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Compositional Meaning in Logic

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Abstract. The Fregean–inspired *Principle of Compositionality of Meaning* (PoC) for formal languages asserts that the meaning of a compound expression is analysable in terms of the meaning of its constituents, taking into account the mode in which these constituents are combined so as to form the compound expression. From a logical point of view, this amounts to prescribing a constraint — that may or may not be respected — on the internal mechanisms that build and give meaning to a given formal system. Within the domain of formal semantics and of the structure of logical derivations, the PoC is often directly reflected by metaproperties such as truth-functionality and analyticity, characteristic of computationally well-behaved logical systems.

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1. The area

This text is a foreword to the special issue of *Logica Universalis* dedicated to recent research results on different aspects of *Compositional Meaning in Logic*, a topic that stems naturally from the theme of compositionality in linguistics and the philosophy of meaning [70], after Gottlob Frege, that must be able to provide, in the realm of logic, a wealth of fundamental tools for formal reasoning and its applications. Such principles are well-reflected in modern approaches to the semantics of programming languages following the seminal work of Dana Scott and Christopher Strachey [62, 67].

The ubiquity of non-classical logics in the formalization of practical reasoning demands the formulation of more flexible theories of meaning and compositionality that allow for the establishment of coherent and inclusive bases for their understanding. Such investigations encompass not just the development of adequate frameworks from the perspectives of Model Theory, Proof Theory and Universal Logic, but also the construction of solid bridges

between the related approaches based on various generalizations of truth-functionality. Applications of broadly truth-functional logics, in their various guises, are of utmost importance in several areas of computer science, mathematics, philosophy and linguistics, where the ever increasing complexity of systems continuously raises new and difficult challenges to compositionality.

This special issue includes 5 papers, covering some relevant topics on the theme of compositionality in logic. Of course, these papers are not intended to all the territory of such a vast area, nor do they touch upon all the research tracks that it comprises. Framing their contributions in the general picture is the main reason why we decided to write this introductory text.

1.1. Semantics

The most usual reading of the PoC to be found in the literature is the *model-theoretic* one, according to which compositionality concerns semantics for languages, and the principle is construed as asserting, in one way or another, that the interpretation of a complex expression is a function of the interpretation of its (immediate) constituents and the interpretation of the syntactic constructors by means of which the complex expression is formed. A straightforward realization of such a principle is given by *truth-functional* semantics, where an interpretation is given by a homomorphism between a syntactic free term algebra and an algebra of truth-values with the same similarity type, so that each constructor from the first algebra finds in the second an adequate operational counterpart. Moreover, on such a reading, by well-known results from universal algebra, the assignment of a given interpretation to atomic constructors is sufficient to provide a unique way of extending such interpretation to all compound terms built from them.

Research-wise, the aim is to depart from the traditional notion of *logical matrix* [73], prevalent throughout the last century since the pioneering work of Jan Łukasiewicz, and a fundamental tool for the study of *algebraic logic* [35] and so-called *many-valued* logics [40], and to experiment with generalizations of the notion of truth-functionality, looking into different ways of controlling each non-standard variety of semantics, while still retaining good computational behavior.

An important and very natural semantics of this genus is the so-called *non-deterministic semantics*, introduced and studied by Arnon Avron and collaborators in the last decade [8, 9]. This variety of non-deterministic approach, when applied to ordinary multi-valued matrices, extends the notion of truth-functionality in producing the so-called *Nmatrices*, with the following behavior: in an Nmatrix, the truth-value of a complex formula is chosen non-deterministically from some non-empty set of options. Thus, the interpretation $\widetilde{\diamond}$ of an n-ary connective $\diamond$ is a function $\widetilde{\diamond} : \mathcal{V}^n \longrightarrow 2^{\mathcal{V}} \setminus \{\emptyset\}$, where $\mathcal{V}$ is the set of logical values of the given logic. A legal valuation v in an Nmatrix is a mapping from formulas to logical values in $\mathcal{V}$ such that $v(\diamond(\psi_1, \ldots, \psi_n)) \in \tilde{\diamond}(v(\psi_1), \ldots, v(\psi_n))$. Since their inception, Nmatrices have proven to be a truly powerful tool. Their use preserves all the advantages of

ordinary many-valued matrices, but they are applicable to a much wider range of logics. A particularly useful feature of the semantic framework of Nmatrices is its *modularity*: the semantics of a proof system based on an Nmatrix is often obtained by joining the semantics of its component rules, while the main effect of each of the rules is to reduce the degree of non-determinism of the underlying logical operations. This procedure has indeed been successfully applied, in recent years, to provide simple and modular semantics for a myriad of non-classical logics, even when further extended with *partiality* into the notion of *PNmatrix* [10].

Other analogous samples of the same non-deterministic genus, which have been studied by João Marcos and collaborators, go in two different directions, being either more relaxed and inclusive (such as *society semantics*), thus less comprehensive, or more strict and restrictive (such as *possible-translations semantics*), thus more far-reaching, on what concerns the class of legal valuations. Particularly interesting is to notice that non-determinism also takes place when truth-functionality is abandoned in favor of more classic-like viewpoints, such as the one that construes logical systems, semantically, as well-defined non-truth-functional collections of two-valued mappings on the set of formulas. The so-called *theory of valuations* [28] that underlies the latter stance has been extensively studied by Jean-Yves Béziau and others, but many results that make these ideas more computationally treatable are only nowadays emerging [20]. Finally, the task of generalizing even the basic notion of a truth-value and the very approach to logical entailment — thus, to the question of 'what a logic is' — has been recently proposed and investigated by Heinrich Wansing, Yaroslav Shramko [64] and others.

1.2. Proof systems

A *proof-theoretic* reading of the PoC is also to be found in the literature. According to this reading, a proof system is said to be compositional if the proof of a compound expression may always be constructed by suitably manipulating proofs that only involve its constituents. Not surprisingly, in the approach advocated by Gerhard Gentzen, one way of guaranteeing such a version of the principle, for a proof system given in terms of sequent rules, is exactly by proving *admissibility of the cut rule*. In the standard structural approach, after cut is eliminated from a certain proof, a certain form of *subformula principle* is observable in the new proof, according to which each proof step, in a goal-directed approach, reduces the complexity of some specific expression that was already present in the proof. Such ideas, in fact, may be generalized to other formal systems, such as concurrent and reactive systems, whose properties would be compositionally verifiable in terms of the properties of their components. The most important task, given such a version of the PoC, would consist in checking whether a given proof system may be considered to be *analytic*, in guaranteeing that derivations of correct inferences may always be found whose construction involves a steady decrease in a certain *complexity measure* associated to the intervening expressions.

Work on proof-theoretical generalizations of the PoC can be categorized as falling into two main classes:

- work that generalizes Gentzen-type deductive systems by adding syntactical information to these systems, be they generic labels (labeled deductive systems of several types), different levels of sequents (e.g., hypersequents), terms (e.g., CHI), or a proof-search strategy (e.g., focusing);
- work that constrains the deductive system via a strongly adequate semantics for it (as in the bivalent approach, or in the non-deterministic approach).

Work on labeled deductive systems [36, 71] is concerned with labeling disciplines as an enabling technology. Labeling creates a strong bridge between model theory and proof theory, as it is based on adding semantic information to the deductive systems in question. The bivalent approach, in particular, presupposes the simplest logically meaningful labeling strategy, and there are several competing recipes for lifting this approach so as to be applicable to multiple-valued semantics, including those of the non-deterministic stripe.

In another thread, one may stay syntactically closer to the original structural Gentzen systems, being based on focusing disciplines for some of the logics that generalize the notion of truth-functionality.

An overarching theme relating these approaches is the idea of Curry-Howard correspondences, which, on one hand are 'labeled deductive systems' where the labels are simply lambda-calculus terms, but on the other hand, try to stay closer to the original Gentzen systems for Natural Deduction, but also to generalizations like hypersequents and n-sided sequents, and as such are amenable to focusing methods.

A third thread of investigation, here, is more foundational and all-embracing. It concentrates on generalized proof-theoretic properties encompassing the approaches based on bivaluations and on Nmatrices as setting constraints not only on the classes of legal valuations but also on the deductive systems that characterize them. The semantic characterization of both syntactic and proof-theoretical properties of each non-standard variety of semantics of interest is intended to work as a bridge between proof theory and model theory. In particular, one of the main methods for developing Gentzen-type systems for non-classical logics is precisely to translate many-valued semantics (deterministic or non-deterministic) into a bivalent framework.

1.3. Universal logic

Yet a third approach to the PoC — and to a general theory of meaning, from the logical viewpoint — is the one belonging to the field of *Universal Logic*, whose import is illustrated by but not exhausted in both Model Theory and Proof Theory. In this setting it is quite natural to consider more relaxed readings of the PoC, from the viewpoint of abstract consequence relations,

that allow for more generous, yet equally useful, versions of compositionality. According to such readings, from the model-theoretic perspective, certain non-deterministic varieties of semantics are found to be suggestive and acceptable, and the effective assignment of an interpretation to complex terms may be guaranteed by a suitable combination of the interpretations of other related expressions which are not necessarily their immediate nor their most basic subterms. From the viewpoint of the adequacy results that connect Model Theory to Universal Logic, though, generalizations of the notion of truth-functionality are demanded in order to accommodate such less standard semantical approaches. Otherwise, from the proof-theoretic perspective, an extended version of analyticity may still be obtained in many non-standard situations, for instance, if convenient logical frameworks and proof strategies are associated to a given proof system, often employing non-canonical syntactical measures of complexity and more liberal versions of the subformula principle. From the viewpoint of theorem proving, it is necessary, for each logic or class of logics dealt with in such generalized approaches, to study the expressiveness and the limitations of the associated cut-free or cut-based proof formalisms and proof systems, and how they may contribute to proof-search.

Overall, the foundational task of categorizing, in Universal Logic terms, the reach of each specific class of 'broadly truth-functional logics' immediately imposes itself.

1.4. Applications

A number of the logical systems that have been addressed on investigating the PoC came to the fore as meaningful generalizations of the works on applied logic, i.e., applications of logic to formalise and reason about complex and interacting systems. These applications can be broadly organised into a number of different areas such as: (i) the development and use of modal [15, 16], temporal and other non-classical logics for different and heterogenous applications, ranging from security in computer science, especially of the reactive type, to modelling of non-conventional systems such as quantum systems, (ii) the development and use of logics for reasoning under uncertainty and inconsistency [42, 22], such as paraconsistent and fuzzy logics with applications to knowledge representation and databases, but also to probabilistic reasoning, and (iii) combinations of logical systems [38, 37, 18], such as fuzzy modal logics, which allow one to capture different compositional aspects of complex systems.

2. The context

The whole idea for compiling this volume came up as a natural consequence of the series of *GeTFun Workshops* (see `sqig.math.ist.utl.pt/GeTFun`):

- *GeTFun 1.0* was held in Rio de Janeiro, Brazil, in April 2013, jointly with the *4th World Congress and School on Universal Logic (UniLog 2013)*;
- *GeTFun 2.0* was held in Vienna, Austria, in July 2014, and colocated with the *Vienna Summer of Logic 2014*;
- *GeTFun 3.0* was held in Natal, Brazil, in September 2015, as part of the event *NAT@Logic 2015*; and
- *GeTFun 4.0* was held in Coimbra, Portugal, in July 2016, in affiliation with the *8th International Joint Conference on Automated Reasoning (IJCAR 2016).*

These international events, aimed at dissemination and discussion of new research results among experts in the field, were organized as part of the EU FP7 Marie Curie PIRSES-GA-2012-318986 project *GeTFun: Generalizing Truth-Functionality*, which ran between 2013 and 2016, and gathered researchers from 17 institutions in Austria, Brazil, Germany, Israel, Italy, Poland, Portugal, Romania, Ukraine, and United Kingdom, with complementary expertise and in several cases with a significant history of fruitful collaborative work.

The project, as well as the workshops, were dedicated to the study of various well-motivated ways in which the attractive properties (and metaproperties) enjoyed by truth-functional logics may be generalized in order to cover more extensive logical grounds. Specific topics of interest included generalizations of metalogical notions such as truth-functionality, nondeterminism in semantics, logical bivalence versus algebraic many-valuedness, analyticity, the subformula principle and complexity measures, semantic effectiveness, decision procedures for non-classical logics, cut-free versus cut-based proof formalisms, proof strategies and proof-search, modular logical specification formalisms, rule invertibility, structurality, and logical harmony.

Overall, the impact and relevance of such a line of research is to be measured directly by its foundational character with respect to a better and deeper understanding of meaning in logics modeling complex phenomena and, of necessity, suitable general forms of compositional reasoning. Some highlights of the GeTFun project, covering topics such as truth-values, valuation semantics, bivalence, proof-systems and focusing, controlled nondeterminism, distance-based reasoning, information sources, paraconsistency, labeled and hybrid logic, *inter alia*, can be found in [2, 10, 12, 20, 19, 21, 24, 25, 29, 32, 45, 44, 55, 56, 58, 74, 50, 7, 3, 27, 26, 59, 11, 47, 57, 43, 49, 60, 54, 1, 66].

Next, we shall briefly frame each of the contributed articles in this special issue within the landscape of this broad research area.

3. The papers

As we remarked above, this special issue includes 5 contributed papers, each covering some relevant topics on the theme of compositionality in logic. We provide a short introduction to each of them, in their order of appearance.

3.1. Self-extensional three-valued paraconsistent logics

The paper [5] authored by Arnon Avron from Tel Aviv University, Israel, is a contribution to the understanding of paraconsistency in many-valued logics. It was known from [6] that there is no self-extensional three-valued paraconsistent logic with a reasonable implication connective. Now, the author shows that there is exactly one such logic (dubbed PE3) in the language with negation, conjunction and disjunction.

The logic PE3 is further shown to be strongly maximally paraconsistent in its language, and endowed with a sound and complete analytic cut-free sequent system.

This result marks an important and clear-cut distinction between truth-functional meaning in logic as provided by logical matrices, and more general approaches adopting *generalized logical matrices* (as used to define order-preserving logics) or non-deterministic matrices, for which forms of three-valued self-extensional paraconsistency in richer languages, including implications, are known to exist [61, 4].

3.2. Modal multilattice logic

The paper [41] coauthored by Norihiro Kamide from Teikyo University, Japan, and Yaroslav Shramko from Kryvyi Rih University, Ukraine, is exemplary in the way it spans from semantic ideas relating to generalized truth-values toward Gentzenian sequent calculi, by exploration of interesting applications involving multi-modalities.

While attempts at combining many-valued logics with modal operators [15, 16] have a tradition that reaches back, at least, to the work of Melvin Fitting [33, 34], the paper approaches the task from the perspective of generalized truth-values stemming from multilattices, in the trend initiated by the study of *first-degree entailment* and logical *bilattices* by J. Michael Dunn [31] and Nuel Belnap [14, 13].

The modal multilattice logics obtained are provided with adequate sequent calculi, namely in the case of $S4$-like modalities, enjoying suitable cut-elimination properties that witness their decidability. The logics are further endowed with complete Kripke-style semantics.

3.3. Sequent systems for negative modalities

The paper [48] coauthored by Ori Lahav from Max Planck Institute, Germany, João Marcos from UFRN, Brazil, and Yoni Zohar from Tel Aviv University, Israel, is an interesting contribution to a deeper understanding of

non-classical negations (see [39]) which takes advantage of the wealth of results about normal modal logics [15, 16]. Further, the results come as applications of recent fundamental results on compositionality and proof-theory of sequent systems by Ori Lahav and Arnon Avron [46].

The approach taken is based on the semantic definition of negative modalities, using the usual Kripke-style semantics, following the abstract principles introduced in [30]. The setting is enriched with the consideration of corresponding adjustment connectives, following the principles underlying the so-called logics of *formal inconsistency* and of *formal undeterminedness* of [23, 53].

The modularity of the semantic approach taken, inherent to the modal setting, allows the authors to provide sequent calculi with cut-admissibility and corresponding, generalized, analyticity results, incrementally, for the class of all frames and various restrictions defined in terms of additional frame properties: seriality, reflexivity, transitivity, symmetry, functionality, and combinations thereof. The approach allows for a neat classification of the obtained negations in terms of their paraconsistency/paracompleteness and an assessment of their relative strengths, namely with respect to the recovery of classical negation.

3.4. Locally tabular $\neq$ locally finite

The paper [52] coauthored by Sérgio Marcelino from IT and Universidade de Lisboa, Portugal, and Umberto Rivieccio from UFRN, Brazil, presents some fundamental research related to the abstract characterization of the notion of finite-valuedness, with immediate impact on the decidability and complexity of the logics at hand.

The paper's contribution can be better understood in the wider setting of [51], a recent article that provides the first thorough abstract characterization of several variants of the notion of finite-valuedness in logic. In particular, the article isolates the *strongly finite* logics of [73] as those whose consequence relation satisfies two properties: *finite determinedness* and *local tabularity*. These properties, together with the well known property of *cancellation* [63], capture precisely the logics definable by a single finite logical matrix.

Herein, the authors clarify the precise relationship between local tabularity and a similar (but arguably more complex) property used by Ryszard Wójcicki in [72]. Both properties assert the existence of finitely many equivalence classes on syntactical fragments with finitely many variables but with respect to two distinct relations one can associate to a logic: the *Frege relation* of interderivability, in the case of local tabularity, and the *Tarski congruence* (see [35]), in the case of local finiteness. The authors show that although the two notions coincide for most well-behaved logics, namely when they are finite-valued, algebraizable, or selfextensional, still local tabularity is a strictly weaker property.

3.5. A note on Two's company: "The humbug of many logical values"

The paper [65], authored by Daniel Skurt from Ruhr-Universität Bochum, Germany, is a contribution to implementing the separation of truth-values in Łukasiewicz finite-valued logics that may allow for their bivalent reduction.

In fact, the ideas underlying Roman Suszko's 'infamous' rejection of the PoC [68, 69] can be reconciled with the advantages of many-valuedness by reequating the very notion of truth-functionality [17] in a way that is shown by Carlos Caleiro, João Marcos, and Marco Volpe in [20] to be fully implementable, as long as the syntactical resources of the logic can be used to distinguish its finitely many truth-values. Under these circumstances, the reduction yields a clausal presentation for an equivalent non-truth-functional bivalent semantics which can be used as a basis for producing decision procedures for the logics, namely via tableaux, branching-like or cut-based, by adopting generalized versions of analyticity and the subformula property based on suitably defined complexity measures.

In [65], the author shows how to systematically produce syntactic separators for Łukasiewicz finite-valued logics by adopting a witty representation of binary prints by means of *Gray codes*, as used in *error-correcting codes* and *coding theory* in general. The method is arguably applicable to other finite-valued logics which may be able to express certain piecewise linear functions on the $[0, 1]$ interval.

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