Mellin and Wiener-Hopf operators in a non-classical boundary value problem describing a Levy process

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Mellin and Wiener-Hopf operators in a non-classical boundary value problem describing a Lévy process

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Abstract

This research, into non-classical boundary value problems, is motivated by the study of stochastic processes, restricted to a domain, that can have discontinuous trajectories. We demonstrate that the singularities, for example delta functions, that might be expected at the boundary, can be mitigated, using current probability theory, by what amounts to the inclusion of a carefully chosen potential.

To make this general problem more tractable, we consider a particular operator, $\mathcal{A}$, which is chosen to be the generator of a certain stable Lévy process restricted to the positive half-line. We are able to represent $\mathcal{A}$ as a (hyper-) singular integral and, using this representation and other methods, deduce simple conditions for its boundedness, between Bessel potential spaces. Moreover, from energy estimates, we prove that, under certain conditions, $\mathcal{A}$ has a trivial kernel.

A central feature of this research is our use of Mellin operators to deal with the leading singular terms that combine, and cancel, at the boundary. Indeed, after considerable analysis, the problem is reformulated in the context of an algebra of multiplication, Wiener-Hopf and Mellin operators, acting on a Lebesgue space. The resulting generalised symbol is examined and, it turns out, that a certain transcendental equation, involving gamma and trigonometric functions with complex arguments, plays a pivotal role. Following detailed consideration of this transcendental equation, we are able to determine when our operator is Fredholm and, in that case, calculate its index. Finally, combining information on the kernel with the Fredholm index, we establish precise conditions for the invertibility of $\mathcal{A}$. 
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Chapter 1
Introduction

1.1 Preamble

We begin by defining some key concepts central to this research. References for all the results stated in this section without proof can be found in a combination of [14] and [37].

The Fourier Transform on the Schwartz space, \( S(\mathbb{R}) \), of rapidly decaying infinitely differentiable functions \( u \) is given by

\[
(Fu)(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi x} u(x) \, dx, \quad \xi \in \mathbb{R}.
\]  

(1.1)

The Fourier transform is invertible on the Schwartz space, and its inverse \( F^{-1} \) is given by

\[
(F^{-1}v)(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} v(\xi) \, d\xi, \quad x \in \mathbb{R}.
\]

\( F^{\pm1} \) can be extended to \( S'(\mathbb{R}) \), the space of tempered distributions corresponding to the Schwartz space.

For any \( y, s \in \mathbb{R} \), we define

\[
\langle y \rangle := (1 + y^2)^{1/2},
\]

\[
I^s := F^{-1} (\xi)^s F.
\]

We define the Bessel potential space

\[
H^s_p(\mathbb{R}) := \{ f : f \in S'(\mathbb{R}), \| f | H^s_p(\mathbb{R}) \| := \| I^s f \|_{L_p} < \infty \}.
\]

We note that for any \( s \in \mathbb{R} \) and \( 1 < p < \infty \), \( H^s_p(\mathbb{R}) \) is a Banach space. Moreover, both the Schwartz space and the space of infinitely smooth functions
with compact support are dense in $H^s_p(\mathbb{R})$.

We are interested in the half-line $\mathbb{R}_+$ and, accordingly, we define

$$\tilde{H}^s_p(\mathbb{R}_+) := \{ u : u \in H^s_p(\mathbb{R}), \ \text{supp} \ u \subseteq \mathbb{R}_+ \}.$$  

Of course, by definition, we have $\tilde{H}^s_p(\mathbb{R}_+) \subset H^s_p(\mathbb{R})$.

Let $C^\infty_0(\mathbb{R}_+)$ denote the space of infinitely differentiable functions on $\mathbb{R}_+$ with compact support in $\mathbb{R}_+$, and let $\chi_{\mathbb{R}_\pm}$ denote the characteristic function of the sets $\mathbb{R}_\pm$ respectively.

It will also be useful to let $r_+$ denote the restriction operator from $\mathbb{R}$ to $\mathbb{R}_+$. In addition, we let $l_+$ denote an arbitrary extension operator from $\mathbb{R}_+$ to $\mathbb{R}$ and $e_+$ be the particular extension by zero. We now define

$$H^s_p(\mathbb{R}_+) := \{ r_+ u : u \in H^s_p(\mathbb{R}) \},$$

with norm

$$\| u \|_{H^s_p(\mathbb{R}_+)} := \inf \{ \| u_0 \|_{H^s_p(\mathbb{R})} : u_0 \in H^s_p(\mathbb{R}), r_+ u_0 = u \}. \quad (1.2)$$

Of course, if $u \in H^s_p(\mathbb{R}_+)$ then

$$r_+ l_+ u = u,$$

so that $r_+ l_+$ acts as the identity on the space $H^s_p(\mathbb{R}_+)$. Assuming $s > 1 + 1/p$, it will also convenient to define

$$H^s_{p,0}(\mathbb{R}_+) := \{ u \in H^s_p(\mathbb{R}_+) : \frac{du}{dx} \big|_{x=0} = 0 \}. \quad (1.3)$$

When working with the Fourier transform, we define

$$D := i \frac{\partial}{\partial x},$$

so that, for example, for a given $u \in S(\mathbb{R})$ we have $\mathcal{F}_{x \rightarrow \xi}(D^k u) = \xi^k \mathcal{F} u$, for all $k \in \mathbb{N}$.

Suppose $1 < p < \infty$. We say that $a \in L_\infty(\mathbb{R})$ is a Fourier $L_p$ multiplier if, for all $u \in L_2(\mathbb{R}) \cap L_p(\mathbb{R})$, we have $\mathcal{F}^{-1} a \mathcal{F} u \in L_p(\mathbb{R})$ and

$$\| \mathcal{F}^{-1} a \mathcal{F} u \|_p \leq C_p \| u \|_p,$$
where the constant $C_p$ is independent of $u$. The set of all such Fourier multipliers is denoted by $\mathfrak{M}_p$.

If $a \in \mathfrak{M}_p$, then the operator $\mathcal{F}^{-1}a \mathcal{F} : L_2(\mathbb{R}) \cap L_p(\mathbb{R}) \to L_p(\mathbb{R})$ extends continuously to a bounded operator on $L_p(\mathbb{R})$. This extension is called a Fourier convolution operator with symbol $a$, and is denoted by $W^0(a)$. Moreover, we let $W(a) = r_+ W^0(a) e_+$ denote the corresponding Wiener-Hopf operator.

We now define the operator $Z_p : L_p(\mathbb{R}^+) \to L_p(\mathbb{R})$ by

$$ (Z_p u)(y) := \sqrt{2\pi} e^{-y/p} u(e^{-y}), \quad y \in \mathbb{R}, $$

for any $u \in L_p(\mathbb{R}^+)$. It is easy to show that

$$ \|Z_p u\|_p = \sqrt{2\pi} \|u\|_{L_p(\mathbb{R}^+)}, $$

so that $Z_p$ is bounded. In addition, $Z_p$ is invertible and its inverse $Z_p^{-1} : L_p(\mathbb{R}) \to L_p(\mathbb{R}^+)$ is given by

$$ (Z_p^{-1} v)(x) := \frac{1}{\sqrt{2\pi}} x^{-1/p} v(-\log(x)), \quad x \in \mathbb{R}^+, $$

for any $v \in L_p(\mathbb{R})$.

The operator $\mathcal{M}_p := \mathcal{F} Z_p$ is called the Mellin transform, and it is given explicitly by

$$ (\mathcal{M}_p u)(\eta) = \int_0^\infty x^{1/p-1-in} u(x) \, dx, \quad \eta \in \mathbb{R}, \quad (1.4) $$

and moreover, its inverse, $\mathcal{M}_p^{-1} = Z_p^{-1} \mathcal{F}^{-1}$, can be written as

$$ (\mathcal{M}_p^{-1} v)(x) = \frac{1}{2\pi} \int_{-\infty}^\infty x^{-1/p+in} v(\eta) \, d\eta, \quad x \in \mathbb{R}^+. $$

Let $b \in \mathfrak{M}_p$. Then the operator

$$ M^0(b) := \mathcal{M}_p^{-1} b \mathcal{M}_p \quad (1.5) $$

is bounded on $L_p(\mathbb{R}^+)$, and is called the Mellin convolution operator with symbol $b$.

We will be particularly interested in integral operators of the form

$$ (M \varphi)(x) = \int_0^\infty K\left(\frac{x}{y}\right) \varphi(y) \frac{dy}{y}, \quad \varphi \in L_p(\mathbb{R}^+), \quad (1.6) $$
where the kernel $K$ satisfies the integrability condition

$$
\int_0^\infty |K(t)|t^{1/p-1} \, dt < \infty.
$$

(1.7)

It is easy to show (see, for example, p. 174, [14], for the case $p = 2$) that

$$
\mathcal{M}_p(M\varphi) = \mathcal{M}_p(K) \cdot \mathcal{M}_p(\varphi).
$$

If the kernel $K$ satisfies the integrability condition (1.7), then the associated integral operator $M$ is a Mellin convolution operator, say $M^0(b)$, where the symbol $b$ is given by the formula

$$
b = \mathcal{M}_p(K).
$$

(1.8)

That is, the symbol of the integral operator $M$ is the Mellin transform of its kernel $K$.

We will need to consider fractional powers of complex numbers. To make the complex argument, $\arg(\cdot)$, single valued we insist that

$$
-\pi < \arg z \leq \pi, \quad z \in \mathbb{C} \setminus \{0\}.
$$

(1.9)

That is, we assume that the cut in the complex plane is along the negative horizontal axis.

Thus, given $z \in \mathbb{C} \setminus \{0\}$, we can write

$$
z = |z| \exp(i \arg z), \quad -\pi < \arg z \leq \pi,
$$

and define

$$
\log z := \log|z| + i \arg z.
$$

Finally, for any $z \in \mathbb{C} \setminus \{0\}$ and $\gamma \in \mathbb{C}$ we define

$$
z^\gamma := \exp(\gamma \log z).
$$

(1.10)

It is immediately clear from definition (1.10), that for any $z \in \mathbb{C} \setminus \{0\}$ and $\beta, \gamma \in \mathbb{R}$,

$$
z^\beta z^\gamma = z^{\beta+\gamma}.
$$

However, suppose that $z_1, z_2 \in \mathbb{C}$ and $\gamma \in \mathbb{R}$. Then the relationship

$$
(z_1 z_2)^\gamma = z_1^\gamma z_2^\gamma
$$

does not hold universally.
Remark 1.1. Suppose \( \nu \in \mathbb{R} \). Then, for \( z_1, z_2 \in \mathbb{C} \),

\[
(z_1 z_2)^\nu = z_1^\nu z_2^\nu,
\]

provided

\[
-\pi < \arg z_1 + \arg z_2 \leq \pi.
\]

Of course, condition (1.12) is automatically satisfied if \( z_1 \) and \( z_2 \) have (non-trivial) imaginary parts of opposite sign.

Suppose \( \nu, \xi \in \mathbb{R} \). The following useful results are immediate consequences of Remark 1.1:

\[
(1 + \xi^2)^\nu = (1 - i\xi)^\nu (1 + i\xi)^\nu
\]

\[
(1 + \xi^2)^\nu = (\xi - i)^\nu (\xi + i)^\nu
\]

\[
(1 + i\xi)^\nu = (i\xi)^\nu (1 - i/\xi)^\nu \quad (\xi \neq 0).
\]
1.2 The problem

1.2.1 Introduction

This research, into non-classical boundary value problems, is motivated by the study of stochastic processes, restricted to a domain, that can have discontinuous trajectories. The context for this work is a large class of Markov processes – namely the so-called Feller and (hence) Lévy processes. (For more details on Feller and Lévy processes, see Appendix B.) It follows from a well known result of Ph. Courrège that the generator of a Feller process in $\mathbb{R}^n$ is a pseudodifferential operator.

Suppose $G \subset \mathbb{R}^n$ be an open domain with a $C^\infty$-smooth boundary $\partial G$ and let $A$ be a pseudodifferential operator on $\mathbb{R}^n$. Then the central questions of this research are, firstly, how to define $A$ on $G$ and, secondly, under what conditions is $A$, or perhaps some perturbation of $A$, invertible?

Let $r_G$ denote the operator of restriction from $\mathbb{R}^n$ to $G$ and $e_G$ be the operator of extension, by zero, from $G$ to $\mathbb{R}^n$. Further suppose that $X(\mathbb{R}^n)$ is some function space on $\mathbb{R}^n$. Then, we will show, drawing on some work in probability theory, that the “correct” operator of interest is given by

$$A_G u := r_G A e_G u + u r_G A(\chi_{\mathbb{R}^n \setminus G}), \quad u \in r_G X(\mathbb{R}^n),$$

where $\chi_{\mathbb{R}^n \setminus G}$ denotes the characteristic function of the complement of $G$ in $\mathbb{R}^n$.

1.2.2 Pseudodifferential operators

A vector of the form $\gamma = (\gamma_1, \ldots, \gamma_n)$, where each component $\gamma_i$ is a non-negative integer, is called a multi-index of order $|\gamma| = \gamma_1 + \cdots + \gamma_n$.

Given a multi-index $\gamma$, we define

$$D^\gamma := D_{x_1}^{\gamma_1} \cdots D_{x_n}^{\gamma_n}, \quad \text{where} \quad D_{x_k} = i \frac{\partial}{\partial x_k}, \quad k = 1, \ldots, n.$$

Then any linear partial differential operator with constant coefficients

$$\mathcal{P}(D) = \sum_{|\gamma| \leq N} c_\gamma D^\gamma, \quad c_\gamma \in \mathbb{C},$$
can be represented in the form
\[ \mathcal{P}(D) = \mathcal{F}^{-1} \sum_{|\gamma| \leq N} c_\gamma \xi^\gamma \mathcal{F}. \]

For example, if \( \Delta \) denotes the Laplacian, then
\[ -\Delta = \mathcal{F}^{-1} |\xi|^2 \mathcal{F}, \]
and we say that the symbol of \( -\Delta \) is \( |\xi|^2 \).

An operator \( \mathcal{P}(D) \) is a pseudodifferential operator if it can be written in the form
\[ \mathcal{P}(D) = \mathcal{F}^{-1} \mathcal{P}(\xi) \mathcal{F}, \]
where, of course, the symbol \( \mathcal{P}(\xi) \) is not necessarily a polynomial in \( \xi \).

For example, the generator of the (so-called) symmetric \( 2\alpha \)-stable Lévy process in \( \mathbb{R}^n \) is the fractional Laplacian \([10], [47]\) which can be defined as
\[ (-\Delta)^\alpha := \mathcal{F}^{-1} |\xi|^{2\alpha} \mathcal{F}, \quad 0 < \alpha < 1. \]
(1.13)

Unlike the Laplacian, the fractional Laplacian is a non-local operator. In a discrete setting, say the lattice \( \mathbb{Z}^n \), we can think of the fractional Laplacian as random walk in which a particle may experience arbitrarily long jumps, albeit with a small probability, \([47]\).

There are at least ten equivalent definitions of the fractional Laplacian, see \([31]\), and it can be very useful to interchange them. Indeed, suppose \( u \in H_2^\alpha(\mathbb{R}^n) \), then we can also write
\[ (-\Delta)^\alpha u(x) = \lim_{\epsilon \to 0} c_{n,\alpha} \int_{y \in \mathbb{R}^n, |x-y| > \epsilon} \frac{u(x) - u(y)}{|x-y|^{n+2\alpha}} dy, \]
(1.14)
where the constant \( c_{n,\alpha} \) depends only on \( n \) and \( \alpha \). In passing, we make special note of an excellent survey paper, \([10]\), that details applications as disparate as crystal dislocation, finance and water waves, where the fractional Laplacian is playing an important role.

### 1.2.3 Pseudodifferential operators on a domain

Let \( G \subset \mathbb{R}^n \) be an open domain with a \( C^\infty \)-smooth boundary \( \partial G \) and let \( A \) be a pseudodifferential operator on \( \mathbb{R}^n \). A central question of this research
is how to define $A$ on $G$. The major difficulty lies in finding an appropriate
way of extending functions defined on $G$ to the full Euclidean space $\mathbb{R}^n$.

Of course, starting from $G$, an obvious way forward is to extend any func-
tions defined on $G$ by zero to the whole of $\mathbb{R}^n$, apply the pseudodifferential
operator $A$ and then restrict back to $G$.

To make this more precise, suppose that $Y(\mathbb{R}^n)$, $Z(\mathbb{R}^n)$ are functions spaces
defined on $\mathbb{R}^n$. Let $r_G$ denote the operator of restriction from $\mathbb{R}^n$ to $G$. Then
for any function space $Y(\mathbb{R}^n)$, we define

$$Y(G) := r_G Y(\mathbb{R}^n).$$

(1.15)

Similarly, we define $e_G$ to be the operator of extension, by zero, from $G$ to
$\mathbb{R}^n$. Hence, we can write

$$A_G u := r_G A e_G u, \quad u \in Y(G).$$

(1.16)

Unfortunately, there are some significant difficulties with this simple ap-
proach. For example, it may be the case that the extended function $e_G u$
is discontinuous on the boundary of $G$, and this may cause singularities in
$A e_G u$. Thus, even when $A : Y(\mathbb{R}^n) \to Z(\mathbb{R}^n)$ and $u \in Y(G)$, it may happen
that

$$A_G u \not\in Z(G).$$

A detailed example illustrating this phenomenon is given in Appendix A.

**Remark 1.2.** We note that if $A$ is a differential operator there is no need to
use the extension operator $e_G$, since $A$ is a local operator in this case.

However, this begs the question what happens if we do choose to use the ex-
tension operator $e_G$ with the differential operator $A$. In this scenario, the
singularities of $A e_G u$ will be of the type $\delta$-functions, and their derivatives,
supported on $\partial G$. In particular, they will disappear when the restriction op-
erator, $r_G$, is applied.

1.2.4 Transmission property

There is a well developed theory, notably [5], of elliptic boundary value prob-
lems for pseudodifferential operators that have the transmission property.
(See also [21].) This covers a wide class of operators, including partial differ-
ential operators.
Assuming $A_G$ is as defined in (1.16), we say that a pseudodifferential operator $A$ satisfies the transmission property on $G$ if

$$A_G : \mathcal{C}^\infty(G) \cap \mathcal{C}_0^\infty(\mathbb{R}^n) \to \mathcal{C}^\infty(G).$$

There is an equivalent condition on the symbol of the operator $A$ described in local coordinate systems in a neighbourhood of $\partial G$.

Unfortunately, the fractional Laplacian does not satisfy the transmission condition. Indeed, let $r_+$ and $e_+$ denote the restriction and extension operators (respectively) in the case $G = (0, \infty) \subset \mathbb{R}$. Then, in Appendix A, we show for a certain function $u \in r_+ C_0^\infty(\mathbb{R})$, that

$$r_+ (\Delta)^\alpha e_+ u \notin L^1_{\text{loc}}(\mathbb{R}_+), \quad \frac{1}{2} < \alpha < 1.$$ 

Even worse, this pathology is characteristic of Feller processes. The generator of a subordinate diffusion does not have the transmission property [26]. So, in our context of Feller processes, the transmission property is not applicable.

### 1.2.5 Modified transmission property

In a series of recent papers, see for example [22, 23], Grubb has proposed a more general transmission property. $A$ is said to satisfy the $\mu$–transmission condition on $G$ if

$$A_G : x_n^\mu \mathcal{C}^\infty(G) \to \mathcal{C}^\infty(G).$$

In Appendix C, we consider the operator, $(I - \Delta)^\alpha$, acting on the domain $G = \mathbb{R}_+^n$, $n \geq 2$. It can be shown that this operator satisfies the $\mu$–transmission condition with $\mu = \alpha$. To accommodate the singularities that arise at the boundary, this approach seeks solutions to the Dirichlet problem in the so-called Hörmander space, $H^{\alpha(t+2\alpha)}(\mathbb{R}_+^n)$, defined for $t \geq 0$, as

$$H^{\alpha(t+2\alpha)}(\mathbb{R}_+^n) := F^{-1}(\langle \xi' \rangle - i\xi_n)^{-\alpha} F(e_+ H^{t+\alpha}(\mathbb{R}_+^n)),$$

where $\xi = (\xi', \xi_n)$.

If $-\frac{1}{2} < t + \alpha < \frac{1}{2}$ then, see Section 2.8.7, p.158, [45], we can identify $e_+ H^{t+\alpha}(\mathbb{R}_+^n)$ with $\tilde{H}^{t+\alpha}(\mathbb{R}_+^n)$. Hence,

$$H^{\alpha(t+2\alpha)}(\mathbb{R}_+^n) = \tilde{H}^{t+2\alpha}(\mathbb{R}_+^n) \quad \text{if} \quad -\frac{1}{2} < t + \alpha < \frac{1}{2}.$$ 

On the other hand, if $t + \alpha > \frac{1}{2}$ then functions from $e_+ H^{t+\alpha}(\mathbb{R}_+^n)$ may have a jump at $x_n = 0$. This gives rise to a singularity at $x_n = 0$ when the operator
\( \mathcal{F}^{-1}(\langle \xi' \rangle - i\xi_n)^{-\alpha} \mathcal{F} \) is applied.

In summary, Grubb presents a useful approach, combining the power of Wiener-Hopf factorisation with a more general transmission condition, that works well for fractional powers of elliptic operators, providing the domain of the operator is assumed to be a Hörmander space. (In this research, we will seek solutions in more conventional function spaces.)

1.2.6 No transmission property

We have seen, in a prior section, that the (unmodified) transmission property is too restrictive. Let us now consider a pseudodifferential operator, \( A \), without the transmission property, but restrict the domain of \( A \) to functions \( u \) such that \( e_Gu \) does not have a “discontinuity” at the boundary \( \partial G \).

To this end, we define:

\[
\widetilde{Y}(G) := \{ u \in Y(\mathbb{R}^n) : \text{supp } u \subseteq \overline{G} \} = \{ u \in Y(\mathbb{R}^n) : u|_{\mathbb{R}^n \setminus \overline{G}} = 0 \}.
\]

Of course, if \( Y(\mathbb{R}^n) = L^p(\mathbb{R}^n) \), we can identify the spaces \( \widetilde{L}^p(G) \) and \( L^p(G) \).

On the other hand, if \( Y(\mathbb{R}^n) = C^m(\mathbb{R}^n) \) where \( m \in \mathbb{N} \cup \{0\} \), then \( u \in \widetilde{C}^m(G) \) implies

\[
\partial^\gamma u|_{\partial G} = 0, \quad |\gamma| \leq m.
\]

(1.17)

In other words, if we take the domain of \( A \) to be \( \widetilde{C}^m(G) \), then (1.17) shows there are some implicit boundary conditions on \( \partial G \).

Nonetheless, given \( A : Y(\mathbb{R}^n) \to Z(\mathbb{R}^n) \), let us consider

\[
r_GA : Y(G) \to Z(G).
\]

(1.18)

The theory of boundary value problems for elliptic pseudodifferential operators without the transmission property has its origins in the work of M.I. Vishik and G.I. Eskin in the 1960’s. The main analytical tool is the Wiener-Hopf method.

Consider the following boundary value problem. Find \( u \in Y(G) \) such that

\[
\begin{align*}
  r_GAu &= f \\
  Bu|_{\partial G} &= g,
\end{align*}
\]

(1.19)
where $B$ is a pseudodifferential operator, $f \in Z(G)$ and $g$ belongs to some appropriate function space on $\partial G$.

Then the boundary value problem (1.19) cannot be well-posed if $B$ is a differential operator. Indeed, suppose $Y(G)$ consists of all functions smooth enough for $Bu|_{\partial G}$ to be well-defined. Then all $\partial G$-traces of the corresponding derivatives of $u$ equal 0, because $u|_{\mathbb{R}^n \setminus G} = 0$.

The properties of the boundary value problem (1.19) depend strongly on the choice of the space $Y(\mathbb{R}^n)$. Indeed, the number of implicit boundary conditions we impose, by assuming that the domain of $A$ is $\tilde{Y}(G)$, increases as the smoothness of the functions in the domain $\tilde{Y}(G)$ increases.

As a consequence, (1.19) is unlikely to be well-posed if $Bu|_{\partial G} = g$ is a (so-called) Wentzell boundary condition from the theory of Feller processes.

### 1.2.7 Balayage Dirichlet problem

In 1938, M. Riesz [36] showed that the correct boundary condition for the Dirichlet problem for the fractional Laplacian is not the customary $u|_{\partial G} = g$ but rather the balayage condition:

\[
\begin{aligned}
(−\Delta)^{\alpha} u &= 0 \quad \text{in } G \\
u|_{\mathbb{R}^n \setminus G} &= g.
\end{aligned}
\]  

(1.20)

Heuristically, the difference between the Laplacian and the fractional Laplacian can be described very simply in terms of stochastic processes. For the Laplacian, inside a smooth boundary, the process paths follow Brownian motion and are therefore continuous (almost surely). On the other hand, the paths for the fractional Laplacian are right continuous with left limits, or càdlàg. In this case, the first hitting of $G^c$ will not occur (almost surely) at the boundary $\partial G$, but rather somewhere in $\overline{G}^c$. That is to say, the process will jump over the boundary (almost surely).

Let us now generalise boundary value problem (1.20) by replacing $(-\Delta)^{\alpha}$ with $A$ to obtain

\[
\begin{aligned}
r_G Av &= 0 \\
v|_{\mathbb{R}^n \setminus G} &= g.
\end{aligned}
\]  

(1.21)

Take any $v_0 \in Y(\mathbb{R}^n)$ satisfying the “boundary condition”

$v_0|_{\mathbb{R}^n \setminus G} = g$. 

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Then \( u := v - v_0 \in Y(\mathbb{R}^n) \) and \( u|_{\mathbb{R}^n \setminus G} = g - g = 0 \). In other words, \( u \in \tilde{Y}(G) \) and (1.21) is equivalent to
\[
r_G Au = f, \quad (f := -r_G Av_0).
\]

So, finally, the balayage Dirichlet problem is simply about the invertibility of the operator
\[
r_G A : \tilde{Y}(G) \to Z(G),
\]
and now there is no contradiction between the implicit and explicit boundary conditions.

### 1.2.8 Symmetric stable Lévy process in \( G \)

The generator of the symmetric \( 2\alpha \)-stable Lévy process in \( \mathbb{R}^n \) is the fractional Laplacian
\[
(-\Delta)^\alpha = \mathcal{F}^{-1}|\xi|^{2\alpha}\mathcal{F}, \quad 0 < \alpha < 1.
\]
Suppose \( u, v \in S(\mathbb{R}) \). Then the Dirichlet form, see [33], associated with \( (-\Delta)^\alpha \) is given by
\[
E^{(\alpha)}(u, v) = \int_{\mathbb{R}^n} (-\Delta)^\alpha u(x) \overline{v(x)} \, dx = \int_{\mathbb{R}^n} |\xi|^{2\alpha} \mathcal{F}u(\xi) \overline{\mathcal{F}v(\xi)} \, d\xi,
\]
or, equivalently,
\[
E^{(\alpha)}(u, v) = \frac{c_{n,\alpha}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))(\overline{v(x)} - \overline{v(y)})}{|x - y|^{n+2\alpha}} \, dx \, dy,
\]
where, see for example [31], the positive constant \( c_{n,\alpha} \) is given by
\[
c_{n,\alpha} = \frac{2^{2\alpha} \Gamma(\alpha + \frac{n}{2})}{\pi^{n/2} \Gamma(-\alpha)}.
\]

Let us now consider the domain \( G \). Dirichlet forms relating to the symmetric stable Lévy process on a domain have been studied by a number of authors. See, for example, Bogdan et al. [4], Chen and Kim [6], Chen and Song [7], Guan and Ma [24, 25]. It turns out that Dirichlet form on \( G \) is given by
\[
E^{(\alpha)}_G(u, v) = \frac{c_{n,\alpha}}{2} \int_{G} \int_{G} \frac{(u(x) - u(y))(\overline{v(x)} - \overline{v(y)})}{|x - y|^{n+2\alpha}} \, dx \, dy. \tag{1.23}
\]

The generator of the Dirichlet form (1.23) is known as the regional fractional Laplacian, and can be written as
\[
\Lambda^\alpha_G u(x) := \lim_{\epsilon \downarrow 0} c_{n,\alpha} \int_{y \in G, |x - y| > \epsilon} \frac{u(x) - u(y)}{|x - y|^{n+2\alpha}} \, dy, \quad x \in G. \tag{1.24}
\]
For more details see, for example, [24, 25]. Of course, if $G$ is the whole of $\mathbb{R}^n$, then $\Lambda^\alpha_{\mathbb{R}^n} = (-\Delta)^\alpha$, and we simply recover equation (1.14).

Using a weak representation of (1.24), Guan and Ma [24] consider a Dirichlet boundary value problem in $G$. They show that, subject to several technical qualifications, there exists a unique (low regularity) solution in $H^\alpha(\overline{G}) \cap C_b(\overline{G})$ for all $\alpha$ in the range $0 < \alpha < 1$, where $C_b(\overline{G})$ denotes the space of continuous bounded functions on $\overline{G}$.

### 1.2.9 Regional fractional Laplacian $\Lambda^\alpha_G$

Let us now compare the regional fractional Laplacian, $\Lambda^\alpha_G$, with the operator $(-\Delta)^\alpha_G = r_G(-\Delta)^\alpha e_G$. To this end, we define

$$\kappa^\alpha_G := \Lambda^\alpha_G - (-\Delta)^\alpha_G. \quad (1.25)$$

Then, for $x \in G$, we have

$$(\kappa^\alpha_G u)(x) = c_{n,\alpha} \lim_{\epsilon \searrow 0} \int_{y \in G, |x-y| > \epsilon} \frac{u(x) - u(y)}{|x-y|^{n+2\alpha}} dy - c_{n,\alpha} \int_{y \in \mathbb{R}^n} \frac{e_G(x) - e_G(y)}{|x-y|^{n+2\alpha}} dy$$

$$= -c_{n,\alpha} \int_{y \in \mathbb{R}^n \setminus G} \frac{e_G(x) - e_G(y)}{|x-y|^{n+2\alpha}} dy$$

$$= -c_{n,\alpha} u(x) \int_{y \in \mathbb{R}^n \setminus G} \frac{1}{|x-y|^{n+2\alpha}} dy,$$

since if $y \in G$ then $e_G(y) = u(y)$, and $e_G(y) = 0$ if $y \in \mathbb{R}^n \setminus G$.

In other words, we can consider the regional fractional Laplacian to be

$$\Lambda^\alpha_G = (-\Delta)^\alpha_G + \kappa^\alpha_G. \quad (1.26)$$

where the singular potential

$$\kappa^\alpha_G(x) = -c_{n,\alpha} \int_{y \in \mathbb{R}^n \setminus G} \frac{1}{|x-y|^{n+2\alpha}} dy, \quad x \in G. \quad (1.27)$$

### 1.2.10 Boundary value problems for $\Lambda^\alpha_G$

It is worth noting, given (1.26), that the regional fractional Laplacian resembles the Schrödinger operator

$$H = \Delta + V,$$
for some potential $V$. This might suggest that a good approach for solving a boundary value problem for the regional fractional Laplacian would be to start by analysing the operator $(-\Delta)_G^\alpha$, and then consider the added potential term as a perturbation. However, as we have seen, this does not work since the only well-posed and useful “probabilistic” boundary value problem for $(-\Delta)_G^\alpha$ is the balayage Dirichlet problem.

In fact, the more useful method is to consider $\Lambda_G^\alpha$ as the unperturbed operator. It is less than singular than both $(-\Delta)_G^\alpha$ and $\kappa_G^\alpha$ because their leading singular terms cancel at the boundary $\partial G$ when they are added together. (See Appendix A for full details.) The cancellation of singularities does not happen if $\kappa_G^\alpha$ is multiplied by a constant different from 1. So that, for example, $\Lambda_G^\alpha + 2\kappa_G^\alpha$ is as singular as $\Lambda_G^\alpha$ and $\kappa_G^\alpha$.

In summary, $\Lambda_G^\alpha$ can be defined on a larger function space than $(-\Delta)_G^\alpha$ and, therefore, it is possible to remove at least some of the implicit homogeneous boundary conditions discussed earlier.

Finally, if $x \in G$, then, from equation (1.27),

$$\kappa_G^\alpha(x) = -c_{n,\alpha} \int_{\mathbb{R}^n} \frac{\chi_{\mathbb{R}^n \setminus G}(y)}{|x - y|^{n+2\alpha}} \, dy$$

$$= c_{n,\alpha} \int_{\mathbb{R}^n} \frac{(\chi_{\mathbb{R}^n \setminus G}(x) - \chi_{\mathbb{R}^n \setminus G}(y))}{|x - y|^{n+2\alpha}} \, dy$$

$$= (-\Delta)^\alpha(\chi_{\mathbb{R}^n \setminus G})(x),$$

where $\chi_{\mathbb{R}^n \setminus G}$ denotes the characteristic function of the set $\mathbb{R}^n \setminus G$.

Hence, we can write the regional fractional Laplacian, purely in terms of $(-\Delta)^\alpha$, as

$$\Lambda_G^\alpha = r_G(-\Delta)^\alpha e_G + r_G((-\Delta)^\alpha \chi_{\mathbb{R}^n \setminus G}) I. \quad (1.28)$$

**Remark 1.3.** In the preceding analysis, we have made various attempts to answer the central question concerning the nature of a pseudodifferential operator $A$ acting on a domain $G$. Our conclusion, and essentially the beginning of this research, is that the formulation described in equation (1.28) offers many advantages. It is interesting to note that this formulation arises in a natural way in the theory of Lévy processes. On the other hand, it could
be argued that insufficient progress has been made using the theory of partial differential equations on dealing with (discontinuous) Lévy processes in domains.

1.2.11 Transcendental equation

Later in the research we will establish that, for a particular choice of the pseudodifferential operator $A$, the corresponding operator defined (effectively) using (1.28) is Fredholm. This analysis depends critically on the examination of a certain transcendental equation that depends on $\alpha$, $p$, $s$ and a variable $\xi$ which ranges over $\mathbb{R}$. (For more details, see Chapter 9, equations (9.3), (9.4) and (9.5).)

Although, in general, this transcendental equation involves complex-valued arguments, the special case when all arguments are real, corresponding to $\xi = 0$, turns out to be particularly important. (See equation (9.7).) Intriguingly, this simplified transcendental equation also arises, albeit in a slightly different context, when $A$ is taken to be the fractional Laplacian.

Indeed, let us now consider equation (1.28), in the case $n = 1$ and $G = \mathbb{R}_+$. We denote $r_G$, $e_G$, $\Lambda_G^\alpha$ by $r_+$, $e_+$ and $\Lambda_+^\alpha$ respectively. Let $\theta$ denote the Heaviside step function.

Then, see equation (2.34), p. 23, [14],

$$F_{x \to \xi} e_+ x^\lambda = e^{i\pi(\lambda+1)/2} \cdot \frac{\Gamma(\lambda + 1)}{\sqrt{2\pi}} \cdot (\xi + i0)^{-\lambda-1}, \quad \lambda > -1,$$

where $(\xi + i0)^\mu := |\xi|^\mu e^{i\mu \pi \theta(-\xi)}$, for any $\mu \in \mathbb{R}$.

Noting that $\text{sgn}(\xi) = 2 \theta(\xi) - 1$, we can write

$$Fe_+ x^\lambda = \frac{\Gamma(\lambda + 1)}{\sqrt{2\pi}} \left\{ - \sin \frac{\lambda \pi}{2} + i \text{sgn}(\xi) \cos \frac{\lambda \pi}{2} \right\} \cdot |\xi|^{-\lambda-1},$$

and hence deduce

$$F|x|^\lambda = -\frac{2 \Gamma(\lambda + 1)}{\sqrt{2\pi}} \sin \frac{\lambda \pi}{2} \cdot |\xi|^{-\lambda-1}.$$ 

Thus, after some calculation, we have the simple result that

$$r_+ (\Delta)^\alpha e_+ x^\lambda = \left( \frac{\Gamma(2\alpha - \lambda) \Gamma(1 + \lambda) \sin \pi (\alpha - \lambda)}{\pi} \right) \cdot x^{\lambda-2\alpha}, \quad (1.29)$$
and

\[ r_+(-\Delta)^\alpha \chi_{\mathbb{R}_\pm} = \left( \pm \frac{\Gamma(2\alpha) \sin \pi \alpha}{\pi} \right) \cdot x^{-2\alpha}. \]

Hence, from equation (1.28),

\[ \Lambda_+^\alpha x_+ = r_+(-\Delta)^\alpha \epsilon_+ x_+ + x_+^{\lambda r_+}((-\Delta)^\alpha \chi_{\mathbb{R}_-}) = 0, \]

if and only if

\[ \Gamma(2\alpha - \lambda)\Gamma(1 + \lambda) \sin \pi(\alpha - \lambda) = \Gamma(2\alpha) \sin \pi \alpha. \quad (1.30) \]

Of course, as expected, given the homogeneity of \(|\xi|^{2\alpha}\), we have the immediate solution \(\lambda = 0\).

Later, in this research, we consider an “inhomogeneous” variant of the fractional Laplacian. We shall see that calculations involving the Fourier transform are considerably more complex. However, the (simplified) transcendental equation (1.30) will reappear and, moreover, will play a critical role. (See Chapter 9 and, particularly, equation (9.7).)
1.2.12 Problem statement

Equation (1.28) provides the central motivation for this current research. Indeed, given a pseudo-differential operator $A$, acting on $\mathbb{R}^n$, we will, in general, consider the operator

$$A_G := r_G Ae_G + r_G(A\chi_{\mathbb{R}^n \setminus G})I. \quad (1.31)$$

We shall demonstrate now that, under certain conditions, the operator $r_G Ae_G + (r_G A\chi_{\mathbb{R}^n \setminus G})I$ is “less singular” than $r_G Ae_G$ itself. Let $U \in Y(\mathbb{R}^n)$ be any extension of $u \in Y(G)$. Then, by definition, $r_G U = u$ and

$$r_G Ae_G u + (r_G A\chi_{\mathbb{R}^n \setminus G})u = r_G Ae_G u + (r_G A\chi_{\mathbb{R}^n \setminus G})r_G U$$

$$= r_G Ae_G u + r_G(UA\chi_{\mathbb{R}^n \setminus G})$$

$$= r_G(Ae_G u + UA\chi_{\mathbb{R}^n \setminus G})$$

$$= r_G(A(U\chi_G) + A(U\chi_{\mathbb{R}^n \setminus G}) + [UI, A] \chi_{\mathbb{R}^n \setminus G})$$

$$= r_G(AU + [UI, A] \chi_{\mathbb{R}^n \setminus G}),$$

where $[UI, A]$ is the commutator given by

$$[UI, A] = UA - A(UI).$$

Of course, $[UI, A]$ is of lower order than $A$, if $U$ has a degree of smoothness.

In summary, the perturbed operator has the simple form

$$r_G Ae_G u + (r_G A\chi_{\mathbb{R}^n \setminus G})u = r_G(AU + [UI, A] \chi_{\mathbb{R}^n \setminus G}). \quad (1.32)$$

Moreover, it can be determined purely in terms of a restriction, to the domain $G$, of an appropriate interaction between $A$ and $U$ over $\mathbb{R}^n$.

Remark 1.4. It is easy to see that the representation on the right-hand side of (1.32) is independent of the extension $U$. Indeed, suppose that $U_1, U_2 \in Y(\mathbb{R}^n)$ are two extensions of $u \in Y(G)$. Then, by definition, $r_G U_1 = u = r_G U_2$. Moreover,

$$r_G(AU_1 + [U_1 I, A] \chi_{\mathbb{R}^n \setminus G}) - r_G(AU_2 + [U_2 I, A] \chi_{\mathbb{R}^n \setminus G})$$

$$= r_G(A(U_1 - U_2) + (U_1 - U_2)A\chi_{\mathbb{R}^n \setminus G}) - r_G(A(U_1 - U_2)\chi_{\mathbb{R}^n \setminus G})$$

$$= r_G(A(U_1 - U_2) - r_G A(U_1 - U_2)\chi_{\mathbb{R}^n \setminus G})$$

$$= r_G(A(U_1 - U_2)\chi_G$$

$$= 0.$$
The genesis of this research is the simple realisation that adding the potential term, $r_G(A\chi_{\mathbb{R}^n\setminus\partial G})$, actually does improve the situation. This is because the leading singular terms of $r_G Ae_G$ and $r_G(A\chi_{\mathbb{R}^n\setminus\partial G})I$ cancel each other at $\partial G$. Appendix A details a simple illustrative example of this phenomenon.

So, the resulting (perturbed) operator $A_G$ is “less singular” than either the truncated operator $r_G Ae_G$ or the multiplier $r_G(A\chi_{\mathbb{R}^n\setminus\partial G})$ when viewed separately. (In passing, we note that the cancellation of singularities will not occur if the coefficient of $r_G(A\chi_{\mathbb{R}^n\setminus\partial G})u$ is a constant different from 1.)

Given the above analysis, the underlying problem, in abstract terms, is to develop a theory of boundary value problems for operators which are sums of pseudodifferential operators and “fine-tuned potentials”, which when taken together are less singular than the corresponding pseudodifferential operators considered on their own. In this research, to make the problem more tractable, we shall consider a particular elliptic pseudodifferential operator chosen because it possesses two important characteristics, namely:

(a) All the richness intrinsic to the fractional Laplacian;

(b) Representative of a large class of operators.

For additional simplicity, we shall restrict our attention to one spatial dimension. However, it is worth remarking that even one-dimensional problems of this kind can have important applications in various fields including financial mathematics (non-Gaussian market models).

Finally, suppose $0 < \alpha < 1$. Let $A$ denote the pseudodifferential operator of order $2\alpha$, with symbol

$$A(\xi) = (1 + \xi^2)^\alpha.$$  \hspace{1cm} (1.33)

Our (simplified) problem is to investigate the solvability of the equation

$$Au := r_+ Ae_+ u + u r_+ A(\chi_{\mathbb{R}_-}) = f,$$  \hspace{1cm} (1.34)

where $u \in H^s_p(\mathbb{R}_+)$ for a given $f \in H^{s-2\alpha}_p(\mathbb{R}_+)$ where we shall assume that

$$1 < p < \infty.$$  \hspace{1cm} (1.35)

Remark 1.5. By definition, the operator $A$, and hence $A$, depend on the (fixed) parameter $\alpha$. For ease of reading, and in common with many of our key references (e.g. [14]), we shall suppress this dependence in our notation.
1.2.13 Outline of this research

From equations (1.33) and (1.34), we note our operator of interest, $A$, is defined via the Fourier transform. In Chapter 2, we formulate $A$, acting on a restriction of the Schwartz space $S(\mathbb{R})$, as a singular integral. This representation plays a part in Chapter 3, where we establish conditions under which the operator $A$, acting on Bessel potential spaces, is bounded. Moreover, at least for the case $p = 2$, we also determine sufficient conditions for $A$ to have a trivial kernel. Later, in Chapters 7 and 8, this latter result is generalised to any $p$ satisfying the constraint $1 < p < \infty$.

We begin with the case of lower regularity, where $1/p < s < 1 + 1/p$. In Chapters 4 and 5, we reformulate the problem in $L_p(\mathbb{R}_+)$, in terms of an operator algebra containing multiplication, Mellin and Wiener-Hopf operators. Our goal is to establish precise conditions under which $A$ is Fredholm, and then calculate its index. A significant part of the index calculation, see Chapter 6, involves a certain transcendental equation, which includes terms containing gamma functions with complex arguments.

The analysis for $1/p < s < 1 + 1/p$ is completed in Chapter 7, where we determine the range of parameters for which the Fredholm index is zero. Given this, and the trivial kernel results, we are able to establish the conditions under which $A$ is invertible. Of course, Fredholm theory in the multi-dimensional case requires invertibility of the one-dimensional operator. (Indeed suppose, for example, that the one-dimensional operator has a non-trivial but finite dimensional kernel. An element of this kernel can be defined with a finite number of arbitrary constants. Hence, the corresponding $n-$dimensional operator may not be Fredholm, as allowing the first $(n - 1)$ variables to range over $\mathbb{R}^{n-1}$ may, in turn, create a finite number of arbitrary functions that could give rise to a kernel of infinite dimension over $\mathbb{R}^n$.)

In Chapter 8, the case of higher regularity, namely $1 + 1/p < s < 2 + 1/p$, is examined using the methods established previously. To improve readability, and also because of its significant technical complexity, we delay detailed examination of the transcendental equation until Chapter 9. Finally, areas of possible future research are discussed in Chapter 10.
1.3 Key results

Suppose \( \epsilon > 0 \), and let \( \chi_\epsilon \) denote the characteristic function of the interval \((\epsilon, \infty)\). Let the space \( H^s_{p,0}(\mathbb{R}_+) \) be as defined in (1.3).

The operator \( A \) can be represented as a (hyper-)singular integral operator.

**Theorem 1.6.** Suppose \( 0 < \alpha < 1 \). Let \( v \in S(\mathbb{R}) \) and define \( u := r_d v \).

Then, for \( x > 0 \),

\[
(\mathcal{A}u)(x) = u(x) + \lim_{\epsilon \to 0} \int_0^\infty (u(x) - u(y)) \chi_\epsilon(|x - y|) m_\alpha(|x - y|) dy. \tag{1.36}
\]

Moreover,

\[
(\mathcal{A}u, u) = \int_0^\infty |u|^2 dx + \frac{1}{2} \int_0^\infty \int_0^\infty |u(x) - u(y)|^2 m_\alpha(|x - y|) dy dx,
\]

where

\[
m_\alpha(w) = \frac{\alpha}{\Gamma(1 - \alpha)} \frac{2^{1+\alpha}}{\sqrt{\pi}} |w|^{-\frac{1}{2} - \alpha} K_{\frac{1}{2}+\alpha}(|w|),
\]

and \( K_\nu \) is a modified Bessel function of the second kind of order \( \nu \). Finally, \( m_\alpha(w) \) is \( O(|w|^{-1-2\alpha}) \) for small \( |w| \) and \( O(e^{-|w|}) \) as \( |w| \to \infty \).

Under certain conditions \( A \) is bounded and has a trivial kernel.

**Theorem 1.7.** Suppose \( 1 < p < \infty \) and \( 0 < \alpha < 1 \).

(a) If \( 2\alpha - 1 + 1/p < s < 1 + 1/p \) then \( A : H^s_p(\mathbb{R}_+) \to H^{s-2\alpha}_p(\mathbb{R}_+) \) is bounded.

(b) If \( 1 + 1/p < s < 2 + 1/p \) then \( A : H^s_{p,0}(\mathbb{R}_+) \to H^{s-2\alpha}_p(\mathbb{R}_+) \) is bounded.

Moreover, if \( p = 2 \) and \( 0 < \alpha < \frac{1}{2}, \frac{1}{2} < s < 1 + \frac{1}{2} \), then \( A : H^s_2(\mathbb{R}_+) \to H^{s-2\alpha}_2(\mathbb{R}_+) \) has a trivial kernel. Similarly, if \( p = 2 \) and \( 0 < \alpha < 1, 1 + \frac{1}{2} < s < 2 + \frac{1}{2} \) then \( A : H^s_{2,0}(\mathbb{R}_+) \to H^{s-2\alpha}_2(\mathbb{R}_+) \) has a trivial kernel.

**Remark 1.8.** The condition that \( p = 2 \) for \( A \) to have a trivial kernel is not as restrictive as it might appear. Under appropriate conditions, we will be able to determine sufficient conditions for \( A \) to have a trivial kernel for any \( p \) in the range \( 1 < p < \infty \), using the result (above) for \( p = 2 \).
Let \( \tau := s - 1/p \). Then, it turns out that the following transcendental equation, see (1.30), will play a pivotal role in our analysis:

\[
\Gamma(2\alpha - \tau) \Gamma(\tau + 1) \sin \pi (\alpha - \tau) = \Gamma(2\alpha) \sin \pi \alpha.
\]

Indeed, if \( 0 < \alpha < \frac{1}{2} \) and \( 0 < \tau < 1 \), we prove that equation (1.30) has no solution. On the other hand, if \( 0 < \alpha < 1 \) and \( 1 < \tau < 2 \), we prove that equation (1.30) has a unique solution of the form \( \tau = 1 + \alpha_c \), where \( \alpha_c \) only depends on \( \alpha \) and satisfies \( 0 < \alpha_c < \alpha \).

Finally, via a calculation of the Fredholm index, we establish conditions for the invertibility of \( \mathcal{A} \).

**Theorem 1.9.** Suppose \( 0 < \alpha < \frac{1}{2} \), \( 1 < p < \infty \) and \( 1/p < s < 1 + 1/p \). Then the operator \( \mathcal{A} : H^s_p(\mathbb{R}_+) \to H^{s-2\alpha}_p(\mathbb{R}_+) \) is invertible.

**Theorem 1.10.** Suppose \( 0 < \alpha < 1 \), \( 1 < p < \infty \) and \( 1 + 1/p < s < 1 + 1/p + \alpha_c \). Then the operator \( \mathcal{A} : H^s_{p,0}(\mathbb{R}_+) \to H^{s-2\alpha}_p(\mathbb{R}_+) \) is invertible.

On the other hand, if \( 0 < \alpha < 1 \), \( 1 < p < \infty \) and \( 1 + 1/p + \alpha_c < s < 2 + 1/p \), then \( \mathcal{A} \) has a trivial kernel and is Fredholm with index equal to \(-1\).
Chapter 2

Singular integral representation

In this chapter we examine the operator \( \mathcal{A} \) given in equation (1.34) in more detail. We consider its action on the restriction of the Schwartz space \( S(\mathbb{R}) \) to the positive half-line, and formulate a singular integral representation.

Suppose \( \epsilon > 0 \), and let \( \chi_\epsilon \) denote the characteristic function of the interval \( (\epsilon, \infty) \).

2.1 Main result

Theorem 2.1. Suppose \( 0 < \alpha < 1 \). Let \( v \in S(\mathbb{R}) \) and define \( u := r_+ v \). Then, for \( x > 0 \),

\[
(\mathcal{A}u)(x) = u(x) + \lim_{\epsilon \searrow 0} \int_0^\infty (u(x) - u(y)) \chi_\epsilon(|x - y|) m_\alpha(|x - y|) \, dy. \tag{2.1}
\]

Moreover,

\[
(\mathcal{A}u, u) = \int_0^\infty |u|^2 \, dx + \frac{1}{2} \int_0^\infty \int_0^\infty |u(x) - u(y)|^2 m_\alpha(|x - y|) \, dy \, dx, \tag{2.2}
\]

where

\[
m_\alpha(w) = \frac{\alpha}{\Gamma(1 - \alpha)} \frac{2^{\frac{1}{2} + \alpha}}{\sqrt{\pi}} |w|^{-\frac{1}{2} - \alpha} K_{\frac{1}{2} + \alpha}(|w|),
\]

and \( K_\nu \) is a modified Bessel function of the second kind of order \( \nu \). Finally, \( m_\alpha(w) \) is \( O(|w|^{-1 - 2\alpha}) \) for small \( |w| \) and \( O(e^{-|w|}) \) as \( |w| \to \infty \).

Remark 2.2. Consider the integral operator representation for \( \mathcal{A} \) given by equation (2.1) in Theorem 2.1. Suppose \( x > 0 \) is fixed and let \( \epsilon \searrow 0 \). Then,
in a neighbourhood of $x$ the integrand has a singularity which is (generally) of order $-2\alpha$. In particular, if $\frac{1}{2} < \alpha < 1$ then the integral is hypersingular. Nonetheless, even in this case, the limit as $\epsilon \downarrow 0$ does exist and is finite. It turns out that this is due to a cancellation, arising from the fact that the weight $m_\alpha$ is symmetric about $x$.

On the other hand, the double integral in equation (2.2) in the inner product $\langle Au, u \rangle$ has a weaker singularity of order $-2\alpha + 1$. We will show that, for all $0 < \alpha < 1$, the double integral exists and is finite.

**Definition 2.3.** Suppose $0 < \epsilon < 1$. Then we define

$$m_{\alpha, \epsilon}(y) := \chi_\epsilon(y)m_\alpha(y), \quad y \geq 0.$$  

In particular, given $\epsilon$, the function $m_{\alpha, \epsilon}$ is bounded. Given $m_{\alpha, \epsilon}$, we further define

$$(\mathcal{A}_\epsilon u)(x) := u(x) + \int_0^\infty (u(x) - u(y))m_{\alpha, \epsilon}(|x - y|) \, dy \quad (x > 0).$$

### 2.2 Supporting lemmas

An infinitely differentiable function $f : (0, \infty) \to \mathbb{R}$ with $f \geq 0$ is said to be a Bernstein function if

$$(-1)^k \frac{d^k f}{dx^k} \leq 0, \quad k = 1, 2, 3, \ldots,$$

and completely monotone if

$$(-1)^k \frac{d^k f}{dx^k} \geq 0, \quad k = 1, 2, 3, \ldots.$$  

It is easy to verify directly from this definition that $f(x) = (1 + x)^{\alpha}$ is a Bernstein function if $0 < \alpha < 1$.

Any Bernstein function $g : (0, \infty) \to \mathbb{R}$ can be written in the standard Lévy-Khinchine representation, see equation (12), p. 6, [28],

$$g(x) = a + bx + \int_0^\infty (1 - e^{-xs}) \tau(ds),$$

where $\tau$ is a Radon measure on $(0, \infty)$ such that $\int_{0^+}^\infty \min\{s, 1\} \tau(ds) < \infty$.  

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From 3.434, p. 361, [17], we have
\[
\int_0^\infty \frac{e^{-\nu s} - e^{-\mu s}}{s^{\rho+1}} ds = \frac{\mu^\rho - \nu^\rho}{\rho} \Gamma(1 - \rho),
\]
provided \(\mu > 0, \nu > 0\) and \(\rho < 1\). Taking \(\nu = 1, \mu = 1 + x\) and \(\rho = \alpha\) gives
\[
\int_0^\infty \frac{e^{-s} - e^{-(1+x)s}}{s^{\alpha+1}} ds = \frac{(1 + x)^\alpha - 1}{\alpha} \Gamma(1 - \alpha).
\]
Rearranging
\[
(1 + x)^\alpha = 1 + \frac{\alpha}{\Gamma(1 - \alpha)} \int_0^\infty (1 - e^{-xs}) e^{-s^\alpha - 1} ds.
\]
In other words, in the standard form representation of the Bernstein function \((1 + x)^\alpha\), for \(0 < \alpha < 1\), we take \(a = 1, b = 0\) and
\[
\tau(ds) = \frac{\alpha}{\Gamma(1 - \alpha)} e^{-s^\alpha - 1} ds.
\]

**Remark 2.4.** If we take \(x = \lambda/m^{1/\alpha}\) in equation (2.3), and make the change of variable \(s \to sm^{1/\alpha}\) in the right-hand side, we obtain the following result:
\[
(\lambda + m^{1/\alpha})^\alpha - m = \frac{\alpha}{\Gamma(1 - \alpha)} \int_0^\infty (1 - e^{-\lambda s}) e^{-m^{1/\alpha} s^{\alpha - 1}} ds,
\]
as given in Example 5.9, p. 97, [18], for (so-called) relativistic stable subordinators. For more information see Appendix B.

We say that a function \(\psi : \mathbb{R} \to \mathbb{R}\) is **negative definite** if for all \(N \in \mathbb{N}\) and \(\xi_1, \xi_2, \ldots, \xi_N \in \mathbb{R}\) we have
\[
\psi(0) \geq 0; \quad \sum_{j,k=1}^N \psi(\xi_j - \xi_k) \lambda_j \lambda_k \leq 0, \quad \forall \lambda_j \in \mathbb{C} \text{ such that } \sum_{j=1}^N \lambda_j = 0.
\]
In particular, we note that \(\psi(\xi) = \xi^2\) is negative definite.

Now since \((1 + x)^\alpha\) is a Bernstein function, we see immediately that \((1 + \xi^2)^\alpha\), for \(0 < \alpha < 1\), is also a continuous negative definite function. See, for example, [28]. Moreover, from equation (2.3) and Lemma 2.1, p. 7, [28], we have the general representation
\[
(1 + \xi^2)^\alpha = 1 + \int_\mathbb{R} (1 - \cos y \xi) m_\alpha(y) \, dy.
\]
Lemma 2.5. Suppose $s > 0$, then

$$1 - e^{-s\xi^2} = \int_{\mathbb{R}} (1 - \cos \xi y) \frac{1}{\sqrt{4\pi s}} \exp \left( -\frac{y^2}{4s} \right) dy.$$

Proof. By a change of variable from the standard formula $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$, we have

$$\int_0^\infty \exp(-q^2 y^2) dy = \frac{\sqrt{\pi}}{2q}, \quad q > 0.$$

Hence

$$\int_{\mathbb{R}} \frac{1}{\sqrt{4\pi s}} \exp \left( -\frac{y^2}{4s} \right) dy = 2 \int_0^\infty \frac{1}{\sqrt{4\pi s}} \exp \left( -\frac{y^2}{4s} \right) dy = 1.$$

Therefore, it remains to show that

$$e^{-s\xi^2} = \int_{\mathbb{R}} \cos \xi y \cdot \frac{1}{\sqrt{4\pi s}} \exp \left( -\frac{y^2}{4s} \right) dy.$$

But from 3.896 2, p. 488, [17], we have

$$\int_{\mathbb{R}} e^{-q^2 y^2} \cos[p(y + \lambda)] dy = \frac{\sqrt{\pi}}{q} e^{-\frac{p^2}{4q^2}} \cos p\lambda.$$

So, taking $q = 1/(2\sqrt{s})$, $p = \xi$ and $\lambda = 0$

$$\int_{\mathbb{R}} \cos \xi y \cdot \exp \left( -\frac{y^2}{4s} \right) dy = \sqrt{\pi} \cdot 2\sqrt{s} \exp \left( -\frac{\xi^2}{4} \right)$$

$$= \sqrt{4\pi s} \exp (-s\xi^2), \quad \text{as required.}$$

Lemma 2.6. Suppose $0 < \alpha < 1$. Then

$$(1 + \xi^2)^\alpha = 1 + \int_{\mathbb{R}} (1 - \cos y\xi) m_\alpha(y) dy,$$

where

$$m_\alpha(y) = \frac{\alpha}{\Gamma(1 - \alpha)} \frac{2^{\frac{1}{2} + \alpha}}{\sqrt{\pi}} |y|^{\frac{1}{2} - \alpha} K_{\frac{1}{2} + \alpha}(|y|),$$

and $K_\nu$ is a modified Bessel function of the second kind of order $\nu$. See, for example, Chapter 10, [34].
Proof. From equation (2.3)

$$(1 + x)^\alpha = 1 + \frac{\alpha}{\Gamma(1 - \alpha)} \int_0^\infty (1 - e^{-xs}) e^{-s} s^{-\alpha - 1} ds.$$ 

Now define

$$\tau(s) := \frac{\alpha}{\Gamma(1 - \alpha)} e^{-s} s^{-\alpha - 1},$$

so that

$$(1 + \xi^2)^\alpha = 1 + \int_0^\infty (1 - e^{-s\xi^2}) \tau(s) ds.$$ 

From Lemma 2.5, we have

$$1 - e^{-s\xi^2} = \int_{\mathbb{R}} (1 - \cos \xi y) \frac{1}{\sqrt{4\pi s}} \exp \left( - \frac{y^2}{4s} \right) dy.$$

Hence

$$(1 + \xi^2)^\alpha = 1 + \int_0^\infty \int_{\mathbb{R}} (1 - \cos \xi y) \frac{1}{\sqrt{4\pi s}} \exp \left( - \frac{y^2}{4s} \right) \tau(s) dy ds = 1 + \int_{\mathbb{R}} (1 - \cos \xi y) \{ \int_0^\infty \frac{1}{\sqrt{4\pi s}} \exp \left( - \frac{y^2}{4s} \right) \tau(s) ds \} dy.$$ 

From 3.478 4, p. 372, [17], we have

$$\int_0^\infty x^{\nu-1} \exp(-\beta x^p - \gamma x^{-p}) \, dx = \frac{2}{p} \left( \frac{\gamma}{\beta} \right)^{\frac{\nu}{2}} K_p(2\sqrt{\beta\gamma}),$$

provided $\beta, \gamma > 0$ and $p \neq 0$. We take $p = 1$, $\beta = 1$, $\gamma = y^2/4$ and $\nu = -(\alpha + \frac{1}{2})$. Hence

$$\int_0^\infty s^{-(\alpha + \frac{1}{2})-1} \exp \left( - s - \frac{y^2}{4s} \right) \, ds = 2 \left( \frac{y^2}{4} \right)^{-(\alpha + \frac{1}{2})} K_{-(\alpha + \frac{1}{2})}(|y|).$$

So, finally

$$m_\alpha(y) = \frac{\alpha}{\Gamma(1 - \alpha)} \frac{1}{\sqrt{4\pi}} 2^{2\alpha + \frac{1}{2}} |y|^{-\frac{1}{2} - \alpha} K_{-(\alpha + \frac{1}{2})}(|y|)$$

$$= \frac{\alpha}{\Gamma(1 - \alpha)} \frac{1}{\sqrt{\pi}} 2^{\frac{\alpha}{2} - \alpha} |y|^{-\frac{\alpha}{2} - \alpha} K_{\frac{1}{2} + \alpha}(|y|),$$

noting that $K_\nu(x) = K_{-\nu}(x)$ for $x > 0$, $\nu \in \mathbb{R}$. (See 10.27.3, [34].)

\[\square\]
We now give a detailed consideration of the function $m_\alpha$.

By Lemma 2.6, for $y > 0$

$$m_\alpha(y) = c_\alpha y^{-\frac{1}{2}+\alpha} K_{\frac{1}{2}+\alpha}(y),$$

where the constant $c_\alpha$ only depends on $\alpha$. From 10.25.2, 10.27.4 and 10.31.1 [34], $m_\alpha(y) \in C^\infty([1, \infty))$. Moreover, from 10.40.2, [34], for $y \geq \frac{1}{2}$, the function $m_\alpha(y)$, together with its derivatives, is bounded and $O(e^{-y})$ as $y \to \infty$.

On the other hand, if $\alpha \neq \frac{1}{2}$ then, from 10.25.2 and 10.27.4, [34]

$$m_\alpha(y) = c_\alpha y^{-\frac{1}{2}-\alpha} \left(y^{-\frac{1}{2}-\alpha} \phi_\alpha(y) + y^{\frac{1}{2}+\alpha} \psi_\alpha(y)\right)$$

$$= c_\alpha \left(y^{-1-2\alpha} \phi_\alpha(y) + \psi_\alpha(y)\right),$$

where $\phi_\alpha, \psi_\alpha \in C^\infty([0, 2])$.

Similarly, for $\alpha = \frac{1}{2}$, from 10.31.1, [34]

$$m_\alpha(y) = c_1 y^{-1} K_1(y)$$

$$= c_1 y^{-1} \left(y^{-1} \phi_\frac{1}{2}(y) + \vartheta(y) \log y + \psi_\frac{1}{2}(y)\right)$$

$$= c_1 y^{-2} \left(\phi(y) + y \vartheta(y) \log y\right),$$

where $\phi, \vartheta \in C^\infty([0, 2])$.

Let $\varphi \in C^\infty_0(\mathbb{R})$ be such that

$$\varphi(x) = \begin{cases} 1 & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| > 2. \end{cases}$$

If $\alpha \neq \frac{1}{2}$ then

$$m_\alpha(y) = \varphi(y)m_\alpha(y) + (1 - \varphi(y))m_\alpha(y)$$

$$= \varphi(y) \left(c_\alpha y^{-1-2\alpha} \phi_\alpha(y) + \psi_\alpha(y)\right) + (1 - \varphi(y))m_\alpha(y)$$

$$= y^{-1-2\alpha} \left[c_\alpha \varphi(y) \phi_\alpha(y)\right] + \left[c_\alpha \varphi(y) \psi_\alpha(y) + (1 - \varphi(y))m_\alpha(y)\right],$$

with a similar result for $\alpha = \frac{1}{2}$.

Given the above analysis, the following remark details the essential characteristics of the function $m_\alpha$. 


Remark 2.7. From Lemma 2.5, it is easy to see that $m_\alpha(y) = m_\alpha(|y|)$ and $m_\alpha(y) > 0$ for all finite $y$. Moreover, for $y > 0$

$$m_\alpha(y) = \begin{cases} y^{-1-2\alpha}\phi_1(y) + \phi_2(y), & \alpha \neq \frac{1}{2} \\ y^{-2}(\phi_3(y) + y\phi_4(y)\log y), & \alpha = \frac{1}{2} \end{cases}$$

where $\phi_1, \phi_2, \phi_3, \phi_4 \in C^\infty(\mathbb{R})$ and, together with their derivatives, are bounded and $O(e^{-y})$ as $y \to \infty$.

Finally, for $0 < \alpha < 1$, we have $m_\alpha(y) = O(|y|^{-1-2\alpha})$ for small $|y|$ and $m_\alpha(y) = O(e^{-|y|})$ as $|y| \to \infty$.

We will refer to Remark 2.7 several times both in this chapter and Chapter 3, where it will play a central role on the discussion on boundedness of the operator $\mathcal{A}$.

Lemma 2.8. Suppose $v \in S(\mathbb{R})$ and (fixed) $x \in \mathbb{R}$. Then

$$\int_\mathbb{R} (v(x) - v(y)) m_{\alpha,\epsilon}(|x-y|) \, dy = -\frac{1}{2} \int_\mathbb{R} (v(x+y) + v(x-y) - 2v(x)) m_{\alpha,\epsilon}(|y|) \, dy,$$

where $m_{\alpha,\epsilon}(y)$ is given in Definition 2.3.

Proof. Suppose $v \in S(\mathbb{R})$ and (fixed) $x \in \mathbb{R}$. Then

$$\int_\mathbb{R} (v(x) - v(y)) m_{\alpha,\epsilon}(|x-y|) \, dy$$

$$= -\int_\mathbb{R} (v(y) - v(x)) m_{\alpha,\epsilon}(|x-y|) \, dy$$

$$= -\int_\mathbb{R} (v(x+z) - v(x)) m_{\alpha,\epsilon}(|z|) \, dz \quad (z = y - x).$$

But

$$\int_\mathbb{R} (v(x+z) - v(x)) m_{\alpha,\epsilon}(|z|) \, dz$$

$$= \int_\mathbb{R} (v(x-w) - v(x)) m_{\alpha,\epsilon}(|w|) \, dw \quad (w = -z)$$

$$= \int_\mathbb{R} (v(x-z) - v(x)) m_{\alpha,\epsilon}(|z|) \, dz \quad (z = w).$$
Hence,

\[ \int_\mathbb{R} (v(x) - v(y)) m_{\alpha,\epsilon}(|x - y|) dy \]

\[ = -\frac{1}{2} \left\{ \int_\mathbb{R} (v(x + z) - v(x)) m_{\alpha,\epsilon}(|z|) dz + \int_\mathbb{R} (v(x - z) - v(x)) m_{\alpha,\epsilon}(|z|) dz \right\} \]

\[ = -\frac{1}{2} \left\{ \int_\mathbb{R} (v(x + z) + v(x - z) - 2v(x)) m_{\alpha,\epsilon}(|z|) dz \right\} \]

\[ = -\frac{1}{2} \left\{ \int_\mathbb{R} (v(x + y) + v(x - y) - 2v(x)) m_{\alpha,\epsilon}(|y|) dy \right\}. \]

\[ \square \]

**Remark 2.9.** It turns out that we could take \( \epsilon = 0 \) in the right-hand side of the equation in Lemma 2.8 as there is, in fact, only a weak singularity at the origin. Indeed, by Remark 2.7, \( m_{\alpha}(y) = O(|y|^{-1 - 2\alpha}) \) for small \( |y| \) and hence the integrand is \( O(|y|^{2 - 1 - 2\alpha}) = O(|y|^{-1 - 2\alpha}) \). Provided \( 0 < \alpha < 1 \) then \( 1 - 2\alpha > -1 \), and hence the (improper) integral exists.

**Lemma 2.10.** Suppose \( 0 < \epsilon < 1 \) and \( v \in S(\mathbb{R}) \). Let \( u = r_\epsilon v \). Then, for \( x > 0 \),

\[ \int_0^\infty (u(x) - u(y)) m_{\alpha,\epsilon}(|x - y|) dy \]

\[ = \int_0^x (2u(x) - u(x + w) - u(x - w)) m_{\alpha,\epsilon}(|w|) dw + \int_x^\infty (u(x) - u(x + w)) m_{\alpha,\epsilon}(|w|) dw. \]

**Proof.** Let \( w := y - x \). Then

\[ \int_0^\infty (u(x) - u(y)) m_{\alpha,\epsilon}(|x - y|) dy \]

\[ = \int_{-x}^\infty (u(x) - u(w + x)) m_{\alpha,\epsilon}(|w|) dw \]

\[ = \int_{-x}^x (u(x) - u(w + x)) m_{\alpha,\epsilon}(|w|) dw + \int_x^\infty (u(x) - u(w + x)) m_{\alpha,\epsilon}(|w|) dw. \]

But

\[ \int_{-x}^0 (u(x) - u(w + x)) m_{\alpha,\epsilon}(|w|) dw = \int_{0}^x (u(x) - u(x - z)) m_{\alpha,\epsilon}(|z|) dz, \]

and the required result follows immediately. \( \square \)
Lemma 2.11. Suppose $0 < \epsilon < 1$ and $0 < \alpha < 1$. Let $v \in S(\mathbb{R})$ and define $u := r_+ v$. Then there exists a strictly positive constant $C$ and a function $g(x)$, both independent of $\epsilon$, such that
\[
| (A_{\epsilon} u)(x) | \leq \chi_{[0,1]}(x) g(x) + C \quad (x > 0),
\]
where $g(x)$ is $O(1)$ as $x \searrow 0$ for $0 < \alpha < \frac{1}{2}$, and is $O(x^{1-2\alpha})$ as $x \searrow 0$ for $\frac{1}{2} \leq \alpha < 1$.

Proof. We now define:
\[
M_1 := \int_1^\infty w m_\alpha(w) dw;
\]
\[
h(x) := \int_x^1 w m_\alpha(w) dw \quad (0 < x < 1);
\]
\[
M_2 := \int_0^\infty w^2 m_\alpha(w) dw.
\]
Then $M_1, M_2 < \infty$ and
\[
h(x) = \begin{cases} 
O(1) & \text{if } 0 < \alpha < \frac{1}{2} \\
O(x^{1-2\alpha}) & \text{if } \frac{1}{2} \leq \alpha < 1
\end{cases} \quad \text{as } x \searrow 0.
\]
Moreover, noting that $u := r_+ v$, $(v \in S(\mathbb{R}))$, we define
\[
V_0 := \sup_{x \geq 0} |v(x)|;
\]
\[
V_1 := \sup_{x \geq 0} |v'(x)|;
\]
\[
V_2 := \sup_{x \geq 0, 0 < w \leq x} |(2v(x) - v(x + w) - v(x - w))/w^2|.
\]
Clearly, $V_0, V_1, V_2 < \infty$.

Our goal now is to determine point-wise estimates for $(A_{\epsilon} u)(x)$. Suppose, initially that $0 < x < 1$. Then, from Definition 2.3 and Lemma 2.10,
\[
| (A_{\epsilon} u)(x) | \leq |u(x)| + I_1 + I_2,
\]
where
\[
I_1 := \int_0^x (2u(x) - u(x + w) - u(x - w)) m_{\alpha, \epsilon}(|w|) dw;
\]
\[
I_2 := \int_x^\infty (u(x) - u(x + w)) m_{\alpha, \epsilon}(|w|) dw.
\]
But
\[ I_1 \leq \int_0^x w^2 V_2 m_{\alpha,\epsilon}(w)dw \leq V_2 M_2. \]

On the other hand,
\[ I_2 \leq \int_1^x wV_1 m_{\alpha,\epsilon}(w)dw + \int_1^\infty wV_1 m_{\alpha,\epsilon}(w)dw \leq V_1 h(x) + V_1 M_1. \]

In summary, for \(0 < x < 1\),
\[ |(A_\epsilon u)(x)| \leq g(x) + V_0 + V_1 M_1 + V_2 M_2, \quad (2.4) \]
where \(g(x) := V_1 h(x)\).

Now suppose that \(x \geq 1\). Then, from Lemma 2.10,
\[ |(A_\epsilon u)(x)| \leq |u(x)| + I_1 + I_2, \]
where \(I_1\) and \(I_2\) are as defined previously but now, of course, the value of \(x\) is in a different range.

Now
\[ I_1 \leq \int_0^x w^2 V_2 m_{\alpha,\epsilon}(w)dw \leq V_2 M_2 \quad \text{(as previously)}. \]

On the other hand,
\[ I_2 \leq \int_1^\infty wV_1 m_{\alpha,\epsilon}(w)dw \leq V_1 M_1. \]

In summary, for \(x \geq 1\),
\[ |(A_\epsilon u)(x)| \leq V_0 + V_1 M_1 + V_2 M_2. \quad (2.5) \]

**Remark 2.12.** Estimates (2.4) and (2.5) are independent of \(\epsilon\).

\[ \square \]

**Lemma 2.13.** Suppose \(v \in S(\mathbb{R})\) and \(0 < \alpha < 1\). Then
\[ |v(x + y) + v(x - y) - 2v(x)| m_\alpha(|y|) \]
is integrable over \(\mathbb{R} \times \mathbb{R}\).
Proof. Firstly, we integrate with respect to \(x\) and define
\[
I(y) := \int_{\mathbb{R}} |v(x + y) + v(x - y) - 2v(x)| \, m_\alpha(|y|) \, dx.
\]

Now if \(|y| \leq 1\), then \(m_\alpha(|y|) = O(|y|^{-1 - 2\alpha})\). On the other hand, if \(|y| > 1\) then \(m_\alpha(|y|) = O(e^{-|y|})\). Hence, for certain positive constants \(C_1\) and \(C_2\),
\[
I(y) \leq \int_{\mathbb{R}} C_1 \chi_{[-1,1]}(y) \frac{|v(x + y) + v(x - y) - 2v(x)|}{|y|^{1+2\alpha}} \, dx
+ \int_{\mathbb{R}} C_2 \chi_{[\mathbb{R}[-1,1]}(y) \frac{|v(x + y) + v(x - y) - 2v(x)|}{e^{|y|}} \, dx
\leq C_1 \chi_{[-1,1]}(y) |y|^{2-1-2\alpha} \int_{\mathbb{R}} \sup_{z \in [x-1,x+1]} |v''(z)| \, dx
+ C_2 \chi_{[\mathbb{R}\setminus[-1,1]}(y) e^{-|y|} \int_{\mathbb{R}} 4|v(x)| \, dx.
\]

But
\[
\int_{\mathbb{R}} \sup_{z \in [x-1,x+1]} |v''(z)| \, dx \leq C_3 \int_{\mathbb{R}} \frac{1}{1 + x^2} \, dx = \pi C_3,
\]
and
\[
\int_{\mathbb{R}} |v(x)| \, dx \leq C_4,
\]
for certain positive constants \(C_3\) and \(C_4\). Hence
\[
I(y) \leq C \left( \chi_{[-1,1]}(y) |y|^{1-2\alpha} + \chi_{\mathbb{R}\setminus[-1,1]}(y) e^{-|y|} \right)
\in L_1(\mathbb{R}),
\]
and the required result now follows directly from Tonelli’s theorem.

Lemma 2.14. Suppose \(m_\alpha(y)\) is as defined in Lemma 2.6, and the pseudo-differential operator \(A\) has symbol \((1 + \xi^2)^\alpha\), where \(0 < \alpha < 1\). Then, for all \(v \in S(\mathbb{R})\),
\[
(Av)(x) = v(x) + \lim_{\epsilon \searrow 0} \int_{\mathbb{R}} (v(x) - v(y)) \, m_{\alpha,\epsilon}(|x - y|) \, dy.
\]
Proof. Let \( \mathcal{F} \) denote \( \mathcal{F}_{x \to \xi} \). Then

\[
\mathcal{F}(Av)(\xi) = (1 + \xi^2)^\alpha (\mathcal{F}v)(\xi)
\]

\[
= (\mathcal{F}v)(\xi) + \left[ \int_R (1 - \cos \gamma \xi) m_\alpha(y) \, dy \right] (\mathcal{F}v)(\xi) \quad \text{by Lemma 2.6}
\]

\[
= (\mathcal{F}v)(\xi) - \frac{1}{2} \int_R \{ e^{i\xi y} + e^{-i\xi y} - 2 \} m_\alpha(y) \, dy
\]

\[
= (\mathcal{F}v)(\xi) - \frac{1}{2} \int_R \mathcal{F}((v(\cdot + y) + v(\cdot - y) - 2v(\cdot))) m_\alpha(y) \, dy
\]

\[
= (\mathcal{F}v)(\xi) - \frac{1}{2} \mathcal{F} \left( \int_R (v(\cdot + y) + v(\cdot - y) - 2v(\cdot)) m_\alpha(y) \, dy \right)(\xi)
\]

\[
= (\mathcal{F}v)(\xi) + \mathcal{F} \left( \lim_{\epsilon \to 0} \int_R (v(x) - v(y)) m_{\alpha, \epsilon}(|x - y|) \, dy \right)(\xi) \quad \text{by Lemma 2.8},
\]

where we have used Lemma 2.13 to justify the change in order of \( \mathcal{F} \) and integration with respect to \( y \).

So now applying the inverse transform \( \mathcal{F}_{\xi \to x}^{-1} \) to both sides

\[
(Av)(x) = v(x) + \lim_{\epsilon \to 0} \int_R (v(x) - v(y)) m_{\alpha, \epsilon}(|x - y|) \, dy. \tag{2.6}
\]

This completes the proof of the lemma.

\( \square \)

**Lemma 2.15.** Suppose \( m_\alpha(y) \) is as defined in Lemma 2.6 and the pseudodifferential operator \( A \) has symbol \( (1 + \xi^2)^\alpha \), where \( 0 < \alpha < 1 \). Then for all \( v \in S(\mathbb{R}) \) and \( x > 0 \),

\[
(r_+ A(\chi_{\mathbb{R}+}v))(x) = -\int_{-\infty}^{0} v(y) m_\alpha(|x - y|) \, dy.
\]

**Proof.** Choose any \( w \in C_0^\infty(\mathbb{R}_+) \), and define \( \chi_n \in C_0^\infty(\mathbb{R}) \) such that

\[
\chi_n(x) := \begin{cases} 1 & x \in [-n, -\frac{1}{n}] \\ 0 & x \not\in [-n, 0]. \end{cases}
\]

Then, from Lemma 2.14

\[
(r_+ A(\chi_n v), w) = -\int_{\mathbb{R}_+} w(x) \left( \int_{-\infty}^{0} \chi_n(y) v(y) m_\alpha(|x - y|) \, dy \right) \, dx. \tag{2.7}
\]
But $\chi_n v \to \chi_{\mathbb{R}_-} v$ in $L_1(\mathbb{R}) \hookrightarrow S'(\mathbb{R})$, and hence $A(\chi_n v) \to A(\chi_{\mathbb{R}_-} v)$ in $S'(\mathbb{R})$.

On the other hand, since $x > 0$, the right-hand side of equation (2.7) converges to

$$-\int_{\mathbb{R}_+} w(x) \left( \int_{-\infty}^0 v(y) m_\alpha(|x - y|) \, dy \right) \, dx.$$ 

Thus, letting $n \to \infty$, we obtain

$$(r_+ A(\chi_{\mathbb{R}_-}), w) = -\int_{\mathbb{R}_+} w(x) \left( \int_{-\infty}^0 v(y) m_\alpha(|x - y|) \, dy \right) \, dx.$$ 

But since $w \in C_0^\infty(\mathbb{R}_+)$ was arbitrary

$$(r_+ A(\chi_{\mathbb{R}_-})) (x) = -\int_{-\infty}^0 v(y) m_\alpha(|x - y|) \, dy.$$ 

This completes the proof of the lemma. \qed

\textbf{Lemma 2.16.} Suppose $m_\alpha(y)$ is as defined in Lemma 2.6 and the pseudodifferential operator $A$ has symbol $(1 + \xi^2)^\alpha$, where $0 < \alpha < 1$. Then for $x > 0$,

$$(r_+ A(\chi_{\mathbb{R}_-})) (x) = -\int_{-\infty}^0 m_\alpha(|x - y|) \, dy.$$ 

\textbf{Proof.} Choose any $w \in C_0^\infty(\mathbb{R}_+)$, and let $\chi_n \in C_0^\infty(\mathbb{R})$ be as defined in the proof of Lemma 2.15. Then, from Lemma 2.15

$$(r_+ A(\chi_n \chi_{\mathbb{R}_-}), w) = -\int_{\mathbb{R}_+} w(x) \left( \int_{-\infty}^0 \chi_n(y) \chi_{\mathbb{R}_-}(y) m_\alpha(|x - y|) \, dy \right) \, dx. \tag{2.8}$$

But $\chi_n \chi_{\mathbb{R}_-} \to \chi_{\mathbb{R}_-}$ in $L_1(\mathbb{R}, \frac{dx}{1 + x^2}) \hookrightarrow S'(\mathbb{R})$, and hence $A(\chi_n \chi_{\mathbb{R}_-}) \to A(\chi_{\mathbb{R}_-})$ in $S'(\mathbb{R})$.

On the other hand, the right-hand side of equation (2.8) converges to

$$-\int_{\mathbb{R}_+} w(x) \left( \int_{-\infty}^0 m_\alpha(|x - y|) \, dy \right) \, dx.$$ 

Thus, letting $n \to \infty$, we obtain

$$(r_+ A(\chi_{\mathbb{R}_-}), w) = -\int_{\mathbb{R}_+} w(x) \left( \int_{-\infty}^0 m_\alpha(|x - y|) \, dy \right) \, dx.$$ 

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But since $w \in C_0^\infty(\mathbb{R}^+)$ was arbitrary

$$\left( r_+ A(\chi_{\mathbb{R}^-}) \right)(x) = -\int_{-\infty}^0 m_\alpha(|x - y|) \, dy.$$

This completes the proof of the lemma. $\square$

### 2.3 Proof of Theorem 2.1

**Lemma 2.17.** Suppose $0 < \alpha < 1$. Let $v \in S(\mathbb{R})$ and define $u := r_+ v$. Then, for $x > 0$,

$$(Au)(x) = u(x) + \lim_{\epsilon \downarrow 0} \int_0^\infty (u(x) - u(y)) \, m_{\alpha, \epsilon}(|x - y|) \, dy.$$  

**Proof.** We have $v \in S(\mathbb{R})$ and $u := r_+ v$. Moreover, from equation (1.32),

$$r_+ A e_+ u + u r_+ A(\chi_{\mathbb{R}^-}) = r_+ A v + r_+ [vI, A] \chi_{\mathbb{R}^-}.$$  

From Lemma 2.16,

$$r_+ (vA \chi_{\mathbb{R}^-})(x) = -r_+ \int_{-\infty}^0 v(x)m_\alpha(|x - y|) \, dy \quad (x > 0).$$

On the other hand, by Lemma 2.15

$$r_+ A(v \chi_{\mathbb{R}^-}) = -r_+ \int_{-\infty}^0 v(y)m_\alpha(|x - y|) \, dy.$$  

Thus, combining these results

$$r_+ [vI, A] \chi_{\mathbb{R}^-}(x) := r_+ \{ v(x)(A \chi_{\mathbb{R}^-})(x) - A(v \chi_{\mathbb{R}^-})(x) \}$$

$$= -r_+ \int_{-\infty}^0 (v(x) - v(y))m_\alpha(|x - y|) \, dy. \quad (2.9)$$

Finally, from Lemma 2.14,

$$r_+ A e_+ u + u r_+ A(\chi_{\mathbb{R}^-}) = r_+ A v + r_+ [vI, A] \chi_{\mathbb{R}^-}$$

$$= r_+ v + r_+ \lim_{\epsilon \downarrow 0} \int_0^\infty (v(x) - v(y))m_{\alpha, \epsilon}(|x - y|) \, dy$$

$$= u + \lim_{\epsilon \downarrow 0} \int_0^\infty (u(x) - u(y)) \, m_{\alpha, \epsilon}(|x - y|) \, dy \quad (x > 0).$$

This completes the proof of the lemma. $\square$
**Lemma 2.18.** Suppose $0 < \alpha < 1$. Let $v \in S(\mathbb{R})$ and define $u := r_+ v$. Then,

$$
(Au, u) = \int_0^\infty |u|^2 \, dx + \frac{1}{2} \int_0^\infty \int_0^\infty |u(x) - u(y)|^2 m_\alpha(|x - y|) \, dy \, dx.
$$

**Proof.** Suppose $0 < \epsilon < 1$. From Remark 2.7, $m_\alpha(|w|)$ is $O(|w|^{-1-2\alpha})$ for small $|w|$, and $O(e^{-|w|})$ for large $|w|$. Moreover, $m_\alpha(w) > 0$ for all finite $w \geq 0$.

From Definition 2.3,

$$
m_{\alpha, \epsilon}(w) = \chi_\epsilon(w) m_\alpha(w) \quad w \geq 0;
$$

$$
(A_\epsilon u)(x) = u(x) + \int_0^\infty (u(x) - u(y)) m_{\alpha, \epsilon}(|x - y|) \, dy \quad (x > 0).
$$

Hence

$$
(A_\epsilon u, u) = \int_0^\infty |u|^2 \, dx + \int_0^\infty \int_0^\infty \overline{u(x)}(u(x) - u(y)) m_{\alpha, \epsilon}(|x - y|) \, dy \, dx. \quad (2.10)
$$

Interchanging the roles of $x$ and $y$

$$
(A_\epsilon u, u) = \int_0^\infty |u|^2 \, dx + \int_0^\infty \int_0^\infty (-1)u(y)(u(x) - u(y)) m_{\alpha, \epsilon}(|x - y|) \, dy \, dx. \quad (2.11)
$$

Using Fubini’s theorem, and adding equations (2.10) and (2.11),

$$
(A_\epsilon u, u) = \int_0^\infty |u|^2 \, dx + \frac{1}{2} \int_0^\infty \int_0^\infty |u(x) - u(y)|^2 m_{\alpha, \epsilon}(|x - y|) \, dy \, dx. \quad (2.12)
$$

Our method of proof is to take the limit in equation (2.12) as $\epsilon \searrow 0$. For the left-hand side we use the Dominated Convergence Theorem, and for the right-hand side we use the Monotone Convergence Theorem.

Firstly, consider the left-hand side.

(i) $\lim_{\epsilon \searrow 0} (A_\epsilon u)(x) \to (Au)(x) \quad (x > 0)$ (Lemma 2.14);

(ii) $|A_\epsilon u(x)| \leq \chi_{[0,1]}(x) g(x) + C \quad (C \geq 0) \quad$ Lemma 2.11,

where $g(x)$ is $O(1)$ as $x \searrow 0$ for $0 < \alpha < \frac{1}{2}$, and is $O(x^{1-2\alpha})$ as $x \searrow 0$ for $\frac{1}{2} \leq \alpha < 1$.  

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Hence, \( u(x)\left[ \chi_{[0,1]}(x)g(x) + C \right] \in L_1([0, \infty)) \).

Therefore, by the Dominated Convergence Theorem,

\[
\int_0^\infty u(x)(A_\epsilon u)(x) \, dx \to \int_0^\infty u(x)(Au)(x) \, dx \quad \text{as } \epsilon \searrow 0.
\]

On the other hand, for the right-hand side as \( \epsilon \searrow 0 \),

\[
\int_0^\infty \int_0^\infty |u(x) - u(y)|^2 m_{\alpha,\epsilon}(|x-y|) \, dy \, dx \to \int_0^\infty \int_0^\infty |u(x) - u(y)|^2 m_\alpha(|x-y|) \, dy \, dx,
\]

by a routine application of the Monotone Convergence Theorem. This completes the proof of the lemma.

\( \square \)
Chapter 3

Trivial kernel

3.1 Main result

Theorem 3.1. Suppose $1 < p < \infty$ and $0 < \alpha < 1$.

(a) If $2\alpha - 1 + 1/p < s < 1 + 1/p$ then $A : H^s_p(\mathbb{R}_+) \rightarrow H^{s-2\alpha}_p(\mathbb{R}_+)$ is bounded.

(b) If $1 + 1/p < s < 2 + 1/p$ then $A : H^s_{p,0}(\mathbb{R}_+) \rightarrow H^{s-2\alpha}_p(\mathbb{R}_+)$ is bounded.

Moreover, if $p = 2$ and $0 < \alpha < \frac{1}{2}$, $\frac{1}{2} < s < 1 + \frac{1}{2}$, then $A : H^s_2(\mathbb{R}_+) \rightarrow H^{s-2\alpha}_2(\mathbb{R}_+)$ has a trivial kernel. Similarly, if $p = 2$ and $0 < \alpha < 1$, $1 + \frac{1}{2} < s < 2 + \frac{1}{2}$ then $A : H^s_{2,0}(\mathbb{R}_+) \rightarrow H^{s-2\alpha}_2(\mathbb{R}_+)$ has a trivial kernel.

Remark 3.2. The condition that $p = 2$ for $A$ to have a trivial kernel is not as restrictive as it might appear. Under appropriate conditions, we will be able to determine sufficient conditions for $A$ to have a trivial kernel for any $p$ in the range $1 < p < \infty$, using the result (above) for $p = 2$.

3.2 Supporting lemmas

Lemma 3.3. Let $\gamma > 0$ and $s > \gamma + 1/p - 1$. Then the multiplication operator $x^{-\gamma} I : \tilde{H}^s_p(\mathbb{R}_+) \rightarrow \tilde{H}^{s-\gamma}_p(\mathbb{R}_+)$ is bounded.

Proof. Suppose initially that $s = \gamma$. Then $\tilde{H}^{s-\gamma}_p(\mathbb{R}_+) = L_p(\mathbb{R}_+)$ and the required result follows directly from the proof of Proposition 1, Section 2.8.6, [45]. In other words, if $u \in \tilde{H}^s_p(\mathbb{R}_+)$ then

$$||x^{-\gamma}u||_p \leq C_{\gamma,p}||u||_{\gamma,p}.$$
Hence, \( x^\gamma \) is bounded. Secondly, suppose that \( s = \gamma + 1 \), and let \( u \in H_p^{\gamma+1}(\mathbb{R}_+) \). Then we will show that \( x^{-\gamma}I : H_p^{\gamma+1}(\mathbb{R}_+) \rightarrow H_p^1(\mathbb{R}_+) \) is bounded. Note that \( H_p^{\gamma+1}(\mathbb{R}_+) \hookrightarrow H_\gamma^p(\mathbb{R}_+) \) and \( \partial u \in \tilde{H}_\gamma^p(\mathbb{R}_+) \). Therefore, using an equivalent norm on \( H_p^{\gamma+1}(\mathbb{R}_+) \), see Chapter 1, p. 6, [46],

\[
\|x^{-\gamma}u\|_{1,p} \leq \text{const} \left\{ \|x^{-\gamma}u\|_p + \|\partial(x^{-\gamma}u)\|_p \right\} \\
\leq \text{const} \left\{ \|x^{-\gamma}u\|_p + \|x^{-\gamma}\partial u\|_p + \gamma \|x^{-(\gamma+1)}u\|_p \right\} \\
\leq \text{const} \left\{ \|u\|_{\gamma,p} + C_{\gamma,p} \|\partial u\|_{\gamma,p} + C_{\gamma+1,p} \|u\|_{\gamma+1,p} \right\} \\
\leq \text{const} \left\{ \|u\|_{\gamma+1,p} \right\}.
\]

Hence, \( x^{-\gamma}I : H_p^{\gamma+1}(\mathbb{R}_+) \rightarrow H_p^1(\mathbb{R}_+) \) is bounded. In the same way, the result for \( s = \gamma + m \), for any \( m \in \mathbb{N} \) follows by induction.

Moreover, the proof of the lemma for any \( s \geq \gamma \), the first case, follows by interpolation. See, for example, Chapter 1, [45].

We now consider the second case \(-1 + \gamma + 1/p < s < \gamma\).

Firstly suppose that \( 0 \leq s < \gamma \). Then from the first case, it is clear that \( x^{-s}I : \tilde{H}_p^s(\mathbb{R}_+) \rightarrow L_p(\mathbb{R}_+) \) is bounded. Hence, it is sufficient to show that \( x^{-(\gamma-s)}I : L_p(\mathbb{R}_+) \rightarrow \tilde{H}_p^{s-\gamma}(\mathbb{R}_+) \) is bounded. Since \(-1 + 1/p < s - \gamma < 0\), this operator is adjoint to \( x^{-(\gamma-s)}I : \tilde{H}_p^{s-\gamma}(\mathbb{R}_+) \rightarrow L_p(\mathbb{R}_+) \) which is bounded by the first case because \( \gamma - s > 0 \).

Secondly, suppose \( \gamma + 1/p - 1 < s < 0 \). Then \( 1/p - 1 < s, s - \gamma < 0 \), i.e. \( 0 < -s, \gamma - s < 1/p' \), and the spaces \( H_p^{s-\gamma}(\mathbb{R}_+) = \left( \tilde{H}_p^{s-\gamma}(\mathbb{R}_+) \right)' \), \( H_p^{-s}(\mathbb{R}_+) = \left( \tilde{H}_p^{s}(\mathbb{R}_+) \right)' \) can be identified with \( \tilde{H}_p^{s-\gamma}(\mathbb{R}_+) \) and \( \tilde{H}_p^{-s}(\mathbb{R}_+) \) respectively.

The adjoint of the operator \( x^{-\gamma}I : \tilde{H}_p^s(\mathbb{R}_+) \rightarrow \tilde{H}_p^{s-\gamma}(\mathbb{R}_+) \) is the operator \( x^{-\gamma}I : \tilde{H}_p^{s-\gamma}(\mathbb{R}_+) \rightarrow \tilde{H}_p^{s}(\mathbb{R}_+) \). Since \( s < 0 \), one has \( \gamma - s > \gamma \) and the latter operator is bounded by the first case. Hence the former operator is also bounded.

This completes the proof of the lemma.

\[ \square \]

**Lemma 3.4.** Let \( \gamma \geq 0, \phi \in r_+S(\mathbb{R}) \) and \( s \geq 0 \). Then

\[
x^\gamma \phi I : \tilde{H}_p^s(\mathbb{R}_+) \rightarrow \tilde{H}_p^s(\mathbb{R}_+)
\]

is bounded.
Proof. The result is clearly true for \( s = 0 \).

We now use proof by induction on \( s \). Suppose result is true for \( s = m \in \mathbb{N} \cup \{0\} \), for all \( \phi \in r_+S(\mathbb{R}) \) and all \( \gamma \geq 0 \). Then, we shall prove it is also true for \( s = m + 1 \).

Let \( u \in \tilde{H}^{m+1}_p(\mathbb{R}_+) \). Then, using the inductive hypothesis,

\[
\|x^\gamma \phi u\|_{m+1,p} \leq \text{const} \left\{ \|x^\gamma \phi u\|_{m,p} + \|\frac{d}{dx}(x^\gamma \phi u)\|_{m,p} \right\}
\]

\[
\leq \text{const} \left\{ \|u\|_{m,p} + \|x^\gamma \phi u\|_{m,p} + \|x^\gamma \phi' u\|_{m,p} + \|x^{\gamma-1} \phi u\|_{m,p} \right\}
\]

\[
\leq \text{const} \left\{ \|u\|_{m,p} + \|u'\|_{m,p} + \|x^{\gamma-1} \phi u\|_{m,p} \right\}.
\]

It remains to consider the term \( \|x^{\gamma-1} \phi u\|_{m,p} \).

If \( \gamma - 1 \geq 0 \), then by the inductive hypothesis,

\[
\|x^{\gamma-1} \phi u\|_{m,p} \leq \text{const} \|u\|_{m,p}.
\]

Finally, if \( \gamma - 1 < 0 \), then by Lemma 3.3,

\[
\|x^{\gamma-1} \phi u\|_{m,p} \leq \text{const} \|\phi u\|_{m+1-\gamma,p} \leq \text{const} \|u\|_{m+1,p}.
\]

In summary, for \( s = m + 1 \) and all \( \phi \in r_+S(\mathbb{R}), \gamma > 0 \) and \( u \in \tilde{H}^{m+1}_p(\mathbb{R}_+) \), we have

\[
\|x^{\gamma-1} \phi u\|_{m,p} \leq \text{const} \|u\|_{m+1,p}.
\]

This completes the proof by induction for \( s \in \mathbb{N} \).

Hence, by interpolation, the required result follows for all \( s \geq 0 \).

\[ \square \]

**Lemma 3.5.** Suppose \( 1 < p < \infty \), \( k \in \mathbb{N} \cup \{0\} \) and \( s \geq 0 \). Then for any \( \beta > 0 \) and \( \epsilon > 0 \), the map

\[
u \mapsto e^{-\beta x^{\epsilon} \log^k x} \cdot \nu,
\]

from \( \tilde{H}^s_p(\mathbb{R}_+) \to \tilde{H}^s_p(\mathbb{R}_+) \) is bounded.

**Proof.** We proceed by induction on \( s \).

Suppose \( k \in \mathbb{N} \cup \{0\} \) and let \( s = 0 \). Since the function \( e^{-\beta x^{\epsilon} \log^k x} \) is bounded for \( x \geq 0 \), the map \( u(x) \mapsto e^{-\beta x^{\epsilon} \log^k x} \cdot u(x) \), from \( L_p(\mathbb{R}_+) \to L_p(\mathbb{R}_+) \) is bounded.
Now suppose the result is true for some \( s = m \in \mathbb{N} \cup \{0\} \). We shall prove it is also true for \( s = m + 1 \). Suppose \( u \in \tilde{H}^{m+1}_p(\mathbb{R}_+) \) and let us define

\[
F(x) := (e^{-\beta x} x^s \log^k x) \cdot u(x).
\]

Then, a routine calculation gives

\[
F'(x) = T_1 + T_2 + T_3 + T_4,
\]

where

\[
T_1 := -\beta e^{-\beta x} x^s \log^k x \cdot u(x) = (e^{-\beta x} x^s \log^k x) \cdot u; \\
T_2 := e^{-\beta x} x^{s+1} \log^k x \cdot u(x) = (e^{-\beta x} x^s \log^k x) \cdot x^{-1} u; \\
T_3 := e^{-\beta x} x^{1+k} \log^{k-1} x \cdot u(x) = (k e^{-\beta x} x^{k-1} \log^k x) \cdot x^{-1} u \quad (k \geq 1); \\
T_4 := e^{-\beta x} x^{s+1} \log^k x \cdot u'(x) = (e^{-\beta x} x^s \log^k x) \cdot u'.
\]

For each of \( T_1, T_2, T_3 \) and \( T_4 \), the function in parentheses on the right-hand side is of the correct form for the inductive hypothesis. Moreover, taking \( s = m + 1 \) and \( \gamma = 1 \) in Lemma 3.3,

\[
\|x^{-1} u\|_{m,p} \leq \text{const} \|u\|_{m+1,p}.
\]

Then, using the inductive hypothesis, but only including the term \( T_3 \) if \( k \geq 1 \),

\[
\|(e^{-\beta x} x^s \log^k x) \cdot u\|_{m+1,p}
\]

\[
\leq \text{const} \left\{ \|F\|_{m,p} + \|F'\|_{m,p} \right\}
\]

\[
\leq \text{const} \left\{ \|F\|_{m,p} + \|T_1\|_{m,p} + \|T_2\|_{m,p} + \|T_3\|_{m,p} + \|T_4\|_{m,p} \right\}
\]

\[
\leq \text{const} \left\{ \|u\|_{m,p} + \|u\|_{m+1,p} + \|x^{-1} u\|_{m,p} + \|u'\|_{m,p} \right\}
\]

\[
\leq \text{const} \left\{ \|u\|_{m,p} + \|u\|_{m+1,p} + \|u'\|_{m,p} \right\}
\]

This completes the proof by induction for \( s = 0, 1, 2, 3, \ldots \). Hence, by interpolation, the required result holds for all \( s \geq 0 \). 

\[\square\]

**Corollary 3.6.** Suppose \( 1 < p < \infty \), \( k \in \mathbb{N} \cup \{0\} \) and \( s \geq 0 \). Then for any \( \beta > 0 \) and \( 0 < \epsilon < s + 1 - 1/p \), the map

\[ u \mapsto e^{-\beta x} \log^k x \cdot u, \]

from \( \tilde{H}^s_p(\mathbb{R}_+) \to \tilde{H}^{s-\epsilon}_p(\mathbb{R}_+) \) is bounded.

**Proof.** Suppose \( u \in \tilde{H}^s_p(\mathbb{R}_+) \). We write

\[ e^{-\beta x} \log^k x \cdot u = x^{-\epsilon} \left\{ e^{-\beta x} x^s \log^k x \cdot u \right\}, \]

and the required result now follows directly from Lemmas 3.5 and 3.3. 

\[\square\]
3.3 Proof of Theorem 3.1

The proof of the boundedness of the operator $\mathcal{A}$ is given in Lemma 3.7. For $\frac{1}{2} < s < 1 + \frac{1}{2}$ and $1 + \frac{1}{2} < s < 2 + \frac{1}{2}$ respectively, Lemmas 3.11 and 3.12 establish sufficient conditions for $\mathcal{A}$ to have a trivial kernel.

**Lemma 3.7.** Suppose $1 < p < \infty$ and $0 < \alpha < 1$.

(a) If $2\alpha - 1 + 1/p < s < 1 + 1/p$ then $\mathcal{A} : H^s_p(\mathbb{R}_+) \to H^{s-2\alpha}_p(\mathbb{R}_+)$ is bounded.
(b) If $1 + 1/p < s < 2 + 1/p$ then $\mathcal{A} : H^s_{p,0}(\mathbb{R}_+) \to H^{s-2\alpha}_p(\mathbb{R}_+)$ is bounded,

where the space $H^s_{p,0}(\mathbb{R}_+)$ is as defined in (1.3), Section 1.1.

**Proof.** Suppose initially that $1/p < s < 1 + 1/p$ or $1 + 1/p < s < 2 + 1/p$. Our first step is to show that $\mathcal{A} : r_+ H^s_p(\mathbb{R}_+) \to H^{s-2\alpha}_p(\mathbb{R}_+)$ is bounded, and to do this we use the representation for $\mathcal{A}$ given in (1.34). Firstly, suppose that $\alpha \neq \frac{1}{2}$. Then, from Lemma 2.16 and Remark 2.7,

$$(r_+ A(\chi_{\mathbb{R}_-}))(x) = x^{-2\alpha} \phi(x) + \psi(x),$$

where $\phi, \psi \in C^\infty(\mathbb{R})$, and their derivatives, are bounded and $O(e^{-x})$ as $x \to \infty$. Hence, from Lemmas 3.3 and 3.4, $\mathcal{A}$ is bounded from $r_+ H^s_p(\mathbb{R}_+)$ to $H^{s-2\alpha}_p(\mathbb{R}_+)$, provided $s > 2\alpha - 1 + 1/p$.

On the other hand, if $\alpha = \frac{1}{2}$ then again, from Lemma 2.16 and Remark 2.7,

$$(r_+ A(\chi_{\mathbb{R}_-}))(x) = x^{-1}\{\phi(x) + x \vartheta(x) \log x\} = x^{-1}\{\phi(x) + \vartheta(x) e^{x^2/2} \cdot e^{-x/2} x \log x\},$$

where $\phi, \vartheta \in C^\infty(\mathbb{R})$, and their derivatives, are bounded and $O(e^{-x})$ as $x \to \infty$. With the additional use of Lemma 3.5, the boundedness of $\mathcal{A}$ from $r_+ H^s_p(\mathbb{R}_+)$ to $H^{s-1}_p(\mathbb{R}_+)$ now follows as in the case $\alpha \neq \frac{1}{2}$.

The case $s < 1/p$ follows similarly, providing that, in our use of Lemma 3.3, we note the constraint that $s > 2\alpha - 1 + 1/p$. Also, if $s < 0$ and hence $\alpha < \frac{1}{2}$, we use Theorem 4.2.2(ii), p. 203, [46] in place of Lemma 3.4. But for $-1 + 1/p < s < 1/p$, we can identify $e_+ H^s_p(\mathbb{R}_+)$ with $\tilde{H}^s_p(\mathbb{R}_+)$, see Section 2.8.7, p. 158, [45], and the proof for $2\alpha - 1 + 1/p < s < 1/p$ is thus complete.
It remains to consider the case \( s \geq 1/p \).

We let \( \eta(x) \in C_0^\infty(\mathbb{R}) \) be such that
\[
\eta(x) = \begin{cases} 
1 & \text{if } |x| \leq 1 \\
0 & \text{if } |x| > 2.
\end{cases}
\]

Suppose \( u \in H^s_p(\mathbb{R}_+) \) or \( u \in H^s_{p,0}(\mathbb{R}_+) \), as \( 1/p < s < 1 + 1/p \) or \( 1 + 1/p < s < 2 + 1/p \) respectively. Then we can define
\[
u_0(x) := u(x) - u(0)r_+\eta(x),
\]
and hence write
\[
u(x) = \nu_0(x) + u(0)r_+\eta(x).
\]
Then, by construction, \( \nu_0(0) = 0 \). Moreover, if we assume \( u'(0) = 0 \), then \( \nu'_0(0) = 0 \). Therefore, see, for example, Lemma 1.15, p. 55, [41], we have \( \nu_0 \in r_+\tilde{H}^s_p(\mathbb{R}_+) \). Hence, it remains to consider \( A \) acting on \( r_+\eta \).

But, from equation (1.32), it is therefore enough to show that \( r_+[\eta I, A]_{\chi_{\mathbb{R}_-}} \) is bounded from the one-dimensional subspace of \( H^s_p(\mathbb{R}_+) \), or \( H^s_{p,0}(\mathbb{R}_+) \) if \( 1 + 1/p < s < 2 + 1/p \), spanned by \( \eta \) to \( H^{s-2\alpha}_p(\mathbb{R}_+) \).

Let \( \psi_1 \) be any smooth function defined on \( \mathbb{R} \) such that \( \psi_1(x) = 0 \) if \( x \leq \frac{1}{2} \), and \( \psi_1(x) = 1 \) if \( x \geq 1 \). From equation (2.9), for \( x > 0 \) we have
\[
r_+[\eta I, A]_{\chi_{\mathbb{R}_-}}(x) = -r_+\int_{-\infty}^0 (\eta(x) - \eta(y))m_\alpha(x - y)\,dy
\]
\[
= -r_+\int_{-\infty}^0 (\eta(x) - \eta(y))\psi_1(x - y)\,m_\alpha(x - y)\,dy,
\]
since \( \eta(x) - \eta(y) = 1 - 1 = 0 \) if \( x - y < 1 \). (Indeed, \( x > 0, y < 0 \) and \( x - y < 1 \) implies that \( 0 < x < 1 \) and \( -1 < y < 0 \).)

We note that \( \psi_1(x)m_\alpha(x) \) is smooth on \( \mathbb{R}_+ \) and decays exponentially as \( x \to \infty \). Hence, \( r_+[\eta I, A]_{\chi_{\mathbb{R}_-}} \in H^{s-2\alpha}_p(\mathbb{R}_+) \) as required. Finally, boundedness follows immediately since the linear operator \( r_+[\eta I, A]_{\chi_{\mathbb{R}_-}} \) is defined on a one-dimensional space. This completes the proof for the ranges \( 1/p < s < 1 + 1/p \) and \( 1/p < s < 2 + 1/p \).

Finally, to complete the proof of the lemma, we note that boundedness for the exceptional value \( s = 1/p \) follows directly by interpolation. See, for example,
Remark 3.8. Suppose $1 < p < \infty$ and $1 + 1/p < s < 2 + 1/p$. Then, from the proof of Lemma 3.7, given any $u \in H_{p,0}^s(\mathbb{R}_+)$ we can write

$$u = u_0 + u(0)r_+\eta,$$

where $e_+u_0 \in \tilde{H}_p^s(\mathbb{R}_+)$ and $\eta \in C_0^\infty(\mathbb{R})$, with $\eta'(0) = 0$.

Since $e_+C_0^\infty(\mathbb{R}_+)$ is dense in $\tilde{H}_p^s(\mathbb{R}_+)$, see Section 2.10.3, p. 231, [44], this allows us to approximate $u$ arbitrarily closely by a sequence $\{u_n\}_{n=1}^\infty \subset r_+C_0^\infty(\mathbb{R})$ with $u_n(0) = u(0)$ and, importantly, $u_n'(0) = 0$ for each $n$.

Remark 3.9. Suppose $0 < \alpha < 1$. From Theorem 2.1, we have

$$m_\alpha(y) = \frac{\alpha}{\Gamma(1 - \alpha)} \frac{2^{\frac{1}{2} + \alpha}}{\sqrt{\pi}} |y|^{-\frac{1}{2} - \alpha} K_{\frac{1}{2} + \alpha}(|y|).$$

Now for any $u \in H_2^\alpha(\mathbb{R}_+)$, we define the functional

$$I(u) := \left\{ \int_0^\infty |u(x)|^2 \, dx + \frac{1}{2} \int_0^\infty \int_0^\infty |u(x) - u(y)|^2 m_\alpha(|x-y|) \, dy \, dx \right\}^{\frac{1}{2}}. \quad (3.1)$$

From Remark 4.2, p. 62, [14],

$$\|u\|_{\alpha,2}^+ := \left\{ \int_0^\infty |u(x)|^2 \, dx + \int_0^\infty \int_0^\infty \frac{|u(x) - u(y)|^2}{|x-y|^{1+2\alpha}} \, dy \, dx \right\}^{\frac{1}{2}},$$

is an equivalent norm on $H_2^\alpha(\mathbb{R}_+)$. Moreover, see Remark 2.7, we have

$$I(u) \leq \text{const} \|u\|_{\alpha,2}^+.$$

It is easy to show that $I(\cdot)$ is, in fact, a norm on $H_2^\alpha(\mathbb{R}_+)$ for $0 < \alpha < 1$.

Lemma 3.10. Suppose $0 < \alpha < 1$ and $u \in H_2^\alpha(\mathbb{R}_+)$. Further let the sequence $\{u_n\}_{n \geq 1}$ in $r+S(\mathbb{R})$ be such that

$$\|u_n - u\|_{\alpha,2}^+ \to 0 \quad \text{as} \quad n \to \infty.$$

Then

$$\lim_{n \to \infty} I(u_n) = I(u).$$
Proof. From Remark 3.9,

\[ |I(u_n) - I(u)| \leq I(u_n - u) \leq \text{const} \|u_n - u\|_{\alpha,2}^+ \to 0 \text{ as } n \to \infty. \]

Therefore,

\[ \lim_{n \to \infty} I(u_n) = I(u). \]

\[ \square \]

Lemma 3.11. Suppose \( 0 < \alpha < \frac{1}{2}, \frac{1}{2} < s < 1 + \frac{1}{2} \) and \( u \in H^s_2(\mathbb{R}_+) \). Then

\[ (Au, u) = (I(u))^2. \]

In particular, if \( u \in \text{Ker } A \) then \( u = 0 \).

Proof. Since \( s > \frac{1}{2} > \alpha \), we have the continuous embedding

\[ H^s_2(\mathbb{R}) \hookrightarrow H^\alpha_2(\mathbb{R}). \]

Moreover, as \( 0 < \alpha < \frac{1}{2} \) we have \( \alpha > 2\alpha - 1 + \frac{1}{2} \) and thus from Lemma 3.7,

\[ A : H^\alpha_2(\mathbb{R}_+) \to H^{-\alpha}_2(\mathbb{R}_+) \]

is bounded.

Let \( f \in H^\alpha_- (\mathbb{R}_+), g \in H^\alpha_+ (\mathbb{R}_+) \) then, from the Plancherel theorem and the Cauchy-Schwartz inequality, we have the estimate

\[ |(f, g)_{\mathbb{R}_+}| = |(e_+ f, e_+ g)_{\mathbb{R}}| \leq \|e_+ f\|_{-\alpha,2}\|e_+ g\|_{\alpha,2} \leq \text{const} \|f\|_{\alpha,2}\|g\|_{\alpha,2}. \]

Since \( 0 < \alpha < \frac{1}{2} \), \( C^\infty_0(\mathbb{R}_+) \) is dense in \( H^\alpha_2(\mathbb{R}_+) \), see Section 2.9.3, p. 220, [44], and there exists a sequence \( \{u_n\}_{n \geq 1} \) in \( C^\infty_0(\mathbb{R}_+) \) such that

\[ \|u_n - u\|_{\alpha,2}^+ \to 0 \text{ as } n \to \infty. \]

Hence

\[ (Au, u) - (Au_n, u_n) = (A(u - u_n), u) + (Au_n, u - u_n) \to 0 \text{ as } n \to \infty. \]

That is,

\[ \lim_{n \to \infty} (Au_n, u_n) = (Au, u), \]

and from Lemma 3.10,

\[ \lim_{n \to \infty} I(u_n) = I(u). \]

Hence, from Lemma 2.18,

\[ (Au, u) = (I(u))^2. \]

Finally, if \( Au = 0 \) then, see Remark 3.9, we have \( u = 0 \).

\[ \square \]
Lemma 3.12. Suppose $0 < \alpha < 1, 1 + \frac{1}{2} < s < 2 + \frac{1}{2}$ and $u \in H_{2,0}^s(\mathbb{R}_+)$. Then
\[(Au, u) = (I(u))^2.\]

In particular, if $u \in \ker A$ then $u = 0$.

Proof. For $0 < \alpha < 1$, we define
\[\beta := \begin{cases} \alpha & \text{if } 0 < \alpha < \frac{1}{2} \\ \alpha - \frac{1}{2} & \text{if } \frac{1}{2} \leq \alpha < 1 \end{cases}\]
so that $0 \leq \beta < \frac{1}{2}$. As previously, if $f \in H_{2,0}^{-\beta}(\mathbb{R}_+), g \in H_{2,0}^\beta(\mathbb{R}_+)$ then, from the Plancherel theorem and the Cauchy-Schwartz inequality, we have the estimate
\[|\langle f, g \rangle_{\mathbb{R}_+}| = |\langle e_+ f, e_+ g \rangle_{\mathbb{R}_+}| \leq \|e_+ f\|_{-\beta, 2} \|e_+ g\|_{\beta, 2} \leq \text{const} \|f\|_{\alpha, 2} \|g\|_{\beta, 2}.\]

Moreover, since $0 < \alpha < 1$ and $1 + \frac{1}{2} < s < 2 + \frac{1}{2}$ from Lemma 3.7, the operator
\[A : H_{2,0}^s(\mathbb{R}_+) \to H_{2,0}^{s-2\alpha}(\mathbb{R}_+)\]
is bounded. In addition, $H_{2,0}^s(\mathbb{R}_+) \hookrightarrow H_{2}^\beta(\mathbb{R}_+)$ and $H_{2,0}^{s-2\alpha}(\mathbb{R}_+) \hookrightarrow H_{2,0}^{-\beta}(\mathbb{R}_+)$. From Remark 3.8, there exists a sequence $\{u_n : u_n(0) = u(0), u_n'(0) = 0\}_{n \geq 1}$ in $r_+C_0^\infty(\mathbb{R})$ such that
\[\|u_n - u\|_{s, 2}^+ \to 0 \quad \text{as } n \to \infty.\]

Therefore, as $H_{2}^\beta(\mathbb{R}_+) \hookrightarrow H_{2}^{\beta}(\mathbb{R}_+)$,
\[\|u_n - u\|_{\beta, 2}^+ \to 0 \quad \text{as } n \to \infty.\]

Hence
\[(Au, u) - (Au_n, u_n) = (A(u - u_n), u) + (Au_n, u - u_n) \to 0 \quad \text{as } n \to \infty.\]

That is,
\[\lim_{n \to \infty} (Au_n, u_n) = (Au, u),\]
and, since $H_{2,0}^s(\mathbb{R}_+) \hookrightarrow H_{2}^\alpha(\mathbb{R}_+)$, from Lemma 3.10,
\[\lim_{n \to \infty} I(u_n) = I(u).\]

Hence, from Lemma 2.18,
\[(Au, u) = (I(u))^2.\]

Finally, if $Au = 0$ then, see Remark 3.9, we have $u = 0$. \qed
Chapter 4
Operator algebra - Part I

4.1 Introduction

This chapter details the first step in describing our problem in the context of an operator algebra of multiplication, Mellin and Wiener-Hopf operators acting on $L_p(\mathbb{R}_+)$. The results calculated here act as the starting point for the second, and final, step given in Chapter 5.

Throughout this chapter we assume the problem constraints $0 < \alpha < \frac{1}{2}$, $1 < p < \infty$ and $1/p < s < 1 + 1/p$. Moreover, we suppose that $u \in H^s_p(\mathbb{R}_+)$. (However, where appropriate, we shall also prove variants of certain results that apply in the case of higher regularity, namely $1 + 1/p < s < 2 + 1/p$.)

The discontinuity of the function $e_+u$, at $x = 0$, gives rise to a delta function on the boundary. For, if $\delta$ denotes the Dirac delta function then, see Lemma 4.1, we have:

$$(D - i)e_+u = e_+(D - i)u + iu(0)\delta.$$  

Terms, as above, involving the trace value $u(0)$ pose a significant difficulty. However, it will be seen that we can combine such terms with the “added” potential to form expressions including the factor $(u(x) - u(0))$. These differences can then be reformulated as a composition of certain multiplication, Mellin and Wiener-Hopf operators. Such conversions are a significant part of the analysis of this present chapter.

Since our ultimate objective is a reformulation of our problem in $L_p(\mathbb{R}_+)$, we introduce

$$u_s := (D + i)^s e_+(D - i)u.$$
From Lemma 4.3, \( u_s \in L_p(\mathbb{R}) \) with \( \text{supp} \ u_s \subseteq \mathbb{R}_+ \). Therefore, we have

\[ e_+ r_+ u_s = u_s. \]

This relationship will prove essential in dealing with the Wiener-Hopf operators. For example, we show in Lemma 4.21 that

\[ u = W(c)(r_+ u_s), \]

where the Wiener-Hopf operator \( W(c) \) has symbol \( c(\xi) = (\xi - i)^{-1}(\xi + i)^{1-s} \).

As an intermediate step in expressing our problem in an operator algebra acting on \( L_p(\mathbb{R}_+) \), our goal in this chapter is to reformulate equation (1.34) in the form

\[ \tilde{a}_0(x)u(0) + \sum_{j=1}^{N} \tilde{a}_j(x) M^0(\tilde{b}_j)(r_+ \tilde{C}_j e_+)(r_+ u_s) + \tilde{K} u = f, \quad (4.1) \]

where the operator \( \tilde{K} : H^s_p(\mathbb{R}_+) \to H^{s-2\alpha}_p(\mathbb{R}_+) \) is compact.

In doing so, we provide precise determinations of the multiplication symbols \( \{\tilde{a}_k\}_{k=0}^N \), the Mellin symbols \( \{\tilde{b}_j\}_{j=1}^N \) and the symbols \( \{\tilde{c}_j\}_{j=1}^N \) of the pseudodifferential operators \( \{\tilde{C}_j\}_{j=1}^N \). Since, our ultimate goal is to calculate the Fredholm index of the corresponding operator, along the way we will effectively discard any compact operators - as the Fredholm index is invariant under compact perturbations.

Finally, we note that, by hypothesis, \( f \in H^{s-2\alpha}_p(\mathbb{R}_+) \). In Chapter 5, we apply the operator \( r_+(D - i)^{s-2\alpha} l_+ \) to each side of equation (4.1), to obtain our required formulation in \( L_p(\mathbb{R}_+) \). (See, in particular, Lemma 4.5.) In this sense, the value of the results from the current chapter will only be apparent later. Accordingly, they are described here as interim results.

### 4.2 Problem reformulation

As an initial step in reformulating equation (1.34), we define

\[ A^-(D) := A(D)(D - i)^{-1}. \quad (4.2) \]
Since $A$ has order $2\alpha$, $A^-$ is a pseudodifferential operator of order $2\alpha - 1$. Of course, as $0 < \alpha < \frac{1}{2}$, $A^-$ has negative order. We now recast equation (1.34) in terms of the operator $A^-$. In passing, and looking ahead to the case of higher regularity, we also define

$$A^-(D) := A(D)(D - i)^{-2}. \quad (4.3)$$

From Lemma 4.1,

$$r_+ A e_+ u = r_+ A^- (D - i) e_+ u = r_+ A^- e_+ (D - i) u + i u(0) r_+ A^- \delta. \quad (4.4)$$

Moreover,

$$r_+ A(\chi_{\mathbb{R}^-}) = r_+ A^- (D - i) \chi_{\mathbb{R}^-} = r_+ A^- (-i \delta - i \chi_{\mathbb{R}^-}),$$

since $D(\chi_{\mathbb{R}^-}) = -D(\chi_{\mathbb{R}^+}) = -i \delta$. (See Example 1.3, p. 10, [14].)

Hence, with these substitutions, equation (1.34) becomes

$$r_+ A^- e_+ (D - i) u - i(u(x) - u(0)) r_+ A^- \delta - i u(x) r_+ A^- (\chi_{\mathbb{R}^-}) = f. \quad (4.4)$$

Let us now define

$$u_s := (D + i)^{s-1} e_+ (D - i) u. \quad (4.5)$$

Then, we can write

$$r_+ A^- e_+ (D - i) u = r_+ A^- (D + i)^{1-s} (D + i)^{s-1} e_+ (D - i) u \quad (4.6)$$

where

$$A_s(D) := A^- (D + i)^{1-s}. \quad (4.6)$$

Hence, equation (1.34) becomes

$$r_+ A_s u_s - i(u(x) - u(0)) r_+ A^- \delta - i u(x) r_+ A^- (\chi_{\mathbb{R}^-}) = f. \quad (4.7)$$

We will see subsequently that the function $u$ appearing in the potential term, $-iu(x) r_+ A^- (\chi_{\mathbb{R}^-})$ in equation (4.7), can also be expressed appropriately in terms of $u_s$. Moreover, it turns out that the difference $u(x) - u(0)$ can be described in terms of the composition of a multiplication, Mellin and Wiener-Hopf operators. Finally, we are able to calculate both $r_+ A^- \delta$ and $r_+ A^- (\chi_{\mathbb{R}^-})$ explicitly, using special functions.
4.3 Supporting lemmas

It will now be convenient to introduce certain functions. We have

\[ M(a, b, z) := 1 + \sum_{k=1}^{\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!} , \]

as defined in 13.1.2, [1] or 9.210, [17]. (We use the notation \((a)_k = a(a + 1) \cdots (a + k - 1)\) for any \(k \in \mathbb{N}\).)

Following 13.1.3, [1] and 9.210 2, [17], we also introduce the confluent hypergeometric function of the second kind

\[ U(a, b, z) := \frac{\Gamma(1 - b)}{\Gamma(a - b + 1)} M(a, b, z) + \frac{\Gamma(b - 1)}{\Gamma(a)} z^{1-b} M(a - b + 1, 2 - b, z) , \]

for \(a > 0\) and \(b > 0\), provided \(b \notin \mathbb{N}\). In the exceptional case that \(b \in \mathbb{N}\), the corresponding expression for \(U(a, b, z)\) includes a logarithmic term. (See, for example, 13.1.6, [1]).

**Lemma 4.1.** Suppose \(1 < p < \infty\) and \(1/p < s < 1 + 1/p\). If \(u \in H^s_p(\mathbb{R}_+)\) then
\[
(D - i)e^+ u = e^+ (D - i)u + iu(0) \delta .
\]

**Proof.** We first show that if \(v \in r_+ C^\infty_0(\mathbb{R})\), then \((e^+ v)' = v(0) \delta + e^+ v'\). Take any \(\varphi \in S(\mathbb{R})\). Then

\[
\langle (e^+ v)', \varphi \rangle = -\langle e^+ v, \varphi' \rangle = -\int_{-\infty}^{\infty} (e^+ v)(t) \varphi'(t) \, dt = -\int_{0}^{\infty} v(t) \varphi'(t) \, dt = -[v(t) \varphi(t)]_{0}^{\infty} + \int_{0}^{\infty} v'(t) \varphi(t) \, dt = v(0) \varphi(0) + \int_{-\infty}^{\infty} (e^+ v')(t) \varphi(t) \, dt = \langle v(0) \delta + e^+ v', \varphi \rangle ,
\]

which gives the required result, since \(\varphi \in S(\mathbb{R})\) was arbitrary.
Since $1/p < s < 1 + 1/p$, the value $u(0)$ is well-defined. (See Section 2.9, [44].) Therefore, by continuity

$$(D - i)e_+ u = De_+ u - ie_+ u$$

$$= i (e_+ u)' - ie_+ u$$

$$= i u(0)\delta + e_+(Du) - ie_+ u$$

$$= e_+(D - i)u + i u(0)\delta.$$

This completes the proof of the lemma.

\[ \square \]

If $1 + 1/p < s < 2 + 1/p$, we have the following equivalent of Lemma 4.1.

**Lemma 4.2.** Suppose $1 < p < \infty$ and $1 + 1/p < s < 2 + 1/p$. If $u \in H^s_p(\mathbb{R}_+)$ then

$$(D - i)^2 e_+ u = e_+(D - i)^2 u - u(0)\delta' + 2u(0)\delta - u'(0)\delta.$$

**Proof.** We first show that if $v \in r_+ C^\infty_0(\mathbb{R})$, then $(e_+ v)'' = v(0)\delta' + v'(0)\delta + e_+ v'''$. From the proof of Lemma 4.1,

$$(e_+ v)' = v(0)\delta + e_+ v'$$

$$= v(0)\delta' + v'(0)\delta + e_+ v''.$$

Since $1 + 1/p < s < 2 + 1/p$, the values $u(0)$ and $u'(0)$ are well-defined. (See, for example, Section 2.9, [44]). Therefore, by continuity

$$(D - i)^2 e_+ u = D^2 e_+ u - 2iDe_+ u - e_+ u$$

$$= -(e_+ u)'' + 2(e_+ u)' - e_+ u$$

$$= [e_+ u'' - u(0)\delta' - u'(0)\delta] + 2[e_+ u' + u(0)\delta] - e_+ u$$

$$= -e_+ (u'' - 2u' + u) - u(0)\delta' + 2u(0)\delta - u'(0)\delta$$

$$= e_+(D - i)^2 u - u(0)\delta' + 2u(0)\delta - u'(0)\delta,$$

as required.

\[ \square \]

**Lemma 4.3.** Suppose $1 < p < \infty$, $1/p < s < 1 + 1/p$ and $u \in H^s_p(\mathbb{R}_+)$. Let

$$u_s := (D + i)^{s-1} e_+(D - i)u.$$  

Then $u_s \in L_p(\mathbb{R})$ with supp $u_s \subseteq \mathbb{R}_+.$
Proof. Let \( u_{(-1)} = (D - i)u \). Since \( u \in H^s_p(\mathbb{R}_+) \), we have \( u = r_+u_0 \) for some \( u_0 \in H^s_p(\mathbb{R}) \). Hence

\[
u_{(-1)} = (D - i)u = (D - i)r_+u_0 = r_+(D - i)u_0.
\]

But \( (D - i)u_0 \in H^{s-1}_p(\mathbb{R}) \) and, therefore, \( u_{(-1)} \in H^{s-1}_p(\mathbb{R}_+) \).

Since, by hypothesis, \( \frac{1}{p} - 1 < s - 1 < \frac{1}{p} \), from Section 2.10.3, p. 232, [44], we have \( e_+u_{(-1)} \in H^{s-1}_p(\mathbb{R}_+) \) and, of course, \( \text{supp} \; e_+u_{(-1)} \subseteq \mathbb{R}_+ \). Now since \( u_s = (D + i)^{s-1}e_+u_{(-1)} \), then, see Theorem 1.9, p. 52, [41], we have \( u_s \in L^p(\mathbb{R}) \) and \( \text{supp} \; u_s \subseteq \mathbb{R}_+ \). This completes the proof of the lemma.

The following counterpart of Lemma 4.3 applies in the case of higher regularity, namely \( 1 + 1/p < s < 2 + 1/p \).

**Lemma 4.4.** Suppose \( 1 < p < \infty \), \( 1 + 1/p < s < 2 + 1/p \) and \( u \in H^s_p(\mathbb{R}_+) \). Let \( u_s := (D + i)^{s-2}e_+(D - i)^2u \). Then \( u_s \in L^p(\mathbb{R}) \) with \( \text{supp} \; u_s \subseteq \mathbb{R}_+ \).

Proof. The proof follows the method used in the proof of Lemma 4.3.

**Lemma 4.5.** Suppose \( 1 < p < \infty \) and \( \sigma, \nu \in \mathbb{R} \). Let \( l_+ : H^\sigma_p(\mathbb{R}_+) \to H^\sigma_p(\mathbb{R}_+) \) be an arbitrary extension operator. Then \( \Lambda^\nu_+ = r_+(D - i)^\nu l_+ \) is bounded from \( H^\sigma_p(\mathbb{R}_+) \) to \( H^{\sigma-\nu}_p(\mathbb{R}_+) \), and does not depend on the choice of extension \( l_+ \).

Moreover,

\[
(r_+(D - i)^\nu l_+)r_+ = r_+(D - i)^\nu.
\]

Proof. From Theorem 1.12, p. 54, [41], the pseudodifferential operator \( (D - i)^\nu \) is bounded from \( H^\sigma_p(\mathbb{R}) \) to \( H^{\sigma-\nu}_p(\mathbb{R}) \). In addition, its symbol \( (\xi - i)^\nu \) admits an analytic continuation with respect to \( \xi \) to the lower complex half-plane such that

\[
|\xi + i\tau - i\nu| \leq (|\xi| + |\tau| + 1)^{\max(0, \nu)}, \quad \tau \leq 0.
\]

Therefore, from Theorem 1.10, p. 53, [41], \( \Lambda^\nu_+ = r_+(D - i)^\nu l_+ \) is continuous from \( H^\sigma_p(\mathbb{R}_+) \) to \( H^{\sigma-\nu}_p(\mathbb{R}_+) \), and does not depend on the choice of the extension \( l_+ \).

Finally, by Remark 1.11, p. 53 [41], we also have

\[
(r_+(D - i)^\nu l_+)r_+ = r_+(D - i)^\nu.
\]

This completes the proof of the lemma.
Lemma 4.6. Suppose \( 1 < p < \infty \) and \( 1/p < s < 1 + 1/p \). If \( \varphi \in r_+S(\mathbb{R}) \) then the map \( T_\varphi : H^s_p(\mathbb{R}^+) \to r_+\tilde{H}^s_p(\mathbb{R}^+) \) given by

\[
(T_\varphi u)(x) = \varphi(x)(u(x) - u(0)) \quad (x > 0),
\]

is bounded.

Proof. Firstly, we note that \( \varphi \in H^s_p(\mathbb{R}_+) \). Moreover, since \( s > 1/p \), \( H^s_p(\mathbb{R}^+) \) is a Banach algebra. (See Section 2.8.3, Remark 3, p. 146, [45].) Hence, \( \varphi u \in H^s_p(\mathbb{R}^+) \) and

\[
\varphi(x)(u(x) - u(0)) \in H^s_p(\mathbb{R}^+).
\]

But \( \varphi(x)(u(x) - u(0))|_{x=0} = 0 \), and hence by Corollary 3.4.3, p. 210, [45],

\[
T_\varphi u \in r_+\tilde{H}^s_p(\mathbb{R}^+).
\]

Finally, we note that

\[
\|T_\varphi u|_{r_+\tilde{H}^s_p(\mathbb{R}^+)}\| = \|\varphi(x)(u(x) - u(0))|_{r_+\tilde{H}^s_p(\mathbb{R}^+)}\|
\leq \text{const} \|\varphi(x)(u(x) - u(0))|_{H^s_p(\mathbb{R}^+)}\|
\leq \text{const} \|\varphi u\|_{s,p} + \|\varphi u(0)\|_{s,p}
\leq \text{const} \|u\|_{s,p},
\]

since \( s > 1/p \), and thus \( |u(0)| \leq \|u\|_{s,p} \) by the Sobolev embedding theorem.

\[\square\]

Remark 4.7. Using the same method of proof as Lemma 4.6, it is easy to show that if \( 1 + 1/p < s < 2 + 1/p \), then the map \( T_\varphi : H^s_{p,0}(\mathbb{R}^+) \to r_+\tilde{H}^s_p(\mathbb{R}^+) \), as defined above, is also bounded.

Lemma 4.8. Suppose \( \alpha < 1 \). Then, for \( x > 0 \),

\[
\mathcal{F}^{-1}(1 + \xi^2)^{\alpha - 1} = \frac{2^\alpha}{\Gamma(1 - \alpha)} x^{-\alpha + \frac{1}{2}} K_{\alpha - \frac{1}{2}}(x)
= \sqrt{\frac{\pi}{2}} \frac{2^\alpha}{\Gamma(1 - \alpha)} e^{-x} U(\alpha, 2\alpha, 2x),
\]

where \( K_{\nu}(x) \) and \( U(a,b,x) \) denote the modified Bessel function and confluent hypergeometric function respectively.
Proof. By definition, for $x > 0$, 
\[ F^{-1}(1 + \xi^2)^{\alpha - 1} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (1 + \xi^2)^{\alpha - 1} e^{-ix\xi} d\xi \]
\[ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (1 + \xi^2)^{\alpha - 1} \cos \xi x d\xi \]
\[ = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} (1 + \xi^2)^{\alpha - 1} \cos \xi x d\xi. \]

From 3.771 2, p. 445, \[\text{[17]},\] we have 
\[ \int_{0}^{\infty} (1 + \xi^2)^{\nu - \frac{1}{2}} \cos \xi x d\xi = \frac{1}{\sqrt{\pi}} \left(\frac{2}{x}\right)^{\nu} \cos(\pi \nu) \Gamma(\nu + \frac{1}{2}) K_{-\nu}(x), \]
provided $x > 0$ and $\nu < \frac{1}{2}$. Hence, taking $\nu = \alpha - \frac{1}{2}$ we have 
\[ F^{-1}(1 + \xi^2)^{\alpha - 1} = 2 \frac{\sqrt{\pi}}{\pi} \cdot \frac{1}{\sqrt{\pi}} \left(\frac{2}{x}\right)^{\alpha - \frac{1}{2}} \cos \pi(\alpha - \frac{1}{2}) \Gamma(\alpha) K_{-\alpha}(x) \]
\[ = \frac{\sqrt{2}}{\pi} \cdot 2^{\alpha - \frac{1}{2}} \sin(\pi \alpha) \Gamma(\alpha) x^{-\alpha + \frac{1}{2}} K_{\alpha - \frac{1}{2}}(x) \quad (K_{-\nu}(x) = K_{\nu}(x)) \]
\[ = \frac{2^{\alpha}}{\Gamma(1 - \alpha)} x^{-\alpha + \frac{1}{2}} K_{\alpha - \frac{1}{2}}(x) \quad \text{(see 5.5.3, [34])} \]
\[ = \frac{2^{\alpha}}{\Gamma(1 - \alpha)} x^{-\alpha + \frac{1}{2}} K_{\alpha - \frac{1}{2}}(x), \quad \text{as required.} \]

Finally, since $K_{\nu}(z) = \sqrt{\pi}(2z)^{\nu} e^{-z} U(\nu + \frac{1}{2}, 2\nu + 1, 2z)$, see 10.39.6, \[\text{[34]},\] we have 
\[ F^{-1}(1 + \xi^2)^{\alpha - 1} = \frac{2^{\alpha}}{\Gamma(1 - \alpha)} x^{-\alpha + \frac{1}{2}} \cdot \sqrt{\pi} (2x)^{\alpha - \frac{1}{2}} e^{-x} U(\alpha, 2\alpha, 2x) \]
\[ = \sqrt{\pi} \cdot \frac{2^{2\alpha}}{\Gamma(1 - \alpha)} e^{-x} U(\alpha, 2\alpha, 2x). \]
This completes the proof of the lemma. 

\[ \square \]

Lemma 4.9. Suppose $\alpha < 1$. Then 
\[ (r_+ A^\nu A^\delta)(x) = C_\alpha e^{-x} U(\alpha + 1, 2\alpha + 1, 2x), \]
where the constant $C_\alpha$ depends only on $\alpha$, and is given by 
\[ C_\alpha = -i \cdot \frac{\alpha 2^{2\alpha}}{\Gamma(1 - \alpha)}. \quad (4.8) \]
Proof. Now, by definition, we have

\[
A^{-}(D)\delta = \mathcal{F}^{-1}A^{-}(\xi)\mathcal{F}\delta \\
= \mathcal{F}^{-1}(1 + \xi^2)^\alpha(\xi - i)^{-1} \cdot (1/\sqrt{2\pi}) \\
= \frac{i}{\sqrt{2\pi}} \mathcal{F}^{-1}(1 + \xi^2)^\alpha(1 + i\xi)^{-1} \\
= \frac{i}{\sqrt{2\pi}} \left\{ \left(1 + \frac{d}{dx}\right) \mathcal{F}^{-1}(1 - i\xi)^{-1}\mathcal{F} \right\} \mathcal{F}^{-1}(1 + \xi^2)^\alpha(1 + i\xi)^{-1} \\
= \frac{i}{\sqrt{2\pi}} \left(1 + \frac{d}{dx}\right) \mathcal{F}^{-1}(1 + \xi^2)^{\alpha - 1}. \tag{4.9}
\]

Using Lemma 4.8, we can write

\[
r_+ A^{-}(D)\delta = \frac{i}{\sqrt{2\pi}} \sqrt{\frac{\pi}{2}} \frac{2^{2\alpha}}{\Gamma(1 - \alpha)} \left(1 + \frac{d}{dx}\right) e^{-x} U(\alpha, 2\alpha, 2x) \\
= \frac{i}{\sqrt{2\pi}} \frac{2^{2\alpha}}{\Gamma(1 - \alpha)} \left(1 + \frac{d}{dx}\right) e^{-x} U(\alpha, 2\alpha, 2x) \\
= \frac{i}{\sqrt{2\pi}} \frac{2^{2\alpha}}{\Gamma(1 - \alpha)} e^{-x} \frac{d}{dx} U(\alpha, 2\alpha, 2x) \\
= \frac{i}{\sqrt{2\pi}} \frac{2^{2\alpha}}{\Gamma(1 - \alpha)} e^{-x} (-2\alpha) U(\alpha + 1, 2\alpha + 1, 2x) \quad \text{(see 13.3.22, [34])} \\
= -i \frac{\alpha 2^{2\alpha}}{\Gamma(1 - \alpha)} e^{-x} U(\alpha + 1, 2\alpha + 1, 2x) \\
= C_\alpha e^{-x} U(\alpha + 1, 2\alpha + 1, 2x).
\]

This completes the proof of the lemma.

\[\square\]

Lemmas 4.10 and 4.11 are the counterparts of Lemma 4.9 for the operator \(A^\equiv\).

Lemma 4.10. Suppose \(\alpha < 1\). Then

\[
(r_+ A^{\equiv}\delta)(x) = \frac{i}{2} i C_\alpha e^{-x} U(\alpha + 1, 2\alpha, 2x),
\]

where the constant \(C_\alpha\), defined in equation (4.8), depends only on \(\alpha\).
Proof. Now, by definition, we have

\[ A^\pm(D)\delta = \mathcal{F}^{-1} A^\pm(\xi) \mathcal{F}\delta \]
\[ = \mathcal{F}^{-1} (1 + \xi^2)^\alpha (\xi - i)^{-2} \cdot (1/\sqrt{2\pi}) \]
\[ = -\frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1} (1 + \xi^2)^\alpha (1 + i\xi)^{-2} \]
\[ = -\frac{1}{\sqrt{2\pi}} \left( \left( 1 + \frac{d}{dx} \right)^2 \mathcal{F}^{-1} (1 - i\xi)^{-2} \mathcal{F} \right) \mathcal{F}^{-1} (1 + \xi^2)^\alpha (1 + i\xi)^{-2} \]
\[ = -\frac{1}{\sqrt{2\pi}} \left( 1 + \frac{d}{dx} \right)^2 \mathcal{F}^{-1} (1 + \xi^2)^{\alpha - 2}. \]

Using Lemma 4.8, and noting that \( \alpha - 2 = (\alpha - 1) - 1 \), we can write

\[
r_+ A^-(D)\delta = -\frac{1}{\sqrt{2\pi}} \sqrt{\frac{\pi}{2}} \frac{2^{2\alpha - 2}}{\Gamma(1 - (\alpha - 1))} \left( 1 + \frac{d}{dx} \right)^2 e^{-x} U(\alpha - 1, 2\alpha - 2, 2x) \]
\[ = -\frac{1}{2} \frac{2^{2\alpha - 2}}{\Gamma(2 - \alpha)} e^{-x} \frac{d^2}{dx^2} U(\alpha - 1, 2\alpha - 2, 2x) \]
\[ = -\frac{1}{2} \frac{2^{2\alpha - 2}}{\Gamma(2 - \alpha)} e^{-x} (\alpha - 1)\alpha 2^2 U(\alpha + 1, 2\alpha, 2x) \quad \text{(See 13.3.23, [34])} \]
\[ = \frac{\alpha 2^{2\alpha - 1}}{\Gamma(1 - \alpha)} e^{-x} U(\alpha + 1, 2\alpha, 2x) \quad \text{(since } \Gamma(2 - \alpha) = (1 - \alpha)\Gamma(1 - \alpha)) \]
\[ = \frac{1}{2} iC_\alpha e^{-x} U(\alpha + 1, 2\alpha, 2x). \]

This completes the proof of the lemma.

\[ \square \]

Lemma 4.11. Suppose \( \alpha < 1 \). Then

\[ (r_+ A^+(\delta' - \delta))(x) = -iC_\alpha e^{-x} U(\alpha + 1, 2\alpha + 1, 2x), \]

where the constant \( C_\alpha \), defined in equation (4.8), depends only on \( \alpha \).

Proof. Firstly, we note that

\[
\mathcal{F}(\delta' - \delta) = \mathcal{F}(\delta') - \mathcal{F}(\delta) \\
= -i^2 \mathcal{F}(\delta') - \mathcal{F}(\delta) \\
= -i\mathcal{F}(D\delta) - \mathcal{F}(\delta) \\
= -(1 + i\xi)\mathcal{F}\delta.
\]
Now, by definition, we have

\[ A^\pm(D)(\delta' - \delta) = \mathcal{F}^{-1} A^\pm(\xi) \mathcal{F}(\delta' - \delta) \]
\[ = \mathcal{F}^{-1} (1 + \xi^2)^\alpha (\xi - i)^{-2} \cdot (-1)(1 + i\xi)(1/\sqrt{2\pi}) \]
\[ = \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1} (1 + \xi^2)^\alpha (1 + i\xi)^{-1} \]
\[ = \frac{1}{\sqrt{2\pi}} \left( \frac{d}{dx} \right) \mathcal{F}^{-1} (1 - i\xi)^{-1} \mathcal{F}^{-1} (1 + \xi^2)^\alpha (1 + i\xi)^{-1} \]
\[ = \frac{1}{\sqrt{2\pi}} \left( 1 + \frac{d}{dx} \right) \mathcal{F}^{-1} (1 + \xi^2)^{\alpha - 1}. \]

Hence, from equation (4.9),

\[ r_+ A^\pm(D)(\delta' - \delta) = -i C_\alpha e^{-x} U(\alpha + 1, 2\alpha + 1, 2x). \]

This completes the proof of the lemma.

\[ \square \]

**Lemma 4.12.** Suppose \( a > 0 \) and \( 0 < b < 3 \). Then, for \( x > 0 \),

\[ e^{-x} U(a, b, 2x) = \begin{cases} 
  x^{1-b} \psi(a, b, x) + \phi(x) & \text{if } b \neq 1, 2 \\
  x^{1-b} \psi(a, b, x) + \vartheta(x) \log x + \phi(x) & \text{if } b = 2 \\
  \vartheta(x) \log x + \phi(x) & \text{if } b = 1,
\end{cases} \]

where \( \vartheta, \phi \in C^\infty(\mathbb{R}) \), and their derivatives, are bounded and \( O(e^{-x}) \) as \( x \to +\infty \). Moreover, \( \psi \in C^\infty_0(\mathbb{R}) \) with \( \psi(a, b, x) = 0 \) for \( x > 2 \). Finally,

\[ \psi(a, b, 0) = 2^{1-b} \frac{\Gamma(b - 1)}{\Gamma(a)} \quad (b \neq 1). \]

**Proof.** Suppose \( 0 < b < 3 \) with \( b \neq 1, 2 \). From 13.1.3, [1], \( U(a, b, 2x) \in C^\infty([1, \infty)) \). Moreover, from 13.5.2, [1], for \( x \geq \frac{1}{2} \) the function \( U(a, b, 2x) \), together with its derivatives, is bounded and \( O(x^{-a}) \) as \( x \to +\infty \).

On the other hand, we can write (see 13.1.3, [1]),

\[ U(a, b, 2x) = F(a, b, x) + x^{1-b}G(a, b, x), \]

where
where $F, G \in C^\infty([0, 2])$. Let $\varphi \in C_0^\infty(\mathbb{R})$ be such that

$$
\varphi(x) = \begin{cases} 
1 & \text{if } |x| \leq 1 \\
0 & \text{if } |x| > 2.
\end{cases}
$$

Then, for $x > 0$, we have

$$
e^{-x}U(a, b, 2x) = \varphi(x)e^{-x}U(a, b, 2x) + (1 - \varphi(x))e^{-x}U(a, b, 2x)
= \varphi(x)e^{-x}(F(a, b, x) + x^{1-b}G(a, b, x)) + (1 - \varphi(x))e^{-x}U(a, b, 2x)
= \{ \varphi(x)e^{-x}F(a, b, x) + (1 - \varphi(x))e^{-x}U(a, b, 2x) \} + x^{1-b}\{ \varphi(x)e^{-x}G(a, b, x) \}
:= \phi(x) + x^{1-b}\psi(a, b, x),$$

where $\phi \in C^\infty(\mathbb{R})$ and, together with its derivatives, is bounded and $O(e^{-x})$ as $x \to +\infty$. Moreover, $\psi \in C_0^\infty(\mathbb{R})$ with $\psi(a, b, x) = 0$ for $x > 2$.

Now, see 13.5.6 and 13.5.8, [1],

$$
\psi(a, b, 0) = G(a, b, 0) = 2^{1-b}\frac{\Gamma(b - 1)}{\Gamma(a)}.
$$

Finally, the proof for each of the remaining cases, $b = 1, 2$, follows in a similar manner, but using the logarithmic solution described in 13.1.6, [1].

In the following two lemmas, we make use of a Mellin integral operator with kernel $K_{2\alpha}$. See Section 1.1 for more details. In addition, the operator $C_{0+}^{2\alpha}$ is discussed in Appendix D.

**Lemma 4.13.** Suppose $0 < \alpha < \frac{1}{2}$ and $u \in r_+C_0^\infty(\mathbb{R})$. Then

$$
x^{-2\alpha}(u(x) - u(0)) = \int_0^\infty K_{2\alpha}\left(\frac{x}{y}\right)h(y) \frac{dy}{y},
$$

where $h(x) = (C_{0+}^{2\alpha}u)(x)$ and

$$
K_{2\alpha}(t) = \frac{\chi_{[1,\infty)}(t)}{\Gamma(2\alpha)t^{2\alpha}(t - 1)^{1-2\alpha}}.
$$

**Proof.** From Appendix D equation (D.6), taking $a = 0$,

$$
u(x) - u(0) = I_0^{2\alpha}C_{0+}^{2\alpha}u(x).
$$
Now consider the operator \((P_{2\alpha}u)(x) = x^{-2\alpha}[u(x) - u(0)]\). We have
\[
(P_{2\alpha}u)(x) = x^{-2\alpha}(I_{0^+}^{2\alpha}C_{0^+}^{2\alpha}u)(x) = x^{-2\alpha}(I_{0^+}^{2\alpha}h)(x) \quad \text{(where } h(x) = (C_{0^+}^{2\alpha}u)(x))
\]
\[
= \frac{1}{\Gamma(2\alpha)} \int_0^x \frac{h(y)}{x^{2\alpha}(x-y)^{1-2\alpha}} \, dy
\]
\[
= \frac{1}{\Gamma(2\alpha)} \int_0^\infty \chi_{[0,x]}(y) \frac{h(y)}{x^{2\alpha}(x-y)^{1-2\alpha}} \, dy
\]
\[
= \frac{1}{\Gamma(2\alpha)} \int_0^\infty \chi_{[1,\infty)}(\frac{x}{y}) \frac{h(y)}{x^{2\alpha}(x-y)^{1-2\alpha}} \, dy
\]
\[
= \int_0^\infty K_{2\alpha}(\frac{x}{y})h(y) \frac{dy}{y}.
\]

\[
\square
\]

**Remark 4.14.** Lemma 4.15 is the counterpart of Lemma 4.13 in the case that \(\frac{1}{2} \leq \alpha < 1\). We note, in particular, that the required boundary condition, \(u'(0) = 0\), means we will not consider the case \(\frac{1}{2} \leq \alpha < 1\) and \(1/p < s < 1 + 1/p\).

**Lemma 4.15.** Suppose \(\frac{1}{2} \leq \alpha < 1\) and \(u \in r_+C_{0}^\infty(\mathbb{R})\) with \(u'(0) = 0\). Then
\[
x^{-2\alpha}(u(x) - u(0)) = \int_0^\infty K_{2\alpha}(\frac{x}{y})h(y) \frac{dy}{y},
\]
where \(h(x) = (C_{0^+}^{2\alpha}u)(x)\) and \(K_{2\alpha}\) is as defined in Lemma 4.13.

**Proof.** From Appendix D equation (D.7), taking \(a = 0\) and \(u'(0) = 0\),
\[
u(x) - u(0) = I_{0^+}^{2\alpha}C_{0^+}^{2\alpha}u(x),
\]
and the proof now follows as Lemma 4.13.

Finally, we note, in passing, that if \(\alpha = \frac{1}{2}\), then
\[
(I_{0^+}^{1}u)(x) = \int_0^x u(y) \, dy; \quad (C_{0^+}^{1}u)(x) = u'(x) - u'(0) = u'(x);
\]
and we simply have
\[
u(x) - u(0) = \int_0^x u'(y) \, dy.
\]

\[
\square
\]
Lemma 4.16. Suppose $1 < p < \infty$. Let $M$ denote the Mellin integral operator with kernel $K$, as defined in (1.6). Then $M$ is bounded on $L_p(\mathbb{R}_+)$ if the function $K(t) t^{-1/p'}$ belongs to $L_1(\mathbb{R}_+)$, where $1/p + 1/p' = 1$.

Proof. By definition, see (1.6), the action of the operator $M$ on $u \in L_p(\mathbb{R}_+)$ is given by

$$(Mu)(t) = \int_0^\infty K\left(\frac{t}{\tau}\right) \frac{u(\tau)}{\tau} d\tau.$$ 

We now define $t/\tau = x$, and hence can write

$$(Mu)(t) = \int_0^\infty K(x) u(t/x)x^{-1} dx. \quad (4.11)$$

By definition,

$$\|u(\cdot/x)\|_p := \left( \int_0^\infty |u(t/x)|^p dt \right)^{1/p} = \left( \int_0^\infty |u(s)|^p x ds \right)^{1/p} \quad (\text{where } s = t/x) = x^{1/p} \|u\|_p.$$ 

Applying the $L_p(\mathbb{R}_+)$ norm to equation (4.11) we have

$$\|Mu\|_p = \left\| \int_0^\infty K(x)u(\cdot/x)x^{-1} dx \right\|_p \leq \int_0^\infty \|K(x)u(\cdot/x)x^{-1}\|_p dx \quad \text{(by Minkowski integral inequality)}$$

$$= \int_0^\infty |K(x)x^{-1}| \cdot \|u(\cdot/x)\|_p dx \quad \text{=} \quad \left( \int_0^\infty |K(x)x^{-1/p'}| dx \right) \cdot \|u\|_p,$$

which completes the proof of the lemma.

Lemma 4.17. Suppose $1 < p < \infty$, $\rho > 1/p - 1$ and $\gamma > 0$. Then the Mellin integral operator $M_{\gamma,\rho}$ with kernel

$$K_{\gamma,\rho}(t) = \frac{\chi_{[1,\infty)}(t)}{t^\rho \Gamma(\gamma) t^\gamma (t - 1)^{1-\gamma}}$$
is bounded on $L_p(\mathbb{R}^+)$. Moreover, see (1.8), $M_{\gamma, \rho}$ has symbol

$$b(y) := (M_p K_{\gamma, \rho})(y) = \frac{B(\rho + 1/p' + iy, \gamma)}{\Gamma(\gamma)},$$

where $M_p$ denotes the Mellin transform.

**Proof.** From Lemma 4.16, to prove boundedness on $L_p(\mathbb{R}^+)$ it is enough to show that

$$\Gamma(\gamma) \int_0^\infty |K_{\gamma, \rho}(t)| t^{-1/p'} dt < \infty.$$

We will make use of the following result, see 5.12.3, [34],

$$\int_0^\infty \frac{t^{a-1} dt}{(1 + t)^{a+b}} = B(a, b), \quad \text{Re } a > 0, \text{ Re } b > 0,$$

where $B(\cdot, \cdot)$ denotes the beta function. Now

$$\Gamma(\gamma) \int_0^\infty |K_{\gamma, \rho}(t)| t^{-1/p'} dt = \int_1^\infty t^{-\rho} t^{-\gamma} (t - 1)^{\gamma-1} t^{-1/p'} dt$$

$$= \int_1^\infty t^{-(\rho+\gamma+1/p')} (t - 1)^{\gamma-1} dt$$

$$= \int_0^\infty (1 + w)^{-(\rho+\gamma+1/p')} w^{\gamma-1} dw \quad (w = t - 1)$$

$$= \int_0^\infty \frac{w^{\gamma-1} dw}{(1 + w)^{\gamma+\rho+1/p'}}$$

$$= B(\gamma, \rho + 1/p')$$

$$= B(\rho + 1/p', \gamma)$$

$$< \infty.$$

From (1.8), to calculate the symbol, we take the Mellin transform of the kernel:

$$(M_p K_{\gamma, \rho})(y) = \int_0^\infty t^{-1/p'-iy} K_{\gamma, \rho}(t) dt$$

$$= \int_0^\infty \frac{w^{\gamma-1} dw}{\Gamma(\gamma) (1 + w)^{\gamma+\rho+1/p'+iy}}$$

$$= \frac{B(\rho + 1/p' + iy, \gamma)}{\Gamma(\gamma)},$$

as required. This completes the proof of the lemma.
Suppose a function $f : \mathbb{R} \to \mathbb{C}$. Then we define the total variation, $V(f)$ as

$$V(f) := \sup \left( \sum_{k=1}^{N} |f(t_k) - f(t_{k-1})| \right),$$

where the supremum is taken over all partitions $-\infty \leq t_0 < t_1 < \cdots < t_N \leq +\infty$ of $\mathbb{R}$. We denote the set of all bounded functions on $\mathbb{R}$ with finite total variation by $BV(\mathbb{R})$. See [11, 37]. We note, in passing, that this set is a Banach space under the norm

$$\|f\|_{BV} := \|f\|_{\infty} + V(f).$$

One important motivation for the study of functions of bounded variation, see, for example, Proposition 4.2.2, p. 200, [37], is the inclusion

$$BV(\mathbb{R}) \subset M_{p}, \quad 1 < p < \infty, \quad (4.12)$$

The following remark describes a useful way to demonstrate that certain functions have bounded variation on $\mathbb{R}$.

**Remark 4.18.** Suppose $f : \mathbb{R} \to \mathbb{C}$ is bounded, and differentiable almost everywhere with $f' \in L_1(\mathbb{R})$. Since we can write

$$f(t_k) - f(t_{k-1}) = \int_{t_{k-1}}^{t_k} f'(t) \, dt \quad \text{for} \quad k = 1, \ldots, N,$$

it is easy to see that

$$V(f) \leq \|f'\|_1.$$

Therefore, $f \in BV(\mathbb{R})$.

**Lemma 4.19.** Suppose $d > c > 0$, and define the function

$$g(y) := \frac{\Gamma(c + iy)}{\Gamma(d + iy)}, \quad y \in \mathbb{R}.$$  

Then $g$ is continuous and bounded on $\mathbb{R}$, and has bounded variation.

**Proof.** Since $c, d > 0$, by 5.2.1, [34], the functions $\Gamma(c + iy), \Gamma(d + iy)$ are continuous for $y \in \mathbb{R}$, and have no zeroes. Hence, $g$ is continuous on $\mathbb{R}$.  

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Moreover, from 5.11.12, [34], we have the following asymptotic
\[ g(y) \sim (iy)^{c-d}, \quad |y| \to \infty. \]
Thus, as \( c - d < 0 \), the function \( g \) is continuous and bounded on \( \mathbb{R} \).

In terms of the digamma function, \( \psi(z) \), see 5.2.2, [34],
\[ g'(y) = ig(y)\left(\psi(c + iy) - \psi(d + iy)\right). \]
From 5.11.2, [34], \( \psi(z) \sim \log z \) as \( |z| \to \infty \), and we have
\[ g'(y) \sim (iy)^{c-d} \log \left(\frac{c + iy}{d + iy}\right) \sim (iy)^{c-d} \left(\frac{c - d}{d + iy}\right). \]
Since \( c - d < 0 \), it clear that \( g' \in L_1(\mathbb{R}) \).

Finally, from Remark 4.18, \( g \) has bounded variation on \( \mathbb{R} \).

\[ \square \]

**Remark 4.20.** Suppose \( \gamma > 0 \). We note from Lemma 4.17, that the kernel \( K_{\gamma,0} \), of the Mellin integral operator \( M_{\gamma,0} \), satisfies the conditions
\[ \text{supp } K_{\gamma,0} \subseteq [1, \infty) \quad \text{and} \quad \int_0^\infty |K_{\gamma,0}(t)|t^{-\epsilon} dt < \infty, \quad \text{for all } \epsilon > 0. \quad (4.13) \]
If, in addition, \( 1 < p < \infty \) and \( \rho > 1/p - 1 \) then, from Lemma 4.19, and its proof, the symbol \( b_{\gamma,\rho}(y) = B(\rho + 1/p' + iy, \gamma)/\Gamma(\gamma) \) is continuous, with bounded variation, as \( y \) varies over \( \mathbb{R} \). Moreover, \( b_{\gamma,\rho}(\pm\infty) = 0 \).

From inclusion (4.12), \( b_{\gamma,\rho} \) is a Fourier \( L_p \)-multiplier. Hence, see equation (1.5), \( M_{\gamma,\rho} = M^0(b_{\gamma,\rho}) \) is a Mellin convolution operator.

We will make extensive use of Remark 4.20 in subsequent chapters.

**Lemma 4.21.** Suppose \( 1/p < s < 1 + 1/p \) and \( u \in H^s_p(\mathbb{R}_+) \). Let \( u_s = (D + i)^{s-1}e_+(D - i)u \). Then
\[ u = (r_+ C_s(D) e_+) (r_+ u_s), \]
where \( C_s(D) \) has the symbol \( c_s(\xi) = (\xi - i)^{-1}(\xi + i)^{1-s} \).
Proof. We have $u_s = (D + i)^{s - 1}e_+(D - i)u$. Hence

$$(D + i)^{1-s}u_s = e_+(D - i)u$$

$$r_+(D + i)^{1-s}u_s = (D - i)u$$

$$r_+(D + i)^{1-s}u_s = (D - i)(r_+u_0)$$

where $u = r_+u_0$, $u_0 \in H_p^s(\mathbb{R})$

$$r_+(D + i)^{1-s}u_s = r_+(D - i)u_0$$

$$(r_+(D - i)^{-1}l_+)r_+(D + i)^{1-s}u_s = (r_+(D - i)^{-1}l_+)r_+(D - i)u_0$$

$$r_+(D - i)^{-1}(D + i)^{1-s}u_s = r_+(D - i)^{-1}(D - i)u_0$$  by Lemma 4.5

$$r_+(D - i)^{-1}(D + i)^{1-s}e_+r_+u_s = r_+u_0$$  since $\text{supp} u_\alpha \subseteq \mathbb{R}_+$ by Lemma 4.3

$$(r_+C_s(D)e_+)(r_+u_s) = u,$$  as required.

\[ \square \]

If $1 + 1/p < s < 2 + 1/p$, we have the following counterpart to Lemma 4.21.

**Lemma 4.22.** Suppose $1 + 1/p < s < 2 + 1/p$ and $u \in H_p^s(\mathbb{R}_+)$. Let $u_s = (D + i)^{s - 2}e_+(D - i)^2u$. Then

$$u = (r_+C_s(D)e_+)(r_+u_s),$$

where $C_s(D)$ has the symbol $c_s(\xi) = (\xi - i)^{-2}(\xi + i)^{2-s}$.

**Proof.** The proof follows as Lemma 4.21, but using Lemma 4.4 instead of Lemma 4.3.

\[ \square \]

**Lemma 4.23.** Suppose $0 < \alpha < \frac{1}{2}$, $1 < p < \infty$, $1/p < s < 1 + 1/p$ and $u \in H_p^s(\mathbb{R}_+)$. Let $h = C_{2a}^s u$ and $u_s = (D + i)^{s - 1}e_+(D - i)u$. Then

$$h = (r_+C(D)e_+)(r_+u_s) + i \frac{u(0)}{\sqrt{2\pi}} r_+F^{-1}(-i\xi)^{2a-1}(\xi - i)^{-1},$$

where $C(D)$ has the symbol $c(\xi) = (-i\xi)^{2a}(\xi + i)^{1-s}(\xi - i)^{-1}$.

**Proof.** From Lemma 4.1 and the definition of $u_s$, we have

$$F(e_+u) = F((D - i)^{-1}(D - i)e_+u)$$

$$= F((D - i)^{-1}e_+(D - i)u) + F((D - i)^{-1}i u(0) \delta)$$

$$= F((D - i)^{-1}(D + i)^{1-s}u_s) + i u(0) (\xi - i)^{-1} \cdot \frac{1}{\sqrt{2\pi}}$$

$$= (\xi - i)^{-1}(\xi + i)^{1-s}F(u_s) + i u(0) (\xi - i)^{-1} \cdot \frac{1}{\sqrt{2\pi}}.$$
Moreover,
\[ i e_+ h(x) = i e_+(C^2_0 u)(x) = e_+ I_0^{1-2\alpha} D u \] (see Appendix D)
\[ = I_+^{1-2\alpha} (e_+ D u) = I_+^{1-2\alpha} (e_+(D - i)u + i e_+ u) = I_+^{1-2\alpha}((D + i)\gamma u_s + i e_+ u). \]

Applying the Fourier transform, see Appendix D,
\[ i \mathcal{F}(e_+ h) = (-i\xi)^{2\alpha - 1} \{ \mathcal{F}((D + i)\gamma u_s) + i \mathcal{F}(e_+ u) \} \]
\[ = (-i\xi)^{2\alpha - 1} \left\{ (\xi + i)^{1-s} \mathcal{F}(u_s) + i (\xi - i)^{-1} (\xi + i)^{1-s} \mathcal{F}(u_s) + i^2 \frac{u(0)}{\sqrt{2\pi}} (\xi - i)^{-1} \right\} \]
\[ = (-i\xi)^{2\alpha - 1} (\xi + i)^{1-s} \mathcal{F}(u_s) \{ 1 + i (\xi - i)^{-1} \} - \frac{u(0)}{\sqrt{2\pi}} (-i\xi)^{2\alpha - 1} (\xi - i)^{-1}. \]

Noting that \( 1 + i (\xi - i)^{-1} = \xi (\xi - i)^{-1} \) we have
\[ \mathcal{F}(e_+ h) = (-i\xi)^{2\alpha} (\xi + i)^{1-s} (\xi - i)^{-1} \mathcal{F}(u_s) + i \frac{u(0)}{\sqrt{2\pi}} (-i\xi)^{2\alpha - 1} (\xi - i)^{-1}. \]

But since \( \text{supp } u_s \subseteq \mathbb{R}_+ \),
\[ h = (r_+ C(D) e_+)(r_+ u_s) + i \frac{u(0)}{\sqrt{2\pi}} r_+ \mathcal{F}^{-1}(-i\xi)^{2\alpha - 1} (\xi - i)^{-1}, \]
which completes the proof of the lemma.

\[ \Box \]

Lemmas 4.25 and 4.26 are the counterparts of Lemma 4.23 for the case of higher regularity, namely \( 1 + 1/p < s < 2 + 1/p \).

**Remark 4.24.** Let \( \theta \) denote the Heaviside step function. We note, in preparation for Lemma 4.25, that if \( \alpha = 1/2 \), then
\[ r_+ \mathcal{F}^{-1}(-i\xi)^{2\alpha - 1} (\xi - i)^{-1} = -r_+ \mathcal{F}^{-1} (1 + i\xi)^{-2} = r_+ \mathcal{F}^{-1} (1 + i\xi)^{-2} \]
\[ = -r_+ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (1 + i\xi)^{-2} e^{-i\xi x} \, d\xi \]
\[ = r_+ \left( \sqrt{2\pi} x e^x \theta(-x) \right) \quad (3.382 \, 6, \, p. \, 349 \, [17]) \]
\[ = 0. \]
Lemma 4.25. Suppose $\frac{1}{2} \leq \alpha < 1$, $1 < p < \infty$, $1+1/p < s < 2+1/p$ and $u \in H^s_p(\mathbb{R}_+)$ with $u'(0) = 0$. Let $h(x) = (C_0^{2\alpha}u)(x)$ and $u_s = (D+i)^{s-2}e_+(D-i)^2 u$. Then

$$h = (r_+ C(D)e_+)(r_+ u_s) + \frac{u(0)}{\sqrt{2\pi}} r_+ \mathcal{F}^{-1}(-i\xi)^{2\alpha-1}(\xi - i)^{-2},$$

where $C(D)$ has the symbol $c(\xi) = (-i\xi)^{2\alpha}(\xi + i)^{2-s}(\xi - i)^{-2}$.

Proof. Firstly, suppose that $\alpha = \frac{1}{2}$. Then, $2\alpha = 1$, and we simply have

$$h(x) = (C_0^{1/2}u)(x) = u'(x) \quad \text{from equation (4.10)}$$

$$= \frac{d}{dx} (r_+ (D-i)^{-2}(D+i)^{2-s}e_+)(r_+ u_s) \quad \text{from Lemma 4.22}$$

$$= (r_+ C(D)e_+)(r_+ u_s),$$

since $\frac{d}{dx} = -iD$ and $Dr_+ = r_+ D$. Noting the result in Remark 4.24, this completes the proof for $\alpha = \frac{1}{2}$.

We now consider the case $\frac{1}{2} < \alpha < 1$. From Lemma 4.2, with $u'(0) = 0$, and the definition of $u_s$ we have

$$\mathcal{F}(e_+ u) = \mathcal{F}((D-i)^{-2}(D-i)^2 e_+ u)$$

$$= \mathcal{F}((D-i)^{-2} e_+(D-i)^2 u) - \mathcal{F}((D-i)^{-2} u(0) (\delta' - 2\delta))$$

$$= \mathcal{F}((D-i)^{-2} (D+i)^{2-s} e_+) + i u(0) (\xi - i)^{-2}(\xi - 2i) \cdot \frac{1}{\sqrt{2\pi}}$$

$$= (\xi - i)^{-2}(\xi + i)^{2-s} \mathcal{F}(u_s) + i u(0) (\xi - i)^{-2}(\xi - 2i) \cdot \frac{1}{\sqrt{2\pi}},$$

since $-\mathcal{F}(\delta' - 2\delta) = i \mathcal{F}(D\delta) + 2\mathcal{F}\delta = i(\xi - 2i) F\delta$.

Now $(e_+ u)' = e_+ u' + u(0)\delta$, and hence $D(e_+ u) = e_+ Du + iu(0)\delta$. Therefore,

$$\mathcal{F}(e_+ Du) = \mathcal{F}(D(e_+ u)) - \mathcal{F}(iu(0)\delta)$$

$$= \xi \mathcal{F}(e_+ u) - \frac{iu(0)}{\sqrt{2\pi}}.$$
Moreover,

\[-e_+ h(x) = -e_+ (C_{0+}^{20} u)(x) = -e_+ I_0^{2-2\alpha} u'' = e_+ I_0^{2-2\alpha} D^2 u = I_0^{2-2\alpha} (e_+ D^2 u)\]

\[= I_0^{2-2\alpha} (e_+ (D - i)^2 u + 2i e_+ D u + e_+ u) = I_0^{2-2\alpha} ((D + i)^2 - 2\alpha u + 2i e_+ D u + e_+ u).\]

Applying the Fourier transform, see Appendix D, and using the expressions recently established for \(\mathcal{F}(e_+ u)\) and \(\mathcal{F}(e_+ D u)\),

\[\mathcal{F}(e_+ h) = -(i \xi)^{2\alpha-2} \left\{ \mathcal{F}((D + i)^2 - 2\alpha u) + 2i \mathcal{F}(e_+ D u) + \mathcal{F}(e_+ u) \right\} \]

\[= -(i \xi)^{2\alpha-2} \left\{ (\xi + i)^{2-\alpha} \mathcal{F}(u_s) + (1 + 2i \xi) \mathcal{F}(e_+ u) + \frac{u(0)}{\sqrt{2\pi}} \right\} \]

\[= -(i \xi)^{2\alpha-2} \left\{ (\xi + i)^{2-\alpha} \mathcal{F}(u_s) + (1 + 2i \xi)(\xi - i)^{-2}(\xi + i)^{2-\alpha} \mathcal{F}(u_s) \right\} \]

\[= -(i \xi)^{2\alpha-2} \left( \frac{u(0)}{\sqrt{2\pi}(\xi - i)^2} \right) \cdot \left[ (1 + 2i \xi) i(\xi - 2i) + 2(\xi - i)^2 \right] \]

But \((\xi + i)^{2-\alpha} \mathcal{F}(u_s) + (1 + 2i \xi)(\xi - i)^{-2}(\xi + i)^{2-\alpha} \mathcal{F}(u_s)\)

\[= (\xi - i)^{-2}(\xi + i)^{2-\alpha} \mathcal{F}(u_s) \left\{ (\xi - i)^2 + 1 + 2i \xi \right\} \]

\[= \xi^2 (\xi - i)^{-2}(\xi + i)^{2-\alpha} \mathcal{F}(u_s),\]

and \((1 + 2i \xi)i(\xi - 2i) + 2(\xi - i)^2 = i \xi\). Thus, we have

\[\mathcal{F}(e_+ h) = -(i \xi)^{2\alpha}(\xi + i)^{2-\alpha}(\xi - i)^{-2} \mathcal{F}(u_s) + \frac{u(0)}{\sqrt{2\pi}} (-i \xi)^{2\alpha-1}(\xi - i)^{-2}.\]

But since \text{supp} \, u_s \subseteq \mathbb{R}_+,

\[h = (r_+ C(D)e_+)(r_+ u_s) + \frac{u(0)}{\sqrt{2\pi}} r_+ \mathcal{F}^{-1}(-i \xi)^{2\alpha-1}(\xi - i)^{-2},\]

which completes the proof of the lemma.
Lemma 4.26. Suppose $0 < \alpha < \frac{1}{2}$, $1 < p < \infty$, $1 + 1/p < s < 2 + 1/p$ and $u \in H^s_p(\mathbb{R}^+)$, with $u'(0) = 0$. Let $h(x) = (C^2_0u)(x)$ and $u_s = (D + i)^{s-2}e_+(D - i)^2u$. Then

$$h = (r_+C(D)e_+)(r_+u_s) + \frac{u(0)}{\sqrt{2\pi}} r_+ \mathcal{F}^{-1}(-i\xi)^{2\alpha-1}(\xi - i)^{-2}.$$  

where $C(D)$ has the symbol $c(\xi) = (-i\xi)^{2\alpha}(\xi + i)^2(\xi - i)^{-2}$.

Proof. As in Lemma 4.25, we have

$$\mathcal{F}(e_+u) = (\xi - i)^{-2}(\xi + i)^{2-s}\mathcal{F}(u_s) + \frac{iu(0)}{\sqrt{2\pi}} (\xi - i)^{-3}(\xi - 2i),$$

and

$$\mathcal{F}(e_+Du) = \xi \mathcal{F}(e_+u) - \frac{iu(0)}{\sqrt{2\pi}}.$$  

Moreover,

$$i e_+h(x) = i e_+(C^2_0u)(x)$$

$$= i e_+I_0^{-2\alpha}u' \quad \text{(See Appendix D)}$$

$$= e_+I_0^{-2\alpha}Du$$

$$= I^{-2\alpha}_+(e_+Du).$$

Applying the Fourier transform, see Appendix D, and using the expression above for $\mathcal{F}(e_+Du)$,

$$\mathcal{F}(e_+h) = -i (-i\xi)^{2\alpha-1}\mathcal{F}(e_+Du)$$

$$= -i (-i\xi)^{2\alpha-1}\left\{\xi \mathcal{F}(e_+u) - \frac{iu(0)}{\sqrt{2\pi}}\right\}$$

$$= (-i\xi)^{2\alpha} \mathcal{F}(e_+u) - (-i\xi)^{2\alpha-1} \frac{u(0)}{\sqrt{2\pi}}.$$  

But, using the expression above for $\mathcal{F}(e_+u)$, and collecting the terms containing $u(0)$,

$$(-i\xi)^{2\alpha}\left\{i \frac{u(0)}{\sqrt{2\pi}} (\xi - i)^{-2}(\xi - 2i)\right\} - (-i\xi)^{2\alpha-1} \frac{u(0)}{\sqrt{2\pi}}$$

$$= \frac{u(0)}{\sqrt{2\pi}} \cdot \frac{(-i\xi)^{2\alpha-1}}{(\xi - i)^2} \cdot \{(-i\xi)i(\xi - 2i) - (\xi - i)^2\}$$

$$= \frac{u(0)}{\sqrt{2\pi}} \cdot \frac{(-i\xi)^{2\alpha-1}}{(\xi - i)^2} \cdot \{\xi^2 - 2i\xi - \xi^2 + 2i\xi + 1\}$$

$$= \frac{u(0)}{\sqrt{2\pi}} \cdot \frac{(-i\xi)^{2\alpha-1}}{(\xi - i)^2}.$$  

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Thus, we have
\[ \mathcal{F}(e_+ h) = (-i\xi)^{2\alpha}(\xi + i)^{2-\alpha}(\xi - i)^{-2}\mathcal{F}(u_+) + \frac{u(0)}{\sqrt{2\pi}}(-i\xi)^{2\alpha-1}(\xi - i)^{-2}. \]

But since \( \text{supp } u_+ \subseteq \mathbb{R}_+ \),
\[ h = (r_+ C(D)e_+)(r_+ u_+) + \frac{u(0)}{\sqrt{2\pi}} r_+ \mathcal{F}^{-1}(-i\xi)^{2\alpha-1}(\xi - i)^{-2}, \]
which completes the proof of the lemma.

\[ \square \]

**Lemma 4.27.** Suppose \( 0 < \alpha < \frac{1}{2} \). Then
\[ a(x) := (r_+ A^- \chi_{\mathbb{R}_-})(x) = C_\alpha \int_{x}^{\infty} e^{-t} U(\alpha + 1, 2\alpha + 1, 2t) \, dt, \]
where the constant \( C_\alpha \) only depends on \( \alpha \), and is given by equation (4.8) in the statement of Lemma 4.9.

**Proof.** We begin by noting the following standard results:
\[ \mathcal{F}(\delta) = 1/\sqrt{2\pi} \quad \text{and} \quad \mathcal{F}(f * g) = \sqrt{2\pi} \mathcal{F}(f) \mathcal{F}(g). \]
See, for example, Chapter I, Section 2, [14], with an appropriate correction for the different constant used in the Fourier transform definitions.

Since
\[ \mathcal{F}^{-1}(1 + \xi^2)^\alpha(\xi - i)^{-1} = \mathcal{F}^{-1}(1 + \xi^2)^\alpha(\xi - i)^{-1} \cdot 1 = \mathcal{F}^{-1}(1 + \xi^2)^\alpha(\xi - i)^{-1} \sqrt{2\pi} \mathcal{F}(\delta) = \sqrt{2\pi} A^- \delta, \]
we have
\[ (1 + \xi^2)^\alpha(\xi - i)^{-1} = \sqrt{2\pi} \mathcal{F}(A^- \delta). \]

Hence, for \( x > 0 \),
\[ (r_+ A^- \chi_{\mathbb{R}_-})(x) = r_+ \mathcal{F}^{-1}(1 + \xi^2)^\alpha(\xi - i)^{-1}\mathcal{F}(\chi_{\mathbb{R}_-}) = r_+ \mathcal{F}^{-1} \sqrt{2\pi} \mathcal{F}(A^- \delta) \mathcal{F}(\chi_{\mathbb{R}_-}) = (A^- \delta \ast \chi_{\mathbb{R}_-})(x) = C_\alpha \int_{x}^{\infty} e^{-t} U(\alpha + 1, 2\alpha + 1, 2t) \chi_{\mathbb{R}_-}(x-t) \, dt \quad \text{by Lemma 4.9} = C_\alpha \int_{x}^{\infty} e^{-t} U(\alpha + 1, 2\alpha + 1, 2t) \, dt. \]

\[ \square \]
The following result is the counterpart of Lemma 4.27 for the operator $A^\pm$.

**Lemma 4.28.** Suppose $0 < \alpha < 1$. Then

$$a(x) := (r_+ A^- \chi_{\mathbb{R}_-})(x) = \frac{1}{2} i C_\alpha \int_x^\infty e^{-t} U(\alpha + 1, 2\alpha, 2t) \, dt,$$

where the constant $C_\alpha$ only depends on $\alpha$, and is given by equation (4.8) in the statement of Lemma 4.9.

**Proof.** Following the method of proof of Lemma 4.27, it is easy to show that

$$(1 + \xi^2)^\alpha (\xi - i)^{-2} = \sqrt{2\pi} \mathcal{F}(A^\pm \delta).$$

The required result now follows, as Lemma 4.27, but now using Lemma 4.10.

**Lemma 4.29.** Suppose $0 < \alpha < \frac{1}{2}$. Then, for $x > 0$, we can write

$$(r_+ A^- \chi_{\mathbb{R}_-})(x) = C_\alpha (\phi_1(x) + x^{1-2\alpha} \phi_2(x)),$$

where $\phi_1, \phi_2 \in C^\infty(\mathbb{R})$, and together with their derivatives, are bounded and $O(e^{-x})$ as $x \to +\infty$.

**Proof.** From Lemma 4.27, for $x > 0$, we have

$$(r_+ A^- \chi_{\mathbb{R}_-})(x) = C_\alpha \int_x^\infty e^{-t} U(\alpha + 1, 2\alpha + 1, 2t) \, dt$$

for some constant $C_\alpha$. Noting that

$$\int_x^\infty t^{-2\alpha} \psi(\alpha + 1, 2\alpha + 1, t) \, dt$$

contributes to both $\phi_1(x)$ and $x^{1-2\alpha} \phi_2(x)$, the required result now follows directly from Lemma 4.12.

**Remark 4.30.** Similarly, if $0 < \alpha < 1$ and $\alpha \neq \frac{1}{2}$, then for $x > 0$, we can write

$$(r_+ A^\pm \chi_{\mathbb{R}_-})(x) = \frac{1}{2} i C_\alpha (\phi_1(x) + x^{2-2\alpha} \phi_2(x)),$$

where $\phi_1, \phi_2 \in C^\infty(\mathbb{R})$, and together with their derivatives, are bounded and $O(e^{-x})$ as $x \to +\infty$. 

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On the other hand, if \( \alpha = \frac{1}{2} \), then for \( x > 0 \), we can write
\[
(r + A^c \chi_{\mathbb{R}_-})(x) = \frac{1}{2} i C_\alpha \left( \vartheta_1(x) \log x + \phi_3(x) \right),
\]
where \( \vartheta_1, \phi_3 \in C^\infty(\mathbb{R}) \), and together with their derivatives, are bounded and \( O(e^{-x}) \) as \( x \to +\infty \).

**Lemma 4.31.** Suppose \( 1 < p < \infty \) and \( \phi \in C^\infty_0(\mathbb{R}) \). Then
\[
\phi I : H^t_p(\mathbb{R}_+) \to H^{t-\epsilon}_p(\mathbb{R}_+)
\]
is compact for all \( t \in \mathbb{R} \) and all \( \epsilon > 0 \).

**Proof.** Since \( \phi I = \phi r_+ l_+ = r_+(\phi l_+) \), it is enough to prove that
\[
\phi I : H^t_p(\mathbb{R}) \to H^{t-\epsilon}_p(\mathbb{R})
\]
is compact for all \( t \in \mathbb{R} \) and all \( \epsilon \). From Section 3.3.1, p. 195, [45], the multiplication operator
\[
\phi I : H^t_p(\mathbb{R}) \to H^t_p(\mathbb{R}) \quad (∋ H^{t-\epsilon}_p(\mathbb{R}))
\]
is bounded.

Suppose \( \text{supp } \phi \subset \Omega \), where \( \Omega \subset \mathbb{R} \) is a bounded open set. Let \( r_\Omega : H^t_p(\mathbb{R}) \to H^t_p(\Omega) \) and \( e_\Omega : H^{t-\epsilon}_p(\Omega) \to \tilde{H}^{t-\epsilon}_p(\Omega) \) denote the operations of restriction to \( \Omega \), and extension by zero from \( \Omega \) respectively. Let \( i_{\Omega,t,p,\epsilon} \) denote the inclusion map
\[
i_{\Omega,t,p,\epsilon} : H^t_p(\Omega) \to H^{t-\epsilon}_p(\Omega).
\]
Then, we have the operator identity
\[
\phi I = e_\Omega i_{\Omega,t,p,\epsilon} r_\Omega \phi I,
\]
where on the left-hand side we note that \( \phi I : H^t_p(\mathbb{R}) \to H^{t-\epsilon}_p(\mathbb{R}) \) and, on the right-hand side, we simply assume \( \phi I : H^t_p(\mathbb{R}) \to H^t_p(\mathbb{R}) \).

But from Section 2.9.1, p. 166, [45], the restriction \( r_\Omega : H^t_p(\mathbb{R}) \to H^t_p(\Omega) \) is bounded and from Section 3.4.3, Remark 2, p. 211, [45], the extension operator \( e_\Omega : H^{t-\epsilon}_p(\Omega) \to \tilde{H}^{t-\epsilon}_p(\Omega) \leftarrow H^{t-\epsilon}_p(\mathbb{R}) \) is also bounded. Moreover, from Section 4.3.2, Remark 1, p. 233, [45], the inclusion map \( i_{\Omega,t,p,\epsilon} \) is compact. Hence, \( \phi I : H^t_p(\mathbb{R}) \to H^{t-\epsilon}_p(\mathbb{R}) \) is compact, as required.

\[\square\]
Lemma 4.32. Suppose $1 < p < \infty$ and $t > 1/p$. Let $a \in H^t_p(\mathbb{R}_+^+)$. Then
\[ aI : H^t_p(\mathbb{R}_+^+) \to H^{t-\epsilon}_p(\mathbb{R}_+^+) \]
is compact for any $\epsilon > 0$.

Proof. Since, by hypothesis, $t > 1/p$, $H^t_p(\mathbb{R}_+^+)$ is a Banach algebra. (See Section 2.8.3, Remark 3, p. 146, [45].) Thus
\[ aI : H^t_p(\mathbb{R}_+^+) \to H^t_p(\mathbb{R}_+^+) \hookrightarrow H^{t-\epsilon}_p(\mathbb{R}_+^+) \]
is a bounded operator for all $\epsilon > 0$. In particular, it has operator norm
\[ \|aI\|_{op} \leq \text{const} \|a\|_{t,p}. \]
Since $C_0^\infty(\mathbb{R})$ is dense in $H^t_p(\mathbb{R})$, we can approximate $a$ arbitrarily closely by a sequence $\{\phi_n\}_{n=1}^\infty \subset C_0^\infty(\mathbb{R})$. Finally, from Lemma 4.31, the operator $\phi_n I$ is compact for each $n \in \mathbb{N}$, and hence $aI$ is compact, as required.

4.4 Interim results

Let $\epsilon > 0$ be a small parameter. We now examine the individual summands in the left-hand side of equation (4.7).

4.4.1 First term

Consider the term $r_+ A_s u_s$. From equations (4.2) and (4.6), $A_s(D) := A(D)(D - i)^{-1}(D + i)^{1-s}$ and hence, we can write
\[ r_+ A_s u_s = (r_+ A(D)(D - i)^{-1}(D + i)^{1-s}e_+) (r_+ u_s), \]
since, by Lemma 4.3, $u_s \in L_p(\mathbb{R})$ and $\text{supp } u_s \subseteq \mathbb{R}_+^+$.

Thus, in the notation of equation (4.1),
\[ \tilde{a}_1(x) = 1; \]
\[ \tilde{b}_1(\xi) = 1; \]
\[ \tilde{c}_1(\xi) = (1 + \xi^2)^\alpha(\xi - i)^{-1}(\xi + i)^{1-s}. \]
4.4.2 Middle term

Now consider the middle term, \(-i(u(x) - u(0))r_+A^{-\delta}\). From Lemma 4.9

\[(r_+ A^{-\delta})(x) = C_\alpha e^{-x} U(\alpha + 1, 2\alpha + 1, 2x),\]

where the constant \(C_\alpha\) only depends on \(\alpha\), and is given by equation (4.8) in the statement of Lemma 4.9 as

\[C_\alpha = -i \frac{\alpha 2^{2\alpha}}{\Gamma(1 - \alpha)}.\]

From Lemma 4.12,

\[(r_+ A^{-\delta})(x) = C_\alpha (\phi(x) + x^{-2\alpha} \psi(\alpha + 1, 2\alpha + 1, x)),\]

where \(\phi \in C^\infty(\mathbb{R})\) and, together with its derivatives, is bounded and \(O(e^{-x})\) as \(x \to +\infty\). Moreover, \(\psi \in C_0^\infty(\mathbb{R})\) with \(\psi(\alpha + 1, 2\alpha + 1, x) = 0\) for \(x > 2\).

Hence, we can write

\[-i(u(x) - u(0))r_+A^{-\delta} = -iC_\alpha (T_{11} + T_{12})u\]

where

\[T_{11}u(x) := \phi(x)(u(x) - u(0))\]

\[T_{12}u(x) := x^{-2\alpha} \psi(\alpha + 1, 2\alpha + 1, x)(u(x) - u(0)).\]

Firstly, we will show that \(T_{11} : H^s_p(\mathbb{R}_+) \to H^{s-\varepsilon}_p(\mathbb{R}_+)\) is compact. Now

\[\phi(x)(u(x) - u(0)) = \phi(x) e^{x/2} \cdot e^{-x/2}(u(x) - u(0)).\]

By Lemma 4.6, \(u \mapsto e^{-x/2}(u(x) - u(0))\) defines a bounded operator from \(H^s_p(\mathbb{R}_+)\) to \(r_+H^{s-\varepsilon}_p(\mathbb{R}_+)\). Moreover, \(\phi(x) e^{x/2} \in H^s_p(\mathbb{R}_+)\), since it and its derivatives are bounded, smooth and \(O(e^{-x/2})\) as \(x \to +\infty\). Finally, the compactness of \(T_{11} : H^s_p(\mathbb{R}_+) \to H^{s-\varepsilon}_p(\mathbb{R}_+)\) follows directly from Lemma 4.32.

It remains to consider

\[-iC_\alpha T_{12}u(x) = -iC_\alpha x^{-2\alpha} \psi(\alpha + 1, 2\alpha + 1, x)(u(x) - u(0)),\]

and it is convenient to write

\[-iC_\alpha x^{-2\alpha} \psi(\alpha + 1, 2\alpha + 1, x)(u(x) - u(0))\]

\[= -iC_\alpha \psi(\alpha + 1, 2\alpha + 1, x) \cdot \{x^{-2\alpha}(u(x) - u(0))\},\]
noting that $\psi \in C^\infty_0(\mathbb{R})$ with $\psi(\alpha + 1, 2\alpha + 1, x) = 0$ for $x > 2$.

On the other hand, from Lemma 4.13 and Appendix D,

$$x^{-2\alpha}(u(x) - u(0)) = \int_0^\infty K_{2\alpha} \left( \frac{x}{y} \right) h(y) \frac{dy}{y} := M_{2\alpha}h$$

where $h(x) = (C_{0+}^{2\alpha} u)(x)$. Moreover, from Lemma 4.23

$$h = (r+C(D)e_+)(r+u_s) + i \frac{u(0)}{\sqrt{2\pi}} r_+ \mathcal{F}^{-1}(-i\xi)^{2\alpha-1}(\xi - i)^{-1},$$

where $C(D)$ has the symbol $c(\xi) = (-i\xi)^{2\alpha}(\xi + i)^{1-s}(\xi - i)^{-1}$. From Lemma 4.17, $M_{2\alpha}$ is a Mellin convolution operator with symbol $b(\xi) = B(1/p' + i\xi, 2\alpha)/\Gamma(2\alpha)$.

Thus, in the notation of equation (4.1), we have

$$\tilde{a}_2(x) = -iC_\alpha \psi(\alpha + 1, 2\alpha + 1, x) \quad (\in C^\infty_0(\mathbb{R}));$$

$$\tilde{b}_2(\xi) = B(1/p' + i\xi, 2\alpha)/\Gamma(2\alpha);$$

$$\tilde{c}_2(\xi) = (-i\xi)^{2\alpha}(\xi + i)^{1-s}(\xi - i)^{-1};$$

and

$$\tilde{a}_0(x) = \tilde{a}_2(x) M^0(\tilde{b}_2) \frac{i}{\sqrt{2\pi}} r_+ \mathcal{F}^{-1}(-i\xi)^{2\alpha-1}(\xi - i)^{-1}. \quad (4.16)$$

### 4.4.3 Final term

It remains to consider the last term, $-iu r_+ A^-(\chi_{\mathbb{R}_-})$. From Lemma 4.29,

$$(r_+ A^- \chi_{\mathbb{R}_-})(x) = C_\alpha \left( \phi_1(x) + x^{1-2\alpha} \phi_2(x) \right),$$

where $\phi_1, \phi_2 \in C^\infty(\mathbb{R})$ and, together with their derivatives, are bounded and $O(e^{-x})$ as $x \to +\infty$.

Hence, we can write

$$-iu r_+ A^-(\chi_{\mathbb{R}_-}) = -iC_\alpha (T_{21} + T_{22} + T_{23}) u$$

where

$$T_{21} u(x) := \phi_1(x) u(x)$$

$$T_{22} u(x) := x^{1-2\alpha} \phi_2(x) (u(x) - u(0))$$

$$T_{23} u(x) := x^{1-2\alpha} \phi_2(x) u(0).$$
We will now show that $T_{21}, T_{22}, T_{23} : H^s_p(\mathbb{R}_+) \to H^{s-2\alpha}_p(\mathbb{R}_+)$ are compact operators.

Firstly, consider $T_{21}$. We note that the compactness of $u \mapsto \phi_1(x)u(x)$ from $H^s_p(\mathbb{R}_+) \to H^{s-\epsilon}_p(\mathbb{R}_+)$ follows immediately from Lemma 4.32.

Secondly, we will show $T_{22}$ is compact. We can write

$$x^{1-2\alpha}\phi_2(x)(u(x) - u(0)) = \phi_2(x)e^{x/2} \cdot x^{1-2\alpha}e^{-x/4} \cdot e^{-x/4}(u(x) - u(0)).$$

By Lemma 4.6, $u \mapsto e^{-x/4}(u(x) - u(0))$ defines a bounded operator from $H^s_p(\mathbb{R}_+) \to r+\tilde{H}^s_p(\mathbb{R}_+)$. Since $1 - 2\alpha > 0$, from Lemma 3.4, the operator $x^{1-2\alpha}e^{-x/4}I$ is bounded on $\tilde{H}^s_p(\mathbb{R}_+)$. Moreover, $\phi_2(x)e^{x/2} \in H^s_p(\mathbb{R}_+)$, since it and its derivatives are bounded, smooth and $O(e^{-x/2})$ as $x \to +\infty$. Finally, the compactness of $T_{22} : H^s_p(\mathbb{R}_+) \to H^{s-\epsilon}_p(\mathbb{R}_+)$ follows directly from Lemma 4.32.

Thirdly, we will show $T_{23}$ is compact. We can write

$$x^{1-2\alpha}\phi_2(x)u(0) = \phi_2(x)e^{x/2} \cdot x^{-2\alpha} \cdot xe^{-x/2}u(0).$$

Let $s' = \max\{s, 1\}$. We note that $xe^{-x/2} \in \tilde{H}^s_p(\mathbb{R}_+)$, since it is smooth, assumes the value zero at $x = 0$ and decays exponentially. Therefore, $u \mapsto xe^{-x/2}u(0)$ defines a bounded operator from $H^s_p(\mathbb{R}_+) \to r+\tilde{H}^s_p(\mathbb{R}_+)$. Since $-2\alpha < 0$, from Lemma 3.3, the operator $x^{-2\alpha}I : H^s_p(\mathbb{R}_+) \to \tilde{H}^{s-2\alpha}_p(\mathbb{R}_+)$ is bounded. As $0 < \alpha < 1/2$, $s' - 2\alpha > 0$. Moreover, $\phi_2(x)e^{x/2}$ and its derivatives are bounded, smooth and $O(e^{-x/2})$ as $x \to +\infty$, and thus the operator $\phi_2(x)e^{x/2}I$ is bounded on $\tilde{H}^{s-2\alpha}_p(\mathbb{R}_+)$ by Lemma 3.4. Finally, $T_{23} : H^s_p(\mathbb{R}_+) \to H^{s-2\alpha}_p(\mathbb{R}_+)$ is bounded and rank one, and is therefore compact.

### 4.4.4 Summary

So, in summary, taking $N = 2$, we have the required representation

$$\tilde{a}_0(x)u(0) + \sum_{j=1}^{2} \tilde{a}_j(x) M^0[\tilde{b}_j](r_+\tilde{C}_je_+)(r_+u_+) + \tilde{K}u = f,$$

where the operator $\tilde{K} : H^s_p(\mathbb{R}_+) \to H^{s-2\alpha}_p(\mathbb{R}_+)$ is compact. The symbols $\tilde{a}_0$ and $(\tilde{a}_j, \tilde{b}_j, \tilde{c}_j)$ for $j = 1, 2$, are given by equations (4.16), (4.14) and (4.15).
respectively.

Purely for convenience, these results are also repeated here:

\[
\begin{align*}
\tilde{a}_0(x) &= \tilde{a}_2(x) M^0(\tilde{b}_2) \frac{i}{\sqrt{2\pi}} r_+ \mathcal{F}^{-1}(-i\xi)^{2\alpha-1}(\xi - i)^{-1}. \\
\tilde{a}_1(x) &= 1; \\
\tilde{b}_1(\xi) &= 1; \\
\tilde{c}_1(\xi) &= (1 + \xi^2)^\alpha(\xi - i)^{-1}(\xi + i)^{1-s}.
\end{align*}
\]

\[
\begin{align*}
\tilde{a}_2(x) &= -iC_\alpha \psi(\alpha + 1, 2\alpha + 1, x) \quad (\in C^\infty_0(\mathbb{R})); \\
\tilde{b}_2(\xi) &= B(1/p' + i\xi, 2\alpha)/\Gamma(2\alpha); \\
\tilde{c}_2(\xi) &= (-i\xi)^{2\alpha}(\xi + i)^{1-s}(\xi - i)^{-1}.
\end{align*}
\]
Chapter 5
Operator algebra - Part II

5.1 Main result

From Section 1.2.12, our (initial) problem is to investigate the solvability of the equation

\[ Au = f, \]

where \( u \in H^s_p(\mathbb{R}^+), \) for a given \( f \in H^{s-2\alpha}_p(\mathbb{R}^+) \), under the assumptions \( 1 < p < \infty \) and lower regularity, namely \( 1/p < s < 1 + 1/p \). Moreover, see Remark 4.14, we further assume that \( 0 < \alpha < \frac{1}{2} \).

In Lemma 4.3 we defined

\[ u_s := (D + i)^{s-1} e_+(D - i)u, \]

so that \( u_s \in L^p(\mathbb{R}) \) with supp \( u_s \subseteq \mathbb{R}^+ \).

**Theorem 5.1.** Suppose \( 1 < p < \infty, \ 1/p < s < 1 + 1/p \) and \( 0 < \alpha < \frac{1}{2} \). Then we can recast the equation \( Au = f \) in the form

\[ (W(c_1) + a_2M^0(b_2)W(c_2) + T)(r_+u_s) = g, \]

where

\[ g := r_+(D - i)^{s-2\alpha}l_+f \quad (\in L^p(\mathbb{R}^+)); \]
\[ c_1(\xi) = (1 + \xi^2)^\alpha(\xi - i)^{s-2\alpha-1}(\xi + i)^{1-s}; \]
\[ a_2(x) = -iC_\alpha \psi(\alpha + 1, 2\alpha + 1, x); \]
\[ b_2(\xi) = B(s - 2\alpha + 1 - 1/p + i\xi, 2\alpha)/\Gamma(2\alpha); \]
\[ c_2(\xi) = (-i\xi)^{2\alpha}(\xi - i)^{s-2\alpha-1}(\xi + i)^{1-s}, \]
and the operator $T$, acting on $L_p(\mathbb{R}^+)$, is compact.

The constant $C_\alpha$ is given in equation (4.8), and the smooth compactly supported function $\psi$ is discussed in Lemma 4.12.

### 5.2 Introduction

We have seen in Section 4.4 that equation (1.34) can be written as (see equation (4.1) with $N = 2$)

$$\tilde{a}_0(x)u(0) + \sum_{j=1}^{2} \tilde{a}_j(x) M^0(\tilde{b}_j)(r_+ \tilde{C}_j e_+)(r_+ u_+) + \tilde{K} u = f,$$

(5.1)

where the operator $\tilde{K} : H^s_p(\mathbb{R}^+) \to H^{s-2\alpha}_p(\mathbb{R}^+)$ is compact and $f \in H^{s-2\alpha}_p(\mathbb{R}^+)$ is a given function.

In this section, we present a formulation in $L_p(\mathbb{R}^+)$ of the form

$$(a_1(x) M^0(b_1) W(c_1) + a_2(x) M^0(b_2) W(c_2) + T)(r_+ u_+) = g,$$

(5.2)

where the operator $T$, acting on $L_p(\mathbb{R}^+)$, is compact. The function $g \in L_p(\mathbb{R}^+)$ is defined by

$$g := r_+(D - i)^{s-2\alpha} l_+ f,$$

(5.3)

where by Lemma 4.5, $g$ does not depend on the choice of the extension $l_+$.

The subsequent analysis will show that, after the application of the operator $r_+(D - i)^{s-2\alpha} l_+$, some of the terms in equation (5.1) represent compact operators on $L_p(\mathbb{R}^+)$. We now consider the action of the operator $r_+(D - i)^{s-2\alpha} l_+$ on the individual summands on the left-hand side of equation (5.1) in turn.

**Remark 5.2.** In this chapter we will make repeated use of Proposition 5.3.4, p. 267, [37], concerning the compactness on $L_p(\mathbb{R}^+)$ of the operator $M^0(b) W(c)$ and the commutator $[M^0(b), W(c)]$. In all cases where we use this result the symbols $b$ and $c$ will be continuous on $\mathbb{R}$ and have bounded variation, thus ensuring the applicability of Proposition 5.3.4, ibid. For more details, see Lemma 4.19 and 5.10.
5.3 Supporting lemmas

Lemma 5.3. Suppose that $\psi_0 \in r_+C_0^\infty(\mathbb{R})$ with $\psi_0(t) = 0$ for $t \geq 2$. Let $\chi \in C_0^\infty(\mathbb{R})$ be such that $\chi(t) = 1$ if $|t| \leq 2$. Then

$$\psi_0 M_{2\alpha,0}r_+h = \psi_0 M_{2\alpha,0}(r_+\chi h)$$

for all $h \in S(\mathbb{R})$.

Proof. Let $K_{2\alpha,0}$ denote the kernel of the Mellin integral operator $M_{2\alpha,0}$.

Firstly, suppose $0 \leq t \leq 2$. Then

$$\psi_0(t)(M_{2\alpha,0}r_+h)(t) := \psi_0(t) \int_0^\infty K_{2\alpha,0}(\tau)r_+h\left(\frac{t}{\tau}\right) \frac{d\tau}{\tau}$$

$$= \psi_0(t)\chi(t) \int_0^\infty K_{2\alpha,0}(\tau)r_+h\left(\frac{t}{\tau}\right) \frac{d\tau}{\tau} \quad (0 \leq t \leq 2)$$

$$= \psi_0(t)\chi(t) \int_1^\infty K_{2\alpha,0}(\tau)r_+h\left(\frac{t}{\tau}\right) \frac{d\tau}{\tau} \quad (\text{supp } K_{2\alpha,0} \subset [1, \infty))$$

$$= \psi_0(t) \int_1^\infty K_{2\alpha,0}(\tau)r_+\chi(t)h\left(\frac{t}{\tau}\right) \frac{d\tau}{\tau} \quad (0 \leq t/\tau \leq 2)$$

$$= \psi_0(t) \int_0^\infty K_{2\alpha,0}(\tau)r_+\chi\left(\frac{t}{\tau}\right)h\left(\frac{t}{\tau}\right) \frac{d\tau}{\tau} \quad (0 \leq \tau \leq 2)$$

$$= \psi_0(t)(M_{2\alpha,0}(r_+\chi h))(t).$$

On the other hand, if $t > 2$ then

$$\psi_0(t)(M_{2\alpha,0}r_+h)(t) = 0 = \psi_0(t)(M_{2\alpha,0}(r_+\chi h))(t).$$

This completes the proof of the lemma. \qed

If $1 + 1/p < s < 2 + 1/p$, for any $\mu, r \in \mathbb{R}$, it will be convenient to define

$$v_{\pm,\mu,r}(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-ix\xi}(-i\xi)^\mu (\xi \pm i)^{-2} d\xi, \quad x \in \mathbb{R}. \quad (5.4)$$
Lemma 5.4. Let $-1 < \mu < 1$ and $r < 2 - \mu$. Then $v_{\pm,\mu}(x)$ is bounded away from $x = 0$ for all finite $x$. Moreover, as $x \to 0$

$$v_{\pm,\mu} = \begin{cases} 
O(1) & \text{if } r < 1 - \mu \\
O(|x|^{1-\mu-r}) & \text{if } r > 1 - \mu.
\end{cases}$$

Finally, $v_{+,-1-\mu} = O(1)$, $v_{-,1} = O(1)$ and, for $\mu \neq 0$, we have $v_{-,\mu,1-\mu} = O(\log |x|)$.

Proof. If $r < 1 - \mu$, then $v_{\pm,\mu,r}$ is the inverse Fourier transform of an integrable function, and hence is continuous and vanishes at infinity.

Now suppose that $1 - \mu \leq r < 2 - \mu$. Initially, we assume that $\epsilon < x < N$, for some $\epsilon > 0$ and $N < \infty$, noting that the case $-N < x < -\epsilon$ follows in a similar manner. Changing the variable of integration, we obtain

$$v_{\pm,\mu,r}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\eta}(-i\eta)^\mu (\eta \pm ix)^{r-2} d\eta. \quad (5.5)$$

Now,

$$\int_{1}^{\infty} e^{-i\eta}(-i\eta)^\mu (\eta \pm ix)^{r-2} d\eta = i \int_{1}^{\infty} \left( \frac{d}{d\eta} e^{-i\eta} \right) (-i\eta)^\mu (\eta \pm ix)^{r-2} d\eta$$

integrating by parts

$$-i e^{-i}(-i)^\mu (1 \pm ix)^{r-2} - \mu \int_{1}^{\infty} e^{-i\eta}(-i\eta)^{\mu-1} (\eta \pm ix)^{r-2} d\eta$$

$$+ i(2 - r) \int_{1}^{\infty} e^{-i\eta}(-i\eta)^\mu (\eta \pm ix)^{r-3} d\eta = O(1), \quad \text{for } \epsilon < x < N.$$

Moreover, it is easy to see that the above estimate also applies in the limit as $x \searrow 0$. The case $\int_{-\infty}^{-1} \cdots$ follows similarly.

On the other hand

$$\int_{-1}^{1} e^{-i\eta}(-i\eta)^\mu (\eta \pm ix)^{r-2} d\eta = O(1), \quad \text{for } \epsilon < x < N.$$

In addition, the above estimate also holds in the limit as $x \searrow 0$, provided we exclude the case $r = 1 - \mu$. See Lemma G.1.

Hence, $v_{\pm,\mu,r}(x)$ is bounded away from $x = 0$ for all finite $x$. Moreover, as $x \to 0$

$$v_{\pm,\mu} = \begin{cases} 
O(1) & \text{if } r < 1 - \mu \\
O(|x|^{1-\mu-r}) & \text{if } r > 1 - \mu.
\end{cases}$$
Finally, by Lemma G.1, \( v_{+,-1} = O(1) \), \( v_{-,0} = O(1) \) and, for \( \mu \neq 0 \), we have \( v_{-,1} = O(\log |x|) \).

**Lemma 5.5.** Let \(-1 < \mu < 1\), \( r < 2 - \mu \) and \( 1 < p < \infty \). Define

\[
v_{\pm,\mu}(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} (-i\xi)^{\mu} \frac{1}{(\xi \pm i)^{2}} d\xi, \quad x \in \mathbb{R}.
\]

Then, for any \( \chi_1 \in C_0^\infty(\mathbb{R}) \),

\[
\chi_1(D \pm i)^r v_{\pm,\mu} \in L_p(\mathbb{R}) \quad \text{if} \quad r < 1 - \mu + 1/p.
\]

**Proof.** Firstly, we note from equation (5.4) and the definition of \( v_{\pm,\mu} \) that

\[
v_{\pm,\mu,r}(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} (-i\xi)^{\mu} (\xi \pm i)^{r-2} d\xi
\]

\[
= \mathcal{F}^{-1}((\xi \pm i)^{r} \frac{(-i\xi)^{\mu}}{(\xi \pm i)^{2}})
\]

\[
= \mathcal{F}^{-1}((\xi \pm i)^{r} v_{\pm,\mu})
\]

\[
= (D \pm i)^r v_{\pm,\mu}.
\]

But from Lemma 5.4, \( v_{\pm,\mu,r} \) is bounded away from \( x = 0 \) for all finite \( x \), and as \( x \to 0 \)

\[
v_{\pm,\mu,r} = \begin{cases} 
O(1) & \text{if } r < 1 - \mu \\
O(|x|^{1-\mu-r}) & \text{if } r > 1 - \mu.
\end{cases}
\]

Moreover, \( v_{+,1} = O(1) \), and

\[
v_{-,1} = \begin{cases} 
O(1) & \text{if } \mu = 0 \\
O(\log |x|) & \text{if } \mu \neq 0,
\end{cases}
\]

as \( x \to 0 \).

Hence, \( \chi_1 v_{\pm,\mu,r} \in L_p(\mathbb{R}) \) if

\[
p(1 - \mu - r) > -1.
\]

So, finally

\[
\chi_1(D \pm i)^r v_{\pm,\mu} \in L_p(\mathbb{R}) \quad \text{if} \quad r < 1 - \mu + 1/p.
\]

This completes the proof of the lemma.
We now consider the case of lower regularity. If \( 1/p < s < 1 + 1/p \), for any \( \mu, r \in \mathbb{R} \), we let

\[
 w_{\pm, \mu, r}(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} (-i\xi)^{-\mu} (\xi \pm i)^{r-1} d\xi, \quad x \in \mathbb{R}. \tag{5.6}
\]

Given definition (5.6), the following corollary and lemma are the counterparts, for lower regularity, of Lemmas 5.4 and 5.5 respectively.

**Corollary 5.6.** Let \( 0 < \mu < 1 \) and \( r < 1 + \mu \). Then \( w_{\pm, \mu, r}(x) \) is bounded away from \( x = 0 \) for all finite \( x \). Moreover, as \( x \to 0 \)

\[
 w_{\pm, \mu, r} = \begin{cases} O(1) & \text{if } r < \mu \\ O(|x|^{\mu-r}) & \text{if } r > \mu, \end{cases}
\]

and \( w_{+, \mu, \mu} = O(1) \) and \( w_{-, \mu, \mu} = O(\log |x|) \).

**Proof.** Let us define \( \tilde{\mu} := -\mu \) and \( \tilde{r} := r - 1 \). Then, if we recast Lemma 5.4 in terms of \( \tilde{\mu} \) and \( \tilde{r} \), we obtain Corollary 5.6, with \( \tilde{\mu} \) and \( \tilde{r} \) in place of \( \mu \) and \( r \) respectively.

**Lemma 5.7.** Let \( 0 < \mu < 1 \), \( r < 1 + \mu \) and \( 1 < p < \infty \). Define

\[
 w_{\pm, \mu}(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} (-i\xi)^{-\mu} \frac{1}{\xi \pm i} d\xi, \quad x \in \mathbb{R}.
\]

Then, for any \( \chi_1 \in C^\infty_0(\mathbb{R}) \)

\[
 \chi_1(D \pm i)^r w_{\pm, \mu} \in L_p(\mathbb{R}) \quad \text{if} \quad r < \mu + 1/p.
\]

**Proof.** Firstly, we note that

\[
 (D \pm i)v_{\pm, \mu} = \mathcal{F}^{-1}(\xi \pm i)(-i\xi)^{\mu} (\xi \pm i)^{-2} = \mathcal{F}^{-1}(-i\xi)^{\mu} (\xi \pm i)^{-1} = w_{\pm, -\mu}.
\]

Now let us define \( \tilde{\mu} := -\mu \) and \( \tilde{r} := r - 1 \). Then, if we recast Lemma 5.5 in terms of \( \tilde{\mu} \) and \( \tilde{r} \), we obtain Lemma 5.7, with \( \tilde{\mu} \) and \( \tilde{r} \) in place of \( \mu \) and \( r \) respectively.

**Lemma 5.8.** Suppose \( a, b, c \in \mathbb{R} \) are such that \( a \geq 0 \) and \( a + b + c \leq 0 \). Let \( m_p \) be given by

\[
 m_p(\xi) := (-i\xi)^a (\xi + i)^b (\xi - i)^c, \quad \xi \in \mathbb{R}.
\]
Then $m_p(\xi)$ is a Fourier multiplier.

**Proof.** The conditions $a \geq 0$ and $a+b+c \leq 0$ ensure that $|m_p(\xi)|$ is bounded.

We now assume that each of $a, b$ and $c$ is non-zero, noting that in the special cases where at least one of these exponents is zero, the same method of proof applies. A routine calculation gives

$$m_p'(\xi) = (-i\xi)^{a-1} (\xi + i)^{b-1} (\xi - i)^{c-1} \{ -ia - (b - c)\xi - i(a + b + c)\xi^2 \}.$$

For $|\xi| \geq 1$, it is easy to see that

$$|\xi m_p'(\xi)| \leq C \left( \sqrt{1 + \xi^2} \right)^{a+(b-1)+(c-1)+2}
= C \left( \sqrt{1 + \xi^2} \right)^{a+b+c}
\leq C,$$

since $a + b + c \leq 0$.

On the other hand, suppose $|\xi| < 1$. Then

$$|\xi m_p'(\xi)| \leq C|\xi|^a \leq C,$$

since we are assuming $a > 0$.

Hence, $m_p(\xi)$ is a Fourier multiplier, by the Mikhlin multiplier theorem.

Following [12, 37], we let $C_0$ denote the algebra of all continuous and piecewise linear functions on $\mathbb{R}$, and $PC_0$ the algebra of all piecewise constant functions on $\mathbb{R}$, with only finitely many discontinuities. Further, for $1 < p < \infty$, let $C_p$ and $PC_p$ represent the closure of $C_0$ and $PC_0$ in $M_p$ respectively.

**Remark 5.9.** We note that a number of the results that we use from [12, 37], require that a given Wiener-Hopf symbol belongs to either $C_p$ or $PC_p$. Fortunately, we have the following inclusions:

(a) The set of all continuous functions on $\mathbb{R}$ with bounded variation is contained in $C_p$. See, for example, Proposition 5.1.2 (iii), p. 261, [37].

(b) The set of all (piecewise) continuous functions on $\mathbb{R}$ with bounded variation is contained in $PC_p$. See, for example, Proposition 5.1.4 (ii), p. 261, [37].

**Lemma 5.10.** Let the function $m_p$ be as defined in Lemma 5.8. Then $m_p \in BV(\mathbb{R})$ and is continuous.
Proof. From Lemma 5.8, the function $m_p$ is continuous and bounded on $\mathbb{R}$.

As in Lemma 5.8, we assume that each of $a$, $b$, $c$ is non-zero, noting that in the remaining special cases, the same method of proof applies. From the proof of Lemma 5.8, we have

$$m'_p(\xi) = (-i\xi)^{a-1} (\xi + i)^{b-1} (\xi - i)^{c-1} \left\{ -ia - (b - c)\xi - i(a + b + c)\xi^2 \right\}.$$ 

As $|\xi| \to 0$, we have $m'_p(\xi) = O(|\xi|^{a-1})$. Since we are assuming that $a > 0$, $m'_p$ is integrable in a neighbourhood of $\xi = 0$.

As $|\xi| \to \infty$, we have $m'_p(\xi) = O(|\xi|^{-2})$ or $m'_p(\xi) = O(|\xi|^{a+b+c-1})$, as either $a + b + c = 0$ or $a + b + c < 0$. Hence, $m'_p(\xi)$ is integrable near $\xi = \pm\infty$, and thus $m'_p \in L_1(\mathbb{R})$.

Finally, from Remark 4.18, we have $m_p \in BV(\mathbb{R})$, as required.

$\square$

**Lemma 5.11.** Suppose $0 < r < 1$, and $c_r : \mathbb{R} \to \mathbb{C}$ is given by

$$c_r(\xi) = (\xi - i)^r - (-i)^r (i\xi)^r - (-i)^r, \quad \xi \in \mathbb{R}.$$ 

Then $c_r(\xi)$ is bounded for all $\xi \in \mathbb{R}$. Moreover,

$$c_r(0) = 0 \quad \text{and} \quad \lim_{\xi \to \pm\infty} c_r(\xi) = -(-i)^r,$$

and $c_r$ is a Fourier $L_p$-multiplier.

**Proof.** The boundedness of $c_r(\xi)$ will follow immediately once the limits of $c_r(\xi)$ at 0 and $\pm\infty$ are established. Of course, it is elementary to verify that $c_r(0) = 0$.

Now suppose $|\xi| > 1$. Then, for $0 < r < 1$,

$$c_r(\xi) = (-i)^r \left\{ (1 + i\xi)^r - (i\xi)^r - 1 \right\}$$

$$= (-i)^r \left\{ (i\xi)^r (1 - i/\xi)^r - (i\xi)^r - 1 \right\}$$

$$= (-i)^r \left\{ (i\xi)^r (-ir)/\xi - 1 + O(|\xi|^{-2+r}) \right\}$$

$$\to -(-i)^r, \quad \text{as} \ |\xi| \to \infty.$$ 

From the Mikhlin multiplier theorem, to show that $c_r$ is a Fourier $L_p$-multiplier, it remains to show that $\xi c'_r(\xi)$ is bounded.
From the definition of \(c_r(\xi)\), we have
\[
c'_r(\xi) = r(\xi - i)^{r-1} - (-i)^r(i\xi)^{r-1},
\]
and thus
\[
\xi c'_r(\xi) = r\{\xi(\xi - i)^{r-1} - (-i)^r(i\xi)^r\}.
\]
Hence \(\xi c'_r(\xi)|_{\xi=0} = 0\).

Now suppose that \(|\xi| > 1\). Then, writing
\[
c_r(\xi) = (-i)^r\{(1 + i\xi)^r - (i\xi)^r - 1\}
\]
we have
\[
c'_r(\xi) = (-i)^r\{i(1 + i\xi)^{r-1} - i(i\xi)^{r-1}\}
\]
Thus,
\[
\xi c'_r(\xi) = (-i)^r\{(i\xi)(1 + i\xi)^{r-1} - (i\xi)^r\}
\]
\[
= (-i)^r\{(i\xi)^r(1 - i/\xi)^{r-1} - (i\xi)^r\}
\]
\[
= (-i)^r\{(i\xi)^r(-i(r - 1))/\xi + O(|\xi|^{-2+r})\}
\]
\[
\rightarrow 0, \quad \text{as } |\xi| \rightarrow \infty.
\]

This completes the proof of the lemma.

\[\square\]

**Lemma 5.12.** Suppose \(1 < p < \infty\), \(\sigma \in \mathbb{R}\) and \(\nu > 0\). Let \(l_+ : H^\sigma_p(\mathbb{R}_+) \rightarrow H^\sigma_p(\mathbb{R}_+)\) be an arbitrary extension operator. Then \(r_+(iD)\nu l_+\) is bounded from \(H^\sigma_p(\mathbb{R}_+)\) to \(H^{\sigma-\nu}_p(\mathbb{R}_+)\), and does not depend on the choice of the extension \(l_+\). Moreover,
\[
(r_+(iD)\nu l_+)r_+ = r_+(iD)\nu.
\]

**Proof.** We follow the approach taken in Lemma 4.5. Let us define the symbol
\[
A_\nu(\xi) := \frac{(i\xi)^\nu}{(1 + \xi^2)^{\nu/2}}.
\]
Then, see Lemma 5.11, with \(a = \nu, b = c = -\nu/2\), it is a routine calculation to show that \(A_\nu\) is a Fourier \(H^\sigma_p\) multiplier. But, directly from the definition of Bessel potential spaces, we have
\[
\mathcal{F}^{-1}(1 + \xi^2)^{\nu/2} \mathcal{F} : H^\sigma_p(\mathbb{R}) \rightarrow H^{\sigma-\nu}_p(\mathbb{R})
\]
is bounded. Thus, the pseudodifferential operator \((iD)^\nu\) is bounded from \(H^\sigma_p(\mathbb{R})\) to \(H^{\sigma-\nu}_p(\mathbb{R})\).
In addition, its symbol \((i\xi)^\nu\) admits an analytic continuation with respect to \(\xi\) to the lower complex half-plane \((\tau < 0)\) such that

\[
|(|i\xi - \tau|)^\nu| \leq (|\xi| + |\tau| + 1)^\nu, \quad \tau \leq 0.
\]

Therefore, from Theorem 1.10, p. 53, [41], \(r_+(iD)^\nu l_+\) is continuous from \(H^{\sigma}_p(\mathbb{R}_+^1)\) to \(H^{\sigma - \nu}_p(\mathbb{R}_+^1)\), and does not depend on the choice of extension \(l_+\).

Finally, by Remark 1.11, p. 53, [41], we also have

\[
(r_+(iD)^\nu l_+)r_+ = r_+(iD)^\nu.
\]

This completes the proof of the lemma.

\[\square\]

**Lemma 5.13.** Let \(B\) be a pseudodifferential operator whose symbol satisfies the condition \(|B(\xi)| \leq C(1 + |\xi|)^\nu\), for certain constants \(C\) and \(\nu\). Suppose \(\varphi \in S(\mathbb{R})\). Then

\[
B(D_t) \varphi \left( \frac{t}{\tau} \right) = \left[ B \left( \frac{D_t}{\tau} \right) \varphi \right] \left( \frac{t}{\tau} \right).
\]

**Proof.**

\[
B(D_t) \varphi \left( \frac{t}{\tau} \right) = \mathcal{F}^{-1}B(\xi)\mathcal{F} \varphi \left( \frac{t}{\tau} \right)
\]

\[
= \mathcal{F}^{-1}B(\xi)\tau (\mathcal{F} \varphi)(\tau \xi) \quad \text{Proposition 2.2.11 (8), p. 100, [19]}
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-it\xi} B(\xi)\tau (\mathcal{F} \varphi)(\tau \xi) \, d\xi
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\eta t/\tau} B(\eta/\tau) (\mathcal{F} \varphi)(\eta) \, d\eta \quad \text{(where \(\eta = \tau \xi\))}
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\eta (t/\tau)} \mathcal{F} \left( B \left( \frac{D_t}{\tau} \right) \varphi \right) \, d\eta
\]

\[
= \left[ B \left( \frac{D_t}{\tau} \right) \varphi \right] \left( \frac{t}{\tau} \right), \quad \text{as required.}
\]

\[\square\]

We have noted previously, see Remark 4.20, that the Mellin integral operator \(M_{\gamma,0}\) with kernel

\[
K_{\gamma,0}(t) = \frac{\chi_{(1,\infty)}(t)}{\Gamma(\gamma) t^\gamma (t-1)^{1-\gamma}}, \quad \gamma > 0,
\]

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is bounded on $L_p(\mathbb{R}^+)$ and $K_{\gamma,0}(t)t^{-\epsilon}$ belongs to $L_1(\mathbb{R}^+)$ for all $\epsilon > 0$.

**Lemma 5.14.** Let $\varphi \in S(\mathbb{R})$. Suppose $K : \mathbb{R}^+ \to \mathbb{R}$ satisfies the two conditions

$$\text{supp } K \subseteq [1, \infty) \quad \text{and} \quad \int_0^\infty |K(\tau)|\tau^{-\epsilon}d\tau < \infty, \quad \text{for all } \epsilon > 0. \quad (5.7)$$

Then, for all $t > 0$,

$$D_t \int_0^\infty K\left(\frac{t}{\tau}\right)r_+\varphi(\tau) \frac{d\tau}{\tau} = \int_0^\infty K(\tau) r_+(D_t\varphi)\left(\frac{t}{\tau}\right) \frac{d\tau}{\tau}.$$  

In other words, on applying the operator $D_t$, we have $K(s) \mapsto s^{-1}K(s)$ and $r_+\varphi(t) \mapsto r_+D_t\varphi(t)$.

**Proof.** Firstly, we note that

$$\int_0^\infty K\left(\frac{t}{\tau}\right)r_+\varphi(\tau) \frac{d\tau}{\tau} = \int_0^\infty K(\tau) r_+\varphi\left(\frac{t}{\tau}\right) \frac{d\tau}{\tau}.$$  

Now let us define

$$F(t, \tau) := \frac{K(\tau)}{\tau} r_+\varphi\left(\frac{t}{\tau}\right),$$

and thus

$$\int_0^\infty |F(t, \tau)|d\tau = \int_0^\infty \left|\frac{K(\tau)}{\tau} r_+\varphi\left(\frac{t}{\tau}\right)\right|d\tau \leq C_{1,\varphi} \int_0^\infty |K(\tau)|\tau^{-1}d\tau < \infty,$$

where the constant $C_{1,\varphi}$ only depend on $\varphi$.

Since supp $K \subseteq [1, \infty)$, from Theorem 16.11, p. 213, [29], to prove that we can differentiate through the integral sign, it remains to show that

$$\frac{\partial}{\partial t} F(t, \tau)$$

is dominated, uniformly for all $t > 0$, by an integrable function over the range $1 \leq \tau < \infty$. But, clearly

$$\left|\frac{\partial}{\partial t} F(t, \tau)\right| = \left|\frac{K(\tau)}{\tau^2} r_+(D_t\varphi)\left(\frac{t}{\tau}\right)\right| \leq C_{2,\varphi} |K(\tau)|\tau^{-2},$$

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where the constant $C_{2, \varphi}$ only depends on $\varphi$.

Hence, for $t > 0$,

\[ D_t \int_0^\infty K(\tau) \frac{r + \varphi(t/\tau)}{\tau^\nu} d\tau = \int_0^\infty K(\tau) D_t \left( \frac{t}{\tau} \right) \frac{d\tau}{\tau} = \int_0^\infty K(\tau) r + (D_t \varphi)(t/\tau) \frac{d\tau}{\tau}. \]

This completes the proof of the lemma.

Lemma 5.14 allows us to change the order of (repeated) differentiation and integration, within a certain class of Mellin integral operators. It will be useful to consider an extension of this result to include “fractional” differentiation.

Suppose $\varphi \in S(\mathbb{R})$ and $\nu > 0$. We write

\[ \nu = [\nu] + \{\nu\}. \]

Then, as (5.8), [40], we define

\[ \mathcal{D}_\nu \varphi := (iD_t)^{[\nu] + 1} I_{-\{\nu\}} \varphi = (iD_t)^{[\nu] + 1} I_{-\{\nu\}} \varphi, \]

where from (5.3) ibid.,

\[ (I_{-\{\nu\}} \varphi)(t) := \frac{1}{\Gamma(1 - \{\nu\})} \int_t^\infty \frac{\varphi(s)}{(s-t)^{1-\{\nu\}}} ds. \]

But from (7.4) ibid.,

\[ \mathcal{F}(\mathcal{D}_\nu \varphi) = (i\xi)^\nu \mathcal{F} \varphi \quad (\nu \geq 0), \]

and thus

\[ (iD_t)^\nu = (iD_t)^{[\nu] + 1} I_{-\{\nu\}}. \quad (5.8) \]

In other words, we can consider the fractional operator $(iD_t)^\nu$ to be the composition of a certain Riemann-Liouville integral of order $1 - \{\nu\}$ with a (conventional) differential operator of order $[\nu] + 1$. 

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Hence, we would now like to show that we can change the order of integration in the following iterated integral:

\[
(I_{1-\nu} M \varphi)(t) = \int_{t}^{\infty} \left( \int_{1}^{\infty} K(\tau) \varphi \left( \frac{s}{\tau} \right) \frac{d\tau}{\tau} \right) ds.
\]

**Lemma 5.15.** Suppose the kernel, \( K \), of a Mellin integral operator satisfies the two conditions in (5.7). Let \( \varphi \in S(\mathbb{R}) \) and \( 0 < \gamma < 1 \). Then, for \( t > 0 \),

\[
I(t) := \int_{1}^{\infty} \frac{|K(\tau)|}{\tau} \left( \int_{t}^{\infty} \frac{1}{(s-t)^{\gamma}} |r_{+} \varphi \left( \frac{s}{\tau} \right)| ds \right) d\tau < \infty.
\]

*Proof.* It is convenient to define

\[
J(\tau,t) := \int_{t}^{\infty} \frac{1}{(s-t)^{\gamma}} |r_{+} \varphi \left( \frac{s}{\tau} \right)| ds.
\]

Let \( w = (s-t)/\tau \), and thus

\[
J(\tau,t) = \int_{0}^{\infty} \frac{1}{(\tau w)^{\gamma}} |r_{+} \varphi (w + t/\tau)| \tau dw = \frac{\tau}{\tau^{\gamma}} \int_{0}^{\infty} w^{-\gamma} |r_{+} \varphi (w + t/\tau)| dw \\
\leq C_{\varphi,\gamma} \tau^{1-\gamma},
\]

since \( 0 < \gamma < 1 \). (The constant \( C_{\varphi,\gamma} \) only depends on \( \varphi \) and \( \gamma \).)

Hence

\[
I(t) \leq \int_{1}^{\infty} \frac{|K(\tau)|}{\tau} \cdot C_{\varphi,\gamma} \tau^{1-\gamma} d\tau = C_{\varphi,\gamma} \int_{1}^{\infty} |K(\tau)| \tau^{-\gamma} d\tau < \infty.
\]

We have the following immediate Corollary of Lemma 5.15.

**Corollary 5.16.** Suppose the kernel, \( K \), of a Mellin integral operator, \( M \), satisfies the two conditions in (5.7). Let \( \varphi \in S(\mathbb{R}) \) and \( 0 < \gamma < 1 \). Then,
from Lemma 5.15 and the Fubini-Tonelli Theorem, we can change the order of integration in the following iterated integral:

\[
(1_{\xi}^{-\nu} M \varphi)(t) = \int_{t}^{\infty} \frac{1}{(s-t)^{\nu}} \left( \int_{1}^{\infty} K(\tau) \varphi\left(\frac{s}{\tau}\right) d\tau \right) ds.
\]

Remark 5.17. Combining Corollary 5.16 with Lemma 5.14, we see that when applying the operator \(r_{+}(iD)^{\nu}l_{+}\) to Mellin integral operators whose kernels satisfy the two conditions in (5.7), we can reverse the order of \(r_{+}(iD)^{\nu}l_{+}\) and (Mellin) integration for all \(\nu > 0\).

With these preparations complete, we can now compute the action of \(r_{+}(iD)^{\nu}l_{+}\) on our class of Mellin integral operators.

**Lemma 5.18.** Suppose the kernel, \(K\), of a Mellin integral operator satisfies the two conditions in (5.7). Let \(\varphi \in S(\mathbb{R})\). Then, for \(\nu > 0\) and \(t > 0\),

\[
r_{+}(iD)^{\nu}l_{+} \int_{0}^{\infty} K\left(\frac{t}{\tau}\right) r_{+}\varphi(\tau) \frac{d\tau}{\tau} = \int_{0}^{\infty} K(\tau) r_{+}(iD)^{\nu}\varphi\left(\frac{t}{\tau}\right) \frac{d\tau}{\tau}.
\]

In other words, on applying the operator \(r_{+}(iD)^{\nu}l_{+}\), we have \(K(s) \mapsto s^{-\nu}K(s)\) and \(r_{+}\varphi(t) \mapsto r_{+}(iD)^{\nu}\varphi(t)\).

Proof. For \(t > 0\),

\[
r_{+}(iD)^{\nu}l_{+} \int_{0}^{\infty} K\left(\frac{t}{\tau}\right) r_{+}\varphi(\tau) \frac{d\tau}{\tau}
\]

\[
= r_{+}(iD)^{\nu}l_{+} \int_{0}^{\infty} K(\tau) r_{+}\varphi\left(\frac{t}{\tau}\right) \frac{d\tau}{\tau}
\]

\[
= \int_{0}^{\infty} K(\tau) r_{+}(iD)^{\nu}l_{+}r_{+}\varphi\left(\frac{t}{\tau}\right) \frac{d\tau}{\tau} \quad \text{by Remark 5.17}
\]

\[
= \int_{0}^{\infty} K(\tau) r_{+}(iD)^{\nu}\varphi\left(\frac{t}{\tau}\right) \frac{d\tau}{\tau} \quad \text{by Lemma 5.12}
\]

\[
= \int_{0}^{\infty} K(\tau) r_{+}\left[\left(\frac{iD}{\tau}\right)^{\nu}\varphi\right]\left(\frac{t}{\tau}\right) \frac{d\tau}{\tau} \quad \text{by Lemma 5.13}
\]

\[
= \int_{0}^{\infty} \frac{K(\tau)}{\tau^{\nu}} r_{+}(iD)^{\nu}\varphi\left(\frac{t}{\tau}\right) \frac{d\tau}{\tau}.
\]

This completes the proof of the lemma.
5.4 Mellin operator boundedness

We have noted previously, see Lemma 4.17, that the Mellin integral operator $M_{\gamma,\rho}$ with kernel

$$K_{\gamma,\rho}(t) = \frac{\chi_{[1,\infty)}(t)}{t^\rho \Gamma(\gamma) t^\gamma (t - 1)^{1-\gamma}}.$$ 

is bounded on $L_p(\mathbb{R}_+)$.

**Lemma 5.19.** Suppose $1 < p < \infty$, $\rho > 1/p - 1$ and $\gamma > 0$. If $t \geq 0$, then

$$M_{\gamma,\rho} : H^1_p(\mathbb{R}_+) \to H^1_p(\mathbb{R}_+)$$

is bounded.

**Proof.** The special case $t = 0$ follows directly from Lemma 4.17.

Now suppose that $t = m \in \mathbb{N}$. Let $\varphi \in H^m_p(\mathbb{R}_+)$. Then, using an equivalent norm on $H^m_p(\mathbb{R}_+)$,

$$\|M_{\gamma,\rho}\varphi\|_{m,p} \leq \text{const} \sum_{k=0}^m \|M_{\gamma,\rho}\varphi^{(k)}\|_p$$

$$= \text{const} \sum_{k=0}^m \|M_{\gamma,\rho+k}\varphi^{(k)}\|_p \quad \text{by Lemma 5.14}$$

$$\leq \sum_{k=0}^m c_k \|M_{\gamma,\rho+k}\| \|\varphi^{(k)}\|_p \quad \text{by Lemma 4.17}$$

$$\leq C_{\gamma,\rho,m,p} \sum_{k=0}^m \|\varphi^{(k)}\|_p \quad \text{for some positive constant } C_{\gamma,\rho,m,p}$$

$$= C_{\gamma,\rho,m,p} \|\varphi\|_{m,p}.$$ 

In other words, the operator $M_{\gamma,\rho} : H^m_p(\mathbb{R}_+) \to H^m_p(\mathbb{R}_+)$ is bounded for any $m \in \mathbb{N}$.

Let $t > 0$. Choose any $m \in \mathbb{N}$ such that $m > t$. Then we have boundedness on $H^1_p(\mathbb{R}_+)$ by interpolation between $H^m_p(\mathbb{R}_+)$ and $H^0_p(\mathbb{R}_+) = L_p(\mathbb{R}_+).$
5.5 Multiplication operator commutator

Suppose $1 < p < \infty$ and $\sigma, \nu \in \mathbb{R}$. Then, see Lemma 4.5,

$$\Lambda_\nu := r_+ (D - i)^{\nu} l_+$$

is bounded from $H_p^\sigma(\mathbb{R}_+)$ to $H_p^{\sigma - \nu}(\mathbb{R}_+)$, and does not depend on the choice of extension $l_+$.

**Lemma 5.20.** Let $1 < p < \infty$ and $\phi \in C_0^\infty(\mathbb{R})$. Then

$$[\Lambda_\nu, \phi I] : H_p^s(\mathbb{R}_+) \to H_p^{s-r}(\mathbb{R}_+)$$

is compact for all $r, s \in \mathbb{R}$.

**Proof.** Since $\phi \in C_0^\infty(\mathbb{R})$, the commutator $[\Lambda_\nu, \phi I]$ is a pseudodifferential operator of order $r - 1$, and thus

$$[\Lambda_\nu, \phi I] : H_p^s(\mathbb{R}_+) \to H_p^{s-r+1}(\mathbb{R}_+)$$

From Lemma 4.31,

$$\phi I : H_p^t(\mathbb{R}_+) \to H_p^{t-\epsilon}(\mathbb{R}_+)$$

is compact for $-\infty < t < +\infty$ and all $\epsilon > 0$. Therefore, with $t = s$,

$$\Lambda_\nu \phi I : H_p^s(\mathbb{R}_+) \to H_p^{s-r-\epsilon}(\mathbb{R}_+)$$

and then taking $t = s - r$,

$$\phi \Lambda_\nu : H_p^s(\mathbb{R}_+) \to H_p^{s-r-\epsilon}(\mathbb{R}_+)$$

are both compact.

In summary,

$$[\Lambda_\nu, \phi I] : H_p^s(\mathbb{R}_+) \to H_p^{s-r+1}(\mathbb{R}_+)$$

is bounded, and

$$H_p^s(\mathbb{R}_+) \to H_p^{s-r-\epsilon}(\mathbb{R}_+)$$

is compact.

Therefore, by interpolation, see [8],

$$[\Lambda_\nu, \phi I] : H_p^s(\mathbb{R}_+) \to H_p^{s-r}(\mathbb{R}_+)$$

is compact for all $r, s \in \mathbb{R}$.

\[\square\]
5.6 Pseudodifferential and Mellin operators

Remark 5.21. Lemma 5.22 and 5.24 describe the action of the operator $\Lambda^r$ on the Mellin integral operator $M_{2\alpha,0}$, since this is sufficient for our purposes. However, it is clear that these results also hold for a wider class of Mellin operators. Indeed, we can replace $M_{2\alpha,0}$ by a general Mellin convolution operator, with symbol $b$, such that $b(\pm\infty) = 0$, and whose kernel, $K$, satisfies the two conditions in (5.7).

Lemma 5.22. Suppose $0 < r < 1$ and $0 < \alpha < 1$. Then

$$\Lambda^r M_{2\alpha,0} = M_{2\alpha,r} \Lambda^r + (-i)^r (M_{2\alpha,0} - M_{2\alpha,r}) + T,$$

where $T : H^r_\nu(\mathbb{R}_+) \to L^p_\nu(\mathbb{R}_+)$ is compact.

On the other hand, if $-1 + 1/p < r < 0$ then

$$\Lambda^r M_{2\alpha,0} = M_{2\alpha,r} \Lambda^r - (-i)^r (M_{2\alpha,0} - M_{2\alpha,r}) \Lambda^{2r} + T,$$

where $T : H^r_\nu(\mathbb{R}_+) \to L^p_\nu(\mathbb{R}_+)$ is compact.

Proof. We first consider the case where $r > 0$.

Suppose $0 < r < 1$. Then, from Lemma 5.11

$$(\xi - i)^r = (-i)^r (i\xi)^r + (-i)^r + c_r(\xi),$$

where $c_r$ is bounded for $\xi \in \mathbb{R}$ and $c_r(0) = 0, c_r(\pm\infty) = -(-i)^r$.

Moreover, from Lemma 5.18, for $\nu > 0$

$$r_+(iD)^\nu l_+ M_{2\alpha,0} = M_{2\alpha,\nu} r_+(iD)^\nu l_+.$$  \hfill (5.10)

Hence

$$\Lambda^r M_{2\alpha,0} = r_+ \{ (-i)^r (iD)^r + (-i)^r + c_r(D) \} l_+ M_{2\alpha,0}$$

$$= M_{2\alpha,r} (-i)^r r_+(iD)^\nu l_+ + M_{2\alpha,0} (-i)^r + r_+ c_r(D) l_+ M_{2\alpha,0}$$

$$= M_{2\alpha,r} \Lambda^r + (-i)^r \{ M_{2\alpha,0} - M_{2\alpha,r} \}$$

$$+ \{ r_+ c_r(D) l_+ M_{2\alpha,0} - M_{2\alpha,r} r_+ c_r(D) l_+ \}.$$ 

Now, for ease of notation, define

$$C_r := r_+ c_r(D) l_+.$$
Hence, we can write
\[ \Lambda_r M_{2\alpha,0} - M_{2\alpha,r} \Lambda_r - (-i)^r (M_{2\alpha,0} - M_{2\alpha,r}) = C_r M_{2\alpha,0} - M_{2\alpha,r} C_r, \quad (5.11) \]
where \( C_r \) has order 0. For the case \( 0 < r < 1 \), it now remains to prove the compactness of the two operators on the right-hand side of equation (5.11).

From Proposition 5.3.4(i), p. 267, \[37\],
\[ \Lambda_r - C_r M_{2\alpha,0} : L_p(\mathbb{R}^+) \to L_p(\mathbb{R}^+) \]
is compact. Therefore,
\[ C_r M_{2\alpha,0} : L_p(\mathbb{R}^+) \to H_p^{-r}(\mathbb{R}^+) \]
is compact. But
\[ C_r M_{2\alpha,0} : H_p^r(\mathbb{R}^+) \to H_p^{-r}(\mathbb{R}^+) \]
is bounded for all \( t \geq 0 \). So, taking \( t > r \), we obtain by interpolation that
\[ C_r M_{2\alpha,0} : H_p^r(\mathbb{R}^+) \to L_p(\mathbb{R}^+) \]
is compact.

Similarly, from Proposition 5.3.4(i), p. 267, \[37\], \( M_{2\alpha,r} C_r \Lambda_r : L_p(\mathbb{R}^+) \to L_p(\mathbb{R}^+) \) is compact. Therefore,
\[ M_{2\alpha,r} C_r : H_p^r(\mathbb{R}^+) \to L_p(\mathbb{R}^+) \]
is compact. This completes the proof for the case \( 0 < r < 1 \).

**Now suppose that** \( r < 0 \).

Suppose \( -1 + 1/p < r < 0 \) and write \( r = -s \) where \( s > 0 \). Our starting point is equation (5.11), where for the pseudodifferential terms we replace \( r \) by \( s \), and for the Mellin operators we replace the pair \( M_{2\alpha,0} \) and \( M_{2\alpha,r} \) by \( M_{2\alpha,0} \) and \( M_{2\alpha,0} \) respectively. Hence
\[ \Lambda_s M_{2\alpha,r} - M_{2\alpha,0} \Lambda_s - (-i)^s (M_{2\alpha,r} - M_{2\alpha,0}) = C_s M_{2\alpha,r} - M_{2\alpha,0} C_s, \quad (5.12) \]

From Proposition 5.3.4(i), p. 267, \[37\],
\[ \Lambda_s C_s M_{2\alpha,r} : L_p(\mathbb{R}^+) \to L_p(\mathbb{R}^+) \]
is compact. Therefore,
\[ C_s M_{2\alpha,r} : L_p(\mathbb{R}_+) \rightarrow H'_p(\mathbb{R}_+) \]
is compact.

Similarly,
\[ C_s M_{2\alpha,0} : L_p(\mathbb{R}_+) \rightarrow H'_p(\mathbb{R}_+) \]
is compact. But, in addition, from Proposition 5.3.4 (ii)1, p. 267, [37],
\[ [M_{2\alpha,0}, C_s] : L_p(\mathbb{R}_+) \rightarrow L_p(\mathbb{R}_+) \]
is compact and so, finally,
\[ M_{2\alpha,0} C_s : L_p(\mathbb{R}_+) \rightarrow H'_p(\mathbb{R}_+) \]
is compact. In summary,
\[ T_s := C_s M_{2\alpha,r} - M_{2\alpha,0} C_s : L_p(\mathbb{R}_+) \rightarrow H'_p(\mathbb{R}_+) \]
is compact. From equation (5.12)
\[ \Lambda_- \Lambda^s M_{2\alpha,r} \Lambda_- = \Lambda_- M_{2\alpha,0} \Lambda^s \Lambda_- + \Lambda_- (-i)^r (M_{2\alpha,0} - M_{2\alpha,r}) \Lambda_- + \Lambda_- T_s \Lambda^r. \]
But since \( r + s = 0 \), we have
\[ M_{2\alpha,r} \Lambda^r = \Lambda_- M_{2\alpha,0} + \Lambda_- (-i)^r (M_{2\alpha,0} - M_{2\alpha,r}) \Lambda_- + \Lambda_- T_s \Lambda^r. \]
Rearranging
\[ \Lambda_- M_{2\alpha,0} = M_{2\alpha,r} \Lambda_- - \Lambda_- (-i)^r (M_{2\alpha,0} - M_{2\alpha,r}) \Lambda_- + T_1, \]
where \( T_1 = -\Lambda_- T_s \Lambda^r \). Since \( T_s : L_p(\mathbb{R}_+) \rightarrow H'_p(\mathbb{R}_+) \) is compact, it follows that \( T_1 : H'_p(\mathbb{R}_+) \rightarrow L_p(\mathbb{R}_+) \) is compact.

Finally, we note that
\[ \Lambda_- (M_{2\alpha,0} - M_{2\alpha,r}) \Lambda_- = [\Lambda_- , (M_{2\alpha,0} - M_{2\alpha,r})] \Lambda_- + (M_{2\alpha,0} - M_{2\alpha,r}) \Lambda^{-2r}. \]
But, since \( r < 0 \), the commutator \([\Lambda_- , (M_{2\alpha,0} - M_{2\alpha,r})]\) is compact on \( L_p(\mathbb{R}_+) \) by Proposition 5.3.4 (ii)1, p. 267, [37]. Hence \([\Lambda_- , (M_{2\alpha,0} - M_{2\alpha,r})] \Lambda_- : H'_p(\mathbb{R}_+) \rightarrow L_p(\mathbb{R}_+) \) is compact.

This completes the proof of the lemma. \( \square \)
Remark 5.23. In passing, we note that there is a minor inaccuracy in the proof of Proposition 5.3.4 (i) p. 267, [37].

The sum in the display formula 9 lines below (5.7) might not, in fact, be identically zero. However, by Proposition 4.2.10, p. 204, [37], it can be made arbitrarily small, so that for any \( \varepsilon > 0 \), we can choose \( f \) such that

\[
\|W(a)M^0(f_b) - W(a)M^0(f)\| < \varepsilon/2.
\]

On the other hand, \( W(a)M^0(f) \) can be represented as the sum of a compact operator and an operator of norm \( \varepsilon/2 \). Therefore, \( W(a)M^0(f_b) \) can be represented as the sum of a compact operator and an operator of arbitrarily small norm. That is, \( W(a)M^0(f_b) \) is also compact. \(^1\)

We note that for \( 0 < \alpha < 1 \) and any \( r > 0 \) the Mellin integral operator \( M_{2\alpha,r} \) has a symbol that vanishes at \( \pm\infty \). For a fixed \( \alpha \), it will be convenient to define a composite Mellin operator to be any linear combination of operators \( M_{2\alpha,r} \), as \( r > 0 \) varies. Clearly, by its construction, any composite Mellin operator also has a symbol that vanishes at \( \pm\infty \).

Lemma 5.24. Suppose \( r > 0 \) and \( 0 < \alpha < 1 \). If

(a) \( r = k \in \mathbb{N} \), then \( \Lambda_k M_{2\alpha,0} = M_{2\alpha,k} \Lambda_k^\alpha + S_{k-1} \);

(b) \( r \notin \mathbb{N} \), then \( \Lambda_r M_{2\alpha,0} = M_{2\alpha,r} \Lambda_r^\alpha + S_r + T \),

where \( T : H_p^r(\mathbb{R}_+) \to L_p(\mathbb{R}_+) \) is compact, and

\[
S_{\sigma} := \sum_{0 \leq \mu \leq \sigma} M_{\mu} \Lambda_{\mu}^\alpha \quad (\sigma \geq 0),
\]

for certain composite Mellin operators \( M_{\mu} \). (The sum in the expression for \( S_{\sigma} \) always has finitely many terms.)

Proof. Firstly, suppose \( r = k \in \mathbb{N} \). Since

\[
(D - i)^k = \sum_{j=0}^{k} \binom{k}{j} (-i)^{k-j} D^j,
\]

\(^1\)This correction to the proof was confirmed in a personal communication from Prof. Steffen Roch on 1st November 2016.

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part (a) follows directly from Lemma 5.14.

Secondly, suppose \( r > 0 \) and \( r \not\in \mathbb{N} \). Then, we write
\[
\Lambda^r_\gamma = \Lambda^{(r)}_\gamma \Lambda^{[r]}_\gamma.
\]
Hence, from part (a) and Lemma 5.22,
\[
\Lambda^r_\gamma M_{2\alpha,0} = \Lambda^{(r)}_\gamma \Lambda^{[r]}_\gamma M_{2\alpha,0}
= \Lambda^{(r)}_\gamma \{ M_{2\alpha, [r]} \Lambda^{[r]}_\gamma + S_{[r]-1} \}
= (M_{2\alpha, r} \Lambda^{(r)}_\gamma + S_0 + T) \Lambda^{[r]}_\gamma + S_{r-1}
= M_{2\alpha, r} \Lambda^r_\gamma + S_{[r]} + T \Lambda^{[r]}_\gamma,
\]
which completes the proof of part (b).

\[\square\]

5.7 Proof of Theorem 5.1

We now consider the action of the operator \( r_+ (D - i)^s 2\alpha I_+ \) on the individual summands on the left-hand side of equation (5.1) in turn.

5.7.1 First term

For the first term, from equation (4.16), we have
\[
\tilde{a}_0(x) = \tilde{a}_2(x) M^0(\tilde{b}_2) \frac{i}{\sqrt{2\pi}} r_+ \mathcal{F}^{-1}(-i\xi)^{2\alpha-1}(\xi - i)^{-1}
= \tilde{a}_2(x) M^0(\tilde{b}_2) r_+ h_1(x),
\]
where
\[
h_1(x) := \frac{i}{\sqrt{2\pi}} \mathcal{F}^{-1}(-i\xi)^{2\alpha-1}(\xi - i)^{-1}.
\]
Note that, from equation (4.15),
\[
\tilde{a}_2(x) = -i C_\alpha \psi(\alpha + 1, 2\alpha + 1, x);
\]
\[
\tilde{b}_2(\xi) = B(1/p' + i\xi, 2\alpha)/\Gamma(2\alpha).
\]
Our goal is to show that
\[
\Lambda^{s-2\alpha}_- \tilde{a}_0(x) = \Lambda^{s-2\alpha}_- \tilde{a}_2(x) M^0(\tilde{b}_2) r_+ h_1(x) \in L_p(\mathbb{R}_+),
\]

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because then the operator

\[ L_p(\mathbb{R}_+) \ni r_+ u_s \mapsto u(0) \Lambda_{s-2\alpha} \tilde{a}_0 \in L_p(\mathbb{R}_+) \]

is bounded on \( L_p(\mathbb{R}_+) \) and has rank one, and is therefore compact.

We note that \( \tilde{a}_2 \in r_+ C_0^\infty(\mathbb{R}) \) and \( \tilde{a}_2(x) = 0 \) for \( x \geq 2 \). Let \( \chi \in C_0^\infty(\mathbb{R}) \) be such that

\[
\chi(t) := \begin{cases} 
1 & \text{if } |t| \leq 2 \\
0 & \text{if } |t| > 3.
\end{cases}
\]

Then, see Lemmas 4.17 and 5.3,

\[
\Lambda_{s-2\alpha} \tilde{a}_2 M^0(\tilde{b}_2)(r_+ h_1) = \Lambda_{s-2\alpha} \tilde{a}_2(x) M^0(\tilde{b}_2)(r_+ \chi h_1).
\]

Since \( h_1 \) is the inverse Fourier transform of an integrable function it is continuous and vanishes at infinity. Hence \( r_+ \chi h_1 \in L_p(\mathbb{R}_+) \). Thus, if \( s - 2\alpha < 0 \), using Lemma 5.19, the required result follows immediately.

It remains to consider the case \( s - 2\alpha \geq 0 \). Set \( \mu = 1 - 2\alpha \), \( r = s - 2\alpha \), so that \( 0 < \mu < 1 \), \( r < 2 - 2\alpha = 1 + \mu \). Then, from Lemma 5.7, for any \( \chi_1 \in C_0^\infty(\mathbb{R}) \),

\[
\chi_1(D - i)^{s-2\alpha} h_1 \in L_p(\mathbb{R}),
\]

subject only to the condition

\[
s < 1 + \frac{1}{p}.
\]

Hence, see Lemma F.1, \( (D - i)^{s-2\alpha} \chi h_1 \in L_p(\mathbb{R}) \), and so, after applying the operator \( r_+(D - i)^{2\alpha-s} \), we have

\[
r_+ \chi h_1 \in H_p^{s-2\alpha}(\mathbb{R}_+).
\]

Therefore, as \( s - 2\alpha \geq 0 \), again from Lemma 5.19,

\[
M^0(\tilde{b}_2) r_+ \chi h_1 \in H_p^{s-2\alpha}(\mathbb{R}_+),
\]

and hence,

\[
\Lambda_{s-2\alpha} \tilde{a}_2(x) M^0(\tilde{b}_2) r_+ h_1(x) \in L_p(\mathbb{R}_+),
\]

as required, since \( \tilde{a}_2 \in r_+ C_0^\infty(\mathbb{R}) \).
5.7.2 Second term

Using (4.14), we have
\[ \tilde{a}_1(x) M^0(\tilde{b}_1) (r_+ \tilde{C}_1 e_+) = r_+ \tilde{C}_1 e_+ \]
where the pseudodifferential operator \( \tilde{C}_1 \) has symbol \( (1+\xi^2)^\alpha (\xi-i)^{-1} (\xi+i)^{1-s} \).

Hence, by Lemma 4.5
\[ r_+(D-i)^{s-2\alpha} l_+ r_+ \tilde{C}_1 e_+ = r_+(D-i)^{s-2\alpha} \tilde{C}_1 e_+. \]

Now \( (D-i)^{s-2\alpha} \tilde{C}_1 \) has symbol \( (1+\xi^2)^\alpha (\xi-i)^{s-2\alpha-1} (\xi+i)^{1-s} \), which is clearly a Fourier \( L_p \) multiplier. (See Lemma 5.8.) Therefore, in the notation of equation (5.2),
\[
\begin{align*}
  a_1(x) &= 1; \\
  b_1(\xi) &= 1; \\
  c_1(\xi) &= (1+\xi^2)^\alpha (\xi-i)^{s-2\alpha-1} (\xi+i)^{1-s}.
\end{align*}
\]

5.7.3 Third term

Now from (4.15) we have
\[
\begin{align*}
  \tilde{a}_2(x) &= -i C_\alpha \psi(\alpha + 1, 2\alpha + 1, x); \\
  \tilde{b}_2(\xi) &= B(1/p' + i\xi, 2\alpha)/\Gamma(2\alpha); \\
  \tilde{c}_2(\xi) &= (-i\xi)^{2\alpha} (\xi + i)^{1-s} (\xi - i)^{-1}.
\end{align*}
\]

Let us define
\[ r := s - 2\alpha, \]
so that we need to consider
\[ -1 + 1/p < r < 1 + 1/p, \]
since \( 0 < \alpha < \frac{1}{2} \) and \( 1/p < s < 1 + 1/p \). We note that the pseudodifferential operator \( \tilde{C}_2 \) has order \(-r\).

If \( r \geq 0 \), then from Lemma 5.19, the operator \( M^0(\tilde{b}_2) (r_+ \tilde{C}_2 e_+) : L_p(\mathbb{R}_+) \to H^r_p(\mathbb{R}_+) \) is bounded.

On the other hand, if \( -1 + 1/p < r < 0 \), we can write
\[ \tilde{c}_2(\xi) = (i\xi)^{-r} \cdot \tilde{c}_0(\xi) \]
where \[ \tilde{c}_0(\xi) := (i\xi)^r (-i\xi)^{2\alpha} (\xi + i)^{1-s}(\xi - i)^{-1}. \]

Since \( r + 2\alpha = s > 0 \), \( \tilde{c}_0(0) = 0 \). Moreover, as \( r = s - 2\alpha \), the operator \( \tilde{C}_0 \), with symbol \( \tilde{c}_0 \), has order 0.

From Lemma 5.18, \( M_{2\alpha,0} r_+ (iD)^{-r} l_+ = r_+ (iD)^{-r} l_+ M_{2\alpha,r} \), and from Lemma 5.12, \( r_+ (iD)^{-r} l_+ : L_p(\mathbb{R}_+) \rightarrow H^r_p(\mathbb{R}_+) \) is bounded. Therefore, the operator \( M^0(\tilde{b}_2) (r_+ \tilde{C}_2 e_+) = M_{2\alpha,0} (r_+ \tilde{C}_2 e_+) \) (see Lemma 4.17)

\[ = M_{2\alpha,0} (r_+ (iD)^{-r} l_+ r_+ \tilde{C}_0 e_+) \] (see Lemma 5.12) \[ = r_+ (iD)^{-r} l_+ M_{2\alpha,r} (r_+ \tilde{C}_0 e_+) \]

is bounded from \( L_p(\mathbb{R}_+) \rightarrow H^r_p(\mathbb{R}_+) \).

So now, using Lemma 3.4, each of the three operators in the identity

\[ \Lambda^r - \tilde{a}_2(x) M^0(\tilde{b}_2) (r_+ \tilde{C}_2 e_+) \]

\[ = [\Lambda^r - \tilde{a}_2(x)] M^0(\tilde{b}_2) (r_+ \tilde{C}_2 e_+) + \tilde{a}_2(x) \Lambda^r M^0(\tilde{b}_2) (r_+ \tilde{C}_2 e_+) \]

is bounded on \( L_p(\mathbb{R}_+) \).

Moreover, the compactness of the operator involving the commutator term follows directly from Lemma 5.20. Thus, it remains to consider \( \tilde{a}_2(x) \Lambda^r M^0(\tilde{b}_2) (r_+ \tilde{C}_2 e_+) \).

Firstly, suppose that \( 0 < r < 1 \). Then, using Lemma 5.22,

\[ \Lambda^r M^0(\tilde{b}_2) (r_+ \tilde{C}_2 e_+) \]

\[ = \Lambda^r M_{2\alpha,0} (r_+ \tilde{C}_2 e_+) \]

\[ = (M_{2\alpha,r} \Lambda^r + (-i)^r (M_{2\alpha,0} - M_{2\alpha,r}) + T) (r_+ \tilde{C}_2 e_+) \]

\[ = M_{2\alpha,r} (r_+ C_2 e_+) + (-i)^r (M_{2\alpha,0} - M_{2\alpha,r}) (r_+ \tilde{C}_2 e_+) + T (r_+ \tilde{C}_2 e_+). \]

From Lemma 5.22, \( T : H^r_p(\mathbb{R}_+) \rightarrow L_p(\mathbb{R}_+) \) is compact. Moreover, the pseudodifferential operator \( \tilde{C}_2 \) has order \( -r \), and hence \( T (r_+ \tilde{C}_2 e_+) \) is compact on \( L_p(\mathbb{R}_+) \).

By Remark 4.20, the symbols of both \( M_{2\alpha,0} \) and \( M_{2\alpha,r} \) take the value zero at \( \pm\infty \). Hence, \( (M_{2\alpha,0} - M_{2\alpha,r}) (r_+ \tilde{C}_2 e_+) \) is compact on \( L_p(\mathbb{R}_+) \), from Proposition 5.3.4 (i), p. 267, [37].
So, in summary, if $0 < r < 1$ then
\[
\Lambda_- M^0(b_2) (r_+ C_2 e_+) = M_{2\alpha,r} (r_+ C_2 e_+) + K_1,
\]
where $C_2$ has symbol
\[
c_2(\xi) = (-i\xi)^{2\alpha} (\xi - i)^{s - 2\alpha - 1}(\xi + i)^{1-s},
\]
and the operator $K_1$, acting on $L_p(\mathbb{R}^+)$, is compact.

Similarly, in the case that $-1 + 1/p < r < 0$, then we can again apply Lemma 5.22, noting that the operator $\Lambda_2 r_+ C_2 e_+$ has order $r < 0$.

\[
\Lambda_- M^0(b_2) (r_+ C_2 e_+)
= \Lambda_- M_{2\alpha,0} (r_+ C_2 e_+)
= (M_{2\alpha,r} \Lambda_- - (-i)^r (M_{2\alpha,0} - M_{2\alpha,r}) \Lambda_2 r_+ C_2 e_+) + T (r_+ C_2 e_+)
= M_{2\alpha,r} (r_+ C_2 e_+) - (-i)^r (M_{2\alpha,0} - M_{2\alpha,r}) \Lambda_2 r_+ C_2 e_+ + T (r_+ C_2 e_+).
\]
The compactness of $T (r_+ C_2 e_+)$ on $L_p(\mathbb{R}^+)$ follows exactly as in the case $0 < r < 1$. Moreover,
\[
(M_{2\alpha,0} - M_{2\alpha,r}) \Lambda_2 r_+ C_2 e_+
\]
is compact on $L_p(\mathbb{R}^+)$, from Proposition 5.3.4 (i), p. 267, [37].

So, in summary, if $-1 + 1/p < r < 0$ then
\[
\Lambda_- M^0(b_2) (r_+ C_2 e_+) = M_{2\alpha,r} (r_+ C_2 e_+) + K_2
\]
where the operator $K_2$, acting on $L_p(\mathbb{R}^+)$, is compact.

The case $1 < r < 1 + 1/p$ follows similarly, except that we now apply Lemma 5.24. In particular, we note that the operator $S_1 r_+ C_2 e_+$ has order $1 - r < 0$. Hence, as in the case $0 < r < 1$ discussed above,
\[
S_1 r_+ C_2 e_+
\]
is compact on $L_p(\mathbb{R}^+)$, from Proposition 5.3.4 (i), p. 267, [37].

Finally, the case $r = 1$ follows in the same way, and for the case $r = 0$, there is nothing to prove.
Hence, using Lemma 4.17, in the notation of equation (5.2) we have,

\[
\begin{align*}
a_2(x) &= -iC_\alpha \psi(\alpha + 1, 2\alpha + 1, x); \\
b_2(\xi) &= B(s - 2\alpha + 1/p' + i\xi, 2\alpha)/\Gamma(2\alpha); \\
c_2(\xi) &= (-i\xi)^{2\alpha} (\xi - i)^{s-2\alpha-1} (\xi + i)^{1-s}.
\end{align*}
\] (5.15)

Note that a routine application of Lemma 5.8 confirms that \( c_2 \) is a Fourier \( L_p \) multiplier.

### 5.7.4 Summary

Our base assumptions are that

\[
1 < p < \infty, \quad 1/p < s < 1 + 1/p \quad \text{and} \quad 0 < \alpha < \frac{1}{2}.
\] (5.16)

So, finally, subject to condition (5.16), the formulation given by equation (5.2) becomes

\[
(W(c_1) + a_2 M^0(b_2) W(c_2) + T)(r_+ u_s) = g,
\] (5.17)

where the operator \( T \), acting on \( L_p(\mathbb{R}_+) \), is compact and

\[
\begin{align*}
g &:= r_+ (D - i)^{s-2\alpha} l_+ f; \\
c_1(\xi) &= (1 + \xi^2)^{\alpha} (\xi - i)^{s-2\alpha-1} (\xi + i)^{1-s}. \\
a_2(x) &= -iC_\alpha \psi(\alpha + 1, 2\alpha + 1, x) \quad \text{(see Lemma 4.12)}; \\
b_2(\xi) &= B(s - 2\alpha + 1/p' + i\xi, 2\alpha)/\Gamma(2\alpha); \\
c_2(\xi) &= (-i\xi)^{2\alpha} (\xi - i)^{s-2\alpha-1} (\xi + i)^{1-s},
\end{align*}
\] (5.18)

and the constant \( C_\alpha \) is given by

\[
C_\alpha = -i \frac{\alpha \cdot 2^{2\alpha}}{\Gamma(1 - \alpha)}.
\]
Chapter 6
Fredholm analysis

6.1 Introduction

Suppose $N \in \mathbb{N}$. Our goal is to establish the conditions under which the operator
\[ \tilde{A} := \sum_{j=1}^{N} a_j M^0(b_j) W(c_j) \]
acting on $L_p(\mathbb{R}_+)$ is Fredholm. It will be sufficient, for our purposes to assume that, for $j = 1, \ldots, N$, we have $a_j$ continuous on $\mathbb{R}^+$, and $b_j, c_j$ continuous on $\mathbb{R}$. In addition, see Lemmas 4.19, 5.10 and Remark 5.9, we may suppose that the symbols $\{b_j, c_j\}_{j=1}^{N}$ have bounded variation.

Later in this chapter, we will apply these general results to the specific set of symbols $\{c_1, a_2, b_2, c_2\}$, derived earlier in Chapter 5, for the case of lower regularity $1/p < s < 1 + 1/p$.

We detail the method originally developed by Duduchava [11, 12], and later reviewed in [37]. Indeed, our starting point is Corollary 5.5.10, p. 290 [37]. To use this important result most effectively, we will combine it with Theorems 5.5.3, 5.5.4 and 5.5.7 as given in pp. 279 to 290, ibid. In preparation for this, the following remark addresses some important points on notation.

Remark 6.1. We adopt the convention that Mellin and Wiener-Hopf operators are given by $M^0(b)$ and $W(c)$, with symbols $b$ and $c$ respectively. However, in [37], this convention is reversed. (So that, for example, in Theorem 5.5.3, p. 279, ibid, the symbol $b$ is used to describe a Wiener-Hopf operator.)
Moreover, in [37], the Fourier transform is defined using the opposite sign convention. (C.f. Equation (1.1) and equation (4.9), p. 199, [37].) So, if we denote this alternative Fourier transform by $F_-$, then, by a routine calculation

$$ F_-^{-1}b(\xi)F_+ = F_-^{-1}b(-\xi)F_+ \quad (6.1) $$

On the other hand, both here and in [37], the Mellin transform is defined identically. (C.f. Equation (1.4) and equation (4.27), p. 203, [37].)

In our case, we note the multiplication symbol $a(x)$ and the Mellin symbol $b(\xi)$ are both continuous on $\mathbb{R}_+$ and $\mathbb{R}$ respectively. On the other hand, whilst the Wiener-Hopf symbols, $c(\xi)$, are continuous for all finite $\xi$, we do allow $c(\infty) \neq c(-\infty)$.

We note that Theorem 5.5.4, p. 281, [37], makes use of the operator $S_\mathbb{R}$, see equation (4.18), p. 201, [37], with the important property that $S_\mathbb{R}S_\mathbb{R} = I$.

Therefore,

$$ \frac{(I \pm S_\mathbb{R})}{2}(I \pm S_\mathbb{R}) = \frac{(I \pm S_\mathbb{R})}{2} \quad \text{and} \quad \frac{(I \pm S_\mathbb{R})}{2}(I \mp S_\mathbb{R}) = 0. $$

Hence, any operator of the form

$$ h_- \frac{(I - S_\mathbb{R})}{2} + h_+ \frac{(I + S_\mathbb{R})}{2}, \quad h_\pm \in \mathbb{C}, $$

is invertible if and only if

$$ h_- \neq 0 \quad \text{and} \quad h_+ \neq 0. $$

Let $\chi_\pm$ denote the characteristic functions of the positive and negative half-lines respectively. Then trivially, any operator of the form

$$ h_-\chi_-I + h_+\chi_+I, \quad h_\pm \in \mathbb{C}, $$

is invertible if and only if

$$ h_- \neq 0 \quad \text{and} \quad h_+ \neq 0. $$

**6.1.1 Loop functions**

Finally, in the light of Theorem 5.5.7, p. 286, [37] and observation (6.1), it will be convenient, in our notation, to define

$$ g_p(\infty, \xi) := g(-\infty) \frac{1 + d(\xi)}{2} + g(+\infty) \frac{1 - d(\xi)}{2} \quad (6.2) $$
where \( d(\xi) := \coth \pi(i/p + \xi) \).

It is easy to verify that
\[
\lim_{\xi \to \pm \infty} g_p(\infty, \xi) = g(\mp \infty).
\]

The function \( g_p(\infty, \xi) \), defined by equation (6.2), traces out an arc of a circle, dependent only on \( p \) and the function values \( g(\pm \infty) \), in the complex plane, as \( \xi \) varies from \(-\infty\) to \(+\infty\). Indeed, if we assume, without loss of generality, that \( g(-\infty) = -1 \) and \( g(\infty) = +1 \), then a routine calculation shows that
\[
|g_p(\infty, \xi) - a| = r,
\]
where the constants \( a \) and \( r \) only depend on \( p \), and are given by
\[
a = i \cot(2\pi/p) \quad \text{and} \quad r = \frac{1}{\sin(2\pi/p)}.
\]

Moreover, if \( g(\mp \infty) = \mp 1 \) then
\[
\Im g_p(\infty, \xi) = \frac{\sin(2\pi/p)}{\cosh(2\pi \xi) - \cos(2\pi/p)},
\]
so that the sign of the imaginary part of \( g_p(\infty, \xi) \) is determined simply by the sign of \( \sin(2\pi/p) \). In other words, if \( 1 < p < 2 \) then the circular arc is below the interval \([-1, 1]\) and if \( 2 < p < \infty \) it is above. Finally, in the special case that \( p = 2 \) the arc degenerates precisely to the interval \([-1, 1]\) in the complex plane.

### 6.2 The contour \( \Gamma_M \) and symbol \( A_{\alpha,p,s} \)

We now follow Duduchava, see p. 520, [12], and using his notation we define
\[
\omega := (x, \xi, \lambda) \quad \text{where} \quad 0 \leq x \leq \infty, \ -\infty \leq \xi, \lambda \leq \infty.
\]

Then we consider the contour \( \Gamma_M \), which can be described as
\[
\Gamma_M := \Gamma_1 \cup \Gamma_2^{+} \cup \Gamma_3^{+} \cup \Gamma_4 \cup \Gamma_3^{-} \cup \Gamma_2^{-}, \quad (6.3)
\]
where the order of the six segments indicates the direction to be taken.
On each of the six segments of $\Gamma_M$, two of the variables in the triple $(x, \xi, \lambda)$ are fixed, whilst the third varies over its permitted range. The precise definition, including orientation, of each segment of the contour $\Gamma_M$ is as follows:

\[
\Gamma_M = \begin{cases} 
\Gamma_1 &= \{(0, \xi, \infty) : -\infty \leq \xi \leq \infty\} \\
\Gamma_2^+ &= \{(x, \infty, \infty) : 0 \leq x \leq \infty\} \\
\Gamma_2^- &= \{(\infty, \infty, \lambda) : \infty \geq \lambda \geq 0\} \\
\Gamma_3^+ &= \{(\infty, \xi, 0) : \infty \geq \xi \geq -\infty\} \\
\Gamma_3^- &= \{(\infty, -\infty, \lambda) : 0 \leq \lambda \leq \infty\} \\
\Gamma_4 &= \{(x, -\infty, \infty) : \infty \geq x \geq 0\}. 
\end{cases}
\]

**Remark 6.2.** There is a typographical error in the statement of Theorem 5.5.7, pp. 286, 287, [37]. The right hand side of the display formula in the second line on p. 287 should read

\[ a(0^+)\chi_I - a(+\infty)\chi_I, \]

*instead of*

\[ a(+\infty)\chi_I - a(0^+)\chi_I. \]

With these preparations complete, we are now in a position to restate Corollary 5.5.10, p. 290, [37], in a more convenient form.

Firstly, see Theorems 5.5.3, 5.5.4, and 5.5.7, [37] it will be convenient to define the following functions:

\[ \text{This was confirmed in a personal communication from Prof. Steffen Roch on 13th October 2016.} \]
\[ S_{\Gamma_3^+}(\lambda) := \sum_{j=1}^{N} a_j(\infty)b_j(\infty)c_j(-\lambda) \quad (\lambda > 0); \] (6.4)

\[ S_{\Gamma_3^-}(\lambda) := \sum_{j=1}^{N} a_j(\infty)b_j(-\infty)c_j(\lambda) \quad (\lambda > 0); \] (6.5)

\[ S_{\Gamma_2^+}(x) := \sum_{j=1}^{N} a_j(x)b_j(\infty)c_j(-\infty) \quad (x > 0); \] (6.6)

\[ S_{\Gamma_2^-}(x) := \sum_{j=1}^{N} a_j(x)b_j(-\infty)c_j(\infty) \quad (x > 0); \] (6.7)

\[ S_{\Gamma_1}(\xi) := \sum_{j=1}^{N} a_j(0)b_j(\xi)c_j(\infty, \xi) \quad (-\infty < \xi < \infty); \] (6.8)

\[ S_{\Gamma_4}(\xi) := \sum_{j=1}^{N} a_j(\infty)b_j(\xi)c_j(0) \quad (-\infty < \xi < \infty). \] (6.9)

In addition, see Theorem 5.5.7, we also define the “intersection” values:

\[ I_{\Gamma_3^+ \cap \Gamma_4} := \sum_{j=1}^{N} a_j(\infty)b_j(\infty)c_j(0); \]

\[ I_{\Gamma_3^- \cap \Gamma_2^+} := \sum_{j=1}^{N} a_j(\infty)b_j(-\infty)c_j(\infty); \]

\[ I_{\Gamma_2^+ \cap \Gamma_3^-} := \sum_{j=1}^{N} a_j(\infty)b_j(\infty)c_j(-\infty); \]

\[ I_{\Gamma_2^- \cap \Gamma_1} := \sum_{j=1}^{N} a_j(0)b_j(-\infty)c_j(\infty); \]

\[ I_{\Gamma_1 \cap \Gamma_2^+} := \sum_{j=1}^{N} a_j(0)b_j(\infty)c_j(-\infty); \]

\[ I_{\Gamma_4 \cap \Gamma_3^-} := \sum_{j=1}^{N} a_j(\infty)b_j(-\infty)c_j(0). \]

Accordingly, we now define the \textit{generalised symbol} \( A_{\alpha,p,s}(\omega) \) by:

\[ A_{\alpha,p,s}(\omega) := \begin{cases} 
S_{\Gamma_1}(\xi) & \text{on } \Gamma_1 \\
S_{\Gamma_2^+}(x) & \text{on } \Gamma_2^+ \\
S_{\Gamma_3^+}(\lambda) & \text{on } \Gamma_3^+ \\
S_{\Gamma_4}(\xi) & \text{on } \Gamma_4.
\end{cases} \]

Then it is easy to see, from the above results, that as the triple \( \omega = (x, \xi, \lambda) \) traverses the contour \( \Gamma_M \), \( A_{\alpha,p,s}(\omega) \) forms a closed loop in the complex plane.

Hence, we can re-write Corollary 5.5.10, p. 290, [37] as:
Theorem 6.3. The operator

$$\tilde{A} = \sum_{j=1}^{N} a_j M^0(b_j)W(c_j)$$

is Fredholm on $L_p(\mathbb{R}_+)$ if and only if

$$\inf_{\omega \in \Gamma_M} |A_{\alpha,p,s}(\omega)| > 0.$$  

Remark 6.4. In the case that $\tilde{A}$ is a Fredholm operator on $L_p(\mathbb{R}_+)$, then, see Theorem 3.2, p. 521, [12], the index of $\tilde{A}$ is given by

$$\text{ind} \tilde{A} = - (\text{winding number of } A_{\alpha,p,s}(\omega)). \quad (6.10)$$

In particular, if the winding number of $A_{\alpha,p,s}(\omega)$ is zero, then $\tilde{A}$ has Fredholm index equal to zero.

We now verify Remark 6.4 in a simple case. Let us define the symbol

$$c_{(n)}(\xi) := (\xi + i)^n(\xi - i)^{-n}, \quad n \in \mathbb{N}.$$  

Then, see Chapter 1, Section 8, [16], the Wiener-Hopf operator $W(c_{(n)})$, acting on $L_p(\mathbb{R}_+)$, has the following properties:

(a) $W(c_{(n)})$ is right-invertible;

(b) $\text{Ker} W(c_{(n)})$ is spanned by the set $\{t^{k-1}e^{-t}\}_{k=1}^{n}$.

Hence, $\dim \text{CoKer} W(c_{(n)}) = 0$, $\dim \text{Ker} W(c_{(n)}) = n$ and $W(c_{(n)})$ is a Fredholm operator with index $n$.

On the other hand, if we now calculate the generalised symbol of $W(c_{(n)})$, using equations (6.8), (6.6), (6.7), (6.4), (6.5) and (6.9) respectively, we obtain:

$$A_{\alpha,p,s}(\omega) := \begin{cases} 
  c_{(n)}(\infty, \xi) & \text{on } \Gamma_1 \\
  c_{(n)}(\mp \infty) & \text{on } \Gamma_2^\pm \\
  c_{(n)}(\mp \lambda) & \text{on } \Gamma_3^\pm \\
  c_{(n)}(0) & \text{on } \Gamma_4.
\end{cases}$$
But since \( c(n)(+\infty) = c(n)(-\infty) \), it is easy to see that as the triple \( \omega = (x, \xi, \lambda) \)
traverses the contour \( \Gamma_M \), the function \( A_{\alpha,p,s}(\omega) \) can be represented by simply
\( c(n)(\lambda) \).

Moreover, a routine calculation shows that

\[
\text{winding number of } c(n)(\lambda) = -n,
\]

and, thus, we have validated the formula given by (6.10), in the special case that \( \tilde{A} = W(c(n)) \).

### 6.3 Supporting lemmas

**Lemma 6.5.** Suppose \( \nu = m - s + \alpha \), and the symbol \( c_1(\xi) \) is given by

\[
c_1(\xi) = (1 + \xi^2)^\alpha (\xi - i)^s e^{-2\alpha - m} (\xi + i)^m - s, \quad m = 1 \text{ or } 2.
\]

Then

\[
\lim_{\xi \to \infty} c_1(\xi) = 1; \quad \lim_{\xi \to -\infty} c_1(\xi) = e^{2\pi i \nu},
\]

and

\[
\lim_{\xi \to 0^+} c_1(\xi) = e^{\pi i \nu}; \quad \lim_{\xi \to 0^-} c_1(\xi) = e^{\pi i \nu}.
\]

Thus, the symbol \( c_1(\xi) \) has a (single) discontinuity at \( \xi = \infty \).

**Proof.** It is easy to see that \( |c_1(\xi)| = 1 \) for all \( \xi \in \mathbb{R} \). Therefore

\[
c_1(\xi) = \exp \left[ 0 + i(s - 2\alpha - m) \arg(\xi - i) + i(m - s) \arg(\xi + i) \right].
\]

As \( \xi \to \infty \), we have \( \arg(\xi \pm i) \to 0 \). Hence \( \lim_{\xi \to \infty} c_1(\xi) = 1 \).

On the other hand,

\[
\lim_{\xi \to -\infty} c_1(\xi) = \exp \left[ i(\xi(\xi - 2\alpha - m)(-\pi) + i(m - s)\pi) \right]
= \exp \left[ i\pi(m - s - s + 2\alpha + m) \right]
= \exp[2\pi i \nu], \quad (\nu = m - s + \alpha).
\]

Moreover

\[
\lim_{\xi \to 0^+} c_1(\xi) = \exp \left[ i(s - 2\alpha - m)(-\pi/2) + i(m - s)(+\pi/2) \right]
= \exp[\pi i \nu]
= \lim_{\xi \to 0^-} c_1(\xi).
\]
This completes the proof of the lemma.

**Lemma 6.6.** Suppose the symbol $c_2(\xi)$ is given by

$$c_2(\xi) = (-i\xi)^{2\alpha} (\xi + i)^{m-s} (\xi - i)^{-m+s-2\alpha}, \quad m = 1 \text{ or } 2.$$ 

Then, if $\nu' := m - s + 2\alpha$,

$$\lim_{\xi \to \infty} c_2(\xi) = \exp[-i\pi\alpha]; \quad \lim_{\xi \to -\infty} c_2(\xi) = \exp[-i\pi\alpha] \cdot \exp[i\pi2\nu'],$$

and

$$\lim_{\xi \to 0^+} c_2(\xi) = 0; \quad \lim_{\xi \to 0^-} c_2(\xi) = 0$$

Thus, the symbol $c_2(\xi)$ has a (single) discontinuity at $\xi = \infty$.

**Proof.** We write $(-i\xi)^{2\alpha} (\xi + i)^{m-s} (\xi - i)^{-m+s-2\alpha}$

$$= \frac{\exp[2\alpha \log |\xi| + i2\alpha \arg(-\xi) + (-m + s - 2\alpha) \log |\xi - i| + i(-m + s - 2\alpha) \arg(\xi - i)]}{\exp[(s - m) \log |\xi + i| + i(s - m) \arg(\xi + i)]}$$

$$= \frac{\exp[2\alpha \log |\xi| + (s - m - 2\alpha) \log |\xi - i|]}{\exp[(s - m) \log |\xi + i|]} \cdot \exp[i2\alpha \arg(-\xi) + i(m - s) \arg(\xi + i) + i(s - m - 2\alpha) \arg(\xi - i)].$$

Hence

$$\lim_{\xi \to \infty} c_2(\xi) = \exp[i2\alpha(-\pi/2) + 0 + 0] = \exp[-i\pi\alpha],$$

and, if $\nu' = m - s + 2\alpha$,

$$\lim_{\xi \to -\infty} c_2(\xi) = \exp[i2\alpha(\pi/2) + i(m - s)\pi + i(s - m - 2\alpha)(-\pi)]$$

$$= \exp[i\pi(2m - 2s + 3\alpha)]$$

$$= \exp[-i\pi\alpha] \cdot \exp[i\pi2\nu'].$$

Finally, we consider the behaviour of $c_2(\xi)$ near $\xi = 0$, and note that

$$\lim_{\xi \to 0^+} |c_2(\xi)| = 0 = \lim_{\xi \to 0^-} |c_2(\xi)|$$

and the required results follow immediately. □
Lemma 6.7. Suppose $1 < p < \infty$ and $\nu \in \mathbb{R}$. Let

$$d_p(\infty, \theta) := \frac{e^{2\pi i \nu}}{2} \left[ 1 + \coth \pi \left( \frac{i}{p} + \theta \right) \right] + \frac{1}{2} \left[ 1 - \coth \pi \left( \frac{i}{p} + \theta \right) \right].$$

Then

$$d_p(\infty, \theta) = e^{\pi \nu i} \sin(\pi(1/p + \nu - i\theta)) \sin(\pi(1/p - i\theta)).$$

Proof. Let $z = \pi(i/p + \theta)$. Then

$$d_p(\infty, \theta) = \frac{e^{\pi \nu i}}{2} \left\{ e^{\pi \nu i} [1 + \coth z] + e^{-\pi \nu i} [1 - \coth z] \right\}$$

$$= \frac{e^{\pi \nu i}}{2(e^z - e^{-z})} \left\{ e^{\pi \nu i} (e^z - e^{-z} + e^z + e^{-z}) + e^{-\pi \nu i} (e^z - e^{-z} - e^z - e^{-z}) \right\}$$

$$= e^{\pi \nu i} \frac{\sinh(\pi \nu i + z)}{\sinh z}$$

$$= e^{\pi \nu i} \frac{\sin(-\pi \nu + iz)}{\sin iz} \quad \text{since} \quad \sin(iw) = i\sin(w) \quad (4.28.8, [34])$$

$$= e^{\pi \nu i} \frac{\sin(\pi(1/p + \nu - i\theta))}{\sin(\pi(1/p - i\theta))}, \quad \text{as required.}$$

\[ \square \]

6.4 Generalised symbol - lower regularity

Suppose that $1 < p < \infty$, $1/p < s < 1 + 1/p$ and $0 < \alpha < \frac{1}{2}$.

We are interested in the solvability of the equation (5.17)

$$(W(c_1) + a_2 M^0(b_2) W(c_2) + T)(r_+ u_s) = g,$$

where the operator $T$, acting on $L_p(\mathbb{R}_+)$, is compact and from (5.18)

$$g := r_+(D - i)^{s-2\alpha} f;$$

$$c_1(\xi) = (1 + \xi^2)^{\alpha}(\xi - i)^{s-2\alpha-1}(\xi + i)^{1-s}.$$  

$$a_2(x) = -iC_\alpha \psi(\alpha + 1, 2\alpha + 1, x) \quad \text{(see Lemma 4.12)};$$

$$b_2(\xi) = B(s - 2\alpha + 1/p' + i\xi; 2\alpha)/\Gamma(2\alpha);$$

$$c_2(\xi) = (-i\xi)^{2\alpha}(\xi - i)^{s-2\alpha-1}(\xi + i)^{1-s}.$$
and the constant $C_\alpha$ is given by
\[ C_\alpha = -i \frac{\alpha 2^{2\alpha}}{\Gamma(1 - \alpha)}. \]

Our immediate goal is to show that the operator
\[ \tilde{A} := W(c_1) + a_2 M^0(b_2) W(c_2) \]
acting on $L_p(\mathbb{R}_+)$ is Fredholm.

Finally, purely for notational convenience, we define
\[ a_1(x) = 1 \quad \text{and} \quad b_1(\xi) = 1. \]

### 6.4.1 Segment $\Gamma_1$

Firstly, we note that
\[ a_2(0) b_2(\xi) = -i C_\alpha \psi(\alpha + 1, 2\alpha + 1, 0) \]
\[ = -i C_\alpha 2^{-2\alpha} \frac{\Gamma(2\alpha)}{\Gamma(\alpha + 1)} \quad (\text{see Lemma 4.12}). \]

Hence,
\[
a_2(0) b_2(\xi) = -i C_\alpha 2^{-2\alpha} \frac{\Gamma(2\alpha)}{\Gamma(\alpha + 1)} \cdot \frac{\Gamma(1 - \alpha)}{\Gamma(\alpha + 1)} \cdot B(s - 2\alpha + 1/p' + i\xi, 2\alpha) \\
= -\frac{\alpha 2^{2\alpha}}{\Gamma(1 - \alpha)} \cdot \frac{2^{-2\alpha}}{\Gamma(\alpha + 1)} \cdot B(s - 2\alpha + 1/p' + i\xi, 2\alpha) \\
= -\frac{1}{\Gamma(1 - \alpha)\Gamma(\alpha)} \cdot B(s - 2\alpha + 1/p' + i\xi, 2\alpha) \quad (\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)) \\
= -\frac{\sin \pi \alpha}{\pi} \cdot B(s - 2\alpha + 1/p' + i\xi, 2\alpha) \quad (5.5.3, [34]).
\]

Therefore, on the segment $\Gamma_1$, for $-\infty \leq \xi \leq \infty$, we have
\[ A_{\alpha,p,s}(\omega) := a_1(0) b_1(\xi) c_{1p}(\infty, \xi) + a_2(0) b_2(\xi) c_{2p}(\infty, \xi) \]
\[ = c_{1p}(\infty, \xi) - \frac{\sin \pi \alpha}{\pi} \cdot B(s - 2\alpha + 1/p' + i\xi, 2\alpha) c_{2p}(\infty, \xi). \]

From Lemmas 6.5 and 6.7, we have
\[ c_{1p}(\infty, \xi) = e^{i\pi \nu \sin[\pi(1/p + \nu - i\xi)]} \frac{\sin \pi(1/p - i\xi)}{\sin \pi(1/p - i\xi)}, \quad \nu = 1 - s + \alpha. \]
Similarly, from Lemmas 6.6 and 6.7, we have
\[ c_{2p}(\infty, \xi) = e^{-\pi \alpha} e^{i \pi \nu' \sin[\pi(1/p + \nu' - i \xi)]} \sin \pi(1/p - i \xi) \], \quad \nu' = 1 - s + 2\alpha. \]

But \( e^{-\pi \alpha} e^{i \pi \nu'} = e^{i \pi \nu} \), and thus \( c_{1p}(\infty, \xi) \) and \( c_{2p}(\infty, \xi) \) have a common factor \( e^{i \pi \nu} \sin \pi(1/p - i \xi) \).

So, we are interested in establishing the precise conditions under which the quadruple \((\alpha, p, s, \xi)\) is not a solution of the following transcendental equation
\[ \sin(\pi(1/p + \nu - i \xi)) \sin(\pi(1/p + \nu' - i \xi)) - \sin(\pi \alpha) B(s - 2\alpha + 1/p' + i \xi, 2\alpha) = 0. \] (6.11)

Let us now define
\[ T_s := \frac{\sin(\pi(1/p + \nu - i \xi))}{\sin(\pi(1/p + \nu' - i \xi))} \quad \nu = 1 - s + \alpha; \quad \nu' = 1 - s + 2\alpha, \] (6.12)
and
\[ T_B := \frac{\sin(\pi \alpha)}{\sin(\pi(1/p + \nu' - i \xi))} B(s - 2\alpha + 1/p' + i \xi, 2\alpha). \] (6.13)
Then, the transcendental equation (6.11) simply becomes
\[ T_s = T_B. \]

6.4.2 Segment \( \Gamma_2^\pm \)

Similarly, on \( \Gamma_2^+ \), for \( 0 \leq x \leq \infty \), we have
\[
A_{\alpha,p,s}(\omega) := a_1(x) b_1(\infty) c_1(-\infty) + a_2(x) b_2(\infty) c_2(-\infty)
= c_1(-\infty) + 0
= e^{2\pi \nu i},
\]
and on \( \Gamma_2^- \), for \( \infty \geq x \geq 0 \),
\[
A_{\alpha,p,s}(\omega) := a_1(x) b_1(-\infty) c_1(+\infty) + a_2(x) b_2(-\infty) c_2(+\infty)
= c_1(+\infty) + 0
= 1.
\]
Hence,
\[ \inf_{\omega \in \Gamma_2^+ \cup \Gamma_2^-} |A_{\alpha,p,s}(\omega)| = 1. \]
6.4.3 Segment $\Gamma_{3}^{\pm}$

On $\Gamma_{3}^{+}$ for $\infty > \lambda \geq 0$,

$$A_{\alpha,p,s}(\omega) := a_{1}(\infty) b_{1}(\infty) c_{1}(-\lambda) + a_{2}(\infty) b_{2}(\infty) c_{2}(-\lambda)$$

$$= c_{1}(-\lambda) + 0$$

$$= c_{1}(-\lambda),$$

and on $\Gamma_{3}^{-}$, for $0 \leq \lambda < \infty$,

$$A_{\alpha,p,s}(\omega) := a_{1}(\infty) b_{1}(-\infty) c_{1}(\lambda) + a_{2}(\infty) b_{2}(-\infty) c_{2}(\lambda)$$

$$= c_{1}(\lambda) + 0$$

$$= c_{1}(\lambda).$$

Note that on $\Gamma_{3}^{+}$ and $\Gamma_{3}^{-}$ the parameter $\lambda$ varies between $0$ and $\infty$ but, of course, in an opposite sense. So, in summary,

$$\inf_{\omega \in \Gamma_{3}^{\pm}} |A_{\alpha,p,s}(\omega)| = 1.$$ 

6.4.4 Segment $\Gamma_{4}$

Finally, on $\Gamma_{4}$, for $-\infty \leq \xi \leq \infty$,

$$A_{\alpha,p,s}(\omega) := a_{1}(\infty) b_{1}(\xi) c_{1}(0) + a_{2}(\infty) b_{2}(\xi) c_{2}(0)$$

$$= c_{1}(0) + 0$$

$$= c_{1}(0).$$

Hence,

$$\inf_{\omega \in \Gamma_{4}} |A_{\alpha,p,s}(\omega)| = 1,$$

and this completes the review of the contour $\Gamma_{M}$.

6.4.5 Summary

Note that the preceding analysis of the segments of the contour has shown that $A_{\alpha,p,s}(\omega)$ is constant on the segments $\Gamma_{2}^{\pm}$ and $\Gamma_{4}$. Therefore, it remains to consider $A_{\alpha,p,s}(\omega)$ on $\Gamma_{1} \cup \Gamma_{3}^{+} \cup \Gamma_{3}^{-}$. But from subsection 6.4.3, we can combine $\Gamma_{3}^{\pm}$ to give a new segment $\Gamma_{3}$ (say), where now the parameter $\lambda$ varies from $-\infty$ to $\infty$. (Note that, as expected, the symbol $A_{\alpha,p,s}(\omega)$ is continuous at $\lambda = 0$ on the new segment $\Gamma_{3}$.)
By construction, we observe that $A_{\alpha,p,s}(\omega)$ is continuous on $\Gamma_1 \cup \Gamma_3$. Indeed, from subsection 6.4.1, on the segment $\Gamma_1$

$$A_{\alpha,p,s}(\omega) = c_{1p}(\infty, \xi) - \frac{\sin \pi \alpha}{\pi} B(s - 2\alpha + 1/p' + i\xi, 2\alpha) c_{2p}(\infty, \xi) \quad (6.14)$$

with limits $c_{1p}(\infty, \pm \infty) = c_1(\mp \infty)$ at $\xi = \pm \infty$ respectively.

The condition that $A_{\alpha,p,s}(\omega) = 0$ on $\Gamma_1$ gives rise to the transcendental equation

$$\sin(\pi(1/p + \nu - i\xi)) \sin(\pi(1/p + \nu' - i\xi)) = \frac{\sin \pi \alpha}{\pi} B(s - 2\alpha + 1/p' + i\xi, 2\alpha),$$

where $\nu = 1 - s + \alpha$ and $\nu' = 1 - s + 2\alpha$.

Finally, on $\Gamma_3$ we have

$$A_{\alpha,p,s}(\omega) = c_1(\lambda) \quad (6.15)$$

with limits $c_1(\pm \infty)$ for $\lambda = \pm \infty$.

**Remark 6.8.** It turns out that we are in the subalgebra described by Duduchava, Section 3.2, p. 524, [12]. In other words, the generators of the algebra are simply of the form $aM^0(b)$ and $W(c)$, rather than $a$, $M^0(b)$ and $W(c)$ individually.
Chapter 7

Index and invertibility

7.1 Main results

Theorem 7.1. For all $\alpha, p, s$ satisfying the conditions $0 < \alpha < \frac{1}{2}$, $1 < p < \infty$ and $1/p < s < 1 + 1/p$, the winding number of the generalised symbol $(A_{\alpha, p, s}, \Gamma_M)$ in the complex plane is 0. Hence, the operator $W(c_1) + a_2 M^0(b_2) W(c_2)$, defined on $L_p(\mathbb{R}_+)$, has Fredholm index equal to zero.

Theorem 7.2. Suppose $0 < \alpha < \frac{1}{2}$, $1 < p < \infty$ and $1/p < s < 1 + 1/p$. Then the operator $\mathcal{A} : H^s_p(\mathbb{R}_+) \to H^{2\alpha-s}_p(\mathbb{R}_+)$ is invertible.

7.2 Proof of Theorem 7.1

The constraints are

$$0 < \alpha < \frac{1}{2}, \ 1 < p < \infty \text{ and } 1/p < s < 1 + 1/p.$$  (7.1)

Let $\alpha, p, s$ fall within their admissible ranges and be fixed. From Chapter 6, we know that the generalised symbol $A_{\alpha, p, s}$ can be represented by a closed contour in the complex plane given by the union of the two curves, $S_1$ and $S_3$.

Indeed, from Section 6.4.3 we have,

$$S_3(\xi) := (1 + \xi^2)^\alpha (\xi - i)^{s-2\alpha-1} (\xi + i)^{1-s}, \quad -\infty \leq \xi \leq \infty.$$  (7.2)

Now

$$S_3(\xi) = (\xi + i)^\alpha (\xi - i)^\alpha (\xi - i)^{s-2\alpha-1} (\xi + i)^{1-s}$$
$$= (\xi + i)^{1-s+\alpha} (\xi - i)^{(1-s+\alpha)}.$$
From Section 6.4.1, for $-\infty \leq \xi \leq \infty$,

$$S_1(\xi) := S_{11}(\xi) - \frac{\sin \pi \alpha}{\pi} B(s - 2\alpha + 1 - 1/p + i\xi, 2\alpha) S_{12}(\xi), \quad (7.3)$$

where

$$S_{11}(\xi) := e^{i\pi \nu} \frac{\sin[\pi(1/p + \nu - i\xi)]}{\sin[\pi(1/p - i\xi)]}, \quad S_{12}(\xi) := e^{i\pi \nu} \frac{\sin[\pi(1/p + \nu + \alpha - i\xi)]}{\sin[\pi(1/p - i\xi)]},$$

and $\nu = 1 - s + \alpha$.

Let us now choose the values

$$\alpha = \frac{2}{5}, \quad p = 2 \quad \text{and} \quad s = \frac{7}{5}, \quad (7.4)$$

as the set of parameters used to define the model contour.

Note that the values for $\alpha$, $p$ and $s$ given in (7.4) satisfy the constraints described in condition (7.1). Moreover, $\nu = 1 - s + \alpha = 0$, and thus

$$S_3(\xi) = 1; \quad S_{11}(\xi) = 1. \quad (7.5)$$

With the chosen values for $\alpha$, $p$ and $s$, equation (7.3) becomes

$$S_1(\xi) = 1 - \frac{\sin 2\pi}{\pi} B\left(\frac{11}{10} + i\xi, \frac{4}{5}\right) \frac{\sin[\pi(\frac{9}{10} - i\xi)]}{\sin[\pi(\frac{1}{2} - i\xi)]}$$

But by Lemma 9.10, with $\sigma = \frac{11}{10}$ and $\alpha = \frac{2}{5}$, we have

$$|B\left(\frac{11}{10} + i\xi, \frac{4}{5}\right)| \leq B\left(\frac{11}{10}, \frac{2}{5}\right) < 1.152 < 1.2.$$ 

Moreover, by Lemma 9.14, with $a = \frac{9}{10}$ and $b = \frac{1}{2}$,

$$\left|\frac{\sin[\pi(\frac{9}{10} - i\xi)]}{\sin[\pi(\frac{1}{2} - i\xi)]}\right| \leq \left(\frac{\cosh 2\pi \xi}{\cosh 2\pi \xi + 1}\right)^{\frac{1}{2}} \leq 1,$$

noting that $\cos 2\pi a > 0$ and $\cos 2\pi b = -1$.

Hence, we have the estimate

$$\left|\frac{\sin 2\pi}{\pi} B\left(\frac{11}{10} + i\xi, \frac{4}{5}\right) \frac{\sin[\pi(\frac{9}{10} - i\xi)]}{\sin[\pi(\frac{1}{2} - i\xi)]}\right| < \left(\frac{1}{\pi}\right)^{\frac{1}{2}} \times 1.2 < \frac{2}{5}. \quad (7.6)$$
So, from (7.5) and (7.6), the model contour, formed by the union of the sections $S_1$ and $S_3$, is wholly contained in the disc of radius $\frac{2}{5}$ centred on the point 1 in the complex plane. Hence, the winding number of the model contour must be zero.

But, given any set of parameters $\alpha, p, s$ satisfying the constraints $0 < \alpha < \frac{1}{2}$, $1 < p < \infty$ and $1/p < s < 1 + 1/p$, the associated contour can be continuously deformed into the model contour, and from Theorem 9.1, does this without ever crossing the origin. Hence, the two contours must have the same winding number, namely, zero.

Therefore, see Remark 6.4, the operator $W(c_1) + a_2 M^0(b_2) W(c_2)$, defined on $L_p(\mathbb{R}_+)$, has Fredholm index equal to zero. This completes the proof of the first theorem.

To complete this discussion on winding number, we now give three numerical examples of symbol plots for fixed $\alpha = 0.25$ and $p = 4$, with $s$ taking the values $0.3$, $0.7$ and $1.1$, in turn. See Figures 7.1, 7.2 and 7.3 respectively. The contour is the union of the curves $S_1$ and $S_3$, where $S_3$ forms part of the unit circle. As expected, in each case, it has winding number equal to zero.

Secondly, we set $s = 0.7$. 

Figure 7.1: Symbol plot for $\alpha = 0.25$, $p = 4$ and $s = 0.3$. 

Secondly, we set $s = 0.7$. 

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Finally, we take $s = 1.1$.

Figure 7.3: Symbol plot for $\alpha = 0.25$, $p = 4$ and $s = 1.1$.

### 7.3 Proof of Theorem 7.2

Suppose $0 < \alpha < \frac{1}{2}$, $1 < p < \infty$ and $1/p < s < 1 + 1/p$. Let $u \in H^s_p(\mathbb{R}_+)$. Then, see Lemma 4.3,

$$u_s := (D + i)^{s-1}e_+(D - i)u,$$

and $u_s \in L_p(\mathbb{R})$ with supp $u_s \subseteq \mathbb{R}_+$. Indeed, see Lemma 4.21, we can also write

$$u = (r_+(D - i)^{-1}(D + i)^{1-s}e_+)u_s.$$
Moreover, from Theorem 7.1, the operator
\[ \tilde{A} := W(c_1) + a_2 M^0(b_2)W(c_2), \]  
(7.7)
defined on \( L_p(\mathbb{R}_+) \), has Fredholm index equal to zero.

By Theorem 5.1,
\[ (r_+(D - i)^{s-2\alpha} l_+) A (r_+(D - i)^{-1}(D + i)^{1-s} e_+) = \tilde{A} + T \]
where \( T \), defined on \( L_p(\mathbb{R}_+) \), is a compact operator.

The operators \( r_+(D - i)^{s-2\alpha} l_+ : H^{s-2\alpha}_p(\mathbb{R}_+) \to L_p(\mathbb{R}_+) \) and \( r_+(D - i)^{-1}(D + i)^{1-s} e_+ : L_p(\mathbb{R}_+) \to H^s_p(\mathbb{R}_+) \) are both invertible. Therefore,
\[ \text{ind} A = \text{ind}(\tilde{A} + T) = \text{ind} \tilde{A} = 0, \]  
(7.8)
since the Fredholm index is stable under compact perturbations.

On the other hand, from Theorem 3.1, the operator \( A : H^{s_2}_2(\mathbb{R}_+) \to H^{s_2-2\alpha}_2(\mathbb{R}_+) \) (is bounded and) has a trivial kernel. The following lemma will allow us to generalise this result from \( p = 2 \) to the full range \( 1 < p < \infty \).

**Lemma 7.3.** \(^1\) Let \( X_1, X_2, Y_1, Y_2 \) be Banach spaces such that \( X_1 (Y_1) \) is continuously and densely embedded into \( X_2 \) (into \( Y_2 \), respectively). Suppose \( A : X_j \to Y_j \) is Fredholm, \( j = 1, 2 \), and
\[ \text{Ind}_{X_1 \to Y_1} A = \text{Ind}_{X_2 \to Y_2} A. \]

Then
\[ \text{Ker}_{X_1 \to Y_1} A = \text{Ker}_{X_2 \to Y_2} A. \]

**Proof.** Since the above embeddings are dense, \( X_j^* \hookrightarrow X_1^*, Y_j^* \hookrightarrow Y_1^* \). The operator \( A^* : Y_j^* \to X_j^* \) is Fredholm, \( j = 1, 2 \). Let
\[ \alpha_j := \dim \text{Ker}_{X_j \to Y_j} A, \quad \beta_j := \dim \text{Ker}_{Y_j^* \to X_j^*} A^*, \quad j = 1, 2. \]

Since
\[ \text{Ker}_{X_1 \to Y_1} A \subseteq \text{Ker}_{X_2 \to Y_2} A, \quad \text{Ker}_{Y_2^* \to X_2^*} A \subseteq \text{Ker}_{Y_1^* \to X_1^*} A, \]  
(7.9)
\(^1\)A stronger version of this result can be found in Lemma 2.17, p. 99, [41]. The idea of the proof is taken from a Dissertation of Vladimir Pilidi.

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we have
\[ \alpha_1 \leq \alpha_2, \quad \beta_1 \geq \beta_2. \]
Since
\[ \alpha_1 - \beta_1 = \text{Ind}_{X_1 \to Y_1} A = \text{Ind}_{X_2 \to Y_2} A = \alpha_2 - \beta_2, \]
we conclude that \( \alpha_1 = \alpha_2, \beta_1 = \beta_2. \) Hence the inclusions (7.9) are in fact equalities.

To complete the proof of the second main result, we now consider (the dimension of) \( \text{Ker} A, \) for the cases \( p > 2 \) and \( p < 2 \) respectively.

**Firstly, suppose** \( p > 2. \) Then, for \( 0 < \delta < 1, \) we define
\[ X_1 := H^{\frac{1}{p} + \delta} (\mathbb{R}_+), \quad Y_1 := H^{\frac{1}{p} + \delta - 2\alpha} (\mathbb{R}_+) \]
and
\[ X_2 := H^{\frac{1}{p} + \delta - 2\alpha} (\mathbb{R}_+), \quad Y_2 := H^{\frac{1}{p} + \delta - 2\alpha} (\mathbb{R}_+). \]
Then \( X_1 (Y_1) \) is continuously and densely embedded into \( X_2 \) (into \( Y_2, \) respectively). Moreover, \( A : X_j \to Y_j \) is Fredholm, \( j = 1, 2, \) and
\[ \text{Ind}_{X_1 \to Y_1} A = \text{Ind}_{X_2 \to Y_2} A \quad (= 0). \]
Therefore, by Lemma 7.3,
\[ \text{Ker}_{X_1 \to Y_1} A = \text{Ker}_{X_2 \to Y_2} A. \]
That is,
\[ \text{Ker}_{X_2 \to Y_2} A = \{0\}. \]

**Secondly, suppose** \( p < 2. \) Then, for \( 0 < \delta < 1, \) we define
\[ X_2 := H^{\frac{1}{p} + \delta} (\mathbb{R}_+), \quad Y_2 := H^{\frac{1}{p} + \delta - 2\alpha} (\mathbb{R}_+) \]
and
\[ X_1 := H^{\frac{1}{p} + \delta - 2\alpha} (\mathbb{R}_+), \quad Y_1 := H^{\frac{1}{p} + \delta - 2\alpha} (\mathbb{R}_+). \]
We can now repeat the argument made above, for the case \( p > 2, \) to show that
\[ \text{Ker}_{X_1 \to Y_1} A = \{0\}. \]

So, finally, the operator \( A : H^{\delta}_p (\mathbb{R}_+) \to H^{\delta - 2\alpha}_p (\mathbb{R}_+) \) is invertible.
Chapter 8

Higher regularity

8.1 Problem definition

Suppose $1 < p < \infty$ and $0 < \alpha < 1$. We now assume $1 + 1/p < s < 2 + 1/p$. Let $A$ denote the pseudodifferential operator of order $2\alpha$, with symbol, see (1.33),

$$A(\xi) = (1 + \xi^2)^\alpha.$$  

Our problem is to investigate the solvability of equation (1.34)

$$r^+ A e^+ u + u r^+ A(\chi_{\mathbb{R}^-}) = f,$$

where $u \in H^s_\mathbb{R}^+$ for a given $f \in H^{s-2\alpha}_\mathbb{R}^+$, subject to the boundary condition

$$u'(0) = 0.$$  \hspace{1cm} (8.1)

8.2 Reformulation

As a first step in reformulating equation (1.34), it will be convenient to define

$$A^\pm(D) := A(D)(D - i)^{-2},$$  \hspace{1cm} (8.2)

where $D = i \frac{\partial}{\partial x}$.

Let $\delta$ denote the Dirac delta function and let $\chi_G$ denote the characteristic function of $G$. Now $\chi_{\mathbb{R}^+}(x) = \delta(x)$, see Example 1.3, p. 10, [14], and $\chi_{\mathbb{R}^+}(x) + \chi_{\mathbb{R}^-}(x) = 1$. Therefore, $\chi_{\mathbb{R}^-}(x) = -\delta(x)$, and we can write

$$r^+ A(\chi_{\mathbb{R}^-}) = r^+ A(D - i)^{-2}(D - i)^2 \chi_{\mathbb{R}^-}$$

$$= r^+ A^\pm(D - i)(-i\delta - i\chi_{\mathbb{R}^-})$$

$$= r^+ A^\pm(\delta' - 2\delta - \chi_{\mathbb{R}^-}).$$
Moreover, from Lemma 4.2, we have the identity,
\[(D - i)^2e_+ u = e_+(D - i)^2 u - u(0) \delta' + 2u(0) \delta - u'(0) \delta.\]
Using the boundary condition (8.1) and equation (8.2)
\[r_+ A^+ e_+ u = r_+(D - i)^2 e_+ u = r_+ A^+ e_+(D - i)^2 u - u(0) r_+ A^+ (\delta' - 2\delta).\]
Hence, we can rewrite equation (1.34) as
\[r_+ A^+ e_+(D - i)^2 u + (u(x) - u(0)) r_+ A^+ (\delta' - 2\delta) - u(x) r_+ A^+ (\chi_{\mathbb{R}_+}) = f. \quad (8.3)\]
We now define
\[A_s(D) := A(D)(D - i)^{-2}(D + i)^{2-s} = A^+(D + i)^{2-s}, \quad (8.4)\]
and hence
\[r_+ A^+ e_+(D - i)^2 u = r_+ A_s(D + i)^{s-2} e_+(D - i)^2 u = r_+ A_s u_s, \quad (8.5)\]
where we set
\[u_s := (D + i)^{s-2} e_+(D - i)^2 u. \quad (8.5)\]
Moreover, from Lemma 4.4, we have \(u_s \in L^p(\mathbb{R})\) with \(\text{supp } u_s \subseteq \overline{\mathbb{R}_+}^+.\)
Then, it follows directly from (8.3), that equation (1.34) becomes
\[r_+ A_s u_s + (u(x) - u(0)) r_+ A^+ (\delta' - 2\delta) - u(x) r_+ A^+ (\chi_{\mathbb{R}_+}) = f. \quad (8.6)\]

### 8.3 Operator algebra - initial step

Our starting point for this section is equation (8.6). We remark that the given function \(f \in H^{s-2\alpha}_{\mathbb{R}_+} \). In a subsequent section, we shall apply the operator \(r_+(D - i)^{s-2\alpha} l_+\) to each side of equation (8.6), since our ultimate goal is a formulation in \(L^p(\mathbb{R}_+).\)

For the time being however, we recast equation (8.6) in the form
\[\tilde{a}_0(x) u(0) + \sum_{j=1}^N \tilde{a}_j(x) M_0(\tilde{b}_j) (r_+ C_j e_+) (r_+ u_s) + \tilde{K} u = f, \quad (8.7)\]
where \( \tilde{K} : H^s_{p,0}(\mathbb{R}_+) \to H^s_{p-2\alpha}(\mathbb{R}_+) \) is compact. (The definition of the space \( H^s_{p,0}(\mathbb{R}_+) \), for \( s > 1 + 1/p \), is given in (1.3).)

In equation (8.7), for \( k = 0, 1, \ldots, N \) the functions \( \tilde{a}_k(x) \) are known. Moreover, for \( j = 1, 2, \ldots, N \), \( \tilde{b}_j \) is the symbol of a Mellin convolution operator and \( \tilde{C}_j \) is a pseudodifferential operator. We shall denote the symbol of \( \tilde{C}_j(D) \) by \( \tilde{c}_j(\xi) \). We now examine the individual summands in the left-hand side of equation (8.6).

8.3.1 First term

Consider the term \( r_+ A_s u_s \). From equation (8.4), \( A_s(D) := A(D)(D - i)^{-2}(D + i)^{2-s} \) and hence, we can write

\[
 r_+ A_s u_s = (r_+ A(D)(D - i)^{-2}(D + i)^{2-s} e_+)(r_+ u_s),
\]

since, by Lemma 4.4, \( u_s \in L_p(\mathbb{R}) \) and \( \text{supp} \ u_s \subseteq \mathbb{R}_+ \).

Thus, in the notation of equation (8.7),

\[
 \begin{align*}
 \tilde{a}_1(x) &= 1; \\
 \tilde{b}_1(\xi) &= 1; \\
 \tilde{c}_1(\xi) &= (1 + \xi^2)^\alpha (\xi - i)^{-2}(\xi + i)^{2-s}.
\end{align*}
\]

8.3.2 Middle term

Now consider the middle term, \( (u(x) - u(0)) r_+ A^= (\delta' - 2\delta) \). It will be convenient to write

\[
 (u(x) - u(0)) r_+ A^= (\delta' - 2\delta) = (u(x) - u(0)) r_+ A^= (\delta' - \delta) - (u(x) - u(0)) r_+ A^= \delta.
\]

**Middle term - first part**

From Lemma 4.11,

\[
 (r_+ A^= (\delta' - \delta))(x) = -i C_\alpha e^{-\alpha} U(\alpha + 1, 2\alpha + 1, 2x),
\]

where the constant \( C_\alpha \) only depends on \( \alpha \), and is given by equation (4.8) in the statement of Lemma 4.9 as

\[
 C_\alpha = -i \frac{\alpha 2^{2\alpha}}{\Gamma(1 - \alpha)}.
\]
From Lemma 4.12,

\[(r_+ A^{\pm}(\delta' - \delta))(x) = -iC_\alpha \left( \phi(x) + \vartheta(x) \log x + x^{-2\alpha} \psi(\alpha + 1, 2\alpha + 1, x) \right),\]

where \( \phi, \vartheta \in C^\infty(\mathbb{R}) \) and, together with their derivatives, are bounded and \( O(e^{-x}) \) as \( x \to +\infty \). (We can set \( \vartheta \) to be identically zero unless \( \alpha = \frac{1}{2} \).)

Moreover, \( \psi \in C_0^\infty(\mathbb{R}) \) with \( \psi(\alpha + 1, 2\alpha + 1, x) = 0 \) for \( x > 2 \).

Hence, we can write

\[(u(x) - u(0))r_+ A^{\pm}(\delta' - \delta) = -iC_\alpha (T_{11} + T_{12} + T_{13})u\]

where

\[T_{11}u(x) := \phi(x)(u(x) - u(0))\]
\[T_{12}u(x) := \vartheta(x) \log x (u(x) - u(0))\]
\[T_{13}u(x) := x^{-2\alpha} \psi(\alpha + 1, 2\alpha + 1, x)(u(x) - u(0)).\]

Firstly, we will show that \( T_{11} : H_{p,0}^s(\mathbb{R}_+) \to H_{p}^{s-\varepsilon}(\mathbb{R}_+) \), is compact. Now

\[\phi(x)(u(x) - u(0)) = \phi(x)e^{x/2} \cdot e^{-x/2}(u(x) - u(0))\]

Since \( u'(0) = 0 \), by Remark 4.7, the map \( u \mapsto e^{-x/2}(u(x) - u(0)) \) defines a bounded operator from \( H_{p,0}^s(\mathbb{R}_+) \) to \( H_p^{s-\varepsilon}(\mathbb{R}_+) \). Moreover, \( \phi(x)e^{x/2} \in H_p^s(\mathbb{R}_+) \), since it and its derivatives are bounded, smooth and \( O(e^{-x/2}) \) as \( x \to +\infty \). The compactness of the operator \( T_{11} : H_{p,0}^s(\mathbb{R}_+) \to H_p^{s-\varepsilon}(\mathbb{R}_+) \) now follows immediately from Lemma 4.32.

Secondly, we will show that \( T_{12} : H_{p,0}^s(\mathbb{R}_+) \to H_p^{s-\varepsilon}(\mathbb{R}_+) \), is compact. Now

\[\vartheta(x) \log x (u(x) - u(0)) = \vartheta(x)e^{x/2} \cdot e^{-x/4} \log x \cdot e^{-x/4}(u(x) - u(0)).\]

Since \( u'(0) = 0 \), by Remark 4.7, the map \( u \mapsto e^{-x/4}(u(x) - u(0)) \) defines a bounded operator from \( H_{p,0}^s(\mathbb{R}_+) \) to \( H_p^{s-\varepsilon}(\mathbb{R}_+) \). Further, from Corollary 3.6, \( e^{-x/4} \log x I \) defines a bounded operator from \( \tilde{H}_p^s(\mathbb{R}_+) \) to \( \tilde{H}_p^{s-\frac{2}{2}}(\mathbb{R}_+) \). Finally, \( \vartheta(x)e^{x/2} \in H_p^{s-\frac{\varepsilon}{2}}(\mathbb{R}_+) \), since it and its derivatives are bounded, smooth and \( O(e^{-x/2}) \) as \( x \to +\infty \). The compactness of the operator \( T_{12} : H_{p,0}^s(\mathbb{R}_+) \to H_p^{s-\varepsilon}(\mathbb{R}_+) \) now follows immediately from Lemma 4.32.
It remains to consider \(-i C_\alpha T_{13} u(x) = -i C_\alpha x^{-2\alpha} \psi(\alpha + 1, 2\alpha + 1, x)(u(x) - u(0))\), and it is convenient to write
\[
-i C_\alpha x^{-2\alpha} \psi(\alpha + 1, 2\alpha + 1, x)(u(x) - u(0)) = -i C_\alpha \psi(\alpha + 1, 2\alpha + 1, x) \cdot \{x^{-2\alpha}(u(x) - u(0))\},
\]
noting that \(\psi \in C_0^\infty(\mathbb{R})\) with \(\psi(\alpha + 1, 2\alpha + 1, x) = 0\) for \(x > 2\).

On the other hand, from Lemmas 4.13 and 4.15,
\[
x^{-2\alpha}(u(x) - u(0)) = \int_0^\infty K_{2\alpha} \left( \frac{x}{y} \right) h(y) \frac{dy}{y} := M_{2\alpha} h,
\]
where \(h(x) = (C_{0\alpha}^a u)(x)\). Moreover, from Lemmas 4.25 and 4.26
\[
h = \begin{cases} \left( r_+ C(D)e_+ \right)(r_+ u_a), & \alpha = \frac{1}{2}; \\ \left( r_+ C(D)e_+ \right)(r_+ u_a) + \frac{u(0)}{\sqrt{2\pi}} r_+ F^{-1}(-i\xi)^{2\alpha - 1}(\xi - i)^{-2}, & \alpha \neq \frac{1}{2}, \end{cases}
\]
where \(C(D)\) has the symbol \(c(\xi) = (-i\xi)^{2\alpha}(\xi + i)^{2-s}(\xi - i)^{-2}\). From Lemma 4.17, \(M_{2\alpha}\) is a Mellin convolution operator with symbol \(b(\xi) = B(1/p' + i\xi, 2\alpha) / \Gamma(2\alpha)\).

Thus, in the notation of equation (8.7), we have
\[
\tilde{a}_2(x) = -i C_\alpha \psi(\alpha + 1, 2\alpha + 1, x);
\]
\[
\tilde{b}_2(\xi) = B(1/p' + i\xi, 2\alpha) / \Gamma(2\alpha);
\]
\[
\tilde{c}_2(\xi) = (-i\xi)^{2\alpha}(\xi + i)^{2-s}(\xi - i)^{-2}.
\]
and for \(\alpha \neq \frac{1}{2}\)
\[
\tilde{a}_0(x) = \tilde{a}_2(x) M^0 b_2 \frac{1}{\sqrt{2\pi}} r_+ F^{-1}(-i\xi)^{2\alpha - 1}(\xi - i)^{-2}.
\]

**Middle term - second part**

From Lemma 4.10
\[
(r_+ A^x \delta)(x) = \frac{1}{2} i C_\alpha e^{-x} U(\alpha + 1, 2\alpha, 2x),
\]
where the constant \(C_\alpha\) only depends on \(\alpha\), and is given by equation (4.8) in the statement of Lemma 4.9 as
\[
C_\alpha = -i \frac{\alpha 2^{2\alpha}}{\Gamma(1 - \alpha)}.
\]
From Lemma 4.12,

\[ (r_+ A^\alpha)(x) = \frac{1}{2} iC_\alpha \left( \phi(x) + \vartheta(x) \log x + x^{1-2\alpha} \psi(\alpha + 1, 2\alpha, x) \right), \]

where \( \phi, \vartheta \in C^\infty(\mathbb{R}) \) and, together with their derivatives, are bounded and \( O(e^{-x}) \) as \( x \to +\infty \). (We can set \( \vartheta \) to be identically zero unless \( \alpha = \frac{1}{2} \).) Moreover, \( \psi \in C^\infty_0(\mathbb{R}) \) with \( \psi(\alpha + 1, 2\alpha, x) = 0 \) for \( x > 2 \).

Hence, we can write

\[ -(u(x) - u(0)) r_+ A^\alpha \delta = -\frac{1}{2} iC_\alpha (T_{21} + T_{22}) u \]

where

\[ T_{21} u(x) := (\phi(x) + \vartheta(x) \log x)(u(x) - u(0)) \]
\[ T_{22} u(x) := x^{1-2\alpha} \psi(\alpha + 1, 2\alpha, x)(u(x) - u(0)). \]

From the earlier part of this section, \( T_{21} : H^s_{p,0}(\mathbb{R}_+) \to H^{s-\epsilon}_{p}(\mathbb{R}_+) \) is compact.

It remains to consider the operator \( T_{22} \).

Firstly, suppose that \( 0 < \alpha \leq \frac{1}{2} \). Then we can write

\[ x^{1-2\alpha} \psi(\alpha + 1, 2\alpha, x)(u(x) - u(0)) \]
\[ = \psi(\alpha + 1, 2\alpha, x) e^{x/2} \cdot x^{1-2\alpha} e^{-x/4} \cdot e^{x/4}(u(x) - u(0)). \]

Since \( u'(0) = 0 \), by Remark 4.7, the map \( u \mapsto e^{-x/4}(u(x) - u(0)) \) defines a bounded operator from \( H^s_{p,0}(\mathbb{R}_+) \) to \( \widetilde{H}^s_{p}(\mathbb{R}_+) \). Further, from Lemma 3.4, the operator \( x^{1-2\alpha} e^{-x/4}I \) is bounded on \( \widetilde{H}^s_{p}(\mathbb{R}_+) \). Now \( \psi(\alpha + 1, 2\alpha, x) e^{x/2} \) is bounded, smooth and has compact support. Finally, for \( 0 < \alpha \leq \frac{1}{2} \), the compactness of the operator \( T_{22} : H^s_{p,0}(\mathbb{R}_+) \to H^{s-\epsilon}_{p}(\mathbb{R}_+) \) now follows immediately from Lemma 4.31.

Secondly, suppose that \( \frac{1}{2} < \alpha < 1 \). Then we can write

\[ x^{1-2\alpha} \psi(\alpha + 1, 2\alpha, x)(u(x) - u(0)) \]
\[ = \psi(\alpha + 1, 2\alpha, x) e^{x/2} \cdot x^{1-2\alpha} e^{-x/2}(u(x) - u(0)). \]

Since \( u'(0) = 0 \), by Remark 4.7, the map \( u \mapsto e^{-x/2}(u(x) - u(0)) \) defines a bounded operator from \( H^s_{p,0}(\mathbb{R}_+) \) to \( \widetilde{H}^s_{p}(\mathbb{R}_+) \). Further, from Lemma 3.3, the operator \( x^{1-2\alpha}I \) is bounded from \( \widetilde{H}^s_{p}(\mathbb{R}_+) \) to \( \widetilde{H}^{s-2\alpha+1}_{p}(\mathbb{R}_+) \). Now
\[ \psi(\alpha + 1, 2\alpha, x)e^{x/2} \] is bounded, smooth and has compact support. Finally, for \( \frac{1}{2} < \alpha < 1 \), the compactness of the operator \( T_{22} : H^s_{p,0}(\mathbb{R}_+) \to H^{s-2\alpha + 1-\epsilon}_{p}(\mathbb{R}_+) \) now follows immediately from Lemma 4.31.

In other words, for \( 0 < \alpha < 1 \), the operator \( T_{22} : H^s_{p,0}(\mathbb{R}_+) \to H^{s-2\alpha}_{p}(\mathbb{R}_+) \) is also compact.

### 8.3.3 Final term

It remains to consider the last term, \(-u(x)r_+ A^-(\chi_{\mathbb{R}_-})\). From Lemma 4.28

\[
(r_+ A^-(\chi_{\mathbb{R}_-}))(x) = \frac{1}{2} iC_\alpha \int_x^\infty e^{-t}U(\alpha + 1, 2\alpha, 2t) \, dt,
\]

where the constant \( C_\alpha \) only depends on \( \alpha \).

Suppose \( 0 < \alpha < 1 \) and \( \alpha \neq \frac{1}{2} \). Then, from Remark 4.30,

\[
(r_+ A^- (\chi_{\mathbb{R}_-}))(x) = \frac{1}{2} iC_\alpha (\phi_1(x) + x^{2-2\alpha} \phi_2(x)),
\]

where \( \phi_1, \phi_2 \in C^\infty(\mathbb{R}) \) and, together with their derivatives, are bounded and \( O(e^{-x}) \) as \( x \to +\infty \).

Hence, we can write

\[
-wr_+ A^-(\chi_{\mathbb{R}_-}) = -\frac{1}{2} iC_\alpha (T_{31} + T_{32} + T_{33})u
\]

where

\[
T_{31}u(x) := \phi_1(x)u(x)
\]

\[
T_{32}u(x) := x^{2-2\alpha} \phi_2(x)(u(x) - u(0))
\]

\[
T_{33}u(x) := x^{2-2\alpha} \phi_2(x)u(0).
\]

Since, by assumption, \( u'(0) = 0 \), the compactness of the operators \( T_{31}, T_{32} \) and \( T_{33} \) now follows in the same manner as Section 4.4.3.

It remains to consider the case \( \alpha = \frac{1}{2} \).

The analysis proceeds as for the case \( \alpha \neq \frac{1}{2} \), but, see Remark 4.30, we need also to consider a term of the form \( \vartheta_1(x) x \log x \), where \( \vartheta_1 \in C^\infty(\mathbb{R}) \) and, together with its derivatives, is bounded and \( O(e^{-x}) \) as \( x \to +\infty \).
For $u \in H^s_{p,0}(\mathbb{R}^+)$, let us consider the map

$$u(x) \mapsto \vartheta_1(x) x \log x \cdot u(x).$$

We write

$$\vartheta_1(x) x \log x \cdot u(x) = T_{34}u(x) + T_{35}u(x),$$

where

$$T_{34}u(x) = \vartheta_1(x)e^{x/2} \cdot e^{-x/4}x \log x \cdot e^{-x/4}(u(x) - u(0))$$

and

$$T_{35}u(x) = \vartheta_1(x)e^{x/2} \cdot e^{-x/4} \log x \cdot x^{-1} \cdot x^2 e^{-x/4}u(0).$$

We will show that $T_{34}$ and $T_{35}$ represent compact operators.

Firstly, consider $T_{34}$. From Remark 4.7, $u \mapsto e^{-x/4}(u(x) - u(0))$ defines a bounded operator from $H^s_{p,0}(\mathbb{R}^+)$ to $H^s_{p}(\mathbb{R}^+)$. By Lemma 3.5, the operator $e^{-x/4}x \log x I$ is bounded on $H^s_{p}(\mathbb{R}^+)$. Moreover, $\vartheta_1(x)e^{x/2} \in H^s_{p}(\mathbb{R}^+)$, since it and its derivatives are bounded, smooth and $O(e^{-x/2})$ as $x \to \infty$. Finally, the compactness of the operator $T_{34} : H^s_{p,0}(\mathbb{R}^+) \to H^s_{p}(\mathbb{R}^+)$, now follows directly from Lemma 4.32.

Secondly, consider $T_{35}$. Since $1 + 1/p < s < 2 + 1/p$, we can choose $\epsilon > 0$ such that $s + \epsilon < 2 + 1/p$. We note that $x^2e^{-x/4} \in H^{s+\epsilon}_{p}(\mathbb{R}^+)$ because it, and its first derivative, take the value 0 at $x = 0$, and it is smooth with exponential decay. By Lemma 3.3, the operator $x^{-1}I$ is bounded from $H^{s+\epsilon}_{p}(\mathbb{R}^+)$ to $H^{s+\epsilon}_{p}(\mathbb{R}^+)$. Further, by Corollary 3.6, the operator $e^{-x/4} \log x I$ is bounded from $H^{s+\epsilon}_{p}(\mathbb{R}^+) \to H^{s+\epsilon}_{p}(\mathbb{R}^+)$ and $\vartheta_1(x)e^{x/2}$ and its derivatives are bounded, smooth and $O(e^{-x/2})$ as $x \to \infty$, and thus the operator $\vartheta_1(x)e^{x/2}I$ is bounded on $H^{s+\epsilon}_{p}(\mathbb{R}^+)$ by Lemma 3.4. Finally, $T_{35} : H^s_{p,0}(\mathbb{R}^+) \to H^s_{p}(\mathbb{R}^+)$ is bounded and rank one, and is therefore compact.

### 8.3.4 Summary

So, in summary, taking $N = 2$, we have the required representation

$$\tilde{a}_0(x)u(0) + \sum_{j=1}^{2} \tilde{a}_j(x) M^0(\tilde{b}_j)(r_+\tilde{C}_j e_+)(r_+u_a) + \tilde{K}u = f,$$

where the symbols $\tilde{a}_0$ and $(\tilde{a}_j, \tilde{b}_j, \tilde{c}_j)$ for $j = 1$ and $j = 2$ are given by equations (8.10) and (8.8), (8.9) respectively, and the operator $\tilde{K} : H^s_{p,0}(\mathbb{R}^+) \to H^s_{p,2\alpha}(\mathbb{R}^+)$ is compact.
8.4 Operator algebra - final step

We have seen in Section 8.3 that equation (1.34) can be written as (see equation (8.7) with \( N = 2 \))

\[
\tilde{a}_0(x)u(0) + \sum_{j=1}^{2} \tilde{a}_j(x) M^0(\tilde{b}_j)(r+c_j e_+)(r+u_s) + \tilde{K}u = f, \tag{8.11}
\]

where the operator \( \tilde{K} : H^s_{p,0}(\mathbb{R}_+) \to H^{s-2\alpha}_p(\mathbb{R}_+) \) is compact and \( f \in H^{s-2\alpha}_p(\mathbb{R}_+) \) is a given function.

In this section, we present a formulation in \( L^p(\mathbb{R}_+) \) of the form

\[
(a_1(x) M^0(b_1) W(c_1) + a_2(x) M^0(b_2) W(c_2) + T)(r+u_s) = g, \tag{8.12}
\]

where the operator \( T \), acting on \( L^p(\mathbb{R}_+) \), is compact. The function \( g \in L^p(\mathbb{R}_+) \) is defined by

\[
g := r_+(D - i)^{s-2\alpha} l_+ f, \tag{8.13}
\]

where by Lemma 4.5, \( g \) does not depend on the choice of the extension \( l_+ \).

The subsequent analysis will show that, after the application of the operator \( r_+(D - i)^{s-2\alpha} l_+ \), some of the terms in equation (8.11) represent compact operators on \( L^p(\mathbb{R}_+) \).

We now consider the action of the operator \( r_+(D - i)^{s-2\alpha} l_+ \) on the individual summands on the left-hand side of equation (8.11) in turn.

8.4.1 First term

We assume \( 0 < \alpha < 1 \) and note, from equation (8.10), that the first term is only present if \( \alpha \neq \frac{1}{2} \). Indeed, in this case, we have

\[
\tilde{a}_0(x) = a_2(x) M^0(\tilde{b}_2) \frac{1}{\sqrt{2\pi} r_+ F^{-1}(-i\xi) 2^{2\alpha-1}(\xi - i)^{-2}}
\]

\[
= \tilde{a}_2(x) M^0(\tilde{b}_2) r_+ h_1(x),
\]

where

\[
h_1(x) := \frac{1}{\sqrt{2\pi} r_+ F^{-1}(-i\xi) 2^{2\alpha-1}(\xi - i)^{-2}}.
\]

Note that, from equation (8.9),

\[
\tilde{a}_2(x) = -i C_\alpha \psi(\alpha + 1, 2\alpha + 1, x);
\]

\[
\tilde{b}_2(\xi) = B(1/p' + i\xi, 2\alpha)/\Gamma(2\alpha).
\]
Our goal is to show that
\[ \Lambda_{-}^{s-2\alpha} \tilde{a}_0(x) = \Lambda_{-}^{s-2\alpha} \tilde{a}_2(x) M^0(\tilde{b}_2) r_+ h_1(x) \in L_p(\mathbb{R}_+), \]
because then the operator
\[ L_p(\mathbb{R}_+) \ni r_+ u_s \mapsto u(0) \Lambda_{-}^{s-2\alpha} \tilde{a}_0 \in L_p(\mathbb{R}_+) \]
is bounded on \( L_p(\mathbb{R}_+) \). Moreover, it has rank one and is therefore compact.

We note that \( \tilde{a}_2 \in r_+ C_0^\infty(\mathbb{R}) \) and \( \tilde{a}_2(x) = 0 \) for \( x \geq 2 \). Let \( \chi \in C_0^\infty(\mathbb{R}) \) be such that
\[ \chi(t) := \begin{cases} 1 & \text{if } |t| \leq 2 \\ 0 & \text{if } |t| > 3. \end{cases} \]
Then, see Lemmas 4.17 and 5.3,
\[ \Lambda_{-}^{s-2\alpha} \tilde{a}_2 M^0(\tilde{b}_2) r_+ h_1 = \Lambda_{-}^{s-2\alpha} \tilde{a}_2 M^0(\tilde{b}_2) (r_+ \chi h_1). \]
Since \( h_1 \) is the inverse Fourier transform of an integrable function it is continuous and vanishes at infinity. Hence \( r_+ \chi h_1 \in L_p(\mathbb{R}_+) \). Thus, if \( s - 2\alpha < 0 \), using Lemma 5.19, the required result follows immediately.

It remains to consider the case \( s - 2\alpha \geq 0 \). Set \( \mu = 2\alpha - 1 \), \( r = s - 2\alpha \), so that \(-1 < \mu < 1\), \( r < 3 - 2\alpha = 2 - (2\alpha - 1) = 2 - \mu \). Then, from Lemma 5.5, for any \( \chi_1 \in C_0^\infty(\mathbb{R}) \),
\[ \chi_1(D - i)^{s-2\alpha} h_1 \in L_p(\mathbb{R}), \]
subject only to the condition
\[ s < 2 + \frac{1}{p}. \tag{8.14} \]
Hence, using the method of proof in Lemma F.1, \( (D - i)^{s-2\alpha} \chi h_1 \in L_p(\mathbb{R}) \), and so, after applying the operator \( r_+ (D - i)^{2\alpha-s} \), we have
\[ r_+ \chi h_1 \in H_p^{s-2\alpha}(\mathbb{R}_+). \]
Therefore, as \( s - 2\alpha \geq 0 \), again from Lemma 5.19,
\[ M^0(\tilde{b}_2) r_+ \chi h_1 \in H_p^{s-2\alpha}(\mathbb{R}_+), \]
and hence,
\[ \Lambda_{-}^{s-2\alpha} \tilde{a}_2(x) M^0(\tilde{b}_2) r_+ h_1(x) \in L_p(\mathbb{R}_+), \]
as required, since \( \tilde{a}_2 \in r_+ C_0^\infty(\mathbb{R}). \)
8.4.2 Second term

We assume $0 < \alpha < 1$. Using (8.8), we have

$$\tilde{a}_1(x) M^0(\tilde{b}_1)(r_+ \tilde{C}_1 e_+) = r_+ \tilde{C}_1 e_+$$

where the pseudodifferential operator $\tilde{C}_1$ has symbol $(1+\xi^2)^\alpha (\xi - i)^{-2} (\xi + i)^{2-s}$.

Hence, by Lemma 4.5

$$r_+(D - i)^{s-2\alpha} l_+ r_+ \tilde{C}_1 e_+ = r_+(D - i)^{s-2\alpha} \tilde{C}_1 e_+.$$ 

Now $(D - i)^{s-2\alpha} \tilde{C}_1$ has symbol $(1 + \xi^2)^\alpha (\xi - i)^{s-2\alpha-2} (\xi + i)^{2-s}$, which is clearly a Fourier $L_p$ multiplier. (See Lemma 5.8.) Therefore, in the notation of equation (8.12),

$$a_1(x) = 1;$$

$$b_1(\xi) = 1;$$

$$c_1(\xi) = (1 + \xi^2)^\alpha (\xi - i)^{s-2\alpha-2} (\xi + i)^{2-s}. \tag{8.15}$$

8.4.3 Third term

Now from (8.9) we have

$$\tilde{a}_2(x) = -iC_\alpha \psi(\alpha + 1, 2\alpha + 1, x);$$

$$\tilde{b}_2(\xi) = B(1/p' + i\xi, 2\alpha)/\Gamma(2\alpha);$$

$$\tilde{c}_2(\xi) = (-i\xi)^{2\alpha} (\xi + i)^{2-s} (\xi - i)^{-2}.$$ 

Let us define

$$r := s - 2\alpha,$$

so that we need to consider

$$-1 + 1/p < r < 2 + 1/p,$$

since $0 < \alpha < 1$ and $1 + 1/p < s < 2 + 1/p$. We note that the pseudodifferential operator $\tilde{C}_2$ has order $-r$.

If $r \geq 0$, then from Lemma 5.19, the operator $M^0(\tilde{b}_2)(r_+ \tilde{C}_2 e_+) : L_p(\mathbb{R}_+) \rightarrow H^r_p(\mathbb{R}_+)$ is bounded.

On the other hand, if $-1 + 1/p < r < 0$, we can write

$$\tilde{c}_2(\xi) = (i\xi)^{-r} \cdot \tilde{c}_0(\xi)$$
where
\[ \tilde{c}_0(\xi) := (i\xi)^r (-i\xi)^{-2} (\xi + i)^{2-s} (\xi - i)^{-2}. \]

Since \( r + 2\alpha = s > 0 \), \( \tilde{c}_0(0) = 0 \). Moreover, as \( r = s - 2\alpha \), the operator \( \tilde{C}_0 \), with symbol \( \tilde{c}_0 \), has order 0.

From Lemma 5.18, \( M_{2\alpha,0} r_+(iD)^{-r} l_+ = r_+(iD)^{-r} l_+ M_{2\alpha,r} \), and from Lemma 5.12, \( r_+(iD)^{-r} l_+ : L_p(\mathbb{R}_+) \to H^s_p(\mathbb{R}_+) \) is bounded. Therefore, the operator
\[ M^0(\tilde{b}_2) (r_+ \tilde{C}_2 e_+) = M_{2\alpha,0} (r_+ \tilde{C}_2 e_+) \] (see Lemma 4.17)
\[ = M_{2\alpha,0} (r_+(iD)^{-r} l_+ r_+ \tilde{C}_0 e_+) \] (see Lemma 5.12)
\[ = r_+(iD)^{-r} l_+ M_{2\alpha,r} (r_+ \tilde{C}_0 e_+) \]
is bounded from \( L_p(\mathbb{R}_+) \to H^s_p(\mathbb{R}_+) \).

So now, using Lemma 3.4, each of the three operators in the identity
\[ \Lambda_-^* \tilde{a}_2(x) M^0(\tilde{b}_2) (r_+ \tilde{C}_2 e_+) \]
\[ = [\Lambda_-^*, \tilde{a}_2(x)] M^0(\tilde{b}_2) (r_+ \tilde{C}_2 e_+) + \tilde{a}_2(x) \Lambda_-^* M^0(\tilde{b}_2) (r_+ \tilde{C}_2 e_+) \]
is bounded on \( L_p(\mathbb{R}_+) \).

Moreover, the compactness of the operator involving the commutator term follows directly from Lemma 5.20. Thus, it remains to consider \( \tilde{a}_2(x) \Lambda_-^* M^0(\tilde{b}_2) (r_+ \tilde{C}_2 e_+) \).

Firstly, suppose that \( 0 < r < 1 \). Then, using Lemma 5.22,
\[ \Lambda_-^* M^0(\tilde{b}_2) (r_+ \tilde{C}_2 e_+) \]
\[ = \Lambda_-^* M_{2\alpha,0} (r_+ \tilde{C}_2 e_+) \]
\[ = (M_{2\alpha,r} \Lambda_-^* + (-i)^r (M_{2\alpha,0} - M_{2\alpha,r}) + T) (r_+ \tilde{C}_2 e_+) \]
\[ = M_{2\alpha,r} (r_+ C_2 e_+) + (-i)^r (M_{2\alpha,0} - M_{2\alpha,r}) (r_+ \tilde{C}_2 e_+) + T (r_+ \tilde{C}_2 e_+). \]

From Lemma 5.22, \( T : H^s_p(\mathbb{R}_+) \to L_p(\mathbb{R}_+) \) is compact. Moreover, the pseudodifferential operator \( \tilde{C}_2 \) has order \(-r\), and hence \( T (r_+ \tilde{C}_2 e_+) \) is compact on \( L_p(\mathbb{R}_+) \).

By Remark 4.20, the symbols of both \( M_{2\alpha,0} \) and \( M_{2\alpha,r} \) take the value zero at \( \pm\infty \). Hence, \( (M_{2\alpha,0} - M_{2\alpha,r}) (r_+ \tilde{C}_2 e_+) \) is compact on \( L_p(\mathbb{R}_+) \), from Proposition 5.3.4 (i), p. 267, [37].

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So, in summary, if $0 < r < 1$ then
\[
\Lambda_r M^0 \tilde{b}_2 (r_+ \tilde{C}_2 e_+) = M_{2\alpha,r} (r_+ C_2 e_+) + K_1,
\]
where $C_2$ has symbol
\[
c_2(\xi) = (-i\xi)^{2\alpha} (\xi - i)^{s-2\alpha-2} (\xi + i)^{2-s},
\]
and the operator $K_1$, acting on $L_p(\mathbb{R}^+)$, is compact.

Similarly, in the case that $-1 + 1/p < r < 0$, then we can again apply Lemma 5.22, noting that the operator $\Lambda_r^2 r_+ \tilde{C}_2 e_+$ has order $r < 0$.

\[
\Lambda_r^2 M^0 \tilde{b}_2 (r_+ \tilde{C}_2 e_+) = \Lambda_r^2 M_{2\alpha,0} (r_+ \tilde{C}_2 e_+)
\]
\[
= \Lambda_r^2 M_{2\alpha,0} (r_+ \tilde{C}_2 e_+) - (-i)^r (M_{2\alpha,0} - M_{2\alpha,r}) \Lambda_r^2 + T (r_+ \tilde{C}_2 e_+)
\]
\[
= M_{2\alpha,r} (r_+ C_2 e_+) - (-i)^r (M_{2\alpha,0} - M_{2\alpha,r}) \Lambda_r^2 (r_+ \tilde{C}_2 e_+) + T (r_+ \tilde{C}_2 e_+).
\]
The compactness of $T (r_+ \tilde{C}_2 e_+)$ on $L_p(\mathbb{R}^+)$ follows exactly as in the case $0 < r < 1$. Moreover,
\[
(M_{2\alpha,0} - M_{2\alpha,r}) \Lambda_r^2 (r_+ \tilde{C}_2 e_+)
\]
is compact on $L_p(\mathbb{R}^+)$, from Proposition 5.3.4 (i), p. 267, [37].

So, in summary, if $-1 + 1/p < r < 0$ then
\[
\Lambda_r^2 M^0 \tilde{b}_2 (r_+ \tilde{C}_2 e_+) = M_{2\alpha,r} (r_+ C_2 e_+) + K_2
\]
where the operator $K_2$, acting on $L_p(\mathbb{R}^+)$, is compact.

The case $1 < r < 2$ follows similarly, except that we now apply Lemma 5.24. In particular, we note that the operator $S_1 r_+ \tilde{C}_2 e_+$ has order $1-r < 0$. Hence, as in the case $0 < r < 1$ discussed above,
\[
S_1 r_+ \tilde{C}_2 e_+
\]
is compact on $L_p(\mathbb{R}^+)$, from Proposition 5.3.4 (i), p. 267, [37].

For the case $2 < r < 2 + 1/p$ we observe that $S_2 r_+ \tilde{C}_2 e_+$ has order $2-r < 0$, and the analysis proceeds as for $1 < r < 2$. 

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Finally, the cases \( r = 1 \) and \( r = 2 \) follow in the same way, and for the case \( r = 0 \), there is nothing to prove.

Hence, using Lemma 4.17, in the notation of equation (5.2) we have,

\[
\begin{align*}
a_2(x) &= -iC_\alpha \psi(\alpha + 1, 2\alpha + 1, x); \\
b_2(\xi) &= B(s - 2\alpha + 1/p' + i\xi, 2\alpha)/\Gamma(2\alpha); \\
c_2(\xi) &= (-i\xi)^{2\alpha}(\xi - i)^{s-2\alpha-2}(\xi + i)^{2-s}.
\end{align*}
\]

Note that a routine application of Lemma 5.8 confirms that \( c_2 \) is a Fourier \( L_p \) multiplier.

### 8.4.4 Summary

Our base assumptions are that

\[
1 < p < \infty, \quad 1 + 1/p < s < 2 + 1/p \text{ and } 0 < \alpha < 1. \tag{8.17}
\]

So, finally, subject to condition (8.17), the formulation given by equation (8.12) becomes

\[
(W(c_1) + a_2 M^0(b_2) W(c_2) + T)(r_+ u_s) = g, \tag{8.18}
\]

where the operator \( T \), acting on \( L_p(\mathbb{R}_+) \), is compact and

\[
\begin{align*}
g : &= r_+(D - i)^{s-2\alpha} t_+ f; \\
c_1(\xi) &= (1 + \xi^2)^\alpha(\xi - i)^{s-2\alpha-2}(\xi + i)^{2-s}. \\
a_2(x) &= -iC_\alpha \psi(\alpha + 1, 2\alpha + 1, x) \quad \text{(see Lemma 4.12)}; \\
b_2(\xi) &= B(s - 2\alpha + 1/p' + i\xi, 2\alpha)/\Gamma(2\alpha); \\
c_2(\xi) &= (-i\xi)^{2\alpha}(\xi - i)^{s-2\alpha-2}(\xi + i)^{2-s},
\end{align*}
\]

and the constant \( C_\alpha \) is given by

\[
C_\alpha = -i \left( \frac{\alpha 2^{2\alpha}}{(1 - \alpha)} \right).
\]

### 8.5 Generalised symbol

We now follow the approach taken in Chapter 6 and examine the generalised symbol \( A_{\alpha,p,s}(\omega) \) defined on the contour \( \Gamma_M \).
8.5.1 Segment $\Gamma_1$

Firstly, we note that
\[
a_2(0) = -iC_\alpha \psi(\alpha + 1, 2\alpha + 1, 0) = -iC_\alpha 2^{-2\alpha} \frac{\Gamma(2\alpha)}{\Gamma(\alpha + 1)} \quad (\text{see Lemma 4.12}).
\]
Hence,
\[
a_2(0) b_2(\xi) = -iC_\alpha 2^{-2\alpha} \frac{\Gamma(2\alpha)}{\Gamma(\alpha + 1)} \cdot \frac{B(s - 2\alpha + 1/p' + i\xi, 2\alpha)}{\Gamma(2\alpha)}
= -\frac{\alpha 2^{2\alpha}}{\Gamma(1 - \alpha) \Gamma(\alpha + 1)} \cdot 2^{-2\alpha} \cdot B(s - 2\alpha + 1/p' + i\xi, 2\alpha)
= -\frac{1}{\Gamma(1 - \alpha) \Gamma(\alpha)} \cdot B(s - 2\alpha + 1/p' + i\xi, 2\alpha) \quad (\Gamma(\alpha + 1) = \alpha \Gamma(\alpha))
= -\frac{\sin \pi \alpha}{\pi} \cdot B(s - 2\alpha + 1/p' + i\xi, 2\alpha) \quad (5.5.3, [34]).
\]
Therefore, on the segment $\Gamma_1$, for $-\infty \leq \xi \leq \infty$, we have
\[
A_{\alpha,p,s}(\omega) := a_1(0) b_1(\xi) c_{1p}(\infty, \xi) + a_2(0) b_2(\xi) c_{2p}(\infty, \xi)
= c_{1p}(\infty, \xi) - \frac{\sin \pi \alpha}{\pi} \cdot B(s - 2\alpha + 1/p' + i\xi, 2\alpha) c_{2p}(\infty, \xi).
\]

From Lemmas 6.5 and 6.7, we have
\[
c_{1p}(\infty, \xi) = e^{i\pi \nu} \frac{\sin[\pi(1/p + \nu - i\xi)]}{\sin \pi(1/p - i\xi)}, \quad \nu = 2 - s + \alpha.
\]
Similarly, from Lemmas 6.6 and 6.7, we have
\[
c_{2p}(\infty, \xi) = e^{-i\pi \alpha} e^{i\pi \nu'} \frac{\sin[\pi(1/p + \nu' - i\xi)]}{\sin \pi(1/p - i\xi)}, \quad \nu' = 2 - s + 2\alpha.
\]
But $e^{-i\pi \alpha} e^{i\pi \nu'} = e^{i\pi(2-s+\alpha)} = e^{i\pi \nu}$, and thus $c_{1p}(\infty, \xi)$ and $c_{2p}(\infty, \xi)$ have a common factor
\[
e^{i\pi \nu} \frac{\sin \pi(1/p - i\xi)}{\sin \pi(1/p - i\xi)}.
\]
So, we are interested in establishing the precise conditions under which the quadruple \((\alpha, p, s, \xi)\) is **not** a solution of the following transcendental equation

\[
\frac{\sin(\pi(1/p + \nu - i\xi))}{\sin(\pi(1/p + \nu' - i\xi))} - \frac{\sin \pi \alpha}{\pi} B(s - 2\alpha + 1/p' + i\xi, 2\alpha) = 0. \tag{8.20}
\]

Let us now define

\[
T_B := \frac{\sin \pi \alpha}{\pi} B(s - 2\alpha + 1 - 1/p + i\xi, 2\alpha), \tag{8.21}
\]

and

\[
T_s := \frac{\sin \pi (1/p + m - s + \alpha - i\xi)}{\sin \pi (1/p + m - s + 2\alpha - i\xi)}, \tag{8.22}
\]

where \(m = 2\) and \(\xi \in \mathbb{R}\).

Then, the transcendental equation (8.20) simply becomes

\[T_s = T_B.\]

### 8.5.2 Segment \(\Gamma_2^\pm\)

Similarly, on \(\Gamma_2^+\), for \(0 \leq x \leq \infty\), we have

\[
A_{\alpha,p,s}(\omega) := a_1(x) b_1(\infty) c_1(-\infty) + a_2(x) b_2(\infty) c_2(-\infty)
\]

\[= c_1(-\infty) + 0
\]

\[= e^{2\pi i \nu i},
\]

and on \(\Gamma_2^-\), for \(\infty \geq x \geq 0\),

\[
A_{\alpha,p,s}(\omega) := a_1(x) b_1(-\infty) c_1(+\infty) + a_2(x) b_2(-\infty) c_2(+\infty)
\]

\[= c_1(+\infty) + 0
\]

\[= 1.
\]

Hence,

\[
\inf_{\omega \in \Gamma_2^+ \cup \Gamma_2^-} |A_{\alpha,p,s}(\omega)| = 1.
\]

### 8.5.3 Segment \(\Gamma_3^\pm\)

On \(\Gamma_3^+\) for \(\infty > \lambda \geq 0\),

\[
A_{\alpha,p,s}(\omega) := a_1(\infty) b_1(\infty) c_1(-\lambda) + a_2(\infty) b_2(\infty) c_2(-\lambda)
\]

\[= c_1(-\lambda) + 0
\]

\[= c_1(-\lambda),
\]

\[\text{140}\]
and on $\Gamma^+_3$, for $0 \leq \lambda < \infty$,

$$A_{\alpha,p,s}(\omega) := a_1(\infty) b_1(\lambda) c_1(\lambda) + a_2(\infty) b_2(\lambda) c_2(\lambda) = c_1(\lambda).$$

Note that on $\Gamma^+_3$ and $\Gamma^-_3$ the parameter $\lambda$ varies between 0 and $\infty$ but, of course, in an opposite sense. So, in summary,

$$\inf_{\omega \in \Gamma^+_3} |A_{\alpha,p,s}(\omega)| = 1.$$

### 8.5.4 Segment $\Gamma_4$

Finally, on $\Gamma_4$, for $-\infty \leq \xi \leq \infty$,

$$A_{\alpha,p,s}(\omega) := a_1(\infty) b_1(0) c_1(0) + a_2(\infty) b_2(0) c_2(0) = c_1(0).$$

Hence,

$$\inf_{\omega \in \Gamma_4} |A_{\alpha,p,s}(\omega)| = 1,$$

and this completes the review of the contour $\Gamma_M$.

### 8.5.5 Summary

Note that the preceding analysis of the segments of the contour has shown that $A_{\alpha,p,s}(\omega)$ is constant on the segments $\Gamma^+_3$ and $\Gamma_4$. Therefore, it remains to consider $A_{\alpha,p,s}(\omega)$ on $\Gamma_1 \cup \Gamma^+_3 \cup \Gamma^-_3$. But from subsection 8.5.3, we can combine $\Gamma^+_3$ to give a new segment $\Gamma_3$ (say), where now the parameter $\lambda$ varies from $-\infty$ to $\infty$. (Note that, as expected, the symbol $A_{\alpha,p,s}(\omega)$ is continuous at $\lambda = 0$ on the new segment $\Gamma_3$.)

By construction, we observe that $A_{\alpha,p,s}(\omega)$ is continuous on $\Gamma_1 \cup \Gamma_3$. Indeed, from subsection 8.5.1, on the segment $\Gamma_1$

$$A_{\alpha,p,s}(\omega) = c_{1p}(\infty, \xi) - \frac{\sin \pi \alpha}{\pi} B(s - 2\alpha + 1/p' + i\xi, 2\alpha) c_{2p}(\infty, \xi) \quad (8.23)$$

with limits $c_{1p}(\infty, \pm \infty) = c_1(\mp \infty)$ at $\xi = \pm \infty$ respectively.
The condition that $A_{\alpha,p,s}(\omega) = 0$ on $\Gamma_1$ gives rise to the transcendental equation
\[
\frac{\sin(\pi(1/p + \nu - i\xi))}{\sin(\pi(1/p + \nu' - i\xi))} = \frac{\sin \pi \alpha}{\pi} B(s - 2\alpha + 1/p' + i\xi, 2\alpha),
\]
where $\nu = 2 - s + \alpha$ and $\nu' = 2 - s + 2\alpha$.

Finally, on $\Gamma_3$ we have
\[
A_{\alpha,p,s}(\omega) = c_1(\lambda)
\] (8.24)
with limits $c_1(\pm\infty)$ for $\lambda = \pm\infty$.

8.6 Index and invertibility

As previously, we set
\[
\tau := s - 1/p.
\]
Then, see Chapter 9, we note the critical importance of the following transcendental equation, see (9.7), which, for convenience, we repeat here:
\[
\Gamma(2\alpha - \tau)\Gamma(\tau + 1) \sin \pi (\alpha - \tau) = \Gamma(2\alpha) \sin \pi \alpha.
\]
From Lemma 9.29, if $0 < \alpha < 1$ is fixed, and $1 < \tau < 2$ is considered to vary, then the above equation has a unique solution for $\tau (= s - 1/p)$ of the form $1 + \alpha_c$, where $\alpha_c$ only depends on $\alpha$ and satisfies $0 < \alpha_c < \alpha$.

For example, if $\alpha = 0.75$, $p = 2$, Figure 8.1 shows that when $s \approx 2.226$, or equivalently $\alpha_c \approx 0.726$, our operator is not Fredholm.

![Figure 8.1: Symbol plot for $\alpha = 0.7500$, $p = 2$ and $s = 2.2260$.](image)
Given $\alpha$, we can readily determine a good estimate for $\alpha_c$ using Mathematica®. Indeed, Figure 8.2 shows the graph of this estimate for $\alpha_c$, as $\alpha$ varies over the range $(0,1)$. The straight line shown on the plot is simply to highlight the fact that $0 < \alpha < \alpha_c$, and as $\alpha$ tends to 1, the difference $\alpha - \alpha_c$ tends to 0.

In the special case that $\alpha = \frac{1}{2}$, equation (9.7) reduces to

$$\tan(\pi \tau) = \pi \tau,$$

and, in this case, we obtain $\alpha_c \approx 0.4303$.

### 8.6.1 Main results

**Theorem 8.1.** For all $\alpha, p, s$ satisfying the conditions $0 < \alpha < 1$, $1 < p < \infty$ and $1 + 1/p < s < 1 + 1/p + \alpha_c$, the winding number of the generalised symbol $(A_{\alpha,p,s}, \Gamma_M)$ in the complex plane is $-1$. Hence, the operator $W(c_1) + a_2 M^0(b_2) W(c_2)$, defined on $L_p(\mathbb{R}^+_+)$, has Fredholm index equal to $1$. On the other hand, if $1 + 1/p + \alpha_c < s < 2 + 1/p$, the operator $W(c_1) + a_2 M^0(b_2) W(c_2)$ has Fredholm index equal to $0$.

**Theorem 8.2.** Suppose $0 < \alpha < 1$, $1 < p < \infty$ and $1 + 1/p < s < 1 + 1/p + \alpha_c$. Then the operator $\mathcal{A} : H^s_{p,0}(\mathbb{R}^+_+) \to H^{s-2\alpha}_{p}(\mathbb{R}^+_+)$ is invertible.
On the other hand, if $0 < \alpha < 1$, $1 < p < \infty$ and $1 + 1/p + \alpha_c < s < 2 + 1/p$, then $A$ has a trivial kernel and is Fredholm with index equal to $-1$.

### 8.7 Proof of Theorem 8.1

The constraints are

$$0 < \alpha < 1, \quad 1 < p < \infty \quad \text{and} \quad 1 + 1/p + \alpha_c < s < 2 + 1/p.$$  \hfill (8.25)

Let $\alpha, p, s$ fall within their admissible ranges and be fixed. From Section 8.5, it is easy to show that the generalised symbol $A_{\alpha,p,s}$ can be represented by a closed contour in the complex plane given by the union of the two curves, $S_1$ and $S_3$.

Indeed, from Section 8.5.3 we have,

$$S_3(\xi) := (1 + \xi^2)^\alpha (\xi - i)^{s-2\alpha-2}(\xi + i)^{2-s}, \quad -\infty \leq \xi \leq \infty.$$  

Now

$$S_3(\xi) = (\xi + i)^{\alpha}(\xi - i)^{\alpha}(\xi - i)^{s-2\alpha-2}(\xi + i)^{2-s} = (\xi + i)^{2-s+\alpha}(\xi - i)^{-(2-s+\alpha)}.$$  

From Section 8.5.1, for $-\infty \leq \xi \leq \infty$,

$$S_1(\xi) := S_{11}(\xi) - \frac{\sin \pi \nu}{\pi} B(s - 2\alpha + 1 - 1/p + i\xi, 2\alpha)S_{12}(\xi),$$

where

$$S_{11}(\xi) := e^{i\pi \nu} \frac{\sin[\pi(1/p + \nu - i\xi)]}{\sin[\pi(1/p - i\xi)]}, \quad S_{12}(\xi) := e^{i\pi \nu} \frac{\sin[\pi(1/p + \nu + \alpha - i\xi)]}{\sin[\pi(1/p - i\xi)]},$$

and $\nu = 2 - s + \alpha$.

#### 8.7.1 The case $s > 1 + 1/p + \alpha_c$

Firstly, we consider the case where $s > 1 + 1/p + \alpha_c$, and we will show that the winding number of the model contour is 0.
Assume $0 < \epsilon << 1$, and let us choose

$$\alpha = \frac{1}{2}, \quad p = 2 \quad \text{and} \quad s = \frac{5}{2} - \epsilon,$$

as the set of parameters used to define the first model contour. Note that these values lie within the set of admissible constraints given in condition (8.25).

Now $s - 2\alpha + 1 - 1/p = 2 - \epsilon$, and hence by Lemma 9.10, with $\sigma = 2 - \epsilon$ and $\alpha = \frac{1}{2}$,

$$\left| \frac{\sin \pi \alpha}{\pi} B(s - 2\alpha + 1 - 1/p + i\xi, 2\alpha) \right| \leq \frac{1}{\pi} \cdot B(2 - \epsilon, 1)$$

$$= \frac{1}{\pi} \cdot \frac{\Gamma(2 - \epsilon)\Gamma(1)}{\Gamma(3 - \epsilon)}$$

$$= \frac{1}{\pi} \cdot \frac{1}{2 - \epsilon}$$

$$< \frac{1}{\pi}, \quad \text{if} \quad 0 < \epsilon << 1.$$

Moreover, $\nu = 2 - s + \alpha = \epsilon$, and thus

$$S_{11}(\xi) = e^{i\pi \epsilon} \frac{\sin \pi \left(\frac{1}{2} + \epsilon - i\xi\right)}{\sin \pi \left(\frac{1}{2} - i\xi\right)}.$$

Since $\sin \pi \left(\frac{1}{2} + \epsilon - i\xi\right) = \sin \pi \epsilon \cos \pi \left(\frac{1}{2} - i\xi\right) + \cos \pi \epsilon \sin \pi \left(\frac{1}{2} - i\xi\right)$ and $\cot \pi \left(\frac{1}{2} - i\xi\right) = i \tanh \pi \xi$, we can write

$$S_{11}(\xi) = e^{i\pi \epsilon} \left( \cos \pi \epsilon + i \tanh \pi \xi \sin \pi \epsilon \right),$$

and similarly

$$S_{12}(\xi) = e^{i\pi \epsilon} \left( -\sin \pi \epsilon + i \tanh \pi \xi \cos \pi \epsilon \right).$$

Hence, we have the following elementary expansions for $0 < \epsilon << 1$

$$S_{11}(\xi) = 1 + \{i\pi (1 + \tanh \pi \xi)\} \epsilon + O(\epsilon^2),$$

and

$$S_{12}(\xi) = i \tanh(\pi \xi) - \{\pi (1 + \tanh(\pi \xi))\} \epsilon + O(\epsilon^2).$$
Combining these estimates

\[ |S_1(\xi) - 1| \leq \{\pi(1 + |\tanh(\pi\xi)|)\} \epsilon + |S_{12}|/\pi + O(\epsilon^2) \]
\[ \leq 1/\pi + 4\pi \epsilon + O(\epsilon^2) \]
\[ < \frac{2}{5} \quad \text{for sufficiently small } \epsilon > 0. \]

On the other hand, for any \( \xi \in \mathbb{R} \), we have

\[ \xi \pm i = \sqrt{1 + \xi^2} \exp(\pm i\theta), \quad \text{where} \quad 0 < \theta < \pi. \]

Hence, since \( 2 - s + \alpha = \epsilon \),

\[ |S_3(\xi) - 1| = |\exp(i2\epsilon\theta) - 1| \]
\[ \leq |\cos 2\epsilon\theta + i \sin 2\epsilon\theta - 1| \]
\[ \leq 2\pi \epsilon + O(\epsilon^2) \]
\[ < \frac{2}{5} \quad \text{for sufficiently small } \epsilon > 0. \]

So, for sufficiently small \( \epsilon \), the model contour, formed by the union of the curves \( S_1 \) and \( S_3 \), is wholly contained in the disc of radius \( \frac{2}{5} \) centred on the point 1 in the complex plane. Hence, for \( s > 1 + 1/p + \alpha_c \), the winding number of the model contour, given by \( \alpha = \frac{1}{2}, \ p = 2 \) and \( s = \frac{5}{2} - \epsilon \), must be 0.

We now give a plot of the model contour, see Figure 8.3, where we take \( \epsilon = \frac{1}{1000} \). As discussed, the contour is contained in a circle with centre 1 with radius \( \frac{2}{5} \).

![Figure 8.3: Symbol plot for \( \alpha = 0.5000, \ p = 2 \) and \( s = 2.5000 - \epsilon \).]
The following four plots, see Figures 8.4, 8.5, 8.6 and 8.7 show $\alpha$ increasing in steps of $\frac{1}{16}$ and, at the same time, $s$ decreasing by $\frac{1}{16}$. In each case, the plot confirms that the winding number of the contour is zero.

Figure 8.4: Symbol plot for $\alpha = 0.5625$, $p = 2$ and $s = 2.4375$.

Figure 8.5: Symbol plot for $\alpha = 0.6250$, $p = 2$ and $s = 2.3750$. 
Finally, we recall that, for $\alpha = 0.75$, we have $\alpha_c \approx 0.726$. Hence, if $p = 2$ and $s = 2.25$ we have $s > 1 + 1/p + \alpha_c \approx 2.226$. As expected, the winding number of the contour in Figure 8.7 is 0.

### 8.7.2 The case $s < 1 + 1/p + \alpha_c$

Let us first return to our example with $\alpha = 0.75$ and $p = 2$. We now take $s = 2.2$, so that $s < 1 + \frac{1}{2} + \alpha_c \approx 2.226$. 

Figure 8.6: Symbol plot for $\alpha = 0.6875$, $p = 2$ and $s = 2.3125$.

Figure 8.7: Symbol plot for $\alpha = 0.7500$, $p = 2$ and $s = 2.2500$. 

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Figure 8.8: Symbol plot for $\alpha = 0.7500$, $p = 2$ and $s = 2.2000$.

Clearly, the winding number of the contour in Figure 8.8 is not zero and, moreover, must take the value $\pm 1$.

We now consider the general case where $s < 1 + 1/p + \alpha_c$, and we will show that the winding number is, in fact, $-1$.

Again assume $0 < \epsilon << 1$, and let us now choose

$$\alpha = \frac{1}{2}, \quad p = 2 \quad \text{and} \quad s = \frac{3}{2} + \epsilon,$$

as the set of parameters used to define the second model contour. Note that these values lie within the set of admissible constraints given in condition (8.25).

Now $s - 2\alpha + 1 - 1/p = 1 + \epsilon$, and hence by Lemma 9.10, with $\sigma = 1 + \epsilon$ and $\alpha = \frac{1}{2}$,

$$\left| \frac{\sin \pi \alpha}{\pi} B(s - 2\alpha + 1 - 1/p + i\xi, 2\alpha) \right| \leq \frac{1}{\pi} \cdot B(1 + \epsilon, 1)$$

$$= \frac{1}{\pi} \cdot \frac{\Gamma(1 + \epsilon)\Gamma(1)}{\Gamma(2 + \epsilon)}$$

$$= \frac{1}{\pi} \cdot \frac{1}{1 + \epsilon}$$

$$< \frac{1}{\pi}, \quad \text{if} \quad \epsilon > 0.$$
Moreover, \( \nu = 2 - s + \alpha = 1 - \epsilon \), and we have the following elementary expansions for \( 0 < \epsilon << 1 \)

\[
S_{11}(\xi) = 1 - \{i \pi (1 + \tanh \pi \xi)\} \epsilon + O(\epsilon^2),
\]

and

\[
S_{12}(\xi) = i \tanh(\pi \xi) + \{\pi (1 + \tanh(\pi \xi))\} \epsilon + O(\epsilon^2).
\]

Combining these estimates

\[
|S_1(\xi) - 1| \leq \{\pi (1 + |\tanh(\pi \xi)|)\} \epsilon + |S_{12}|/\pi + O(\epsilon^2)
\]

\[
\leq 1/\pi + 4\pi \epsilon + O(\epsilon^2)
\]

\[
< \frac{2}{5} \quad \text{for sufficiently small } \epsilon > 0.
\]

On the other hand, since \( 2 - s + \alpha = 1 - \epsilon \), the curve \( S_3 \) traverses, in a clockwise direction, the complete unit circle apart from a small neighbourhood near the point 1 in the complex plane. By choosing \( \epsilon \) sufficiently small, we can ensure that the omitted portion lies wholly within the disk of radius \( \frac{2}{5} \) centred on 1.

Since the model contour, formed by the union of the curves \( S_1 \) and \( S_3 \), forms a closed loop, for sufficiently small \( \epsilon \), it encircles the origin once, in a clockwise direction, plus an additional component that is wholly contained in the disc of radius \( \frac{2}{5} \) centred on the point 1 in the complex plane. Hence, the winding number of the model contour, for \( s < 1 + 1/p + \alpha_c \), is equal to \(-1\).

### 8.7.3 Conclusion

Given any set of parameters \( \alpha, p, s \) satisfying the constraints \( 0 < \alpha < 1 \), \( 1 < p < \infty \) and \( 1 + 1/p < s < 1 + 1/p + \alpha_c \), the associated contour can be continuously deformed into the model contour, and from Theorem 9.2, does this without ever crossing the origin. Hence, the two contours must have the same winding number, namely, \(-1\).

By a similar argument, the winding number is constant, and equal to 0 in the case that \( 1 + 1/p + \alpha_c < s < 2 + 1/p \).

Therefore, see Remark 6.4, the operator \( W(c_1) + a_2 M^0(b_2) W(c_2) \), defined on \( L_p(\mathbb{R}_+) \), has Fredholm index equal to 1 if \( s < 1 + 1/p + \alpha_c \), and index 0 if \( s > 1 + 1/p + \alpha_c \). This completes the proof of the first theorem.
8.8 Proof of Theorem 8.2

Suppose $0 < \alpha < 1$, $1 < p < \infty$ and $1 + 1/p < s < 2 + 1/p$. Then, from Theorem 3.1, the operator $A : H^s_{p,0}(\mathbb{R}+) \to H^{s-2\alpha}_{p}(\mathbb{R}+)$ is bounded, where

$$H^s_{p,0}(\mathbb{R}+) := \{ u \in H^s_{p}(\mathbb{R}+) : u'(0) = 0 \}.$$  

Now let $u \in H^s_{p}(\mathbb{R}+)$. Further choose an arbitrary $u_0 \in H^s_{p}(\mathbb{R}+)$, with $u_0'(0) = 1$. Then, we can write

$$u(x) = (u(x) - u'(0)u_0(x)) + u'(0)u_0(x) = v(x) + u_0(x),$$

where $v(x) := u(x) - u'(0)u_0(x)$ and, clearly, $v'(0) = 0$. That is, $v \in H^s_{p,0}(\mathbb{R}+)$. In other words, $H^s_{p,0}(\mathbb{R}+)$ has co-dimension 1 in $H^s_{p}(\mathbb{R}+)$. We now define

$$u_s := (D + i)^{s-2}e_+(D - i)^2u,$$

so that $u_s \in L_p(\mathbb{R}+)$ with supp $u_s \subseteq \mathbb{R}+$. Indeed, the operator

$$I_s := r_+(D + i)^{s-2}e_+(D - i)^2 : H^s_{p}(\mathbb{R}+) \to L_p(\mathbb{R}+)$$

is an isomorphism for $1 + 1/p < s < 2 + 1/p$. (See Lemma 4.22 and the proof of Lemma 4.21.) Let $L_{p,0}(\mathbb{R}+)$ be the image of $H^s_{p,0}(\mathbb{R}+)$ under $I_s$. Then $L_{p,0}(\mathbb{R}+)$ has co-dimension 1 in $L_p(\mathbb{R}+)$. Hence, we can write

$$L_p(\mathbb{R}+) = L_{p,0}(\mathbb{R}+) \oplus M_{p,0}(\mathbb{R}+),$$

(8.26)

where $M_{p,0}(\mathbb{R}+)$ has dimension 1.

Let $\widetilde{A}_{ext} : L_p(\mathbb{R}+) \to L_p(\mathbb{R}+)$ be the (bounded) operator defined by

$$\widetilde{A}_{ext} := W(c_1) + a_2M(b_2)W(c_2).$$

We have shown, in Section 8.7, that $\widetilde{A}_{ext}$ is Fredholm, and

(i) if $s < 1 + 1/p + \alpha_c$ then $\text{ind} \widetilde{A}_{ext} = 1;$

(ii) if $s > 1 + 1/p + \alpha_c$ then $\text{ind} \widetilde{A}_{ext} = 0.$

Of course, our interest is in the operator

$$\widetilde{A} = W(c_1) + a_2M(b_2)W(c_2) : L_{p,0}(\mathbb{R}+) \to L_p(\mathbb{R}+),$$

(8.27)

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which can usefully be considered as the restriction of $\tilde{A}_{\text{ext}}$ to $L_{p,0}(\mathbb{R}+)$. Let $\tilde{A}_0 : L_p(\mathbb{R}+) \to L_p(\mathbb{R}+)$ be the linear operator defined by

$$
\tilde{A}_0 w = \begin{cases} 
\tilde{A}_{\text{ext}} w, & \text{if } w \in L_{p,0}(\mathbb{R}+) \\
0, & \text{if } w \in M_{p,0}(\mathbb{R}+). 
\end{cases}
$$

Then $\tilde{A}_0 - \tilde{A}_{\text{ext}}$ has rank one, and is therefore compact. In particular, $\text{ind } \tilde{A}_0 = \text{ind } \tilde{A}_{\text{ext}}$. On the other hand, it is clear that $\text{ind } \tilde{A}_0 = \text{ind } \tilde{A} + 1$. Hence,

$$
\text{ind } \tilde{A} = \text{ind } \tilde{A}_{\text{ext}} - 1. \quad (8.28)
$$

Thus, if $s < 1 + 1/p + \alpha_c$ then $\text{ind } \tilde{A} = 0$, and if $s > 1 + 1/p + \alpha_c$ then $\text{ind } \tilde{A} = -1$.

Repeating the argument from Section 7.3,

$$
\text{ind } A = \text{ind } \tilde{A}. \quad (8.29)
$$

To complete the proof of the second main result, we now consider (the dimension of) $\text{Ker } A$, for the cases $p = 2$, $p > 2$ and $p < 2$ respectively.

**Firstly, suppose** $p = 2$. Then, from Theorem 3.1, $\dim \text{Ker } A = 0$, for $1 + \frac{1}{2} < s < 2 + \frac{1}{2}$.

**Secondly, suppose** $p > 2$. Then, for $0 < \delta < \alpha_c$ or $\alpha_c < \delta < 1$, we define

$$
X_1 := H_{2,0}^{1+\frac{1}{2}+\delta}(\mathbb{R}+), \quad Y_1 := H_2^{1+\frac{1}{2}+\delta-2\alpha}(\mathbb{R}+),
$$

and

$$
X_2 := H_{p,0}^{1+\frac{1}{p}+\delta}(\mathbb{R}+), \quad Y_2 := H_p^{1+\frac{1}{p}+\delta-2\alpha}(\mathbb{R}+).
$$

Then $X_1$ ($Y_1$) is continuously and densely embedded into $X_2$ (into $Y_2$, respectively). Moreover, $A : X_j \to Y_j$ is Fredholm, $j = 1, 2$, and

$$
\text{Ind}_{X_1 \to Y_1} A = \text{Ind}_{X_2 \to Y_2} A.
$$

Therefore, by Lemma 7.3,

$$
\text{Ker}_{X_1 \to Y_1} A = \text{Ker}_{X_2 \to Y_2} A.
$$
That is,
\[ \text{Ker}_{X_2 \to Y_2} \mathcal{A} = \{0\}. \]

**Thirdly, suppose** \( p < 2 \). We now define

\[
X_2 := H^{1 + \frac{1}{2} + \delta} (\mathbb{R}_+), \quad Y_2 := H^{1 + \frac{1}{2} + \delta - 2\alpha} (\mathbb{R}_+)
\]

and

\[
X_1 := H^{1 + \frac{1}{p} + \delta} (\mathbb{R}_+), \quad Y_1 := H^{1 + \frac{1}{p} + \delta - 2\alpha} (\mathbb{R}_+).
\]

We can now repeat the argument made above, for the case \( p > 2 \), to show that

\[ \text{Ker}_{X_1 \to Y_1} \mathcal{A} = \{0\}. \]

So, finally, if \( 0 < \alpha < 1, \ 1 < p < \infty \) and \( 1 + 1/p < s < 1 + 1/p + \alpha_c \), then the operator \( \mathcal{A} : H^{s}_{p,0}(\mathbb{R}_+) \to H^{s-2\alpha}_{p} (\mathbb{R}_+) \) is invertible.

On the other hand, if \( 0 < \alpha < 1, \ 1 < p < \infty \) and \( 1 + 1/p + \alpha_c < s < 2 + 1/p \), then \( \mathcal{A} \) has a trivial kernel and is Fredholm with index equal to \(-1\).
Chapter 9

Transcendental equation

9.1 Main results

Following Chapter 6, let \((A_{\alpha,p,s}, \Gamma_M)\) denote the generalised symbol and associated contour of the operator

\[
\tilde{A} := W(c_1) + a_2 M^0(b_2) W(c_2),
\]

defined on \(L_p(\mathbb{R}_+)\). Then, see Theorem 6.3, \(\tilde{A}\) is Fredholm if and only if

\[
\inf_{\omega \in \Gamma_M} |A_{\alpha,p,s}(\omega)| > 0.
\]

In the case \(1/p < s < 1 + 1/p\), from Chapter 6, we have the following constraints on the values of \(\alpha, p\) and \(s\):

\[
0 < \alpha < \frac{1}{2}, \quad 1 < p < \infty \quad \text{and} \quad 1/p < s < 1 + 1/p.
\] (9.1)

Similarly, from Section 8.5, for higher regularity, we will assume

\[
0 < \alpha < 1, \quad 1 < p < \infty \quad \text{and} \quad 1 + 1/p < s < 2 + 1/p.
\] (9.2)

**Theorem 9.1.** For all \(\alpha, p, s\) satisfying the conditions \(0 < \alpha < \frac{1}{2}, 1 < p < \infty\) and \(1/p < s < 1 + 1/p\), we have

\[
\inf_{\omega \in \Gamma_M} |A_{\alpha,p,s}(\omega)| > 0.
\]

**Theorem 9.2.** For all \(\alpha, p, s\) satisfying the conditions \(0 < \alpha < 1, 1 < p < \infty\) and \(1 + 1/p < s < 2 + 1/p\), we have

\[
\inf_{\omega \in \Gamma_M} |A_{\alpha,p,s}(\omega)| > 0,
\]

unless \(\xi = 0\) and \(s = 1 + 1/p + \alpha_c\), where \(\alpha_c\) only depends on \(\alpha\) and satisfies \(0 < \alpha_c < \alpha\).
9.2 Background

We have seen in Chapter 6 that as $\omega$ varies over $\Gamma_M$, the symbol $A_{\alpha,p,s}(\omega)$ forms a closed loop in the complex plane. This loop comprises two components. The first lies on the unit circle, and the second is given in terms of certain transcendental functions.

Let us define

$$T_B := \frac{\sin \pi \alpha}{\pi} B(s - 2\alpha + 1 - 1/p + i\xi, 2\alpha),$$

(9.3)

and

$$T_s := \frac{\sin \pi (1/p + m - s + \alpha - i\xi)}{\sin \pi (1/p + m - s + 2\alpha - i\xi)},$$

(9.4)

where $m = 1$ or 2, as $1/p < s < 1 + 1/p$ or $1 + 1/p < s < 2 + 1/p$ respectively, and $\xi \in \mathbb{R}$. It is easy to see that $T_s$ is independent of the choice of $m$, and it will be convenient for us to assume that $m = 1$.

From Chapter 6 and Section 8.5, for the cases $1/p < s < 1 + 1/p$ and $1 + 1/p < s < 2 + 1/p$ respectively, in order to show that

$$\inf_{\omega \in \Gamma_M} |A_{\alpha,p,s}(\omega)| > 0,$$

it is enough to show that the transcendental equation

$$T_s = T_B,$$

(9.5)

has no solutions for $\alpha, p$ and $s$ varying subject to either the constraints (9.1) or (9.2).

It will be convenient to begin by considering the transcendental equation when $\xi = 0$. Moreover, from the simple relationships

$$\sin(z) = \sin(\tau) \quad \text{and} \quad \Gamma(z) = \Gamma(\tau) \quad (6.1.23, p. 256, [1]),$$

it is easy to see that if $(\alpha_0, p_0, s_0, \xi_0)$ is a solution quadruple, then so is $(\alpha_0, p_0, s_0, -\xi_0)$. Therefore, having the required result for $\xi = 0$, it remains to consider the case $\xi > 0$.

Finally, we note that $T_s$ and $T_B$ depend only the difference $(s - 1/p)$, rather than $s$ and $p$ independently. Accordingly, we define

$$\tau := s - 1/p.$$ 

(9.6)
Of course, we are interested in either $0 < \tau < 1$ or $1 < \tau < 2$.

From equation (9.3) with $\xi = 0$,

$$T_B = \frac{\sin \pi \alpha}{\pi} B(\tau - 2\alpha + 1, 2\alpha) = \frac{\sin \pi \alpha}{\pi} \frac{\Gamma(\tau - 2\alpha + 1)\Gamma(2\alpha)}{\Gamma(\tau + 1)}.$$  

From equation (9.4) with $\xi = 0$,

$$T_s = \frac{\sin \pi (1 - \tau + \alpha)}{\sin \pi (1 - \tau + 2\alpha)} = \frac{\sin \pi (\alpha - \tau)}{\sin \pi (\tau - 2\alpha + 1)}.$$  

Now $\Gamma(z)\Gamma(1 - z) = \pi/\sin \pi z$, see 5.5.3, [34]. Hence, taking $z = \tau - 2\alpha + 1$, we can re-write the equation $T_s = T_B$ as

$$\Gamma(2\alpha - \tau)\Gamma(\tau + 1) \sin \pi (\alpha - \tau) = \Gamma(2\alpha) \sin \pi \alpha. \quad (9.7)$$

If $0 < \alpha < 1$ and $1 < \tau < 2$, it turns out, see Lemma 9.29, that equation (9.7) has a unique solution of the form $\tau = 1 + \alpha_c$, where $\alpha_c$ only depends on $\alpha$ and satisfies $0 < \alpha_c < \alpha$.

**Remark 9.3.** In this chapter we will do several computations involving the function $\arg(\cdot)$. For $z \in \mathbb{C} \setminus \{0\}$ and $c \in \mathbb{R}$, we shall write

$$\arg z \equiv c,$$

to indicate that

$$\arg z = c + 2\pi k,$$

for some $k \in \mathbb{Z}$. (Of course, if $-\pi < c \leq \pi$, then $k = 0$. See (1.9).)

### 9.3 Supporting lemmas

**Remark 9.4.** In the lemmas that follow we will be considering various derivatives of combinations of gamma functions of real and complex arguments. Suppose $z \in \mathbb{C}$. Then, see 6.1.23, p. 256 and 6.3.1, p. 258, [1],

$$\Gamma(z) = \Gamma(\tau); \quad \Gamma'(z) = \Gamma(z)\psi(z),$$

where $\psi$ denotes the digamma function.
Suppose \( x > 0 \). Then from 6.4.1, p. 260, [1], we have
\[
\psi'(x) > 0, \quad \psi''(x) < 0 \quad \text{and} \quad \psi'''(x) > 0.
\]
In particular, the function \( \psi(x) \) is concave and strictly increasing.

For the purposes of Lemmas 9.5, 9.6 and 9.7 we define
\[
f(\tau; \alpha) := \Gamma(2\alpha - \tau)\Gamma(\tau + 1) \sin \pi(\alpha - \tau).
\]

**Lemma 9.5.** Suppose that one of the following three conditions hold:

(a) \( 0 < \alpha < \frac{1}{2} \) and \( 0 < \tau < \alpha \);

(b) \( 0 < \alpha < 1 \) and \( 2\alpha < \tau < 1 + \alpha \);

(c) \( 0 < \alpha < \frac{1}{2} \) and \( 1 + 2\alpha < \tau < 2 \).

Then
\[
f(\tau; \alpha) > 0 \quad \text{and} \quad \frac{\partial}{\partial \tau} f(\tau; \alpha) < 0.
\]

**Proof.** The assertion that \( f(\tau; \alpha) > 0 \) follows immediately from the observation that if \( x \in (-2, -1) \cup (0, \infty) \) then \( \Gamma(x) > 0 \), and if \( x \in (-1, 0) \) then \( \Gamma(x) < 0 \).

Since \( \Gamma'(z) = \Gamma(z)\psi(z) \), we have
\[
\frac{\partial}{\partial \tau} f(\tau; \alpha) = \Gamma(2\alpha - \tau)\Gamma(\tau + 1)\left\{ (\psi(\tau + 1) - \psi(2\alpha - \tau)) \sin \pi(\alpha - \tau) - \pi \cos \pi(\alpha - \tau) \right\}
\]
\[
= f(\tau; \alpha)(\psi(\tau + 1) - \psi(2\alpha - \tau)) - \Gamma(2\alpha - \tau)\Gamma(\tau + 1)\pi \cos \pi(\alpha - \tau).
\]

We will now prove that if one of the conditions (a), (b) or (c) holds then
\[
\psi(\tau + 1) - \psi(2\alpha - \tau) < \psi(\tau + 1 - \alpha) - \psi(\alpha - \tau). \tag{9.8}
\]

Firstly, suppose \( 0 < \alpha < \frac{1}{2} \) and \( 0 < \tau < \alpha \). Then
\[
\psi(\tau + 1) - \psi(2\alpha - \tau) < \psi(\tau + 1 - \alpha) - \psi(\alpha - \tau),
\]
follows directly from the concavity of \( \psi(x) \) for \( x > 0 \), since \( \tau + 1 - \alpha > 1 - \alpha > \alpha > \alpha - \tau > 0 \).
For the remaining two cases we note that
\[ \psi(z + 1) = \psi(z) + 1/z, \quad z \in \mathbb{C} \setminus \{0\}, \]
see 6.3.5, p. 258, [1].

Hence, for \(0 < \alpha < 1\) and \(2\alpha < \tau < 1 + \alpha\),
\[
\psi(\tau + 1) - \psi(2\alpha - \tau) = \psi(\tau + 1) - \psi(1 + 2\alpha - \tau) + 1/(2\alpha - \tau)
< \psi(\tau + 1 - \alpha) - \psi(1 + \alpha - \tau) + 1/(2\alpha - \tau)
= \psi(\tau + 1 - \alpha) - \psi(\alpha - \tau) - 1/(\alpha - \tau) + 1/(2\alpha - \tau)
= \psi(\tau + 1 - \alpha) - \psi(\alpha - \tau) + \{1/(\tau - \alpha) - 1/(\tau - 2\alpha)\}
< \psi(\tau + 1 - \alpha) - \psi(\alpha - \tau),
\]
noting that \(\tau + 1 - \alpha > 1 + \alpha > 1 + \alpha - \tau > 0\).

Finally, for \(0 < \alpha < \frac{1}{2}\) and \(1 + 2\alpha < \tau < 2\),
\[
\psi(\tau + 1) - \psi(2\alpha - \tau)
= \psi(\tau + 1) - \psi(1 + 2\alpha - \tau) + 1/(2\alpha - \tau)
= \psi(\tau + 1) - \psi(2 + 2\alpha - \tau) + 1/(2\alpha - \tau) + 1/(1 + 2\alpha - \tau)
< \psi(\tau + 1 - \alpha) - \psi(2 + \alpha - \tau) + 1/(2\alpha - \tau) + 1/(1 + 2\alpha - \tau)
= \psi(\tau + 1 - \alpha) - \psi(\alpha - \tau)
- 1/(1 + \alpha - \tau) - 1/(\alpha - \tau) + 1/(2\alpha - \tau) + 1/(1 + 2\alpha - \tau)
= \psi(\tau + 1 - \alpha) - \psi(\alpha - \tau)
+ \{1/(\tau - \alpha) - 1/(\tau - 2\alpha)\} + \{1/(\tau - 1 - \alpha) - 1/(\tau - 1 - 2\alpha)\}
< \psi(\tau + 1 - \alpha) - \psi(\alpha - \tau),
\]
noting that \(\tau + 1 - \alpha > 2 + \alpha > 2 + \alpha - \tau > 0\).

Hence, assuming that one of the conditions (a), (b) or (c) holds, then from (9.8),
\[
\psi(\tau + 1) - \psi(2\alpha - \tau) < \psi(\tau + 1 - \alpha) - \psi(\alpha - \tau)
= - (\psi(\alpha - \tau) - \psi(1 - \alpha + \tau))
= \frac{\pi}{\tan \pi(\alpha - \tau)},
\]
since \(\psi(z) - \psi(1 - z) = -\pi/\tan \pi z\), (see 6.3.7, p. 259, [1]).
Since $f(\tau; \alpha) > 0$,

$$\frac{\partial}{\partial \tau} f(\tau; \alpha) < f(\tau; \alpha) \frac{\pi}{\tan \pi(\alpha - \tau)} - \Gamma(2\alpha - \tau) \Gamma(\tau + 1) \pi \cos \pi(\alpha - \tau)$$

$$= \Gamma(2\alpha - \tau) \Gamma(\tau + 1) \left\{ \frac{\pi}{\tan \pi(\alpha - \tau)} \cdot \sin \pi(\alpha - \tau) - \pi \cos \pi(\alpha - \tau) \right\}$$

$$= 0.$$ 

This completes the proof of the lemma.

\[\square\]

**Lemma 9.6.** Suppose $0 < \alpha < \frac{1}{2}$. Then, for any given $\alpha$, there exists a unique $\tau_\alpha$, depending only on $\alpha$, such that

$$f(\tau_\alpha; \alpha) - \Gamma(2\alpha) \sin \pi \alpha = 0 \quad \text{and} \quad 1 < \tau_\alpha < 1 + \alpha.$$ 

**Proof.** Suppose $1 \leq \tau \leq 1 + \alpha$. Hence, $-1 - \alpha \leq -\tau \leq -1$, and thus

$$-1 < -1 + \alpha \leq 2\alpha - \tau \leq 2\alpha - 1 < 0.$$ 

Therefore,

$$-1 < 2\alpha - \tau < 0.$$ 

Hence, for any given $\alpha \in (0, \frac{1}{2})$, the function $f(\tau; \alpha)$ is continuous for all $\tau \in [1, 1 + \alpha]$. Moreover,

$$f(1; \alpha) - \Gamma(2\alpha) \sin \pi \alpha = \Gamma(2\alpha - 1) \cdot (-\sin \pi \alpha) - \Gamma(2\alpha) \sin \pi \alpha$$

$$= \sin \pi \alpha \cdot \Gamma(2\alpha - 1) \{ -1 - (2\alpha - 1) \}$$

$$= 2\alpha \cdot \sin \pi \alpha \cdot (-\Gamma(2\alpha - 1))$$

$$> 0.$$ 

On the other hand,

$$f(1 + \alpha; \alpha) - \Gamma(2\alpha) \sin \pi \alpha = -\Gamma(2\alpha) \sin \pi \alpha < 0.$$ 

The existence of $\tau_\alpha$ now follows directly from the Intermediate Value Theorem, and Lemma 9.5 guarantees its uniqueness.

\[\square\]

**Lemma 9.7.** Suppose $\frac{1}{2} \leq \alpha < 1$. Then, for any given $\alpha$, there exists a unique $\tau_\alpha$, depending only on $\alpha$, such that

$$f(\tau_\alpha; \alpha) - \Gamma(2\alpha) \sin \pi \alpha = 0 \quad \text{and} \quad 2\alpha < \tau_\alpha < 1 + \alpha.$$
Proof. Choose any $\delta$ such that $0 < \delta < 1 - \alpha$.

Further, assume that $\tau$ satisfies $2\alpha + \delta \leq \tau \leq 1 + \alpha$. Then, $-1 - \alpha \leq -\tau \leq -2\alpha - \delta$ and $-1 \leq -1 + \alpha \leq 2\alpha - \tau \leq -\delta < 0$. Therefore,

$$-1 < 2\alpha - \tau < 0.$$ 

Hence, for any given $\alpha \in [\frac{1}{2}, 1)$, the function $f(\tau; \alpha)$ is continuous for all $\tau \in [2\alpha + \delta, 1 + \alpha]$. Moreover,

$$f(2\alpha + \delta; \alpha) - \Gamma(2\alpha) \sin \pi\alpha = -\Gamma(-\delta) \cdot \Gamma(2\alpha + \delta + 1) \cdot \sin \pi(-\alpha - \delta) - \Gamma(2\alpha) \sin \pi\alpha$$

$$> 0,$$ for sufficiently small $\delta$.

On the other hand,

$$f(1 + \alpha; \alpha) - \Gamma(2\alpha) \sin \pi\alpha = -\Gamma(2\alpha) \sin \pi\alpha < 0.$$ 

The existence of $\tau_\alpha$ now follows directly from the Intermediate Value Theorem, and Lemma 9.5 guarantees its uniqueness.

Lemma 9.8. Suppose $z_0 = x + iy$, $z_1 = x + a + iy$ where $a, x, y \in \mathbb{R}$. Then

$$\frac{\partial}{\partial x} \left| \frac{\Gamma(z_0)}{\Gamma(z_1)} \right|^2 = \left| \frac{\Gamma(z_0)}{\Gamma(z_1)} \right|^2 \{\psi(z_0) - \psi(z_1) + \psi(\overline{z_0}) - \psi(\overline{z_1})\}.$$ 

Proof. Since $\Gamma'(z) = \Gamma(z)\psi(z)$, it is easy to show that

$$\frac{\partial}{\partial x} \left( \frac{\Gamma(z_0)}{\Gamma(z_1)} \right) = \frac{\Gamma(z_0)}{\Gamma(z_1)} \{\psi(z_0) - \psi(z_1)\}.$$ 

Finally, since $\Gamma(z) = \Gamma(\overline{z})$, we have

$$\left| \frac{\Gamma(z_0)}{\Gamma(z_1)} \right|^2 = \frac{\Gamma(z_0)\Gamma(\overline{z_0})}{\Gamma(z_1)\Gamma(\overline{z_1})}$$ 

and the required result follows immediately.

Remark 9.9. Under the same hypotheses as Lemma 9.8, we can similarly show that

$$\frac{\partial}{\partial y} \left| \frac{\Gamma(z_0)}{\Gamma(z_1)} \right|^2 = i \left| \frac{\Gamma(z_0)}{\Gamma(z_1)} \right|^2 \{\psi(z_0) - \psi(z_1) - \psi(\overline{z_0}) + \psi(\overline{z_1})\}.$$ 

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Lemma 9.10. Suppose $0 < \alpha < 1$, $\sigma > 0$ and $\xi > 0$. Then
\[
\left| \frac{\Gamma(\sigma + i\xi)}{\Gamma(\sigma + 2\alpha + i\xi)} \right| \leq \frac{\Gamma(\sigma)}{\Gamma(\sigma + 2\alpha)}.
\]

Proof. To simplify the exposition, let us define
\[
R(\sigma + i\xi, 2\alpha) := \frac{\Gamma(\sigma + i\xi)}{\Gamma(\sigma + 2\alpha + i\xi)}.
\]
Then, since $\Gamma(z) = \Gamma(z)$ we have
\[
|R(\sigma + i\xi, 2\alpha)|^2 := \frac{\Gamma(\sigma + i\xi)\Gamma(\sigma - i\xi)}{\Gamma(\sigma + 2\alpha + i\xi)\Gamma(\sigma + 2\alpha - i\xi)}.
\]
Further let
\[
z_0 = \sigma + i\xi; \quad z_1 = \sigma + 2\alpha + i\xi.
\]
Then, since $\overline{\psi(z)} = \psi(z)$ (6.3.9, p. 259, [1]), by Remark 9.9,
\[
\frac{\partial}{\partial \xi} |R(\sigma + i\xi, 2\alpha)|^2 = i |R(\sigma + i\xi, 2\alpha)|^2 \left\{ \psi(z_0) - \psi(z_1) - \psi(\overline{z_0}) + \psi(\overline{z_1}) \right\}
\]
\[
= i |R(\sigma + i\xi, 2\alpha)|^2 \left\{ 2i \Im \psi(z_0) - 2i \Im \psi(z_1) \right\}
\]
\[
= 2 |R(\sigma + i\xi, 2\alpha)|^2 \left\{ \Im \psi(z_1) - \Im \psi(z_0) \right\}
\]
\[< 0, \quad \text{by Lemma 9.15}.
\]
This completes the proof of the lemma.

Lemma 9.11. Suppose $\alpha > 0$ and $\sigma \geq \sigma_{\min} > 0$. Then
\[
\frac{\Gamma(\sigma)}{\Gamma(\sigma + 2\alpha)} \leq \frac{\Gamma(\sigma_{\min})}{\Gamma(\sigma_{\min} + 2\alpha)}.
\]

Proof. It is easy to see that
\[
\frac{\partial}{\partial \sigma} \frac{\Gamma(\sigma)}{\Gamma(\sigma + 2\alpha)} = \frac{\Gamma(\sigma)}{\Gamma(\sigma + 2\alpha)} \left( \psi(\sigma) - \psi(\sigma + 2\alpha) \right) < 0.
\]
This completes the proof of the lemma.

Lemma 9.12. Suppose $\alpha > 0$. Then the function
\[
f(\alpha) := \frac{\Gamma(2\alpha + 1)}{(\Gamma(1 + \alpha))^2}
\]
strictly increases as $\alpha$ increases.
Proof. It is easy to see that
\[ \frac{d}{d\alpha} \frac{\Gamma(2\alpha + 1)}{(\Gamma(1 + \alpha))^2} = \frac{2\Gamma(2\alpha + 1)}{(\Gamma(1 + \alpha))^2} \left( \psi(1 + 2\alpha) - \psi(1 + \alpha) \right) > 0. \]
This completes the proof of the lemma.

\[ \square \]

Lemma 9.13. Suppose \( 0 < \alpha < 1 \). Then the function
\[ g(\alpha) := -\frac{\Gamma(2\alpha)}{\Gamma(\alpha - 1)\Gamma(2 + \alpha)} \]
strictly decreases as \( \alpha \) increases.

Proof. It is easy to see that
\[ \frac{dg}{d\alpha} = \frac{\Gamma(2\alpha)}{\Gamma(\alpha - 1)\Gamma(2 + \alpha)} \cdot \left\{ \psi(2 + \alpha) - 2\psi(2\alpha) + \psi(\alpha - 1) \right\}. \]
Since \( \Gamma(\alpha - 1) < 0 \) for \( 0 < \alpha < 1 \), it is enough to show that \( \psi(2 + \alpha) - 2\psi(2\alpha) + \psi(\alpha - 1) > 0 \). We note the identity
\[ \psi(z + 1) = \psi(z) + \frac{1}{z} \quad (6.3.5, \text{p. 258, [1]}), \]
and that \( \psi(x) \) is increasing for \( x > 0 \). Since \( 2\psi(2\alpha) = \psi(\alpha) + \psi(\alpha + \frac{1}{2}) + \log 4 \), see 6.3.8, p. 259, [1], we have
\[ \psi(2 + \alpha) - 2\psi(2\alpha) + \psi(\alpha - 1) = \psi(2 + \alpha) - [\psi(\alpha) + \psi(\alpha + \frac{1}{2}) + \log 4] + \psi(\alpha - 1) \]
\[ = \psi(2 + \alpha) + \frac{1}{1 - \alpha} - \psi(\alpha + \frac{1}{2}) - \log 4 \]
\[ > \psi(2) + \frac{1}{1 - \alpha} - \psi(\frac{3}{2}) - \log 4. \]
But, see 6.3.2, 6.3.3, p. 258, [1],
\[ \psi(2) - \psi(\frac{3}{2}) - \log 4 = [\psi(1) + 1] - [\psi(\frac{1}{2}) + 2] - \log 4 \]
\[ = [\psi(1) + 1] - [\psi(1) - \log 4 + 2] - \log 4 \]
\[ = -1. \]
Hence, finally,
\[ \psi(2 + \alpha) - 2\psi(2\alpha) + \psi(\alpha - 1) > \frac{1}{1 - \alpha} - 1 \]
\[ = \frac{\alpha}{1 - \alpha} \]
\[ > 0, \]
for \( 0 < \alpha < 1 \). This completes the proof of the lemma.

\[ \square \]
Lemma 9.14. Suppose $a, b \in \mathbb{R}$. Define

$$S(a, b; \xi) := \frac{\sin[\pi(a - i\xi)]}{\sin[\pi(b - i\xi)]}.$$  

Then

$$|S(a, b; \xi)| = \left(\frac{\cosh 2\pi\xi - \cos 2\pi a}{\cosh 2\pi\xi - \cos 2\pi b}\right)^{\frac{1}{2}}.$$  

Proof. From 4.21.37, [34],

$$\sin[\pi(a - i\xi)] = \cosh \pi\xi \sin \pi a - \cos \pi a \sinh \pi\xi.$$  

Therefore $|\sin[\pi(a - i\xi)]|^2$

$$= \cosh^2 \pi\xi \sin^2 \pi a + \cos^2 \pi a \sinh^2 \pi\xi$$
$$= \frac{1}{2} (\cosh 2\pi\xi + 1) \sin^2 \pi a + \frac{1}{2} (\cosh 2\pi\xi - 1) \cos^2 \pi a$$
$$= \frac{1}{2} (\cosh 2\pi\xi - \cos 2\pi a).$$

Hence

$$|S(a, b; \xi)| = \left(\frac{\cosh 2\pi\xi - \cos 2\pi a}{\cosh 2\pi\xi - \cos 2\pi b}\right)^{\frac{1}{2}}.$$  

Lemma 9.15. Suppose $\xi > 0$ and $\sigma > 0$. Then, for fixed $\xi$,

$$\text{Im } \psi(\sigma + i\xi) \text{ decreases as } \sigma \text{ increases.}$$  

Proof. From Section 44.11, p. 455, [35],

$$\text{Im } \psi(\sigma + i\xi) = \sum_{j=0}^{\infty} \frac{\xi}{(j + \sigma)^2 + \xi^2}.$$  

Since

$$\frac{\partial}{\partial \sigma} \left(\frac{\xi}{(j + \sigma)^2 + \xi^2}\right) = \frac{-2(j + \sigma)\xi}{[(j + \sigma)^2 + \xi^2]^2} < 0, \quad j = 0, 1, 2, \ldots,$$

the required result follows immediately.
Lemma 9.16. Suppose $\gamma > 0, \sigma > 0$ and $\xi > 0$. Then

$$\arg B(\sigma + i\xi, \gamma) = \sum_{n=0}^{\infty} \left( \arctan \frac{\xi}{\sigma + \gamma + n} - \arctan \frac{\xi}{\sigma + n} \right) + 2\pi k,$$  \hspace{1cm} (9.9)

for some $k \in \mathbb{Z}$.

Proof. From 6.1.27, p. 256, [1],

$$\arg \Gamma(\sigma + i\xi) \equiv \xi \psi(\sigma) + \sum_{n=0}^{\infty} \left( \frac{\xi}{\sigma + n} - \arctan \frac{\xi}{\sigma + n} \right).$$

Now

$$\arg B(\sigma + i\xi, \gamma) = \arg \left( \frac{\Gamma(\sigma + i\xi) \Gamma(\gamma)}{\Gamma(\sigma + \gamma + i\xi)} \right) \equiv \arg \Gamma(\sigma + \gamma - i\xi) + \arg \Gamma(\sigma + i\xi)$$

$$\equiv -\xi \psi(\sigma + \gamma) - \sum_{n=0}^{\infty} \left( \frac{\xi}{\sigma + \gamma + n} - \arctan \frac{\xi}{\sigma + \gamma + n} \right)$$

$$+ \xi \psi(\sigma) + \sum_{n=0}^{\infty} \left( \frac{\xi}{\sigma + n} - \arctan \frac{\xi}{\sigma + n} \right).$$  \hspace{1cm} (9.10)

We note from 8.363 3, p. 903, [17], that

$$\psi(x) - \psi(y) = \sum_{n=0}^{\infty} \left( \frac{1}{y + n} - \frac{1}{x + n} \right).$$

Using this result, with equation (9.10), we can write

$$\arg B(\sigma + i\xi, \gamma) = \sum_{n=0}^{\infty} \left( \arctan \frac{\xi}{\sigma + \gamma + n} - \arctan \frac{\xi}{\sigma + n} \right) + 2\pi k,$$

as required. \hfill \Box

Lemma 9.17. Suppose $0 < \gamma < 1, \sigma > 0$ and $\xi > 0$. Define

$$S(\gamma, \sigma, \xi) := \sum_{n=0}^{\infty} \left( \arctan \frac{\xi}{\sigma + \gamma + n} - \arctan \frac{\xi}{\sigma + n} \right).$$

Then

$$-\frac{\pi}{2} < -\arctan \frac{\xi}{\sigma} < S(\gamma, \sigma, \xi) < \arctan \frac{\xi}{\sigma + \gamma} - \arctan \frac{\xi}{\sigma} < 0.$$
Proof. Since $0 < \gamma < 1$, we can determine a lower bound for $S(\gamma, \sigma, \xi)$ by writing

$$S(\gamma, \sigma, \xi) = \sum_{n=0}^{\infty} \left( \arctan \frac{\xi}{\sigma + \gamma + n} - \arctan \frac{\xi}{\sigma + n} \right)$$

$$> \sum_{n=0}^{\infty} \left( \arctan \frac{\xi}{\sigma + 1 + n} - \arctan \frac{\xi}{\sigma + n} \right)$$

$$= - \arctan \frac{\xi}{\sigma}.$$ 

On the other hand, to determine an upper bound we note that

$$S(\gamma, \sigma, \xi) = \sum_{n=0}^{\infty} \left( \arctan \frac{\xi}{\sigma + \gamma + n} - \arctan \frac{\xi}{\sigma + n} \right)$$

$$= \arctan \frac{\xi}{\sigma + \gamma} - \arctan \frac{\xi}{\sigma} + \sum_{n=1}^{\infty} \left( \arctan \frac{\xi}{\sigma + \gamma + n} - \arctan \frac{\xi}{\sigma + n} \right)$$

$$< \arctan \frac{\xi}{\sigma + \gamma} - \arctan \frac{\xi}{\sigma}.$$ 

We now give a corollary of Lemma 9.16, in the case $0 < \gamma < 1$.

**Corollary 9.18.** If we assume $0 < \gamma < 1$ in Lemma 9.16, then the integer $k = 0$, and we have the estimate

$$-\pi/2 < \arg B(\sigma + i\xi, \gamma) < 0.$$ 

**Proof.** See equation (9.9) and Lemma 9.17. \qed

On the other hand if $1 \leq \gamma < 2$ we need to add to an extra condition.

**Remark 9.19.** Similarly, if $1 \leq \gamma < 2$ then

$$- \arctan \frac{\xi}{\sigma} - \arctan \frac{\xi}{\sigma + 1} < \arg B(\sigma + i\xi, \gamma) < \arctan \frac{\xi}{\sigma + \gamma} - \arctan \frac{\xi}{\sigma}.$$ 

Moreover, if $\sigma \geq \xi$, then

$$- \arctan \frac{\xi}{\sigma} - \arctan \frac{\xi}{\sigma + 1} = - \arctan \left( \frac{\xi(2\sigma + 1)}{\sigma(\sigma + 1) - \xi^2} \right).$$
so that, in particular, if $1 \leq \gamma < 2$ and $\sigma \geq \xi$ then

$$-\pi/2 < \arg B(\sigma + i\xi, \gamma) < 0,$$

as previously.

**Lemma 9.20.** Suppose $0 < \alpha < \frac{1}{2}$, $\sigma \geq 1 - \alpha$, and $0 < \xi < \frac{1}{4}$. Then

$$-\pi/2 < -4\alpha \xi < \arg B(\sigma + i\xi, 2\alpha) < 0.$$

**Proof.** From equation (9.10),

$$\arg B(\sigma + i\xi, 2\alpha) = -\xi \psi(\sigma + 2\alpha) - \sum_{n=0}^{\infty} \left( \frac{\xi}{\sigma + 2\alpha + n} - \arctan \frac{\xi}{\sigma + 2\alpha + n} \right)$$

$$+ \xi \psi(\sigma) + \sum_{n=0}^{\infty} \left( \frac{\xi}{\sigma + n} - \arctan \frac{\xi}{\sigma + n} \right) + 2\pi k,$$

for some $k \in \mathbb{Z}$. We now use the fact that $\arg B(\sigma + i\xi, 2\alpha) \in (-\pi, \pi]$ to show that $k = 0$.

Let us define

$$t_{\alpha,n} := \frac{\xi}{\sigma + 2\alpha + n} - \arctan \frac{\xi}{\sigma + 2\alpha + n}.$$

Then, it is easy to see that

$$0 < \sum_{n=0}^{\infty} (t_{0,n} - t_{\alpha,n}) < t_{0,0},$$

since $0 < 2\alpha < 1$, and the function $x - \arctan x$ is strictly increasing for $x > 0$. But

$$t_{0,0} = \frac{\xi}{\sigma} - \arctan \frac{\xi}{\sigma},$$

and because $\xi/\sigma < \frac{1}{2}$, we have

$$t_{0,0} < \frac{1}{2} - \arctan \frac{1}{2} < \frac{1}{2\pi}.$$

Using the relationship

$$\psi(z + 1) = \psi(z) + 1/z, \quad z \in \mathbb{C} \setminus \{0\},$$

see 6.3.5, p. 258, [1], we have the estimate

$$0 < \psi(\sigma + 2\alpha) - \psi(\sigma) < \psi(\sigma + 1) - \psi(\sigma) = 1/\sigma < 2.$$
Therefore
\[-\frac{1}{2} < \xi (\psi(\sigma) - \psi(\sigma + 2\alpha)) < 0.\]
Hence, \(k = 0.\)

Noting that \(\psi(x)\) is increasing and \(\psi'(x)\) is decreasing,
\[
\arg B(\sigma + i\xi, 2\alpha) > -\xi \left\{ \psi(\sigma + 2\alpha) - \psi(\sigma) \right\} \\
\geq -\xi \left\{ \psi(1 + \alpha) - \psi(1 - \alpha) \right\} \quad \text{since} \ \sigma \geq 1 - \alpha \\
= -\xi h(\alpha),
\]
where the function
\[
h(\alpha) := \psi(1 + \alpha) - \psi(1 - \alpha) \quad \text{for} \ 0 < \alpha < \frac{1}{2}.
\]
Clearly \(h(0) = 0.\) On the other hand, from 6.3.5, p. 258, [1],
\[
h\left(\frac{1}{2}\right) = \psi\left(\frac{3}{2}\right) - \psi\left(\frac{1}{2}\right) = 2.
\]
Noting that \((-\psi(1 - \alpha))'' = -\psi''(1 - \alpha),\) we have
\[
h''(\alpha) = \psi''(1 + \alpha) - \psi''(1 - \alpha) > 0,
\]
because \(\psi'' > 0\) by 6.4.1, p.260, [1]. Hence \(h\) is convex and, therefore,
\[
h(\alpha) \leq \left(\frac{2 - 0}{\frac{1}{2} - 0}\right) \alpha = 4\alpha.
\]
So, now we have the lower bound
\[
\arg B(\sigma + i\xi, 2\alpha) > -4\alpha \xi.
\]
Finally, combining this result and Corollary 9.18, with \(0 < \gamma = 2\alpha < 1,\) we have the final estimate
\[
-\pi/2 < -4\alpha \xi < \arg B(\sigma + i\xi, 2\alpha) < 0.
\]

\[\square\]

**Lemma 9.21.** Suppose \(0 < \alpha < \frac{1}{2}\) and \(0 < \xi < \frac{1}{4}.\) Then
\[
\arctan(\sin \pi \alpha \tanh \pi \xi) > 4\alpha \xi.
\]
Proof. Firstly, by Jordan’s inequality
\[
\frac{\sin x}{x} \geq \frac{2}{\pi} \quad \text{for } 0 < x < \pi/2.
\]
So, setting \( x = \pi \alpha \) gives
\[
\sin \pi \alpha \geq 2\alpha \quad \text{for } 0 < \alpha < \frac{1}{2}.
\]
(9.11)

Secondly, \( \tanh \pi \xi \) is concave function for \( 0 < \xi < \frac{1}{4} \), with \( \tanh(0) = 0 \) and \( \tanh(\pi/4) > 0.655 > \frac{5}{8} \). Hence
\[
\tanh \pi \xi > \frac{5}{8} \xi \quad \text{for } 0 < \xi < \frac{1}{4}.
\]
(9.12)

Thirdly, \( \arctan y \) is concave function for \( 0 < y < \frac{5}{8} \), with \( \arctan(0) = 0 \) and \( \arctan(\frac{5}{8}) > \frac{1}{2} \). Hence
\[
\arctan y > \frac{4}{5} y \quad \text{for } 0 < y < \frac{5}{8}.
\]
(9.13)

Finally, using estimates (9.11), (9.12) and (9.13) in turn,
\[
\arctan(\sin \pi \alpha \tanh \pi \xi) \geq \arctan(2\alpha \tanh \pi \xi)
\]
> \( \arctan(2\alpha \cdot \frac{5}{8} \xi) \)
\[
= \arctan(5\alpha \xi)
\]
> \( 4\alpha \xi \).

Lemma 9.22. Suppose \( \frac{1}{2} \leq \alpha < 1 \) and \( 1 + \alpha \leq \tau < 2 \). Define \( b := 1 - \tau + 2\alpha \).

Then
\[
\cos \pi \alpha - \cos \pi(\alpha - 2b) \leq 0.
\]

Proof.
\[
\cos \pi \alpha - \cos \pi(\alpha - 2b) = \cos \pi \alpha - \cos \pi(\alpha - 2 + 2\tau - 4\alpha)
\]
\[
= \cos \pi \alpha - \cos \pi(2\tau - 3\alpha)
\]
\[
= -2\sin \pi(\tau - \alpha) \sin \pi(2\alpha - \tau)
\]
\[
= 2\sin \pi(\tau - \alpha) \sin \pi(\tau - 2\alpha).
\]
But since \( 1 + \alpha \leq \tau < 2 \), we have \( 1 \leq \tau - \alpha < 2 \) and thus, \( \sin \pi(\tau - \alpha) \leq 0 \).

Similarly given \( 1 + \alpha \leq \tau < 2 \), we have \( 0 < 1 - \alpha \leq \tau - 2\alpha < 2 - 2\alpha \leq 1 \) and thus \( \sin \pi(\tau - 2\alpha) > 0 \). Therefore,
\[
\cos \pi \alpha - \cos \pi(\alpha - 2b) \leq 0.
\]
\( \square \)
9.4 Proof of Theorems 9.1 and 9.2

**Theorem 9.23.** Suppose $0 < \alpha < \frac{1}{2}$, $0 < \tau < 1$ and $\xi \in \mathbb{R}$. Define

$$T_s := \frac{\sin \pi (1 - \tau + \alpha - i\xi)}{\sin \pi (1 - \tau + 2\alpha - i\xi)}$$  \hspace{1cm} (9.14)

and

$$T_B := \frac{\sin \pi \alpha}{\pi} B(\tau - 2\alpha + 1 + i\xi, 2\alpha).$$  \hspace{1cm} (9.15)

(a) If $\xi = 0$ and $0 < \tau < 1$, then $T_s \neq T_B$.

(b) If $\xi \geq \frac{1}{4}$ and $\alpha \leq \tau < 1$, then $|T_s| > |T_B|$.

(c) If $0 < \xi < \frac{1}{4}$ and $\alpha \leq \tau < 1$, then $\arg T_s < \arg T_B$.

(d) If $\xi > 0$ and $0 < \tau < \alpha$, then $\arg T_s \neq \arg T_B$.

In other words, the transcendental equation $T_s = T_B$ has no solutions for $0 < \alpha < \frac{1}{2}$, $0 < \tau < 1$ and $\xi \in \mathbb{R}$.

**Proof.** See Lemmas 9.25, 9.26, 9.27 and 9.28 for the proof of cases (a), (b), (c) and (d) respectively.

\[\square\]

**Theorem 9.24.** Suppose $0 < \alpha < 1$ and $\xi \in \mathbb{R}$. Let $T_s$ and $T_B$ be as defined previously in (9.14) and (9.15) respectively.

(a) If $\xi = 0$ and $1 < \tau < 2$, then there exists a unique $\alpha_c$ such that $T_s = T_B$, with $\tau = 1 + \alpha_c$.

(b) If $\xi \geq \frac{1}{4}$ and $1 + \alpha \leq \tau < 2$, then $|T_s| > |T_B|$.

(c) If $0 < \xi < \frac{1}{4}$ and $1 + \alpha \leq \tau < 2$, then $\arg T_s \neq \arg T_B$.

(d) If $\xi > 0$ and $1 < \tau < 1 + \alpha$ then $\arg T_s \neq \arg T_B$. 

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In other words, for a given \(0 < \alpha < 1\), the transcendental equation \(T_s = T_B\) has a unique solution in the range \(1 < \tau < 2\) and \(\xi \in \mathbb{R}\), which occurs when \(\xi = 0\) and \(\tau = 1 + \alpha_c\). (In particular, if \(\xi \neq 0\), then the equation \(T_s = T_B\) has no solutions for \(0 < \alpha < 1\) and \(1 < \tau < 2\).)

Proof. See Lemmas 9.29 and 9.30 for the proof of cases (a) and (b) respectively. For case (c) we use Lemma 9.27 if \(0 < \alpha < \frac{1}{2}\), and Lemma 9.31 if \(\frac{1}{2} \leq \alpha < 1\). Finally, for case (d), see Lemma 9.32.

Lemma 9.25. If \(0 < \alpha < \frac{1}{2}, \xi = 0\) and \(0 < \tau < 1\), then \(T_s \neq T_B\).

Proof. Suppose that \(\alpha \in (0, \frac{1}{2})\) is fixed and \(\xi = 0\). We note that if \(\tau = 2\alpha\) then the left-hand side of equation (9.7) becomes infinite, whilst the right-hand side is finite. Moreover, if \(\alpha \leq \tau < 2\alpha\), then the left-hand side is bounded above by zero, whereas the right-hand side is strictly positive. In other words, equation (9.7) can have no solutions for \(\tau\) in the range \(\alpha \leq \tau \leq 2\alpha\).

If \(\tau \in (0, \alpha) \cup (2\alpha, 1)\), it easy to see that

\[
0 < \Gamma(2\alpha - \tau)\Gamma(\tau + 1) \sin \pi(\alpha - \tau) < \infty.
\]

Now suppose that \(0 \leq \tau < \alpha\). It is trivially obvious that \(\tau = 0\) is a (inadmissible) solution of equation (9.7). But since, from Lemma 9.5, the derivative of the left-hand side, with respect to \(\tau\), is strictly negative we see immediately that

\[
\Gamma(2\alpha - \tau)\Gamma(\tau + 1) \sin \pi(\alpha - \tau) < \Gamma(2\alpha) \sin \pi \alpha.
\]

On the other hand, suppose that \(2\alpha < \tau \leq 1\). Then, by Lemma 9.5, the left-hand side of equation (9.7) is a strictly decreasing function of \(\tau\) over this range. But if \(\tau = 1\), then

\[
\Gamma(2\alpha - \tau)\Gamma(\tau + 1) \sin \pi(\alpha - \tau) = \Gamma(2\alpha - 1)\Gamma(2) \sin \pi(\alpha - 1)
= \frac{\Gamma(2\alpha)}{1 - 2\alpha} \sin \pi \alpha
= \frac{\Gamma(2\alpha - 1)\sin \pi \alpha}{\Gamma(\alpha + 1) - \Gamma(\alpha + 1)}
= \frac{\Gamma(2\alpha - 1)}{\Gamma(\alpha + 1)} \sin \pi \alpha
\]

Hence, if \(\xi = 0\) and \(0 < \tau < 1\), then \(T_s \neq T_B\).
Given that we have proved there are no solutions to the transcendental equation for $\xi = 0$, for the case $0 < \tau < 1$, it remains to consider the case $\xi > 0$. (See Section 9.2.)

From equation (9.3) and Remark 4.20, we have

$$0 < |T_B| < \infty \text{ for all finite } \xi > 0, \text{ and } |T_B| \to 0 \text{ as } \xi \to \infty.$$  

On the other hand, from Lemma 9.14

$$0 < |T_s| < \infty \text{ for all finite } \xi > 0, \text{ and } |T_s| \to 1 \text{ as } \xi \to \infty.$$  

**Lemma 9.26.** If $0 < \alpha < \frac{1}{2}$, $\xi \geq \frac{1}{4}$ and $\alpha \leq \tau < 1$, then

$$|T_s| > \frac{2}{\pi} > |T_B|.$$  

*Proof.* Firstly, we find an upper bound for $|T_B|$. Define

$$\sigma := \tau - 2\alpha + 1,$$

so that $\sigma \geq 1 - \alpha$. Now

$$|T_B| = \frac{\sin \pi \alpha}{\pi} |B(\sigma + i\xi, 2\alpha)|$$

$$= \frac{\sin \pi \alpha}{\pi} \Gamma(2\alpha) \frac{\Gamma(\sigma + i\xi)}{\Gamma(\sigma + 2\alpha + i\xi)},$$

$$\leq \frac{\sin \pi \alpha}{\pi} \frac{\Gamma(2\alpha) \cdot \Gamma(\sigma)}{\Gamma(\sigma + 2\alpha)} \Gamma(1 - \alpha) \Gamma(1 + \alpha) \text{ by Lemma 9.10}$$

$$\leq \frac{\sin \pi \alpha}{\pi} \frac{\Gamma(2\alpha) \cdot \Gamma(1 - \alpha)}{\Gamma(1 + \alpha)} \text{ by Lemma 9.11.}$$
Since \((\sin \pi \alpha) / \pi = 1 / (\Gamma(1 - \alpha) \Gamma(\alpha))\), we have

\[
|T_B| \leq \frac{\Gamma(2\alpha)}{\Gamma(1 - \alpha) \Gamma(\alpha)} \cdot \frac{\Gamma(1 - \alpha)}{\Gamma(1 + \alpha)}
= \frac{\Gamma(2\alpha)}{\Gamma(1 + \alpha) \Gamma(\alpha)}
= \frac{2\alpha \Gamma(2\alpha)}{2 \Gamma(1 + \alpha) \Gamma(\alpha)}
= \frac{\Gamma(2\alpha + 1)}{2 (\Gamma(1 + \alpha))^2}
< \frac{\Gamma(2)}{2(\Gamma(\frac{3}{2}))^2} \quad \text{by Lemma 9.12}
= \frac{2}{\pi}, \quad \text{since } \Gamma\left(\frac{3}{2}\right) = \sqrt{\pi}/2.
\]

In other words, we can find a uniform upper bound for \(|T_B|\) by taking \(\alpha = \frac{1}{2}, \tau = \alpha\) and \(\xi = 0\).

Secondly, we determine a lower bound for \(|T_s|\). By Lemma 9.14

\[
|T_s| = \left( \frac{\cosh 2\pi \xi - \cos 2\pi (1 - \tau + \alpha)}{\cosh 2\pi \xi - \cos 2\pi (1 - \tau + 2\alpha)} \right)^{\frac{1}{2}}
\geq \left( \frac{\cosh 2\pi \xi - 1}{\cosh 2\pi \xi + 1} \right)^{\frac{1}{2}}
= \tanh(\pi \xi)
\geq \tanh(\pi/4) \quad \text{for } \xi \geq \frac{1}{4}.
\]

So finally,

\[
|T_s| \geq \tanh(\pi/4) > 0.655 > \frac{2}{\pi} > |T_B|.
\]

That is, for \(\xi \geq \frac{1}{4}\) and \(\alpha \leq \tau < 1\), we have \(|T_s| > \frac{2}{\pi} > |T_B|\), as required.

\(\square\)

**Lemma 9.27.** Let \(0 < \alpha < \frac{1}{2}\) and \(0 < \xi < \frac{1}{4}\). If \(\alpha \leq \tau < 1\) or \(1 + \alpha \leq \tau < 2\), then

\[
\arg T_B > -4\alpha \xi > \arg T_s.
\]

**Proof.** Let us define

\[
\sigma := \tau - 2\alpha + 1,
\]
so that $\sigma \geq 1 - \alpha$, if $\alpha \leq \tau < 1$ or $1 + \alpha \leq \tau < 2$. Hence, from Lemma 9.20,

$$\arg T_B > -4\alpha \xi.$$ 

We now find an upper bound for $\arg T_s$. Let $b := 1 - \tau + 2\alpha$, so that

$$T_s = \frac{\sin \pi (b - \alpha - i\xi)}{\sin \pi (b - i\xi)}.$$ 

Then, since $\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$, see, for example, 4.21.37, [34], a routine calculation gives

$$\text{Im } T_s = -\frac{\sin \pi \alpha \sinh 2\pi \xi}{\cosh 2\pi \xi - \cos 2\pi b} < 0,$$

and

$$\text{Re } T_s = -\frac{\cos \pi (\alpha - 2b) + \cos \pi \alpha \cosh 2\pi \xi}{\cosh 2\pi \xi - \cos 2\pi b}.$$ 

Since $\text{Im } T_s < 0$, we must have $-\pi < \arg T_s < 0$. As $\xi > 0$, $\cosh 2\pi \xi - \cos 2\pi b > 0$, and we can determine an upper bound for $\arg T_s$ by finding an upper bound for $-\cos \pi (\alpha - 2b) + \cos \pi \alpha \cosh 2\pi \xi$.

But $-\cos \pi (\alpha - 2b) + \cos \pi \alpha \cosh 2\pi \xi \leq 1 + \cosh 2\pi \xi$, and thus

$$\arg T_s \leq -\arctan \left( \frac{\sin \pi \alpha \sinh 2\pi \xi}{1 + \cosh 2\pi \xi} \right)$$

$$= -\arctan(\sin \pi \alpha \tanh \pi \xi)$$

$$< -4\alpha \xi \quad \text{by Lemma 9.21.}$$

So, finally

$$\arg T_B > -4\alpha \xi > \arg T_s.$$ 

Lemma 9.28. If $0 < \alpha < \frac{1}{2}$, $\xi > 0$ and $0 < \tau < \alpha$, then

$$\arg T_s \neq \arg T_B.$$ 

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Proof. We have

\[ T_B = \frac{\sin \pi \alpha \quad \Gamma(\tau + 1 - 2\alpha + i\xi) \Gamma(2\alpha)}{\Gamma(\tau + 1 + i\xi)} \, . \]

On the other hand,

\[ T_s = \frac{\sin \pi (1 - \tau + \alpha - i\xi)}{\sin \pi (1 - \tau + 2\alpha - i\xi)} = \frac{\sin \pi (\tau + 1 - \alpha + i\xi)}{\sin \pi (\tau + 1 - 2\alpha + i\xi)} \, . \]

Using the identity \( \sin \pi z = \pi / (\Gamma(z) \Gamma(1 - z)) \), (see 5.5.3, [34]),

\[ T_s = \frac{\Gamma(\tau + 1 - 2\alpha + i\xi) \Gamma(2\alpha - \tau - i\xi)}{\Gamma(\tau + 1 - \alpha + i\xi) \Gamma(\alpha - \tau - i\xi)} \, . \]

Hence, noting that \( \Gamma(z) = \Gamma(z) \), we have

\[ \arg T_B - \arg T_s = \arg \left( \frac{\Gamma(\tau + 1 - 2\alpha + i\xi)}{\Gamma(\tau + 1 - \alpha + i\xi)} \right) - \arg \left( \frac{\Gamma(\tau + 1 - \alpha + i\xi) \Gamma(2\alpha - \tau - i\xi)}{\Gamma(\tau + 1 - \alpha + i\xi) \Gamma(\alpha - \tau - i\xi)} \right) \]

\[ \equiv \arg \left( \frac{\Gamma(\tau + 1 - \alpha + i\xi)}{\Gamma(\tau + 1 - \alpha + i\xi)} \right) + \arg \left( \frac{\Gamma(\alpha - \tau - i\xi)}{\Gamma(\alpha - \tau - i\xi)} \right) \]

\[ \equiv \arg \left( \frac{\Gamma(\tau + 1 - \alpha + i\xi)}{\Gamma(\tau + 1 - \alpha + i\xi)} \right) - \arg \left( \frac{\Gamma(\alpha - \tau + i\xi)}{\Gamma(\alpha - \tau + i\xi)} \right) \quad (\arg z = - \arg \bar{z}) \]

\[ \equiv \arg B(\tau + 1 - \alpha + i\xi, \alpha) - \arg B(\alpha - \tau + i\xi, \alpha) \, . \]

In other words, \( \arg T_B - \arg T_s = T_\Delta + 2\pi k \), for some \( k \in \mathbb{Z} \), where

\[ T_\Delta := \arg B(\tau + 1 - \alpha + i\xi, \alpha) - \arg B(\alpha - \tau + i\xi, \alpha) \, . \]

To find an upper bound for \( T_\Delta \) we note that since \( 0 < \alpha < \frac{1}{2} < 1 \), from Corollary 9.18, \( -\pi/2 < \arg B(\tau + 1 - \alpha + i\xi, \alpha) \), \( \arg B(\alpha - \tau + i\xi, \alpha) < 0 \) and therefore,

\[ T_\Delta < \frac{\pi}{2} \, . \]

We now use the identity (1.625 9, p. 59, [17])

\[ \arctan(x) - \arctan(y) = \arctan \left( \frac{x - y}{1 + xy} \right) \quad \text{if} \quad xy > -1, \]
in conjunction with the result from Lemma 9.16, to compute \( \arg B(\tau + 1 - \alpha + i\xi, \alpha) \) and \( \arg B(\alpha - \tau + i\xi, \alpha) \) in turn.

Since \( 0 < \alpha < \frac{1}{2} < 1 \), from Corollary 9.18, we can write

\[
T_\Delta = \sum_{n=0}^{\infty} \left( \arctan \frac{\xi \alpha}{(\sigma_s + n)(\sigma_s + n + \alpha) + \xi^2} - \arctan \frac{\xi \alpha}{(\sigma_B + n)(\sigma_B + n + \alpha) + \xi^2} \right),
\]

where \( \sigma_s := \alpha - \tau \) and \( \sigma_B := \tau + 1 - \alpha \).

But \( 0 < \alpha - \tau < \alpha < 1 - \alpha < \tau + 1 - \alpha \) and hence

\[
0 < \sigma_s < \sigma_B,
\]

and, thus, \( T_\Delta > 0 \).

In summary, since \( \arg T_B - \arg T_s = T + 2\pi k \), with \( 0 < T_\Delta < \pi/2 \), we have

\[
\arg T_s \neq \arg T_B,
\]

as required.

In the case that \( 1 < \tau < 2 \), the transcendental equation (9.7) always has a unique root \( \tau_c \), where \( 1 < \tau_c < 1 + \alpha \). (Lemmas 9.6 and 9.7 provide the details for \( 0 < \alpha < \frac{1}{2} \) and \( \frac{1}{2} \leq \alpha < 1 \) respectively.)

**Lemma 9.29.** Suppose \( 0 < \alpha < 1 \). If \( \xi = 0 \) and \( 1 < \tau < 2 \), then \( T_s \neq T_B \), unless \( \tau = 1 + \alpha_c \).

**Proof.** From equation (9.7), we can re-write equation \( T_s = T_B \) as

\[
\Gamma(2\alpha - \tau)\Gamma(\tau + 1) \sin \pi(\alpha - \tau) = \Gamma(2\alpha) \sin \pi \alpha.
\]

We define

\[
f(\tau; \alpha) := \Gamma(2\alpha - \tau)\Gamma(\tau + 1) \sin \pi(\alpha - \tau).
\]

**Firstly, suppose** \( 0 < \alpha < \frac{1}{2} \).

If \( 1 < \tau \leq 1 + \alpha \) then, by Lemma 9.6, \( T_s \neq T_B \) unless \( \tau = 1 + \alpha_c \).

If \( 1 + \alpha < \tau < 1 + 2\alpha \) then, by a routine calculation,

\[
f(\tau; \alpha) < 0.
\]
On the other hand, $\Gamma(2\alpha) \sin \pi \alpha > 0$ and hence, $T_s \neq T_B$.

If $\tau = 1 + 2\alpha$ the the left-hand side of equation (9.7) is infinite, the right-hand side is finite and, again, the required result follows.

Finally, if $1 + 2\alpha < \tau < 2$ then $f(\tau; \alpha) > 0$. We now apply Lemma 9.5, and the required result follows if we can show that

$$f(2; \alpha) > \Gamma(2\alpha) \sin \pi \alpha.$$ 

But, for $0 < \alpha < \frac{1}{2}$,

$$f(2; \alpha) = \Gamma(2\alpha - 2) \Gamma(3) \sin (\pi - 2) \\
= \frac{2}{(2\alpha - 2)(2\alpha - 1)} \cdot \Gamma(2\alpha) \sin \pi \alpha \\
> \Gamma(2\alpha) \sin \pi \alpha.$$ 

![Figure 9.1: Graph of $f(\tau, \alpha)$ for $\alpha = 0.5$ and $0 < \tau < 2.$](image)

Now suppose $\frac{1}{2} \leq \alpha < 1$.

If $1 < \tau < 2\alpha$ or $1 + \alpha < \tau < 2$ then, by a routine calculation,

$$f(\tau; \alpha) < 0.$$ 

On the other hand, $\Gamma(2\alpha) \sin \pi \alpha > 0$ and hence, $T_s \neq T_B$. 

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If $\tau = 2\alpha$ the the left-hand side of equation (9.7) is infinite, the right-hand side is finite and, again, the required result follows.

Finally, if $2\alpha < \tau \leq 1 + \alpha$ then, by Lemma 9.7, $T_s \neq T_B$ unless $\tau = 1 + \alpha_c$. \hfill \Box

**Lemma 9.30.** Suppose $0 < \alpha < 1$. If $\xi \geq \frac{1}{4}$ and $1 + \alpha \leq \tau < 2$, then $|T_s| > |T_B|$.

*Proof.* Firstly, we find an upper bound for $|T_B|$. Define

$$\sigma := \tau - 2\alpha + 1,$$

so that $\sigma \geq 2 - \alpha > 0$. Now

$$|T_B| = \frac{\sin \pi \alpha}{\pi} |B(\sigma + i\xi, 2\alpha)|$$

$$= \frac{\sin \pi \alpha}{\pi} \Gamma(2\alpha) \left| \frac{\Gamma(\sigma + i\xi)}{\Gamma(\sigma + 2\alpha + i\xi)} \right|.$$ 

$$\leq \frac{\sin \pi \alpha}{\pi} \Gamma(2\alpha) \cdot \frac{\Gamma(\sigma)}{\Gamma(\sigma + 2\alpha)} \quad \text{by Lemma 9.10}$$

$$\leq \frac{\sin \pi \alpha}{\pi} \Gamma(2\alpha) \cdot \frac{\Gamma(2 - \alpha)}{\Gamma(2 + \alpha)} \quad \text{by Lemma 9.11.}$$

Therefore, using the identity $\sin \pi z = \pi / (\Gamma(z) \Gamma(1 - z))$, see 5.5.3, [34],

$$|T_B| \leq \frac{\Gamma(2\alpha)}{\Gamma(2 + \alpha)} \cdot \frac{\Gamma(2 - \alpha)}{\Gamma(2 + \alpha)} \cdot \frac{\Gamma(\sigma)}{\Gamma(\sigma + 2\alpha)} \cdot (\Gamma(\sigma + 1) = z\Gamma(z))$$

$$= \frac{\Gamma(2\alpha)}{\Gamma(2 + \alpha)} \cdot \frac{\Gamma(2 - \alpha)}{\Gamma(2 + \alpha)} \cdot \frac{\Gamma(\sigma)}{\Gamma(\sigma + 2\alpha)} \cdot \frac{\Gamma(\sigma + 1)}{\Gamma(\sigma + 2\alpha)}$$

$$= -\frac{\Gamma(2\alpha)}{\Gamma(\sigma) \Gamma(\sigma + 2\alpha)}.$$ 

From Lemma 9.13,

$$|T_B| \leq \lim_{\alpha \searrow 0^+} \left( -\frac{\Gamma(2\alpha)}{\Gamma(\sigma) \Gamma(\sigma + 2\alpha)} \right)$$

$$= \lim_{\alpha \searrow 0^+} \left( -\frac{\Gamma(2\alpha + 1)}{2\alpha} \cdot \frac{(\alpha - 1)\alpha}{\Gamma(\alpha + 1)\Gamma(2 + \alpha)} \right)$$

$$= \lim_{\alpha \searrow 0^+} \left( \frac{(\alpha - 1)\Gamma(2\alpha + 1)}{2\Gamma(\alpha + 1)\Gamma(2 + \alpha)} \right) = \frac{1}{2}. $
In other words, we can find a uniform upper bound for $|T_B|$ by taking $\alpha = 0$, $\tau = 1 + \alpha$ and $\xi = 0$.

Secondly, we determine a lower bound for $|T_s|$. As in the proof of Lemma 9.26,

$$|T_s| \geq \tanh(\pi/4) \quad \text{for} \quad \xi \geq \frac{1}{4}. \tag{1}$$

So finally,

$$|T_s| \geq \tanh(\pi/4) > 0.655 > 0.5 \geq |T_B|. \tag{2}$$

That is, for $\xi \geq \frac{1}{4}$ and $1 + \alpha \leq \tau < 2$, we have $|T_s| > |T_B|$, as required.

**Lemma 9.31.** Suppose $\frac{1}{2} \leq \alpha < 1$. If $0 < \xi < \frac{1}{4}$ and $1 + \alpha \leq \tau < 2$, then

$$-\pi < \arg T_s \leq -\pi/2 < \arg T_B < 0. \tag{3}$$

**Proof.** Let $\sigma := \tau + 1 - 2\alpha$. Then

$$\sigma \geq (1 + \alpha) + 1 - 2\alpha = 2 - \alpha > \frac{1}{4} > \xi. \tag{4}$$

Hence, from Remark 9.19 with $\gamma = 2\alpha \geq 1$,

$$-\pi/2 < \arg B(\sigma + i\xi, 2\alpha) < 0. \tag{5}$$

Therefore, $-\pi/2 < \arg T_B < 0. \tag{6}$

We now determine bounds for $\arg T_s$. Let $b := 1 - \tau + 2\alpha$. Then

$$T_s = \frac{\sin \pi(b - \alpha - i\xi)}{\sin \pi(b - i\xi)}. \tag{7}$$

Then, as in the proof of Lemma 9.27, a routine calculation gives

$$\text{Im} T_s = -\frac{\sin \pi\alpha \sinh 2\pi\xi}{\cosh 2\pi\xi - \cosh 2\pi b} < 0, \tag{8}$$

and

$$\text{Re} T_s = \frac{(\cos \pi\alpha - \cos \pi(\alpha - 2b)) + (\cosh 2\pi\xi - 1) \cos \pi\alpha}{\cosh 2\pi\xi - \cosh 2\pi b}. \tag{9}$$

Of course, for $\xi > 0$ we have $\cosh 2\pi\xi - 1 > 0$. Moreover, if $\frac{1}{2} \leq \alpha < 1$ then $\cos \pi\alpha \leq 0$. Hence, $\text{Re} T_s \leq 0$ if $\cos \pi\alpha - \cos \pi(\alpha - 2b) \leq 0$, which follows directly from Lemma 9.22.

So, finally

$$-\pi < \arg T_s \leq -\pi/2, \tag{10}$$

which completes the proof of the lemma.  \[ \square \]
Lemma 9.32. If $0 < \alpha < 1$, $\xi > 0$ and $1 < \tau < 1 + \alpha$, then

\[
\arg T_s \neq \arg T_B.
\]

Proof. We follow the method taken in Lemma 9.28, and repeat the result

\[
\arg T_B - \arg T_s = \arg \left( \frac{\Gamma(\tau + 1 - \alpha + i\xi)}{\Gamma(\tau + 1 + i\xi)} \right) - \arg \left( \frac{\Gamma(\alpha - \tau + i\xi)}{\Gamma(2\alpha - \tau + i\xi)} \right),
\]

but now $\alpha - \tau < 0$. However, we can write

\[
\arg \left( \frac{\Gamma(\alpha - \tau + i\xi)}{\Gamma(2\alpha - \tau + i\xi)} \right) = \arg \left( \frac{(\alpha - \tau + i\xi)\Gamma(\alpha - \tau + i\xi)}{(2\alpha - \tau + i\xi)\Gamma(2\alpha - \tau + i\xi)} \cdot \frac{(2\alpha - \tau + i\xi)}{(\alpha - \tau + i\xi)} \right)
\equiv \arg \left( \frac{\Gamma(1 + \alpha - \tau + i\xi)}{\Gamma(1 + 2\alpha - \tau + i\xi)} \right) + \arg \left( \frac{2\alpha - \tau + i\xi}{\alpha - \tau + i\xi} \right).
\]

Therefore,

\[
\arg T_B - \arg T_s \equiv \left( \arg B(\tau + 1 - \alpha + i\xi, \alpha) - \arg B(1 + \alpha - \tau + i\xi, \alpha) \right)
- \arg \left( \frac{2\alpha - \tau + i\xi}{\alpha - \tau + i\xi} \right).
\]

Noting that $0 < 1 + \alpha - \tau < \tau + 1 - \alpha$, and using the approach of Lemma 9.28,

\[
\arg B(\tau + 1 - \alpha + i\xi, \alpha) - \arg B(1 + \alpha - \tau + i\xi, \alpha) = T_\Delta,
\]

where $0 < T_\Delta < \pi/2$.

On the other hand, by a routine calculation,

\[
\text{Im} \left( \frac{2\alpha - \tau + i\xi}{\alpha - \tau + i\xi} \right) = \frac{-\alpha \xi}{(\alpha - \tau)^2 + \xi^2} < 0,
\]

and thus

\[
0 < -\arg \left( \frac{2\alpha - \tau + i\xi}{\alpha - \tau + i\xi} \right) < \pi.
\]

Therefore,

\[
\arg T_s \neq \arg T_B.
\]

This completes the proof of the lemma.
Chapter 10
Future research

The backdrop for this research is the goal of developing a theory of boundary value problems for operators which are sums of pseudodifferential operators and “fine-tuned potentials”, which are less singular than the original (unperturbed) pseudodifferential operators. As we have seen, the generators of Lévy processes in domains are a rich source of interesting models. Indeed, in this thesis, we have taken a first step by studying, in detail, a particular one-dimensional operator on the half-line. In passing, we note again the usefulness of Mellin operators in this work.

Even in one spatial dimensional, there are further significant opportunities for research. These include both consideration of a wider class of elliptic operators and the analysis of the related problem on a bounded domain. It is worth remarking that even one-dimensional models have important applications in various fields, including non-Gaussian market models in financial mathematics.

The second major future phase of this research is to extend the work done in one dimension to the $n$-dimensional case, where the region of interest becomes a half-space. In addition, it will be interesting to consider the natural generalisation to systems of equations.

In summary, this thesis is simply a (promising) beginning... Many useful techniques have been developed which, no doubt, will be reusable in a wider context. However, there remains much to do, and many challenges lie ahead!
Appendix A

Simple example

We define the homogeneous fractional Laplacian $(-\Delta)^\alpha$ by

$$(-\Delta)^\alpha := \mathcal{F}^{-1}|\xi|^{2\alpha}\mathcal{F}, \quad \text{where } 0 < \alpha < 1.$$ 

Our goal in this appendix is to consider $(-\Delta)^\alpha$, for the range $\frac{1}{2} < \alpha < 1$, acting in one spatial dimension, in some detail. Indeed, we will critically examine both the difficulties caused by truncation, see Section 1.2.3, and, on the other hand, the improvements offered by adding a potential term, as described in Section 1.2.9.

Let $u \in C_0^\infty(\mathbb{R})$, so that $\mathcal{F}u \in S(\mathbb{R})$. Since $2\alpha - 2 > -1$, we have $(|\xi|^{2\alpha}\mathcal{F}u)'' \in L_1(\mathbb{R})$, and hence $\mathcal{F}^{-1}|\xi|^{2\alpha}\mathcal{F}u$ is continuous and $O(|x|^{-2})$ as $|x| \to \infty$. Therefore,

$$(-\Delta)^\alpha : C_0^\infty(\mathbb{R}) \to L_1(\mathbb{R}) \cap C^\infty(\mathbb{R}),$$

since $(-\Delta)^\alpha : C_0^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$. (See, for example, Theorem 3.1, p. 47, [39].)

Let $G = \mathbb{R}_+$. Our first goal is to construct $u \in r_+C_0^\infty(\mathbb{R})$ such that $r_+(-\Delta)^\alpha e_+u \notin L_{1,\text{loc}}(\mathbb{R}_+)$. Indeed, let $U \in C_0^\infty(\mathbb{R})$ be such that

$$U = 1 \text{ if } |x| \leq 1 \text{ and } U = 0 \text{ if } |x| \geq 2,$$

and set

$$u = r_+U.$$

Since $|\xi|^{2\alpha} = |\xi|^{2(\alpha-1)}\xi$, we have

$$r_+(-\Delta)^\alpha e_+u = r_+\mathcal{F}^{-1}|\xi|^{2\alpha}\mathcal{F}(e_+u) = -r_+\frac{d}{dx}\mathcal{F}^{-1}|\xi|^{2(\alpha-1)}\mathcal{F}(e_+u)'.'

But, see Lemma 4.1,

$$(e_+u)' = u(0)\delta + e_+u' = \delta + e_+u'.$$

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Of course, \( e_+ u' \in C_0^\infty(\mathbb{R}) \), and thus \( \mathcal{F}(e_+ u') \in S(\mathbb{R}) \). Hence, \( P(\xi)|\xi|^{2\alpha - 2} \mathcal{F}(e_+ u') \in L_1(\mathbb{R}) \), for any polynomial \( P \). Therefore, \( \mathcal{F}^{-1} P(\xi)|\xi|^{2\alpha - 2} \mathcal{F}(e_+ u') \) is continuous and vanishes at infinity. So,

\[
 w(x) := -r_+ \frac{d}{dx} \mathcal{F}^{-1}|\xi|^{2\alpha - 2} \mathcal{F}(e_+ u'),
\]

and all its derivatives, are continuous and vanish at infinity.

It remains to consider \(-r_+ \frac{d}{dx} \mathcal{F}^{-1}|\xi|^{2(\alpha - 1)} \mathcal{F} \delta\).

Now \( \mathcal{F} \delta = 1/\sqrt{2\pi} \), see Appendix E, and from Section 17.23, p. 1119, [17] or Example 3.1, equation (3.14), p. 38, [14],

\[
 \mathcal{F}(|x|^{-\alpha}) = \frac{\sqrt{\frac{\alpha}{\pi}}}{\Gamma(1 - \alpha)} \sin\left(\frac{\pi a}{2}\right) |\xi|^{\alpha - 1}, \quad 0 < \alpha < 1. \quad (A.1)
\]

Applying the inverse Fourier transform, and taking \( \alpha = 2\alpha - 1 \),

\[
 \mathcal{F}^{-1}|\xi|^{2(\alpha - 1)} \mathcal{F} \delta = \frac{\sqrt{\frac{\alpha}{\pi}}}{\Gamma(2 - 2\alpha)} \frac{|x|^{1 - 2\alpha}}{\sin\left(\frac{\pi (2\alpha - 1)}{2}\right)} \cdot \frac{1}{\sqrt{2\pi}}
\]

\[
 = \frac{\frac{1}{2}}{\Gamma(2 - 2\alpha)} \frac{|x|^{1 - 2\alpha}}{\pi} \Gamma(\alpha - \frac{1}{2}) \Gamma(\frac{3}{2} - \frac{1}{2})
\]

\[
 = \frac{\frac{1}{2}}{\sqrt{2\pi}} 2^{2-2\alpha-\frac{1}{2}} \Gamma(1 - \alpha) \Gamma\left(\frac{3}{2} - \alpha\right) \pi
\]

\[
 = -\frac{2^{2\alpha - 1}}{\pi^\frac{3}{2} \Gamma(1 - \alpha)} \frac{|x|^{1 - 2\alpha}}{1 - 2\alpha}
\]

since \((\alpha - \frac{1}{2})\Gamma(\alpha - \frac{1}{2}) = \Gamma(\alpha + \frac{1}{2})\), see 5.5.1, [34].

Hence

\[
 r_+(-\Delta)^\alpha e_+ u = \frac{2^{2\alpha - 1}}{\pi^\frac{3}{2} \Gamma(1 - \alpha)} \frac{\Gamma\left(\frac{1}{2} + \alpha\right)}{1 - 2\alpha} x^{-2\alpha} + w. \quad (A.2)
\]

But since \( \frac{1}{2} < \alpha < 1 \), we have \( r_+(-\Delta)^\alpha e_+ u \notin L_{1,loc}(\mathbb{R}_+) \), as required.

As before, we assume \( G = \mathbb{R}_+ \). Our second goal is to show that the leading singular terms of \( r_+(-\Delta)^\alpha e_+ u \) and \( \kappa_G^\alpha u \), near \( x = 0 \), cancel each other, and thus, see equation (1.28),

\[
 (-\Delta)_G^\alpha u = r_+(-\Delta)^\alpha e_+ u + \kappa_G^\alpha u,
\]

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is less singular than either of \( r_+(-\Delta)^\alpha e_+u \) and \( \kappa_G^\alpha u \), when considered separately.

Now, from equation (1.27), for \( x > 0 \),

\[
\kappa_G^\alpha(x) = -c_{1,\alpha} \int_{-\infty}^{0} \frac{1}{|x-y|^{1+2\alpha}} \, dy
- c_{1,\alpha} \int_{0}^{\infty} \frac{1}{(x+\tau)^{1+2\alpha}} \, d\tau
= -c_{1,\alpha} \left[ \frac{(x+\tau)^{-2\alpha}}{(-2\alpha)} \right]_0^\infty
= -c_{1,\alpha} \frac{x^{-2\alpha}}{2\alpha}.
\]

But, see for example [31], the positive constant \( c_{n,\alpha} \) is given by

\[
c_{n,\alpha} = -\frac{2^{2\alpha} \Gamma(\alpha + \frac{n}{2})}{\pi^{\frac{n}{2}} \Gamma(-\alpha)}.
\]

Therefore, noting that \((-\alpha)\Gamma(-\alpha) = \Gamma(1-\alpha)\), for \( x > 0 \),

\[
\kappa_G^\alpha(x) = -\frac{2^{2\alpha-1} \Gamma(\alpha + \frac{1}{2})}{\pi^{\frac{3}{2}} \Gamma(1-\alpha)} x^{-2\alpha}.
\] (A.3)

So, comparing equations (A.2) and (A.3), we see that the leading singular terms of \( r_+(-\Delta)^\alpha e_+u \) and \( \kappa_G^\alpha u \), near \( x = 0 \), cancel each other, as required.
Appendix B

Feller and Lévy processes

Useful background on the material in this appendix can be found in [27, 42, 43]. Our starting point is a probability space. Let Ω be a non-empty set, and let A denote a σ-field on Ω. Further, suppose that P is a probability measure defined on A. Then the triple (Ω, A, P) is called a probability space.

Suppose G ⊂ R^n is a Borel set. Then we can define a stochastic process by the quadruple (Ω, A, P, (X_t)_{t≥0}), where X_t : Ω → G, t ≥ 0 is a random variable. We call G the state space. The mappings t → X_t(ω), for ω ∈ Ω, are called the paths of the process. We will be interested in families of stochastic processes indexed by the state space, sometimes called universal processes. More precisely, we will consider for each x ∈ G, the stochastic process (Ω, A, P^x, (X_t)_{t≥0})_{x∈G}, where P^x{X_0 = x} = 1.

Let B_b(G) denote the set of bounded Borel functions on G. Then, given a universal process, we can define a family (T_t)_{t≥0} of operators acting on B_b(G) by

(T_t u)(x) = E^x(u(X_t)).

In particular, given a Borel set A ⊂ G, we can define the transition function p_t(x, A), for x ∈ G, to be

p_t(x, A) := (T_t χ_A)(x) = E^x(χ_A(X_t)) = P^x{X_t ∈ A}.

Intuitively, the transition function p_t(x, A) is the probability of being in the set A at time t, starting at time 0 from a point x ∈ G.

Let B(G) denote the Borel σ-field over G. Then it is easy to see that, for fixed t ≥ 0, the mapping A → p_t(x, A) is a probability measure on B(G).
Hence, the operator $T_t$, $t \geq 0$ can be represented as
\[(T_t u)(x) = \int_G u(y)p_t(x, dy).\]

Similarly, let us define
\[p_{s,t}(x, A) = \int_G p_s(y, A)p_t(x, dy).\]

We call the family $(p_t(x, A))_{t \geq 0, x \in G}$ a semigroup of Markovian kernels if for all $s, t \geq 0, x \in G$, and any Borel set $A \subset G$ we have
\[p_{s,t}(x, A) = p_{s+t}(x, A).\]

In other words, the Chapman-Kolmogorov equations
\[p_{s+t}(x, A) = \int_G p_s(y, A)p_t(x, dy) \quad (B.1)\]
hold. In this case, it follows that $(T_t)_{t \geq 0}$ is a semigroup of linear operators on $B_b(G)$ with
\[T_{s+t} = T_s \circ T_t\]
valid for all $s, t \geq 0$. Since $p_0(\cdot, \{x\}) = \delta_x$, we always have $T_0$ is the identity map. A universal process is called a Markov Process when its transition function satisfies equation (B.1).

Suppose we have a Markov process $((X_t)_{t \geq 0}, P^x)_{x \in G}$, and the associated semigroup of linear operators $T_t : B_b(G) \to B_b(G)$. The it is easy to show that the operator $T_t$ is a contraction on $B_b(G)$. It is also useful to consider the action of $T_t$ on the Banach space $C_\infty(G)$, equipped with the supremum norm, consisting of all continuous functions on $G$ that vanish at infinity. We say that a semigroup $(T_t)_{t \geq 0}$ of linear operators on $C_\infty(G)$ is a Feller semigroup if it meets the following three conditions:

(i) $T_t : C_\infty(G) \to C_\infty(G)$ is a linear contraction;

(ii) $\lim_{t \to 0} \|T_t u - u\|_\infty = 0$, i.e. the semigroup is strongly continuous;

(iii) $0 \leq u \leq 1$ implies $0 \leq T_t u \leq 1$.

A Markov process is called a Feller process when its corresponding semigroup is a Feller semigroup.
Let $X = (X_t)_{t \geq 0}$ be a stochastic process defined on a probability space $(\Omega, \mathcal{A}, P)$. We say that $X$ has independent increments if for each $n \in \mathbb{N}$, and $0 \leq t_1 < t_2 < \cdots < t_{n+1} < \infty$, the random variables $\{X_{t_{j+1}} - X_{t_j}\}_{j=1}^n$ are independent. Moreover, we say that $X$ has stationary increments if, $X_t - X_s$ is equal in distribution to $X_{t-s}$ for $t > s$.

A Lévy process is a Feller process with independent and stationary increments which is continuous in probability. That is, for all $a > 0$ and all $s \geq 0$,

$$\lim_{t \to s} P(|X_t - X_s| > a) = 0.$$ 

It is sometimes helpful to think of a Lévy process as a continuous analogue of a random walk [47]. The most well known examples of Lévy processes are Brownian motion and the Poisson process. Lévy processes whose paths are almost surely non-decreasing are called subordinators. See, for example, [32].

The generator of the symmetric $\alpha$-stable subordinator is the fractional Laplacian, which may be defined as

$$(-\Delta)^\alpha := \mathcal{F}^{-1} |\xi|^{2\alpha} \mathcal{F}, \quad 0 < \alpha < 1.$$ 

For more details, see Example 5.8, p. 96 [18].

Similarly, see Example 5.9, p. 97, [18], the generator of the (killed) relativistic $\alpha$-stable subordinator is the pseudodifferential operator

$$\mathcal{F}^{-1} (1 + |\xi|^2)^\alpha \mathcal{F}, \quad 0 < \alpha < 1.$$ 

This class of Lévy processes is also discussed in Remark 2.4.
Appendix C

Hörmander space

In Section 1.2.4, we briefly reviewed the approach taken in [22, 23], using the \( \mu \)-transmission condition and Hörmander spaces. The goal of this appendix is to consider a particular example to explore these ideas in more detail.

The following material is taken from a presentation entitled “Boundary problems for fractional Laplacians and other fractional-order operators”, given by Gerd Grubb in March 2016. The main changes shown here are due to notational differences, differing Fourier transform conventions and, where appropriate, further explanation.

We suppose that \( 0 < \alpha < 1 \), and let \( A \) be a pseudo-differential operator of fractional order. In general, given an open set \( \Omega \subset \mathbb{R}^n, \ n \geq 2, \) with a suitably smooth boundary, we are interested in the Dirichlet problem

\[
Au = f \quad \text{in } \Omega, \quad \text{supp } u \subset \overline{\Omega},
\]

where \( f \in H^t(\Omega) \) is given. The novel aspect here, in our terms, is that the solution \( u \), if it exists, is sought in the so-called Hörmander space \( H^{\alpha(t+2\alpha)}(\mathbb{R}^n_+) \).

(See Section 1.2.4 for the definition of \( H^{\alpha(t+2\alpha)}(\mathbb{R}^n_+), \ t \geq 0. \))

**Theorem C.1.** Let \( A = (I - \Delta)^\alpha \) on \( \Omega = \mathbb{R}^n_+ \) and suppose \( t \geq 0 \). Then the Dirichlet problem (C.1) has a unique solution \( u \in H^{\alpha(t+2\alpha)}(\mathbb{R}^n_+) \).

**Proof.** The operator \( A = (I - \Delta)^\alpha \) has symbol \( (1 + |\xi|^2)^\alpha = (\xi^2)^\alpha \). We have the factorisation

\[
(1 + |\xi|^2)^\alpha = (\langle \xi' \rangle + i\xi_n)^\alpha (\langle \xi' \rangle - i\xi_n)^\alpha.
\]
where \( \xi = (\xi', \xi_n) \) and \( \xi' = (\xi_1, \ldots, \xi_{n-1}) \).

We set \( \mu_t^\pm := (\langle \xi' \rangle \mp i \xi_n)^t \). (The reason for this counter-intuitive definition will become clear later. See (C.2).) Then we define \( \hat{\Lambda}_\pm^t := F^{-1}(\mu_t^\pm)F \), and we can write
\[
(I - \Delta)^\alpha = \hat{\Lambda}_\pm^a \hat{\Lambda}_\pm^a.
\]

Since \( (\langle \xi' \rangle - i(\xi_n + i\tau))^t = (\langle \xi' \rangle + \tau - i\xi_n)^t \) is analytic for \( \tau > 0 \), from Theorem 1.9, p. 52, [41], for any \( s, t \in \mathbb{R} \),
\[
\hat{\Lambda}_+^t : \tilde{H}^s(\mathbb{R}_+^n) \to \tilde{H}^{s-t}(\mathbb{R}_+^n),
\]
is bounded and, moreover, has inverse \( \hat{\Lambda}_-^t \). In particular, we note that \( \hat{\Lambda}_+^t \) preserves support in \( \mathbb{R}_+^n \).

Similarly, as \( (\langle \xi' \rangle + i(\xi_n - i\tau))^t = (\langle \xi' \rangle + \tau + i\xi_n)^t \) is analytic for \( \tau > 0 \), from Theorem 1.10, p. 53, [41], for any \( s, t \in \mathbb{R} \),
\[
r_+ \hat{\Lambda}_-^t l_+ : H^s(\mathbb{R}_+^n) \to H^{s-t}(\mathbb{R}_+^n),
\]
is bounded, where \( l_+ \) is the extension operator. In particular, we note from Remark 1.11, p. 53, [41], that \( (r_+ \hat{\Lambda}_-^t l_+) r_+ = r_+ \hat{\Lambda}_+^t \).

The model Dirichlet problem is
\[
r_+ (I - \Delta)^\alpha u = f \quad \text{on} \quad \mathbb{R}_+^n, \quad \text{supp} \, u \subset \mathbb{R}_+^n,
\]
where, by hypothesis, \( f \in H^t(\mathbb{R}_+^n) \) for some \( t \geq 0 \), and we seek a solution \( u \in H^{\alpha(t + 2\alpha)}(\mathbb{R}_+^n) \). Now
\[
r_+ (I - \Delta)^\alpha u = r_+ \hat{\Lambda}_-^a \hat{\Lambda}_+^a u = r_+ \hat{\Lambda}_-^a l_+ r_+ \hat{\Lambda}_+^a u.
\]

But, it is easy to see from Remark 1.11, p. 53, [41], that \( (r_+ \hat{\Lambda}_-^t l_+)^{-1} = r_+ \hat{\Lambda}_-^{-t} l_+ \) and thus (C.4) can be reduced to
\[
r_+ \hat{\Lambda}_+^a u = g \quad \text{on} \quad \mathbb{R}_+^n, \quad \text{supp} \, u \subset \mathbb{R}_+^n,
\]
where
\[
g := r_+ \hat{\Lambda}_-^{-a} l_+ f \in H^{t+\alpha}(\mathbb{R}_+^n).
\]

We note that, by Theorem 1.10, p. 53, [41], \( g \) is independent of the choice of the extension \( l_+ \).
Now suppose that (C.5) has two solutions $u_1$ and $u_2$ with $\text{supp } u_1, \text{supp } u_2 \subset \mathbb{R}^n_+$. Let $v = u_1 - u_2$. Then

$$r_+ \hat{\Lambda}^\alpha_+ v = 0 \text{ on } \mathbb{R}^n_+, \text{ and } \text{supp } v \subset \mathbb{R}^n_+.$$ 

Hence, see (C.2), $\hat{\Lambda}^\alpha_+ v = 0$ on $\mathbb{R}^n$, and thus $v = 0$. In other words, if a solution to (C.5) does exist, then it is unique.

But now it is easy to see, by direct substitution, that (C.5) has the solution $u = \hat{\Lambda}^{-\alpha}_+ e_+ g$.

Thus, (C.4) has a unique solution $u$, and it lies in

$$H^{\alpha(t+2\alpha)}(\mathbb{R}^n_+) := \hat{\Lambda}^{-\alpha}_+ (e_+ H^{t+\alpha}(\mathbb{R}^n_+)),$$ (C.6)

which is known as Hörmander’s space. In particular, we note that if $t + \alpha > \frac{1}{2}$ then functions in the space $e_+ H^{t+\alpha}(\mathbb{R}^n_+)$ may have a jump at $x_n = 0$. This gives rise to a singularity when the operator $\hat{\Lambda}^{-\alpha}_+$ is applied.

\[\square\]

**Remark C.2.** The following relationships, see [22, 23], provide a useful characterisation of $H^{\alpha(t+2\alpha)}(\mathbb{R}^n_+)$:

$$H^{\alpha(t+2\alpha)}(\mathbb{R}^n_+) \begin{cases} = \tilde{H}^{t+2\alpha}(\mathbb{R}^n_+) & \text{if } -\frac{1}{2} < t + \alpha < \frac{1}{2}; \\ \subset e_+ x_n^{\alpha} H^{t+\alpha}(\mathbb{R}^n_+) + \tilde{H}^{t+2\alpha-\epsilon}(\mathbb{R}^n_+) & \text{if } t + \alpha > \frac{1}{2}, \end{cases}$$

where the term $-\epsilon$ only applies if $t + \alpha - \frac{1}{2} \in \mathbb{N}$.
Appendix D

Fractional Calculus

The goal of this appendix is summarise some useful components of the Fractional Calculus. The authoritative text on this subject is Samko et al., [40]. However, given that this book is out of print, supplementary technical references are taken from a more recent work by Diethelm [9].

Suppose $n \in \mathbb{N}$. Then, from equation (2.16), p. 33, [40], we have the following formula for the $n$-fold integral

$$\int_a^x \int_a^{\sigma_1} \cdots \int_a^{\sigma_n-1} f(\sigma_n) d\sigma_n \cdots d\sigma_1 = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t) dt,$$

where, of course, $(n-1)! = \Gamma(n)$. This provides the motivation for the following definition.

Let $-\infty \leq a < x < b \leq \infty$. The Riemann-Liouville fractional integral of order $\alpha > 0$ with lower limit $a$ is defined for locally integrable functions $f : [a, b] \to \mathbb{R}$ as

$$(I_a^\alpha f)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(y)}{(x-y)^{1-\alpha}} dy, \quad x > a. \quad \text{(D.1)}$$

See Definition 2.1, equation (2.17), p. 33, [40]. (Also, Definition 2.1, p. 13, [9].)

In particular, as expected, we have

$$(I_a^1 f)(x) = \int_a^x f(y) dy \quad \text{and} \quad (I_a^2 f)(x) = \int_a^x \left( \int_a^y f(t) dt \right) dy. \quad \text{(D.2)}$$

Moreover, fractional integration has the property that

$$I_a^\alpha I_a^\beta f = I_a^{\alpha+\beta} f, \quad \text{for } \alpha, \beta > 0, \quad \text{(D.3)}$$
see equation (2.21), p. 34, [40]. (See also Corollary 2.3, p. 14, [9].)

Suppose $\alpha > 0$ and $n = [\alpha] + 1$. We now define the Caputo fractional derivative of order $\alpha$ with lower limit $a$ as

$$(C^\alpha_a f)(x) := (I^{n-\alpha}_{a+} f^{(n)})(x), \quad (D.4)$$

for sufficiently smooth functions $f$. (See Definition 3.1, p. 49, [9].) In the special case that $\alpha = 1$, we have

$$(C^1_{a+} f)(x) = I^1_{a+} f''(x) = \int_a^x f''(t) \, dt = f'(x) - f'(a). \quad (D.5)$$

Suppose $0 < \alpha < 1$. Then

$$I^\alpha_a C^\alpha_a f = I^1_{a+} (I^{1-\alpha}_{a+} f') = I^1_{a+} f' = \int_a^x f'(t) \, dt = f(x) - f(a).$$

Similarly, if $1 \leq \alpha < 2$, then

$$I^\alpha_a C^\alpha_a f = I^2_{a+} f'' = \int_a^x \left( \int_a^y f''(t) \, dt \right) \, dy = f(x) - f(a) - (x-a)f'(a).$$

In summary,

$$f(x) - f(a) = I^\alpha_a C^\alpha_a f(x) \quad (0 < \alpha < 1), \quad (D.6)$$

and

$$f(x) - f(a) - f'(a)(x-a) = I^\alpha_a C^\alpha_a f(x) \quad (1 \leq \alpha < 2). \quad (D.7)$$

Finally, let $I^\alpha_a := I^\alpha_{(-\infty)+}$. Then, for $0 < \alpha < 1$,

$$\mathcal{F}(I^\alpha_{a+} f) = (\mathcal{F}f)(\xi) \frac{(-i\xi)^\alpha}{(\xi^2 + 1)^\alpha}, \quad (D.8)$$

as given in equation (7.1), p. 137, [40].
Appendix E

Fourier transform results

As previously, we define the *Fourier Transform* on the Schwartz space, $S(\mathbb{R})$, of rapidly decaying infinitely differentiable functions $\varphi$ by

$$(F\varphi) (\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^n} e^{i\xi \cdot x} \varphi(x) \, dx, \quad \xi \in \mathbb{R}.$$  

Let $S'(\mathbb{R})$ denote the corresponding space of tempered distributions. Then, see equation (2.28), p. 22, [14], the Fourier transform of $f \in S'(\mathbb{R})$ is the tempered distribution $\hat{f} \in S'(\mathbb{R})$, such that

$$\langle \hat{f}, \hat{\varphi} \rangle = \langle f, \varphi \rangle \quad \text{for all } \varphi \in S(\mathbb{R}).$$

As [14], for $f \in S'(\mathbb{R})$, we adopt the convention that $Ff := \hat{f}$.

From 17.23, p. 1118, [17], we have

$$F(1) = \sqrt{2\pi} \delta(\xi) \quad \text{(E.1)}$$

$$F(\delta(x)) = 1/\sqrt{2\pi} \quad \text{(E.2)}$$

$$F(\chi_{\mathbb{R}^+}(x)) = \frac{i}{\sqrt{2\pi}} \frac{1}{\xi} + \sqrt{\frac{\pi}{2}} \delta(\xi). \quad \text{(E.3)}$$

Hence, given the identities $\chi_{\mathbb{R}^-} = 1 - \chi_{\mathbb{R}^+}$ and $\text{sgn} = 2\chi_{\mathbb{R}^+} - 1$, we can easily deduce

$$F(\chi_{\mathbb{R}^-}(x)) = -\frac{i}{\sqrt{2\pi}} \frac{1}{\xi} + \sqrt{\frac{\pi}{2}} \delta(\xi) \quad \text{(E.4)}$$

$$F(\text{sgn}(x)) = i\sqrt{\frac{2}{\pi}} \frac{1}{\xi}. \quad \text{(E.5)}$$
Appendix F

Technical lemma

Suppose $0 < \mu < 1$. It will be convenient to define
\[ w_{-\mu}(x) := \mathcal{F}^{-1}(-i\xi)^{-\mu}(\xi - i)^{-1}, \quad x \in \mathbb{R}. \] (F.1)

Further, let $\chi \in C^\infty_0(\mathbb{R})$ be such that
\[ \chi(t) := \begin{cases} 1 & \text{if } |t| \leq 2 \\ 0 & \text{if } |t| > 3. \end{cases} \]

Further take $\chi_1, \chi_2 \in C^\infty_0(\mathbb{R})$ such that $\chi_1\chi = \chi$, $\chi_2\chi_1 = \chi_1$. (That is, $\chi_1 = 1$ on supp $\chi$, and $\chi_2 = 1$ on supp $\chi_1$.)

Given these definitions, the key result of this Appendix is:

**Lemma F.1.** Suppose $0 < \mu < 1$, $1 < p < \infty$ and $r < \mu + 1/p$. Then
\[ (D - i)^r \chi_{w_{-\mu}} \in L_p(\mathbb{R}). \]

**Proof.** We have
\[ (D - i)^r \chi_{w_{-\mu}} = (1 - \chi_1)(D - i)^r \chi_{w_{-\mu}} + \chi_1(D - i)^r \chi_{w_{-\mu}} \]
\[ = (1 - \chi_1)(D - i)^r \chi_{w_{-\mu}} + \chi_1(D - i)^r (\chi - \chi_2)w_{-\mu} \]
\[ + \chi_1(D - i)^r (\chi_2 - 1)w_{-\mu} + \chi_1(D - i)^r w_{-\mu}. \]

From Lemma F.6,
\[ (1 - \chi_1)(D - i)^r \chi_{w_{-\mu}} \text{ and } \chi_1(D - i)^r (\chi_2 - 1)w_{-\mu} \in L_p(\mathbb{R}). \]

Moreover, from Lemma F.8,
\[ \chi_1(D - i)^r (\chi - \chi_2)w_{-\mu} \in L_p(\mathbb{R}). \]
But, from Lemma 5.7,
\[ \chi_1(D - i)^r w_{-\mu} \in L_p(\mathbb{R}), \]
and the required result follows immediately.

Our first task is to prove Lemma F.6. We begin with a definition.

**Definition F.2.** Let \( m \in \mathbb{R} \). Then, we say that \( a \in S^m \), if \( a = a(\xi) \) is smooth on \( \mathbb{R} \) and if
\[ |\partial^\beta a(\xi)| \leq C_\beta (1 + |\xi|)^{m-\beta} \]
for all \( \beta \in \mathbb{N} \cup \{0\} \), \( \xi \in \mathbb{R} \), for certain constants \( C_\beta \), that only depend on \( \beta \).

**Remark F.3.** Let \( A \) be a pseudodifferential operator with symbol \( a \in S^m \), for some \( m \in \mathbb{R} \). Then, for \( u \in S(\mathbb{R}) \),
\[ Au = \mathcal{F}^{-1} a \mathcal{F} u = \mathcal{F}^{-1} (\mathcal{F}(\mathcal{F}^{-1} a) \mathcal{F} u) = k \ast u \]
where
\[ k := \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1} a. \]

**Lemma F.4.** Suppose \( a \in S^m \) for some \( m \in \mathbb{R} \). Then the kernel, \( k(z) \), satisfies
\[ |\partial^\beta k(z)| \leq \text{const} \ |z|^{-N} \]
for \( N > m + 1 + \beta \) and \( z \neq 0 \). Thus, for \( z \neq 0 \), the kernel \( k(z) \) is a smooth function which is rapidly decaying as \( |z| \to \infty \).

**Proof.** Let \( N \in \mathbb{N} \cup \{0\} \). Then, from (F.2),
\[ (iz)^N \partial^\beta k(z) = \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1} \partial^N [(-i\xi)^\beta a(\xi)]. \]
Since \( (-i\xi)^\beta a(\xi) \in S^{m+\beta} \), we have the upper bound
\[ |\partial^N [(-i\xi)^\beta a(\xi)]| \leq C_{N,\beta} (1 + |\xi|)^{m+\beta-N}. \]
and hence, \( \partial^N [(-i\xi)^\beta a(\xi)] \in L_1(\mathbb{R}) \), provided \( N > m + 1 + \beta \).

Therefore, its inverse Fourier transform is bounded. In other words,
\[ (iz)^N \partial^\beta k(z) \in L_\infty(\mathbb{R}) \]
for \( N > m + 1 + \beta \).
**Remark F.5.** For \( u \in S(\mathbb{R}) \),

\[
((1 - \chi_1)A\chi u)(x) = \int_{\mathbb{R}} (1 - \chi_1(x))\chi(y) k(x-y)u(y) \, dy.
\]

By the definition of \( \chi \) and \( \chi_1 \), if \( x = y \) then \( (1 - \chi_1(x))\chi(y) k(x-y) = 0 \). Therefore, from Lemma F.4, the integral kernel \( (1 - \chi_1(x))\chi(y) k(x-y) \) is smooth, bounded and rapidly decaying as \( |x-y| \to \infty \).

Similarly,

\[
(\chi_1A(\chi_2 - 1)u)(x) = \int_{\mathbb{R}} \chi_1(x)(\chi_2(y) - 1) k(x-y)u(y) \, dy,
\]

and the integral kernel \( \chi_1(x)(\chi_2(y) - 1) k(x-y) \) is smooth, bounded and rapidly decaying as \( |x-y| \to \infty \).

**Lemma F.6.** Suppose \( r \in \mathbb{R} \) and \( 1 < p < \infty \). Then

\[
(1 - \chi_1)(D - i)^r \chi w_{-\mu} \text{ and } \chi_1(D - i)^r (\chi_2 - 1) w_{-\mu} \in L_p(\mathbb{R}).
\]

**Proof.** Since \( w_{-\mu}(x) \) is the inverse Fourier transform of an integrable function, it is bounded, continuous and tends to zero as \( |x| \to \infty \).

From Remark F.5, the kernels of the integral operators \( (1 - \chi_1)(D - i)^r \chi \) and \( \chi_1(D - i)^r (\chi_2 - 1) \) are smooth, bounded and rapidly decaying as \( |x-y| \to \infty \). Hence result.

Finally, we prove Lemma F.8. We begin with a simple result.

**Lemma F.7.**

\[
(\chi - \chi_2)w_{-\mu} \in C_0^\infty(\mathbb{R}_+).
\]

**Proof.** By definition,

\[
\mathcal{F}w_{-\mu} = \frac{(-i\xi)^{-\mu}}{\xi - i},
\]

where \( 0 < \mu < 1 \). Since \( w_{-\mu}(x) \) is the inverse Fourier transform of an integrable function, it is continuous for all \( x \in \mathbb{R} \). Now

\[
\mathcal{F}w'_{-\mu} = \frac{(-i\xi)(-i\xi)^{-\mu}}{\xi - i} = \frac{(-i(\xi - i) + 1)(-i\xi)^{-\mu}}{\xi - i} = -i(-i\xi)^{-\mu} + \mathcal{F}w_{-\mu},
\]

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so that, from equation (A.1), for $x > 0$, 
\[ w'_{-\mu}(x) = Cx^{\mu-1} + w_{-\mu}(x), \]

But since $\chi - \chi_2$ equals zero in a neighbourhood of $x = 0$, we see immediately that $(\chi - \chi_2)w'_{-\mu}$ is continuous with compact support in $\mathbb{R}_+$. Finally, since for any $m \in \mathbb{N}$, 
\[ Fw^{(m+1)}_{-\mu} = \frac{(-i\xi)^{m+1}(-i\xi)^{-\mu}}{\xi - i} = \frac{(-i(\xi - i) + 1)(-i\xi)^{m-\mu}}{\xi - i} = -i(-i\xi)^{m-\mu}Fw^{(m)}_{-\mu}, \]
the required result follows directly by induction on $m$.

\[ \square \]

**Lemma F.8.** Suppose $r \in \mathbb{R}$. Then
\[ \chi_1(D - i)^r(\chi - \chi_2)w_{-\mu} \in C^\infty_0(\mathbb{R}). \]

**Proof.** From Lemma F.7, $(\chi - \chi_2)w_{-\mu} \in C^\infty_0(\mathbb{R})$. Hence, see Theorem 3.1, p. 47, [39], 
\[ (D - i)^r(\chi - \chi_2)w_{-\mu} \in C^\infty(\mathbb{R}). \]
Finally, \[ \chi_1(D - i)^r(\chi - \chi_2)w_{-\mu} \in C^\infty_0(\mathbb{R}), \] as required. \[ \square \]
Appendix G

A certain integral

Suppose $-1 < \mu < 1$ and $x > 0$. Then we define

$$I^\pm_{[a,b]}(\mu; x) := \int_a^b e^{-i\xi(-i\xi)^\mu(x \pm ix)^{-\mu-1}} d\xi.$$  

Lemma G.1. Suppose $-1 < \mu < 1$ and $x > 0$. Then

$$I^+_{[-1,1]}(\mu; x) = O(1), \quad x \searrow 0^+,$$

and

$$I^-_{[-1,1]}(\mu; x) = \begin{cases} O(1) & \text{if } \mu = 0 \\ O(\log x) & \text{if } \mu \neq 0 \end{cases} \text{ as } x \searrow 0^+.$$

Proof. We begin the proof with some observations that will prove useful.

Suppose $z_1, z_2 \in \mathbb{C}$ and $\nu \in \mathbb{R}$. Then, if $-\pi < \arg z_1 + \arg z_2 \leq \pi$,

$$(z_1z_2)^\nu = z_1^\nu z_2^\nu, \quad (\text{see Remark 1.11}). \quad (G.1)$$

If $|\arg (1 + b)| < \pi$ and $a > 0$, then

$$\int_0^1 \frac{\xi^{a-1}}{(1 + b\xi)^\nu} = \frac{1}{a} {}_2F_1(\nu, a; 1 + a; -b) \quad (3.194 1, p. 318, [17]). \quad (G.2)$$

$${}_2F_1(a, b; c; z) = (1 - z)^{-a-b} {}_2F_1(c - a, c - b; c; z) \quad (9.131 1, p. 1018, [17]). \quad (G.3)$$
We note that \( \binom{2}{F_1}(a, b; c; z) = \Gamma(c) \binom{2}{F_1}(a, b; c; z) \), see 15.1.2, [34]. Hence, if \(|\arg(-z)| < \pi\), then

\[
z \binom{2}{F_1}(1, 1; c; z) = - (c - 1) \log(-z) + O(1), \quad |z| \to \infty \quad (15.8.8, [34]). \tag{G.4}
\]

If we take \( z = \pm i/x \) in (G.4), then

\[
\frac{\binom{2}{F_1}(1, 1; c; \pm i/x)}{x(c-1)} = \mp i \log x + O(1), \quad x \downarrow 0^+. \tag{G.5}
\]

Since the integrand of \( I_{[-1,1]}^+ (\mu; x) \) admits an analytic continuation to the upper complex half-plane, it is easy to see, from Cauchy’s theorem, that

\[
I_{[-1,1]}^+ (\mu; x) = - \int_{\mathbb{T}_+} e^{-i\xi} (-i\xi)^\mu (\xi + ix)^{-\mu-1} d\xi = O(1) \quad \text{as} \quad x \downarrow 0^+,
\]

where \( \mathbb{T}_+ \) is the upper unit semicircle.

Similarly,

\[
I_{[-1,1]}^- (0; x) = \int_{-1}^{1} e^{-i\xi} (\xi - ix)^{-1} d\xi = \int_{\mathbb{T}_-} e^{-i\xi} (\xi - ix)^{-1} d\xi = O(1) \quad \text{as} \quad x \downarrow 0^+.
\]

On the other hand, we will now show that \( I_{[-1,1]}^- (\mu; x), \mu \neq 0 \) is unbounded as \( x \downarrow 0^+ \). Indeed,

\[
I_{[-1,1]}^- (\mu; x) = \int_{-1}^{1} e^{-i\xi} (-i\xi)^\mu (\xi - ix)^{-\mu-1} d\xi
\]

\[
= \int_{-1}^{1} (-i\xi)^\mu (\xi - ix)^{-\mu-1} d\xi + \int_{-1}^{1} O(\xi)(-i\xi)^\mu (\xi - ix)^{-\mu-1} d\xi
\]

\[
= J_{[-1,1]}^- (\mu; x) + O(1) \quad \text{as} \quad x \downarrow 0^+,
\]

where

\[
J_{[a,b]}^- (\mu; x) := \int_{a}^{b} (-i\xi)^\mu (\xi - ix)^{-\mu-1} d\xi.
\]
We can write
\[ J_{[0,1]}(\mu; x) = (-i)^\mu \int_0^1 \xi^\mu [(-ix)(1 + i\xi/x)]^{-\mu-1} d\xi \]
by (G.1)
\[ = (-i)^\mu (-ix)^{-\mu-1} \int_0^1 \xi^\mu (1 + i\xi/x)^{-\mu-1} d\xi \]
by (G.1)
\[ = (-i)^\mu \frac{(-ix)^{-\mu} 2F_1(1 + \mu, 1 + \mu; 2 + \mu; -i/x)}{-i} \]
\[ = i(-i)^\mu (1 - ix)^{-\mu} \frac{2F_1(1, 1; 2 + \mu; -i/x)}{x(1 + \mu)} \]
\[ = -( -i)^\mu \log x + O(1) \text{ as } x \searrow 0^+ , \]
by (G.5).

Since \(-(-i)^\mu = \exp(i\pi) \exp(-i\pi \mu/2) = \exp(i\pi(2 - \mu)/2)\), we have
\[ J_{[0,1]}(\mu; x) = \exp(i\pi(2 - \mu)/2) \log x + O(1). \tag{G.6} \]

Similarly,
\[ J_{[-1,0]}(\mu; x) = \int_{-1}^0 (-i\eta)^\mu (\eta - ix)^{-\mu-1} d\eta \]
\[ = \int_0^1 (i\xi)^\mu (-\xi - ix)^{-\mu-1} d\xi \]
\[ = i^\mu \int_0^1 \xi^\mu [(-ix)(1 - i\xi/x)]^{-\mu-1} d\xi \]
by (G.1)
\[ = i^\mu \frac{(-ix)^{-\mu} 2F_1(1 + \mu, 1 + \mu; 2 + \mu; i/x)}{-i} \]
\[ = i^{1+\mu} (-ix)^{-\mu} \frac{2F_1(1, 1; 2 + \mu; i/x)}{x(1 + \mu)} \]
\[ = i^{1+\mu} (1 - ix)^{-\mu} \frac{2F_1(1, 1; 2 + \mu; i/x)}{x(1 + \mu)}. \]

Noting that \( \lim_{x \searrow 0^+} (-1 - ix)^{-\mu} = \exp(i\pi \mu) \),
\[ J_{[-1,0]}(\mu; x) = \exp(i\pi(1 + \mu + 2\mu - 1)/2) \log x + O(1) \]
by (G.5)
\[ = \exp(i\pi 3\mu/2) \log x + O(1). \tag{G.7} \]
Thus, combining equations (G.6) and (G.7),
\[ J^{-}_{[-1,1]}(\mu; x) = C_{\mu} \log x + O(1), \]
as \( x \searrow 0^+ \), where
\[ C_{\mu} = -\exp(-i\pi\mu/2) + \exp(i\pi3\mu/2). \]
Of course, given \(-1 < \mu < 1\), the coefficient \( C_{\mu} \neq 0 \) if and only if \( \mu \neq 0 \).

**Remark G.2.** On the other hand, it is easy to show that
\[ J^{+}_{[0,1]}(\mu; x) = (-i)^{1+\mu}(1+ix)^{-\mu} \frac{2F_1(1,1; 2+\mu; i/x)}{x(1+\mu)} \]
\[ = -\exp(-i\pi\mu/2) \log x + O(1), \]
and similarly,
\[ J^{+}_{[-1,0]}(\mu; x) = -i^{\mu}(-1+ix)^{-\mu} \frac{2F_1(1,1; 2+\mu; -i/x)}{x(1+\mu)} \]
\[ = \exp(-i\pi\mu/2) \log x + O(1). \]
Therefore, \( J^{+}_{[-1,1]}(\mu; x) = O(1) \), as \( x \searrow 0^+ \), confirming the result in Lemma G.1.
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