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Seeing-as and Mathematical Creativity

Michael Beaney and Robert Clark

Introduction

Over the last twenty years, aspect perception has been receiving increasing philosophical attention, inspired by the continuing influence of the remarks on seeing-as that form the first twenty pages or so of section xi of Part II of Wittgenstein's *Philosophical Investigations*, first published in 1953. The initial influence of these remarks occurred mainly in aesthetics, Richard Wollheim's *Art and Its Objects* (1968) being a prominent example. But it soon became a standard topic in philosophy of mind and philosophical psychology, and books on Wittgenstein's later philosophy, in particular, would often contain a chapter on aspect perception.¹ The topic tended to be seen, however, as somewhat self-contained, and its significance for areas outside aesthetics and philosophical psychology was rarely discussed.

In 1990 Stephen Mulhall published *On Being in the World*, offering the first full-length account of aspect perception through a comparison of the views of Wittgenstein and Heidegger, which demonstrated its broader relevance. From the early 1990s, after his break with Peter Hacker, with whom he had collaborated on a series of books on Wittgenstein in the 1980s, Gordon Baker began to explore and stress the methodological importance of Wittgenstein's thinking about aspect perception; and after his premature death, a collection of his papers was published in 2004, *Wittgenstein's Method: Neglected Aspects*. Coupled with the emergence of 'New Wittgensteinian' readings led by Cora Diamond and James Conant, and greater attention to Wittgenstein's last writings, as reflected in talk of the 'third Wittgenstein',² these developments culminated in the first collection of essays devoted to aspect perception, *Seeing Wittgenstein Anew*, published in 2010.

In introducing the essays in this collection, the editors William Day and Victor Krebs write that they contain the "recurring discovery ... that there is something to be found in his remarks on aspect-seeing that is crucial to, yet all but overlooked in, the reception of the later Wittgenstein" (2010, p. 1). These remarks, they go on, far from being a detour in Wittgenstein's "long and meandering journeyings" (as he put it in his preface to the *Investigations*), are "the expression of a theme whose figures and turns we might have been hearing, however faintly, all along" (2010, p. 5). They constitute "Wittgenstein's indirect meditation on the difficulties of receiving his (later) philosophical methods" (2010, p. 10).

¹ See e.g. Budd 1989; McGinn 1997.

² See especially Moyal-Sharrock 2004.

These claims are both bold and controversial. Peter Hacker, for example, has long held that the remarks on seeing-as are relatively unimportant. This is reflected, most recently, in the decision made by Hacker and his collaborator on the revised translation of the *Investigations*, Joachim Schulte, to rename Part II of the *Investigations* ‘Philosophy of Psychology – A Fragment’, to emphasize not only its draft character but also its tangential connection to Part I.³ The present essay is written in the conviction that Wittgenstein’s remarks on aspect perception are indeed of fundamental methodological importance, not just to Wittgenstein’s philosophy but to all areas of philosophy. We focus here on their relevance to philosophy of mathematics, for two reasons. First, this relevance has been almost entirely ignored,⁴ yet thinking through the questions raised by aspect perception sheds light on issues in philosophy of mathematics, especially in relation to mathematical creativity. Second, it is still the case that scholars underestimate how important the philosophy of mathematics was to Wittgenstein; earlier drafts of the *Philosophical Investigations*, for example, included material on the subject. His remarks on rule-following are now recognized to be as fundamental to his philosophy of mathematics as they are to his philosophy of mind. The same applies to his remarks on seeing-as, but there is still work to be done to show this.

In this essay we want to link Wittgenstein’s remarks on seeing-as with another aspect of his investigations in the philosophy of mathematics. This involves a contrast between two ways of seeing the development of mathematical concepts over time. Remarking on Cantor’s ‘diagonal proof’ of the uncountability of the real numbers, Wittgenstein points to what he describes as a “dangerous, deceptive thing” (see *RFM*, p. 131). Certain ways of looking at mathematical conceptual development, he suggests, can make us assume we are discovering a ‘fact of nature’, when actually what is going on may rather be thought of as ‘the determination of a concept’ (*ibid.*).⁵

This relates closely to Wittgenstein’s characterization of mathematics as ‘invented’ rather than ‘discovered’ (see, e.g., *RFM*, p. 99). But there is another way of looking at this contrast, which we also want to bring out in what follows. Discussions of aspect perception often focus on the first person singular – what *I* see in the duck-rabbit picture or how *I* see a change of aspect in the Necker cube. Consideration of conceptual development in mathematics and its relationship with aspect perception, however, reveals the importance of recognizing the role of first person *plural* views. *We* – users of mathematics, members in a wide sense of the mathematical community – take certain aspects of mathematics to be thus-and-so rather than otherwise. This enables us to see what is going on in mathematical creativity, in its widest sense, not simply as a Gestalt switch in an individual’s phenomenology but as involving a change in aspect perception within a whole community. Recognition of the

³ In this paper, we quote from the 3rd edition (following the traditional practice of giving page numbers to Part II), though have consulted the 4th edition (as revised by Hacker and Schulte) in checking the translation.

⁴ The only exception is Juliet Floyd’s work. The only paper on philosophy of logic and mathematics in Day and Krebs’ collection is by Floyd (2010).

⁵ See below, *passim*, and especially section 4.

importance of first person *plural* aspect perception, in other words, not only sheds light on the historical development of mathematical concepts but also deepens our appreciation of the role that the community of mathematics users plays in such development.

With this in mind, then, we proceed as follows. In section 1 we consider one of the most important sources – perhaps *the* most important source – of philosophical methodology in Plato’s *Meno*. It is here that mathematical methodology first starts influencing philosophy, and, surprising as it may seem, we can see aspect perception as playing a pivotal role. In section 2 we turn from ancient Greek geometry to ancient Greek arithmetic, examining the conceptual development that occurred in the ‘discovery’ of irrational numbers. In section 3 we look at the emergence of non-Euclidean geometry; and in section 4 we switch back again to arithmetic and consider the ‘discovery’ of transfinite numbers. Recognizing the role that first-person *plural* aspect perception plays, a central theme in our discussion is the way that criteria for the relevant mathematical concepts come apart in these key cases, enabling different aspects of those concepts to be seen, which then allows something like a spontaneous choice to be made as to which aspect to take as primary. As we conclude in the final section, what is going on here is best described neither as ‘discovery’ nor as ‘invention’ of something entirely new. There are facts to be revealed, and creativity to be exhibited, but what is crucial is the opening up of different aspects of something, the perception of which prompts a choice that sooner or later ‘catches on’ in a mathematical community and proves fruitful.

1 Meno’s Paradox and Geometry

Philosophical methodology, and especially philosophical analysis (in its various forms), has its roots in ancient Greek geometry and Plato’s dialogues. The influence of Greek geometry on philosophy is first revealed in Plato’s *Meno*, the dialogue in which Socrates cross-examines a slave boy in an attempt to get him to ‘recollect’ the answer to a geometrical problem. The cross-examining is itself intended to answer what we now know as Meno’s paradox or the paradox of inquiry, which poses a dilemma that threatens the very possibility of searching for knowledge. Either we know something, in which case inquiry is pointless; or we do not, in which case we will not know what to inquire into. Either way, it would seem, there can be no genuine inquiry. Much has been written about this over the centuries, and new aspects have revealed themselves in many of these discussions.⁶ Here we want to explore a further aspect that, we feel, has not been appreciated: an aspect, indeed, in which aspect perception itself is seen as central to the solution to both the geometrical problem and Meno’s paradox.

The geometrical problem that Socrates induces the slave boy to solve is that of constructing a square with twice the area of a given square. Socrates starts by asking

⁶ For the fullest recent account, see Fine 2014.

the boy if he knows what a square is, a geometrical figure with four equal sides and where two equal lines can be drawn through its centre from the midpoints of its sides. (Throughout the interrogation, we are to imagine Socrates drawing lines in the sand to illustrate what is being said: see Diagram 1, where the initial square is represented as ABCD.) The boy says that he does, and when Socrates asks how big the square would be if its sides were two feet long, the boy comes to see that its area would be four square feet. He also recognises that a figure twice as big would have an area of eight square feet. However, when Socrates asks him how long the sides of such a square would be in comparison with the sides of the original square, the boy immediately replies “twice the length”. By being shown a construction of a square with sides of this length, however, the boy realises that its area is sixteen, not eight square feet (AEFG in Diagram 1). Since the sides clearly need to be between two and four feet in length, the boy then suggests that they would be three feet long, but again he is brought to see his error by being shown that this would yield a square with an area of nine, not eight square feet. At this point, the boy is thoroughly perplexed, as Socrates breaks off to comment to Meno. (See *Meno*, 82b-84c.)

Diagram 1

It is here that a move of fundamental importance in Plato's work – and in the history of philosophy generally – is made. With the boy now primed to ‘learn’, Socrates goes on to help him provide the right answer. Drawing lines from the corners of the original square and the other three squares that have the same area, Socrates gets the boy to recognize that the resulting figure is a square which does indeed have

twice the area of the initial square (BHID in Diagram 1). Telling the boy that the line that goes across two opposite corners of a square is called the ‘diagonal’, Socrates formulates the answer to the problem he set, to which the boy assents. (*Meno*, 84d-85b.) This answer can be stated as follows:

(AN) A square with twice the area of a given square can be constructed from the diagonal of the given square (i.e., has sides whose length is the length of the diagonal of the given square).

Knowledge of (AN), Socrates concludes to Meno, has been attained by the boy merely by using his own resources. As Socrates commented to Meno at the time (82e), the fact that the boy initially got the answer wrong shows that he was not being *told* what to believe. Even if Socrates suggested something, it was still the boy who had to accept or reject it, based on what he himself thought. So he must have had something within him, which he was ‘recovering’, that enabled him to solve the problem; and this process of ‘recovering’, Socrates suggests, is recollection (85c-d).

This idea of ‘recollection’ has, of course, been hugely controversial. But the general shape of Socrates’ answer to Meno’s paradox is compelling. We cannot ‘fully’ know the answer to a question prior to inquiry, or else we would be impaled on the first horn of the dilemma; yet there must be something we know, if we are to avoid the second horn. The doctrine of ‘recollection’ is intended to provide some such position between the two horns. What the boy has within him is *some* geometrical and arithmetical knowledge, such as that a square has four equal sides and that $2 \times 2 = 4$. Socrates may need to give him some further terms with which to articulate the solution, as captured in (AN), such as the term ‘diagonal’, but this, too, presupposes that the boy has some knowledge. What is absolutely crucial, however, is that the boy himself must recognize the solution as the solution, and it is here that aspect perception is involved. For the turning-point in the boy’s understanding occurs when he switches from seeing the original square as made up of four smaller squares to seeing it as made up of two triangles. Once this switch has occurred, the way is open to seeing the square on the diagonal as twice as big, and hence the solution to the geometrical problem. This can then be ‘proved’ by showing that the larger square is composed of four triangles, each of which has half the area of the original square; but the key step is the aspect switch.

This switch is something that the boy must make for himself, and this, we suggest, is what lies behind Socrates’ claim about ‘recollection’ – whatever other problems this notion may generate. The boy is capable of seeing this aspect (and perhaps has even seen it on other occasions), and no one can see it for him. It would not be inappropriate for the boy to say something on the lines of: “Of course, I should have realized (‘recollected’) that it can be seen as two triangles, which obviously solves the problem”. The use of such terms as ‘of course’ and ‘obviously’ suggest just

how naturally it is for the boy to come to see the answer.⁷ As well as underlying the particular geometrical example in the *Meno*, aspect perception also provides a model for answering the general paradox of inquiry. For in aspect perception there is indeed something already grasped, in some form, for an aspect (or new aspect) to be recognized. In pursuing an inquiry, we must have enough prior knowledge to do so fruitfully and to be able to recognize an answer when it comes. Yet at the heart of what we ‘learn’ may be the perception of a new aspect (an aspect that we have not previously seen – or at any rate, not in that particular context of inquiry).

2 Incommensurability and Arithmetic

Recognition of the diagonal of a square – and in particular, the ‘incommensurability’ between the diagonal and the side of a square – was to raise philosophically significant issues in arithmetic as well as in geometry. Here the ‘discovery’ of irrational numbers was a key stage in the development of the mathematical concept of a number, and lying at the core of this development was a move that essentially required a shift of conceptual aspect. We elucidate this in the present section.

The problem of incommensurability is mentioned by Aristotle in the *Prior Analytics*. Giving an example of a *reductio* proof, Aristotle refers somewhat cryptically to a proof of incommensurability:

All those who reach a conclusion through the impossible deduce the falsehood by a syllogism, but prove the initial thesis from a hypothesis, when something impossible results from the assumption of the contradictory. For example, one proves that the diagonal is incommensurable because odd numbers turn out to be equal to even ones if one assumes that it is commensurable. Now that odd numbers turn out to be equal to even ones is deduced by syllogism, but that the diagonal is incommensurable is proved from a hypothesis, since a falsehood results because of its contradictory. (*Prior Analytics*, I, 23, 41a23–9)

This can be filled out as follows. Supposing the side of a square to be of unit length, we can use the result that Meno’s slave boy ‘recollected’ (and which is generalized in Pythagoras’ theorem) to deduce that the square formed on the diagonal has an area of 2 square units. Now, Aristotle claims, this diagonal is ‘incommensurable’. That is, the length of the diagonal of the square is not expressible as a ratio of the side of the square – there is no rational number equal to $\sqrt{2}$. The proof, as Aristotle says, proceeds by contradiction.⁸ Here it is in present-day terminology:

Suppose $\sqrt{2}$ is a rational number. That is, $\sqrt{2} = \frac{a}{b}$ for some integers a and b . We can assume that a and b are not both even, for if they are we can divide them both

⁷ One is reminded here of Wittgenstein’s discussion in the *Philosophical Investigations* of exclaiming “Now I can go on!” when we suddenly grasp the rule of a series (§§ 151, 179ff.).

⁸ This is essentially the proof to be found in some editions of Euclid’s *Elements*, though it seems it is a later interpolation; see Heath 1956, vol. 3, p. 2.

by 2 leaving $\sqrt{2} = \frac{a'}{b'}$, repeating if necessary if the new numerator and denominator are both even. So, let $\sqrt{2} = \frac{a}{b}$ where at most one of a and b is even.

Now we argue as follows:

$$\sqrt{2} = \frac{a}{b} \Rightarrow a = \sqrt{2}b \Rightarrow a^2 = 2b^2$$

That is, a^2 is even. So a must be even, since the square of an odd number is odd. Since at most one of a and b is even, it follows that b is odd. But now, since a is even, it must be that $a = 2p$ for some integer p . So $a^2 = 4p^2$. But from above, $a^2 = 2b^2$. So $2b^2 = 4p^2$, or, dividing by 2, $b^2 = 2p^2$. That is, b^2 is even and so b is even.

There is our contradiction. From the supposition that the length of the diagonal of a square ($\sqrt{2}$ units where the side is 1 unit) is commensurable with the side (i.e. that $\sqrt{2} = \frac{a}{b}$ for some integers a and b), we have proved that “odd numbers turn out to be equal to even ones”, as Aristotle puts it – we have proved that our integer b is both odd and even. So “that the diagonal is incommensurable [viz. that $\sqrt{2}$ is irrational] is proved from a hypothesis, since a falsehood results because of its contradictory [that the diagonal is commensurable, viz. that $\sqrt{2}$ is rational]”.

What role did this proof of incommensurability play in the development of the concept of number? It might seem natural to us now to suppose that it forced mathematicians to recognize that numbers could be irrational as well as rational. But there is another way of seeing things that emerges once we appreciate two aspects that the ancient Greeks, prior to the discovery of this proof, took numbers to have. Numbers were viewed both as measures of lengths and also as ratios. The proof of incommensurability shows that one cannot hold both views. The ancient Greeks continued to regard numbers as measures of lengths and hence came to reject the view that all numbers could be expressed as ratios, that is, are rational. But it would also have been possible to have held firm to the view that numbers are ratios and hence to have abandoned the view that all lengths can be numbered.

Another way to express this would be to say that the ancient Greek concept of number, prior to the proof of incommensurability, had two key criteria: (1) numbers are measures of lengths, and (2) numbers are expressible as ratios. (1) was understood as an identity claim: every length can be assigned a number and every number can be represented by a length; in other words, there is a one–one correlation between numbers (in arithmetic) and lengths (in geometry), with a given length chosen as the unit length. (2) is simply the claim that all numbers are rational. In many cases (the ‘rational’ ones), these two criteria give the same result. But the proof of incommensurability showed that in some cases (the ‘irrational’ ones), such as in

considering the diagonal of a unit square, the two criteria come apart, forcing us to decide which of the two criteria is the more important.

We know which way the decision went, of course. The second criterion was rejected in order to keep the first. $\sqrt{2}$ is a number in good standing, though irrational, and every length is (still) measurable numerically. The concept of number was determined accordingly: it was broadened to include irrational as well as rational numbers. We now take all this for granted, but if we go back to the origin of the determination, we can see that it was by no means necessary. At the core of this determination was a choice of conceptual aspect, and although we might find it hard now to see things in any other way, it is important to recognize that the choice was there and that our concept of number might have developed in another way. We should also note that while the choice between the different ways of seeing – of determining the concept – was, we might say, forced by mathematics itself (the *proof* above), the outcome of the choice was not so determined. The choice between criteria, whatever its motivation, does not answer uniquely to intra-mathematical considerations; mathematics itself, we might say, allows either choice, while eventually accepting the choice that is made.

A rationale for the decision is not hard to find. Had mathematicians stuck to the view that numbers are ratios, then not only would they have had to conclude that some lengths cannot be numbered but they would also have had to hold that while some expressions of the form ' $\sqrt{n}$ ' – i.e. 'the square root of n ' – denote numbers (such as ' $\sqrt{25}$ ', representing the length of the hypotenuse of a right-angled triangle the other two sides of which have lengths 3 and 4), other such expressions (such as ' $\sqrt{2}$ ') do not. Of course, they came to hold instead that only some such expressions denote rational numbers; but that does not seem so radical a view as holding that some such expressions, despite their similarity of form, do not denote numbers at all.

Before looking at some other examples involving seeing-as, let us briefly take stock, drawing out some implications for our understanding of mathematical creativity. We have suggested that both aspect perception and what we called aspect conception are involved in mathematics and play a role in mathematical creativity. The former was illustrated in recognizing that a square of twice the area of a given square can be constructed on the diagonal of that square, as shown (implicitly) by the slave boy in Plato's *Meno*. This is aspect perception in a familiar sense. Just as one can switch from seeing the duck-rabbit as a duck to seeing it as a rabbit, so one can switch from seeing a square as made of four smaller squares to seeing it as made up of two triangles (although it may be that one has to 'imagine' lines in the diagram being considered). Arguably, at the heart of the creativity involved in solving this mathematical problem is a switch in aspect perception. The point can be readily generalized: as anyone who has had experience of doing Euclidean geometry knows, solving problems of construction or proof involves looking for different ways of

seeing figures, perhaps by adding appropriate auxiliary lines, to draw on the resources of theorems already proved.⁹

Aspect conception was illustrated in the conceptual creativity involved in broadening the concept of number to include the irrationals as well as rationals. Here, it would seem, there is not so much aspect switch as aspect focus. Two aspects are already present, and the creativity arises from distinguishing them and taking one as primary and rejecting or subordinating the other. We might put the contrast in duck-rabbit terms. In the geometrical case, we see something as a duck but need to see it as a rabbit in order to solve our problem. In the arithmetical case, we see something simultaneously (somehow) as both a duck and a rabbit, but need to take it as just a rabbit, say, to make progress.

In the geometrical case, too, however, both aspects are somehow already there (requiring mere ‘recollection’), and progress occurs by selecting and focusing on one; so the key difference does not lie here. What is different in the arithmetical case is the way that the aspect focusing results in the determination of a concept. One aspect is chosen as the defining criterion for the broader concept (number), with the other aspect relegated as a criterion for the subconcept (rational number). Where, previously, we had one concept and some unclarity as to its applicability, we now have more than one concept (number, rational number, irrational number) and greater clarity as to their applicability and logical interrelations.

A final point may be made at this half-way stage. The creativity involved in each of these two examples itself has many aspects. What we have hoped to show is that one central (though not defining) aspect is the role played by aspect perception, and more specifically, by what we have called aspect focusing. We do not want to generalize and claim that all creativity involves aspect focusing; but this does seem to lie at the heart of the creativity involved in these particular mathematical examples. If that is right, then to understand the creativity involved, we need to ‘recreate’ in ourselves that aspect focusing, in just the same way that the slave boy had to recognize for himself the aspect switch required to solve the geometrical problem. We can only understand the creativity from ‘inside’, by attempting to recover the original situation in which the aspect perception and focusing occurred. This may take some of the ‘mystery’ out of creativity but only by enabling us to experience, first-hand, the wonderful mental operations that are involved in creative processes. What *we* do is to bring creativity back from its mysterious to its everyday use.

3 Straight Lines and Non-Euclidean Geometry

We mentioned the move from first person singular to first person plural aspect perception in the introduction, and we can see this illustrated in the two cases we have just considered. The slave boy in the *Meno* may need to make the aspect switch for

⁹ For a more detailed example of this, see Beaney 2005, App. 1.

himself to appreciate the solution to the geometrical problem Socrates posed, and we may all have to do it for ourselves if we are also to appreciate the solution, but this possibility is rooted in our shared capacities for aspect perception, and this solution only works because of these shared capacities. In the case of the concept of number coming to include both rational and irrational numbers, this is no less a matter of a new way of seeing things – here through aspect focus rather than aspect switch – ‘catching on’ in determining a community’s joint conceptualisation. We will see this illustrated further in the next two sections.

We will look in the present section at the development of non-Euclidean geometry. As in the case of the concept of number just discussed, we can see this development as occurring through criteria for the application of concepts coming apart. We will focus here on two different but connected pairs of criteria: one general, concerning how we see geometry as a whole, and one specific, concerning how we conceptualise ‘straight line’. With regard to the first, what we have is a contrast between seeing geometry as an abstract axiomatic system, with associated apriority, certainty and necessity, and seeing it as descriptive of empirical physical space. From the time of the ancient Greeks to the middle of the nineteenth century, however, these two ways of seeing geometry seemed ineluctably joined. (Euclidean) geometry, it seemed, could be known a priori and yet was also applicable to physical space. In the late eighteenth century, Kant gave an account of how both could be maintained; but although this account was controversial from the beginning, it was only with the advent of non-Euclidean geometry that a diagnosis of the problems became possible.

So how did these two ways of seeing geometry come apart? As in the case of number discussed above, certain intra-mathematical developments seem to have been required. Interestingly, the creative work of Bolyai and Lobachevsky was apparently not sufficient (contrary, perhaps, to conventional ways of telling the story); their work was at first received grudgingly (where it was recognised at all), and it took some time for non-Euclidean geometry to win acceptance.¹⁰ So let us sketch some of the key developments.

Talk of non-Euclidean geometry, as is well known, directs us back to Euclid and his (in)famous fifth postulate. Here is how Euclid formulates it:

That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles. (*Elements*, I, Post. 5)

We will take up the question of what it is for a line to be straight shortly. Compared to the first four postulates, this fifth postulate seems relatively complex. So if it is true, might we not be able to *prove* it from the other axioms (definitions, postulates and common notions)? Proclus was just one influential commentator who thought so,

¹⁰ For a full account, to which we are indebted here, see Gray 1979.

writing that it should be struck from the postulates altogether since it is a theorem (*Commentary*, p. 150).

Skipping millennia of mathematical investigation into this infamous postulate, however, we now know it to be independent of the other Euclidean axioms. In the 1820s, in one of those remarkable apparent coincidences in the history of ideas, János Bolyai and Nikolai Lobachevsky independently developed geometries that denied it. Each seems also to have thought of these alternative geometries as genuine rivals to Euclidean geometry, in the sense of being candidates for providing the true description of empirical space. At any rate, there no longer seemed any a priori reason for regarding one as the ‘true’ geometry.¹¹

We see here the beginning of the idea that the two ways of seeing geometry – as an a priori, axiomatic system and as a true description of physical space – might come apart. But now matters become a little complicated, since Bolyai’s and Lobachevsky’s work did not immediately persuade the larger mathematical community to see matters in this light. There were notable exceptions. As far back as 1817, in a letter to Heinrich Olbers, Gauss had written that “I am becoming more and more convinced that the [physical] necessity of our [Euclidean] geometry cannot be proved, at least not by human reason nor for human reason” (quoted in Kline 1972, p. 872). But Gauss did not go public with such thoughts, and overall the general consensus lagged behind.

Why was this so? Let us just take the case of Lobachevsky. He was able to think creatively of a geometrical system denying the parallel postulate as a possible geometry of space, but his *methods* did not directly deal – or at least, not obviously so – with traditional geometric objects existing in empirically experienced space. His trigonometric formulae did not seem to apply to what we might normally describe as straight lines, ordinary triangles, and the like. They might be true, as it appeared, only of *curved* lines, *curvilinear* triangles, and so on; and these curves, again as it appeared, might simply be embedded within Euclidean space. If so, then there would be no need to change any of the ideas underlying our understanding of space itself. So Lobachevskian ‘geometry’ could easily be relegated to the status of a curiosity, inapplicable to the world as a whole. There remained one final key piece of the jigsaw, which brings us to the second pair of criteria mentioned above, concerning how we conceptualise ‘straight line’.

What is it for a line to be straight? Euclid defined a straight line as “a line which lies evenly with the points on itself” (*Elements*, I, Def. 4). But it is unclear how to understand this definition, without some kind of appeal to visual phenomenology.¹² In 1868, however, Bernhard Riemann supplied a more rigorous answer, which can be summarised as follows. He first defined, analytically, the *length* of a line in terms of

¹¹ Bolyai, for example, spoke of “the impossibility (apart from any supposition), of deciding *a priori* whether [Euclid’s geometry] or some [geometry founded on a contrary hypothesis] (and which one) exists” (quoted in Fauvel and Gray 1987, p. 529).

¹² See e.g. Heath’s discussion in his edition of Euclid’s *Elements*, I, pp. 165–9.

the limit of its (infinitesimal) increment. With length thus defined, a line joining two points can then be defined as ‘straight’ if all nearby lines joining the two points are longer (see Riemann 1868). A straight line, then, is the shortest distance between two points. This may come as no surprise, but importantly for our purposes, this works in *any* space – particularly in the *curvilinear* space of Lobachevsky. Furthermore, it then becomes possible to define the curvature of a space *intrinsically*. Gauss, earlier in the century, had specified how the curvature of a *surface* embedded in space was an intrinsic property of the surface. Riemann, however, showed how *any* space could have its curvature defined intrinsically. This is the final piece of the jigsaw. Following Riemann, there can be seen to be many possibilities for geometries (in the plural). A non-Euclidean geometry might thus turn out to provide the ‘true’ description of empirical space; but whether it does or does not is not a question to be answered *a priori*.

The contrast drawn above between geometry as descriptive of empirical space and geometry as an abstract axiomatic system is reflected, at the more specific level, in the contrast between two ways of seeing a straight line: as a line that *looks* straight (“which lies evenly with the points on itself”) and as what measures the shortest distance between two points (a so-called *geodesic*). Once again, we have two criteria, in this case for what is to count as a straight line, that come apart in conceptual development, in this case in giving rise to non-Euclidean geometry. As ever, there needs to be a mathematical background for such coming apart. But, importantly, the mathematics itself does not dictate which of the criteria we keep and which we drop, just as the background psychological prerequisites for seeing the duck-rabbit in two ways do not give us a reason for seeing it as one rather than the other. We may think we have good reasons for preferring one criterion over the other – in the case of non-Euclidean geometry, perhaps, concerning their applications in physics. But such reasons remain outside of mathematics itself, and we need to look at the wider context, and the way in which the decision to accord primacy to one of the criteria ‘caught on’, in understanding the relevant conceptual developments and hence the mathematical creativity involved.

4 Infinity and Transfinite Arithmetic

It was in discussing the way in which ideas of numerical infinity developed that Wittgenstein drew a distinction that has informed our approach in this paper, namely, between discovering matters of fact and determining concepts. In this section we explain what Wittgenstein had in mind here by taking precisely this case of the development of our mathematical concept of infinity. We have talked of how crucial the ‘catching on’ of a certain way of seeing things can be for conceptual development in a mathematical community. We consider here an actual historical case of an individually creative way of seeing that did *not* catch on in the mathematical community, and contrast it with a way of determining the concept of infinity that did catch on, in spite of Wittgenstein’s strictures on it. What goes on in the development

of mathematical concepts, we suggest, is neither wholly ‘discovery’ nor wholly ‘invention’; there is a distinctive role for creativity to play. We will draw out further implications in the concluding section that follows.

The way of seeing infinity that did not catch on is due to the seventeenth-century mathematician John Wallis (1616–1703). Wallis, it should be noted, was no outsider to the mathematical community of his time. He influenced Isaac Newton, and his tenure as Savilian Professor of Geometry at Oxford (from 1649 until his death) overlapped with Newton’s Lucasian Professorship at Cambridge (1669–1702). Nevertheless, the development we outline has been largely forgotten.

In his *Arithmetica Infinitorum* of 1655, Wallis claimed that infinity (which he was first to denote using the symbol ‘ ∞ ’) was less than any negative number.¹³ He arrived at this surprising result by considerations to do with a certain kind of continuity of operation along the ‘number line’ (another invention of his, which did catch on). In brief explanation, suppose we divide a fixed positive number by an ever decreasing positive variable; then the quotient will increase as this variable divisor diminishes. It made sense to Wallis to consider this procedure as being carried on continuously, decreasing the divisor to zero and then successively through decreasing negative quantities. Considering the increase in the quotient to continue seamlessly as this procedure is carried out, and noting that a positive dividend divided by a negative divisor gives a negative quotient, we will have, taking 1 as our dividend, for instance,

$$\begin{aligned} \dots \frac{1}{10} < \frac{1}{5} < \frac{1}{2} < \frac{1}{1} < \frac{1}{0.5} < \frac{1}{0.2} < \frac{1}{0.1} < \frac{1}{0.01} < \frac{1}{0.001} < \dots < \frac{1}{0} < \dots \\ \dots \frac{1}{0} < \dots < \frac{1}{-0.1} < \frac{1}{-0.2} < \frac{1}{-0.5} < \frac{1}{-1} < \dots \end{aligned}$$

But $\frac{1}{-1} = -1$. So $\frac{1}{0} < -1$. That is, $\infty < -1$. In words, infinity is less than negative one.

Nowadays ∞ does not count as a number at all, although the notation survives, in talk of limits, for example. ‘ $\lim_{x \rightarrow 0} \left(\frac{1}{x} \right) = \infty$ ’ is acceptable contemporary notation. Overall, though, Wallis’ ‘ ∞ ’ is not a part of present-day mathematics. It did not catch on.

Before looking further at Wallis’ ‘proof’ that negative numbers are larger than infinity, let us turn to another attempt at determining a concept of infinity, that of Georg Cantor (1845–1918). Cantor’s development of the concept did, in many ways, catch on, and his line of thinking is nowadays well known. Here is a criticism that Wittgenstein made of it, however, involving the distinction mentioned above:

The dangerous, deceptive thing about the idea: “The real numbers cannot be arranged in a series”, or again “The set ... is not denumerable” is that it makes the

¹³ See also Scott 1938, pp. 44–5.

determination of a concept—concept formation—look like a fact of nature. (*RFM*, II, 19; p. 131)

An (infinite) set is said to be ‘denumerable’ if it can be put into one–one correspondence with the set $\mathbf{N}$ of natural (‘counting’) numbers, $\{1, 2, 3, \dots\}$. This relies on the so-called ‘Hume-Cantor Principle’,¹⁴ according to which two sets have the same number of members if and only if their members can be one–one correlated. The set Cantor claims to be non-denumerable, a claim criticised by Wittgenstein for the way it makes concept formation look like a fact of nature, is the set of real numbers $\mathbf{R}$ (the rational as well as irrational numbers – all the numbers on Wallis’ number line, in other words). And the idea of the real numbers ‘being arranged in a series’ is part of Cantor’s ‘proof’¹⁵ of this set’s non-denumerability. For suppose $\mathbf{R}$ is denumerable. Then its members can be put into one–one correspondence with the members of $\mathbf{N}$. Assume that this correspondence has been made explicit, with the real numbers listed in order, in their decimal expansion. Every real number then has a position on the list: it will be either the first, or the second, or the third, and so on. But now consider a number which differs from the first number on the list at the first place of decimals, from the second at the second place, from the third at the third place, and so on. It is clear both that there *is* such a number and that it is nowhere on the list. So the list does not, after all, contain every real number, contrary to the assumption. Hence no such list can be made and $\mathbf{R}$ is not denumerable. This is a version of Cantor’s so-called ‘diagonal argument’ (which first appeared in Cantor 1891).

Cantor symbolised the cardinality (the number of members) of the infinite set $\mathbf{N}$ by ‘ $\aleph_0$ ’ (‘aleph-zero’) and the cardinality of the infinite set $\mathbf{R}$ by ‘ $\aleph_1$ ’ (‘aleph-one’), and called the series beginning with $\aleph_0$ and $\aleph_1$ the ‘transfinite cardinal numbers’. The details need not concern us here. Generally speaking, the notion of ‘infinity-as- $\aleph_0$ ’ is nowadays accepted by mathematicians; Wallis’ notion of ‘infinity-as- ∞ ’, on the other hand, is not accepted, as we saw.

Should we accept current practice and go with Cantor? What about Wittgenstein’s complaints? Is there anything to be said at all for Wallis’ determination of the concept of infinity? We can clarify matters by considering these questions in the light of our account of the way that criteria for the use of concepts can come apart in conceptual development. Here the governing thought is very simple: two criteria for the use of a certain mathematical concept may coincide in finite cases, but when we turn to infinite cases, they do indeed come apart, forcing us to make a choice as to which to give primacy.

¹⁴ Often known as just ‘Hume’s Principle’, we call it here the ‘Hume-Cantor Principle’ to do justice to the role that it played in Cantor’s thinking (in extending its application from finite to infinite sets).

¹⁵ ‘Proof’ is put in scare quotes here to keep equality of treatment *pro tem* with Wallis’ ‘proof’ above.

Let us consider Wallis first. In the move from dividing by a very small positive number to dividing by a negative number, passing through (and assuming a continuity involved in) dividing by zero, the implication is that we move from small to smaller. But we end up with negative numbers *larger* than ∞ . It seems Wallis proves that $1 < \infty < -1$ by using the fact that $-1 < 0 < 1$. This seems to be a contradiction. Consider, now, the following two criteria for the use of ‘<’ (i.e., for our concept of being less than):

- (i) so long as a and b have the same sign, i.e. are either both positive or both negative, $a < b \Rightarrow \frac{1}{a} > \frac{1}{b}$ (if a is less than b , then the reciprocal of a is greater than that of b);
- (ii) $a < b$ and $b < c \Rightarrow a < c$ (if a is less than b and b is less than c , then a is less than c , i.e. the relation represented by ‘<’ is transitive).

A little thought should convince us that in the finite case – if we do not have Wallis’ ‘ ∞ ’ to deal with ($\frac{1}{0}$, recall) – we can keep both of these criteria. If we are to accept Wallis’ notion of infinity, however, then we can keep one of these only, but not both. Wallis effectively denies (ii) in order to keep (i) and avoid the contradiction. The overall consensus in the mathematical community at large, however, was that the loss of transitivity here is too great a price to pay – as likewise, as it turned out, would be the loss of criterion (i) even keeping criterion (ii). Wallis’ creative move, in short, turned out to be a step too far.

What now of Cantor? Consider what has become known as ‘Galileo’s paradox’ (see Galileo 1954 [1638], pp. 31–3). Every square number is a natural number but not every natural number is a square number. So the set of square numbers is a proper subset of the set of natural numbers. In short, and informally, the set of square numbers is a part of the set of natural numbers. Since a whole is always bigger than any of its parts, the set of natural numbers is therefore bigger than the set of square numbers. That there are *more* natural numbers than there are square numbers seems obvious. However, the set of square numbers can clearly be put into one–one correspondence with the set of natural numbers, which means – according to Cantor – that there are the same number of square numbers as there are natural numbers. As in the case of Wallis, Cantor’s understanding of infinity seems to involve a contradiction.

Here, too, though, we can offer a diagnosis in terms of two criteria coming apart when we move from the finite to the infinite. Consider the following two criteria for what it is for two sets to have the same number of members:

- (a) neither is bigger than the other (in the sense of ‘bigger than’ for which a set is always bigger than any proper subset of itself);¹⁶

¹⁶ Cf. Euclid, ‘Common Notions ... 5. The whole is greater than the part.’ (Euclid 1956, p. 155)

(b) their members can be one–one correlated.

For finite sets, these two criteria coincide; in the case of infinite sets, however, they come apart and cannot both be maintained. Cantor gave primacy to (b) and rejected (a) for infinite sets in order to avoid contradiction. And, in effect, he seems to have won the day: transfinite sets Cantor-style are currently accepted by most of the mathematical community.

At the heart of both Wallis' and Cantor's creativity, then, is a preference for one criterion for the use of a mathematical concept over another. Wallis' choice ran aground, however, on the issue of the transitivity of the relation of being less than, which led to its not catching on in the mathematical community. Cantor's choice, on the other hand, enabled him to develop the theory of transfinite numbers, which is now firmly established as a thriving field of mathematics.

Let us end this section with a brief look at what we can learn from Wittgenstein's strictures on Cantor. Wittgenstein's stress on the way in which a certain ("dangerous, deceptive") way of seeing Cantor's procedure tends to confuse the determination of a concept with the discovery of a fact of nature should not be taken to suggest either that Cantor chose the wrong criterion to uphold or that some other creative way such as Wallis' might already be the correct (though as yet undiscovered) way of determining the concept of infinity. On the contrary, seeing the determination of a concept as different from the discovery of a fact of nature emphasises for us the ways in which such determinations are *not* decided prior to being made; the facts are not simply there already, waiting to be uncovered or discovered by our mathematical research. Instead, there is essential space available for creativity. This creativity may be constrained by the situation in which the mathematician works, but it is genuine nevertheless, one of the ways in which it is manifested being in the seeing, highlighting and developing of certain aspects of our use of mathematical concepts. David Hilbert once remarked that "No one shall be able to drive us from the paradise that Cantor created for us" (1967 [1925], p. 376). Wittgenstein is no snake that might lead to our expulsion, but there is nothing to stop us leaving of our own accord, should we so wish.

5 Conclusion

In the last two sections we have considered two further examples of mathematical creativity from the more recent history of mathematics, one from geometry and one from arithmetic. In both cases, we argued, the creativity can be understood in terms of a choice being made of which of two criteria for the use of the relevant mathematical concept(s) to adopt as primary. In the case of the development of non-Euclidean geometry, there were two pairs of criteria involved: geometry was seen both as descriptive of empirical space and as an axiomatic, a priori system, and a straight line was seen both as a line that 'looks' straight and as measuring the shortest distance

between two points. The two criteria in each pair needed to come apart and the first of the two dropped in favour of the second, respectively, in order for non-Euclidean geometry to be developed. In the case of the concept of infinity, we compared Wallis with Cantor. Wallis had to choose between two criteria governing the use of the relation of being less than, and Cantor had to choose between two criteria governing the concept of two sets having the same number of members. Once again, the two criteria had to come apart for the relevant conceptual change to occur, but while Cantor's choice caught on in the mathematical community, enabling transfinite arithmetic to be developed, Wallis' choice did not catch on.

Aspect perception offers a useful model for understanding mathematical creativity, at least in the kinds of cases we have considered. The geometrical problem discussed in section 1 can be readily interpreted as requiring a switch in aspect perception to solve it. The other cases, we suggested, might be seen as involving aspect conception rather than aspect perception, but there are important similarities, as indicated by the very fact that talk of 'seeing as' can be readily carried over. In his discussion of seeing-as, Wittgenstein himself emphasized that "There are here hugely many interrelated phenomena and possible concepts", with lots of these concepts 'crossing' (*PI*, II, xi, pp. 199, 211). Although there are clearly differences between experiencing aspect switching in cases such as the duck-rabbit and the conceptual aspect focusing that we have described in the last three sections, some of the things that Wittgenstein says in discussing the former also apply to the latter. And he makes various remarks that show that mathematics was never far from his mind.¹⁷

Perhaps the most important similarity between aspect perception (in cases such as the duck-rabbit) and aspect conception (in the cases we have considered) is the sense we may have both that something remains the same and that something changes (see, e.g., *PI*, II, xi, pp. 195–6, 201–2, 212). When we switch from seeing the duck-rabbit as a duck to seeing it as a rabbit, the picture itself does not change but yet we experience it in a quite different way. So, too, we may recognize that a concept has two criteria and be able to switch from thinking about one to thinking about the other while nevertheless taking it to be the same concept. Of course, we may come to drop one of the criteria in favour of the other, but even here we might still take it that the concept remains essentially the same.

However, it is at this point that Wittgenstein would insist that we see what is going on instead as *concept-determination*. To quote his own words, albeit in a slightly different context, "Here it is *difficult* to see that what is at issue is the fixing of concepts" (*PI*, II, xi, p. 204). As Wittgenstein also emphasizes, though, such concept-determination needs to be embedded in a new practice for it to catch on within the relevant community: "Something new (spontaneous, 'specific') is always a language-game" (*PI*, II, xi, p. 224). The new language-game will be one in which mastery of the appropriate techniques and the requisite agreement of judgements, of

¹⁷ In *PI*, II, xi, issues in understanding mathematics are alluded to, mentioned or discussed on pp. 196, 201, 203, 209, 220, 224, 225–7, and geometrical figures provide examples on virtually every page.

which Wittgenstein frequently speaks, has been established (cf. e.g. *PI*, §§ 241–2; II, xi, pp. 208–9, 225–7).

In fact, the analogy between aspect perception and aspect conception might be pressed even further than we have so far suggested. For imagine the following visual representation of Galileo’s paradox:

This diagram can be seen in two ways – as showing that the first set is bigger than the second (by seeing that there are ‘gaps’ in the second set where members of the first set are ‘missing’) and as showing that the two sets have the same number of members (by seeing that they can be one–one correlated as the lines indicate). So, too, in the case of the two different criteria for a line to be ‘straight’, one can imagine a line on the surface of the earth going from north pole to south pole. Were we to walk along this line, we would be taking the shortest route from north pole to south pole; yet viewed three-dimensionally from the moon, say, it would seem to us to be curved and not straight. In both cases we might be able to switch from one aspect to the other at will just as in the case of the duck-rabbit.

What this shows is something else that Wittgenstein discussed in his writings on the foundations of mathematics: the role that ‘pictures’ play in our doing and thinking about mathematics, and the dangers that lurk in this. Since pictures can very easily be seen in different ways (as he also stressed), one should not be surprised that consideration of aspect perception has implications for our understanding of mathematics. Such phrases as “Look at it this way” and “You have to see it like this” are readily used in teaching people mathematics (see e.g. *PI*, §144; II, xi, p. 202).

Now it may be that a mathematician who is creative in the way we have suggested may not be aware that this is how their creativity might be seen. They may not see themselves as having made some kind of spontaneous choice as to which of two criteria to take as primary. But this does not invalidate our account. The creativity that someone exhibits is rarely appreciated by the person at the time, and it is invariably the case that any explanation of that creativity can only be offered much later. We have an explanation of this, too, in what we have said: that any new conceptual development takes a while to catch on. We have to see what focusing on just one of the criteria leads to before we get a clear sense of its independence from the other criterion and hence become able to offer a diagnosis of the development. In doing so we must put ourselves into a position to recognise the two possibilities, analogous to the position someone is in when they are able to see both the duck and

the rabbit in the duck-rabbit picture. So even if the mathematician may not appreciate the aspect switch, we must do so ourselves if we are to explain the creativity.

Developing aspect perception may thus be an important skill to foster in those seeking to understand creativity. At the end of section 2 we said (alluding to *PI*, §116) that what *we* do is bring creativity back from its mysterious to its everyday use. This is not to downplay the creativity. We can appreciate both the newness but also how it arose out of the old. The creativity exhibited in the cases we have considered does indeed fall somewhere between ‘invention’ and ‘discovery’ – or perhaps better put, involves aspects of both. A concept is determined through an inventive choice that is made, but that leads to the kind of developments in which talk of ‘discovering’, say, new numbers is not inappropriate. They may not be ‘facts of nature’, but they are rooted in the establishment of a new language-game.

Of course, the account we have offered does not even begin to answer all questions about mathematical creativity or conceptual development. What determines the extra-mathematical moves we have described? How does mathematics maintain its necessity if human choice (even partly) determines its concepts? How can mathematics, a creative human endeavour, nevertheless successfully be applied to physical nature? However, by taking aspect perception as a model for understanding a certain kind of mathematical creativity, and elucidating the way in which concept-determination occurs through criteria for a concept coming apart and one being adopted as primary, we hope to have offered a new and fruitful way of seeing mathematical creativity.

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