Towards First-Order Temporal Resolution

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Abstract. In this paper we show how to extend clausal temporal resolution to the ground eventuality fragment of monodic first-order temporal logic, which has recently been introduced by Hodkinson, Wolter and Zakharyaschev. While a finite Hilbert-like axiomatization of complete monodic first order temporal logic was developed by Wolter and Zakharyaschev, we propose a temporal resolution-based proof system which reduces the satisfiability problem for ground eventuality monodic first-order temporal formulae to the satisfiability problem for formulae of classical first-order logic.

1 Introduction

We consider the first-order temporal logic over the natural numbers $TL(\mathbb{N})$ in a first-order temporal language $\mathcal{TL}$. The language $\mathcal{TL}$ is constructed in the standard way (see i.e. [Fis97, HWZ00]) from a classical (non-temporal) first-order language $\mathcal{L}$ and a set of future-time temporal operators '$\Diamond$' (*sometime*), '$\Box$' (*always*), '$\bigcirc$' (*in the next moment*), '$\mathcal{U}$'(*until*) and '$\mathcal{W}$' (*unless*, or *weak until*). Here, $\mathcal{L}$ does not contain equality or functional symbols.

Formulae in $\mathcal{TL}$ are interpreted in *first-order temporal structures* of the form $\mathfrak{M} = \langle D, \mathcal{I} \rangle$, where D is a non-empty set, the *domain* of $\mathfrak{M}$, and $\mathcal{I}$ is a function associating with every moment of time $n \in \mathbb{N}$ an interpretation of predicate and constant symbols of $\mathcal{L}$ over D. First-order (nontemporal) structures corresponding to each point of time n will be denoted by $\mathfrak{M}_n = \langle D, I_n \rangle$ where $I_n = \mathcal{I}(n)$. Intuitively, the interpretations of $\mathcal{TL}$-formulae are sequences of *worlds* such as $\mathfrak{M}_0, \mathfrak{M}_1, \ldots, \mathfrak{M}_n \ldots$. An *assignment* in D is a function $\mathfrak{a}$ from the set $\mathcal{L}_{\mathbf{v}}$ of individual variables of $\mathcal{L}$ to D. We require that (individual) variables and constants of $\mathcal{TL}$ are *rigid*, that is neither assignments nor interpretations of constants depend on worlds.

The *truth-relation* $\mathfrak{M}_n \models^{\mathfrak{a}} \varphi$ (or simply $n \models^{\mathfrak{a}} \varphi$, if $\mathfrak{M}$ is understood) in the structure $\mathfrak{M}$ for the assignment $\mathfrak{a}$ is defined inductively in usual way under the following semantics of temporal operators:

$$
\begin{array}{lll}
n \models^{\mathfrak{a}} \bigcirc\varphi & \text{iff} & n+1 \models^{\mathfrak{a}} \varphi; \\
n \models^{\mathfrak{a}} \Diamond\varphi & \text{iff} & \text{there exists a } m \geq n \text{ such that } m \models^{\mathfrak{a}} \varphi; \\
n \models^{\mathfrak{a}} \Box\varphi & \text{iff} & m \models^{\mathfrak{a}} \varphi \text{ for all } m \geq n; \\
n \models^{\mathfrak{a}} \varphi\,\mathcal{U}\,\psi & \text{iff} & \text{there exists a } m \geq n \text{ such that } m \models^{\mathfrak{a}} \psi \text{ and} \\
 & & \text{for every } k \in \mathbb{N}, \text{ if } n \leq k < m \text{ then } k \models^{\mathfrak{a}} \varphi : \\
n \models^{\mathfrak{a}} \varphi\,\mathcal{W}\,\psi & \text{iff} & n \models^{\mathfrak{a}} \varphi\,\mathcal{U}\,\psi \text{ or } n \models^{\mathfrak{a}} \Box\varphi.
\end{array}
$$

A formula φ is said to be *satisfiable* if there is a first-order structure $\mathfrak{M}$ and an assignment $\mathfrak{a}$ such that $\mathfrak{M}_0 \models^{\mathfrak{a}} \varphi$. If $\mathfrak{M}_0 \models^{\mathfrak{a}} \varphi$ for every structure $\mathfrak{M}$ and for all assignments, then φ is said to be *valid*. Note that formulae here are interpreted in the initial world $\mathfrak{M}_0$; that is an alternative but equivalent definition to the one used in [HWZ00].

2 Divided Separated Normal Form

Our method works on temporal formulae transformed into a normal form. This normal form follows the spirit of Separated Normal Form (SNF) [Fis91, FDP01] and First-Order Separated Normal Form (SNF$_f$) [Fis92, Fis97]. However, we go even further.

One of the main aims realized in SNF/SNF$_f$ was inspired by Gabbay's separation result [Gab87]. In accordance with this aim, formulae in SNF/SNF$_f$ comprise implications with present-time formulae on the left-hand side and (present or) future formulae on the right-hand side. The transformation into the separated form is based upon the well-known *renaming* technique [PG86], which preserves satisfiability and admits the extension to temporal logic in (Renaming Theorems [Fis97]).

Another intention was to reduce most of the temporal operators to a core set. This concerns the removal of temporal operators represented as *maximal* fixpoints, i.e. $\square$ and $\mathcal{W}$ (Maximal Fixpoint Removal Theorems [Fis97]). Note that the $\mathcal{U}$ operator can be represented as a combination of operators based upon maximal fixpoints and the $\lozenge$ operator (which is retained within SNF/SNF$_f$). This transformation is based upon the simulation of fixpoints using QPTL [Wol82].

Now we add one additional aim, namely to divide the temporal part of a formula from its (classical) first-order part in such way that the temporal part is as simple as possible. The modified normal form is called Divided Separated Normal Form or DSNF for short. A Divided SNF problem is a triple $< \mathcal{U}, \mathcal{S}, \mathcal{T} >$ where $\mathcal{S}$ and $\mathcal{U}$ are the *universal part* and the *initial part*, respectively, given by finite sets of nontemporal first-order formulae (that is, without temporal operators), and $\mathcal{T}$ is the *temporal part* given by a finite set of *temporal clauses*. All formulae are written in $\mathcal{L}$ extended by a set of predicate and propositional symbols. A temporal clause has one of the following forms:

$$P(\bar{x}) \Rightarrow \bigcirc \bigwedge_{i=1}^{n} Q_i(\bar{x}) \quad \text{(\textit{predicate step clause})},$$

$$p \Rightarrow \bigcirc \bigwedge_{j=1}^{m} q_j \quad \text{(\textit{proposition step clause})},$$

$$P(\bar{x}) \Rightarrow \bigcirc \lozenge Q(\bar{x}) \quad \text{(\textit{predicate eventuality clause})},$$

$$p \Rightarrow \bigcirc \lozenge q \quad \text{(\textit{proposition eventuality clause})}$$

where P, Q, Q_i are predicate symbols, p, q, q_j are propositional symbols, and $\Rightarrow$ is a substitute for implication. Sometimes temporal clauses are called *temporal rules* to make distinctions between their left- and right-hand sides. Without loss of generality we suppose that there are no two different temporal step rules with the same left-hand sides and there are no two different eventuality rules with the same right-hand sides. An atom $Q(\bar{x})$ or q from the right-hand side of an eventuality rule is called an *eventuality atom*.

We call examples of DSNF *temporal problems*. The semantics of a temporal problem is defined under the supposition that the universal and temporal parts are closed by the outermost prefixes $\Box\forall$, the initial part is closed only by universal quantifiers. In what follows we will not distinguish between a finite set of formulae $\mathcal{X}$ and the conjunction $\bigwedge \mathcal{X}$ of formulae in it. Thus the temporal formula corresponding to a temporal problem $< \mathcal{U}, \mathcal{S}, \mathcal{T} >$ is $(\Box\forall\mathcal{U}) \wedge \forall\mathcal{S} \wedge (\Box\forall\mathcal{T})$.

So, when we consider the satisfiability or the validity of a temporal problem we implicitly mean the corresponding formula, as above.

Given the results about the renaming of subformulae and the removal of temporal operators mentioned above, we can state the general theorem about translation into DSNF as follows.

Theorem 1. *Any first-order temporal formula φ in $\mathcal{TL}$ can be translated into a temporal problem $< \mathcal{U}, \mathcal{S}, \mathcal{T} >$ (i.e. DSNF of φ) in a language $\mathcal{TL}' \supseteq \mathcal{TL}$ extended by new propositional and predicate symbols such that φ is satisfiable if, and only if, $< \mathcal{U}, \mathcal{S}, \mathcal{T} >$ is satisfiable.*

For any formula φ its DSNF representation $< \mathcal{U}, \mathcal{S}, \mathcal{T} >$ can be constructed in polynomial time in the length of φ. (As a whole the transformation of φ into $< \mathcal{S}, \mathcal{U}, \mathcal{T} >$ is similar to the familiar depth-reducing reductions of first-order formulae via the introduction of new names.)

Example 1. Let us consider the following formula: $\varphi = (\exists x \Box\neg Q(x)) \wedge (\bigcirc \exists y Q(y))$. After transformation φ to a normal form, we get the following temporal problem:

$$\mathcal{S} = \left\{ \begin{array}{l} s1.\ p_1 \\ s2.\ p_3 \end{array} \right\}, \quad \mathcal{T} = \left\{ \begin{array}{l} t1.\ p_1 \Rightarrow \bigcirc p_2, \\ t2.\ P_1(x) \Rightarrow \bigcirc (P_2(x) \wedge P_1(x)) \end{array} \right\},$$

$$\mathcal{U}_0 = \left\{ \begin{array}{lll} u1.\ P_3(x) \supset P_2(x), & u3.\ P_2(x) \supset \neg Q(x), & u5.\ p_3 \supset \exists x P_3(x) \\ u2.\ P_3(x) \supset P_1(x), & u4.\ p_2 \supset \exists y Q(y), & \end{array} \right\}.$$

3 The monodic fragment and merged temporal step rules

Following [HWZ00] we consider the set of all $\mathcal{TL}$-formulae φ such that any subformula of φ of the form $\Diamond\psi$, $\Box\psi$, $\bigcirc\psi$, $\psi_1 \,\mathcal{U}\, \psi_2$, $\psi_1 \,\mathcal{W}\, \psi_2$ has at most one free variable. Such formulae are called *monodic*, and the set of monodic $\mathcal{L}$-formulae is denoted by $\mathcal{T}_1\mathcal{L}$. In spite of its relative narrowness the monodic fragment provides a way for quite realistic applications. For example, temporal extensions of the spatial formalism RCC-8 [Wol00] lie within the monodic fragment. Another example is the verification of properties of relational transducers for electronic commerce [AVFY00] which are expressed in the monodic language again.

The decidability of $\mathcal{T}_1\mathcal{L}$ was proved in [HWZ00] while, in [WZ01], a finite Hilbert-style axiomatization of the monodic fragment of $TL(\mathbb{N})$ has been constructed. However no deduction-based decision procedure for this class has yet been proposed.

The notion 'monodic' is transferred from temporal formulae to temporal problems as follows. A problem $< \mathcal{U}, \mathcal{S}, \mathcal{T} >$ is *monodic problem* if all predicates occurring in its temporal part $\mathcal{T}$ are monadic. Every monodic formula is translated into DSNF given by the monodic problem. In Example 1 both the formula φ and its DSNF problem are monodic.

The key role in propositional temporal resolution is played by so-called *merged step clauses* [FDP01]. In the case of the monodic fragment, we can define an analogue of propositional merged step clauses and so formulate for monodic problems a calculus which is analogous to the propositional temporal resolution calculus (up to replacing the propositional merged clauses by the first-order merged clauses defined below) such that this calculus is complete for the so-called *ground eventuality monodic fragment* defined in the next section.

Next we introduce the notions of colour schemes and constant distributions. Let $\mathcal{P} =< \mathcal{U}, \mathcal{S}, \mathcal{T} >$ be a temporal problem. Let C be the set of constants occurring in $\mathcal{P}$. Let $\mathcal{T}^{\mathbf{P}} = \{P_i(x) \Rightarrow \bigcirc R_i(x), \mid 1 \leq i \leq K\}$ and $\mathcal{T}^{\mathbf{p}} = \{p_j \Rightarrow \bigcirc r_j \mid 1 \leq j \leq k\}$ be the sets of all predicate step rules and all propositional step rules of $\mathcal{T}$, respectively. It is supposed that $K \geq 0$ and $k \geq 0$; if $K = 0$ ($k = 0$) it means that the set $\mathcal{T}^{\mathbf{P}}$ ($\mathcal{T}^{\mathbf{p}}$) is empty. (The expressions $R_i(x)$ and r_j denote finite conjunctions of atoms $\bigwedge\limits_{l} Q_{il}(x)$ and $\bigwedge\limits_{l} q_{jl}$, respectively.)

Let $\{P_1, \ldots, P_K, P_{K+1} \ldots, P_M\}$, $0 \leq K \leq M$, and $\{p_1, \ldots, p_k, p_{k+1} \ldots, p_m\}$, $0 \leq k \leq m$, be sets of all (monadic) predicate symbols and propositional symbols, respectively, occurring in $\mathcal{T}$. Let Δ be the set of all mappings from $\{1, \ldots, M\}$ to $\{0, 1\}$, and Θ be the set of all mappings from $\{1, \ldots, m\}$ to $\{0, 1\}$. An element $\delta \in \Delta$ ($\theta \in \Theta$) is represented by the sequence $[\delta(1), \ldots, \delta(M)] \in \{0, 1\}^M$ ($[\theta(1), \ldots, \theta(m)] \in \{0, 1\}^m$). Let us call elements of Δ and Θ predicate and propositional *colours*, respectively. Let Γ be a subset of Δ, and θ be an element of Θ, and ρ be a map from C to Γ. A triple (Γ, θ, ρ) is called a *colour scheme*, and ρ is called a *constant distribution*.

Note 1. The notion of the colour scheme came, of course, from the well known method of the decidability proof for the monadic class in classical first-order logic (see, for example, [BGG97]). In our case we construct quotient structures based only on the predicates and propositions which occur in the temporal part of the problem, because only these symbols are really responsible for the satisfiability of temporal constraints. Besides, we have to consider so-called constant distributions, because unlike the classical case we cannot eliminate constants replacing them by existentially bounded variables – the monodicity property would be lost.

For every colour scheme $\mathcal{C} = \langle \Gamma, \theta, \rho \rangle$ let us construct the formulae $\mathcal{F}_{\mathcal{C}}$, $\mathcal{A}_{\mathcal{C}}$, $\mathcal{B}_{\mathcal{C}}$ in the following way. In the beginning for every $\gamma \in \Gamma$ and for θ introduce the conjunctions:

$$F_\gamma(x) = \bigwedge_{\gamma(i)=1} P_i(x) \wedge \bigwedge_{\gamma(i)=0} \neg P_i(x), \qquad F_\theta = \bigwedge_{\theta(i)=1} p_i \wedge \bigwedge_{\theta(i)=0} \neg p_i,$$

$$A_\gamma(x) = \bigwedge_{\gamma(i)=1 \& i \leq K} P_i(x), \qquad A_\theta = \bigwedge_{\theta(i)=1 \& i \leq k} p_i,$$

$$B_\gamma(x) = \bigwedge_{\gamma(i)=1 \& i \le K} R_i(x), \qquad B_\theta = \bigwedge_{\theta(i)=1 \& i \le k} r_i.$$

Now $\mathcal{F}_\mathcal{C}$, $\mathcal{A}_\mathcal{C}$, $\mathcal{B}_\mathcal{C}$ are of the following forms

$$\mathcal{F}_\mathcal{C} = \bigwedge_{\gamma \in \Gamma} \exists x F_\gamma(x) \wedge F_\theta \wedge \bigwedge_{c \in C} F_{\rho(c)}(c) \wedge \forall x \bigvee_{\gamma \in \Gamma} F_\gamma(x),$$

$$\mathcal{A}_\mathcal{C} = \bigwedge_{\gamma \in \Gamma} \exists x A_\gamma(x) \wedge A_\theta \wedge \bigwedge_{c \in C} A_{\rho(c)}(c) \wedge \forall x \bigvee_{\gamma \in \Gamma} A_\gamma(x),$$

$$\mathcal{B}_\mathcal{C} = \bigwedge_{\gamma \in \Gamma} \exists x B_\gamma(x) \wedge B_\theta \wedge \bigwedge_{c \in C} B_{\rho(c)}(c) \wedge \forall x \bigvee_{\gamma \in \Gamma} B_\gamma(x).$$

We can consider the formula $\mathcal{F}_\mathcal{C}$ as a 'categorical' formula specification of a quotient structure given by a colour scheme. In turn, the formula $\mathcal{A}_\mathcal{C}$ represents the part of this specification which is 'responsible' just for 'transferring' temporal requirements from the current world (quotient structure) to its immediate successors. The clause $(\Box\forall)(\mathcal{A}_\mathcal{C} \Rightarrow \bigcirc\mathcal{B}_\mathcal{C})$ is then called a *merged step rule*. Note that if both sets $\{i \mid i \le K, \gamma \in \Gamma, \gamma(i) = 1\}$ and $\{i \mid i \le k, \theta(i) = 1\}$ are empty the rule $(\mathcal{A}_\mathcal{C} \Rightarrow \bigcirc\mathcal{B}_\mathcal{C})$ degenerates to (**true** $\Rightarrow \bigcirc$**true**).

Example 2. Let us return to the temporal problem obtained in the example 1. The temporal part produces the following set of step merged clauses

1. $(p_1 \wedge \forall x P_1(x)) \Rightarrow \bigcirc(p_2 \wedge \forall x(P_2(x) \wedge P_1(x)))$,
2. $(p_1 \wedge \exists x P_1(x)) \Rightarrow \bigcirc(p_2 \wedge \exists x(P_2(x) \wedge P_1(x)))$,
3. $(\forall x P_1(x)) \Rightarrow \bigcirc(\forall x(P_2(x) \wedge P_1(x)))$,
4. $(\exists x P_1(x)) \Rightarrow \bigcirc(\exists x(P_2(x) \wedge P_1(x)))$.

For this problem $K = M = 2, k = m = 1$. The problem does not contain any constants, and in this case the colour schemes are defined as pairs of the form (Γ, θ).

The first merged rule corresponds to the colour scheme $(\{[1,\_,\_]\},[1,\_,\_])$ (the subformula $\exists x P_1(x) \wedge \forall x P_1(x)$ is reduced to $\forall x P_1(x)$). The second rule corresponds to $(\{[1,\_,\_],[0,\_,\_]\},[1,\_,\_])$ (as usual the value of the empty conjunction $\bigwedge_{i \in \emptyset} P_i(x)$ is **true**). The third and the fourth rules correspond to $(\{[1,\_,\_]\},[0,\_,\_])$ and $(\{[1,\_,\_],[0,\_,\_]\},[0,\_,\_])$, respectively.

The set of merged step rules for a problem $< \mathcal{U}, \mathcal{S}, \mathcal{T} >$ is denoted by $\mathbf{m}\mathcal{T}$.

4 Resolution procedure for monodic induction free problems

A problem $< \mathcal{U}, \mathcal{S}, \mathcal{T} >$ is called *induction free* if $\mathcal{T}$ does not contain eventuality rules. In this section a derivation system based on a *step resolution rule* is given which is complete for the induction free monodic fragment.

Definition 1 (step resolution rule). *Let* $\mathbf{m}\mathcal{T}$ *be the set of merged rules of a problem* $<\mathcal{U}, \mathcal{S}, \mathcal{T}>$, $(\mathcal{A} \Rightarrow \bigcirc\mathcal{B}) \in \mathbf{m}\mathcal{T}$. *Then the step resolution inference rule w.r.t.* $\mathcal{U}$ *is the rule*

$$\frac{\mathcal{A} \Rightarrow \bigcirc\mathcal{B}}{\neg\mathcal{A}} \ (\bigcirc_{res})$$

with the side condition that the set $\mathcal{U} \cup \{\mathcal{B}\}$ *is unsatisfiable.* [1]

Note 2. The test whether the side condition is satisfied does not involve temporal reasoning and can be given to any first-order proof search procedure.

By Step$(\mathcal{U}, \mathbf{m}\mathcal{T})$ we denote the set of all formulae which are obtained by the step resolution rule w.r.t $\mathcal{U}$ from a merged clause $\mathcal{A} \Rightarrow \bigcirc\mathcal{B}$ in $\mathbf{m}\mathcal{T}$. Since $\mathbf{m}\mathcal{T}$ is finite the set Step$(\mathcal{U}, \mathbf{m}\mathcal{T})$ is also finite.

Lemma 1 (soundness of step resolution). *Let* $<\mathcal{U}, \mathcal{S}, \mathcal{T}>$ *be a temporal problem, and* $\neg\mathcal{A} \in$ *Step*$(\mathcal{U}, \mathbf{m}\mathcal{T})$. *Then* $<\mathcal{U}, \mathcal{S}, \mathcal{T}>$ *is satisfiable if, and only if,* $<\mathcal{U} \cup \{\neg\mathcal{A}\}$, $\mathcal{S}, \mathcal{T}>$ *is satisfiable.*

We describe a proof procedure for $<\mathcal{U}, \mathcal{S}, \mathcal{T}>$ by a binary relation $\triangleright$ on (universal) sets of formulae, which we call a *transition* or *derivation relation*. In this section we define the derivation relations by the condition that each step $\mathcal{U}_i \triangleright \mathcal{U}_{i+1}$ consists of adding to the set $\mathcal{U}_i$ (to the *state* $\mathcal{U}_i$) a formula from Step$(\mathcal{U}_i, \mathbf{m}\mathcal{T})$. (In the next section this relation will be extended by a new *sometime resolution rule*). A finite sequence $\mathcal{U}_0 \triangleright \mathcal{U}_1 \triangleright \mathcal{U}_2 \triangleright \ldots \triangleright \mathcal{U}_n$, where $\mathcal{U}_0 = \mathcal{U}$, is called a *(theorem proving) derivation* for $<\mathcal{U}, \mathcal{S}, \mathcal{T}>$. [2]

Definition 2 (termination rule and fair derivation). *A theorem proving derivation* $\mathcal{U} = \mathcal{U}_0 \triangleright \mathcal{U}_1 \triangleright \ldots \triangleright \mathcal{U}_n$, $n \geq 0$, *for a problem* $<\mathcal{U}, \mathcal{S}, \mathcal{T}>$ *is successfully terminated if the set* $\mathcal{U}_n \cup \mathcal{S}$ *is unsatisfiable. The theorem proving derivation for a problem* $<\mathcal{U}$, $\mathcal{S}, \mathcal{T}>$ *is called* fair *if it either successfully terminates or, for any* $i \geq 0$ *and a formula* $\neg\mathcal{A} \in$ *Step*$(\mathcal{U}_i, \mathbf{m}\mathcal{T})$, *there is* $j \geq i$ *such that* $\neg\mathcal{A} \in \mathcal{U}_j$.

Note 3. We intentionally do not include in our consideration the classical concept of redundancy (see [BG01]) and deletion rules over sets of first-order formulae $\mathcal{U}_i$ because the main purpose of this paper is just new developments within temporal reasoning.

As we can see only the universal part is modified during the derivation, the temporal and initial parts of the problem remain unchanged.

Following [FDP01] we base our proof of completeness on a *behavior graph* for the problem $<\mathcal{U}, \mathcal{S}, \mathcal{T}>$. Since, in this section, we are interested only in induction free problems we consider only so-called *eventuality free* behaviour graphs.

[1] The side condition provides the rule with the second (implicit) premise **true** $\Rightarrow \bigcirc\neg\mathcal{B}$ giving this rule a usual resolution form.

[2] In reality we can keep the states $\mathcal{U}_i$ in the form which is the most suitable for applying a first-order theorem prover procedure. For example, for a classical resolution-based procedure they could be saturated sets of clauses [BG01].

Definition 3 (eventuality free behaviour graph). *Given a problem* $\mathcal{P} = <\mathcal{U}, \mathcal{S}, \mathcal{T}>$ *we construct a finite directed graph* G *as follows. Every node of* G *is a one-tuple* $(\mathcal{C})$ *where* $\mathcal{C}$ *is a colour scheme for* $\mathcal{T}$ *such that the set* $\mathcal{U} \cup \mathcal{F}_{\mathcal{C}}$ *is satisfiable.*

For each node $(\mathcal{C})$, $\mathcal{C} = (\Gamma, \theta, \rho)$, *we construct an edge in* G *to a node* $(\mathcal{C}')$, $\mathcal{C}' = (\Gamma', \theta', \rho')$, *if* $\mathcal{U} \wedge \mathcal{F}_{\mathcal{C}'} \wedge \mathcal{B}_{\mathcal{C}}$ *is satisfiable. They are the only edges originating from* $(\mathcal{C})$.

A node $(\mathcal{C})$ *is designated as an initial node of* G *if* $\mathcal{S} \wedge \mathcal{U} \wedge \mathcal{F}_{\mathcal{C}}$ *is satisfiable.*

The eventuality free behaviour graph H *of* $\mathcal{P}$ *is the full subgraph of* G *given by the set of nodes reachable from the initial nodes.*

It is easy to see that there is the following relation between behaviour graphs of two temporal problems when one of them is obtained by extending the universal part of another one.

Lemma 2. *Let* $\mathcal{P}_1 = <\mathcal{U}_1, \mathcal{S}, \mathcal{T}>$ *and* $\mathcal{P}_2 = <\mathcal{U}_2, \mathcal{S}, \mathcal{T}>$ *be two* $\mathcal{TL}$ *problems such that* $\mathcal{U}_1 \subseteq \mathcal{U}_2$. *Then the behaviour graph* H_2 *of* $\mathcal{P}_2$ *is a subgraph of the behaviour graph* H_1 *of* $\mathcal{P}_1$.

Proof The graph H_2 is the full subgraph of H_1 given by the set of nodes whose interpretations satisfy $\mathcal{U}_2$ and which are reachable from the initial nodes of H_1 whose interpretations also satisfy $\mathcal{U}_2$. ❑

In the remainder of this section we will refer to an eventuality free behaviour graph simply as a behaviour graph.

Definition 4 (suitable pairs). *Let* $(\mathcal{C}, \mathcal{C}')$ *where* $\mathcal{C} = (\Gamma, \theta, \rho)$, $\mathcal{C}' = (\Gamma', \theta', \rho')$ *be an (ordered) pair of colour schemes for* $\mathcal{T}$. *An ordered pair of predicate colours* (γ, γ') *where* $\gamma \in \Gamma$, $\gamma' \in \Gamma'$ *is called* suitable *if the formula* $\mathcal{U} \wedge F_{\gamma'}(x) \wedge B_{\gamma}(x)$ *is satisfiable. Similarly, the ordered pair of propositional colours* (θ, θ') *is suitable if* $\mathcal{U} \wedge F_{\theta'} \wedge B_{\theta}$ *is satisfiable. The ordered pair of constant distributions* (ρ, ρ') *is called suitable if, for every* $c \in C$, *the pair* $(\rho(c), \rho'(c))$ *is suitable.*

Lemma 3. *Let* H *be the behaviour graph of a problem* $<\mathcal{U}, \mathcal{S}, \mathcal{T}>$ *with an edge from a node* $(\mathcal{C})$ *to a node* $(\mathcal{C}')$ *of* H, *where* $\mathcal{C} = (\Gamma, \theta, \rho)$ *and* $\mathcal{C}' = (\Gamma', \theta', \rho')$. *Then*

- *for every* $\gamma \in \Gamma$ *there exists* $\gamma' \in \Gamma'$ *such that the pair* (γ, γ') *is suitable;*
- *for every* $\gamma' \in \Gamma'$ *there exists* $\gamma \in \Gamma$ *such that the pair* (γ, γ') *is suitable;*
- *the pair of propositional colours* (θ, θ') *is suitable;*
- *the pair of constant distributions* (ρ, ρ') *is suitable.*

Proof To prove the first item it is enough to note that satisfiability of the expression $\mathcal{U} \wedge \mathcal{F}_{\mathcal{C}'} \wedge \mathcal{B}_{\mathcal{C}}$ implies satisfiability of $\mathcal{U} \wedge (\forall x \bigvee_{\gamma' \in \Gamma'} F_{\gamma'}(x)) \wedge \exists x B_{\gamma}(x)$. This, in turn, implies satisfiability of its logical consequence $\mathcal{U} \wedge \bigvee_{\gamma' \in \Gamma'} \exists x (F_{\gamma'}(x) \wedge B_{\gamma}(x))$. So, one of the members of this disjunction must be satisfiable. The second item follows from the satisfiability of the formula $\mathcal{U} \wedge (\forall x \bigvee_{\gamma \in \Gamma} B_{\gamma}(x)) \wedge \exists x F_{\gamma'}(x)$. Other items are proved similarly. ❑

Let H be the behaviour graph of a problem $< \mathcal{U}, \mathcal{S}, \mathcal{T} >$ and $\Pi = (\mathcal{C}_0), \dots, (\mathcal{C}_n), \dots$ be a path in H where $\mathcal{C}_i = (\Gamma_i, \theta_i, \rho_i)$. Let $\mathcal{G}_0 = \mathcal{S} \cup \{\mathcal{F}_{\mathcal{C}_o}\}$ and $\mathcal{G}_n = \mathcal{F}_{\mathcal{C}_n} \wedge \mathcal{B}_{\mathcal{C}_{n-1}}$ for $n \geq 1$. From classical model theory, since the language $\mathcal{L}$ is countable and does not contain equality the following lemma holds.

Lemma 4. *Let κ be a cardinal, $\kappa \geq \aleph_0$. For every $n \geq 0$, if a set $\mathcal{U} \cup \{\mathcal{G}_n\}$ is satisfiable, then there exists an $\mathcal{L}$-model $\mathfrak{M}_n = \langle D, I_n \rangle$ of $\mathcal{U} \cup \{\mathcal{G}_n\}$ such that for every $\gamma \in \Gamma_n$ the set $D_{(n,\gamma)} = \{a \in D \mid \mathfrak{M}_n \models F_\gamma(a)\}$ is of cardinality κ.*

Definition 5 (run). *By a* run *in Π we mean a function from $\mathbb{N}$ to $\bigcup\limits_{i \in \mathbb{N}} \Gamma_i$ such that for every $n \in \mathbb{N}$, $r(n) \in \Gamma_n$ and the pair $(r(n), r(n+1))$ is suitable.*

It follows from the definition of H that for every $c \in C$ the function r_c defined by $r_c(n) = \rho_n(c)$ is a run in Π.

Theorem 2. *An* induction free *problem $< \mathcal{U}, \mathcal{S}, \mathcal{T} >$ is satisfiable if, and only if, there exists an infinite path $\Pi = (\mathcal{C}_0), \dots, (\mathcal{C}_n), \dots$ through the behaviour graph H for $< \mathcal{U}, \mathcal{S}, \mathcal{T} >$ where $(\mathcal{C}_0)$ is an initial node of H.*

Proof ($\Rightarrow$) Let $\mathfrak{M} = \langle D, \mathcal{I} \rangle$ be a model of $< \mathcal{U}, \mathcal{S}, \mathcal{T} >$. Let us define for every $n \in \mathbb{N}$ the node $(\mathcal{C})$, $\mathcal{C} = (\Gamma_n, \theta_n, \rho_n)$, as follows.

For every $a \in D$ let $\gamma_{(n,a)}$ be a map from $\{1, \dots, M\}$ to $\{0, 1\}$, and let θ_n be a map from $\{1, \dots, M\}$ to $\{0, 1\}$ such that

$$\gamma_{(n,a)}(i) \risingdotseq \begin{cases} 1, & \text{if } \mathfrak{M}_n \models P_i(a), \\ 0, & \text{if } \mathfrak{M}_n \not\models P_i(a) \end{cases} \qquad \theta_n(i) \risingdotseq \begin{cases} 1, & \text{if } \mathfrak{M}_n \models p_i, \\ 0, & \text{if } \mathfrak{M}_n \not\models p_i \end{cases}$$

for every $1 \leq i \leq M$.

Now we define $\Gamma_n \risingdotseq \{\gamma_{(n,a)} \mid a \in D\}$, and $\rho_n(c) \risingdotseq \gamma_{(n, c^{\mathcal{I}(n)})}$ for every $c \in C$. (Recall that, in accordance with our semantics, all constants are "rigid", that is $c^{I(u)} = c^{I(v)}$ for every $u, v \in \mathbb{N}$.) According to the construction $(\Gamma_n, \theta_n, \rho_n)$ given above we can conclude thet the sequence $(\mathcal{C}_0), \dots, (\mathcal{C}_n), \dots$ where $\mathcal{C}_n = (\Gamma_n, \theta_n, \rho_n)$, $n \in \mathbb{N}$, is a path through H.

Proof ($\Leftarrow$) Following [HWZ00] take a cardinal $\kappa \geq \aleph_0$ exceeding the cardinality of the set $\Re$ of all runs in Π. Let us define a domain $D = \{\langle r, \xi \rangle \mid r \in \Re, \xi < \kappa\}$. Then for every $n \in \mathbb{N}$ and for every $\delta \in \Delta$

$$\|\{\langle r, \xi \rangle \in D \mid r(n) = \delta\}\| = \begin{cases} \kappa, & \text{if } \delta \in \Gamma_n, \\ 0, & \text{otherwise.} \end{cases}$$

So, for every $n \in \mathbb{N}$ it follows that $D = \bigcup\limits_{\gamma \in \Gamma_n} D_{(n,\gamma)}$ where $D_{(n,\gamma)} = \{\langle r, \xi \rangle \in D \mid r(n) = \gamma\}$. Hence by Lemma 4, for every $n \in \mathbb{N}$ there exists an $\mathcal{L}$-structure $\mathfrak{M}_n = \langle D, I_n \rangle$ which satisfies $\mathcal{U} \cup \{\mathcal{G}_n\}$. Moreover, we can suppose that $c^{I_n} = \langle r_c, 0 \rangle$ and $D_{(n,\gamma)} = \{\langle r, \xi \rangle \in D \mid \mathfrak{M}_n \models F_\gamma(\langle r, \xi \rangle)\}$ for every $\gamma \in \Gamma_n$. A first-order temporal model that we sought is $\mathfrak{M} = \langle D, \mathcal{I} \rangle$ where $\mathcal{I}(n) = I_n$ for all $n \in \mathbb{N}$. To be convinced of that let us show validity of an arbitrary step rule $\Box(P_i(x) \Rightarrow \bigcirc R_i(x))$ in $\mathfrak{M}$. Namely, let us show that, for every $n \geq 0$ and for every $\langle r, \xi \rangle \in D$, if $\mathfrak{M}_n \models P_i(\langle r, \xi \rangle)$, then $\mathfrak{M}_{n+1} \models R_i(\langle r, \xi \rangle)$. Suppose $r(n) = \gamma \in \Gamma_n$ and $r(n+1) = \gamma' \in \Gamma_{n+1}$, that is $\langle r, \xi \rangle \in D_{(n,\gamma)}$ and $\langle r, \xi \rangle \in D_{(n+1,\gamma')}$. If $\mathfrak{M}_n \models P_i(\langle r, \xi \rangle)$ then $\gamma(i) = 1$. It follows that $R_i(x)$ is embedded in $B_\gamma(x)$ (if we consider $R_i(x)$ and $F_{\gamma'}(x)$ as sets). Since the pair (γ, γ') is suitable it follows that the conjunction $R_i(x)$ is embedded in $F_{\gamma'}(x)$. Together with $\mathfrak{M}_{n+1} \models F_{\gamma'}(\langle r, \xi \rangle)$ this implies that $\mathfrak{M}_{n+1} \models R_i(\langle r, \xi \rangle)$. ❑

Corollary 1 (completeness of the step resolution). *If an* induction free *problem* $\mathcal{P} = < \mathcal{U}, \mathcal{S}, \mathcal{T} >$ *is unsatisfiable, then every* fair *theorem proving derivation for* $< \mathcal{U}, \mathcal{S}, \mathcal{T} >$ *succesfully terminates.*

Proof Let $\mathcal{U} = \mathcal{U}_0 \rhd \ldots \rhd \mathcal{U}_i \rhd \ldots \rhd \mathcal{U}_n$ be a fair theorem proving derivation for a problem $\mathcal{P} = < \mathcal{U}, \mathcal{S}, \mathcal{T} >$. The proof proceeds by induction on the number of nodes in the behaviour graph H of $\mathcal{P}$, which is finite. If H is empty then the set $\mathcal{U} \cup \mathcal{S}$ is unsatisfiable. In this case the derivation is successfully terminated because the set $\mathcal{U}_n \cup \mathcal{S}$ includes $\mathcal{U} \cup \mathcal{S}$ and therefore it is unsatisfiable too.

Now suppose H is not empty. Since $< \mathcal{U}, \mathcal{S}, \mathcal{T} >$ is unsatisfiable Theorem 2 tells us that all paths through H starting from initial nodes are finite. Let $(\mathcal{C})$ be a node of H which has no successors. In this case the set $\mathcal{U} \cup \{\mathcal{B}_\mathcal{C}\}$ is unsatisfiable. Indeed, suppose $\mathcal{U} \cup \{\mathcal{B}_\mathcal{C}\}$ is satisfiable, and $\langle D', I' \rangle$ is a model of $\mathcal{U} \cup \{\mathcal{B}_\mathcal{C}\}$. Then following the proof of the previous theorem we can define a colour scheme $\mathcal{C}'$ such that $\langle D', I' \rangle \models \mathcal{F}_{\mathcal{C}'}$. Since $\mathcal{B}_\mathcal{C} \wedge \mathcal{F}_{\mathcal{C}'}$ is satisfiable there is an edge from the node $(\mathcal{C})$ to the node $\mathcal{C}'$ in the contradiction with the choice of $(\mathcal{C})$ as having no successor. Since the derivation is fair, there is a step when $\neg \mathcal{A}_\mathcal{C}$ is included to a state $\mathcal{U}_i \supseteq \mathcal{U}$. This implies removing the node $(\mathcal{C})$ from the behaviour graph H_i of the problem $< \mathcal{U}_i, \mathcal{S}, \mathcal{T} >$ because the set $\{\mathcal{F}_\mathcal{C}, \neg \mathcal{A}_\mathcal{C}\}$ is not satisfiable. By lemma 2 it follows that H_i is a proper subgraph of H.

Now we can apply induction hypothesis to the problem $< \mathcal{U}_i, \mathcal{S}, \mathcal{T} >$ and to the fair derivation $\mathcal{U}_i \rhd \ldots \rhd \mathcal{U}_n$. ❑

Example 3. Let us return to Example 2. We can apply step resolution (w.r.t. $\mathcal{U}_0$) to the second clause because the set $\mathcal{U}_0 \cup \{p_2 \wedge \forall x(P_2(x) \wedge P_1(x))\}$ is unsatisfiable:

$$\frac{(p_1 \wedge \forall x P_1(x)) \Rightarrow \bigcirc(p_2 \wedge \forall x(P_2(x) \wedge P_1(x)))}{\neg(p_1 \wedge \forall x P_1(x))} \; (\bigcirc_{res})$$

5 Resolution procedure for ground eventuality monodic problems

A problem $< \mathcal{U}, \mathcal{S}, \mathcal{T} >$ is called a *ground eventuality* problem if $\mathcal{T}$ contains only propositional eventuality rules. In this section a derivation system based on the step resolution rule defined above and on a new *sometime resolution rule* defined below is given which is complete for the ground eventuality monodic fragment.

Definition 6 (sometime resolution rule). *Let* $\mathbf{m}\mathcal{T}$ *be the set of merged rules of a problem* $< \mathcal{U}, \mathcal{S}, \mathcal{T} >$, $\{\mathcal{A}_1 \Rightarrow \bigcirc\mathcal{B}_1, \ldots \mathcal{A}_n \Rightarrow \bigcirc\mathcal{B}_n\}$ *is a subset of* $\mathbf{m}\mathcal{T}$*, and* $p \Rightarrow \bigcirc\Diamond q$ *is a propositional eventuality rule in* $\mathcal{T}$*. Then the sometime resolution inference rule w.r.t.* $\mathcal{U}$ *is the rule*

$$\frac{\mathcal{A}_1 \Rightarrow \bigcirc\mathcal{B}_1, \ldots, \mathcal{A}_n \Rightarrow \bigcirc\mathcal{B}_n \qquad p \Rightarrow \bigcirc\Diamond q}{\neg(\bigvee_{i=1}^{n} \mathcal{A}_i) \vee \neg p} \; (\Diamond_{res})$$

where the following (loop) side condition *has to be satisfied*

$$\mathcal{U} \cup \{\mathcal{B}_m\} \vdash \neg q \wedge \bigvee_{i=1}^{n} \mathcal{A}_i \quad \textit{for all } 1 \leq m \leq n.$$

Under the side condition given above $\bigvee_{i=1}^{n} \mathcal{A}_i$ can be considered as an *invariant formula* that provides the derivability of $\Box\bigcirc\neg q$ from $\mathcal{U} \cup \mathcal{T}$. Again, as in the case of step resolution, the test of whether the side conditions are satisfied does not involve temporal reasoning and can be given to any first-order proof search procedure.

By Res$(\mathcal{U}, \mathcal{T})$ we denote the set of all formulae which are obtained by the sometime resolution rule w.r.t $\mathcal{U}$ from a set of merged clauses $\mathcal{A}_1 \Rightarrow \bigcirc\mathcal{B}_1, \ldots \mathcal{A}_n \Rightarrow \bigcirc\mathcal{B}_n$ in $\mathbf{m}\mathcal{T}$ and an eventuality clause $p \Rightarrow \bigcirc\Diamond q$ in $\mathcal{T}$. Since $\mathcal{T}$ and $\mathbf{m}\mathcal{T}$ are finite the set Step$(\mathcal{U}, \mathcal{T})$ is also finite (up to renaming bound variables). The sometime resolution rule is sound in the sense similar to the soundness of the step resolution rule (see Lemma 1).

To take into account eventuality clauses we modify the notion of the behaviour graph given in the previous section by introducing an additional (eventuality) component to every node.

Definition 7 (ground eventuality behaviour graph).

Given a problem $\mathcal{P} =< \mathcal{U}, \mathcal{S}, \mathcal{T} >$ we construct a finite directed graph G as follows. Every node of G is a two-tuple $(\mathcal{C}, E)$ where

- *$\mathcal{C}$ is a colour scheme for $\mathcal{T}$ such that the set $\mathcal{U} \wedge \mathcal{F}_{\mathcal{C}}$ is satisfiable;*
- *E is a subset of eventuality atoms occurring in $\mathcal{T}$. It will be called the* eventuality set *of the node $(\mathcal{C}, E)$.*

For each node $(\mathcal{C}, E)$, $\mathcal{C} = (\Gamma, \theta, \rho)$, we construct an edge in G to a node $(\mathcal{C}')$, $\mathcal{C}' = (\Gamma', \theta', \rho')$, if $\mathcal{U} \wedge \mathcal{F}_{\mathcal{C}'} \wedge \mathcal{B}_{\mathcal{C}}$ is satisfiable and $E' = E^1 \cup E^2$ where

$$E^1 = \{q \mid q \in E \textit{ and } F_{\theta'} \not\vdash q\},$$
$$E^2 = \{q \mid \textit{there exists an eventuality rule } (p \Rightarrow \bigcirc\Diamond q) \in \mathcal{T} \textit{ such that } F_\theta \vdash p \textit{ and } F_{\theta'} \not\vdash q\}.$$

They are the only edges originating from $(\mathcal{C}, E)$. A node $(\mathcal{C}, \emptyset)$ is designated as an initial node of G if $\mathcal{S} \wedge \mathcal{U} \wedge \mathcal{F}_{\mathcal{C}}$ is satisfiable. The eventuality free behaviour graph *H of $\mathcal{P}$ is the full subgraph of G given by the set of nodes reachable from the initial nodes.*

Let H be the behaviour graph of a problem $\mathcal{P}$, $\mathfrak{n}, \mathfrak{n}'$ be nodes of a graph H. We denote the relation "$\mathfrak{n}'$ is an immediate successor of $\mathfrak{n}$" by $\mathfrak{n} \to \mathfrak{n}'$, and the relation "$\mathfrak{n}'$ is a successor of $\mathfrak{n}$" by $\mathfrak{n} \to^+ \mathfrak{n}'$.

A node $\mathfrak{n}$ of H is called *step inference node* if it has no successors. A node $\mathfrak{n}'$ of H is called *sometime inference node* if it is not a step inference node and there is an eventuality atom q in $\mathcal{P}$ such that for every successor $\mathfrak{n}' = (\mathcal{C}', E')$, $q \in E'$ holds.

Lemma 5 (existence of a model).

Let $\mathcal{P}$ be a problem, H be the behaviour graph of $\mathcal{P}$ such that the set of initial nodes of H is not empty and the following condition is satisfied:

$$\forall \mathfrak{n} \forall q \exists \mathfrak{n}' (\mathfrak{n} \to^+ \mathfrak{n}' \wedge q \notin E') \tag{1}$$

where $\mathfrak{n}, \mathfrak{n}'$ are nodes of H, $\mathfrak{n} = (\mathcal{C}, E)$, $\mathfrak{n}' = (\mathcal{C}', E')$, and q belongs to the set of eventuality atoms of $\mathcal{P}$. Then $\mathcal{P}$ has a model.

Proof We can construct a model for $\mathcal{P}$ as follows. Let $\mathfrak{n}_0$ be an initial node of H and $q_1, \dots, q_m$ be all eventuality atoms of $\mathcal{P}$. Let Π be a path $\mathfrak{n}_0, \dots, \mathfrak{n}_1, \dots, \mathfrak{n}_m, \dots,$ $\mathfrak{n}_{m+1}, \dots, \mathfrak{n}_{2m}, \dots$, where $\mathfrak{n}_{km+j} = (\mathcal{C}_{km+j}, E_{km+j})$ is a successor of $\mathfrak{n}_{km+j-1}$ in H such that $q_j \notin E_{km+j}$ (for every $k \geq 0$, $1 \leq j \leq m$).

Let us take the sequence $(\mathcal{C}_0), \dots, (\mathcal{C}_1), \dots, (\mathcal{C}_m), \dots, (\mathcal{C}_{m+1}), \dots, (\mathcal{C}_{2m}), \dots$ induced by Π. Now let us consider this sequence as an infinite path in the eventuality free behaviour graph for the induction free problem $< \mathcal{U}, \mathcal{S}, \mathcal{T}^* >$ where $\mathcal{T}^*$ is obtained from $\mathcal{T}$ by removing all (propositional) eventuality rules.[3] Then the first-order temporal model $\mathfrak{M} = \langle D, \mathcal{I} \rangle$ constructed by the theorem 2 for $< \mathcal{U}, \mathcal{S}, \mathcal{T}^* >$ from the sequence $(\mathcal{C}_0), \dots, (\mathcal{C}_1), \dots, (\mathcal{C}_m), \dots, (\mathcal{C}_{m+1}), \dots, (\mathcal{C}_{2m}), \dots$ is a model for $\mathcal{P} = < \mathcal{U}, \mathcal{S}, \mathcal{T} >$.

Indeed, all nontemporal clauses and all step clauses of $\mathcal{P}$ are satisfied on this structure immediately by the definition of $< \mathcal{U}, \mathcal{S}, \mathcal{T}^* >$. Let us take an arbitrary eventuality clause $p_j \Rightarrow \bigcirc \Diamond q_j$ of $\mathcal{T}$, a moment of time $l \in \mathbb{N}$ and the l-th element $(\mathcal{C}, E)$ on Π. If $F_{\mathcal{C}} \not\vdash p_j$ then $p_j \Rightarrow \bigcirc \Diamond q_j$ is satisfied at the moment l, i.e. $\mathfrak{M}_l \models (p_j \supset \bigcirc \Diamond q_j)$. If $F_{\mathcal{C}} \not\vdash p_j$ we take a node $\mathfrak{n}_{km+j}$ which is a successor of $(\mathcal{C}, E)$ on Π. By the construction of Π it follows that $q_j \notin E_{km+j}$. We conclude that there exists a successor $(\mathcal{C}', E')$ of $(\mathcal{C}, E)$ along the path to $\mathfrak{n}_{km+j}$ such that $F_{\mathcal{C}'} \vdash q_j$, otherwise $l_j \in E_{km+j}$ would hold. It implies that $p_j \Rightarrow \bigcirc \Diamond q_j$ is satisfied at the moment l as well. ❑

To provide the completeness of the sometime resolution rule for the problems which contain more than one eventuality atom such problems have to be *augmented* in the following way.

Definition 8 (augmented problem). *Let us introduce for every eventuality atom q occurring in $\mathcal{T}$ a new propositional symbol w_q. An* augmented proplem $\mathcal{P}^{aug}$ *is a triple* $< \mathcal{U}^{aug}, \mathcal{S}, \mathcal{T}^{aug} >$ *where*

$$\begin{aligned} \mathcal{U}^{aug} &= \mathcal{U} \cup \{ w_q \supset (p \vee q) \mid (p \Rightarrow \bigcirc \Diamond q) \in \mathcal{T} \}, \\ \mathcal{T}^{aug} &= \mathcal{T} \cup \{ p \Rightarrow \bigcirc w_q \mid (p \Rightarrow \bigcirc \Diamond q) \in \mathcal{T} \}. \end{aligned}$$

The necessity for the augmentation even in the propositional case was shown in [DF00]. It is obvious that the augmentation is invariant with respect to satisfiability.[4]

Now we extend the notion of the derivation relation introduced in the previous section as follows: each step $\mathcal{U}_i \rhd \mathcal{U}_{i+1}$ consists of the adding to the set $\mathcal{U}_i$ a formula from $\text{Step}(\mathcal{U}_i, \mathbf{m}\mathcal{T})$ or from $\text{Res}(\mathcal{U}_i, \mathcal{T})$. Correspondingly, the notion of the (fair) theorem proving derivation is modified.

Theorem 3 (completeness of the step+sometime resolution). *If a ground eventuality problem $\mathcal{P} = < \mathcal{U}, \mathcal{S}, \mathcal{T} >$ is unsatisfiable, then every* fair *theorem proving derivation for $\mathcal{P}^{aug} = < \mathcal{U}^{aug}, \mathcal{S}, \mathcal{T}^{aug} >$ is succesfully terminated.*

[3] To retain the set of propositional symbols of $\mathcal{T}^*$ the same as of $\mathcal{T}$ we can add to $\mathcal{T}^*$ degenerates step rules of the form $p \Rightarrow \bigcirc \mathbf{true}$.

[4] Both the augmentation and including degenerates rules (see the previous footnote) can result in the violation of the condition that there are no different step rules with the same left-hand sides. However this violation is eliminated easy.

Proof The proof proceeds by induction on the number of nodes in the behaviour graph H of $\mathcal{P}^{aug}$, which is finite. The cases when H is empty graph or there exists a node $\mathfrak{n}$ in H which has no successors are considered in the same way as in the proof of Corollary 1.

Now we consider another possibility when H is not empty and every node in H has a successor. It is enough to prove that there is a formula $\psi \in \mathrm{Res}(\mathcal{U}, \mathcal{T})$ such that for some node $(\mathcal{C}, E)$ of H the formula $\mathcal{U} \wedge \mathcal{F}_{\mathcal{C}} \wedge \psi$ is unsatisfiable.

In this case because $\mathcal{P}$ is unsatisfiable the following condition (the negation of the condition (1) of the existence of a model given in Lemma 5) holds:

$$\exists \mathfrak{n} \exists q \forall \mathfrak{n}' (\mathfrak{n} \to^{+} \mathfrak{n}' \supset q \in E') \tag{2}$$

where $\mathfrak{n}, \mathfrak{n}'$ are nodes of H, $\mathfrak{n} = (I, E)$, $\mathfrak{n}' = (I', E')$, and q belongs to the set of eventuality atoms of $\mathcal{P}$.

Let $\mathfrak{n}_0 = (\mathcal{C}_0, E_0)$ be the node defined by the first existential quantifier of the condition (2). Let q_0 be the eventuality atom defined by the second existential quantifier of the condition (2). Let $p \Rightarrow \bigcirc \lozenge q_0$ be the eventuality rule containing q_0 (on the right).

Let $\mathfrak{I}$ be a finite nonempty set of indexes, $\{\mathfrak{n}_i \mid i \in \mathfrak{I}\}$ be the set of all successors of $\mathfrak{n}_0$. (It is possible, of course, that $0 \in \mathfrak{I}$.) Let $\mathfrak{n}_{i_1}, \ldots \mathfrak{n}_{i_k}$ be the set of all immediate successors of $\mathfrak{n}_0$, $\mathfrak{n}_{i_j} = (\mathcal{C}_{i_j}, E_{i_j})$ for $1 \leq j \leq k$. To simplify denotations in this proof we will represent merged rules $\mathcal{A}_{\mathcal{C}_i} \Rightarrow \bigcirc \mathcal{B}_{\mathcal{C}_i}$ ($\mathcal{A}_{\mathcal{C}_{i_j}} \Rightarrow \bigcirc \mathcal{B}_{\mathcal{C}_{i_j}}$) simply as $\mathcal{A}_i \Rightarrow \bigcirc \mathcal{B}_i$ ($\mathcal{A}_{i_j} \Rightarrow \bigcirc \mathcal{B}_{i_j}$), and formulae $\mathcal{F}_{\mathcal{C}_i}$ ($\mathcal{F}_{\mathcal{C}_{i_j}}$) simply as $\mathcal{F}_i$ ($\mathcal{F}_{i_j}$).

Consider two cases depending on whether the merged rule $\mathcal{A}_0 \Rightarrow \bigcirc \mathcal{B}_0$ (or any of $\mathcal{A}_i \Rightarrow \bigcirc \mathcal{B}_i$, $i \in \mathfrak{I}$) is degenerated or not.

1. Let $\mathcal{A}_0 = \mathcal{B}_0 =$ **true** . It implies, that $\mathcal{U} \vdash \neg q_0$. Indeed, since $q_0 \in E_{i_j}$ for all $1 \leq j \leq k$ then $\mathcal{F}_{i_j} \not\vdash q_0$ in accordance with the definition of the ground eventuality behaviour graph. Again similar to the proof of the Corollary 1, suppose that $\mathcal{U} \cup \{q_0\}$ is satisfiable, and $\langle D', I' \rangle$ is a model of $\mathcal{U} \cup \{q_0\}$. Then we can construct a colour scheme $\mathcal{C}'$ such that $\langle D', I' \rangle \models \mathcal{F}_{\mathcal{C}'}$ and therefore $\mathcal{F}_{\mathcal{C}'} \vdash q_0$. Since $\mathfrak{n}_{i_1}, \ldots \mathfrak{n}_{i_k}$ is the set of all immediate successors of $\mathfrak{n}_0$ and $\mathcal{B}_0 =$ **true** it holds that there exists j, $1 \leq j \leq k$, such that $\mathcal{C}_{i_j} = \mathcal{C}'$. We conclude that $q_0 \notin E_{i_j}$ because of $\mathcal{F}_{\mathcal{C}'} \vdash q_0$. It contradicts the choice of the node $\mathfrak{n}_0$. So, $\mathcal{U} \vdash \neg q_0$, and the following sometime resolution inference is realized
$$\frac{\mathbf{true} \Rightarrow \bigcirc \mathbf{true} \qquad p \Rightarrow \bigcirc \lozenge q_0}{\neg p} \; (\lozenge_{res})$$
The behaviour graph for the problem $< \mathcal{U} \cup \{\neg p\}, \mathcal{S}, \mathcal{T} >$ is a proper subgraph of H. Indeed, if $F_{\theta_0} \vdash p$ then $\mathfrak{n}_0$ has to be removed from H. If $F_{\theta_0} \not\vdash p$ then a predecessor $(\mathcal{C}, E)$, $\mathcal{C} = (\Gamma, \theta, \rho)$, of the node $\mathfrak{n}_0$ such that $F_\theta \vdash p$ has to be removed from H. The set of such predecessors is not empty because the eventuality set of every initial node of H is empty.
The same argument holds if one of $\mathcal{A}_i \Rightarrow \bigcirc \mathcal{B}_i$, $i \in \mathfrak{I}$, is degenerate.
2. Let neither $\mathcal{A}_0 \Rightarrow \bigcirc \mathcal{B}_0$ nor any $\mathcal{A}_i \Rightarrow \bigcirc \mathcal{B}_i, i \in \mathfrak{I}$, are degenerate. We are going to prove now that in this case the sometime resolution rule
$$\frac{\{\mathcal{A}_i \Rightarrow \bigcirc \mathcal{B}_i \mid i \in \{0\} \cup \mathfrak{I}\} \qquad p \Rightarrow \bigcirc \lozenge q_0}{(\bigwedge_{i \in \{0\} \cup \mathfrak{I}} \neg \mathcal{A}_i) \vee \neg p} \; (\lozenge_{res})$$

is applied. We have to check the side conditions for the sometime resolution rule.

– By arguments similar those given in item 1 we conclude that the sets $(\mathcal{U}\cup\{\mathcal{B}_i\}\cup\{q_0\})$ for all $i\in\{0\}\cup\mathfrak{I}$ are unsatisfiable. It implies that $\mathcal{U}\cup\{\mathcal{B}_i\}\vdash\neg q_0$ for all $i\in\{0\}\cup\mathfrak{I}$.
– Let us show that $\mathcal{U}\cup\{\mathcal{B}_i\}\vdash\bigvee\limits_{j\in\{0\}\cup\mathfrak{I}}\mathcal{A}_j$ for all $i\in\{0\}\cup\mathfrak{I}$. Consider the case $i=0$, for other indexes arguments are the same. Suppose that $(\mathcal{U}\cup\{\mathcal{B}_0\}\cup\{\bigwedge\limits_{1\le j\le k}\neg\mathcal{A}_j\})$ is satisfied in a structure $\langle D',I'\rangle$. Let $\mathcal{C}'$ be a colour scheme of $\langle D',I'\rangle$, that is $\langle D',I'\rangle\models\mathcal{F}_{\mathcal{C}'}$. Then there is a node $\mathfrak{n}_{i_j}=(\mathcal{C}_{i_j},E_{i_j})$, $1\le j\le k$, which is an immediate successor of $\mathfrak{n}_0$, such that $\mathcal{C}_{i_j}=\mathcal{C}'$, and hence $\langle D',I'\rangle\models\mathcal{A}_{i_j}$. However it contradicts the choice of the structure $\langle D',I'\rangle$.

After applying the $(\Diamond_{res})$ rule given above we add to $\mathcal{U}$ its conclusion, which is equivalent to the set of formulae $\{\neg\mathcal{A}_i\vee\neg p\mid i\in\{0\}\cup\mathfrak{I}\}$. To prove that the behaviour graph of the extended problem will contain less nodes than H we have to consider two cases depending on whether $q_0\in E_0$ or not.

(a) Let us suppose $q_0\notin E_0$. Then $\mathcal{F}_0\vdash p$ because $q_0\in E_{i_1}$ and there is an adge from $(\mathcal{C}_0,E_0)$ to $(\mathcal{C}_{i_1},E_{i_1})$. In this case the node $\mathfrak{n}_0$ has to be removed from H. Recall that $\mathcal{F}_0\vdash\mathcal{A}_0$ by the definition of $\mathcal{A}_0$.

(b) Let us suppose $q_0\in E_0$. Since the eventuality set of every initial node is empty there exists a predecessor $\mathfrak{n}'_0$ of $\mathfrak{n}_0$ and a path $\mathfrak{n}'_0=(\mathcal{C}'_0,E'_0)\ldots$, $\mathfrak{n}'_m=(\mathcal{C}'_m,E'_m)$ from $\mathfrak{n}'_0$ to $\mathfrak{n}_0$, $m\ge 1$, $\mathcal{C}'_m=(\Gamma'_m,\theta'_m,\rho'_m)$, such that $\mathfrak{n}'_m=\mathfrak{n}_0$, $F_{\theta'_0}\vdash p$, and $q_0\in E'_j$ for all $1\le j\le m$. The last condition implies

$$F_{\theta'_j}\vdash\neg q_0 \text{ for all } 1\le j\le m. \tag{3}$$

That is just the place where we have to involve in our arguments the augmenting pair for $p\Rightarrow\bigcirc\Diamond q_0$. Let this pair be presented by the following clauses

$$p\Rightarrow\bigcirc w_0\quad\in\mathcal{T}^{aug}, \tag{4}$$

$$w_0\Rightarrow q_0\vee p\quad\in\mathcal{U}^{aug}. \tag{5}$$

From the clause (4) it follows that $F_{\theta'_1}\vdash w_0$. From the clause (5) and the condition (3) it follows that for all $1\le j\le m$ it holds $F_{\theta'_j}\vdash p$. It implies that $F_{\theta_0}\vdash p$, in particular, since $\mathfrak{n}'_m=\mathfrak{n}_0$. So, $\mathcal{F}_{\mathcal{C}_0}\wedge\neg p$ is unsatisfiable. Therefore $\mathfrak{n}_o$ has to be removed from the behaviour graph after extending $\mathcal{U}^{aug}$ by the formula $\neg\mathcal{A}_0\vee\neg p$ (the same as every node $\mathfrak{n}_i, I\in\mathfrak{I}$ is removed after including the formula $\neg\mathcal{A}_i\vee\neg p$.

Now all possible cases related to the properties of H have been considered.

❑

Lemma 6 (existence of a model, one eventuality case). *Let $\mathcal{P}$ be a problem such that $\mathcal{P}$ contains the only eventuality atom q_0. Let H be the behaviour graph of $\mathcal{P}$ such that the set of initial nodes of H is not empty and the following condition is satisfied:*

$$\forall\mathfrak{n}(q_0\notin E\supset\exists\mathfrak{n}'(\mathfrak{n}\to^{+}\mathfrak{n}'\wedge q_0\notin E')) \tag{6}$$

where $\mathfrak{n},\mathfrak{n}'$ are nodes of H, $\mathfrak{n}=(\mathcal{C},E)$, $n'=(\mathcal{C}',E')$. Then $\mathcal{P}$ has a model.

Proof We use model construction of the proof of Lemma 5 taking $m = 1$. ❑

Corollary 2 (completeness of the one eventuality case). *If a ground eventuality problem $\mathcal{P} =< \mathcal{U}, \mathcal{S}, \mathcal{T} >$ is unsatisfiable, and $\mathcal{T}$ contains at most one eventuality atom then every* fair *theorem proving derivation for $\mathcal{P} =< \mathcal{U}, \mathcal{S}, \mathcal{T} >$ is succesfully terminated.*

Proof This corollary is obtained by analysing the proof of Theorem 3 given above. Firstly, using Lemma 6 and supposing q_0 to be the only eventuality atom of $\mathcal{T}$ we can strengthen the condition (2) to the following $\exists\mathfrak{n}(q_0 \notin E \wedge \forall\mathfrak{n}'(\mathfrak{n} \to^+ \mathfrak{n}' \supset q_0 \in E'))$ where $\mathfrak{n}, \mathfrak{n}'$ are nodes of the eventuality graph H for the problem $\mathcal{P}$, $\mathfrak{n} = (\mathcal{C}, E)$, $\mathfrak{n}' = (\mathcal{C}', E')$. This immediately implies that the case 2(b) of the previous proof, where augmentation has been required, is excluded from the consideration. ❑

6 Conclusion

It has been known for a long time that first-order temporal logic over the natural numbers is incomplete [Sza86], that is there exists no finitary inference system which is sound and complete for the logic, or equivalently, the set of valid formulae of the logic is not recursively enumerable. The monodic fragment is the only known today fragment of first-order temporal logic among not only decidable but even recursively enumerable fragments which has a transparent syntactical definition and a finite inference system.

The method developed in this paper covers a special subsclass of the monodic fragment, namely the subclass of the *ground eventuality* monodic problems. Nevertheless this subclass is still interesting w.r.t. both its theoretical properties and possible area of applications. The first statement is confirmed in particular by the fact that if we slightly extend its boundaries admitting a binary relation in the step rules then its recursive enumerability will be lost. The second is justified in particular by the observation that the temporal specifications for verifying properties of transducers considered in [Spi00] are proved to be not simply monodic but monodic ground eventuality problems.

One of the essential advantages of the method given above follows from the complete separation of the classical first-order component. As a result classical first-order resolution can be applied as a basic tool in the temporal proof search (to solve side and termination conditions, which are expressed in classical first-order logic). That immediately gains access to all benefits, both theoretical and practical, of resolution based decision procedures [FLHT01], because the first-order formulae produced by temporal rules are very simple and they cannot change the decidability/undecidability of the initial fragment. Future work includes extending these results to wider fragments of first-order temporal logic, and implementing this approach.

It might also be interesting to decompose the present separated and ‘global’ temporal inferences into a mix of resolution-like ‘local’ rules. That will involve revision of the resolution method without skolemization for classical logic developed in [Zam87].

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