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Positive Filtering for Positive Takagi-Sugeno Fuzzy Systems under ℓ_1 Performance

Xiaoming Chen

James Lam

Hak-Keung Lam

Abstract

In this paper, the positive filtering problem is addressed for positive Takagi-Sugeno (T-S) fuzzy systems under the ℓ_1 -induced performance. To estimate the output of positive T-S fuzzy systems, error-bounding positive filters are constructed. A new performance characterization is first established to guarantee the asymptotic stability of the filtering error system with the ℓ_1 -induced performance. Moreover, sufficient conditions expressed by linear programming problems are derived to design the required filters. Finally, a numerical example is presented to show the effectiveness of the derived theoretical results.

Keywords: Filtering; Linear Lyapunov functions; Linear programming; ℓ_1 -induced performance; Positive systems; T-S fuzzy systems.

1 Introduction

Positive systems can be found in different application fields such as physical, engineering and social sciences [1], [2]. The variables of positive systems are non-negative because they denote the concentrations or amounts of material in application fields. Positive systems have special structures and possess many unique features. Therefore, many new problems appear and some previous approach used for general systems are no longer applicable to positive systems. In recent years, such systems have been studied in the literature [3], [4], [5], [6]. For example, for a given transfer function, a positive state-space representation has been proposed in [7]. The state-feedback controller has been designed and moreover the controller synthesis results have been expressed by the linear matrix inequality (LMI) problems and the linear programming problems in [8] and [9]. The problem of controllability and reachability has been investigated for positive systems in [10], [11] and [12]. Stability analysis results for compartmental dynamic systems has been presented in [13], [14]. Positive observer has been designed for positive systems in [15]. In [16], [17], the model reduction problem for positive systems has been solved. In addition, the analysis and synthesis problems for some special positive systems have been addressed, like positive systems with time delays [18], [19] and 2-D positive systems [5], [20].

We note that existing research has been conducted mainly for positive linear systems and actually nonlinearities commonly exist in many practical systems. Nonlinearity is difficult to tackle and some results derived for linear positive systems cannot be directly used for nonlinear positive systems. A nonlinear system can be approximated by the T-S fuzzy model, which provides an efficient method to tackle some

Xiaoming Chen (xmingchen@hku.hk), James Lam (james.lam@hku.hk) are with the Department of Mechanical Engineering, The University of Hong Kong, Pokfulam Road, Hong Kong.

Hak-Keung Lam (hak-keung.lam@kcl.ac.uk) is with the Department of Informatics, King's College London, London, WC2R 2LS, United Kingdom.

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problems for nonlinear systems [21], [22], [23], [24]. Through the modeling approach, nonlinear systems can be transformed into a framework composed of subsystems. As a result, some research approaches applicable to linear systems can be used for nonlinear systems. Recently many authors have focused their interest on positive T-S fuzzy systems [25].

On the other hand, it is noted that previous approaches derived for the filtering problem of general systems cannot be directly used for positive systems. The reason lies in that these approaches cannot guarantee the positivity of the filter. The constraint that the filter should be positive makes the filtering problem for positive systems more complicated and cannot be easily tackled with existing approaches. It should be mentioned that in [26], a positivity preserved filter is designed for positive systems. Nevertheless, the design of positivity preserved filter for positive systems is still an open problem and remains challenging. In addition, most of the existing results about positive systems were derived with the quadratic Lyapunov function and correspondingly many results are treated under the LMI framework [27]. In recent years, many researchers derived some results based on the linear Lyapunov function [1], [9], [28], [29], [30], [31], [32], [33], [34]. By using a linear Lyapunov function, a new approach is proposed to investigate the filtering problem for positive systems.

In this paper, the problem of ℓ_1 -induced filtering is investigated for positive T-S fuzzy systems and the positivity is preserved in the filters. More specifically, a pair of ℓ_1 -induced fuzzy positive filters is first proposed such that the output of positive T-S fuzzy systems is estimated. Then, we establish sufficient conditions to design the desired positive filters. It should be noted that the results are expressed by linear programming problems.

The rest of this paper is organized as follows. Some important preliminaries and the problem formulation part are introduced in Section 2. In Section 3, the positive filter design procedure is proposed for positive T-S fuzzy systems. An example is provided in Section 4 to show the application of the theoretical results. The results are finally concluded in Section 5.

Notation: Let $\mathbb{R}$ denote the set of real numbers; $\mathbb{R}^n$ is the n -column real vectors; $\mathbb{R}^{n \times m}$ denotes the set of all real matrices of dimension $n \times m$. For a matrix $A \in \mathbb{R}^{m \times n}$, $[A]_{ij}$ denotes the element located at the i th row and the j th column; $[A]_{r,i}$ and $[A]_{c,j}$ denote the i th row, and the j th column, respectively. $A \geq 0$ (respectively, $A >> 0$) means that for all i and j , $[A]_{ij} \geq 0$ (respectively, $[A]_{ij} > 0$). The notation $A \geq B$ (respectively, $A >> B$) means that the matrix $A - B \geq 0$ (respectively, $A - B >> 0$). Let $\bar{\mathbb{R}}_+^n$ denote the nonnegative orthants of $\mathbb{R}^n$. The superscript “ T ” represents matrix transpose. $\|\cdot\|$ denotes the Euclidean norm for vectors. The 1-norm of a vector $x(k) = (x_1(k), x_2(k), \dots, x_n(k))$ is defined as $\|x(k)\|_1 \triangleq \sum_{i=1}^n |x_i(k)|$.

The induced 1-norm of a matrix $Q \triangleq [q_{ij}] \in \mathbb{R}^{m \times n}$ is denoted by $\|Q\|_1 \triangleq \max_{1 \leq j \leq n} (\sum_{i=1}^m |q_{ij}|)$. The ℓ_1 -norm of an infinite sequence x is defined as $\|x\|_{\ell_1} \triangleq \sum_{k=0}^{\infty} \|x(k)\|_1$. The space of all vector-valued functions defined on $\bar{\mathbb{R}}_+^n$ with finite ℓ_1 norm is denoted by $\ell_1(\bar{\mathbb{R}}_+^n)$. If the dimensions of matrices are not explicitly stated, it is assumed that the matrices have compatible dimensions for algebraic operations. Vector $\mathbf{1} = [1, 1, \dots, 1]^T$.

2 Problem Formulation

Consider the following fuzzy system described by the i th rule as follows:

Model Rule i : IF $\theta_1(k)$ is M_{i1} and $\theta_2(k)$ is M_{i2} and $\dots$ and $\theta_g(k)$ is M_{ig} , THEN

$$\begin{cases} x(k+1) &= A_i x(k) + B_{wi} w(k), \\ y(k) &= C_i x(k) + D_{wi} w(k), \end{cases} \quad (1)$$

where $x(k) \in \mathbb{R}^n, w(k) \in \mathbb{R}^m$ and $y(k) \in \mathbb{R}^q$ denote the system state, disturbance input and output, respectively. $i = 1, 2, \dots, r$, is the number of rules and $\theta_1(k), \theta_2(k), \dots, \theta_g(k)$ are the premise variables. M_{ie} ($i = 1, 2, \dots, r; e = 1, 2, \dots, g$) represents the fuzzy sets. Then, we have the final fuzzy system:

$$\begin{cases} x(k+1) &= \sum_{i=1}^r h_i(\theta(k))(A_i x(k) + B_{wi} w(k)), \\ y(k) &= \sum_{i=1}^r h_i(\theta(k))(C_i x(k) + D_{wi} w(k)), \end{cases} \quad (2)$$

where

$$h_i(\theta(k)) = \mu_i(\theta(k)) / \sum_{i=1}^r \mu_i(\theta(k)), \quad \mu_i(\theta(k)) = \prod_{e=1}^g M_{ie}(\theta_e(k)),$$

and $M_{ie}(\theta_e(k)) \in [0, 1]$ represents the grade of membership of $\theta_e(k)$ in M_{ie} . For all k we have

$$\sum_{i=1}^r h_i(\theta(k)) = 1, \quad h_i(\theta(k)) \geq 0, \quad i = 1, 2, \dots, r.$$

Here, the following definition is given, which will be used in the sequel.

Definition 1 *System (2) is a discrete-time positive system if for all $x(0) \geq 0$ and input $w(k) \geq 0$, we have $x(k) \geq 0$ and $y(k) \geq 0$ for $k \in \mathbb{N}$.*

Next, some useful results are introduced.

Lemma 1 ([25]) *The discrete-time system (2) is positive if and only if*

$$A_i \geq 0, \quad B_{wi} \geq 0, \quad C_i \geq 0, \quad D_{wi} \geq 0, \quad i = 1, 2, \dots, r.$$

Proposition 1 *System (2) with input $w(k) = 0$ is asymptotically stable if there exists a vector $p_i \geq 0$ (or $p_i > 0$) satisfying*

$$p_i^T A_j - p_j^T < 0, \quad (3)$$

where $i, j = 1, 2, \dots, r$.

Proof: Consider the linear Lyapunov function $V(x(k)) = \left(\sum_{i=1}^r h_i(\theta(k)) p_i \right)^T x(k)$ and we have

$$\Delta V(x(k)) = \left(\sum_{i=1}^r h_i(\theta(k+1)) p_i \right)^T x(k+1) - \left(\sum_{i=1}^r h_i(\theta(k)) p_i \right)^T x(k) \quad (4)$$

$$= \sum_{i=1}^r \sum_{j=1}^r h_i(\theta(k+1)) h_j(\theta(k)) (p_i^T A_j x(k)) - \left(\sum_{j=1}^r h_j(\theta(k)) p_j \right)^T x(k) \quad (5)$$

$$= \sum_{i=1}^r \sum_{j=1}^r h_i(\theta(k+1)) h_j(\theta(k)) (p_i^T A_j - p_j^T) x(k) \quad (6)$$

From (3), $\Delta V(x(k)) < 0$ holds and therefore system (2) is asymptotically stable. $\square$

Then, the definition of ℓ_1 -induced performance is introduced. We say that a stable positive system (2) has ℓ_1 -induced performance at the level γ if, under zero initial conditions,

$$\sup_{w \neq 0, w \in \ell_1(\mathbb{R}_+^m)} \frac{\|y\|_{\ell_1}}{\|w\|_{\ell_1}} < \gamma, \quad (7)$$

where $\gamma > 0$ is a given scalar.

In the following, we consider the stable fuzzy system:

$$\begin{cases} x(k+1) &= \sum_{i=1}^r h_i(\theta(k))(A_i x(k) + B_i w(k)), \\ y(k) &= \sum_{i=1}^r h_i(\theta(k))(C_i x(k) + D_i w(k)), \\ z(k) &= \sum_{i=1}^r h_i(\theta(k))L_i x(k), \end{cases} \quad (8)$$

where $x(k) \in \mathbb{R}^n$, $w(k) \in \mathbb{R}^l$, $y(k) \in \mathbb{R}^q$ and $z(k) \in \mathbb{R}^q$ denote the state vector, disturbance signal, measurement and the signal to be determined, respectively. System (8) is positive if for all $x(0) \geq 0$ and all input $w(k) \geq 0$, we have $x(k) \geq 0$, $y(k) \geq 0$ and $z(k) \geq 0$ for $k \in \mathbb{N}$.

It is noted that we cannot obtain the information of the transient output by designing conventional filters, since they only give an estimate of the output asymptotically. To design a filter which can be used to estimate the output at all times, we intend to find a lower-bounding estimate $\check{z}(k)$ and an upper-bounding one $\hat{z}(k)$. With the two estimates, the signal $z(k)$ can be encapsulated at all times. In the following, a pair of filters is proposed as follows:

Filter Rule i : IF $\theta_1(t)$ is M_{i1} and $\theta_2(t)$ is M_{i2} and ... and $\theta_g(t)$ is M_{ig} , THEN

$$\begin{cases} \check{x}(k+1) &= \sum_{i=1}^r h_i(\theta(k))(\underline{A}_{fi} \check{x}(k) + \underline{B}_{fi} y(k)), \\ \check{z}(k) &= \sum_{i=1}^r h_i(\theta(k))\underline{C}_{fi} \check{x}(k), \end{cases} \quad (9)$$

and

$$\begin{cases} \hat{x}(k+1) &= \sum_{i=1}^r h_i(\theta(k))(\overline{A}_{fi} \hat{x}(k) + \overline{B}_{fi} y(k)), \\ \hat{z}(k) &= \sum_{i=1}^r h_i(\theta(k))\overline{C}_{fi} \hat{x}(k), \end{cases} \quad (10)$$

where $\hat{x}(k) \in \mathbb{R}^n$, $\check{x}(k) \in \mathbb{R}^n$, $\hat{z}(k) \in \mathbb{R}^q$ and $\check{z}(k) \in \mathbb{R}^q$. $\underline{A}_{fi}$, $\overline{A}_{fi}$, $\underline{B}_{fi}$, $\overline{B}_{fi}$, $\underline{C}_{fi}$ and $\overline{C}_{fi}$ are filtering parameters.

First, the lower-bounding case is considered. With new variables $\check{x}_e(k) = x(k) - \check{x}(k)$, $\check{\xi}(k) = [x^T(k), \check{x}_e^T(k)]^T$ and $\check{e}(k) = z(k) - \check{z}(k)$, from systems (8) and (9), we have the augmented system as follows:

$$\begin{cases} \check{\xi}(k+1) &= \sum_{i=1}^r \sum_{j=1}^r h_i(\theta(k))h_j(\theta(k))(\underline{A}_{\xi ij} \check{\xi}(k) + \underline{B}_{\xi ij} w(k)), \\ \check{e}(k) &= \sum_{i=1}^r h_i(\theta(k))\underline{C}_{\xi i} \check{\xi}(k), \end{cases} \quad (11)$$

where

$$\underline{A}_{\xi ij} = \begin{bmatrix} A_i & 0 \\ A_i - \underline{B}_{fi}C_j - \underline{A}_{fi} & \underline{A}_{fi} \end{bmatrix}, \quad \underline{B}_{\xi ij} = \begin{bmatrix} B_i \\ B_i - \underline{B}_{fi}D_j \end{bmatrix}, \quad \underline{C}_{\xi i} = \begin{bmatrix} L_i - \underline{C}_{fi} & \underline{C}_{fi} \end{bmatrix}.$$

The filter (9) is designed to approximate $z(k)$ with $\check{z}(k)$. Consequently, the estimate $\check{z}(k)$ is required to be positive, which implies that the filter (9) is supposed to be a positive system. From Lemma 1, we see that that $\underline{A}_{fi} \geq 0$, $\underline{B}_{fi} \geq 0$ and $\underline{C}_{fi} \geq 0$ are needed. In the following, the fuzzy positive lower-bounding filtering (FPLF) problem is formulated.

Fuzzy Positive Lower-bounding Filtering (FPLF): Given a stable fuzzy positive system (8), find a fuzzy positive filter (9) with $\underline{A}_{fi} \geq 0$, $\underline{B}_{fi} \geq 0$ and $\underline{C}_{fi} \geq 0$ such that the filtering error system (11) is

positive, asymptotically stable and satisfies the performance $\|\hat{e}\|_{\ell_1} < \gamma_l \|w\|_{\ell_1}$ under zero initial conditions.

Similarly, one may define $\hat{x}_e(k) = \hat{x}(k) - x(k)$, $\hat{\xi}(k) = [x^T(k), \hat{x}_e^T(k)]^T$ and $\hat{e}(k) = \hat{z}(k) - z(k)$, the filtering error system is formulated as follows:

$$\begin{cases} \hat{\xi}(k+1) &= \sum_{i=1}^r \sum_{j=1}^r h_i(\theta(k)) h_j(\theta(k)) (\bar{A}_{\xi ij} \hat{\xi}(k) + \bar{B}_{\xi ij} w(k)), \\ \hat{e}(k) &= \sum_{i=1}^r h_i(\theta(k)) \bar{C}_{\xi i} \hat{\xi}(k), \end{cases} \quad (12)$$

where

$$\bar{A}_{\xi ij} = \begin{bmatrix} A_i & 0 \\ \bar{A}_{fi} + \bar{B}_{fi} C_j - A_i & \bar{A}_{fi} \end{bmatrix}, \quad \bar{B}_{\xi ij} = \begin{bmatrix} B_i \\ \bar{B}_{fi} D_j - B_i \end{bmatrix}, \quad \bar{C}_{\xi i} = \begin{bmatrix} \bar{C}_{fi} - L_i & \bar{C}_{fi} \end{bmatrix}.$$

Next, the fuzzy positive upper-bounding filtering (FPUF) problem is established.

Fuzzy Positive Upper-bounding Filtering (FPUF): *Given a stable fuzzy positive system (8), find a fuzzy positive filter (10) with $\bar{A}_{fi} \geq 0$, $\bar{B}_{fi} \geq 0$ and $\bar{C}_{fi} \geq 0$ such that the filtering error system (12) is positive, asymptotically stable and satisfies the performance $\|\hat{e}\|_{\ell_1} < \gamma_u \|w\|_{\ell_1}$ under zero initial conditions.*

3 Main Results

In this section, a pair of fuzzy positive error-bounding filters is designed to bound the signal $z(k)$ with the ℓ_1 -induced performance. First, the performance characterization is established for system (2). Then, we present sufficient conditions for the design of lower-bounding filter. Finally, we obtain parallel results for upper-bounding case.

The following result is first derived to serve as a characterization on the asymptotic stability of system (2) with the ℓ_1 -induced performance in (7).

Lemma 2 *The fuzzy positive system (2) is asymptotically stable and satisfies $\|y\|_{\ell_1} < \gamma \|w\|_{\ell_1}$ if there exists a vector $p_i \geq 0$ satisfying*

$$\mathbf{1}^T C_j + p_i^T A_j - p_j^T << 0, \quad (13)$$

$$\mathbf{1}^T D_{wj} + p_i^T B_{wj} - \gamma \mathbf{1}^T << 0, \quad (14)$$

where $i, j = 1, 2, \dots, r$.

Proof: First, we assume that $x(k) \equiv 0$. From (13), system (2) is asymptotically stable. Moreover, when $x(k) \equiv 0$, $y(k) = \sum_{i=1}^r h_i(\theta(k)) D_{wi} w(k)$ holds and from (14), we have $\|y\|_{\ell_1} < \gamma \|w\|_{\ell_1}$.

Then, we assume that there exists a k such that $x(k) \neq 0$. From (13) and Proposition 1, system (2) is asymptotically stable.

Consider the Lyapunov function $V(x(k)) = \left(\sum_{i=1}^r h_i(\theta(k)) p_i \right)^T x(k)$ and the following equations hold.

$$\begin{aligned}
\Delta V(x(k)) &= \left(\sum_{i=1}^r h_i(\theta(k+1)) p_i \right)^T x(k+1) - \left(\sum_{i=1}^r h_i(\theta(k)) p_i \right)^T x(k) \\
&= \sum_{i=1}^r \sum_{j=1}^r h_i(\theta(k+1)) h_j(\theta(k)) (p_i^T A_j x(k) + p_i^T B_{wj} w(k)) - \left(\sum_{j=1}^r h_j(\theta(k)) p_j \right)^T x(k) \\
&= \sum_{i=1}^r \sum_{j=1}^r h_i(\theta(k+1)) h_j(\theta(k)) ((p_i^T A_j - p_j^T) x(k) + p_i^T B_{wj} w(k)).
\end{aligned}$$

Let

$$\begin{aligned}
J &= \|y(k)\|_1 - \gamma \|w(k)\|_1 \\
&= \mathbf{1}^T y(k) - \gamma \mathbf{1}^T w(k) \\
&= [\mathbf{1}^T y(k) - \gamma \mathbf{1}^T w(k) + \Delta V(k)] - \Delta V(k) \\
&= \sum_{i=1}^r \sum_{j=1}^r h_i(\theta(k+1)) h_j(\theta(k)) ((\mathbf{1}^T C_j + p_i^T A_j - p_j^T) x(k) \\
&\quad + (\mathbf{1}^T D_{wj} + p_i^T B_{wj} - \gamma \mathbf{1}^T) w(k)) - \Delta V(k) \\
&= \sum_{i=1}^r \sum_{j=1}^r h_i(\theta(k+1)) h_j(\theta(k)) ((\mathbf{1}^T C_j + p_i^T A_j - p_j^T + \varepsilon \mathbf{1}^T) x(k) \\
&\quad + (\mathbf{1}^T D_{wj} + p_i^T B_{wj} - \gamma \mathbf{1}^T) w(k)) - \varepsilon \mathbf{1}^T x(k) - \Delta V(k),
\end{aligned}$$

where $\varepsilon > 0$ is sufficiently small such that $\mathbf{1}^T C_j + p_i^T A_j - p_j^T + \varepsilon \mathbf{1}^T < 0$ holds.

From (13) and (14), we have the inequality

$$J + \varepsilon \mathbf{1}^T x(k) + \Delta V(k) < 0,$$

which equals

$$\mathbf{1}^T y(k) + \varepsilon \mathbf{1}^T x(k) < \gamma \mathbf{1}^T w(k) - \Delta V(k).$$

Then, the following inequality holds

$$\sum_{k=0}^s \mathbf{1}^T y(k) + \varepsilon \sum_{k=0}^s \mathbf{1}^T x(k) < \gamma \sum_{k=0}^s \mathbf{1}^T w(k) - V(s+1).$$

Due to the asymptotical stability of system (2), when $s \rightarrow \infty$ we have

$$\sum_{k=0}^{\infty} \mathbf{1}^T y(k) + \varepsilon \sum_{k=0}^{\infty} \mathbf{1}^T x(k) \leq \gamma \sum_{k=0}^{\infty} \mathbf{1}^T w(k), \quad (15)$$

which implies

$$\|y\|_{\ell_1} < \gamma \|w\|_{\ell_1}. \quad (16)$$

The whole proof is completed. $\square$

The stability characterization of the lower-bounding augmented system (11) is proposed as follows.

Lemma 3 System (11) is asymptotically stable if there exists a vector $p_i \geq 0$ (or $p_i \gg 0$) satisfying

$$p_i^T \underline{A}_{\xi jt} - p_t^T < 0, \quad (17)$$

where $i, j, t = 1, 2, \dots, r$.

Proof: Consider the Lyapunov function $V(\check{\xi}(k)) = \left(\sum_{i=1}^r h_i(\theta(k)) p_i \right)^T \check{\xi}(k)$ and we have

$$\Delta V(\check{\xi}(k)) = \left(\sum_{i=1}^r h_i(\theta(k+1)) p_i \right)^T \check{\xi}(k+1) - \left(\sum_{i=1}^r h_i(\theta(k)) p_i \right)^T \check{\xi}(k) \quad (18)$$

$$= \sum_{i=1}^r \sum_{j=1}^r \sum_{t=1}^r h_i(\theta(k+1)) h_j(\theta(k)) h_t(\theta(k)) (p_i^T \underline{A}_{\xi jt} \check{\xi}(k)) - \left(\sum_{t=1}^r h_t(\theta(k)) p_t \right)^T \check{\xi}(k) \quad (19)$$

$$= \sum_{i=1}^r \sum_{j=1}^r \sum_{t=1}^r h_i(\theta(k+1)) h_j(\theta(k)) h_t(\theta(k)) (p_i^T \underline{A}_{\xi jt} - p_t^T) \check{\xi}(k) \quad (20)$$

From (17), we have $\Delta V(\check{\xi}(k)) < 0$. Therefore, system (11) is asymptotically stable. $\square$

Next, the performance characterization result of lower-bounding filtering error system (11) is provided.

Theorem 1 The filtering error system (11) is asymptotically stable and satisfies $\|\check{e}\|_{\ell_1} < \gamma_l \|w\|_{\ell_1}$ if there exists a vector $p_i \geq 0$ satisfying

$$\mathbf{1}^T \underline{C}_{\xi t} + p_i^T \underline{A}_{\xi jt} - p_t^T < 0, \quad (21)$$

$$p_i^T \underline{B}_{\xi jt} - \gamma_l \mathbf{1}^T < 0, \quad (22)$$

where $i, j, t = 1, 2, \dots, r$.

Based on the performance characterization, the following theorem is presented to design the desired lower-bounding filter.

Theorem 2 Given a stable fuzzy discrete-time positive system (8), a lower-bounding filter (9) exists such that the filtering error system (11) is positive, asymptotically stable and satisfies $\|\check{e}\|_{\ell_1} < \gamma_l \|w\|_{\ell_1}$ if there exist vectors $p_{1i} \geq 0$, $p_{2i} \geq 0$ and matrices $\underline{M}_{A_{fij}}$, $\underline{M}_{B_{fij}}$, $\underline{C}_{fi} \geq 0$ satisfying

$$\left[\underline{M}_{A_{fij}} \right]_{gv} \geq 0, \quad (23)$$

$$\left[\underline{M}_{B_{fij}} \right]_{gs} \geq 0, \quad (24)$$

$$L_i - \underline{C}_{fi} \geq 0, \quad (25)$$

$$p_{2ig}^T \left[A_j \right]_{gv} - \left[\underline{M}_{B_{fij}} \right]_{r,g} \left[C_t \right]_{c,v} - \left[\underline{M}_{A_{fij}} \right]_{gv} \geq 0, \quad (26)$$

$$p_{2ig}^T \left[B_j \right]_{gs} - \left[\underline{M}_{B_{fij}} \right]_{r,g} \left[D_t \right]_{c,s} \geq 0, \quad (27)$$

$$\mathbf{1}^T (L_t - \underline{C}_{ft}) + p_{1i}^T A_j + p_{2i}^T A_j - \sum_{g=1}^n \left[\underline{M}_{B_{fij}} \right]_{r,g} C_t - \sum_{g=1}^n \left[\underline{M}_{A_{fij}} \right]_{r,g} - p_{1t}^T < 0, \quad (28)$$

$$\mathbf{1}^T \underline{C}_{ft} + \sum_{g=1}^n \left[\underline{M}_{A_{fij}} \right]_{r,g} - p_{2t}^T < 0, \quad (29)$$

$$p_{1i}^T B_j + p_{2i}^T B_j - \sum_{g=1}^n \left[\underline{M}_{B_{fij}} \right]_{r,g} D_t - \gamma_l \mathbf{1}^T < 0, \quad (30)$$

where $i, j, t = 1, 2, \dots, r; g, v = 1, \dots, n; s = 1, \dots, m$. Moreover, a suitable set of $\underline{A}_{fi}$ and $\underline{B}_{fi}$ is given by

$$\begin{bmatrix} \underline{A}_{fi} \end{bmatrix}_{gv} = p_{2jg}^{-1} \begin{bmatrix} \underline{M}_{A_{fji}} \end{bmatrix}_{gv}, \quad \begin{bmatrix} \underline{B}_{fi} \end{bmatrix}_{gs} = p_{2jg}^{-1} \begin{bmatrix} \underline{M}_{B_{fji}} \end{bmatrix}_{gs}. \quad (31)$$

Proof: Note that $p_{2i} \geq 0$, it follows from (23), (24) and (31) that $\underline{A}_{fi} \geq 0$ and $\underline{B}_{fi} \geq 0$. Together with $\underline{C}_{fi} \geq 0$, it implies that the fuzzy lower-bounding filter (9) is positive.

From (31) and $p_{2i} \geq 0$, (26)–(27) become

$$\begin{bmatrix} A_j \end{bmatrix}_{gv} - \begin{bmatrix} \underline{B}_{fj} \end{bmatrix}_{r,g} \begin{bmatrix} C_t \end{bmatrix}_{c,v} - \begin{bmatrix} \underline{A}_{fj} \end{bmatrix}_{gv} \geq 0, \quad (32)$$

$$\begin{bmatrix} B_j \end{bmatrix}_{gs} - \begin{bmatrix} \underline{B}_{fj} \end{bmatrix}_{r,g} \begin{bmatrix} D_t \end{bmatrix}_{c,s} \geq 0, \quad (33)$$

which indicates that

$$\begin{aligned} A_i - \underline{B}_{fi} C_j - \underline{A}_{fi} &\geq 0, \\ B_i - \underline{B}_{fi} D_j &\geq 0. \end{aligned}$$

Together with $\underline{A}_{fi} \geq 0$, $\underline{B}_{fi} \geq 0$, $\underline{C}_{fi} \geq 0$ and (25), from (12), it shows that the filtering error system (11) is positive.

Then, from (31), we have

$$\sum_{g=1}^n \begin{bmatrix} \underline{M}_{A_{fij}} \end{bmatrix}_{r,g} = p_{2i}^T \underline{A}_{fj}, \quad \sum_{g=1}^n \begin{bmatrix} \underline{M}_{B_{fij}} \end{bmatrix}_{r,g} = p_{2i}^T \underline{B}_{fj}. \quad (34)$$

With (34), inequalities (28)–(30) equal to

$$\begin{aligned} \mathbf{1}^T (L_t - \underline{C}_{ft}) + p_{1i}^T A_j + p_{2i}^T A_j - p_{2i}^T \underline{B}_{fj} C_t - p_{2i}^T \underline{A}_{fj} - p_{1t}^T &<< 0, \\ \mathbf{1}^T \underline{C}_{ft} + p_{2i}^T \underline{A}_{fj} - p_{2t}^T &<< 0, \\ p_{1i}^T B_j + p_{2i}^T B_j - p_{2i}^T \underline{B}_{fj} D_t - \gamma_l \mathbf{1}^T &<< 0, \end{aligned}$$

which further implies that

$$\mathbf{1}^T \begin{bmatrix} L_t - \underline{C}_{ft} & \underline{C}_{ft} \end{bmatrix} + p_i^T \begin{bmatrix} A_j & 0 \\ A_j - \underline{B}_{fj} C_t - \underline{A}_{fj} & \underline{A}_{fj} \end{bmatrix} - p_t^T << 0, \quad (35)$$

$$p_i^T \begin{bmatrix} B_j \\ B_j - \underline{B}_{fj} D_t \end{bmatrix} - \gamma_l \mathbf{1}^T << 0, \quad (36)$$

where $p_i^T = \begin{bmatrix} p_{1i}^T & p_{2i}^T \end{bmatrix}$.

Therefore, by Theorem 1, the whole proof is completed. $\square$

For the upper-bounding case, the parallel result is presented as follows.

Theorem 3 *The filtering error system in (12) is asymptotically stable and satisfies $\|\hat{e}\|_{\ell_1} < \gamma_u \|w\|_{\ell_1}$ if there exists a vector $p_i \geq 0$ satisfying*

$$\mathbf{1}^T \overline{C}_{\xi t} + p_i^T \overline{A}_{\xi jt} - p_t^T << 0, \quad (37)$$

$$p_i^T \overline{B}_{\xi jt} - \gamma_u \mathbf{1}^T << 0, \quad (38)$$

where $i, j, t = 1, 2, \dots, r$.

Similarly as for the lower-bounding case, the following theorem is introduced to design the upper-bounding filter and the proof is omitted here.

Theorem 4 *Given a stable fuzzy discrete-time positive system (8), an upper-bounding filter (10) exists such that the closed-loop system (12) is positive, asymptotically stable and satisfies $\|\hat{e}\|_{\ell_1} < \gamma_u \|w\|_{\ell_1}$ if there exist vectors $p_{1i} \geq 0$, $p_{2i} \geq 0$ and matrices $\overline{M}_{A_{fij}}$, $\overline{M}_{B_{fij}}$, $\overline{C}_{fi} \geq 0$ satisfying*

$$\begin{bmatrix} \overline{M}_{A_{fij}} \end{bmatrix}_{gv} \geq 0, \quad (39)$$

$$\begin{bmatrix} \overline{M}_{B_{fij}} \end{bmatrix}_{gs} \geq 0, \quad (40)$$

$$\overline{C}_{fi} - L_i \geq 0, \quad (41)$$

$$\begin{bmatrix} \overline{M}_{A_{fij}} \end{bmatrix}_{gv} + \begin{bmatrix} \overline{M}_{B_{fij}} \end{bmatrix}_{r,g} \begin{bmatrix} C_t \end{bmatrix}_{c,v} - p_{2ig}^T \begin{bmatrix} A_j \end{bmatrix}_{gv} \geq 0, \quad (42)$$

$$\begin{bmatrix} \overline{M}_{B_{fij}} \end{bmatrix}_{r,g} \begin{bmatrix} D_t \end{bmatrix}_{c,s} - p_{2ig}^T \begin{bmatrix} B_j \end{bmatrix}_{gs} \geq 0, \quad (43)$$

$$\mathbf{1}^T (\overline{C}_{ft} - L_t) + p_{1i}^T A_j - p_{2i}^T A_j + \sum_{g=1}^n \begin{bmatrix} \overline{M}_{B_{fij}} \end{bmatrix}_{r,g} C_t + \sum_{g=1}^n \begin{bmatrix} \overline{M}_{A_{fij}} \end{bmatrix}_{r,g} - p_{1t}^T < 0, \quad (44)$$

$$\mathbf{1}^T \overline{C}_{ft} + \sum_{g=1}^n \begin{bmatrix} \overline{M}_{A_{fij}} \end{bmatrix}_{r,g} - p_{2t}^T < 0, \quad (45)$$

$$p_{1i}^T B_j - p_{2i}^T B_j + \sum_{g=1}^n \begin{bmatrix} \overline{M}_{B_{fij}} \end{bmatrix}_{r,g} D_t - \gamma_u \mathbf{1}^T < 0, \quad (46)$$

where $i, j, t = 1, 2, \dots, r$; $g, v = 1, \dots, n$; $s = 1, \dots, m$. Moreover, a suitable set of $\overline{A}_{fi}$ and $\overline{B}_{fi}$ is given by

$$\begin{bmatrix} \overline{A}_{fi} \end{bmatrix}_{gv} = p_{2jg}^{-1} \begin{bmatrix} \overline{M}_{A_{fji}} \end{bmatrix}_{gv}, \quad \begin{bmatrix} \overline{B}_{fi} \end{bmatrix}_{gs} = p_{2jg}^{-1} \begin{bmatrix} \overline{M}_{B_{fji}} \end{bmatrix}_{gs}. \quad (47)$$

4 Illustrative Example

An illustrative example is presented in the following to demonstrate the effectiveness of the theoretical results.

Consider system (8) with the system matrices:

$$\begin{aligned} A &= \begin{bmatrix} 0.15 + 0.1 \sin x_1(k) & 0.5 \\ & 0.35 \end{bmatrix}, \quad B = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, \\ C &= \begin{bmatrix} 0.2 & 0.5 \end{bmatrix}, \quad D = 0.8, \quad L = \begin{bmatrix} 0.2 & 0.1 \end{bmatrix}. \end{aligned}$$

The fuzzy model for the nonlinear system is as follows:

$$\begin{cases} x(k+1) = \sum_{i=1}^2 h_i(\theta(k))(A_i x(k) + B_i w(k)), \\ y(k) = \sum_{i=1}^2 h_i(\theta(k))(C_i x(k) + D_i w(k)), \\ z(k) = \sum_{i=1}^2 h_i(\theta(k))L_i x(k), \end{cases}$$

with

$$\begin{aligned} A_1 &= \begin{bmatrix} 0.05 & 0.5 \\ 0.35 & 0.15 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.25 & 0.5 \\ 0.35 & 0.15 \end{bmatrix}, \\ B_1 &= B_2 = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, \quad C_1 = C_2 = \begin{bmatrix} 0.2 & 0.5 \end{bmatrix}, \\ D_1 &= D_2 = 0.8, \quad L_1 = L_2 = \begin{bmatrix} 0.2 & 0.1 \end{bmatrix}, \end{aligned}$$

where

$$h_1(\theta(k)) = \frac{1 + \sin x_1(k)}{2}, \quad h_2(\theta(k)) = \frac{1 - \sin x_1(k)}{2}.$$

For $\gamma_l = 0.5$, by solving the conditions in Theorem 2 via Yalmip, a feasible solution is achieved with

$$\begin{aligned} p_{11} &= p_{12} = \begin{bmatrix} 0.6257 & 0.7104 \end{bmatrix}^T, \\ p_{21} &= p_{22} = \begin{bmatrix} 0.8924 & 0.9063 \end{bmatrix}^T, \end{aligned}$$

which further yields the matrices of the lower-bounding filter as

$$\begin{aligned} \underline{A}_{f1} &= \begin{bmatrix} 0.0035 & 0.3518 \\ 0.1942 & 0.0531 \end{bmatrix}, \quad \underline{A}_{f2} = \begin{bmatrix} 0.1696 & 0.3468 \\ 0.2464 & 0.0534 \end{bmatrix}, \\ \underline{B}_{f1} &= \begin{bmatrix} 0.0260 \\ 0.0883 \end{bmatrix}, \quad \underline{B}_{f2} = \begin{bmatrix} 0.0361 \\ 0.0870 \end{bmatrix}, \quad \underline{C}_{f1} = \underline{C}_{f2} = \begin{bmatrix} 0.1809 & 0.0865 \end{bmatrix}. \end{aligned}$$

Similarly, for $\gamma_u = 0.5$, by solving the conditions in Theorem 4, a feasible solution is achieved with

$$\begin{aligned} p_{11} &= \begin{bmatrix} 0.8554 & 0.9498 \end{bmatrix}^T, \quad p_{12} = \begin{bmatrix} 0.8557 & 0.9497 \end{bmatrix}^T, \\ p_{21} &= \begin{bmatrix} 1.8449 & 1.9120 \end{bmatrix}^T, \quad p_{22} = \begin{bmatrix} 2.0480 & 1.9032 \end{bmatrix}^T, \end{aligned}$$

which further yields the matrices of the upper-bounding filter as

$$\begin{aligned} \overline{A}_{f1} &= \begin{bmatrix} 0.0990 & 0.4665 \\ 0.3590 & 0.0637 \end{bmatrix}, \quad \overline{A}_{f2} = \begin{bmatrix} 0.2589 & 0.4664 \\ 0.3334 & 0.0637 \end{bmatrix}, \\ \overline{B}_{f1} &= \begin{bmatrix} 0.1760 \\ 0.2983 \end{bmatrix}, \quad \overline{B}_{f2} = \begin{bmatrix} 0.1762 \\ 0.2982 \end{bmatrix}, \\ \overline{C}_{f1} &= \begin{bmatrix} 0.2617 & 0.1578 \end{bmatrix}, \quad \overline{C}_{f2} = \begin{bmatrix} 0.2620 & 0.1576 \end{bmatrix}. \end{aligned}$$

In this example, we use the following external disturbance

$$w(k) = \begin{cases} 0.3, & 5 \leq k \leq 10, \\ 0, & \text{otherwise.} \end{cases} \quad (48)$$

Figure 1 show the output $z(k)$ of the original system, the lower estimate $\hat{z}(k)$ and the upper estimate $\hat{z}(k)$.

5 Conclusion

In this paper, the problem of positive filtering for positive T-S fuzzy systems under ℓ_1 performance has been addressed. Novel performance characterization of the filtering error system has been established. Based on the novel characterization, sufficient conditions have been developed for the existence of positive error-bounding filters. Moreover, all the derived conditions are expressed by linear programming problems. Finally, an example has been presented to demonstrate the effectiveness of the proposed approach.

Figure 1: Output $z(k)$ and its estimates.

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