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Diffusion and signatures of localization in stochastic conformal field theory

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We define a simple model of conformal field theory in random space-time environments, which we refer to as stochastic conformal field theory. This model accounts for the effects of dilute random impurities in strongly interacting critical many-body systems. On one hand, surprisingly, although impurities are separated by macroscopic distances, we find that the infinite-time steady state is factorized on microscopic lengths, a signature of the emergence of localization. The stationary state also displays vanishing energy current and strong uncorrelated spatial fluctuations of local observables. On the other hand, at finite times, the transient shows a crossover from ballistic to diffusive energy propagation. In this regime and a Markovian limit, concentrating on current-generating initial states with a temperature imbalance, we show that the energy current and density satisfy simple dissipative hydrodynamic equations. We describe the space-time scales at which nonequilibrium currents exist. We show that a light-cone effect persists in the presence of impurities although a momentum burst propagates transiently on a diffusive scale only.

I. INTRODUCTION

One of the most interesting and active areas of current research is the physics of randomness in extended, interacting quantum systems. Since the work of Anderson [1], it is well known that the effects of random external fields and impurities may have important physical consequences, such as a vanishing conductivity and the absence of ergodicity [2]. The situation is more complex in extended systems where strong interactions play a crucial role, leading to many-body localization (MBL) [3].

In this letter, we ask the related question of the interplay between emergent collective behaviours in interacting quantum critical systems and randomness due to dilute impurities. On the one hand, perhaps the most striking effect of strong interactions in extended systems is the emergence near criticality of large-scale collective behaviours and large-distance correlations, with associated ballistic propagation of energy. On the other hand, in many-body localized states local degrees of freedom become largely uncorrelated, precluding transport. Can dilute impurities strongly affect collective behaviours and break ballisticity? May signatures of localization emerge? To our knowledge these questions have not been addressed yet in the literature.

Conformal field theory (CFT) is the most powerful theory for emerging collective behaviour at critical points [4], hence the foremost playground for understanding new many-body physics. We introduce the concept of stochastic CFT, and we use it in order to gain insight into the physics of diffusion and the emergence of localization in one-dimensional quantum critical systems. Stochastic CFT, inspired by open quantum Brownian motion [5], is the combination of fundamental notions of CFT such as ballistic energy transport, with randomness in space and time. It can be interpreted as an effective model for the interplay of many-body collective behaviours with dilute, randomly placed impurities. It explicitly describes quantum evolution in a random environment of classically fluctuating impurities.

Interestingly, dynamics in this model displays the passage from ballistic, to diffusive, and then to localized physics. For this purpose, we show how some of the important tools of dissipative quantum mechanics [6] can be extended to CFT. This provides a precise notion of diffusivity, which is in line with that used in recent works on quantum chains [7] (but which does not necessarily correspond to de Gennes’ phenomenological theory [8]). Surprisingly, we also show that after a long time an important signature of localization emerges: we find that the stationary state of macroscopically separated random impurities in interacting, critical many-body systems, is correlated only at microscopic length scales. Randomness (or more precisely irregularity) is important: regularly placed and fluctuating impurities would not give rise to microscopic correlation lengths. This gives an analytically tractable framework in which to extract certain principles at the roots of such localization effects in interacting systems.

II. RANDOM IMPURITIES AND STOCHASTIC CFT

Heuristics. Consider a quasi-one-dimensional metal at low energies. The Luttinger liquid theory predicts that the electronic behaviour is described by a one-dimensional CFT. As is well known, this supports ballistic transport of various quantities such as the energy, and thus an infinite conductivity.

In real, non-ideal, systems, electrons are subject to diffusive processes due to scattering on impurities or with other degree freedoms, and this modifies the dynamics. Two types of scattering may occur: elastic, inducing momentum, but no energy, relaxation, or inelastic, inducing energy relaxation and phase decoherence. In the mesoscopic regime the mean free path is much smaller than the decoherence length so that coherence effects are still
important, and the system is described by elastic scattering events separated by ballistic propagations. We concentrate on this regime.

At low energies and low defect densities, any physical impurity will be described by a conformal defect. Consider the local observables representing conserved currents, such as the energy current. These are chiral: despite the strong interaction between electrons, the local energy is ballistically transported by independent right and left-moving collective energy packets. The associated local observables are the components $T^+(x) = \langle h(x) + p(x) \rangle/2$ and $T^-(x) = \langle h(x) - p(x) \rangle/2$ of the stress-energy tensor, where $h(x)$ is the energy density and $p(x)$ is the momentum density.

The model we consider is that where defects are positioned randomly in space, and also “on” or “off” in a random fashion in time. This is the simplest model of a classically fluctuating environment: the quantum system is connected to an environment only via sparse impurities, activated and deactivated in a random fashion due to classical fluctuations. We expect the model mimics the quasi-classical cumulative effect of random interference that would occur upon scattering off fixed, time-independent impurities with nontrivial reflection and transmission. One may further generalize the model to random velocity fields. CFT with randomly placed, stochastic conformal defects or random velocity fields is what we refer to as stochastic CFT.

Whereas introducing stochastic impurities aims at mimicking destructive interferences, the use of CFT allows us to extract the relevant collective modes in the spirit of the renormalization group. The emergence of conformal invariance at low energy provides a universal description of transport phenomena in these critical systems encompassed in the chiral nature of energy transport. It ultimately leads to the description via random trajectories given below. It applies to interacting systems as our model deals with CFT with arbitrary central charge. Note however that although appropriate to critical energy transport in the presence of impurities, this description does not provide a direct way to disentangle the effect randomness on non chiral observables, which are typically present in interacting models.

**Time evolution and random trajectories.** Many of the conclusions below are valid under quite general setup for stochastic CFT. However, in order to be precise, it is convenient to specify a model of stochastic CFT.

We place defects uniformly in space $\mathbb{R}$, and each defect is activated and deactivated in a random fashion in time, independently of each other. The random set of positions $X \subset \mathbb{R}$ is a Poisson process of intensity $2\nu$, so that the probability of finding a defect between $x$ and $x + dx$ is $P([x, x + dx] \cap X \neq \emptyset) = 2\nu dx$. For each $x \in X$, the random set of activation / deactivation events $\{t_j^x, j = 0, 1, 2, \ldots\} \subset \mathbb{R}^+$ (with $t_0 = 0$) is an independent Poisson process of intensity $\nu$. The description starts at time $t = 0$, and each defect starts either in an activated or deactivated state with equal probabilities, $b^x \in \{0, 1\}$ with probabilities $1/2, 1/2$. We denote by $T^x$ the union of all activation time intervals for the defect at position $x$; this is $T^x = \cup_{k=0}^\infty [t_k^x + br_{k+1}, t_{k+1}^x + br_{k+2}]$. We also denote by $\mathcal{D} := \{(x, t) : x \in X, t \in T^x\}$ the subset of space-time $\mathbb{R}^2$ where an activated defect lies. This is a union of infinitely-many finite vertical segments lying in the upper half plane (which definiteness we consider closed). As an example, let $p(x, \delta) dx$ be the probability that a segment intersects some horizontal line $t = \text{const.}$ between $x$ and $x + dx$, and lasts for a time longer than $\delta$ after $t$ (i.e. it ends at a time bigger than $t + \delta$). One can show that

$$p(x, \delta) dx = \eta e^{-\nu \delta} dx.$$  

Given a sample $\mathcal{S}$ of random defects, there are associated time-dependent evolution operators $U_{t,s}$ describing evolution of states from any time $s$ to $t > s$ within this defect configuration. By construction they satisfy $U_{t,s}\rho_{s,0} = \rho_{t,0}$ and the time-evolved observables are $\langle \cdot \rangle_t := \langle \cdot \rangle_t^{U_{t,0}}$. Averages in the time-evolved state $\langle \cdot \rangle_t$ are obtained, in the Heisenberg picture, as averages of time-evolved observables in the initial state, $\langle \cdot \rangle_t := \langle U_{t,0}^\dagger \cdot U_{t,0} \rangle_0$.

Since $\mathcal{D}$ is random, so is $U_{t,s}$ for any $t > s$. Since $U_{t,s}$ represents evolution from time $s$ to time $t$, it is completely determined by the defects that are active within that time period. Let $H_\epsilon$ be the Hamiltonian representing instantaneous evolution within the defect configuration exactly at time $t$. Then $i\partial_t U_{t,0} = H_\epsilon U_{t,0}$ and therefore $\partial_t O(x, t) = iU_{t,0}^\dagger [H_\epsilon, O(x)] U_{t,0}$. The vector space spanned by right/left-moving densities $T^\epsilon(x) (x \in \mathbb{R}$ and $\epsilon \in \{\pm\})$ is the space of observables on which we wish to describe time evolution. For every $x$ that lies away from an active defect at time $t$, that is $x \in \mathbb{R} : t \notin T^x$, we have chiral evolution, $i[H_\epsilon, T^\epsilon(x)] = -\epsilon \partial_x T^\epsilon(x)$ (here and below, the Fermi velocity is set to unity). Thus we find $\partial_T T^\epsilon(x, t) + \epsilon \partial_x T^\epsilon(x, t) = 0$ for all $(x, t) \in \mathbb{R}^2 \setminus \mathcal{D}$. That is, the time evolved operator $T^\epsilon(x, t)$ is obtained by drawing a trajectory starting from $(x, t)$ and going backward in time towards the left (right) if $\epsilon > 0$ ($\epsilon < 0$), as long as this diagonal does not intersect any active impurity.

On both sides of a defect, we impose conformal bound-
ary conditions. On the energy sector, a conformal boundary condition at the point \( x_0 \) is the operator equality \( T^\epsilon(x_0) = T^{-\epsilon}(x_0) \). It implies that, on both sides of the defect, the trajectory performs a reflection against the defect: we map \( T^\epsilon(x_0) \mapsto T^{-\epsilon}(x_0) \) and then continue the chiral backward evolution using the new, reflected operator, which will then go away from the defect. This is repeated as many times as active impurities are hit, until time 0 is reached, where the resulting operator can be evaluated within the initial state.

The full time evolution is therefore described in terms of trajectories: for every \( \mathcal{D} \) and every \( x, t \), the trajectory

\[
s \in [0, t] \mapsto (\epsilon_s, x_s), \quad \epsilon_t = \epsilon, \; x_t = x
\]

is obtained by evolution backward from \( (\epsilon, x) \) at time \( t \), chiral between defects and reflecting on defects. Time evolution is described by the transport equation:

\[
T^\epsilon(x, t) = T^{\epsilon t}(x, s), \quad \forall \; s < t.
\]

Taking \( s = 0 \), we have \( T^\epsilon(x, t) = T^0(x_0) \). Therefore,

\[
\left\langle \prod_{j=1}^N T_{\epsilon}^{(j)}(x(j)) \right\rangle_t = \left\langle \prod_{j=1}^N T_0^{(j)}(x_0^{(j)}) \right\rangle_0.
\]

Since time evolution is random, the quantum average at time \( t \) is also a random value. The measure on \( \mathcal{D} \) induces a probability measure on the trajectories.

We therefore need to evaluate the stochastic expectation of random operators,

\[
\Phi_{t,s}[\mathcal{O}] := \mathbb{E}\left[ U_{t,s}^\dagger \mathcal{O} U_{t,s} \right].
\]

In general, \( \Phi_{t,s} \) is not a unitary transformation. However, if \( U_{t,s} \) are unitary, it is a completely positive (CP) map and it defines a dissipative dynamics [6, 9]. It can be defined on density matrices by duality. Thanks to (3), we may express the map \( \Phi_{t,s}^* \) on product of stress tensor components in terms of averages over trajectories:

\[
\Phi_{t,s}^* \left[ \prod_{j=1}^N T_{\epsilon}^{(j)}(x(j)) \right] = \mathbb{E}\left[ \prod_{j=1}^N T^{(j)}(x_j^{(j)}) \right],
\]

where on the right-hand side, the expectation is over the trajectories ending at \( (\epsilon^{(j)}, x^{(j)}) \) at time \( t \).

Inverting the order of the quantum and stochastic averages, we evaluate the combined average in the time-evolved state as follows:

\[
\mathbb{E}[\langle \mathcal{O} \rangle_t] = \langle \Phi_{t,0}^*[\mathcal{O}] \rangle_0,
\]

which can be evaluated using eqs. (4,6).

We make three remarks, which are further developed in the Supplementary Material (SM): (i) The evolution (3) is almost surely unitary on products of stress tensor components. The events, of measure zero, on which unitary is broken correspond to cases in which two infinitely close nearby trajectories split because one hits an activated impurity while the other does not. The physical explanation of this effect is that at the microscopic level, the operations of activation and deactivation are high-energy operations, involving states that do not lie in the low-energy region of the spectrum described by CFT. Non-universal, high-energy effects are therefore involved that are not encoded by the simple activation / deactivation of conformal defects. However, since these events are extremely rare, they do not affect the results, which are thus universal. (ii) The dynamics (5) is non-Markovian because the time evolution operators \( U_{t,s} \) and \( U_{s,u} \) (for \( t < s < u \)) are not independent, but it becomes Markovian in the limit \( \nu^{-1} \to 0 \). (iii) The dynamics (3) is a transport equation and the above construction can be generalized to other random fields.

III. SIGNATURES OF LOCALISATION AT INFINITE TIME

The steady state is naturally defined as the large-time limit of the stochastic average of the quantum state,

\[
\langle \mathcal{O} \rangle_{\text{stat}} := \lim_{t \to \infty} \mathbb{E}[\langle \mathcal{O} \rangle_t].
\]

Recall that the stochastic average mimics the many interference effects of the large number of partial transmission and reflection events on impurities. It is possible to provide rather general arguments for the physical properties of the steady state arising, at large times, from stochastic CFT. The main results are as follows.

We consider an initial state, possibly inhomogeneous, with well defined asymptotics in the far right and left,

\[
\left\langle \prod_{j=1}^N T_{\epsilon}^{(j)}(x(j)) \right\rangle_{\text{r,l}} := \lim_{x \to \pm \infty} \left\langle \prod_{j=1}^N T^{(j)}(x(j) + x) \right\rangle.
\]

Assume that each initial asymptotic state \( (\cdots)_{\text{r,l}} \) is clustering at large distances. These are translation invariant, we denote the one-point functions as \( \langle T^\epsilon(x) \rangle_{\text{r,l}} \).

Microscopic clustering. The steady state is factorized on microscopic scales because backwards random trajectories almost surely split far apart. Since small time evolution is almost surely ballistic, factorization also occurs in time-dependent correlations, a fortiori at larger times as well. Formally, \( \langle \mathcal{O} \rangle_{\text{stat}} = \sum_{x \in \mathbb{R}} \langle \mathcal{O} \rangle_{\text{micro}} \) where \( \mathcal{O} \) is the component of \( \mathcal{O} \) supported at the position \( x \):

\[
\left\langle \prod_{j=1}^N T_{\epsilon}^{(j)}(x(j), t(j)) \right\rangle_{\text{stat}} = \prod_{j=1}^N \left( \langle T^{(j)}(x^{(j)}) \rangle_{\text{micro}} \right).
\]

Because backwards trajectories have equal probability to be on the left or right and to be either left or right-moving types, the microscopic-scale state \( (\cdots)_{\text{micro}} \) is given by

\[
\langle T^\epsilon(x) \rangle_{\text{micro}} = \frac{1}{4} \sum_{\eta \in \{\pm, \pm\}} \langle T^\epsilon \rangle_\eta.
\]
Microscopic clustering indicates that the state obtained at large times is the scaling limit of a localized state: the steady state has a correlation length that is zero in the renormalized distances of CFT, thus finite in units of lattice spacing; and clustering also holds under time separation. Microscopic clustering occurs even though impurities are separated by macroscopic distances: randomness at the macroscopic scale makes correlations microscopic. This occurs independently from the correlation properties of the initial state (under the above weak conditions). These are strong signatures of localization of the stationary state. Note that in quantum states with microscopic correlation length but with ballistic components in transport, the real-time current correlations are large on light-cone rays; this ballistic effect is not seen in our model further pointing to absence of transport in the steady state. Note also that if impurities were distributed in space-time in a regular fashion, trajectories would not almost surely split apart, and thus microscopic clustering would not occur.

Vanishing transport. It is clear from the above results that the mean energy current is zero at every point, \( \langle p(x) \rangle_{\text{stat}} = 0 \). This is true no matter the properties of the initial state: it may be a non-equilibrium, current-carrying state with \( \langle T^c \rangle_{\tau,1} \) dependent on \( \epsilon \), or a spatially unbalanced state with \( \langle T^c \rangle_{\tau,1} \) dependent on \( r,1 \). Another related proposition holds. One may look not only at the stochastic average of the quantum states, but also at the stochastic fluctuations. These represent, within the stochastic CFT modelling, quantum fluctuations occurring at large times due to interference effects. It is possible to show that the current is zero on spatial average, almost surely at large time, \( \int_a^b dx \langle p(x) \rangle_{t \to \infty} = 0 \). Yet, the local currents are far from being zero before the stochastic averages, and in particular the integration of the squares of the quantum-average momenta is nonzero: \( \int_a^b dx \langle (p(x))_{t \to \infty} \rangle^2 > 0 \) almost surely. The spatially integrated current is self-averaging, but locally very rough: there are strong local currents within each microscopic cell, and these can be in any direction, independently from one cell to the other.

IV. CROSSOVER FROM BALLISTIC TO DIFFUSIVE TRANSPORT

Before reaching the localized stationary state at asymptotically large time, the system does not display localization: rather a transient occurs that is dominated by diffusion. For simplicity consider the asymptotic regime \( \nu \gg \eta \), or equivalently when the observation time and the mean impurity spacing become very large simultaneously, \( t, \eta^{-1} \to \infty \), \( \eta \) fixed. The dynamics (5) becomes Markovian because the impurities are renewed at each time. One may thus expect

\[
\partial_t \mathcal{O}(t) = i[H_{\text{cft}}, \mathcal{O}(t)] + \mathcal{L}_{\text{imp}}^*(\mathcal{O})(t),
\]

with \( \mathcal{L}_{\text{imp}}^* \) the Lindblad operator [10] generating the dissipative dynamics due to scattering off impurities.

In this limit, elastic scattering events occur independently at a rate \( \eta \). It can be shown (see the SM) that the energy and momentum density satisfy the following hydrodynamic-type equations:

\[
\begin{align*}
\partial_t h(x,t) + \partial_x p(x,t) &= 0, \\
\partial_t p(x,t) + \partial_x h(x,t) &= -2\eta p(x,t).
\end{align*}
\]

The first equation codes for local energy conservation, characteristic of elastic scattering. The second includes a friction term, due to the scattering events \( p(x) \to -p(x) \) at rate \( \eta \) and net momentum density deficit of \( -2p(x) \). If the momentum density vanishes at infinity (the only case we consider), then the total energy \( E := \int dx h(x,t) \) is constant in time. The total momentum \( P := \int dx p(x,t) \) exponentially approaches a constant finite value, determined by the energy density at infinity. Eqs. (12,13) were first obtained in the context of the kinetic theory of gases a long time ago [11].

Eqs. (12,13) code for a ballistic-to-diffusive crossover: the evolution is ballistic at short time but diffusive at larger time. Indeed, they imply

\[
\partial_t^2 p(x,t) + 2\eta \partial_x p(x,t) - \partial_x^2 p(x,t) = 0.
\]

At small time the higher-derivative terms dominate, giving the relativistic wave equation \( \partial_t^2 p(x,t) - \partial_x^2 p(x,t) = 0 \). At large time, the momentum profile smoothes out and \( \partial_t^2 p \ll \eta \partial_x p \), giving the diffusion equation, \( \partial_t p(x,t) = \frac{D_F}{\eta} \partial_x^2 p(x,t) \), with a diffusion constant \( D_F = 1/\eta \) [12].

The effects are clearly seen in the case where the initial configuration produces an energy carrying non-equilibrium flow. We look at an initial state with unbalanced left (right) energy densities \( h_l \) (\( h_r \)), as in the non-equilibrium CFT setup [13]. This produces an initial momentum burst, and, in absence of dissipative processes, the state evolves toward a current-carrying steady...
state with mean energy flow $\frac{1}{2}(h_l - h_r)$. Momentum relaxation modifies this picture. Eq. (14) can be integrated with initial conditions, $p_{t=0}(x) = 0$ and $\partial_t p_{t=0}(x) = (h_l - h_r) \delta(x)$. It gives $p(x, t) = (h_l - h_r) G(x, t)$ with

$$G(x, t) = e^{-\eta t} \int \frac{dk}{2\pi} \frac{\sin(t \sqrt{k^2 - \eta^2})}{\sqrt{k^2 - \eta^2}} \cos(kx),$$

(15)

with ballistic ($k > \eta$) and diffusive ($k < \eta$) components.

This non-equilibrium state is current-carrying inside the light-cone only: $p(x, t)$ is nontrivial inside the light-cone ($p(x, t) \neq 0$ for $x^2 - t^2 < 0$), but vanishes outside ($p(x, t) = 0$ for $x^2 - t^2 > 0$). This echoes the fact that momentum relaxation does not introduce processes propagating at speeds faster than the Fermi velocity. Inside the light-cone along the ray $x = vt$, ($v < 1$), the flow decreases exponentially: $p(x = vt, t) \sim \text{const} e^{(\sqrt{1-v^2} - 1)\eta t}$. Deep inside the light-cone the diffusive component of the flow is dominating. Near the light-cone the diffusive and ballistic components are of the same order, although both are decaying, and they compensate outside the light-cone. There are contact singularities on the light cone, which are smoothed by irrelevant operators [14]. As a consequence, the momentum density has a Gaussian-like bell profile with width of order $\sqrt{t}$, much smaller than the ballistic propagation length $t$: a momentum burst $\partial_t p_{t=0}(x) \propto \delta(x)$ propagates diffusively only. On the ballistic scale of order $t$, the momentum flow remains concentrated at the origin.

V. CONCLUSION

We have defined a model of stochastic conformal field theory as an effective way to deal with quantum interferences due to multiple elastic scattering on impurities in one-dimensional critical extended interacting many-body system. We have shown that (1) the steady state is localized, and (2) the transient is formed by a crossover from ballistic to diffusive transport due to elastic momentum relaxation. We thus find, over time, the passage from ballistic to diffusive and then to localized physics. Note that it was recently argued, based on a single-particle picture, that “continuous systems” delocalize [15]; however this does not apply to the class of emergent collective behaviors described by CFT.

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[12] This is $D_F = \nu_F^2/\eta$ with a Fermi velocity $\nu_F \neq 1$.