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A Multiperiod Bank Run Model for Liquidity Risk*

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Abstract. We present a new dynamic bank run model for liquidity risk where a financial institution finances its risky assets by a mixture of short- and long-term debt. The financial institution is exposed to insolvency risk at any time until maturity and to illiquidity risk at a finite number of rollover dates. We compute both insolvency and illiquidity default probabilities in this multiperiod setting using a structural credit risk model approach. Firesale rates can be determined endogenously as expected debt value over current asset value. Numerical results illustrate the impact of various input parameters on the default probabilities.

JEL Classification: G01, G32, G33

1. Introduction

The credit crisis of 2007–08 has dramatically shown that credit risk not only reduced insolvency risk but also intertwined with liquidity aspects. In particular, debt runs are mentioned as one of the main reasons for the crisis, for example, in Brunnermeier (2009) and Gorton (2008). The failures of Northern Rock and DSB bank are just two examples of bankruptcies due

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to a run by short-term creditors. After approaching the Bank of England for a liquidity support facility, Northern Rock was the first bank to fail from a bank run during the credit crisis in 2007. As their funding problems were made public, depositors had withdrawn their savings as quickly as possible. As a result, the bank failed due to this panic run.¹ Similarly, the Dutch DSB bank suffered from a run by depositors, who had withdrawn €600 million in 12 days, corresponding to about a quarter of the bank's assets. The bank was placed under supervision of De Nederlandsche Bank in October 2009, and afterward was declared bankrupt.² Furthermore, the failures of Bear Stearns and Lehman Brothers were partly due to a run by short-term creditors. Both institutions had capital cushions well above the Basel II minimal capital requirements, but had financed their long-duration risky assets mostly through short-term debt. Thereby, they were heavily exposed to liquidity risk. In fact, Christopher Cox (2008), then chairman of the US Securities and Exchange Commission, explained in an open letter, "[...]the fate of Bear Stearns was the result of a lack of confidence, not a lack of capital. [...]Counterparty withdrawals and credit denials, resulting in a loss of liquidity - not inadequate capital - caused Bear's demise."³

It is now understood that short-duration financing, for example, through commercial papers and repo transactions, increases the exposure to panic runs that were among the main causes of the credit crisis of 2007–08. In fact, it has been shown in Adrian and Shin (2008, 2010) that the use of very short-term financing dramatically increased during the financial crisis of 2007–08 and the period immediately before that.

In this article, we are concerned with the following set of questions. What is the contribution of liquidity risk to a firm's total default risk? How do fluctuations in the firm fundamentals impact the rollover decision of short-term creditors, and thus total default risk? How does the default probability of a financial institution depend on its financing structure, that is, on the ratio of short-term debt over total debt, and on the maturity structure or rollover frequency of short-term debt?

To answer these questions, we construct a structural credit risk model in continuous time with multiple rollover dates for short-term funding. More specifically, we consider a financial institution financing its risky assets using short- and long-term debt. Short-term debt earns lower return than long-term debt, but the short-term creditors of the financial institution have the choice not to renew their funding at certain rollover dates. We do not use a

¹ For more details, we refer to Shin (2009).

² See the BBC report by Paul Lewis dated November 7 2009.

³ See also Morris and Shin (2009).

staggered debt structure as in He and Xiong (2012b), who solve for the unique threshold equilibrium, that is, they endogenously determine the unique threshold value such that each maturing creditor would decide to run on the firm if the asset value falls below this threshold. Instead, we extend the existing static bank run models to a multiperiod setting. This multiperiod structure is important for practical applications as banks usually finance themselves by a combination of repo transactions, commercial papers, and very short-duration contracts with high rollover frequencies. Thus, the decision of other creditors maturing at the same time can have a dramatic impact on the current default risk. Our model provides a method to analyze this effect. In this way, our multiperiod model extends the game-theoretic approach of Morris and Shin (2009), who propose a two-period model with only one rollover decision, toward real-life applications.

The decision of a short-term creditor, whether to roll over or not, obviously depends on the default probability of the financial institution. The financial institution is exposed to both insolvency risk and illiquidity risk. *Insolvency risk* is defined as the risk of deterioration in the credit quality of some investments resulting in unexpected losses. In contrast, we define *illiquidity risk* as the risk of a default due to a run by short-term creditors, although it would otherwise have been solvent. In case several short-term creditors choose not to roll over their funding, the value of the available liquid assets might not be sufficient to pay off short-term creditors, leading to a default due to illiquidity of the financial institution. In particular, this means that the rollover decisions of short-term creditors today will not only depend on their expectations about the rollover behavior of other creditors today (as in the static bank run models) but also on their expectations about possible bank runs at future decision dates. Thus, our model represents a fully dynamic coordination problem or rollover game between short-term creditors. In this way, it allows us to study how both future aggregate rollover risk and future insolvency risk affect the current decision of short-term creditors, and thus today's default probability. Hence, our model provides an appropriate setting to answer the above questions.

To explicitly compute the total default probability of the financial institution, we use a 1st-passage time approach. Insolvency happens at the 1st-passage time when the asset value process falls below an *insolvency barrier*, which depends on the firm's capital structure. We calculate the total default probability by first specifying a *bank run barrier*. When the asset value at a rollover date falls below this barrier, short-term creditors will decide to run on the financial institution. Note that this does not necessarily lead to a default of the financial institution as the latter might be able to payoff short-term creditors. To determine the bank run barrier, we

compare the expected future return from rolling over with the expected return from running on the financial institution at each decision date. As both depend on the future rollover decisions, we have to proceed backward in time to solve this optimization problem. Moreover, we define an *illiquidity barrier* at each rollover date as the critical asset value when a *successful run* occurs, that is, a run that leads to the default of the financial institution. The total default probability can then be computed as the probability that the asset value falls below the insolvency barrier at any time until maturity, or in case of a bank run, by the probability that the firm is not able to buffer the losses connected with paying off short-term debt at some rollover date. The insolvency default probability can be computed analytically as in the classical [Black and Cox \(1976\)](#) model. The difference between total default probability and insolvency default probability then defines the illiquidity default probability.

Due to the dynamic coordination problem between short-term creditors, we cannot compute the total default probability by simple analytical formulas as in the classical 1st-passage time model of [Black and Cox \(1976\)](#). Instead, we implement our model in a binomial tree framework. We present numerical results showing that illiquidity risk is increasing in the volatility of the risky assets and in the market return rate. These results extend the qualitative observations in [Morris and Shin \(2009\)](#) and complement results in the recent paper of [Huang and Ratnovski \(2011\)](#). The latter show that short-term creditors are easily inveigled by negative public signals to withdraw their funding, and thus can trigger inefficient liquidations.

A main empirical implication of our model is that total default risk is increasing in the rollover frequency and in the short-term debt ratio, that is, in the ratio of short-term debt over total debt. This is due to the fact that bank runs become more likely when there are more possibilities to run on the firm. Moreover, a bank run is more likely to trigger a default when more short-term debt has to be paid off. In this way, our article supports the results in [Brunnermeier and Oehmke \(forthcoming\)](#) and [Adrian and Shin \(2008, 2010\)](#) who show that excessive reliance on short-term financing results in unnecessary rollover risk and inefficient debt runs. Recently, [Acharya, Gale, and Yorulmazer \(forthcoming\)](#) showed that rollover risk is indeed an important factor in the capital structure, especially for those financial institutions with heavy maturity mismatch. Their results imply that debt capacity is lower when short-term debt is more frequently rolled over. The implications of our model support these findings. Moreover, the decomposition of the total default probability in its insolvency and illiquidity components, as presented in our article, can also provide new insight into the optimal capital structure.

The firesale rate is one of two important variables that mainly determine the size of illiquidity risk in our model. It represents the rate by which the risky asset can be sold prematurely (or equivalently, the cash that can be borrowed by pledging one unit of risky asset as collateral). The firesale rate can be implicitly determined from the leverage and has been proved to be time varying. In our model, the firesale rate can be defined endogenously at each rollover date as the ratio of the expected debt value over the asset value. By this definition, not only the uncertainty of the fundamental (i.e., the asset value) but also the future rollover risk and insolvency risk, implicitly captured in the expected debt value, are channeled into the firesale rate. It has been suggested that besides interest rates, leverage (and thereby, firesale rates) can also be used as monetary policy tools (see, e.g., [Ashcraft, Gârleanu, and Pedersen 2010](#); [Morris and Shin, 2009](#); and [Geanakoplos, 2009](#)). When liquidity is tight, central banks should not only take care of lowering interest rates but also provide lending with more generous haircut to enhance liquidity in the market. Our empirical results show that the total default probability is decreasing when firesale rates (and thus, leverage) are increasing. Hence, the predictions of our model imply that in terms of regulation on firm level, as also suggested in [Morris and Shin \(2009\)](#), the single-minded focus on capital requirements needs to give way to a broader range of balance sheet indicators, including the ratio of liquid assets to total assets and short-term liabilities to total liabilities.

Our article is organized as follows. In Section 2, we describe the model setup, our main assumptions, and explain how the insolvency and illiquidity barriers can be derived. Here, we also state our main theoretical result about the existence and uniqueness of a bank run barrier in Theorem 1. Moreover, we present the decomposition of the total default probability in its insolvency and illiquidity components in Theorem 2. We explain how our model can be empirically tested and provide a sensitivity analysis to study the impact of various input parameters on the default probabilities in Section 3. Here, we also discuss the main empirical prediction of our model. Section 4 discusses the endogenous derivation of firesale rates, competitions among banks, and further extensions of our model, as well as the related literature. Finally, Section 5 summarizes our results and concludes. The proofs of our theoretical results are provided in Appendix A. We provide an online supplement that covers an analytical characterization of the default probabilities in terms of the solutions of partial differential equation Dirichlet problems as well as a detailed explanation of how our model can numerically be implemented using binomial tree methods.

2. Multiperiod Bank Run Model

2.1 FINANCING STRUCTURE AND ASSUMPTIONS

Suppose a financial institution finances its risky assets, such as loans or risky securities, by short- and long-term debt. We model the value process $(V(t))_{t \geq 0}$ of the risky assets by a geometric Brownian motion

$$V(t) = V_0 \cdot \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right) \quad (1)$$

with drift μ and volatility $\sigma > 0$, where $(W_t)_{t \geq 0}$ is a standard Brownian motion representing random shocks to the asset value. We further assume that the financial institution also holds a cash amount M on the asset side, which is invested at the (continuously compounded) risk-free rate r .

2.1.a Long- and short-term debt

We now turn to the liabilities side of the financial institution. At time $t_0 = 0$, long-term creditors invest an amount L_0 until time T . The promised (continuously compounded) rate of return for long-term debt is r_L per annum. Thus, if there is no default, the value of long-term debt at maturity is

$$L_T = e^{r_L T} L_0.$$

At initiation time $t_0 = 0$, short-term creditors invest an amount S_0 at the (continuously compounded annual) rate r_S . Assume that short-term creditors can decide whether they want to renew their funding or not at $n - 1$ rollover dates $t_1, \dots, t_{n-1}$.⁴ Let $t_n = T$ be the final maturity of short-term debt. For simplicity, we assume the rollover dates to be equidistant, such that $\Delta t = T/n$ can be understood as the rollover frequency of short-term debt. The short-term rate r_S is fixed at the beginning of the debt contract in t_0 and is assumed to be constant for all time periods $[t_i, t_{i+1}]$. If some short-term creditors decide not to renew their funding, the financial institution will sell the corresponding short-term bonds to new creditors, if it does not default due to a bank run at that time point. If not too many short-term creditors decide to run away, the financial institution should always be able to find some new creditors in the market. Note that by this assumption, the face value of short-term debt at any time t_i is known in advance.

⁴ We will discuss the generalization to a continuous time setting, where short-term creditors can run at any time $t \in [0, T]$, in Section 4.

Remark 1

One can easily extend our model to the case where r_S is a deterministic function of time. Taking into account stochastic fluctuations of the short-term rate, however, would complicate the numerical solution to our model considerably. Instead of the single Brownian term driving the asset value process, we would then have to consider an additional stochastic factor driving the short-term rate. One can in principle implement such a setting in a two-dimensional binomial tree framework. However, for the message of this article, we think this would not lead to any substantial new insights.

Notation 1

To simplify notations, we will, in the following, denote all parameters at time t_i only by the subindex i , for example, $V_i = V(t_i)$.

If short-term debt is rolled over at time t_i , that is, if the face value S_i is invested a new until time t_{i+1} , then the face value of short-term debt at time t_{i+1} is

$$S_{i+1} = e^{r_S \Delta t} S_i = e^{r_S(i+1)\Delta t} S_0.$$

The rollover decision of each short-term creditor surely depends on her beliefs whether the firm will survive a bank run or not, that is, on the likelihood that she will get her face value of debt back. This likelihood clearly depends on the funds that can be raised. At time $t_i < T$, the risky assets can be liquidated prematurely only at a firesale price. We denote by $\psi_i \in [0, 1]$ the firesale rate representing the cash that can be borrowed by pledging one unit of risky asset as collateral.⁵ The value of the cash amount at time t_i is given by $e^{rt_i} M$. Intuitively, the ratio between the funds that can be raised at time t_i and the principle of short-term debt

$$\frac{\psi_i V_i + e^{rt_i} M}{S_i} \quad (2)$$

represents the likelihood that the short-term creditors get back their face value of the debt. The higher the value of Equation (2), the more funds the firm can raise, and the more likely the short-term creditors are to get their debt back. Because the creditors at most get their debt back, the above ratio needs to be cut off at 1. Hence,

$$\lambda_i = \min \left\{ 1, \frac{\psi_i V_i + e^{rt_i} M}{S_i} \right\} \quad (3)$$

⁵ We will discuss in Subsection 4.1 how the firesale rate can be endogenously determined in our model.

can be interpreted as the survival probability from a bank run. Here, we implicitly assume that there are sufficiently many creditors such that the individual rollover decision only affects the survival probability from a bank run at time t_i by a negligible small amount.

The decision at time t_i ($i = 1, \dots, n - 1$) of short-term creditors to rollover or not of course also depends on the return they can earn in the outside market. We assume the (continuously compounded annual) outside rate r^* to be constant for all time periods. It is a variable assumed to be known and given in the market. For our model to be meaningful, we assume the outside return to be strictly smaller than the short-term debt rate. Otherwise, short-term creditors would directly choose the outside option instead of investing their debt at the risky rate r_S .

Finally, we assume that short-term debt holders who decide not to roll over at some time point t_i , where no successful run occurs, cannot return to the financial institution at a later time point $t_j > t_i$. Without this assumption, short-term creditors would just switch from the financial institution to the outside option and back, depending on whether the market rate r^* or the short-term rate r_S is higher, in which case our model would reduce to a two-period model. It can, however, be argued that, when a short-term creditor decides not to roll over her debt at a certain time point, she should be able to return to the bank only under different conditions, that is, for a different short-term rate $\tilde{r}_S \neq r_S$. Incorporating this feature in our model is beyond the scope of this article as we will discuss in Remark 2 below.

Remark 2

Superposition of bank run models. In real life, one is faced with a superposition of processes of the type we are explaining here. Namely, at each date t_i , one can observe new data for $r_{S,i}$, r_i^* , etc., which reflect the current market situation, and which influence the rollover decision of short-term creditors as well as the investment decisions of new debt holders. If $r_i^* > r_{S,i-1}$ at a decision date t_i , short-term creditors will run on the financial institutions. To avoid this, the financial institution has to offer a new short-term rate $r_{S,i} > r^*$ at each decision date t_i for $i = 1, \dots, n - 1$, to keep its creditors. The financial institution, therefore, always needs to be able to buffer losses of size $e^{(r_{S,i} - r^*)(n-i)\Delta t}$. Hence, at each decision date t_i , one actually faces a new multiperiod rollover problem of the type discussed in this article with data adjusted to the specified date. When incorporating this feature in our model, the computation of the expected returns from rolling over and from running on the firm would be formidably complex. It might be possible in principle;

however, in our view, this would complicate our model considerably by adding only very little new insight.⁶

We summarize our assumptions on short-term creditors as follows.

Assumption 1

- (a) *Short-term creditors can only decide whether to roll over their funding or not at dates $t_1, t_2, \dots, t_{n-1}$, and short-term debt can only be rolled over until the next decision date.*
- (b) *Each short-term creditor believes that the firm will survive a bank run with probability*

$$\lambda_i = \min \left\{ 1, \frac{\psi_i V_i + e^{rt_i} M}{S_i} \right\},$$

where $\psi_i \in [0, 1]$ is the firesale rate at time t_i .

- (c) *The outside return rate r^* is strictly smaller than the short-term debt rate r_S .*
- (d) *Short-term debt holders who decide not to roll over at some time point t_i , where no successful run occurs, cannot return to the financial institution at a later time point $t_j > t_i$.*

For simplicity of derivations, we further assume zero recovery rate in case of a default. Note, however, that our results can also be derived with nonzero recovery by a straightforward extension of our arguments.

Assumption 2

The recovery rate is zero.

2.1.b Balance sheet before and after a run

We will now explain how a bank run affects the balance sheet decomposition of the financial institution. The balance sheet at any time $t \in (t_{i-1}, t_i]$ before a run occurs is demonstrated in Table I below.

Suppose now that there is a bank run at time t_i but the firm is still solvent. Then the firm is forced to pledge a fraction θ_i of its assets as collateral at the firesale price $\theta_i \psi_i V_i$ to some lending banks in order to pay off S_i to short-term creditors, that is, θ_i is chosen such that

$$S_i = \theta_i \psi_i V_i + e^{rt_i} M.$$

The remaining $(1 - \theta_i)$ risky assets are kept on the balance sheet. We denote by r_c the rate at which funds $\theta_i \psi_i V_i$ can be raised. The rate r_c must be

⁶ Compare also Remark 1 for a related issue, which similarly applies to the outside rate r^* .

Table I. Balance sheet at time $t \in (t_{i-1}, t_i]$ before a bank run

The table illustrates the balance sheet decomposition at any time $t \in (t_{i-1}, t_i]$ before a bank run has occurred. The firm holds risky assets $V(t)$ and cash $e^{rt}M$ on the asset side. On the liabilities side, the firm holds long-term debt with face value L_T and maturity T and short-term debt with face value S_i that can be rolled over at dates $t_i < \dots < t_{n-1}$ until final maturity T . Cash is invested at the risk-free rate r . Long- and short-term debt pays a return r_L and r_S , respectively.

Assets	Liabilities
Risky assets with market value $V(t)$ and maturity T	Long-term debt with maturity T and face value L_T
Cash $e^{rt}M$ that is invested in the risk-free rate r	Short-term debt with maturity t_i and face value S_i that can be rolled over at dates $t_i, \dots, t_{n-1}$ until final maturity T

understood as the interest rate at which banks are willing to lend money to the firm as emergency funding, that is, in a distressed state, until maturity T . Hence, it seems reasonable to assume that $r_c > r_L$ as the collateralized debt can also be understood as long-term debt. As for r_S and r^* , we assume r_c to be constant for all decision dates $t_1, \dots, t_{n-1}$. Thus, the balance sheet changes after the bank run as illustrated in Table II below. The total cash amount $\theta_i \psi_i V_i + e^{rt_i}M$ that can be raised at time t_i is used to pay off short-term debt. Thus, there is no more cash left on the asset side of the balance sheet. Instead, the firm holds $(1 - \theta_i)V(t)$ risky assets at time t . Moreover, the firm has the opportunity to receive the risky assets $\theta_i V(T)$ at maturity if it is able to pay off collateralized debt at time T . On the liabilities side, the firm has issued long-term debt with maturity T and face value L_T . Moreover, it has to payoff the collateralized debt, that is, it has to pay off $\theta_i \psi_i V_i \cdot e^{r_c(T-t_i)} = (S_i - e^{rt_i}M)e^{r_c(T-t_i)}$ to receive the risky assets. Otherwise, the lending banks will keep the risky assets $\theta_i V(T)$ at maturity T .

2.2 COMPUTATION OF DEFAULT BARRIERS

2.2.a Insolvency barrier

We now turn to the computation of default barriers in the above setting. Similar to Black and Cox (1976), the financial institution’s default due to insolvency at any time $t \in [0, T]$ is determined by the time-varying *insolvency barrier*

$$\alpha(t) = e^{-r(T-t)}(S_0 e^{r_S T} + L_0 e^{r_L T} - M e^{r T}) \cdot \rho, \tag{4}$$

Table II. Balance sheet at time $t \in (t_i, T]$ after a bank run at time t_i

The table illustrates the balance sheet decomposition at time $t \in (t_i, T]$ after a bank run at time t_i . At maturity, the firm has to pay off long-term debt L_T . Moreover, it has to pay off the collateralized debt $\theta_i \psi_i V_i \cdot e^{r_c(T-t_i)}$, where r_c is the interest rate at which the collateralized debt has been raised at time t_i , and ψ_i is the firesale rate at time t_i . When paying off collateralized debt, the firm receives the risky assets $\theta_i V(T)$. Otherwise, the lending banks will keep the risky assets $\theta_i V_T$ as collateral. There is no cash left on the balance sheet as it has been used to pay off short-term creditors at time t_i .

Assets	Liabilities
Risky assets $(1 - \theta_i)V(t)$	Long-term debt with maturity T and face value L_T
Option to receive risky assets $\theta_i V(T)$ at time T when paying off collateralized debt	Collateralized debt $\theta_i \psi_i V_i e^{r_c(t-T)}$ with maturity T

with $\rho \in [0, 1]$ being a safety covenant that determines how much of the asset value is available to compensate creditors and equity holders, according to a predescribed seniority when the firm bankrupts.

2.2.b Bank run barrier at time t_{n-1}

In the computation of the bank run barrier, we are motivated by an idea in [Morris and Shin \(2009\)](#). At each rollover date, the short-term creditors face a binary decision problem. They have to decide whether they rollover their debt or not, depending on the corresponding expected returns from both decisions. Hence, we need to determine the critical asset value at each rollover date at which short-term creditors decide not to roll over their funding. We call this value the *bank run barrier* at time t_i , and denote it by β_i . For its computation, we proceed backward, starting with the final rollover date t_{n-1} , because the default probability at later dates will influence the rollover decision at early dates. Hence, for the terminal time period $[t_{n-1}, T]$, we consider the expected returns from rolling over and from the outside option. We use R_{n-1}^* to denote the expected outside return short-term creditors earn at t_n by investing in the market if they decide not to roll over their debt at time t_{n-1} . Recall that the market rate is denoted by r^* . The expected outside return at time t_{n-1} is given by the product of the market return and the survival probability from a bank run

$$e^{R_{n-1}^* \Delta t} = e^{r^* \Delta t} \cdot \lambda_{n-1}, \tag{5}$$

as the short-term debt is paid back (such that it then can be invested in the market) only if the firm has survived from a bank run. Due to Assumption 2,

short-term creditors get nothing in case of a default. Hence, the expected outside return rate R_{n-1}^* at time t_{n-1} is defined as

$$R_{n-1}^* = r^* + \frac{\ln \lambda_{n-1}}{\Delta t}. \quad (6)$$

If the short-term creditor decides to roll over her debt at time t_{n-1} , she earns r_S at time t_n provided that the firm does not default due to insolvency in $(t_{n-1}, t_n]$, or due to a bank run at time t_{n-1} . Thus, the expected return from rolling over at time t_{n-1} is given by the product of the short-term debt return, the survival probability from insolvency within the time period $(t_{n-1}, t_n]$, and the survival probability from a bank run at t_{n-1} , that is,

$$e^{R_{n-1}^S \Delta t} = e^{r_S \Delta t} \cdot \mathbb{P} \left(\inf_{t_{n-1} < s \leq t_n} (V(s) - \alpha(s)) \geq 0 \mid V_{n-1} \right) \cdot \lambda_{n-1}. \quad (7)$$

Note that we did not multiply the right-hand side (RHS) of Equations (5) and (7) by the survival probability from insolvency at time t_{n-1} because in case of an insolvency default at t_{n-1} , there is no decision to be made for short-term creditors as they lose their debt anyway. Explicitly, the expected return rate from rolling over is

$$R_{n-1}^S = r_S + \frac{\ln \mathbb{P} \left(\inf_{t_{n-1} < s \leq t_n} (V(s) - \alpha(s)) \geq 0 \mid V_{n-1} \right)}{\Delta t} + \frac{\ln \lambda_{n-1}}{\Delta t}. \quad (8)$$

Table III summarizes the timing of the problem and the returns that can be earned in the final time period.

We assume that each short-term creditor wants to maximize her return over the whole time period $[0, T]$. Thus, a run at time t_{n-1} occurs if the expected rollover return R_{n-1}^S is smaller than the expected outside return R_{n-1}^* . From this, we can derive the bank run barrier β_{n-1} at time t_{n-1} by setting the expected outside return equal to the expected return from rolling over. Hence, the bank run barrier β_{n-1} at time t_{n-1} is defined as the critical value V_{n-1} where the RHS of Equation (5) is equal to the RHS of Equation (7) for $V_{n-1} = \beta_{n-1}$. Note that when setting the expected outside return, given by equation (5), equal to the expected return from rolling over, given by equation (7), the survival probability from a bank run drops out of the equation. Therefore, at the last rollover date t_{n-1} , the survival probability from a bank run will not influence the bank run barrier. This will be different at the previous rollover dates as we will discuss below. In the terminal time period $[t_{n-1}, t_n]$, short-term creditors can earn a maximal return rate

Table III. Investments and returns in the final time period

The table demonstrates the timing of the decision problem short-term creditors face at time t_{n-1} . The left column shows the investment opportunities at time t_{n-1} and the remaining columns provide the corresponding returns at time t_n depending on whether there is a bank run at time t_{n-1} and no insolvency default in $[t_{n-1}, t_n]$ (Column 4) or an insolvency default in $[t_{n-1}, t_n]$ (Column 5), and on whether there is no bank run at time t_{n-1} and no insolvency default in $[t_{n-1}, t_n]$ (Column 2) or an insolvency default in $[t_{n-1}, t_n]$ (Column 3).

Investment at t_{n-1}	Return at time t_n in case of			
	No bank run at t_{n-1} and		Bank run at t_{n-1} and	
	No insolvency in $[t_{n-1}, t_n]$	Insolvency in $[t_{n-1}, t_n]$	No insolvency in $[t_{n-1}, t_n]$	Insolvency in $[t_{n-1}, t_n]$
Rollover at rate r_S	$e^{r_S \Delta t}$	0	$\lambda_{n-1} \cdot e^{r_S \Delta t}$	0
Run and invest in r^*	$e^{r^* \Delta t}$	$e^{r^* \Delta t}$	$\lambda_{n-1} \cdot e^{r^* \Delta t}$	$\lambda_{n-1} \cdot e^{r^* \Delta t}$

$$\max\{R_{n-1}^*, R_{n-1}^S\}. \quad (9)$$

2.2.c Bank run barrier at time t_{n-2}

Now, we consider the returns over the time period $[t_{n-2}, t_n]$. At time t_{n-2} , short-term creditors have to worry not only about the return for the next period, that is, up to time t_{n-1} , but also about the final time period $[t_{n-1}, t_n]$, as they want to maximize their total return. Thus, the expected return from rolling over at time t_{n-2} is

$$e^{R_{n-2}^S \cdot 2\Delta t} = \mathbb{E}\left[e^{\max\{R_{n-1}^*, R_{n-1}^S\} \Delta t} | V_{n-2}\right] \cdot e^{r_S \Delta t} \cdot \mathbb{P}\left(\inf_{t_{n-2} < s \leq t_{n-1}} (V(s) - \alpha(s)) \geq 0 | V_{n-2}\right) \cdot \lambda_{n-2}. \quad (10)$$

The 1st term is the expected return from the final period and the product of the remaining terms is the expected return from the period $[t_{n-2}, t_{n-1}]$. Note that the expected return rate R_{n-1}^S in the conditional expectation depends also on λ_{n-1} (compare with Equation (8)). Hence, the expected return rate from rolling over at time t_{n-2} depends not only on the survival probability from a bank run at time t_{n-2} but also on the survival probability from a bank run at time t_{n-1} . If short-term creditors decide to run at t_{n-2} , they obtain the expected outside return

$$e^{2R_{n-2}^* \Delta t} = e^{2r^* \Delta t} \cdot \lambda_{n-2}. \quad (11)$$

A run occurs if the expected rollover return equals the expected outside return. This condition defines the bank run barrier at time t_{n-2} . Again, the survival probability from a bank run at time t_{n-2} drops out of the equation when setting the RHS of Equation (10) equal to the RHS of Equation (11). However, as already mentioned, the RHS of Equation (10) also depends on λ_{n-1} through the term R_{n-1}^S in the conditional expectation. Hence, the bank run barrier at time t_{n-2} does depend on the survival probability from a bank run at the future rollover date t_{n-1} . Summing up, in the time period $[t_{n-2}, t_n]$, short-term creditors can earn a maximal return rate of

$$\max\{R_{n-2}^*, R_{n-2}^S\}, \quad (12)$$

where R_{n-2}^S and R_{n-2}^* are implicitly defined via Equations (10) and (11), respectively.

2.2.d Bank run barrier at any rollover date t_i

We now consider an arbitrary rollover date t_i . The expected outside return from running at time t_i is defined as

$$e^{R_i^* (n-i)\Delta t} = e^{r^* (n-i)\Delta t} \cdot \lambda_i, \quad (13)$$

and the corresponding expected return rate is

$$R_i^* = r^* + \frac{\ln \lambda_i}{(n-i)\Delta t}. \quad (14)$$

The expected return from rolling over the debt at time t_i is defined as

$$\begin{aligned} e^{R_i^S (n-i)\Delta t} &= \mathbb{E} \left[e^{\max\{R_{i+1}^*, R_{i+1}^S\} (n-i-1)\Delta t} | V_i \right] \\ &\quad \cdot e^{r_S \Delta t} \cdot \mathbb{P} \left(\inf_{t_i < s \leq t_{i+1}} (V(s) - \alpha(s)) \geq 0 | V_i \right) \cdot \lambda_i, \end{aligned} \quad (15)$$

and its return rate is given by:

$$\begin{aligned} R_i^S &= \frac{\ln \mathbb{E} \left[e^{\max\{R_{i+1}^*, R_{i+1}^S\} (n-i-1)\Delta t} | V_i \right]}{(n-i)\Delta t} \\ &\quad + \frac{r_S}{(n-i)} + \frac{\ln \mathbb{P} \left(\inf_{t_i < s \leq t_{i+1}} (V(s) - \alpha(s)) \geq 0 | V_i \right)}{(n-i)\Delta t} + \frac{\lambda_i}{(n-i)\Delta t} \end{aligned} \quad (16)$$

for $i = 1, \dots, n-1$. Note that the conditional expectation in the 1st term implicitly depends on the survival probabilities $\lambda_{i+1}, \dots, \lambda_{n-1}$ from bank runs at future rollover dates $t_{i+1}, \dots, t_{n-1}$. Thus, if short-term creditors decide to roll over their debt, their exposure to illiquidity risk is higher than when running on the firm, which is quite intuitive from economic motivation. More specifically, if short-term creditors run at time t_i , they only have to worry about a bank run at that time. If short-term creditors roll over at t_i , however, they are exposed to a possible bank run at t_i and to those at future rollover dates, that is, at $t_{i+1}, \dots, t_{n-1}$. The bank run barrier β_i can be computed recursively and is given as the unique value V_i for which the RHS of Equation (13) equals the RHS of Equation (15). Our main result is stated in the following theorem.

Theorem 1

Recursive Computation of Bank Run Barrier under Assumptions 1 and 2, the bank run barrier β_i at t_i can be computed recursively and is the unique solution of the equation in β_i

$$e^{r^*(n-i)\Delta t} = \mathbb{E} \left[e^{\max\{R_{i+1}^*, R_{i+1}^S\} (n-i-1)\Delta t} | V_i = \beta_i \right] \cdot e^{r_S \Delta t} \cdot \mathbb{P} \left(\inf_{t_i < s \leq t_{i+1}} (V(s) - \alpha(s)) \geq 0 | V_i = \beta_i \right) \quad (17)$$

for $i = 1, \dots, n-1$. The bank run barrier β_i is always higher than the insolvency barrier α_i at the same time point.

Remark 3

Equation (17) is obtained by setting the RHS of Equation (13) and the RHS of Equation (15) equal, where λ_i drops out on both sides. The RHS of Equation (17), however, does depend on the survival probabilities from bank runs at the future decision dates $t_{i+1}, \dots, t_{n-1}$. For example, for $i = n-2$, Equation (17) can be reformulated as

$$1 = \mathbb{E}[\lambda_{n-1} \cdot \max \left\{ e^{(r_S - r^*)\Delta t}, e^{(r_S - r^*)2\Delta t} \cdot \mathbb{P} \left(\inf_{t_{n-1} < s \leq t_n} (V(s) - \alpha(s)) \geq 0 | V_{n-1} \right) \right\} | V_{n-2}] \cdot \mathbb{P} \left(\inf_{t_{n-2} < s \leq t_{n-1}} (V(s) - \alpha(s)) \geq 0 | V_{n-2} \right).$$

Hence, the bank run barrier β_{n-2} at time t_{n-2} is independent of λ_{n-2} but depends on the survival probability λ_{n-1} from a run at time t_{n-1} .

2.2.e Illiquidity barrier

In case of a bank run, the firm only defaults if the funds it can raise are not sufficient to pay off short-term debt. Thus, we define the *illiquidity barrier*, denoted by β_i , as

$$\bar{\beta}_i = \min\{\beta_i, (S_i - e^{rt_i}M)/\psi_i\}. \quad (18)$$

The 2nd term in the minimum function is the maximal asset value V_i such that $\lambda_i = 1$. Thus, the illiquidity barrier is the critical asset value V_i such that there is a run by short-term creditors and the firm is not able to pay off short-term debt.

2.2.f Bank run scenarios

Depending on the asset value V_i at a rollover date t_i , the following situations are possible.

- (i) If $V_i \leq \alpha_i$, the firm will default due to insolvency.
- (ii) If $\alpha_i < V_i \leq \bar{\beta}_i$, there is a bank run at time t_i and the funds that can be raised are not sufficient to pay off short-term creditors, that is, $\psi_i V_i + e^{rt_i}M < S_i$ such that $\lambda_i < 1$. Hence, the firm defaults due to illiquidity.
- (iii) If $\bar{\beta}_i < V_i \leq \beta_i$, there is a bank run at time t_i , but the firm can raise enough funds to pay off short-term debt. However, in doing so, the firm has to buffer an additional loss. Instead of paying interest rate r_S for short-term debt, the firm has to pay interest $r_c > r_S$ for the collateralized debt. More specifically, the firm has to pay off collateralized debt $(S_i - e^{rt_i}M)e^{r_c(T-t_i)}$ at maturity T . Hence, the firm remains solvent in $[t_i, T]$ only if $V(t) > \bar{\alpha}_i(t)$ for all $t \in [t_i, T]$, where the new insolvency barrier $\bar{\alpha}_i$ at any time $t \in [t_i, T]$ is defined as

$$\bar{\alpha}_i(t) \equiv e^{-r(T-t)}((S_i - e^{rt_i}M)e^{r_c(T-t_i)} + L_0 e^{r_L T}) \cdot \rho. \quad (19)$$

Otherwise, the firm will default. As this bankruptcy is actually triggered by the bank run at time t_i , we interpret this default too as a default due to illiquidity. Hence, in the case $\bar{\beta}_i < V_i \leq \beta_i$ we have to distinguish two subcases.

- (a) If $\inf_{t_i < s \leq t} (V(s) - \bar{\alpha}_i(s)) > 0$, the firm remains solvent until maturity T after the bank run at time t_i .
- (b) If $\inf_{t_i < s \leq t} (V(s) - \bar{\alpha}_i(s)) \leq 0$, the firm is not able to buffer the additional loss from interest rate payments connected with the collateralized debt at maturity T , and thus, defaults due to illiquidity problems.

Figure 1 summarizes the different scenarios that can occur in our model. Distinguishing these scenarios is important for the computation of the default probabilities as we will see in Section 2.3.

2.3 DECOMPOSITION OF DEFAULT PROBABILITIES

In this section, we compute the total default probability PD^{total} and decompose it into an insolvency part PD^{ins} and an illiquidity part PD^{ill} . Therefore, we use the insolvency barrier α_t for $t \in [0, T]$, the bank run barriers β_i , and the illiquidity barriers $\bar{\beta}_i$ for $i = 1, 2, \dots, s, n-1$, which we computed in Section 2.2. As discussed above, a bank run does not necessarily lead to a default in our model. Examining the different scenarios that lead to a default (illustrated in Figure 1), we can compute the survival probability PS^{total} that the financial institution will stay alive from $t \in (t_{i-1}, t_i]$ ($i = 1, 2, \dots, s, n$) to t_n as

Figure 1. Possible scenarios in case of a bank run. The flowchart demonstrates the different scenarios that can occur in case of a bank run at time t_i . Here, α_i is the insolvency barrier for balance sheet I in Table I, and $\bar{\alpha}_i$ is the insolvency barrier for balance sheet II in Table II after pledging risky assets as collateral.

$$\text{PS}^{\text{total}}(t) = \mathbb{E} \left[\mathbf{1}_{\left\{ \inf_{t \leq s \leq t_n} \{V(s) - \alpha(s)\} \geq 0 \right\}} \cdot \prod_{j=i}^{n-1} \mathbf{1}_{\{V_j \geq \bar{\beta}_j \text{ and } \inf_{t_j \leq u \leq t_n} \{V(u) - \bar{\alpha}_j(u)\} \geq 0\}} | V(t) \right], \quad (20)$$

where $\bar{\alpha}_j$ is defined by Equation (19), and with the convention that in case of $i=n$, the empty product equals one. The 1st term in the conditional expectation means that the asset value must stay above the insolvency barrier in the time period $[t, T]$. The 2nd term basically means that the financial institution must stay above all illiquidity barriers at each decision node, and that the firm is able to buffer the additional losses from higher interest rate payments connected with the collateralized debt. From Equation (20), we can easily calculate the *total default probability* $\text{PD}^{\text{total}}(t)$ for any time $t \in [0, T]$ as

$$\text{PD}^{\text{total}}(t) = 1 - \text{PS}^{\text{total}}(t). \quad (21)$$

The *default probability due to insolvency* at any time $t \in [0, T]$ can be computed analytically as in Black and Cox (1976) as the probability that the asset value will fall below the insolvency barrier at any time between time t and maturity T , that is,

$$\begin{aligned} \text{PD}^{\text{ins}}(t) &= \mathbb{P} \left(\inf_{t \leq s \leq T} (V(s) - \alpha(s)) < 0 | V(t) \right) \\ &= \Phi \left(\frac{\ln \frac{\alpha(T)}{V(t)} - (\mu - \frac{1}{2} \sigma^2)(T - t)}{\sigma \sqrt{T - t}} \right) \\ &\quad + \left(\frac{V(t)}{\alpha(t)} \right)^{1 - 2\frac{\mu - r}{\sigma^2}} \Phi \left(\frac{\ln \left(\frac{\alpha^2(t)}{V(t)\alpha(T)} \right) + (\mu - \frac{1}{2} \sigma^2)(T - t)}{\sigma \sqrt{T - t}} \right). \end{aligned} \quad (22)$$

The *default probability due to illiquidity* at time $t \in [0, T]$ can then be derived as the difference between the total default probability and the insolvency default probability, that is,

$$\text{PD}^{\text{ill}}(t) = \text{PD}^{\text{total}}(t) - \text{PD}^{\text{ins}}(t). \quad (23)$$

This is the probability that the asset value at some decision date will be less than the bank run barrier. In addition, the firm is not able to pay off short-term creditors with the capital it can raise at that time or will default in $[t, T]$ as it cannot buffer the additional loss from higher interest rate payments.

Given the above default probability PD^{total} and the expected rollover return R_0^S , computed by equation (16), we can calculate the expected values of short- and long-term debt.

Theorem 2

Under Assumptions 1 and 2, the expected value at time t of long-term debt with initial investment L_0 is

$$D^L(t) = e^{-r(T-t)} \{L_0 e^{r_L T} (1 - \text{PD}^{\text{total}}(t))\}. \quad (24)$$

At each date t_i (for $i = 0, \dots, n$), the expected value of short-term debt with initial investment S_0 is

$$D_i^S = e^{-r(T-t_i)} \{S_0 e^{r_S t_i} e^{\max\{R_i^S, R_i^*\}(T-t_i)}\}. \quad (25)$$

Between two rollover dates $t \in (t_{i-1}, t_i)$, the expected value of short-term debt is

$$D^S(t) = e^{-r(T-t)} \left\{ S_0 e^{r_S t_i} \mathbb{E} \left[e^{\max\{R_i^S, R_i^*\}(T-t_i)} | V(t) \right] \mathbb{P} \left(\inf_{t < s \leq t_i} (V(s) - \alpha(s)) \geq 0 | V(t) \right) \right\}. \quad (26)$$

Therefore, the expected value of short-term debt is a process that is left continuous with right limits. At each rollover date t_i (for $i = 1, \dots, n-1$), the process satisfies

$$D_i^S = \max \left\{ \lambda_i D_{i,+}^S, e^{-r(T-t_i)} \{S_0 e^{r_S t_i} e^{R_i^*(T-t_i)}\} \right\}, \quad (27)$$

where $D_{i,+}^S = \lim_{t \downarrow t_i} D^S(t)$. Here, R_i^S and R_i^* are defined recursively by Equations (16) and (14), respectively, with the convention $\lambda_0 = 1$.

For Equations (24–26), the terms in the braces are the values of long- and short-term debt, respectively, at terminal time T conditional on the current state. As an example, let us examine Equation (26) more closely. The 1st term $S_0 e^{r_S t_i} = S_i$ is the principal value of short-term debt at the rollover date t_i . Short-term creditors make their decision whether to rollover their debt or not at time t_i and obtain the maximal interest rate $\max\{R_i^S, R_i^*\}$ for the time period $[t_i, T]$. This is, of course, conditional on the firm being solvent in the time period $[t, t_i]$, which explains the last term in the brace.

Moreover, as stated in the theorem, short-term debt value is a process that is left continuous with right limits. Before the next decision date t_i , the expected debt value depends on the survival probability λ_i from a bank run at time t_i . Immediately after the rollover date, that is, at time t_i^+ , whether or not the firm survived a run is known, leading to a jump in the debt value resulting in Equation (27).

3. Numerical Results

3.1 DATA DESCRIPTION

We empirically test our model by applying it to the example of Merrill Lynch. Therefore, we calibrate the parameters of our model to the data of Merrill Lynch on October 10 2008, just before the US government announced a large intervention in the financial sector.⁷ We obtain the balance sheet information for Merrill Lynch from Table 2 in [Veronesi and Zingales \(2010\)](#). For the risk-free rate in our model, we choose the average 1-year Treasury rate during the second half of 2008, which is $r = 1.56\%$.⁸

For long-term debt in our model, we choose a maturity of $T = 3$ years, that is, we consider debt issued on October 10 2008 and expiring on September 10 2011. The 3-year CDS spread on October 10 2008 for Merrill Lynch is 4.30% and the 3-year treasury rate on that date is 1.87%. Thus, the interest rate for 3-year debt for Merrill Lynch is approximately equal to $r_L = 4.30\% + 1.87\% = 6.17\%$. The balance sheet of Merrill Lynch shows long-term debt of \$232.5 bn and other liabilities of \$272.0 bn which we will also consider as long-term. The average coupon rate of long-term debt outstanding is reported as 3.26%, and the average maturity is 4.9 years. The 5-year CDS spread on October 10 2008 is stated as 3.98%, and the 5-year treasury rate is 2.77%, implying an interest rate of approximately 6.75% for 5-year debt for Merrill Lynch. From these data, we can compute the present value of long-term debt principal and coupons as

$$\sum_{i=1}^5 3.26\% \cdot \$ (232.5 + 272.0) \text{ bn} \cdot e^{-0.0675 \cdot i} + \$ (232.5 + 272.0) \text{ bn} \cdot e^{-0.0675 \cdot 5} = \$427.5 \text{ bn} .$$

As long-term debt in our model is a zero-coupon bond, we set L_T equal to the principal of an equivalent zero-coupon bond with 3-year maturity, that is,

$$L_T = \$427.5 \text{ bn} \cdot e^{0.0617 \cdot 3} = \$514.4 \text{ bn} .$$

Moreover, the balance sheet of Merrill Lynch reports short-term debt of \$242.9 bn and deposits of \$90.0 bn. As both do not pay any coupons, we set

⁷ Under this plan, the largest US commercial banks received a \$125 bn equity infusion and a 3-year government guarantee on new unsecured bank debt issues. Compare [Veronesi and Zingales \(2010\)](#).

⁸ Compare Federal Reserve statistical releases. See also [He and Xiong \(2012b\)](#).

$$S_1 = \$(242.9 + 90.0) \text{ bn} = \$332.9 \text{ bn}.$$

We assume a rollover frequency of $\Delta t = 3$ month, that is, we have eleven rollover dates until maturity as T itself is no decision date. The 6-month CDS spread on October 10 2008 for Merrill Lynch is 4.88%, and the 6-month treasury rate on that date is 0.84%. As this is the shortest maturity CDS spread reported, we set the short-term rate in our model equal to $r_S = 5.72\%$. Thus, we obtain

$$S_0 = \$332.9 \text{ bn} \cdot e^{-0.0572 \cdot 1/4} = \$328.2 \text{ bn}.$$

We compute the firesale rates at the decision points from leverage data for the high-yield market based on quarterly data between September 30 2008 and March 31 2011 as provided by the Bank of America/Merrill Lynch. Leverage ratios $\mathcal{L}_i$ for dates t_i reported in Table IV have been computed as Net Debt over LTM EBITDA, that is, as earnings before interest, taxes, depreciation, and amortization for 12 consecutive months prior to the date of measurement. Motivated by recent work of Geanakoplos (2009), we define the haircut rate $\mathcal{H}$ as the reciprocal of leverage, that is, $\mathcal{H}(t) = 1/\mathcal{L}(t)$. This means that when the firm, for some reason, is forced to raise capital by pledging its risky assets as collateral, it has to cut off $1/\mathcal{L}(t)$ of the market value of the assets, that is, the capital which the firm can raise is

$$\psi(t)V(t) = (1 - \mathcal{H}(t))V(t) = \left(1 - \frac{1}{\mathcal{L}(t)}\right)V(t). \quad (28)$$

We apply this relationship between leverage ratios and firesale rates to determine ψ_i for every rollover date $i = 1, \dots, n-1$ from the leverage data of Merrill Lynch. The results are reported in Table IV.

We set the initial asset value in our model equal to the value for total assets reported in Table 2 of Veronesi and Zingales (2010), that is, $V_0 = \$875.8 \text{ bn}$. The actual volatility for Merrill Lynch is reported as 177.94%. This is the annualized daily standard deviation of daily returns estimated during the period July–September 2008. From this, we can estimate an annual volatility of $\sigma = 177.94\%/\sqrt{250} = 11.25\%$. As our model assumes risk-neutral investors, we set the drift rate equal to the risk-free rate, that is, $\mu = 1.56\%$.

It has been shown in Chen (2010) that the historical average of the recovery rates of bonds is about 40%. Therefore, we set the safety covenant equal to $\rho = 60\%$.

We set the cash amount $M = 0$ and the outside return r^* equal to the risk-free rate of $r = 1.56\%$. Finally, we set the interest rate r_c for collateralized

Table IV. Leverage and firesale rates for Merrill Lynch

The table shows the leverage ratios $\mathcal{L}_i$ for Merrill Lynch between September 30 2008 and March 31 2011. The firesale rates are computed from the leverage ratios as $\psi_i = 1 - 1/\mathcal{L}_i$.

Date t_i	Leverage ratio $\mathcal{L}_i$	Implied firesale rate $\psi_i = 1 - 1/\mathcal{L}_i$
September 30 2008	3.8	0.73
December 31 2008	3.7	0.73
March 31 2009	3.9	0.75
June 30 2009	3.9	0.74
September 30 2009	4.2	0.76
December 31 2009	3.9	0.74
March 31 2010	3.6	0.72
June 30 2010	3.4	0.71
September 30 2010	3.4	0.70
December 31 2010	3.5	0.72
March 31 2011	3.6	0.72

debt equal to $r_c = 8\%$ as this should be interpreted as the interest rate in a distressed state.

3.2 SENSITIVITY ANALYSIS

Our numerical results are based on the binomial tree implementation described in the [online supplement](#). To ensure a sufficiently close approximation of the continuous time case, we chose a high number $m=250$ of interim dates in our calculations.

Figure 2 shows two simulations of a bank run by short-term creditors for the example of Merrill Lynch. Maturity is $T=3$ years and we consider eleven rollover dates, that is, every 3 months. The light solid curves describe two simulated paths for the asset value process following a geometric Brownian motion. The dashed and the dotted graphs illustrate the illiquidity barrier and the bank run barrier, respectively. The dark solid graph is the insolvency barrier. The eleven rollover dates for short-term funding are marked. Note that the first decision date is at time $t=3$ months, and there is no bank run barrier before that date. Moreover, there does not exist a bank run barrier at maturity $T=3$ years as short-term creditors do not have to face a decision at that date. At the rollover date $t_6 = 18$ months, one of the simulated paths is less than the bank barrier, such that the short-term creditors would decide to run on the financial institution at that date. By pledging risky assets as collateral to raise new funds, the insolvency barrier changes to

Figure 2. Default simulation with eleven rollover dates. The figure shows a bank run scenario. The light solid curves describe two simulated paths for the asset value process following a geometric Brownian motion. The dashed and the dotted graphs illustrate the illiquidity barrier and the bank run barrier, respectively, where we used $n - 1 = 11$ rollover dates (i.e., every 3 months). At the marked decision date $t_6 = 18$ months, one simulated asset value is less than the bank run barrier, such that short-term creditors would decide to run on the financial institution at that date. The insolvency barrier changes after the run to the new barrier $\bar{\alpha}_6$ (described by the dash-dotted graph in the figure). The simulated path falls below this new insolvency barrier between the 8th and 9th rollover date. Thus, in that case, the bank run at time t_6 would trigger a default due to illiquidity. The other simulated path falls below the bank run barrier at t_8 . The new insolvency barrier $\bar{\alpha}_8(t)$ almost coincides with $\bar{\alpha}_6(t)$ and, therefore, is omitted in the figure. The simulated path always stays above this barrier until maturity such that the firm would survive the bank run. For the computation of the barriers, we used the parameters derived for Merrill Lynch in Section 3.1. For the safety covenant, we used $\rho = 70\%$. For the simulations in the binomial tree, $m = 250$ interim time steps are used.

$$\begin{aligned}\bar{\alpha}_6(t) &= 0.65 \cdot e^{-0.0156(3-t)} \cdot (\$327.6 \text{ bn} \cdot e^{0.0572 \cdot 1.5} e^{0.08 \cdot 1.5} + \$514.4 \text{ bn} \cdot e^{0.0617 \cdot 3}) \\ &= e^{-0.0156(3-t)} \cdot \$715.7 \text{ bn}.\end{aligned}$$

This barrier is illustrated in the figure as the dashed-dotted line. The simulated asset value falls below this barrier between the 8th and the 9th

rollover date triggering a default due to illiquidity.⁹ This default event cannot be captured in a classical structural credit risk model such as Black and Cox (1976) as the financial institution is still solvent at that time, that is, the simulated asset value stays above the insolvency barrier $\alpha(t)$ until that date. In the Black and Cox (1976) model, a default due to insolvency would occur later in time, namely at the 10th rollover date, when the simulated asset value falls below the insolvency barrier $\alpha(t)$. By taking liquidity risk into account, however, our novel approach can indeed account for this kind of default scenario as illustrated in Figure 2. The other simulated path falls below the bank run barrier at $t_8 = 24$ months. The new insolvency barrier $\bar{\alpha}_8(t)$ almost coincides with $\bar{\alpha}_6(t)$. Therefore, we did not include it in the figure. This path, however, always stays above the new insolvency barrier, such that the firm would survive the bank run in that case.

Figure 3 (a) shows the decomposition of the total default probability into its insolvency and illiquidity components for increasing initial asset value V_0 . For very low V_0 , the financial institution will almost surely default due to insolvency, that is, $\text{PD}^{\text{ins}} = 1$. In these cases, the reason for a default is clearly insolvency and not any liquidity problem. In particular, the default is caused by the event that the asset value falls below the insolvency barrier $\alpha(t)$ at some time $t \in (0, T]$. The insolvency barrier increases in time from 577 to 586 (compare Panel B), such that $\text{PD}^{\text{ins}} = 1$ for initial asset values less than 577 and $\text{PD}^{\text{ill}} = 0$ in these situations. For higher initial asset values, a default due to the fact that the asset value falls below the insolvency barrier becomes more and more unlikely, while the probability that the financial institution will default due to illiquidity problems increases up to a critical point. For initial asset values higher than this critical point, the probability that the asset value process stays above the illiquidity barrier and the firm is able to pay off collateralized debt at maturity is increasing again, such that the illiquidity default probability, and thus, also total default risk decreases.

Figure 3 (b) illustrates the time-dependent bank run, illiquidity, and insolvency barrier for the case of eleven rollover dates. As proved theoretically in Theorem 1, the figure shows that the bank run barrier is always higher than the insolvency barrier. Moreover, by definition, the illiquidity barrier lies between the bank run barrier and the insolvency barrier.

⁹ Note that the simulated path of the asset value process falls below the illiquidity barrier at the 9th rollover date. However, since the balance sheet has changed after the bank run at date t_6 , such that, there is no more short-term debt left, the illiquidity barrier is irrelevant after the bank run, and the drop below that barrier does not necessarily lead to a default.

Figure 3. Default barriers and influence of initial asset value on default probability. (a) illustrates how the total default probability PD^{total} , the insolvency probability PD^{ins} , and the illiquidity probability PD^{ill} vary with increasing initial asset value V_0 . The parameters are chosen for the example of Merrill Lynch (see Section 3.1). We use a safety covenant of $\rho = 60\%$. The number of rollover dates is fixed as $n - 1 = 11$. (b) Shows how the individual barriers change with time. For the simulations in the binomial tree, $m = 250$ interim time steps are used.

As illustrated in Figure 4 (a), we get that the total default risk is increasing in volatility σ . This is also intuitive as higher volatility leads to higher fluctuations in the asset value, and thus increases the total default risk.

Furthermore, we studied the influence of the outside option return r^* on the total default probability. As shown in Figure 4 (b), total default risk is increasing in the outside option return r^* . This is due to the fact that running on the financial institution becomes more attractive. The higher the outside option return, the more likely are the short-term creditors to run.

3.3 EMPIRICAL PREDICTIONS

It has been shown in Adrian and Shin (2008, 2010) that the leverage ratio, and in particular, the use of very short-term financing for the largest US investment banks, dramatically increased during the financial crisis 2007–08 and shortly before. This means that financial institutions borrowed short-term on the interbank market and invested in long-term risky assets. Due to this *maturity mismatch*, short-term debt had to be rolled over frequently, and banks had to rely on the possibility to refinance themselves on a short-term basis on the interbank market. Hence, they were heavily exposed to rollover risk. The recent work of Brunnermeier and Oehmke (forthcoming) provides theoretical support for this empirical phenomenon of a maturity mismatch.

Figure 4. Influence of volatility and rollover frequency on default probabilities. The figures show how volatility σ (Panel A) and outside option return r^* (Panel B) influence the default probabilities. We use the parameters described in Section 3.1 with a safety covenant of $\rho = 60\%$. In Panel A, we vary the volatility σ between 10% and 20%. In Panel B, we vary the outside return r^* between 1.5% and 5.5%. For the simulations in the binomial tree, $m = 250$ interim time steps are used.

The authors show that in equilibrium, financial institutions choose maturity structures that are inefficiently short-term. As high rollover frequencies make early liquidation more likely, this excessive reliance on short-term financing results in unnecessary rollover risk and inefficient debt runs. This emphasizes the importance of quantifying these risks and imposing regulatory restrictions on short-term financing to preserve financial stability.

Our model provides a method to explicitly quantify this rollover risk, and our empirical results highlight the importance of the maturity structure for financial stability. More specifically, Figure 5 (a) shows the influence of the number of rollover dates on the default probabilities in our model. The illiquidity PD is monotonically increasing with increasing rollover frequency. This can be explained by the fact that more rollover dates mean that short-term creditors have more possibilities to run on the financial institution, implying that the probability that the firm will encounter illiquidity problems is increasing. Hence, both the total default probability and the illiquidity probability should increase with the number of rollover dates. The insolvency probability, in contrast, is almost unaffected by the rollover frequency.

Figure 5 (b) shows that the total default probability as well as the illiquidity probability are increasing in the short-term debt ratio. The reason for this is that the influence of the short-term debt ratio on the illiquidity default probability is mainly through the survival probability λ_i from a bank run.

Figure 5. Influence of outside option return and short-term debt ratio on default probability. The figures show the dependence of the total default probability PD^{total} , the insolvency probability PD^{ins} , and the illiquidity probability PD^{ill} on the rollover frequency $n - 1$ (Panel A) and on the ratio of short-term debt S_T over total debt $S_T + L_T$ (Panel B). We use the parameters specified in Section 3.1. In Panel A, we vary the number $n - 1$ of rollover dates between one and eleven. For this figure, we use a constant firesale rate of $\psi = 70\%$. In graph (b), we fixed the sum of short- and long-term debt face value ($S_T + L_T$) to \$800 bn. For the simulations in the binomial tree, $m = 250$ interim time steps are used.

For small short-term debt ratio, λ_i will always be one, because the financial institution will almost surely be able to pay off short-term creditors. In these cases, the firm will quite likely survive a bank run. For a high short-term debt ratio, however, the survival probability λ_i from a bank run will be very small, such that a default due to illiquidity, as a consequence of a bank run, becomes very likely.

In this way, our article supports the theoretical results of Brunnermeier and Oehmke (forthcoming) and the empirical findings of Adrian and Shin (2008, 2010). Moreover, the empirical predictions of our model provide support for the reform package “Basel III” introduced by the Basel Committee of Banking Supervision to enhance financial stability. As one building block in BCBS (2010a, b), minimum global liquidity standards have been suggested to improve banks’ resilience to acute short-term stress and to improve longer term funding. In particular, a *liquidity coverage ratio*, to ensure that a financial institution holds enough liquid assets, and a *net stable funding ratio* have been introduced. The latter ensures that the ratio of the available amount of stable funding over the required amount of stable funding is more than 100%. Hence, it aims at reducing the above discussed maturity mismatch problem.

A consequence of this maturity mismatch and the heavy reliance on short-term financing during the financial crisis 2007–08 and shortly before was that banks, which experienced financial distress, encountered severe difficulties in refinancing themselves on the interbank market. Hence, they were forced to sell their risky assets exactly when prices were low. The resulting loss spiral has been discussed in Brunnermeier (2009). Moreover, other traders might force the distressed financial institutions to sell their *toxic* assets at firesale prices, and thereby, to reduce their leverage ratios. This enforces the loss spiral even further as has been theoretically shown in Brunnermeier and Pedersen (2009) and empirically confirmed in Adrian and Shin (2008).

Therefore, it has been suggested that, besides interest rates, leverage (and thereby, firesale rates) can be used as monetary policy tools (see, e.g., Ashcraft, Gârleanu, and Pedersen, 2010; Morris and Shin, 2009; and Geanakoplos, 2009). When liquidity is tight, central banks can either lower interest rates or provide lending with more generous haircut to enhance liquidity in the market. In Figure 6 (a), we increased the firesale rate ψ from 60% to 70%. Here, we chose the firesale rate to be constant for all decision dates. The figure shows that the total default probability is decreasing with increasing firesale rates, that is, with decreasing haircuts. Thereby, lowering haircuts has a similar effect as decreasing interest rates (see Panel B in Figure 6). Hence, our result supports the above suggestion. If central banks are lending money at a generous haircut (i.e., at a higher firesale rate), this means that the firm can negotiate a better haircut with the central bank when failing to borrow against the risky assets from other financial institution, which in turn increases liquidity in the market.

In the figure, we used a safety covenant of $\rho = 55\%$ as suggested in He and Xiong (2012b). For higher safety covenants, the total default probability is mainly influenced by the likelihood that the asset value falls below the insolvency barrier α , or in case of a bank run, below the new insolvency barrier $\bar{\alpha}$. The firesale rate then only has a very small influence. The intuition underlying this is that the firm might be able to raise enough funds to pay off short-term debt, but in doing so, the firm is exposed to bankruptcy risk. If it is not able to pay back collateralized debt, it will default. This probability increases significantly when the safety covenant ρ increases.

4. Extensions and Discussion

4.1 ENDOGENOUS FIRESALE RATE

The firesale price $\psi_i V_i$ at time t_i in our multiperiod bank run model is defined as the funds that can be raised by pledging the risky asset as collateral to the

Figure 6. Two monetary tools. The figure exhibits the sensitivity of the total default probability with respect to the firesale rate ψ and with respect to the interest rate r_c for collateralized debt. We use the parameters of Section 3.1. In Panel A, we fix the rate r_c for collateralized debt to 2% and increase the firesale rate from 60% to 70% (constant for all decision dates). In Panel B, we fix $\psi = 60\%$ and vary the interest rate r_c for collateralized debt between 2% and 10%. The safety covenant is $\rho = 55\%$. For the simulations in the binomial tree, $m = 250$ interim time steps are used.

lending banks. As already discussed in Section 3.1, the determination of the firesale rate ψ depends on the concept of leverage $\mathcal{L}$, which is given as the ratio of total assets over equity, that is,

$$\mathcal{L}(t) = \frac{V(t)}{V(t) - D(t)},$$

where $D(t)$ denotes the debt value at time t . Due to Equation (28), the firesale price at any rollover date t_i in our model can be defined endogenously as

$$\psi_i V_i = D_{i,+}, \quad (29)$$

where $D_{i,+} = D(t_i^+)$ and t_i^+ is the time right after the rollover date t_i . At a given rollover time, $D_{i,+}$ is the expected debt value conditional on the asset value V_i without taking into account the survival probability from a bank run at time t_i . We compute the total debt value as $D_{i,+} = D_{i,+}^S + D_{i,+}^L$, where the values of short- and long-term debt are defined as in Theorem 2. Note that $D^S(t)$ is a left continuous process with right limits. Before the next rollover date t_i , the debt value depends on the survival probability from a bank run at t_i . Immediately after the decision date t_i , however, whether the firm survived a run or not is known. This leads to a jump in the debt value. Since the expected debt value is increasing in the asset value, the firesale price

will also be higher when the asset value at a rollover date t_i is higher. Moreover, the firesale price is affected by the future rollover risk, and hence, by illiquidity risk. The more likely a bank run is to occur in the future, the lower will be the firesale price. This endogenous determination of the firesale rate is of course only forward looking and does not consider the current illiquidity risk, that is, the risk from a bank run at time t_i . However, it provides reasonable results while keeping numerical tractability.

Figure 7 shows that leverage is time varying and pro-cyclical, that is, it is high in boom times and low in recession times. An excellent empirical study on leverage has been carried out by Adrian and Shin (2010). To determine the leverage, and thereby, the firesale rate, in practice, is a challenging task. Earlier work on firesale rates include Shleifer and Vishny (1992), Bernanke, Gertler, and Gilchrist (1996, 1999), and Holmstrom and Tirole (1997). These papers emphasize the asymmetric information between a borrower and a lender, leading to a principal agent problem. In contrast to the aforementioned literature, Geanakoplos (1997, 2003, 2009) argues that leverage, which is endogenously determined in general equilibrium theory, is not determined by asymmetric information but by the distribution of collateral payoffs. The determination of leverage in the abovementioned papers, from a practical modeling point of view, unfortunately has to be compromised by an additional constraint, such as the margin requirement in Gârleanu and Pedersen (2011) and Ashcraft, Gârleanu, and Pedersen (2010), or the VaR constraint in Danielsson, Shin, and Zigrand (2009). Deriving endogenous firesale rates by general equilibrium theory is beyond the scope of this article. It should be noted, however, that it is in principle possible to incorporate these more microfounded determinations of firesale prices in our model. However, including these frameworks in our setting will insert some new parameters, for example, the level of risk tolerance of market participants, the determination of which might be problematic for practical applications.

4.2 COMPETITION AMONG BANKS

In our article, we consider a bank run model with one bank. Hence, the question arises whether such a monopolistic bank model can fully capture all the salient features of the real world.¹⁰ In real markets, competition among banks will influence the interest rates r_S and r_L , offered by a financial institution. To study the effect of competing firms, one can consider a market with K financial institutions B_k issuing short- and long-term debt at rates r_S^k and r_L^k , respectively, to finance risky assets for $k = 1, \dots, K$. Suppose that

¹⁰ We thank an anonymous referee for pointing out this question to us.

Figure 7. Average leverage over time. The figure shows the time evolution of both the leverage and the firesale rate for the high-yield market based on quarterly data between December 31 1996 and March 31 2011, as provided by the Bank of America/Merrill Lynch. Leverage has been computed as net debt over LTM EBITDA, that is, the earnings before interest, taxes, depreciation, and amortization for 12 consecutive months prior to the date of measurement. Firesale rate is computed as $1-1/\text{leverage}$.

these firms do not lend money. Instead, assume that there exists a representative creditor who wants to invest an amount S_0 short term and an amount L_0 long term in the market. Then, the creditor has to determine the portion $\omega_k \in [0, 1]$ of S_0 , which she will invest for a short term in institution B_k at rate r_S^k . Similarly, she has to determine the portion $\nu_k \in [0, 1]$ of L_0 , which she will invest for a long term in the same institution at rate r_L^k . The creditor determines ω_k and ν_k to maximize her return subject to the constraint that the risk (measured by some appropriate risk measure) of her position is below some exogenously given level. The financial institution B_k , on the contrary, has to finance its risky assets with short-term debt $\omega_k S_0$ and long-term debt $\nu_k L_0$. Equilibrium markets should clear, which means that the financial institution B_k issues short- and long-term debt at the lowest possible rates r_S^k and r_L^k such

that it is able to finance its risky assets using liabilities $\omega_k S_0 + v_k L_0$. Note that in equilibrium, r_S^k and r_L^k will always be higher than the risk-free rate r , which can be interpreted as the outside rate r^* in this framework. Otherwise, the creditor will not participate in the debt market but instead invest in the risk-free rate. Hence, the assumptions of our model are still satisfied in this setting.

Solving such a general equilibrium model leads to a complex numerical problem since the constraint of the creditor, that the risk of her position does not exceed a given level, depends on the default probabilities of each firm in the market. The latter can be computed in our structural approach for given parameters $r_S^k, r_L^k, \omega_k S_0$, and $v_k L_0$, for every $k = 1, \dots, K$. In general, such a problem should be numerically solvable, though only with considerable computational effort. Such an equilibrium approach offers a way to endogenize r_S and r_L as well as the capital structure characterized by S_0 and L_0 . However, it does not change the general results of our article; for example, the total default probability is decreasing with increasing short-term rate r_S since short-term creditors will more likely decide to roll over their debt. Moreover, since our article is mainly motivated by classical structural credit risk models, where these parameters are exogenous, we do not investigate this idea further.

4.3 FURTHER EXTENSIONS

As already discussed in Remark 2, in real life, one observes new data for the short-term rate r_S , the market rate r^* , asset volatility, etc., at each decision date. These will surely influence the rollover decision of short-term creditors as well as the investment decisions of new creditors. Hence, in reality, one actually faces a superposition of processes of the type we discuss in this article. Incorporating these features in our current setting would significantly complicate our model. Moreover, analyzing the economic implications in such a general setting will be much more complicated. For a real-life application, however, taking these possibilities into account seems to be very important. We will leave this problem for future research.

In our model, short-term creditors can decide whether to roll over their debt or not at a finite number of rollover dates $t_1, \dots, t_{n-1}$. Hence, it seems natural to consider the extension of our model to the continuous bank run case, that is, to the situation where creditors can decide to run on the firm at any time. This extension, however, is not as simple as it seems to be at first sight. The main difficulty arises from the formulation of the value function for short-term creditors, that is, the expected value of short-term debt $D^S(t)$ at any time $t \in [0, T]$, or alternatively, from the specification of the expected

rollover return $R^S(t)$ at any time $t \in [0, T]$. In contrast to standard equations of Bellman type, in our dynamic programming equation, short-term creditors need to update their value function D^S , or equivalently, R_i^S at each rollover date by updating their beliefs about the survival probability from a bank run. Thus, the expected return R_i^S in our model is always discounted by λ_i . Due to this feature, we cannot derive a corresponding free-boundary problem or optimal stopping time problem for short-term creditors when the rollover frequency tends to infinity.

Furthermore, it would be interesting to study the impact of liquidity risk on *systemic risk* in our multiperiod bank run model. Here, systemic risk is understood as the risk that financial distress, initially affecting only a single institution or a small number of firms, can spread to a large part of a certain sector or even to the whole economy. Classical bank run models based on spillover effects, for example, through interbank lending as in [Allen and Gale \(2000\)](#), could be extended to our definition of total credit risk. Hence, one could study the effect that the default due to illiquidity of a financial institution leads to defaults of its counterparties, and thereby increases systemic risk in the financial sector.¹¹ Related work in this context has been carried out in [Cont, Moussa, and Santos \(2010\)](#), who provide a quantitative model for default contagion in a network structure, where firms can default because of insufficiently many liquid assets.

4.4 RELATED LITERATURE

There already exists extensive theoretical literature on potential causes for bank runs due to liquidity risk. The classical models of [Bryant \(1980\)](#), [Diamond and Dybvig \(1983\)](#), and [Rochet and Vives \(2004\)](#), for example, provide evidence for the fact that runs can occur due to self-fulfillment of depositor's expectations concerning the behavior of other depositors. Thus, bank runs are a result of coordination problems among short-duration depositors' rollover decisions. [Morris and Shin \(2000, 2004\)](#) and [Goldstein and Pauzner \(2005\)](#) use global games methods to model the coordination problem in rollover decisions. All these papers, however, do not consider the influence of future default risk on today's decisions. [Guimaraes \(2006\)](#) and [He and Xiong \(2012a, b\)](#) extend these models of static coordination problems to dynamic ones. In [He and Xiong \(2012b\)](#), the firm's debt expirations follow a Poisson distribution with infinite time horizon. The authors endogenously determine the unique threshold value such that each maturing creditor would decide to run on the firm if the asset value falls below this

¹¹ We refer to [Morris and Shin \(2004\)](#) for related models on market runs.

threshold. In their model as well as in our setting, the decision of a short-term creditor whether to roll over her debt or not after expiration depends also on her expectation about the rollover decisions of creditors maturing at different times. However, in contrast to our setting, their model cannot capture the impact of the rollover decision of other creditors maturing at the same time. This effect can indeed be significant, in particular, as shown in Adrian and Shin (2010, 2008), that the use of very short-term financing dramatically increased during the financial crisis 2007–08 and shortly before.

Our model bears similarities with Morris and Shin (2009) in that at the rollover dates, the coordination problem between short-term creditors is a binary decision in terms of global games. The concept of global games has been introduced by Carlsson and van Damme (1993) and has been successfully applied to dynamic coordination problems in Morris and Shin (1998, 2003). In Morris and Shin (2009), the authors solve for the unique equilibrium in the coordination problem faced by short-term creditors. Using global game arguments for the rollover decision between short-term creditors, they prove that in equilibrium, the ratio of raised funds to the principle of the due short-term debt measures the likelihood of a bank run. In our model, we interpret this ratio as the survival probability from a bank run (compare Assumption 1). This ratio is connected with the firesale rate, which can be determined endogenously in our model (compare with Section 4.1), and which plays an important role in deriving the illiquidity barrier.

Our article contributes to the existing literature in the following way. Our model is, to the best of our knowledge, the first structural credit risk model that takes into account both insolvency risk and illiquidity risk and that can accommodate multiple rollover times for short-term debt. Our model is fully dynamic in the sense that at each rollover date, the short-term creditors' decision, whether to renew their funding or not, does not only depend on future insolvency risk but also on future illiquidity risk, represented by the possibility of bank runs at later decision dates. Unlike the two-period game theoretic approach of Morris and Shin (2009) with a single rollover date, our model can accommodate multiple rollover dates. Thus, a run can happen at any of these rollover dates. In fact, from the valuation of both short- and long-term debt in our model, it is also clear that debt value decreases with increasing rollover frequency. This cannot be revealed in models with only one decision date as in Morris and Shin (2009). The multiple maturity structure makes our model more realistic in comparison to previous approaches because in practice, banks usually finance themselves by a combination of repo transactions, commercial papers, and very short-duration contracts with high rollover frequencies. Moreover, insolvency can happen at any time until maturity in our setting. On the one hand, these features make

our model ready for practical application. On the other hand, they also complicate the model significantly. Mathematically, the extension to a multiperiod setting can be compared to switching from European to Bermudan option pricing problems.

Another important distinction from [Morris and Shin \(2009\)](#) is that in their model, when short-term creditors choose not to roll over their debt, they can always get the face value of their debt back and go to the market to earn the return r^* . This, however, is only true when the financial institution is healthy and has enough cash to pay back the creditors. Under such an assumption, the incentive of the short-term creditors not to renew their funding clearly increases. In our article, we model the expected return from running on the firm depending on the financial institution's condition. If it is healthy, the likelihood that creditors will get the face value of their debt back is high; if it is in distress (with significant haircut of the asset value), the survival probability of a bank run is low such that the expected outside return is also lower.

The results in [Huang and Huang \(2003\)](#) and [Bao, Pan, J. and Wang \(2011\)](#) suggest that illiquidity risk can explain a significant fraction of credit spreads. Theoretical models, such as [Ericsson and Renault \(2006\)](#), in which illiquidity is caused by debt trading distress, and [Morellec \(2001\)](#), in which illiquidity is caused by asset trading restriction, reveal that the spread is increasing with liquidity risk. In their approach, illiquidity stems from an exogenously given liquidity shock. In our model, we can explicitly quantify the add-on to the insolvency default probability that is solely due to liquidity risk. Our model shows the same results as the aforementioned literature, but allows us to distinguish the contribution of each risk component to total default risk. Hence, from a risk management perspective, our novel approach enables the corresponding risks to be hedged more effectively.

5. Conclusion

In this article, we presented a new multiperiod bank run model that includes illiquidity risk and insolvency risk. Using a structural credit risk modeling approach, similar to the classical [Black and Cox \(1976\)](#) model for insolvency risk, and incorporating illiquidity risk through the possibility of short-term creditors to run on the firm, we succeeded in splitting total default probability into an insolvency component and an illiquidity component. We studied their dependencies on the individual model parameters and showed that total default risk is increasing in volatility and in the ratio of short-term funding over total debt. These results are in accordance with previously derived

implications by [Morris and Shin \(2009\)](#) for the situation with only one rollover date. Moreover, we showed that illiquidity risk is increasing in the number of rollover dates and that increasing leverage in the market will decrease default probabilities. The latter, in particular, provides evidence that leverage should be used as an additional monetary policy tool (besides interest rates) to enhance liquidity in the market, which has also been pointed out by [Ashcraft, Gârleanu, and Pedersen \(2010\)](#).

Supplementary Material

[Supplementary data](#) are available at *Review of finance* online.

Appendix A

A.1. Proofs of Main Results

A.1.1 PROOF OF THEOREM 1

We first prove the existence and uniqueness of a solution to [Equation \(17\)](#). For $i = n - 1$, [Equation \(17\)](#) simplifies to

$$e^{(r^* - r_S)\Delta t} = \mathbb{P}\left(\inf_{t_{n-1} < s \leq t_n} (V(s) - \alpha(s)) \geq 0 \mid V_{n-1}\right). \quad (\text{A.1})$$

The RHS is strictly increasing and continuous as a function of the asset value $V_{n-1} \in [\alpha_{n-1}, \infty)$ and takes values in $[0, 1]$. As $r^* < r_S$ by Assumption 1, there exists a unique solution to [Equation \(A.1\)](#).

For $i = n - 2$, [Equation \(17\)](#) can be reformulated as

$$\begin{aligned} 1 = \mathbb{E}[\lambda_{n-1} \cdot \max \left\{ e^{(r_S - r^*)\Delta t}, e^{(r_S - r^*)2\Delta t} \right. \\ \cdot \mathbb{P}\left(\inf_{t_{n-1} < s \leq t_n} (V(s) - \alpha(s)) \geq 0 \mid V_{n-1}\right) \Big\} \mid V_{n-2}] \\ \cdot \mathbb{P}\left(\inf_{t_{n-2} < s \leq t_{n-1}} (V(s) - \alpha(s)) \geq 0 \mid V_{n-2}\right). \end{aligned}$$

Note that the RHS of the above equation is a strictly increasing and continuous function in $V_{n-2} \in [\alpha_{n-2}, \infty)$ as the survival probability from a bank run and the survival probability from insolvency are continuous and increasing and continuous and strictly increasing, respectively. Moreover, the RHS takes its minimum value zero for $V_{n-2} \rightarrow \alpha_{n-2}$ and we have

$$\begin{aligned}
\lim_{V_{n-2} \rightarrow \infty} \mathbb{E}[\lambda_{n-1} \cdot \max & \left\{ e^{(r_S - r^*)\Delta t}, e^{(r_S - r^*)2\Delta t} \right. \\
& \cdot \mathbb{P}\left(\inf_{t_{n-1} < s \leq t_n} (V(s) - \alpha(s)) \geq 0 \mid V_{n-1} \right) \Bigg] \Bigg| V_{n-2} \\
& \cdot \mathbb{P}\left(\inf_{t_{n-2} < s \leq t_{n-1}} (V(s) - \alpha(s)) \geq 0 \mid V_{n-2} \right) = e^{(r_S - r^*) \cdot 2\Delta t} > 1
\end{aligned}$$

since $r^* < r_S$ by Assumption 1. Hence, there must exist a unique solution to the above equation. For any $1 \leq i \leq n-2$, we have

$$\begin{aligned}
1 = \mathbb{E}[\max & \left\{ e^{(r_S - r^*)\Delta t}, e^{r_S 2\Delta t} \mathbb{P}\left(\inf_{t_{i+1} < s \leq t_{i+2}} (V(s) - \alpha(s)) \geq 0 \mid V_{i+1} \right) \right. \\
& \cdot \mathbb{E}\left[e^{\max\{R_{i+2}^*, R_{i+2}^S\} (n-i-2)\Delta t} \mid V_{i+1} \right] \Bigg] \cdot \lambda_{i+1} \mid V_i \\
& \cdot \mathbb{P}\left(\inf_{t_i < s \leq t_{i+1}} (V(s) - \alpha(s)) \geq 0 \mid V_i \right).
\end{aligned}$$

By backward induction, the RHS is a strictly increasing and continuous function in V_i with minimum value zero and maximum value greater than one for $V_i \rightarrow \infty$. Thus, there must exist a unique solution to Equation (17).

To prove the last statement, note that the RHS of the above equation always takes its minimum value for $V_i = \alpha_i$. Thus, it is straightforward from the above that the bank run barrier is always higher than the insolvency barrier at the same rollover date.

A.1.2 PROOF OF THEOREM 2

The derivations of $D^L(t)$ and $D^S(t)$ directly follow from the derivation of PD^{total} . Therefore, we only prove the sample path property of $D^S(t)$. Since R_i^S is measurable with respect to V_i , it is obvious that

$$\lim_{t \uparrow t_i} D^S(t) = D_i^S.$$

On the other hand, by the recursive computation of the bank run barrier in Theorem 1 (compare also with Equation (15)), we have

$$\begin{aligned}
D_{i,+}^S &= \lim_{t \downarrow t_i} D^S(t) \\
&= e^{-r(T-t_i)} S_0 e^{rSt_i} \mathbb{E} \left[e^{\max\{R_{i+1}^S, R_{i+1}^*\}(T-t_{i+1})} | V_i \right] \mathbb{P} \left(\inf_{t_i < s \leq t_{i+1}} (V(s) - \alpha(s)) \geq 0 | V_i \right) \\
&= e^{-r(T-t_i)} S_0 e^{rSt_i} e^{R_i^S(T-t_i)} / \lambda_i.
\end{aligned} \tag{A.2}$$

Therefore, if short-term creditors roll over their debt, the expected debt value is $D_{i,+}^S \cdot \lambda_i$, and if they run on the firm, the debt value is determined by the outside return for the remaining time period $[t_i, T]$. As short-term creditors will choose the maximal return, the expected debt value at time t_i is given by

$$\begin{aligned}
D_i^S &= \max \left\{ e^{-r(T-t_i)} S_0 e^{rSt_i} e^{R_i^S(T-t_i)}, e^{-r(T-t_i)} S_0 e^{rSt_i} e^{R_i^*(T-t_i)} \right\} \\
&= \max \left\{ \lambda_i D_{i,+}^S, e^{-r(T-t_i)} S_0 e^{rSt_i} e^{R_{i+1}^*(T-t_i)} \right\},
\end{aligned} \tag{A.3}$$

which completes the proof.

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