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Control design for interval type-2 polynomial fuzzy-model-based systems with time-varying delay

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Abstract: In this paper, the problems of stabilization for interval type-2 polynomial fuzzy systems with time-varying delay and parameter uncertainties are investigated. The objective is to design a state-feedback interval type-2 polynomial fuzzy controller such that the closed-loop control system is asymptotically stable. The conditions for the existence of such a controller are delay dependent and membership function dependent in terms of sum-of-squares (SOS). Based on a basic lemma to deal with the delay terms, we formulate and solve the problem with more flexibility due to imperfect premise matching that the number of rules and premise membership functions are not necessary the same between the interval type-2 polynomial fuzzy model and interval type-2 polynomial fuzzy controller. Piecewise linear membership functions approximations enclosing the original lower and upper membership functions are employed to facilitate the stability analysis. A numerical example indicates the effectiveness of the derived results.

1 Introduction

Type-1 fuzzy sets were first introduced in 1965 [1], which have been widely applied to innumerable control related issues. Among all the fuzzy-model-based control approaches, Takagi-Sugeno (T-S) fuzzy model [2] has played an important role to conduct system analysis and control synthesis [3–7] for nonlinear systems in a systematic form for the last three decades. Based on the T-S fuzzy model, a fuzzy controller is proposed to connect the fuzzy plant, which is an average weighted sum of local linear subsystems with weights determined by type-1 fuzzy membership functions, to form a closed-loop fuzzy-model-based control system.

Type-2 fuzzy sets were then introduced in 1975 [8]. Since then, it has attracted great attention and many fruitful results have been presented in both theory and practice (see, e.g. [9–15]). One motivation for studying such a class of systems is that type-2 fuzzy sets are better in representing and capturing uncertainties [16, 17], especially when the nonlinear plant inevitably suffers the parameter uncertainties while type-1 fuzzy sets do not contain uncertain information. Generally speaking, type-2 fuzzy systems could be regarded as a bunch of type-1 fuzzy systems, where a lot of successful applications can be found in the literature such as robot manipulators [18], face recognition [19], image processing [20], energy markets [21], linguistic summarization [22] and so on.

However, the general type-2 fuzzy sets and models were not favorable for fuzzy-model-based control framework, which motivates the development of modeling and analysis tools for this class of nonlinear control systems. An interval type-2 T-S fuzzy system was introduced in [23], where the parameters are obtained through learning. A systematic modeling method for nonlinear systems subject to uncertainties based on the sector nonlinear technique [24] was proposed in the first time in [25] to support the system analysis and control synthesis. By using the lower and upper membership functions, the nonlinearity and uncertainties can be captured. Due to the existence of uncertainties, the membership grades of the fuzzy model become uncertain in values that the analysis techniques developed for type-1 fuzzy-model-based control systems cannot be applied.

The work in [25] proposed an interval type-2 fuzzy controller to control the nonlinear plant and utilized the lower and upper membership functions to facilitate the stability analysis of interval type-2

fuzzy-model-based control systems. It requires that the interval type-2 fuzzy controller shares the same lower and upper membership functions, and the same number of rules as those of the interval type-2 fuzzy model. In [26], these constraints were eliminated to reduce implementation complexity and increase design flexibility for the interval type-2 fuzzy controller. The work [25] and [26] developed stability analysis techniques to obtain stability conditions, which provides the foundation for the development of interval type-2 fuzzy-model-based control systems. They have then been applied to various control strategies, such as output feedback control [27], model predictive control [28], observer-based control [29] and etc.

Polynomial fuzzy model [30] was first proposed in 2009 to model and control nonlinear systems based on sum-of-squares (SOS) approach. Polynomial fuzzy system could be regarded as a powerful extension and generalization of the traditional T-S fuzzy system, as polynomial terms are adopted in describing the dynamic of the nonlinear system in the consequent of the fuzzy rules and when the order of the polynomials is zero the polynomial fuzzy model is reduced to traditional T-S fuzzy model. Because of the polynomial terms, the original LMI approach for stability analysis and control synthesis could not be conducted. Instead, numerical results could be found based on SOS approach with, for example, the third-party toolbox SOSTOOLS [31]. Some of the recent research on polynomial fuzzy-model-based systems are listed as follows, for example, [32] studied stability analysis via approximated membership functions considering sector nonlinearity of control input, [33] studied stability analysis with mismatched premise membership functions and [34] studied stability analysis using switching polynomial Lyapunov function. Most of the existing works on polynomial fuzzy systems are with type-1 membership functions and very few works have been reported to study the interval type-2 polynomial fuzzy-model-based systems, which is the key motivation of this paper.

On the other hand, it is well known that most practical dynamic systems inherently involve time delays. Without taking the limitations into consideration, techniques developed may result in performance degradation or even instability of the closed-loop control system in practice. In recent years, fuzzy-model-based system with time delays has been probed widely through delay-independent approach [35–37] and delay-dependent approach [38–45]. The former approach (delay-independent approach) does not involve time

delay into the analysis, and the resulting stable fuzzy-model-based system would remain stable for other values of the delay. While the latter approach (delay-dependent approach) normally considers the information of the delay, and various inequalities would be applied to approximate the bound of the delay-related terms like the delay itself and its derivative. Less conservative result is expected as more information of the delay is involved in delay-dependent approach. Just to name a few more recent results for delay-dependent approach, in [46], the authors used delay partitioning approach to reduce the conservatism of delayed T-S fuzzy systems. In [47], the authors dealt with the delay with input-output approach and two-term approximation where time-varying delay was treated as a kind of uncertainty to design filter. These methods could be even combined together to get less conservative results. Although we get plenty of meaningful results on fuzzy-model-based systems with delays, there is still room left for us to make further extension, especially for interval type-2 polynomial fuzzy-model-based systems.

In this paper, we investigate the stabilization problem of nonlinear plant subject to parameter uncertainties and time varying but bounded time delay. To approach the problem, an interval type-2 polynomial fuzzy-model-based control approach is adopted. An interval type-2 polynomial fuzzy model with time delay is employed to represent the nonlinear system that the nonlinearity and parameter uncertainties are captured by the lower and upper membership functions. An interval type-2 polynomial fuzzy controller is employed to control the nonlinear plant. To investigate the stability of the closed-loop system, there are mainly two challenges to face: 1) the grades of membership functions are uncertain in values due to the existence of uncertainties that the parallel distributed compensation (PDC) concept facilitating the stability analysis of the traditional type-1 fuzzy-model-based control system cannot be applied, 2) the time-varying delay and its derivative will complicate the system dynamics, which makes the stability analysis more difficult and many existing results for T-S fuzzy systems with time delays cannot be directly applied to polynomial fuzzy systems with time delays. To our best knowledge, this could be the first time to study the stabilization problem of time-delayed systems under interval type-2 polynomial fuzzy-model-based control framework.

To deal with difficulty on the stability analysis due to uncertainties appearing in the membership functions, the information of the lower and upper membership functions will be utilized. An interpolation technique [26] is employed to represent and reconstruct the lower and upper membership functions through interpolation among sample points. Consequently, the system stability is guaranteed if the interval type-2 polynomial fuzzy-model-based control system with time delay is stable at all these sample points. To deal with the time delay, a Lyapunov-Krasovskii functional candidate is employed where the delay information such as the bound of the delay is included as integration limit. SOS-based stability conditions are obtained to determine the stability of the closed-loop system and synthesize the interval type-2 fuzzy controller.

The remainder of this paper is organized as follows. Section 2 is devoted to the mathematical model of the concerned system and some preliminaries. Synthesis of state-feedback interval type-2 polynomial fuzzy controller are presented in Section 3. A numerical simulation in Section 4 is to illustrate the feasibility and effectiveness of the proposed results and Section 5 concludes the paper.

2 Problem Formulation and Preliminaries

In this section, we will provide the preliminaries in support of the stability analysis and design of the control strategy. An interval type-2 polynomial fuzzy model is introduced to represent the nonlinear plant with time delay and parameter uncertainties. An interval type-2 polynomial fuzzy controller is then introduced to control the nonlinear plant by closing the feedback loop. The control problem is formulated as a stabilization problem, which will be investigated based on the Lyapunov approach. To facilitate the stability analysis, the concept of piecewise linear membership functions [26] is recalled which is to represent the lower and upper membership functions in a favorable form. The piecewise linear membership

functions approximations are able to approximate the original membership functions and bring the information of the membership functions to the stability analysis. Due to the characteristic of the piecewise linear membership functions, the stability of the system can be guaranteed by checking the system stability conditions at some sample points.

2.1 Interval Type-2 Polynomial Fuzzy Model with Time-Varying Delay

Consider a nonlinear system with time-varying delays and parameter uncertainties represented by the following interval type-2 polynomial fuzzy model with lower and upper membership functions.

Plant Rule i : IF $\theta_1(x(t))$ is $\tilde{M}_{i1}$, $\theta_2(x(t))$ is $\tilde{M}_{i2}$ $\cdots$ and $\theta_\Psi(x(t))$ is $\tilde{M}_{i\Psi}$, THEN

$$\begin{cases} \dot{x}(t) = A_i(x(t))x(t) + A_{di}(x(t))x(t-d(t)) + B_i(x(t))u(t) \\ x(t) = \varphi(t), t = [-\bar{d}, 0), \end{cases} \quad (1)$$

where $\tilde{M}_{i\alpha}$ is an interval type-2 fuzzy set of rule i , $\alpha = 1, 2, \dots, \Psi$ and $i = 1, 2, \dots, p$. $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^l$ is the input vector, $d(t)$ is the time-varying delay and satisfies $d(t) \in (0, \bar{d}]$, $\bar{d}(t) \leq m$, $\bar{d}$ and m are known positive numbers, $\varphi(t)$ is the initial sequence. $A_i(x(t)) \in \mathbb{R}^{n \times n}$, $B_i(x(t)) \in \mathbb{R}^{n \times l}$, $A_{di}(x(t)) \in \mathbb{R}^{n \times n}$ are known polynomial matrices as system matrices, input matrices and delayed-state matrices, respectively. The firing strength of rule i is the interval sets as follows:

$$W_i(x(t)) = [\underline{w}_i(x(t)), \bar{w}_i(x(t))], i = 1, 2, \dots, p, \quad (2)$$

where

$$\underline{w}_i(x(t)) = \prod_{\alpha=1}^{\Psi} \mu_{\tilde{M}_{i\alpha}}(\theta_\alpha(x(t))) \geq 0, \quad (3)$$

$$\bar{w}_i(x(t)) = \prod_{\alpha=1}^{\Psi} \bar{\mu}_{\tilde{M}_{i\alpha}}(\theta_\alpha(x(t))) \geq 0, \quad (4)$$

in which $\mu_{\tilde{M}_{i\alpha}}(\theta_\alpha(x(t)))$ and $\bar{\mu}_{\tilde{M}_{i\alpha}}(\theta_\alpha(x(t)))$ denote the lower and upper membership functions, respectively, satisfying the property $\bar{\mu}_{\tilde{M}_{i\alpha}}(\theta_\alpha(x(t))) \geq \mu_{\tilde{M}_{i\alpha}}(\theta_\alpha(x(t))) \geq 0$, and $\underline{w}_i(x(t))$ and $\bar{w}_i(x(t))$ denote the lower and upper grade of membership respectively. The inferred interval type-2 polynomial fuzzy model is defined as follows:

$$\begin{aligned} \dot{x}(t) = \sum_{i=1}^p \tilde{w}_i(x(t)) [A_i(x(t))x(t) + B_i(x(t))u(t) \\ + A_{di}(x(t))x(t-d(t))], \end{aligned} \quad (5)$$

where

$$\tilde{w}_i(x(t)) = \underline{w}_i(x(t))\underline{\alpha}_i(x(t)) + \bar{w}_i(x(t))\bar{\alpha}_i(x(t)) \geq 0, \forall i, \quad (6)$$

$$\sum_{i=1}^p \tilde{w}_i(x(t)) = 1, \quad (7)$$

in which $\underline{\alpha}_i(x(t)) \in [0, 1]$, $\bar{\alpha}_i(x(t)) \in [0, 1]$ are nonlinear functions with the property that $\underline{\alpha}_i(x(t)) + \bar{\alpha}_i(x(t)) = 1$ for all i .

2.2 Interval Type-2 Polynomial Fuzzy Controller

An interval type-2 polynomial fuzzy controller of c rules is employed to control the nonlinear plant subject to time delay and parameter uncertainties represented by the interval type-2 polynomial fuzzy model (5), where the j -th control rule is of the following format:

Controller Rule j : IF $\sigma_1(x(t))$ is $\tilde{N}_{j1}$, $\sigma_2(x(t))$ is $\tilde{N}_{j2} \dots$ and $\sigma_\Omega(x(t))$ is $\tilde{N}_{j\Omega}$, THEN

$$u(t) = K_j(x(t))x(t), \quad (8)$$

where $\tilde{N}_{j\beta}$ is an interval type-2 fuzzy set of rule j , $\beta = 1, 2, \dots, \Omega$ and $j = 1, 2, \dots, c$, $K_j(x(t))$ are the polynomial feedback gains to be determined. The firing strength of rule j is the interval sets as follows:

$$M_j(x(t)) = [\underline{m}_j(x(t)), \overline{m}_j(x(t))], \quad j = 1, 2, \dots, c, \quad (9)$$

where

$$\underline{m}_j(x(t)) = \prod_{\beta=1}^{\Omega} \mu_{\tilde{N}_{j\beta}}(\sigma_\beta(x(t))) \geq 0, \quad (10)$$

$$\overline{m}_j(x(t)) = \prod_{\beta=1}^{\Omega} \bar{\mu}_{\tilde{N}_{j\beta}}(\sigma_\beta(x(t))) \geq 0, \quad (11)$$

in which $\mu_{\tilde{N}_{j\beta}}(\sigma_\beta(x(t)))$ and $\bar{\mu}_{\tilde{N}_{j\beta}}(\sigma_\beta(x(t)))$ denote the lower and upper membership functions respectively satisfying the property $\bar{\mu}_{\tilde{N}_{j\beta}}(\sigma_\beta(x(t))) \geq \mu_{\tilde{N}_{j\beta}}(\sigma_\beta(x(t))) \geq 0$, and $\underline{m}_j(x(t))$ and $\overline{m}_j(x(t))$ denote the lower and upper grade of membership respectively. The inferred interval type-2 polynomial fuzzy controller is defined as follows:

$$u(t) = \sum_{j=1}^c \tilde{m}_j(x(t)) K_j(x(t)) x(t), \quad (12)$$

where

$$\tilde{m}_j(x(t)) = \frac{\underline{m}_j(x(t)) \underline{\beta}_j(x(t)) + \overline{m}_j(x(t)) \overline{\beta}_j(x(t))}{\sum_{k=1}^c (\underline{m}_k(x(t)) \underline{\beta}_k(x(t)) + \overline{m}_k(x(t)) \overline{\beta}_k(x(t)))} \geq 0, \quad \forall j, \quad (13)$$

$$\sum_{j=1}^c \tilde{m}_j(x(t)) = 1, \quad (14)$$

in which $\underline{\beta}_j(x(t)) \in [0, 1]$, $\overline{\beta}_j(x(t)) \in [0, 1]$ are predefined functions with the property that $\underline{\beta}_j(x(t)) + \overline{\beta}_j(x(t)) = 1$ for all j .

Remark 1. Throughout this paper, we consider the case that the fuzzy rules and the premise membership functions of the plant and controller could be different, i.e., $p \neq c$, $\tilde{w}_i(x(t)) \neq \tilde{m}_i(x(t))$, which is referred as imperfect premise matching. This setting would lead to more design flexibility and lower implementation cost when choosing less rules and simpler membership functions of the controller. In addition, the value of $\tilde{w}_i(x(t))$ is uncertain, which is not practical to be implemented in the interval type-2 fuzzy controller, which suggests that $\tilde{m}_i(x(t))$ is used instead.

2.3 Interval Type-2 Polynomial Fuzzy-Model-Based Control System

With the plant and controller expressions in (5) and (12), and the property of $\sum_{i=1}^p \tilde{w}_i(x(t)) = 1$, $\sum_{j=1}^c \tilde{m}_j(x(t)) = 1$, $\sum_{i=1}^p \sum_{j=1}^c \tilde{w}_i(x(t)) \tilde{m}_j(x(t)) = 1$, we can have the interval type-2 polynomial fuzzy-model-based control systems as follows:

$$\begin{aligned} \dot{x}(t) = & \sum_{i=1}^p \sum_{j=1}^c \tilde{h}_{ij}(x(t)) [(A_i(x(t)) + B_i(x(t)) K_j(x(t))) x(t) \\ & + A_{di}(x(t)) x(t - d(t))], \end{aligned} \quad (15)$$

where $\tilde{h}_{ij}(x(t)) \triangleq \tilde{w}_i(x(t)) \tilde{m}_j(x(t))$.

In addition $\tilde{h}_{ij}(x(t))$ could be reconstructed as follows [25]:

$$\tilde{h}_{ij}(x(t)) = \underline{\gamma}_{ij}(x(t)) \bar{h}_{ij}(x(t)) + \underline{\gamma}_{ij}(x(t)) \underline{h}_{ij}(x(t)) \quad (16)$$

in which $\underline{\gamma}_{ij}(x(t)) \in [0, 1]$, $\bar{\gamma}_{ij}(x(t)) \in [0, 1]$ are functions with the property that $\underline{\gamma}_{ij}(x(t)) + \bar{\gamma}_{ij}(x(t)) = 1$, and $\bar{h}_{ij}(x(t))$ and $\underline{h}_{ij}(x(t))$ are the piecewise linear membership function approximations of the upper and lower bound of $\tilde{h}_{ij}(x(t))$ with definitions below from [26]:

$$\bar{h}_{ij}(x(t)) = \sum_{k=1}^q \sum_{i_1=1}^2 \cdots \sum_{i_n=1}^2 \prod_{r=1}^n v_{ri_r k}(x_r(t)) \bar{\delta}_{ij i_1 i_2 \dots i_n k}, \quad (17)$$

$$\underline{h}_{ij}(x(t)) = \sum_{k=1}^q \sum_{i_1=1}^2 \cdots \sum_{i_n=1}^2 \prod_{r=1}^n v_{ri_r k}(x_r(t)) \underline{\delta}_{ij i_1 i_2 \dots i_n k}, \quad (18)$$

where $0 \leq \underline{\delta}_{ij i_1 i_2 \dots i_n k} \leq \bar{\delta}_{ij i_1 i_2 \dots i_n k} \leq 1$ are scalars to be determined according to $\tilde{h}_{ij}(x(t))$,

$$0 \leq \underline{h}_{ij}(x(t)) \leq \tilde{h}_{ij}(x(t)) \leq \bar{h}_{ij}(x(t)) \leq 1, \quad (19)$$

$v_{ri_r k}(x_r(t)) \in [0, 1]$, $i_r = 1, 2$ and $v_{r1k}(x_r(t)) + v_{r2k}(x_r(t)) = 1$, otherwise $v_{ri_r k}(x_r(t)) = 0$, $x(t) \in \Psi_k$, $\bigcup_{k=1}^q \Psi_k = \Psi$ is the state space of interest.

Remark 2. The nonlinearity and uncertainties of the interval type-2 polynomial fuzzy-model-based control system are considered to be embedded in the interval type-2 membership functions $\tilde{h}_{ij}(x(t))$. Although $\tilde{h}_{ij}(x(t)) \triangleq \tilde{w}_i(x(t)) \tilde{m}_j(x(t))$, which is uncertain in value, can be characterized by the known lower and upper membership functions $\underline{w}_i(x(t))$ and $\bar{w}_i(x(t))$ from the interval type-2 polynomial fuzzy model and $\underline{m}_j(x(t))$ and $\bar{m}_j(x(t))$ from the interval type-2 polynomial fuzzy controller, these membership functions cannot benefit the stability analysis due to the cross terms generated by them cannot be handled by PDC-based analysis technique. It is necessary to reconstruct them using membership functions of favorable form to make easy the stability analysis. Piecewise linear membership functions $\underline{h}_{ij}(x(t))$ and $\bar{h}_{ij}(x(t))$ in (16) are to serve this purpose and have to satisfy the condition in (19). Referring to (16), we would like to enclose the region bounded by $\underline{h}_{ij}(x(t))$ and $\bar{h}_{ij}(x(t))$ (termed as footprint of uncertainty (FOU) using the terminology in type-2 fuzzy sets) by $\underline{h}_{ij}(x(t))$ and $\bar{h}_{ij}(x(t))$. Consequently, through interpolation, the FOU can be reconstructed by (16).

Remark 3. With the above definitions, in the further stability analysis, we could use scalars $\underline{\delta}_{ij i_1 i_2 \dots i_n k}$ and $\bar{\delta}_{ij i_1 i_2 \dots i_n k}$ to characterize $\underline{h}_{ij}(x(t))$ and $\bar{h}_{ij}(x(t))$ through $\prod_{r=1}^n v_{ri_r k}(x_r(t))$ which are independent of i and j . In a word, the stability conditions involving the membership function information could be achieved by scalars $\underline{\delta}_{ij i_1 i_2 \dots i_n k}$ and $\bar{\delta}_{ij i_1 i_2 \dots i_n k}$.

3 Main Results

The stability of the interval type-2 polynomial fuzzy-model-based control system with time-varying delay (5) is investigated in this section based on a Lyapunov-Krasovskii functional candidate. To bring the information of the membership functions into the stability analysis, we will develop SOS-based stability conditions depending on the piecewise linear membership functions. In order to bring the information of time-delay, its upper bounds, and time derivative will be considered in the stability analysis. As a result, the SOS-based stability conditions are membership-function and time-delay dependent. If there exists a feasible solution to the SOS-based stability conditions, an interval type-2 polynomial fuzzy controller can

be obtained which can stabilize the nonlinear plant subject to the prescribed bounds of the parameter uncertainties and time-varying delay under consideration.

For simplification reason, in the following analysis, we denote $\tilde{w}_i(x(t))$, $\tilde{m}_j(x(t))$, $\tilde{h}_{ij}(x(t))$, $\tilde{h}_{ij}(x(t))$ and $\tilde{h}_{ij}(x(t))$ as $\tilde{w}_i$, $\tilde{m}_j$, $\tilde{h}_{ij}$, $\tilde{h}_{ij}$ and $\tilde{h}_{ij}$, respectively.

Before proceeding to the stability analysis, we need to revisit a fundamental lemma to be used to deal with the delayed term in the following proof.

Lemma 1. [48] For matrix $N = \begin{bmatrix} -R & L \\ * & -R \end{bmatrix} \leq 0$, $d(t) \in (0, \bar{d}]$, and a vector function $\dot{x} : [-\bar{d}, 0) \rightarrow \mathbb{R}^n$ such that the integration in the following inequality is well defined, then it holds that

$$-\bar{d} \int_{t-\bar{d}}^t \dot{x}^T(s) R \dot{x}(s) ds \leq v^T(t) W v(t), \quad (20)$$

where

$$W = \begin{bmatrix} -R & R+L & -L \\ * & -2R-L-L^T & R+L \\ * & * & -R \end{bmatrix}, \quad (21)$$

$$v^T(t) = [x^T(t) \ x^T(t-d(t)) \ x^T(t-\bar{d})]. \quad (22)$$

Theorem 1. Given a constant m , positive scalar $\bar{d}$, predefined scalars $\delta_{ij i_1 i_2 \dots i_n k}$ and $\underline{\delta}_{ij i_1 i_2 \dots i_n k}$ defined in (17) and (18), $K = \{k_1, k_2, \dots, k_q\}$ which is the set of row numbers that the entries of the entire row of $B_i(x)$ and $A_{di}(x)$ are all zeros and $\tilde{x} = (x_{k_1}, x_{k_2}, \dots, x_{k_q})$. If there exist polynomial matrices $X(\tilde{x})$, $\bar{Y}_{ij}(x)$, $\bar{Q}(x)$, $\bar{Z}(x)$ and $\bar{T}(x)$ and $N_j(x)$ of appropriate dimensions such that the following SOS-based conditions hold:

$$\begin{aligned} & v_1^T (X(\tilde{x}) - \varepsilon_1(\tilde{x})I) v_1 \text{ is SOS;} \\ & v_1^T (\bar{Q}(x) - \varepsilon_2(x)I) v_1 \text{ is SOS;} \\ & v_1^T (\bar{Z}(x) - \varepsilon_3(x)I) v_1 \text{ is SOS;} \\ & v_2^T (\bar{Y}_{ij}(x) - \varepsilon_4(x)I) v_2 \text{ is SOS;} \\ & v_3^T \left(\begin{bmatrix} \bar{Z}(x) & -\bar{T}(x) \\ * & \bar{Z}(x) \end{bmatrix} - \varepsilon_5(x)I \right) v_3 \text{ is SOS;} \\ & -v_4^T (\Xi_{ij} - \begin{bmatrix} \bar{Y}_{ij}(x) & 0 \\ 0 & 0 \end{bmatrix} + \varepsilon_6(x)I) v_4 \text{ is SOS;} \\ & -v_5^T (\sum_{i=1}^p \sum_{j=1}^c (\delta_{ij i_1 i_2 \dots i_n k} \Xi_{ij} \\ & + (\bar{\delta}_{ij i_1 i_2 \dots i_n k} - \delta_{ij i_1 i_2 \dots i_n k}) \begin{bmatrix} \bar{Y}_{ij}(x) & 0 \\ 0 & 0 \end{bmatrix} \\ & + \varepsilon_7(x)I) v_5 \text{ is SOS,} \end{aligned} \quad (23)$$

where

$$\Xi_{ij} = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} \\ * & \Xi_{22} & \Xi_{23} & \Xi_{24} \\ * & * & \Xi_{33} & \Xi_{34} \\ * & * & * & \Xi_{44} \end{bmatrix}, \quad (24)$$

with

$$\Xi_{11} = A_i(x)X(\tilde{x}) + B_i(x)N_j(x) \quad (25)$$

$$+ (A_i(x)X(\tilde{x}) + B_i(x)N_j(x))^T + \bar{Q}(x) - \bar{Z}(x)/\bar{d} - \sum_{k \in K} \frac{\partial X(\tilde{x})}{\partial x_k} A_i^k(x)x, \quad (26)$$

$$\Xi_{12} = A_{di}(x)X(\tilde{x}) + (\bar{Z}(x) + \bar{T}(x))/\bar{d}, \quad (27)$$

$$\Xi_{13} = -\bar{T}(x)/\bar{d}, \quad (28)$$

$$\Xi_{14} = \sqrt{\bar{d}}(X(\tilde{x})A_i(x)^T + N_j(x)^T B_i(x)^T), \quad (29)$$

$$\Xi_{22} = (m-1)\bar{Q}(x) - (2\bar{Z}(x) + \bar{T}(x) + \bar{T}(x)^T)/\bar{d}, \quad (30)$$

$$\Xi_{23} = (\bar{Z}(x) + \bar{T}(x))/\bar{d}, \quad (31)$$

$$\Xi_{24} = \sqrt{\bar{d}}X(\tilde{x})A_{di}(x)^T, \quad (32)$$

$$\Xi_{33} = -\bar{Z}(x)/\bar{d}, \quad (33)$$

$$\Xi_{34} = 0, \quad (34)$$

$$\Xi_{44} = \bar{Z}(x) - 2X(\tilde{x}), \quad (35)$$

then the interval type-2 polynomial fuzzy-model-based control system (15) is asymptotically stable. Moreover the feedback gains of the interval type-2 polynomial fuzzy controller gains can be obtained by $K_j(x) = N_j(x)X(\tilde{x})^{-1}$ for all j .

Proof: Consider a candidate of Lyapunov-Krasovskii functional as

$$V(t) = V_1(t) + V_2(t) + V_3(t), \quad (36)$$

$$V_1(t) = x^T(t)X(\tilde{x})^{-1}x(t), \quad (37)$$

$$V_2(t) = \int_{t-d(t)}^t x^T(s)Qx(s)ds, \quad (38)$$

$$V_3(t) = \int_{-\bar{d}t+\theta}^0 \int_t^s \dot{x}^T(s)Z\dot{x}(s)ds d\theta. \quad (39)$$

Along the trajectories of the closed-loop control system, the corresponding time derivative of $V(t)$ is given by

$$\begin{aligned} \dot{V}_1(t) &= 2x^T(t)X(\tilde{x})^{-1}\dot{x}(t) + x^T(t)\dot{X}(\tilde{x})^{-1}x(t) \\ &= 2 \sum_{i=1}^p \sum_{j=1}^c \tilde{w}_i \tilde{m}_j x^T(t)X(\tilde{x})^{-1} \\ &\quad \times [(A_i(x(t)) + B_i(x(t))K_j(x))x(t) \\ &\quad + A_{di}(x(t))x(t-d(t))] \\ &\quad - x^T(t)X(\tilde{x})^{-1} \\ &\quad \times \sum_{i=1}^p \sum_{k \in K} \frac{\partial X(\tilde{x})}{\partial x_k} A_i^k(x(t))x(t)X(\tilde{x})^{-1}x(t), \\ \dot{V}_2(t) &= x^T(t)Qx(t) - (1-d'(t))x^T(t-d(t))Qx(t-d(t)) \\ &\leq x^T(t)Qx(t) - (1-m)x^T(t-d(t))Qx(t-d(t)), \\ \dot{V}_3(t) &= \bar{d}\dot{x}^T(t)Z\dot{x}(t) - \int_{t-\bar{d}}^t \dot{x}^T(s)Z\dot{x}(s)ds, \end{aligned}$$

where $A_i^k(x(t))$ denotes the k -th row of $A_i(x(t))$.

By applying Lemma 1, $\dot{V}_3(t)$ can be expressed as

$$\begin{aligned} \dot{V}_3(t) &\leq \bar{d} \sum_{i=1}^p \sum_{j=1}^c \tilde{w}_i \tilde{m}_j \\ &\quad \times [(A_i(x(t)) + B_i(x(t))K_j(x(t)))x(t) \\ &\quad + A_{di}(x(t))x(t-d(t))]^T Z \\ &\quad \times [(A_i(x(t)) + B_i(x(t))K_j(x(t)))x(t) \\ &\quad + A_{di}(x(t))x(t-d(t))] \\ &\quad + \frac{1}{\bar{d}} v^T(t) \begin{bmatrix} -Z & Z+T & -T \\ * & -2Z-T-T^T & Z+T \\ * & * & -Z \end{bmatrix} v(t), \end{aligned}$$

with $v^T(t) = [x^T(t) \ x^T(t-d(t)) \ x^T(t-\bar{d})]$ and subject to $\begin{bmatrix} -Z & T \\ * & -Z \end{bmatrix} \leq 0$.

$$\text{Rewrite } \dot{V}(t) \text{ with } \Omega_{ij} = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & -T/\bar{d} \\ * & \Lambda_{22} & (Z+T)/\bar{d} \\ * & * & -Z/\bar{d} \end{bmatrix},$$

where

$$\begin{aligned} \Lambda_{11} &= X(\tilde{x})^{-1}(A_i(x(t)) + B_i(x(t))K_j(x(t))) \\ &\quad + (A_i(x(t)) + B_i(x(t))K_j(x(t)))^T X(\tilde{x})^{-1} \\ &\quad + Q - Z/\bar{d} + \bar{d}(A_i(x(t)) + B_i(x(t))K_j(x(t)))^T \\ &\quad \times Z(A_i(x(t)) + B_i(x(t))K_j(x(t))) \\ &\quad - X(\tilde{x})^{-1} \sum_{i=1}^p \sum_{k \in K} \frac{\partial X(\tilde{x})}{\partial x_k} A_i^k(x(t))x(t)X(\tilde{x})^{-1}, \\ \Lambda_{12} &= (Z+T)/\bar{d} + X(\tilde{x})^{-1}A_{di}(x(t)) \\ &\quad + \bar{d}(A_i(x(t)) + B_i(x(t))K_j(x(t)))^T Z A_{di}(x(t)), \\ \Lambda_{22} &= (m-1)Q - (2Z+T+T^T)/\bar{d} \\ &\quad + \bar{d}A_{di}(x(t))^T Z A_{di}(x(t)). \end{aligned}$$

Then we get

$$\begin{aligned} \dot{V}(t) &\leq \sum_{i=1}^p \sum_{j=1}^c \tilde{h}_{ij} v^T(t) \Omega_{ij} v(t) \\ &= \sum_{i=1}^p \sum_{j=1}^c (\underline{h}_{ij} + (\tilde{h}_{ij} - \underline{h}_{ij})) v^T(t) \Omega_{ij} v(t) \\ &\leq \sum_{i=1}^p \sum_{j=1}^c \underline{h}_{ij} v^T(t) \Omega_{ij} v(t) + (\bar{h}_{ij} - \underline{h}_{ij}) v^T(t) Y_{ij} v(t) \\ &= \sum_{i=1}^p \sum_{j=1}^c v^T(t) (\underline{h}_{ij} \Omega_{ij} + (\bar{h}_{ij} - \underline{h}_{ij}) Y_{ij}) v(t), \quad (40) \end{aligned}$$

with

$$\begin{aligned} Y_{ij} &\geq 0, \\ Y_{ij} &\geq \Omega_{ij}. \end{aligned}$$

It is required that $\dot{V}(t)$ in (40) is negative definite for system stability. From (40), $\dot{V}(t) < 0$ can be obtained if the following condition holds:

$$\sum_{i=1}^p \sum_{j=1}^c v^T(t) (\underline{h}_{ij} \Omega_{ij} + (\bar{h}_{ij} - \underline{h}_{ij}) Y_{ij}) v(t) < 0. \quad (41)$$

Rewriting the above condition (41) with (17) and (18), and using the fact that $\sum_{k=1}^q \sum_{i_1=1}^2 \cdots \sum_{i_n=1}^2 \prod_{r=1}^n v_{r i_r k}(x_r(t)) = 1$, one can get the following condition

$$\sum_{i=1}^p \sum_{j=1}^c (\delta_{ij i_1 i_2 \dots i_n k} \Omega_{ij} + (\bar{\delta}_{ij i_1 i_2 \dots i_n k} - \delta_{ij i_1 i_2 \dots i_n k}) Y_{ij}) < 0. \quad (42)$$

By using Schur Complement and congruence transformation, we can get the conditions as stated in the theorem with $K_j(x)X(\tilde{x}) = N_j(x)$, $\tilde{Q}(x) = X(\tilde{x})QX(\tilde{x})$, $\tilde{Z}(x) = X(\tilde{x})ZX(\tilde{x})$, $\tilde{T}(x) = X(\tilde{x})TX(\tilde{x})$, $\bar{Y}_{ij}(x) = \text{diag}\{X(\tilde{x}), X(\tilde{x}), X(\tilde{x})\} \times Y_{ij} \times \text{diag}\{X(\tilde{x}), X(\tilde{x}), X(\tilde{x})\}$. After getting $K_j(x)$, $\tilde{Q}(x)$, $\tilde{Z}(x)$, $\tilde{T}(x)$ and $\bar{Y}_{ij}(x)$, we need to make sure that the above five equations still hold. $X(\tilde{x})$ is a polynomial matrix of zero degree would be one of the many cases that guarantees the above equations. $\square$

Remark 4. The conditions in (41) and (42) are equivalent. In (41), the condition is terms of the lower and upper membership functions

$\underline{h}_{ij}$ and $\bar{h}_{ij}$, which are continuous functions so that the number of stability conditions in (41) is actually infinite, which is impractical to be solved numerically. By rewriting (41) in (42), the stability condition is no longer depending on the lower and upper membership functions $\underline{h}_{ij}$ and $\bar{h}_{ij}$ but their sample points $\delta_{ij i_1 i_2 \dots i_n k}$ and $\bar{\delta}_{ij i_1 i_2 \dots i_n k}$. Due to the advantage of piecewise linear membership functions, satisfying the stability condition in (41) is implied by satisfying the conditions in (42), which requires the checking only at the sample points so that the number of stability conditions become finite. It could be noted that dividing the region of x into more partitions could further reduce the conservatism. The more local upper and lower bounds of the membership functions involved in could lead to more relaxed results while the computational burden would be heavier.

As Theorem 1 involves the information of the membership functions in control design, it is a membership function dependent method. By removing the membership function information in the analysis, we can get the following membership function independent theorem.

Theorem 2. Given a constant m , positive scalar $\bar{d}$, if there exist polynomial matrices $X(\tilde{x})$, $\tilde{Q}(x)$, $\tilde{Z}(x)$, $\tilde{T}(x)$ and $N_j(x)$ of appropriate dimensions such that the following SOS-based conditions hold:

$$\begin{aligned} v_1^T (X(\tilde{x}) - \varepsilon_1(\tilde{x})I) v_1 &\text{ is SOS;} \\ v_1^T (\tilde{Q}(x) - \varepsilon_2(x)I) v_1 &\text{ is SOS;} \\ v_1^T (\tilde{Z}(x) - \varepsilon_3(x)I) v_1 &\text{ is SOS;} \\ v_3^T \begin{bmatrix} \tilde{Z}(x) & -\tilde{T}(x) \\ * & \tilde{Z}(x) \end{bmatrix} - \varepsilon_5(x)I &\text{ is SOS;} \\ -v_4^T (\Xi_{ij} + \varepsilon_6(x)I) v_4 &\text{ is SOS,} \end{aligned} \quad (43)$$

the definitions of the variables are given in Theorem 1. Then the interval type-2 polynomial fuzzy-model-based control system (15) is asymptotically stable. Moreover the feedback gains of the interval type-2 polynomial fuzzy controller can be obtained by $K_j(x) = N_j(x)X(\tilde{x})^{-1}$ for all j .

Remark 5. Compared to Theorem 1, Theorem 2 removes conditions involving membership function information and slack matrices. From this point of view, Theorem 2 is then more conservative than Theorem 1, which will be testified in next section. While Theorem 2 is not as complicated as Theorem 1 and would consume less time to get a solution.

Remark 6. The Lyapunov-Krasovskii functional we choose for Theorem 1 and Theorem 2 consists of three parts: $V_1(t)$, $V_2(t)$ and $V_3(t)$. $V_1(t)$ is a quadratic functional commonly used in stability analysis. $V_2(t)$ and $V_3(t)$ are chosen to include the delay information (bound of the delay and the derivative of delay) in the analysis. Both theorems could be modified to tackle systems without time-varying delay by removing $V_2(t)$ and $V_3(t)$ in $V(t)$ following the similar derivation.

4 Numerical Example

In this section, a numerical example is presented to demonstrate the potential and validity of our developed theoretical results. For a time-delayed nonlinear system represented by an interval type-2 polynomial fuzzy model, Theorem 1 and Theorem 2 are employed to design an interval type-2 polynomial fuzzy controller to ensure the stability of the interval type-2 polynomial fuzzy-model-based system. Moreover, we want to show that our membership function dependent approach (Theorem 1) is less conservative than membership function independent approach (Theorem 2). Details of membership function information, feedback gains of the controller, and the state response and control input of the system could be found below.

Fig. 1: Membership function information with three-rule model and two-rule controller for Case 1 with $x_1 = \{-10, -8, \dots, 8, 10\}$, $d_1 = 0.7978$ and $d_2 = 0.6099$. Solid lines are for the original lower membership functions and dashed lines are for the original upper membership functions. Dotted lines are for the approximated lower piecewise linear membership functions and dashed-dot lines are for the approximated upper piecewise linear membership functions.

Fig. 2: Membership function information with three-rule model and two-rule controller for Case 2 with $x_1 = \{-10, -9, \dots, 9, 10\}$, $d_1 = 0.2461$ and $d_2 = 0.2499$. Solid lines are for the original lower membership functions and dashed lines are for the original upper membership functions. Dotted lines are for the approximated lower piecewise linear membership functions and dashed-dot lines are for the approximated upper piecewise linear membership functions.

Consider a nonlinear plant subject to parameter uncertainties and time delay represented by a three-rule interval type-2 polynomial fuzzy model in the form of (5) with $A_1 = \begin{bmatrix} 0 & -x_1 + 1 \\ 1 & 2 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 \\ 1 & -0.01x_1^2 + 2 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0 & 1 \\ 1 & -0.01x_1 \end{bmatrix}$, $B_1 = \begin{bmatrix} 0.5 & 1 \end{bmatrix}^T$,

Fig. 3: Membership function information with three-rule model and two-rule controller for Case 3 with $x_1 = \{-10, -9.5, \dots, 9.5, 10\}$, $d_1 = 0.1384$ and $d_2 = 0.0600$. Solid lines are for the original lower membership functions and dashed lines are for the original upper membership functions. Dotted lines are for the approximated lower piecewise linear membership functions and dashed-dot lines are for the approximated upper piecewise linear membership functions.

$$B_2 = \begin{bmatrix} 0.5 & 1 \end{bmatrix}^T, B_3 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T, A_{d1} = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.1 \end{bmatrix}, A_{d2} = \begin{bmatrix} -0.2 & 0 \\ 0 & -0.1 \end{bmatrix}, A_{d3} = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.2 \end{bmatrix}, x = [x_1 \ x_2]^T, \varphi(t) = 0 \text{ when } t \in [-d, 0).$$

The membership functions for the plant (5) are chosen as $\tilde{w}_1(x_1) = 1 - 1/(1 + e^{-(x_1+4+\eta(t))})$, $\tilde{w}_2(x_1) = 1 - \tilde{w}_1(x_1) - \tilde{w}_3(x_1)$, $\tilde{w}_3(x_1) = 1/(1 + e^{-(x_1-4-\eta(t))})$. Due to parameter uncertainty $\eta(t) \in [-0.1, 0.1]$, the grades of membership function becomes uncertain in value.

The lower and upper membership functions to be approximated by the piecewise linear membership functions for the three-rule plant (5) are chosen as $w_1(x_1) = 1 - 1/(1 + e^{-(x_1+4+d_1)})$, $\bar{w}_2(x_1) = 1 - w_1(x_1) - w_3(x_1)$, $w_3(x_1) = 1/(1 + e^{-(x_1-4-d_1)})$, $\bar{w}_1(x_1) = 1 - 1/(1 + e^{-(x_1+4-d_1)})$, $\bar{w}_2(x_1) = 1 - \bar{w}_1(x_1) - \bar{w}_3(x_1)$, $\bar{w}_3(x_1) = 1/(1 + e^{-(x_1-4+d_1)})$.

A two-rule interval type-2 polynomial fuzzy controller in the form of (12) is employed to control the nonlinear plant. The lower and upper membership functions are chosen as $\underline{m}_1(x_1) = 1 - 1/e^{-(x_1+d_2)/2}$, $\bar{m}_1(x_1) = 1 - 1/e^{-(x_1-d_2)/2}$, $\underline{m}_2(x_1) = 1 - \bar{m}_1(x_1)$, $\bar{m}_2(x_1) = 1 - \underline{m}_1(x_1)$. From (13), we can get $\tilde{m}_j(x_1)$ by setting $\bar{\beta}_1 = \bar{\beta}_2 = 0.5$.

We consider region of interest to be $x_1 \in [-10, 10]$ and the membership grades outside this region are capped. The sample points of $\underline{h}_{ij}$ and $\bar{h}_{ij}$ are set as Case 1: $x_1 = \{-10, -8, \dots, 8, 10\}$, Case 2: $x_1 = \{-10, -9, \dots, 9, 10\}$ and Case 3: $x_1 = \{-10, -9.5, \dots, 9.5, 10\}$, respectively. To determine $\bar{h}_{ij}(x_1)$ and $\underline{h}_{ij}(x_1)$, we consider Case 2 for demonstration purposes. $\Psi_k : \underline{x}_{1,k} \leq x_1 \leq \bar{x}_{1,k}$, where $\underline{x}_{1,k} = k - 11$ and $\bar{x}_{1,k} = k - 10$, $k = 1, 2, \dots, 19, 20$. By choosing $v_{11k}(x_1) = 1 - (x_1 - \underline{x}_{1,k})/(\bar{x}_{1,k} - \underline{x}_{1,k})$ and $v_{12k}(x_1) = 1 - v_{11k}(x_1)$, and the scalars $\bar{\delta}_{ij1k} = \bar{w}_i(\underline{x}_{1,k})\bar{m}_j(\underline{x}_{1,k})$, $\bar{\delta}_{ij2k} = \bar{w}_i(\bar{x}_{1,k})\bar{m}_j(\bar{x}_{1,k})$, $\underline{\delta}_{ij1k} = \underline{w}_i(\underline{x}_{1,k})\underline{m}_j(\underline{x}_{1,k})$ and $\underline{\delta}_{ij2k} = \underline{w}_i(\bar{x}_{1,k})\underline{m}_j(\bar{x}_{1,k})$ for all k , we can define $\bar{h}_{ij}(x_1)$ and $\underline{h}_{ij}(x_1)$ as $\bar{h}_{ij}(x(t)) = \sum_{k=1}^{20} (v_{11k}\bar{\delta}_{ij1k} + v_{12k}\bar{\delta}_{ij2k})$ and $\underline{h}_{ij}(x(t)) = \sum_{k=1}^{20} (v_{11k}\underline{\delta}_{ij1k} + v_{12k}\underline{\delta}_{ij2k})$.

Fig. 4: State response of the closed-loop control system with $x(0) = [-10 \ 10]^T$, $d(t) = 0.1\bar{d}(1 + 9 \sin^2 t)$, $m = 0.9\bar{d} = 0.0010$.

Fig. 6: Control input of the closed-loop control system with $x(0) = [-10 \ 10]^T$, $d(t) = 0.1\bar{d}(1 + 9 \sin^2 t)$, $m = 0.9\bar{d} = 0.0010$.

Fig. 5: State response of the closed-loop control system with $x(0) = [-10 \ 10]^T$, $d(t) = 0.1\bar{d}(1 + 9 \sin^2 t)$, $m = 0.9\bar{d} = 0.0100$.

Fig. 7: Control input of the closed-loop control system with $x(0) = [-10 \ 10]^T$, $d(t) = 0.1\bar{d}(1 + 9 \sin^2 t)$, $m = 0.9\bar{d} = 0.0100$.

$+ v_{12k}\delta_{ij2k}$). The same idea applies to Cases 1 and 3 to determine $\bar{h}_{ij}(x_1)$ and $\bar{h}_{ij}(x_1)$.

Figs. 1-3 show the membership function information of $\tilde{h}_{ij}$, $\bar{h}_{ij}$ and $\bar{h}_{ij}$ for Cases 1-3, respectively. Table 1 presents the parameter values of d_1 and d_2 for the three cases. The introduction of d_1 and d_2 is for the purpose of obtaining fitter membership function approximations for different sampling intervals.

Table 1 Parameter values for sampling interval, d_1 and d_2 for Cases 1-3

	Case 1	Case 2	Case 3
sampling interval	2.0000	1.0000	0.5000
d_1	0.7978	0.2461	0.1384
d_2	0.6099	0.2499	0.0600

Remark 7. As one can see from Figs. 1-3, the smaller sampling interval for x_1 could lead the approximated membership functions closer to the original ones. For each case, one could choose even larger values of d_1 and d_2 to include the information of the membership functions into the stability analysis, while this is not recommended as larger d_1 and d_2 would increase the conservatism. The values of d_1 and d_2 listed in Table 1 are the smallest values for each case making sure the original membership functions are enclosed by the approximated piecewise linear membership functions.

For Cases 1-3, the stability conditions in Theorem 1 and Theorem 2 are employed to stabilize the nonlinear plant. We choose $\varepsilon_1(\tilde{x}) =$

Table 2 Comparison of maximum delay bound for different values of m

Considered results	$m = 0.0010$	$m = 1.0000$
Theorem 1	0.0869s	0.0795s
Theorem 2	0.0215s	0.0175s

$\varepsilon_2(x) = \varepsilon_3(x) = \varepsilon_4(x) = \varepsilon_5(x) = \varepsilon_6(x) = \varepsilon_7(x) = 0.001$. The solution could be found by using the third-party MATLAB toolbox SOSTOOLS [31]. Table 2 summarizes the maximum allowable delay bound for the polynomial system with feasible solutions by Theorem 1 and Theorem 2 for Case 2. The reason that Case 2 is used to present the simulation results is a compromise between better approximated membership functions (smaller sampling interval) and lower computational burden (larger sampling interval). It could be seen from Table 2 that for the same m (upper bound of the delay derivative) larger maximum delay bound could be obtained by Theorem 1 compared to Theorem 2, which means Theorem 1 can still stabilize the system when Theorem 2 fails to do so for a system with larger time delay. In this sense, our membership function dependent approach (Theorem 1) presents less conservative results compared to the membership function independent approach (Theorem 2).

With the above settings and $d(t) = 0.1\bar{d}(1 + 9\sin^2 t)$, for $m = 0.9\bar{d} = 0.0010$, the feedback gains obtained by Theorem 1 are $K_1(x_1) = [-3.1755 \times 10^{-5}x_1^2 + 4.9220 \times 10^{-3}x_1 - 0.6202 - 2.9771 \times 10^{-2}x_1^2 + 0.1319x_1 - 50.4205]$, $K_2(x_1) = [-9.9620 \times 10^{-5}x_1^2 - 6.0112 \times 10^{-5}x_1 - 0.3184 - 1.0688 \times 10^{-2}x_1^2 - 9.3118 \times 10^{-3}x_1 - 49.8524]$ with $X(\tilde{x}) = \begin{bmatrix} 12.3052 & 0.0868 \\ 0.0868 & 0.2004 \end{bmatrix}$; for $m = 0.9\bar{d} = 0.0100$, the feedback gains obtained by Theorem 1 are $K_1(x_1) = [-1.0764 \times 10^{-5}x_1^2 + 1.2492 \times 10^{-3}x_1 - 0.8623 - 9.8473 \times 10^{-3}x_1^2 + 0.1177x_1 - 82.9392]$, $K_2(x_1) = [-1.9279 \times 10^{-5}x_1^2 + 8.6310 \times 10^{-5}x_1 - 0.6785 - 4.8110 \times 10^{-3}x_1^2 - 6.5791 \times 10^{-3}x_1 - 77.3896]$ with $X(\tilde{x}) = \begin{bmatrix} 8.6423 & 0.0163 \\ 0.0163 & 0.0514 \end{bmatrix}$.

Figs. 4-5 give the state response of the closed-loop control system which is asymptotically stable with the initial state $x(0) = [-10, -10]^T$ with $m = 0.0010$ and $m = 0.0100$, respectively. Figs. 6-7 show the control input of the closed-loop control system with the initial state $x(0) = [-10, -10]^T$ with $m = 0.0010$ and $m = 0.0100$, respectively. As one can see from the figures, our theorem is able to stabilize the system for different values of the delay bound.

Remark 8. In the simulation, we set $X(\tilde{x})$, $\tilde{Q}(x)$, $\tilde{Z}(x)$ and $\tilde{T}(x)$ as polynomials of degree 0, and set $N_j(x)$ and $\tilde{Y}_{ij}(x)$ as polynomials with monomials in x_1 of degree 2. The degrees of the polynomials could be modified according to the users. Normally, the higher order would result in less conservative results and longer computational time.

Remark 9. The software we use for numerical simulation are listed as follows: MATLAB R2012b, SOSTOOLS v3.0 and SeDuMi v1.3.

Remark 10. The design process of the proposed method would start from considering a practical nonlinear system and then representing it by an interval type-2 polynomial fuzzy model. We then choose appropriate fuzzy rules and membership functions for the interval type-2 polynomial fuzzy controller based on the needs. After applying the theorems developed in the paper, we can get the gains of the controller and finally implement the controller to the nonlinear system.

5 Conclusion

The stability of interval type-2 polynomial fuzzy-model-based control systems with time-varying delay and parameter uncertainties under imperfect premise matching has been investigated. A state-feedback interval type-2 polynomial fuzzy controller has been proposed to ensure the asymptotic stability of the closed-loop time-delayed systems under interval type-2 polynomial fuzzy-model-based control framework. To facilitate the membership function dependent stability analysis, piecewise linear membership functions have been employed to approximate the original upper and

lower membership functions. More design flexibility and practicality could be achieved, because it is not required that the polynomial fuzzy controller and polynomial fuzzy plant have the same premise membership function and/or number of fuzzy rules. The stability conditions come in SOS form based on a dedicated chosen delay-dependent Lyapunov-Krasovskii functional. A numerical example is presented to show the effectiveness of the proposed approach. It is mentioned that the results on the stability analysis and control synthesis of the interval type-2 polynomial fuzzy-model-based systems with time-varying delay are established via a common Lyapunov functional. The results may be extended with a fuzzy basis dependent Lyapunov functional, which will be an interesting topic in the future. In addition, the problem of actuator saturation for the systems also deserves further attention.

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