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Itô Stochastic Differential Equations as 2-Jets

John Armstrong¹ and Damiano Brigo²

¹ Dept. of Mathematics, King's College London

² Dept. of Mathematics, Imperial College London
180 Queen's Gate, SW7 2AZ,
`damiano.brigo@imperial.ac.uk`

Abstract. We explain how Itô Stochastic Differential Equations on manifolds may be defined as 2-jets of curves and show how this relationship can be interpreted in terms of a convergent numerical scheme. We use jets as a natural language to express geometric properties of SDEs. We explain that the mainstream choice of Fisk-Stratonovich-McShane calculus for stochastic differential geometry is not necessary. We give a new geometric interpretation of the Itô-Stratonovich transformation in terms of the 2-jets of curves induced by consecutive vector flows. We discuss the forward Kolmogorov equation and the backward diffusion operator in geometric terms. In the one-dimensional case we consider percentiles of the solutions of the SDE and their properties. In particular the median of a SDE solution is associated to the drift of the SDE in Stratonovich form for small times.

1 Introduction

This paper is a summary of the preprint by Armstrong and Brigo (2016) [1] and examines the geometry of stochastic differential equations using a coordinate free approach. We suggest for the first time, to the best of our knowledge, that SDEs on manifolds can be interpreted as 2-jets of curves driven by Brownian motion, and show convergence of a jet-based numerical scheme.

We will use the language of jets to give geometric expressions for many important concepts that arise in stochastic analysis. These geometric representations are in many ways more elegant than the traditional representations in terms of the coefficients of SDEs. In particular we will give coordinate free formulations of the following: Itô's lemma; the diffusion operators; Itô SDEs on manifolds and Brownian motion on Riemannian manifolds.

We discuss how our formulation is related to the Stratonovich formulation of SDEs. We will prove that sections of the bundle of n -jets of curves in a manifold correspond naturally to n -tuples of vector fields in the manifold. This correspondence shows that Itô calculus and Stratonovich calculus can both be interpreted as simply a choice of coordinate system for the space of sections of the bundle of n -jets of curves, and provides a new interpretation of the Itô-Stratonovich transformation.

We will also see that the 2-jet defining an Itô SDE can help in studying quantiles of the SDE solution. Moreover we will show that the drift vector of the

Stratonovich formulation can be similarly interpreted as a short-time approximation to the median. This observation is related to the coordinate independence of 2-jets and vector fields and the coordinate independence of the notion of median.

Coordinate free formulations of SDEs have been considered before. In particular SDEs have been described either in terms of “second order tangent vectors/diffusors” and Schwartz morphism, see for example [7], or via the Itô bundle of Belopolskaja and Dalecky, see for example [8] or the appendix in [4]. We briefly explore the relationship of the jet approach with these earlier approaches in the appendix of [1], but in general we have replaced the structure of second order tangent vectors with the more familiar and standard geometric concept of two jets.

We conclude by remarking that our work has numerous applications, including a novel notion of optimal projection for SDEs ([3, 2]) that allows to approximate in a given submanifold the solution of a SDE evolving on a larger space. We already applied the new projection to stochastic filtering in [2].

2 SDEs as 2-jets & Ito-Stratonovich transformation

Suppose that at every point $x \in \mathbb{R}^n$ we have an associated smooth curve $\gamma_x : \mathbb{R} \rightarrow \mathbb{R}^n$ with $\gamma_x(0) = x$. Example: $\gamma_{(x_1, x_2)}^E(t) = (x_1, x_2) + t(-x_2, x_1) + 3t^2(x_1, x_2)$. This specific example has zero derivatives with respect to t from the third derivative on. We stop at second order in t in the example since this will be enough to converge to classical stochastic calculus, but our theory is general. We have plotted $t \mapsto \gamma_x^E(t)$ in a grid of possible “centers” $x = (x_1, x_2)$ in figure 1.

Given such a γ , a starting point x_0 , a Brownian motion W_t and a time step δt we can define a discrete time stochastic process using the following recurrence relation:

$$X_0 := x_0, \quad X_{t+\delta t} := \gamma_{X_t}(W_{t+\delta t} - W_t) \quad (1)$$

In Figure 1 we have plotted the trajectories of process for γ^E , the starting point $(1, 0)$, a fixed realization of Brownian motion and a number of different time steps. Rather than just plotting a discrete set of points for this discrete time process, we have connected the points using the curves in $\gamma_{X_t}^E$.

As the figure suggests, these discrete time stochastic processes (1) converge in some sense to a limit as the time step tends to zero.

We will use the following notation for the limiting process:

$$\text{Coordinate free SDE: } X_t \curvearrowright \gamma_{X_t}(dW_t), \quad X_0 = x_0. \quad (2)$$

For the time being, let us simply treat equation (2) as a short-hand way of saying that equation (1) converges in some sense to a limit. We will explore the limit question shortly. Note that it will not converge for arbitrary γ 's but for nice γ such as γ^E or more general γ 's with sufficiently good regularity.

An important feature of equation (1) is that it makes no reference to the vector space structure of $\mathbb{R}^n$ for our state space X . We could define the same identical scheme in a manifold. We have maintained this in the formal notation

Fig. 1. Discrete time trajectories for γ^E for a fixed W_t and X_0 with different values for δt

used in equation (2). By avoiding using the vector space structure on $\mathbb{R}^n$ we will be able to obtain a coordinate free understanding of stochastic differential equations.

Now, using $\mathbb{R}^n$ coordinates if we are in an Euclidean space or a coordinate chart if we are in a manifold, consider the (component-wise) Taylor expansion of γ_x . We have:

$$\gamma_x(t) = x + \gamma'_x(0)t + \frac{1}{2}\gamma''_x(0)t^2 + R_x t^3, \quad R_x = \frac{1}{6}\gamma'''_x(\xi), \quad \xi \in [0, t],$$

where $R_x t^3$ is the remainder term in Lagrange form. Substituting this Taylor expansion in our Equation (1) we obtain

$$\delta X_t = \gamma'_{X_t}(0)\delta W_t + \frac{1}{2}\gamma''_{X_t}(0)(\delta W_t)^2 + R_{X_t}(\delta W_t)^3, \quad X_0 = x_0. \quad (3)$$

Classic stochastic analysis and properties of Brownian motion suggests that we can replace the term $(\delta W_t)^2$ with δt and we can ignore terms of order $(\delta W_t)^3$ and above, see [1] for more details. So we expect that under reasonable conditions, in the chosen coordinate system, the recurrence relation given by (1) and expressed in coordinates by (3) will converge to the same limit as the numerical scheme:

$$\delta \bar{X}_t = \gamma'_{\bar{X}_t}(0)\delta W_t + \frac{1}{2}\gamma''_{\bar{X}_t}(0)\delta t, \quad \bar{X}_0 = x_0. \quad (4)$$

Defining $a(X) := \gamma''_X(0)/2$ and $b(X) := \gamma'_X(0)$ we have that this last equation can be written as

$$\delta \bar{X}_t = a(\bar{X}_t)\delta t + b(\bar{X}_t)\delta W_t. \quad (5)$$

It is well known that this last scheme (Euler scheme) does converge in some appropriate sense to a limit ([10]) and that this limit is given by the solution to the Itô stochastic differential equation:

$$d\tilde{X}_t = a(\tilde{X}_t)dt + b(\tilde{X}_t)dW_t, \quad \tilde{X}_0 = x_0. \quad (6)$$

In [1] we have proven the following

Theorem 1. *Given the curved scheme (1), in any coordinate system where this scheme can be expressed as (3) with the coordinate expression of $h \mapsto \gamma_x(h)$ smoothly varying in x with Lipschitz first and second derivatives with respect to h , and with uniformly bounded third derivative with respect to h , we have that the three schemes (3) (4) and (5) converge to the solution of the Itô SDE (6). This solution depends only on the two-jet of the curve γ .*

In which sense Equation (1) and its limit are coordinate free? It is important to note that the coefficients of equation (6) only depend upon the first two derivatives of γ . We say that two smooth curves $\gamma : \mathbb{R} \rightarrow \mathbb{R}^n$ have the same k -jet ($k \in \mathbb{N}, k > 0$) if they are equal up to order $O(t^k)$ in one (and hence all) coordinate system. The k -jet can then be defined for example as the equivalence class of all curves that are equal up to order $O(t^k)$ in one and hence all coordinate systems. Other definitions are possible, based on operators. Using the jets terminology, we say that the coefficients of equation (6) (and (5)) are determined by the 2-jet of the curve γ .

Given a curve γ_x , we will write $j_2(\gamma_x)$ for the two jet associated with γ_x . This is formally defined to be the equivalence class of all curves which are equal to γ_x up to $O(t^2)$.

Since we stated that, under reasonable regularity conditions, the limit in the symbolic equation (2) depends only on the 2-jet of the driving curve, we may rewrite equation (2) as:

$$\text{Coordinate-free 2-jet SDE:} \quad X_t \curvearrowright j_2(\gamma_{X_t})(dW_t), \quad X_0 = x_0. \quad (7)$$

This may be interpreted either as a coordinate free notation for the classical Itô SDE given by equation (6) or as a shorthand notation for the limit of the manifestly coordinate-free process given by the discrete time equation (1). The reformulation of Itô's lemma in the language of jets we are going to present now shows explicitly that also the first interpretation will be independent of the choice of coordinates. The only issue one needs to consider are the bounds needed to ensure existence of solutions. The details of transferring the theory of existence and uniqueness of solutions of SDEs to manifolds are considered in, for example, [6], [5], [9] and [7]. As we just mentioned, we can now give an appealing coordinate free version of Itô's formula. Suppose that f is a smooth mapping from $\mathbb{R}^n$ to itself and suppose that X satisfies (1). It follows that $f(X)$ satisfies $(f(X))_{t+\delta t} = (f \circ \gamma_{X_t})(\delta W_t)$. Taking the limit as $\delta t \rightarrow 0$ we have:

Lemma 1. *[Coordinate free Itô's lemma] If the process X_t satisfies*

$$X_t \curvearrowright j_2(\gamma_{X_t})(dW_t) \text{ then } f(X_t) \text{ satisfies } f(X)_t \curvearrowright j_2(f \circ \gamma_{X_t})(dW_t).$$

We can interpret Itô's lemma geometrically as the statement that the transformation rule for jets under coordinate change is composition of functions.

So far we assumed that our SDEs were driven by scalar Brownian motion. We now briefly summarize our discussion in [1] generalizing the theory to SDEs driven by d independent Brownian motions W_t^α with $\alpha \in \{1, 2, \dots, d\}$. Consider

jets of functions of the form $\gamma_x : \mathbb{R}^d \rightarrow \mathbb{R}^n$. Just as before we can consider the candidate 2-jet scheme as a difference equation of the form:

$$X_{t+\delta t} := \gamma_{X_t}(\delta W_t^1, \dots, \delta W_t^d).$$

Again, the limiting behaviour of such difference equations will only depend upon the 2-jet $j_2(\gamma_x)$ and can be denoted by (7), where it is now understood that dW_t is the vector Brownian motion increment. We obtain a straightforward generalization of Theorem 1, showing that the 2-jet scheme above, in any well-behaving coordinate system, converges in $L^2(\mathbb{P})$ to the classic Itô SDE with the same coefficients (with Einstein summation convention):

$$dX_t^i = \frac{1}{2} \partial_\alpha \partial_\beta \gamma^i dW_t^\alpha dW_t^\beta + \partial_\alpha \gamma^i dW_t^\alpha = \frac{1}{2} \partial_\alpha \partial_\beta \gamma^i g_E^{\alpha\beta} dt + \partial_\alpha \gamma^i dW_t^\alpha \quad (8)$$

Here x^α are the standard $\mathbb{R}^d$ orthonormal coordinates. Our equation should be interpreted with the convention that $dW_t^\alpha dW_t^\beta = g_E^{\alpha\beta} dt$ where g_E is equal to 1 if α equals β and 0 otherwise. We choose to write g_E instead of using a Kronecker δ because one might want to choose non orthonormal $\mathbb{R}^d$ coordinates and so it is useful to notice that g_E represents the symmetric 2-form defining the Euclidean metric on $\mathbb{R}^d$. Using a Kronecker δ would incorrectly suggest that this term transforms as an endomorphism rather than as a symmetric 2-form.

Guided by the above discussion, we introduce the following

Definition 1. (Coordinate free Itô SDEs driven by Brownian motion). *A Itô SDE on a manifold M is a section of the bundle of 2-jets of maps $\mathbb{R}^d \rightarrow M$ together with d Brownian motions W_t^i , $i = 1, \dots, d$.*

Suppose now that f is a function mapping the manifold M to $\mathbb{R}$. We can define a differential operator acting on *functions* in terms of a 2-jet associated with γ_x as follows.

Definition 2. (Backward diffusion operator via 2-jets). *The Backward diffusion operator for the Itô SDE defined by the 2-jet associated with the map γ_x is defined on suitable functions f as*

$$\mathcal{L}_{\gamma_x} f := \frac{1}{2} \Delta_E(f \circ \gamma_x) = \frac{1}{2} \partial_\alpha \partial_\beta (f \circ \gamma_x) g_E^{\alpha\beta}. \quad (9)$$

Here Δ_E is the Laplacian defined on $\mathbb{R}^d$.

$\mathcal{L}_{\gamma_x}$ acts on functions defined on M . In [1] we further express the forward Kolmogorov or Fokker-Planck equation in terms of 2-jets, highlighting the coordinate-free interpretation of the backward and forward diffusion operators. Furthermore, we see that both the Itô SDE (8) and the backward diffusion operator use only part of the information contained in the 2-jet: only the diagonal terms of $\partial_\alpha \partial_\beta \gamma^i$ (those with $\alpha = \beta$) influence the SDE and even for these terms it is only their average value that is important. The same consideration applies to the backward diffusion operator. This motivates our definition of weakly equivalent

and strongly equivalent 2-jets given in [1], where weak equivalence is defined between γ^1 and γ^2 if $\mathcal{L}_{\gamma_x^1} = \mathcal{L}_{\gamma_x^2}$, while strong equivalence requires, in addition, the same 1-jets: $j_1(\gamma^1) = j_1(\gamma^2)$. With strong equivalence, given the same realization of the driving Brownian motions W_t^α , the solutions of the SDEs will be almost surely the same (under reasonable assumptions to ensure pathwise uniqueness). When the 2-jets are weakly equivalent, the transition probability distributions resulting from the dynamics of the related SDEs are the same even though the dynamics may be different for any specific realisation of the Brownian motions. For this reason one can define a diffusion process on a manifold as a smooth selection of a second order linear operator $\mathcal{L}$ at each point that determines the transition of densities. A diffusion can be realised locally as an SDE, but not necessarily globally.

Recall that the top order term of a quasi linear differential operator is called its symbol. In the case of a second order quasi linear differential operator D which maps $\mathbb{R}$ -valued functions to $\mathbb{R}$ -valued functions, the symbol defines a section of S^2T , the bundle of symmetric tensor products of tangent vectors, which we will call g_D . In local coordinates, if the top order term of D is $Df = a^{ij}\partial_i\partial_j f + \text{lower order}$, then g_D is given by $g_D(X_i, X_j) = a^{ij}X_iX_j$. We are using the letter g to denote the symbol for a second order operator because, in the event that g is positive definite and $d = \dim M$, g defines a Riemannian metric on M . In these circumstances we will say that the SDE/diffusion is *non-singular*. Thus we can associate a canonical Riemannian metric $g_{\mathcal{L}}$ to any non-singular SDE/diffusion.

Definition 3. *A non-singular diffusion on a manifold M is called a Riemannian Brownian motion if $\mathcal{L}(f) = \frac{1}{2}\Delta_{g_{\mathcal{L}}}(f)$.*

Note that given a Riemannian metric h on M there is a unique Riemannian Brownian motion (up to diffusion equivalence) with $g_{\mathcal{L}} = h$. This is easily checked with a coordinate calculation.

This completes our definitions of the key concepts in stochastic differential geometry and indicates some of the important connections between stochastic differential equations, Riemannian manifolds, second order linear elliptic operators and harmonic maps. We emphasize that all our definitions are coordinate free and we have worked exclusively with Itô calculus. However, it is more conventional to perform stochastic differential geometry using Stratonovich calculus. The justification usually given for this is that Stratonovich calculus obeys the usual chain rule so the coefficients of Stratonovich SDEs can be interpreted as vector fields. For example one can immediately see if the trajectories to a Stratonovich SDE almost surely lie on particular submanifold by testing if the coefficients of the SDE are all tangent to the manifold. We would argue that the corresponding test for Itô SDEs is also perfectly simple and intuitive: one checks whether the 2-jets lie in the manifold. As is well known, the probabilistic properties of the Stratonovich integral are not as nice as the properties of the Itô integral, and the use of Stratonovich calculus on manifolds comes at a price. Our results allows us to retain the probabilistic advantages of Itô calculus while working directly with geometry. However, we will now further clarify the relationship of our jets approach with Stratonovich calculus.

For simplicity, let us assume for a moment that the SDE driver is one dimensional Brownian motion. Thus to define an SDE on a manifold, one must choose a 2-jet of a curve at each point of the manifold. One way to specify a k -jet of a curve at every point in a neighbourhood is to first choose a chart for the neighbourhood and then consider curves of the form $\gamma_x(t) = x + \sum_{i=1}^k a_i(x)t^i$ where $a_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$. As we have already seen in (1), these coefficient functions a_i depend upon the choice of chart in a relatively complex way. For example for 2-jets the coefficient functions are not vectors but instead transform according to Itô's lemma. We will call this the *standard representation* for a family of k -jets.

An alternative way to specify the k -jet of a curve at every point is to choose k vector fields $A_1, \dots, A_k$ on the manifold. One can then define $\Phi_{A_i}^t$ to be the vector flow associated with the vector field A_i . This allows one to define curves at each point x as $\gamma_x(t) = \Phi_{A_k}^{t^k}(\Phi_{A_{k-1}}^{t^{k-1}}(\dots(\Phi_{A_1}^t(x))\dots))$ where t^k denotes the k -th power of t . We will call this the *vector representation* for a family of k -jets. It is not immediately clear that all k -jets of curves can be written in this way. In [1] we prove that this is indeed the case. The standard and vector representations simply give us two different coordinate systems for the infinite dimensional space of families of k -jets. Let us apply this to SDEs seen as 2-jets.

Lemma 2. *Suppose that a family of 2-jets of curves is given in the vector representation as $\gamma_x(t) = \Phi_A^{t^2}(\Phi_B^t(x))$ for vector fields A and B . Choose a coordinate chart and let A^i, B^i be the components of the vector fields in this chart. Then the corresponding standard representation for the family of 2-jets is (see [1] for a proof) $\gamma_x(t) = x + a(x)t^2 + b(x)t$ with $a^i = A^i + \frac{1}{2} \frac{\partial B^i}{\partial x^j} B^j$, $b^i = B^i$.*

As we have already discussed, the standard representation of a 2-jet corresponds to conventional *Itô calculus*. What we have just demonstrated is that the vector representation of a 2-jet corresponds to *Fisk-Stratonovich-McShane calculus* [11] (Stratonovich from now on). Moreover we have given a geometric interpretation of how the coordinate free notion of a 2-jet of a curve is related to the vector fields defining a Stratonovich SDE, and of the Itô-Stratonovich transformation. A much richer discussion on Itô and Stratonovich SDEs, in relation to our result above, and on why the Itô Stratonovich transformation is not enough to work with SDE on manifolds is presented in [1].

We conclude with a new interpretation of the drift of one-dimensional Stratonovich SDEs as median. This is part of more general results on quantiles of SDEs solutions that are given in full detail in [1]. We begin by noticing that the definition of the α -percentile depends only upon the ordering of $\mathbb{R}$ and not its vector space structure. As a result, for continuous monotonic f and X with connected state space, the median of $f(X)$ is equal to f applied to the median of X . If f is strictly increasing, the analogous result holds for the α percentile. This has the implication that the trajectory of the α -percentile of an $\mathbb{R}$ valued stochastic process is invariant under smooth monotonic coordinate changes of $\mathbb{R}$. In other words, percentiles have a coordinate free interpretation. The mean does not. This raises the question of how the trajectories of the percentiles can be related to the coefficients of the stochastic differential equation. In [1] we calculate this relationship.

We summarize the main result proven in that paper for the one-dimensional SDE with non-vanishing diffusion term b :

$$dX_t = a(X_t, t) dt + b(X_t, t) dW_t, \quad X_0 = x_0. \quad (10)$$

Theorem 2. *For sufficiently small t , the α -th percentile of the solutions to (10) is given by:*

$$x_0 + b_0 \sqrt{t} \Phi^{-1}(\alpha) + \left(a_0 - \frac{1}{2} b_0 b'_0 (1 - \Phi^{-1}(\alpha)^2) \right) t + O(t^{3/2}) \quad (11)$$

so long as the coefficients of (10) are smooth, the diffusion coefficient b never vanishes, and further regularity conditions specified in [1] hold. In this formula a_0 and b_0 denote the values of $a(x_0, 0)$ and $b(x_0, 0)$ respectively. In particular, the median process is a straight line up to $O(t^{\frac{3}{2}})$ with tangent given by the drift of the Stratonovich version of the Itô SDE (10). The $\Phi(1)$ and $\Phi(-1)$ percentiles correspond up to $O(t^{\frac{3}{2}})$ to the curves $\gamma_{X_0}(\pm\sqrt{t})$ where γ_{X_0} is any representative of the 2-jet that defines the SDE in Itô form.

The theorem above has given us the median as a special case, and a link between the median and the Stratonovich version of the SDE. By contrast the mean process has tangent given by the drift of the Itô SDE as the Itô integral is a martingale. In [1] we derive the mode interval equations too and discuss their relationship with median and mean.

References

1. J. Armstrong and D. Brigo. Coordinate free stochastic differential equations as jets. <http://arxiv.org/abs/1602.03931>, 2016.
2. J. Armstrong and D. Brigo. Optimal approximation of SDEs on submanifolds: the Ito-vector and Ito-jet projections. <http://arxiv.org/abs/1610.03887>, 2016.
3. John Armstrong and Damiano Brigo. Extrinsic projection of Itô SDEs on submanifolds with applications to non-linear filtering. In: *Nielsen, F., Critchley, F., & Dodson, K. (Eds), Computational Information Geometry for Image and Signal Processing*, Springer Verlag, 2016.
4. Z. Brzeźniak and K. D. Elworthy. Stochastic differential equations on Banach manifolds. *Methods Funct. Anal. Topology*, 6(1):43–84, 2000.
5. K. D. Elworthy. *Stochastic differential equations on manifolds*. Cambridge University Press, Cambridge, New York, 1982.
6. K. D. Elworthy. Geometric aspects of diffusions on manifolds. In *École d'Été de Probabilités de Saint-Flour XV–XVII, 1985–87*, pages 277–425. Springer, 1988.
7. M. Emery. *Stochastic calculus in manifolds*. Springer-Verlag, Heidelberg, 1989.
8. Yuri E Gliklikh. *Global and Stochastic Analysis with Applications to Mathematical Physics*. Theoretical and Mathematical Physics. Springer, London, 2011.
9. K. Itô. Stochastic differential equations in a differentiable manifold. *Nagoya Math. J.*, (1):35–47, 1950.
10. Peter E. Kloeden and Eckhard Platen. *Numerical solution of stochastic differential equations*. Applications of mathematics. Springer, Berlin, New York, Third printing, 1999.
11. R. L. Stratonovich. A new representation for stochastic integrals and equations. *SIAM Journal on Control*, 4(2):362–371, 1966.