite non-degenerate symmetric bilinear form satisfying (3.1), and thus must be a multiple of the Shapovalov form. By scaling the inclusion $V_\nu \subset V$, we shall always assume that the restricted form is the Shapovalov form. The above comments extend to the case of n-fold tensor products.

We shall need the following result.

Lemma 13.1. *Let ω be a fundamental weight, α_i be the corresponding simple root, and let $\eta = 2\omega - \alpha_i$.*

(1) *Define an equivalence relation on $W \cdot \omega$ generated by $\lambda \sim \mu$ if $\lambda + \mu \in W \cdot \eta$. Then this equivalence relation has a single equivalence class.*
(2) *η appears as a weight in $V_{2\omega}$ with multiplicity 1. The weight η appears as a weight in $V_\omega \otimes V_\omega$ with multiplicity 2.*
(3) *$V_{2\omega}$ occurs as an irreducible summand of $V_\omega \otimes V_\omega$ with multiplicity 1.*

Proof. We prove (1). It is clear from the definition that if $\lambda \sim \mu$ then $w \cdot \lambda \sim w \cdot \mu$ for any $w \in W$. Since $\omega + s_i \cdot \omega = \eta$, we have $\omega \sim s_i \omega$. Let $W_P \subset W$ be the parabolic subgroup that is the stabilizer of ω. Then clearly $\omega \sim v \cdot \omega$ for any $v \in W_P$. Since W is generated by s_i and W_P, it follows that $\omega \sim w \cdot \omega$ for any $w \in W_P$. (1) follows.

The first statement of (2) follows from the fact that $2\omega - \eta = \alpha_i$ is a simple root. The second statement of (2) follows by inspection of the weights of V_ω. (3) is straightforward. $\square$

13.1. **Type A_n.** We shall establish the criterion used in Proposition 3.2. First suppose n is even. Let $V_{\omega_i,\mathbb{R}}$ be a fundamental representation, and let $v^+_{\omega_i} \in V_{\omega_i,\mathbb{R}}$ be the highest weight vector, and $v = \dot{w} \cdot v^+_{\omega_i}$ an extremal weight vector. The weight space with weight $\dot{w} \cdot (n+1)\omega_i$ is extremal (and one-dimensional) in $V_{(n+1)\omega_i,\mathbb{R}}$, and $V_{(n+1)\omega_i,\mathbb{R}}$ is an irreducible representation for PSL_{n+1}. Thus for y as in Proposition 3.2,

$$\langle v, y \cdot v^+_{\omega_i} \rangle^{n+1} = \langle v^{\otimes(n+1)}, y \cdot (v^+_{\omega_i})^{\otimes(n+1)} \rangle > 0.$$

Since n is even, this implies that $\langle v, y \cdot v^+_{\omega_i} \rangle > 0$.

For odd n, let us fix $w \in W$, and consider the set of signs $a_i = \mathrm{sign}(\langle \dot{w} \cdot v_{\omega_i}, y \cdot v_{\omega_i} \rangle)$. We want to prove that the a_i are all $+1$. Note that a sum of (not necessarily distinct) fundamental weights, $\omega_{i_1} + \cdots + \omega_{i_k}$, is allowable precisely if it is trivial on the center of SL_{n+1}, that is if $i_1 + \cdots + i_k$ is divisible by $n+1$. Let $(i_1, i_2, \ldots, i_k)$ be such a sequence of indices, for which $\omega_{i_1} + \cdots + \omega_{i_k}$ is allowable. Then the weight $w(\omega_{i_1} + \cdots + \omega_{i_k})$ is an extremal weight of the representation $V_{\omega_{i_1}+\cdots+\omega_{i_k},\mathbb{R}}$

of PSL_{n+1}, and we have

$$\begin{aligned}\langle \dot w\cdot v^+_{\omega_{i_1}}, y\cdot v^+_{\omega_{i_1}}\rangle\langle \dot w\cdot v^+_{\omega_{i_2}}, y\cdot v^+_{\omega_{i_2}}\rangle\cdots\langle \dot w\cdot v^+_{\omega_{i_k}}, y\cdot v^+_{\omega_{i_k}}\rangle\\ = \langle \dot w\cdot (v^+_{\omega_{i_1}}\otimes\ldots\otimes v^+_{\omega_{i_k}}), y\cdot (v^+_{\omega_{i_1}}\otimes\ldots\otimes v^+_{\omega_{i_k}})\rangle > 0.\end{aligned}$$

Therefore $a_{i_1}\ldots a_{i_k} = +1$ if $i_1+\ldots+i_k = n+1$. In particular $a_i a_1^{n+1-i} = +1$, implying that $a_i = +1$ for even i, and $a_i = a_1$ for odd i.

We now show that $a_1 = +1$. Let $V = V_{\omega_1,\mathbb{R}} = \mathbb{R}^{n+1}$ with standard basis $\{v_1,\ldots,v_{n+1}\}$, and let $Z = V^{\otimes(n+1)}$. If we take $v^+_{\omega_1} = v_1$ then the Shapovalov form on V is the standard symmetric bilinear form given by $\langle v_i, v_j\rangle = \delta_{i,j}$. Let us consider $U = V_{(n+1)\omega_1,\mathbb{R}} = \mathrm{Sym}^{n+1}(V)$, which occurs with multiplicity 1 in Z and has standard basis $\{v_{i_1}\ldots v_{i_{n+1}} \mid 1\le i_1\le i_2\le\ldots\le i_{n+1}\le n+1\}$ of symmetrized tensors,

$$v_{i_1}\ldots v_{i_{n+1}} = \frac{1}{(n+1)!}\sum_{\sigma\in S_{n+1}} v_{\sigma(i_1)}\otimes v_{\sigma(i_2)}\otimes\cdots\otimes v_{\sigma(i_{n+1})}.$$

These are clearly orthonormal for the tensor Shapovalov form restricted to U, which is the Shapovalov form $\langle\ ,\ \rangle_U$ of U. We have $\dot w\cdot v_1 = v_k$ for some k. Consider the vector $z = v_1^n v_k\in U$ which has weight $\dot w\cdot\omega_1 + n\omega_1$. Clearly $\langle z, x\cdot v^+_{(n+1)\omega_1}\rangle_U = \langle v_1^n v_k, x\cdot v_1^{n+1}\rangle_U > 0$ for all totally positive $x\in U^-_{>0}$, and z lies in a 1-dimensional weight space of U. Therefore our assumptions imply that

$$0 < \langle z, y\cdot v^+_{(n+1)\omega_1}\rangle_U = \langle v_1^n v_k, y\cdot v_1^{n+1}\rangle_Z = \langle v_1, y\cdot v_1\rangle^n\langle v_k, y\cdot v_1\rangle = \langle v_k, y\cdot v_1\rangle.$$

Since $\langle v_k, y\cdot v_1\rangle_U = \langle \dot w\cdot v^+_{\omega_1}, y\cdot v^+_{\omega_1}\rangle_U$ this says precisely that $a_1 = +1$.

13.2. **Type B_n.** The approach we use for the other Dynkin types can also be applied in this case, but we shall proceed using a different approach. The adjoint group of type B_n is $SO_{2n+1}(\mathbb{C})$. We realize SO_{2n+1} as subgroup of SL_{2n+1} following Berenstein and Zelevinsky in [2] by setting

$$SO_{2n+1}(\mathbb{C}) = \{A\in SL_{2n+1}(\mathbb{C})\ | AJA^t = J\},$$

for the symmetric bilinear form

$$J = \begin{pmatrix} &&&&1\\ &&&-1&\\ &&\iddots&&\\ &-1&&&\\ 1&&&& \end{pmatrix},$$

where the signs along the antidiagonal are alternating. Let $\tilde e_i, \tilde f_i$ be the usual Chevalley generators of $\mathfrak{sl}_{2n+1}$. Then we can take $e_i = \tilde e_i + \tilde e_{2n+1-i}$ and $f_i = \tilde f_i + \tilde f_{2n+1-i}$ to be Chevalley generators of $SO_{2n+1}(\mathbb{C})$, and we have a corresponding pinning. Let $\tilde T$ denote the maximal torus of diagonal matrices in $SL_{2n+1}(\mathbb{C})$ with character group $X^*(\tilde T) = \mathbb{Z}\langle\tilde\varepsilon_1,\ldots,\tilde\varepsilon_{2n+1}\rangle/(\sum\tilde\varepsilon_i)$, where $\tilde\varepsilon_i(t)$ is the i-th diagonal entry of t. The maximal torus T of $SO_{2n+1}(\mathbb{C})$ in this embedding looks like

$$T = \left\{ t = \begin{pmatrix} t_1 & & & & & & \\ & \ddots & & & & & \\ & & t_n & & & & \\ & & & 1 & & & \\ & & & & t_n^{-1} & & \\ & & & & & \ddots & \\ & & & & & & t_1^{-1} \end{pmatrix} \middle| \; t_i \in \mathbb{C}^* \right\}.$$

The restriction of characters from $\tilde{T}$ to T gives a map $\mathrm{Res} = \mathrm{Res}_T^{\tilde{T}} : X^*(\tilde{T}) \to X^*(T)$, whose kernel is precisely generated by the characters $\tilde{\varepsilon}_i + \tilde{\varepsilon}_{2n-i+2}$ for $1 \leq i \leq n$ and $\tilde{\varepsilon}_{n+1}$.

By work of Berenstein and Zelevinsky the above realisation of $SO_{2n+1}(\mathbb{C})$ as a subgroup of $SL_{2n+1}(\mathbb{C})$ has the desirable property that the standard totally nonnegative part of $SO_{2n+1}(\mathbb{C})$ is the intersection of $SO_{2n+1}(\mathbb{C})$ with the totally nonnegative part of $SL_{2n+1}(\mathbb{C})$.

To prove Proposition 3.3 in this setting we need to consider an element $y \in U^-_{\mathbb{R}} \subset SO_{2n+1}(\mathbb{C})$ satisfying the condition (3.2), and for any such element show that y is a totally positive element of U^-. By [2, Corollary 7.2] this means we can equivalently show that y is totally positive in $\tilde{U}^-$, i.e. as a lower uni-triangular element of SL_{2n+1}. To prove total positivity in $\tilde{U}^-$ we use the 'Chamber Ansatz' of Berenstein, Fomin, and Zelevinsky [1].

In the Weyl group $\tilde{W}$ of SL_{2n+1} let $w = (s_1 s_{2n})(s_2 s_{2n-1}) \dots (s_n s_{n+1}) s_n$. Then multiplying w with itself n times gives a reduced expression for the longest element w_0 of $\tilde{W}$. By the BFZ Chamber Ansatz we can associate to this reduced expression a set of 'chamber minors' which suffice to check the total positivity of any element of $\tilde{U}^- \subset SL_{2n+1}(\mathbb{C})$.

The chamber minors can be worked out graphically using the pseudo-line arrangement for the reduced expression. For w the pseudo-line arrangement is illustrated in Figure 1. We concatenate n copies of this pseudo-line arrangement together to get the relevant pseudo-line arrangement for w_0, which we call our *preferred* pseudo-line arrangement for w_0.

Recall that a *chamber* of a pseudo-line arrangement is either a bounded connected component of the complement, or it is a connected region at the right or left-hand end of the arrangement which is bounded by two adjacent lines and an imagined third, vertical, line connecting up the ends of the two adjacent lines. To every chamber in the arrangement we associate a set $J \subset \{1, \dots, 2n+1\}$ called a *chamber set*, by recording the numbers of the lines running below the chamber. For example, the chamber in Figure 1 which is bounded by lines 1, $n+1$ and $2n+1$ has associated to it the set $J = \{1, n+2, n+3, \dots, 2n+1\}$. We always order the elements of a chamber set by size, so let $j_1 < \dots < j_k$ be the elements of J. Then one associates a minor to J by setting

$$\tilde{\Delta}_J(y) := \Delta_J(y^T) = \left\langle y^T \cdot e_{j_1} \wedge \dots \wedge e_{j_k}, v^+_{\omega_k} \right\rangle = \left\langle e_{j_1} \wedge \dots \wedge e_{j_k}, y \cdot v^+_{\omega_k} \right\rangle,$$

where Δ_J is the 'chamber minor' as defined in [1]. Here $k = |J|$ and $v^+_{\omega_k} = e_1 \wedge \cdots \wedge e_k$ is the highest weight vector of the irreducible representation $V_{\tilde{\omega}_k} = \bigwedge^k \mathbb{C}^{2n+1}$ of

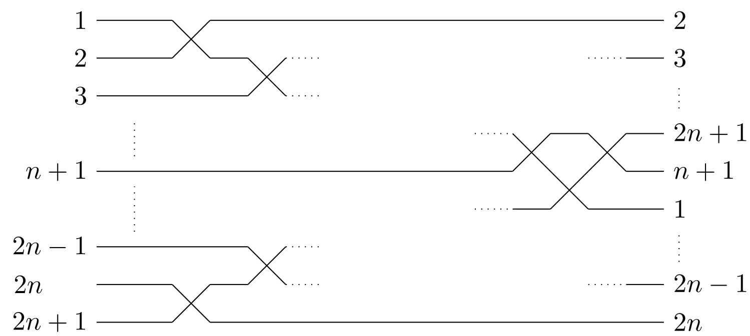

FIGURE 1. Pseudo-line arrangement for w.

$SL_{2n+1}(\mathbb{C})$. The weight $\tilde{\varepsilon}_J := \tilde{\varepsilon}_{j_1} + \ldots + \tilde{\varepsilon}_{j_k}$ associated to the chamber set J, which is the weight of the vector $e_{j_1} \wedge \ldots \wedge e_{j_k}$, is called a *chamber weight*.

By our assumption, y lies in $SO_{2n+1}(\mathbb{C})$ and we know that matrix coefficients of y of a certain type (3.2) are positive. Indeed, a chamber minor $\tilde{\Delta}_J(y) = \left\langle e_{j_1} \wedge \cdots \wedge e_{j_k},\ y \cdot v^+_{\tilde{\omega}_k} \right\rangle$ is a matrix coefficient of this allowable type precisely if $v = e_{j_1} \wedge \cdots \wedge e_{j_k}$ lies in a 1-dimensional weight space of the restricted representation, $Res^{SL_{2n+1}}_{SO_{2n+1}} V_{\tilde{\omega}_k}$. Whether or not this is the case depends entirely on the chamber weight $\tilde{\varepsilon}_J$, and we make the following definition.

Definition 13.2. A weight $\tilde{\varepsilon}$ of $\bigwedge^k \mathbb{C}^{2n+1}$ is said to satisfy the *restriction property* if for $\varepsilon = \tilde{\varepsilon}|_{\mathfrak{h}_{SO_{2n+1}}}$ the ε-weight space in the restricted representation

$$\mathrm{Res}^{SL_{2n+1}}_{SO_{2n+1}} \left(\bigwedge^k \mathbb{C}^{2n+1} \right)$$

is 1-dimensional. Equivalently, this is the case if for any weight $\tilde{\eta}$ of $\bigwedge^k \mathbb{C}^{2n+1}$ the condition $\mathrm{Res}(\tilde{\varepsilon}) = \mathrm{Res}(\tilde{\eta})$ implies $\tilde{\varepsilon} = \tilde{\eta}$.

To prove Proposition 3.3, it now remains to check that the restriction property holds for all of the chamber weights associated to our preferred pseudo-line arrangement. This will be done in two steps. First we will introduce a combinatorial condition on chamber sets which we show is equivalent to the restriction property on corresponding chamber weights. Then we will demonstrate that the chamber sets in our preferred arrangement for w_0 all satisfy this combinatorial condition.

Definition 13.3. Let J be a subset of $\{1, \ldots, 2n+1\}$. We say that J is *good* if

- either $|J| \leq n$ and for each $j \in J$ we have $2n - 2 + j \notin J$,
- or $|J| \geq n+1$ and for each $j \notin J$ we have $2n - 2 + j \in J$.

Note that if J is good then its complement J^c is good and vice versa.

Lemma 13.4. *A subset J of $\{1, \ldots, 2n+1\}$ is good if and only if the chamber weight $\tilde{\varepsilon}_J$ has the restriction property.*

Proof. Suppose R is another subset of $\{1, \ldots, 2n+1\}$ for which $\tilde{\varepsilon}_R = \tilde{\varepsilon}_{r_1} + \cdots + \tilde{\varepsilon}_{r_k}$ is a weight of $\bigwedge^k \mathbb{C}^{2n+1}$, and that $\tilde{\varepsilon}_R$ and $\tilde{\varepsilon}_J$ have the same restriction to $\mathfrak{h}_{SO_{2n+1}}$. Therefore $\tilde{\varepsilon}_J - \tilde{\varepsilon}_R$ lies in the kernel of the restriction map which is spanned by

$$\tilde{\varepsilon}_{n+1}, \qquad \tilde{\varepsilon}_i + \tilde{\varepsilon}_{2n+2-i} \qquad i = 1, \ldots, n.$$

For parity reasons, it follows that $\tilde{\varepsilon}_{n+1}$ either occurs as a summand in both $\tilde{\varepsilon}_R$ and $\tilde{\varepsilon}_J$, or it occurs in neither. Therefore

$$\tilde{\varepsilon}_J - \tilde{\varepsilon}_R = \sum_{i=1}^{n} c_i(\tilde{\varepsilon}_i + \tilde{\varepsilon}_{2n+2-i}) \tag{13.1}$$

for some coefficients c_i. Moreover necessarily, $c_i = 1$ if $i \in J \setminus R$, while $c_i = -1$ if $i \in R \setminus J$ and $c_i = 0$ otherwise. Therefore if (13.1) holds, then

$$\begin{aligned} \tilde{\varepsilon}_J &= \sum_{i, c_i = 1} (\tilde{\varepsilon}_i + \tilde{\varepsilon}_{2n+2-i}) + \sum_{j \in J \cap R} \tilde{\varepsilon}_j, \\ \tilde{\varepsilon}_R &= \sum_{i, c_i = -1} (\tilde{\varepsilon}_i + \tilde{\varepsilon}_{2n+2-i}) + \sum_{j \in J \cap R} \tilde{\varepsilon}_j. \end{aligned}$$

If J is good we will now deduce from these observations that the first sums in $\tilde{\varepsilon}_J$ and $\tilde{\varepsilon}_R$ above are zero and $\tilde{\varepsilon}_J = \tilde{\varepsilon}_R$, proving one direction of the Lemma.

Namely suppose first $|J| \le n$. Then because J is good, $j \in J$ implies $2n+2-j \notin J$. Therefore the coefficient c_j for $j \in J$ must equal 0, and therefore j must lie in R. In other words, we have $J \subset R$. Since $|R| = |J| = k$ we see that $R = J$ and the two weights $\tilde{\varepsilon}_J$ and $\tilde{\varepsilon}_R$ agree.

Similarly, if J is good and $|J| \ge n+1$ then $i \notin J$ implies $2n+2-i \in J$. In this case the analogous argument shows that the complement of J is contained in the complement of R, $J^c \subset R^c$. Again $|R| = |J| = k$ implies $R = J$.

It remains to prove that if J is not good, then there is a weight $\tilde{\varepsilon}_R$ as above, for which (13.1) holds and for which $\tilde{\varepsilon}_R \neq \tilde{\varepsilon}_J$.

If $|J| \le n$ then since J is *not* good, there is a $i \in J$ for which $2n+2-i \in J$ as well. Let $J' := J \setminus \{i, 2n+2-i\}$. Then

$$\tilde{\varepsilon}_J = (\tilde{\varepsilon}_i + \tilde{\varepsilon}_{2n+2-i}) + \sum_{j \in J'} \tilde{\varepsilon}_j.$$

Since $|J' \cup \{i\}| \le n-1$ and there are n pairs of the type $\{k, 2n+2-k\}$ in $\{1, \dots, 2n+1\}$, there must be at least one pair $\{i', 2n+2-i'\}$ for which neither element lies in $J' \cup \{i\}$, by the pigeon hole principle. Moreover, the pair $\{i', 2n+2-i'\}$ is in the complement of J. It follows that for $R := J' \cup \{i', 2n+2-i'\}$, the weight

$$\tilde{\varepsilon}_R = (\tilde{\varepsilon}_{i'} + \tilde{\varepsilon}_{2n+2-i'}) + \sum_{j \in J'} \tilde{\varepsilon}_j.$$

is different from $\tilde{\varepsilon}_J$ but has the same restriction. Therefore J does not have the restriction property.

If the subset has cardinality $\ge n+1$ and is not good, then in fact it arises as the complement J^c of a set J which is not good and of cardinality $|J| \le n$, thus reducing this case to the previous one. Namely $\tilde{\varepsilon}_{J^c}$ and $\tilde{\varepsilon}_{R^c}$, where R^c is the complement of the weight R constructed above, are weights which have the same restriction, but are distinct. Therefore the restriction property does not hold for such sets J^c. □

Lemma 13.5. *If J is a chamber set for our preferred reduced expression $w_0 = w^n$, then J is good in the sense of Definition 13.3.*

Proof. The pseudo-line arrangement is made up of n copies of the one in Figure 1. The j-th copy is illustrated in Figure 13.2. Consider the involution on $\{1, \dots, 2n+1\}$ which sends i to $\bar{i} := 2n+2-i$. Let L_j denote the set of labels of lines that run partly or entirely below the middle line in the j-th diagram, which is shown in

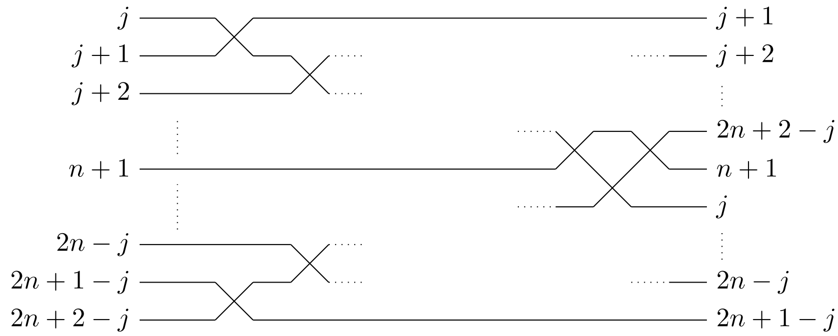

FIGURE 2. The j-th segment of the pseudo-line arrangement for $w_0 = w^n$ in the proof for type B_n.

Figure 13.2. Here by the middle line we mean the line labelled $n+1$. Since only lines j and $2n+2-j$ cross the middle line, we have $|L_j| = n+1$. Moreover by construction, the image $\overline{L_j}$ of L_j under the above involution is always the set U_j labelling all of the lines which run partly or entirely above the middle line. Clearly, the chambers lying below the middle line have chamber sets contained in L_j. The only chamber set among these to contain both j and $2n+2-j$ is L_j itself. This set is good, since it has cardinality $n+1$ and every label appears either in L_j or $\overline{L_j}$. All other chamber sets below the $n+1$-line have cardinality $\leq n$ and contain at most one of every pair $\{k, 2n+2-k\}$. Therefore these chamber sets are also good. The remaining chamber sets are the ones lying above the middle line. They all have cardinality $\geq n+1$. By inspection, they must contain either j or $2n+2-j$. And they all contain $L_j \setminus \{j, 2n+2-j\}$. Therefore they have non-empty intersection with each pair $\{k, 2n+2-k\}$. This implies that these chamber sets are also good. $\square$

Proposition 3.3 in the case of $SO_n(\mathbb{C})$ now follows from these two lemmas. Namely, if y satisfies the conditions of Proposition 3.3, then the minor $\tilde{\Delta}_J(y)$ is known to be positive if J is good (Lemma 13.4). On the other hand, we have found a reduced expression for the longest element of the Weyl group of SL_{2n+1} for which the pseudo-line arrangement has the property that all chamber sets are good (Lemma 13.5). By the BFZ chamber ansatz, if there is a reduced expression of the longest element such that all the chamber minors of y are positive then y is totally positive in $\tilde{U}^-$ (within SL_{2n+1}). Therefore y is also totally positive in U^- (within SO_{2n+1}), by [2, Corollary 7.2].

13.3. **Type C_n.** The order of the weight lattice modulo the root lattice (index of connection) is 2. Let us choose a basis $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in \mathfrak{h}^*_{\mathbb{R}}$ so that the long simple root is $\alpha_1 = 2\varepsilon_1$, and the short simple roots are $\alpha_k = \varepsilon_k - \varepsilon_{k-1}$ for $2 \leq k \leq n$. A weight $\lambda = \sum_{i=1}^n a_i \varepsilon_i$ is allowable if and only if $\sum_{i=1}^n a_i$ is even. Note that $\omega_{n-1} = \varepsilon_{n-1} + \varepsilon_n$ is allowable, while $\omega_n = \varepsilon_n$ is not. Suppose $1 \leq i \leq n-2$ is chosen so that ω_i is not allowable. Consider $V_{\omega_i+\omega_n,\mathbb{R}} \subset V_{\omega_i,\mathbb{R}} \otimes V_{\omega_n,\mathbb{R}}$. (Note that since ω_n is also not allowable and the index of connection is 2, $\omega_i + \omega_n$ is allowable.) Then for any $w \in W$ we have

$$\langle \dot{w} \cdot v^+_{\omega_i}, y \cdot v^+_{\omega_i} \rangle \langle \dot{w} \cdot v^+_{\omega_n}, v^+_{\omega_n} \rangle = \langle \dot{w} \cdot (v^+_{\omega_i} \otimes v^+_{\omega_n}), y \cdot (v^+_{\omega_i} \otimes v^+_{\omega_n}) \rangle > 0,$$

by assumption (3.2) on y. It remains to show that $\langle \dot{w} \cdot v_{\omega_n}, y \cdot v_{\omega_n} \rangle > 0$, since then $\langle \dot{w} \cdot v_{\omega_i}, y \cdot v_{\omega_i} \rangle > 0$ for all $w \in W$ and fundamental weights ω_i, whereby y has to be totally positive, because of Proposition 3.2.

We now consider $V = V_{\omega_n,\mathbb{R}}$, which is $2n$-dimensional with weights $\pm\varepsilon_k$ for $1 \leq k \leq n$. Then $2\omega_n - \alpha_n = \omega_{n-1}$, and we can apply Lemma 13.1 with $\omega = \omega_n$ and $\eta = \omega_{n-1}$. Note that $2\omega_n$ is allowable.

We may now proceed as in the proof for A_n for n odd. We consider the inclusion $U = V_{2\omega_n,\mathbb{R}} \subset V \otimes V = Z$ and look at a vector $z \in U$ with weight $\nu = \lambda + \mu \in W \cdot \omega_{n-1}$. We first argue that z can be chosen so that $\langle z, x \cdot v^+_{2\omega_n} \rangle > 0$ for all totally positive $x \in U^-_{>0}$. In [2, Corollary 7.2], Berenstein and Zelevinsky show that there is an inclusion $Sp_{2n}(\mathbb{C}) \to SL_{2n}(\mathbb{C})$ such that the image of the totally positive part $U^-_{>0}$ of the unipotent of Sp_{2n} lies in the totally nonnegative part of SL_{2n}. Now, V is the standard representation of SL_{2n} and $V \otimes V$ contains the irreducible representation $\mathrm{Sym}^2(V)$. The restriction of $\mathrm{Sym}^2(V)$ to $Sp_{2n}(\mathbb{C})$ contains the representation U, and $v^+_{2\omega_n}$ is exactly the highest-weight vector of $\mathrm{Sym}^2(V)$. By Remark 3.4, we can choose weight vectors $z \in \mathrm{Sym}^2(V)$ such that $\langle z, x \cdot v^+_{2\omega_n} \rangle > 0$ for all x which are totally positive in the unipotent of SL_{2n}. It follows that $\langle z, x \cdot v^+_{2\omega_n} \rangle > 0$ for $x \in U^-_{>0}$.

Under the inclusion $U \subset Z$, the vector z is a linear combination of $v_\lambda \otimes v_\mu$ and $v_\mu \otimes v_\lambda$ by Lemma 13.1(2). Here $\lambda, \mu \in W \cdot \omega_n$, and if $\lambda = w \cdot \omega_n$, then $v_\lambda = \dot{w} \cdot v_{\omega_n} \in V$ and similarly for μ. We have $z = Av_\lambda \otimes v_\mu + Bv_\mu \otimes v_\lambda$ for numbers A, B. Using Lemma 13.1(3), we obtain that

$$\begin{aligned} 0 &< \langle z, y \cdot v^+_{2\omega_n} \rangle_U \\ &= \langle Av_\lambda \otimes v_\mu + Bv_\mu \otimes v_\lambda, y \cdot (v^+_{\omega_n} \otimes v^+_{\omega_n}) \rangle_Z \\ &= (A + B) \langle v_\lambda, y \cdot v^+_{\omega_n} \rangle_V \, \langle v_\mu, y \cdot v^+_{\omega_n} \rangle_V \end{aligned}$$

Now A, B do not depend on y. If we let $y = x \in U^-_{>0}$, we deduce that $A + B > 0$ since $\langle v_\lambda, x \cdot v^+_{\omega_n} \rangle_V$ and $\langle v_\mu, x \cdot v^+_{\omega_n} \rangle_V$ are both positive. It follows that $\langle v_\lambda, y \cdot v^+_{\omega_n} \rangle$ and $\langle v_\mu, y \cdot v^+_{\omega_n} \rangle$ have the same sign whenever $\lambda, \mu \in W \cdot \omega_n$ and $\lambda + \mu \in W \cdot \omega_{n-1}$. By Lemma 13.1(1), for any $\lambda \in W \cdot \omega_n$, we have that $\langle v_\lambda, y \cdot v^+_{\omega_n} \rangle$ has the same sign as $\langle v^+_{\omega_n}, y \cdot v^+_{\omega_n} \rangle = 1$. This concludes the proof in the C_n case.

13.4. **Type D_n.** We take as simple roots $\alpha_1 = \varepsilon_1 + \varepsilon_2$ and $\alpha_k = \varepsilon_k - \varepsilon_{k-1}$ for $2 \leq k \leq n$. Let us consider the (spin) representation $V = V_{\omega_1,\mathbb{R}}$ with highest weight $1/2(\varepsilon_1 + \varepsilon_2 + \cdots \varepsilon_n)$. The argument is the same as for C_n (using also Remark 3.4), after applying Lemma 13.1 to $\omega = \omega_1$ and $\eta = 2\omega_1 - \alpha_1 = \omega_3$. Note that $2\omega_1$ is allowable.

13.5. **Type E_6.** The index of connection of E_6 is 3, which is odd. The proof for A_n with n even can be applied here essentially verbatim.

13.6. **Type E_7.** We fix a labelling of the Dynkin diagram by letting 7 label the minuscule node (at the end of the long leg), and 6 be the unique node adjacent to 7. The argument is the same as for C_n (using also Remark 3.4), after applying Lemma 13.1 to $\omega = \omega_7$ and $\eta = 2\omega_7 - \alpha_7 = \omega_6$. Note that $2\omega_7$ is allowable.

13.7. **Types E_8, F_4, and G_2.** The adjoint group is simply-connected, so there is nothing to prove here.

Appendix A. Quantum Schubert positivity implies affine Schubert positivity in type C

We shall need the quantum Chevalley formula of $qH^*(G/B)$, due to Peterson [32] and Fulton-Woodward [11].

For $w \in W$, define $\pi_P(w) := w_1$, where $w = w_1 w_2$ with $w_1 \in W^P$ and $w_2 \in W_P$. Also we have that 2ρ is the sum of positive roots and set $2\rho_P := \sum_{\alpha \in \Delta_{P,+}} \alpha$. Let $Q_P^\vee$ be the sublattice of $Q^\vee$ spanned by the simple coroots $\alpha_j^\vee$ for $j \in I_P$, and let $\eta_P : Q^\vee \to Q^\vee/Q_P^\vee$ be the natural projection. We let $w \gtrdot v$ denote a cover in Bruhat order.

Theorem A.1 (Quantum equivariant Chevalley formula [32, 11])**.** *Let $i \in I^P$ and $w \in W^P$. Then we have in $qH^*(G/P)$*

$$\sigma_{s_i}^P \, \sigma_w^P = \sum_\alpha \langle \alpha^\vee, \omega_i \rangle \, \sigma_{wr_\alpha}^P + \sum_\alpha \langle \alpha^\vee, \omega_i \rangle \, q_{\eta_P(\alpha^\vee)} \, \sigma_{\pi_P(wr_\alpha)}^P \tag{A.1}$$

where the first summation is over $\alpha \in \Delta_+^P$ such that $wr_\alpha \gtrdot w$ and $wr_\alpha \in W^P$, and the second summation is over $\alpha \in \Delta_+^P$ such that $\ell(\pi_P(wr_\alpha)) = \ell(w)+1-\langle \alpha^\vee, 2(\rho-\rho_P)\rangle$.

It is known that in the second summation, we only need to sum over α such that $\ell(r_\alpha) = \langle \alpha^\vee, 2\rho \rangle - 1$.

We now let G be of type C_n. We choose conventions so that the $\alpha_1, \ldots, \alpha_{n-1}$ are short and α_n is long. One may check that the positive coroots $\alpha^\vee$ satisfying $\ell(r_\alpha) = \langle \alpha^\vee, 2\rho \rangle - 1$ are exactly those of the form $\alpha_i^\vee + \alpha_{i+1}^\vee + \cdots + \alpha_j^\vee$.

Proposition A.2. *Conjecture 7.2 holds in type C_n.*

Proof. Since we already know that we have $X_{>0}^{\mathrm{af}} = X_{>0}$ and $X_{>0}^{\mathrm{af}} \subseteq X_{>0}^{\mathrm{qSchubert}}$, it suffices to show that any quantum Schubert positive point is also affine Schubert positive. By Theorem 6.4, it suffices to show that if $x \in X_{>0}^{\mathrm{qSchubert}}$ then $q_i(x) > 0$ for each $i \in I$.

Now let $i \in I$, and let v_i be the longest element in W^{P_i}, where P_i is the maximal parabolic labeled by i. Let us consider the product $\sigma^{s_i}\,\sigma^{v_i}$ and apply Theorem A.1 for the base $P = B$. Since $v_i\alpha < 0$ for any $\alpha \in \Delta_+^{P_i}$, we see that the first summation of (A.1) is empty. We note that $\ell(v_i r_\alpha) = \ell(v_i) - \ell(r_\alpha)$ if and only if $r_\alpha \in W^{P_i}$. The only such coroots $\alpha^\vee$ which also satisfy $\ell(r_\alpha) = \langle \alpha^\vee, 2\rho \rangle - 1$ are $\alpha_i^\vee$ and $\beta_i^\vee := \alpha_i^\vee + \cdots + \alpha_n^\vee$ in the case $1 \leq i \leq n-1$, and if $i = n$ we only have $\alpha_n^\vee$. Thus we obtain

$$\begin{aligned} \sigma^{s_i}\,\sigma^{v_i} &= q_i \sigma^{v_i s_i} + q_i q_{i+1} \cdots q_n \sigma^{v_i r_{\beta_i^\vee}} && \text{for } i \neq n \\ \sigma^{s_n}\,\sigma^{v_n} &= q_n \sigma^{v_i s_i} \end{aligned}$$

It follows that if $\sigma^w(x) > 0$ for all $w \in W$ then $q_i(x) > 0$ for all i. $\square$

References

[1] A. Berenstein, S. Fomin, and A. Zelevinsky, *Parametrizations of canonical bases and totally positive matrices*, Adv. Math. **122** (1996), no. 1, 49–149.

[2] A. Berenstin and A. Zelevinsky, *Total positivity in Schubert varieties*, Comment. Math. Helv. 72 (1997), no. 1, 128–166.

[3] R. Bezrukavnikov, M. Finkelberg, and I. Mirković, *Equivariant K-homology of affine Grassmannian and Toda lattice*, Compos. Math. 141 (2005), no. 3, 746–768.

[4] A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. of Math. (2) 57 (1953), 115–207.

[5] A. Buch, P.-E. Chaput, L. Mihalcea, and N. Perrin, *Finiteness of cominuscule quantum K-theory*, Annales scientifiques de l'ENS 46, fascicule 3 (2013), 477–494.
[6] R. Chhaibi, *Modèle de Littelmann pour cristaux géométriques, fonctions de Whittaker sur des groupes de Lie et mouvement brownien*, Doctoral Thesis at Université Paris VI, http://arxiv.org/pdf/1302.0902.pdf
[7] I. Ciocan-Fontanine, *Quantum cohomology of flag varieties*, Internat. Math. Res. Notices (1995), no. 6, 263–277.
[8] D. A. Cox and S. Katz, *Mirror symmetry and algebraic geometry*, American Mathematical Society, Providence, RI, 1999.
[9] A. Edrei, *Proof of a conjecture of Schoenberg on the generating function of a totally positive sequence*, Canadian J. Math. 5 (1953), 86–94.
[10] S. Fomin, *Total positivity and cluster algebras*, Proceedings of the International Congress of Mathematicians, vol. 2, 2010, 125–145.
[11] W. Fulton and C. Woodward, *On the quantum product of Schubert classes*, J. Algebraic Geom. 13 (2004), no. 4, 641–661.
[12] S. Galkin, *The conifold point*, preprint arXiv:1404.7388 [math.AG]
[13] V. Ginzburg, *Perverse sheaves on a loop group and Langland's duality*, `arXiv:alg-geom/9511007`.
[14] V. Ginzburg, *Loop Grassmannian cohomology, the principal nilpotent and Kostant theorem*, `arXiv:math/9803141`.
[15] V. Ginzburg, *Variations on themes of Kostant*, Transform. Groups 13 (2008), no. 3-4, 557–573.
[16] A. Givental and B. Kim, *Quantum cohomology of flag manifolds and Toda lattices*, Comm. Math. Phys. 168 (1995), 609–641.
[17] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977. xvi+496 pp.
[18] B. Kim, *Quantum cohomology of flag manifolds G/B and quantum Toda lattices*, Ann. of Math. (2) 149 (1999), no. 1, 129–148.
[19] ———, *Quantization and representation theory*, Proc. Oxford Conference on Group Theory and Physics, Oxford, 1977, pp. 287–316.
[20] B. Kostant, *The solution to a generalized Toda lattice and representation theory*, Adv. in Math. 34 (1979), no. 3, 195–338.
[21] ———, *Flag manifold quantum cohomology, the Toda lattice, and the representation with highest weight ρ*, Selecta Math. (N.S.) 2 (1996), 43–91.
[22] ———, *Quantum cohomology of the flag manifold as an algebra of rational functions on a unipotent algebraic group*, Deformation theory and symplectic geometry (Ascona, 1996), Math. Phys. Stud., vol. 20, Kluwer Acad. Publ., Dordrecht, 1997, pp. 157–175.
[23] S. Kumar, *Kac-Moody Groups, their Flag Varieties and Representation Theory*, Progress in Mathematics, 204. Birkhuser Boston, Inc., Boston, MA, 2002. xvi+606 pp.
[24] S. Kumar and M.V. Nori, *Positivity of the cup product in cohomology of flag varieties associated to Kac-Moody groups*, Internat. Math. Res. Notices 1998, no. 14, 757–763.
[25] T. Lam and M. Shimozono, *Quantum cohomology of G/P and homology of affine Grassmannian*, Acta. Math. 204 (2010), 49–90.
[26] G. Lusztig, *Singularities, character formulas, and a q-analogue for weight multiplicities*, in Analyse et topologie sur les espaces singuliers, Asterisque 101-102 (1982), 208–229.
[27] ———, *Total positivity in reductive groups*, Lie theory and geometry: in honor of Bertram Kostant (G. I. Lehrer, ed.), Progress in Mathematics, vol. 123, Birkhaeuser, Boston, 1994, pp. 531–568.
[28] ———, *Total positivity and canonical bases*, Algebraic groups and Lie groups, Cambridge Univ. Press, Cambridge, 1997, pp. 281–295.
[29] H. Minc, *Nonnegative matrices*, John Wiley & Sons Inc., New York, 1988.
[30] I. Mirković and K. Vilonen, *Geometric Langlands duality and representations of algebraic groups over commutative rings*, Ann. of Math. (2) 166 (2007), no. 1, 95–143.
[31] S. Mitchell, *Quillen's theorem on buildings and the loops on a symmetric space*, L'Ensignement Mathématique, 34 (1988), 123–166.
[32] D. Peterson, *Quantum cohomology of G/P*, Lecture Course, M.I.T., Spring Term, 1997.
[33] A. Pressley and G. Segal, *Loop groups*, Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1986. viii+318 pp.

[34] D. Quillen, unpublished.

[35] K. Rietsch, *Quantum cohomology of Grassmannians and total positivity*, Duke Math. J. **113** (2001), no. 3, 521–551.

[36] K. Rietsch, *Totally positive Toeplitz matrices and quantum cohomology of partial flag varieties*, J. Amer. Math. Soc. 16 (2003), no. 2, 363–392.

[37] ——, *A mirror construction for the totally nonnegative part of the Peterson variety*, Nagoya Mathematical Journal (2006), 105–142,

[38] ——, *A mirror symmetric construction for $qH_T^*(G/P)_{(q)}$*, Advances in Mathematics **217** (2008), 2401–2442. arXiv:math.AG/0511124v2

[39] ——, *A mirror symmetric solution to the quantum Toda lattice*, Communications in Mathematical Physics, Volume 309, Issue 1, pp.23-49, DOI: 10.1007/s00220-011-1308-8

[40] I.R. Shafarevich, *Basic algebraic geometry. 1. Varieties in projective space. Second edition.* Springer-Verlag, Berlin, 1994. xx+303 pp.

[41] Z. Yun and X. Zhu, *Integral homology of loop groups via Langlands dual groups*, Represent. Theory 15 (2011), 347–369.

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