Non-properly embedded $H$-planes in $\mathbb{H}^2 \times \mathbb{R}$

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Abstract For any $H \in (0, \frac{1}{2})$, we construct complete, non-proper, stable, simply-connected surfaces embedded in $\mathbb{H}^2 \times \mathbb{R}$ with constant mean curvature $H$.

1 Introduction

In their ground breaking work [2], Colding and Minicozzi proved that complete minimal surfaces embedded in $\mathbb{R}^3$ with finite topology are proper. Based on the techniques in [2], Meeks and Rosenberg [5] then proved that complete minimal surfaces with positive injectivity embedded in $\mathbb{R}^3$ are proper. More recently, Meeks and Tinaglia [7]
proved that complete constant mean curvature surfaces embedded in $\mathbb{R}^3$ are proper if they have finite topology or have positive injectivity radius.

In contrast to the above results, in this paper we prove the following existence theorem for non-proper, complete, simply-connected surfaces embedded in $\mathbb{H}^2 \times \mathbb{R}$ with constant mean curvature $H \in (0, 1/2)$. The convention used here is that the mean curvature function of an oriented surface $M$ in an oriented Riemannian three-manifold $N$ is the pointwise average of its principal curvatures.

The catenoids in $\mathbb{H}^2 \times \mathbb{R}$ mentioned in the next theorem are defined at the beginning of Sect. 2.1.

**Theorem 1.1** For any $H \in (0, 1/2)$ there exists a complete, stable, simply-connected surface $\Sigma_H$ embedded in $\mathbb{H}^2 \times \mathbb{R}$ with constant mean curvature $H$ satisfying the following properties:

1. The closure of $\Sigma_H$ is a lamination with three leaves, $\Sigma_H$, $C_1$ and $C_2$, where $C_1$ and $C_2$ are stable catenoids of constant mean curvature $H$ in $\mathbb{H}^3$ with the same axis of revolution $L$. In particular, $\Sigma_H$ is not properly embedded in $\mathbb{H}^2 \times \mathbb{R}$.

2. Let $K_L$ denote the Killing field generated by rotations around $L$. Every integral curve of $K_L$ that lies in the region between $C_1$ and $C_2$ intersects $\Sigma_H$ transversely in a single point. In particular, the closed region between $C_1$ and $C_2$ is foliated by surfaces of constant mean curvature $H$, where the leaves are $C_1$ and $C_2$ and the rotated images $\Sigma_H(\theta)$ of $\Sigma$ around $L$ by angle $\theta \in [0, 2\pi)$.

When $H = 0$, Rodríguez and Tinaglia [10] constructed non-proper, complete minimal planes embedded in $\mathbb{H}^2 \times \mathbb{R}$. However, their construction does not generalize to produce complete, non-proper planes embedded in $\mathbb{H}^2 \times \mathbb{R}$ with non-zero constant mean curvature. Instead, the construction presented in this paper is related to the techniques developed by the authors in [3] to obtain examples of non-proper, stable, complete planes embedded in $\mathbb{H}^3$ with constant mean curvature $H$, for any $H \in [0, 1)$.

There is a general conjecture related to Theorem 1.1 and the previously stated positive properness results. Given $X$ a Riemannian three-manifold, let $\text{Ch}(X) := \inf_{S \in S} \frac{\text{Area}(\partial S)}{\text{Vol}(S)}$, where $S$ is the set of all smooth compact domains in $X$. Note that when the volume of $X$ is infinite, $\text{Ch}(X)$ is the Cheeger constant.

**Conjecture 1.2** Let $X$ be a simply-connected, homogeneous three-manifold. Then for any $H \geq \frac{1}{2} \text{Ch}(X)$, every complete, connected $H$-surface embedded in $X$ with positive injectivity radius or finite topology is proper. On the other hand, if $\text{Ch}(X) > 0$, then there exist non-proper complete $H$-planes in $X$ for every $H \in [0, \frac{1}{2} \text{Ch}(X))$.

By the work in [2], Conjecture 1.2 holds for $X = \mathbb{R}^3$ and it holds in $\mathbb{H}^3$ by work in progress in [6]. Since the Cheeger constant of $\mathbb{H}^2 \times \mathbb{R}$ is 1, Conjecture 1.2 would imply that Theorem 1.1 (together with the existence of complete non-proper minimal planes embedded in $\mathbb{H}^2 \times \mathbb{R}$ found in [10]) is a sharp result.

**2 Preliminaries**

In this section, we will review the basic properties of $H$-surfaces, a concept that we next define. We will call a smooth oriented surface $\Sigma_H$ in $\mathbb{H}^2 \times \mathbb{R}$ an $H$-surface if

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it is embedded and its mean curvature is constant equal to $H$; we will assume that $\Sigma_H$ is appropriately oriented so that $H$ is non-negative. We will use the cylinder model of $\mathbb{H}^2 \times \mathbb{R}$ with coordinates $(\rho, \theta, t)$; here $\rho$ is the hyperbolic distance from the origin (a chosen base point) in $\mathbb{H}^2_0$, where $\mathbb{H}^2_t$ denotes $\mathbb{H}^2 \times \{t\}$. We next describe the $H$-catenoids mentioned in the Introduction.

The following $H$-catenoids family will play a particularly important role in our construction.

### 2.1 Rotationally invariant vertical $H$-catenoids $C^H_d$

We begin this section by recalling several results in [8,9]. Given $H \in (0, \frac{1}{2})$ and $d \in [-2H, \infty)$, let

$$\eta_d = \cosh^{-1} \left( \frac{2dH + \sqrt{1 - 4H^2 + d^2}}{1 - 4H^2} \right)$$

and let $\lambda_d : [\eta_d, \infty) \to [0, \infty)$ be the function defined as follows.

$$\lambda_d(\rho) = \int_{\eta_d}^{\rho} \frac{d + 2H \cosh r}{\sqrt{\sinh^2 r - (d + 2H \cosh r)^2}} dr. \quad (1)$$

Note that $\lambda_d(\rho)$ is a strictly increasing function with $\lim_{\rho \to \infty} \lambda_d(\rho) = \infty$ and derivative $\lambda'_d(\eta_d) = \infty$ when $d \in (-2H, \infty)$.

In [8] Nelli, Sa Earp, Santos and Toubiana proved that there exists a 1-parameter family of embedded $H$-catenoids $\{C^H_d \mid d \in (-2H, \infty)\}$ obtained by rotating a generating curve $\lambda_d(\rho)$ about the $t$-axis. The generating curve $\hat{\lambda}_d$ is obtained by doubling the curve $(\rho, 0, \lambda_d(\rho))$, $\rho \in [\eta_d, \infty)$, with its reflection $(\rho, 0, -\lambda_d(\rho))$, $\rho \in [\eta_d, \infty)$.

Note that $\hat{\lambda}_d$ is a smooth curve and that the necksize, $\eta_d$, is a strictly increasing function in $d$ satisfying the properties that $\eta_{-2H} = 0$ and $\lim_{d \to \infty} \eta_d = \infty$.

If $d = -2H$, then by rotating the curve $(\rho, 0, \lambda_d(\rho))$ around the $t$-axis one obtains a simply-connected $H$-surface $E_{-2H}$ that is an entire graph over $\mathbb{H}^2_0$. We denote by $-E_H$ the reflection of $E_H$ across $\mathbb{H}^2_0$.

We next recall the definition of the mean curvature vector.

**Definition 2.1** Let $M$ be an oriented surface in an oriented Riemannian three-manifold and suppose that $M$ has non-zero mean curvature $H(p)$ at $p$. The **mean curvature vector at** $p$ is $\mathbf{H}(p) := H(p)N(p)$, where $N(p)$ is its unit normal vector at $p$. The mean curvature vector $\mathbf{H}(p)$ is independent of the orientation on $M$.

Note that the mean curvature vector $\mathbf{H}$ of $C^H_d$ points into the connected component of $\mathbb{H}^2 \times \mathbb{R} - C^H_d$ that contains the $t$-axis. The mean curvature vector of $E_H$ points upward while the mean curvature vector of $-E_H$ points downward.

In order to construct the examples described in Theorem 1.1, we first obtain certain geometric properties satisfied by $H$-catenoids. For example, in the following lemma, we show that for certain values of $d_1$ and $d_2$, the catenoids $C^H_{d_1}$ and $C^H_{d_2}$ are disjoint.
Given \( d \in (-2H, \infty) \), let \( b_d(t) := \lambda_d^{-1}(t) \) for \( t \geq 0 \); note that \( b_d(0) = \eta_d \). Abusing the notation let \( b_d(t) := b_d(-t) \) for \( t \leq 0 \).

**Lemma 2.1 (Disjoint \( H \)-catenoids)** Given \( d_1 > 2 \), there exist \( d_0 > d_1 \) and \( \delta_0 > 0 \) such that for any \( d_2 \in [d_0, \infty) \), then

\[
\inf_{t \in \mathbb{R}} (b_{d_2}(t) - b_{d_1}(t)) \geq \delta_0.
\]

In particular, the corresponding \( H \)-catenoids are disjoint, i.e. \( C_{d_1}^H \cap C_{d_2}^H = \emptyset \).

Moreover, \( b_{d_2}(t) - b_{d_1}(t) \) is decreasing for \( t > 0 \) and increasing for \( t < 0 \). In particular,

\[
\sup_{t \in \mathbb{R}} (b_{d_2}(t) - b_{d_1}(t)) = b_{d_2}(0) - b_{d_1}(0) = \eta_{d_2} - \eta_{d_1}.
\]

The proof of the above lemma requires a rather lengthy computation that is given in the Appendix.

We next recall the well-known mean curvature comparison principle.

**Proposition 2.2 (Mean curvature comparison principle)** Let \( M_1 \) and \( M_2 \) be two complete, connected embedded surfaces in a three-dimensional Riemannian manifold. Suppose that \( p \in M_1 \cap M_2 \) satisfies that a neighborhood of \( p \) in \( M_1 \) locally lies on the side of a neighborhood of \( p \) in \( M_2 \) into which \( H_2(p) \) is pointing. Then \( |H_1|(p) \geq |H_2|(p) \). Furthermore, if \( M_1 \) and \( M_2 \) are constant mean curvature surfaces with \( |H_1| = |H_2| \), then \( M_1 = M_2 \).

### 3 The examples

For a fixed \( H \in (0, 1/2) \), the outline of construction is as follows. First, we will take two disjoint \( H \)-catenoids \( C_1 \) and \( C_2 \) whose existence is given in Lemma 2.1. These catenoids \( C_1, C_2 \) bound a region \( \Omega \) in \( \mathbb{H}^2 \times \mathbb{R} \) with fundamental group \( \mathbb{Z} \). In the universal cover \( \tilde{\Omega} \) of \( \Omega \), we define a piecewise smooth compact exhaustion \( \Delta_1 \subset \Delta_2 \subset \cdots \subset \Delta_n \subset \cdots \) of \( \tilde{\Omega} \). Then, by solving the \( H \)-Plateau problem for special curves \( \Gamma_n \subset \partial \Delta_n \), we obtain minimizing \( H \)-surfaces \( \Sigma_n \) in \( \Delta_n \) with \( \partial \Sigma_n = \Gamma_n \). In the limit set of these surfaces, we find an \( H \)-plane \( \Sigma_H \subset \mathbb{H}^2 \times \mathbb{R} \).

#### 3.1 Construction of \( \tilde{\Omega} \)

Fix \( H \in (0, 1/2) \) and \( d_1, d_2 \in (2, \infty) \), \( d_1 < d_2 \), such that by Lemma 2.1, the related \( H \)-catenoids \( C_{d_1}^H \) and \( C_{d_2}^H \) are disjoint; note that in this case, \( C_{d_1}^H \) lies in the interior of the simply-connected component of \( \mathbb{H}^2 \times \mathbb{R} - C_{d_2}^H \). We will use the notation \( C_{i} := C_{d_i}^H \). Recall that both catenoids have the same rotational axis, namely the \( t \)-axis, and recall that the mean curvature vector \( H_i \) of \( C_i \) points into the connected component of \( \mathbb{H}^2 \times \mathbb{R} \).
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$\mathbb{H}^2 \times \mathbb{R} - C_i$ that contains the $t$-axis. We emphasize here that $H$ is fixed and so we will omit describing it in future notations.

Let $\Omega$ be the closed region in $\mathbb{H}^2 \times \mathbb{R}$ between $C_1$ and $C_2$, i.e., $\partial \Omega = C_1 \cup C_2$ (Fig. 1-left). Notice that the set of boundary points at infinity $\partial_{\infty} \Omega$ is equal to $S^1_{\infty} \times \{-\infty\} \cup S^1_{\infty} \times \{\infty\}$, i.e., the corner circles in $\partial_{\infty} \mathbb{H}^2 \times \mathbb{R}$ in the product compactification, where we view $\mathbb{H}^2$ to be the open unit disk $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ with base point the origin $\vec{0}$.

By construction, $\Omega$ is topologically a solid torus. Let $\tilde{\Omega}$ be the universal cover of $\Omega$. Then, $\partial \tilde{\Omega} = \tilde{C}_1 \cup \tilde{C}_2$ (Fig. 1-right), where $\tilde{C}_1, \tilde{C}_2$ are the respective lifts to $\tilde{\Omega}$ of $C_1, C_2$. Notice that $\tilde{C}_1$ and $\tilde{C}_2$ are both $H$-planes, and the mean curvature vector $H$ points outside of $\tilde{\Omega}$ along $\tilde{C}_1$ while $H$ points inside of $\tilde{\Omega}$ along $\tilde{C}_2$. We will use the induced coordinates $(\rho, \tilde{\theta}, t)$ on $\tilde{\Omega}$ where $\tilde{\theta} \in (-\infty, \infty)$. In particular, if

$$\Pi: \tilde{\Omega} \to \Omega$$

is the covering map, then $\Pi(\rho_o, \tilde{\theta}_o, t_o) = (\rho_o, \theta_o, t_o)$ where $\theta_o \equiv \tilde{\theta}_o \mod 2\pi$.

Recalling the definition of $b_i(t), i = 1, 2$, note that a point $(\rho, \theta, t)$ belongs to $\Omega$ if and only if $\rho \in [b_1(t), b_2(t)]$ and we can write

$$\tilde{\Omega} = \{(\rho, \tilde{\theta}, t) \mid \rho \in [b_1(t), b_2(t)], \tilde{\theta} \in \mathbb{R}, t \in \mathbb{R}\}.$$

3.2 Infinite bumps in $\tilde{\Omega}$

Let $\gamma$ be the geodesic through the origin in $\mathbb{H}^2_0$ obtained by intersecting $\mathbb{H}^2_0$ with the vertical plane $\{\theta = 0\} \cup \{\theta = \pi\}$. For $s \in [0, \infty)$, let $\varphi_s$ be the orientation preserving hyperbolic isometry of $\mathbb{H}^2_0$ that is the hyperbolic translation along the geodesic $\gamma$ with $\varphi_s(0, 0) = (s, 0)$. Let

$$\tilde{\varphi}_s: \mathbb{H}^2 \times \mathbb{R} \to \mathbb{H}^2 \times \mathbb{R}, \quad \tilde{\varphi}_s(\rho, \theta, t) = (\varphi_s(\rho, \theta), t)$$

be the related extended isometry of $\mathbb{H}^2 \times \mathbb{R}$.
Let $\mathcal{C}_d$ be an embedded $H$-catenoid as defined in Sect. 2.1. Notice that the rotation axis of the $H$-catenoid $\hat{\varphi}_{\delta_0}(\mathcal{C}_d)$ is the vertical line $\{(s_0, 0, t) \mid t \in \mathbb{R}\}$.

Let $\delta := \inf_{t \in \mathbb{R}}(b_2(t) - b_1(t))$, which gives an upper bound estimate for the asymptotic distance between the catenoids; recall that by our choices of $\mathcal{C}_1, \mathcal{C}_2$ given in Lemma 2.1, we have $\delta > 0$. Let $\delta_1 = \frac{1}{2} \min\{\delta, \eta_1\}$ and let $\delta_2 = \delta - \frac{\delta_1}{2}$. Let $\bar{\mathcal{C}}_1 := \hat{\varphi}_{\delta_1}(\mathcal{C}_1)$ and $\bar{\mathcal{C}}_2 := \hat{\varphi}_{-\delta_2}(\mathcal{C}_2)$. Note that $\delta_1 + \delta_2 > \delta$.

**Claim 3.1** The intersection $\Omega \cap \bar{\mathcal{C}}_i$, $i = 1, 2$, is an infinite strip.

**Proof** Given $t \in \mathbb{R}$, let $\mathbb{H}^2_2$ denote $\mathbb{H}^2 \times \{t\}$. Let $\tau^1_i := \mathcal{C}_i \cap \mathbb{H}^2_2$ and $\hat{\tau}^1_i := \hat{\mathcal{C}}_i \cap \mathbb{H}^2_2$. Note that for $i = 1, 2$, $\tau^1_i$ is a circle in $\mathbb{H}^2_2$ of radius $b_i(t)$ centered at $(0, 0, t)$ while $\hat{\tau}^1_i$ is a circle in $\mathbb{H}^2_2$ of radius $b_i(t)$ centered at $p_{1,i} := (\delta_1, 0, t)$ and $\tau^2_i$ is a circle in $\mathbb{H}^2_2$ of radius $b_2(t)$ centered at $p_{2,i} := (-\delta_2, 0, t)$. We claim that for any $t \in \mathbb{R}$, the intersection $\hat{\tau}^1_i \cap \Omega$ is an arc with end points in $\tau^1_i$, $i = 1, 2$. This result would give that $\Omega \cap \bar{\mathcal{C}}_i$ is an infinite strip. We next prove this claim.

Consider the case $i = 1$ first. Since $\delta_1 < \eta_1 \leq b_1(t)$, the center $p_{1,i}$ is inside the disk in $\mathbb{H}^2_2$ bounded by $\tau^1_i$. Since the radii of $\tau^1_i$ and $\hat{\tau}^1_i$ are both equal to $b_1(t)$, then the intersection $\tau^1_i \cap \hat{\tau}^1_i$ is nonempty. It remains to show that $\tau^1_i \cap \tau^2_i = \emptyset$, namely that $b_1(t) + \delta_1 < b_2(t)$. This follows because

$$\delta_1 < \delta = \inf_{t \in \mathbb{R}}(b_2(t) - b_1(t)).$$

This argument shows that $\Omega \cap \bar{\mathcal{C}}_1$ is an infinite strip.

Consider now the case $i = 2$. Since $\delta_2 < \delta < b_2(t)$, the center $p_{2,i}$ is inside the disk in $\mathbb{H}^2_2$ bounded by $\tau^2_i$. Since the radii of $\tau^2_i$ and $\hat{\tau}^2_i$ are both equal to $b_2(t)$, then the intersection $\tau^2_i \cap \hat{\tau}^2_i$ is nonempty. It remains to show that $\tau^2_i \cap \tau^1_i = \emptyset$, namely that $b_2(t) - \delta_2 > b_1(t)$. This follows because

$$b_2(t) - b_1(t) \geq \inf_{t \in \mathbb{R}}(b_2(t) - b_1(t)) = \delta > \delta_2$$

This completes the proof that $\Omega \cap \bar{\mathcal{C}}_2$ is an infinite strip and finishes the proof of the claim. \hfill $\Box$

Now, let $\mathcal{Y}^+ := \Omega \cap \bar{\mathcal{C}}_2$ and let $\mathcal{Y}^- := \Omega \cap \bar{\mathcal{C}}_1$. In light of Claim 3.1 and its proof, we know that $\mathcal{Y}^+ \cap \mathcal{C}_1 = \emptyset$ and $\mathcal{Y}^- \cap \mathcal{C}_2 = \emptyset$.

![Fig. 2](image-url) The position of the bumps $\mathcal{B}^\pm$ in $\hat{\Omega}$ is shown in the picture. The small arrows show the mean curvature vector direction. The $H$-surfaces $\Sigma_n$ are disjoint from the infinite strips $\mathcal{B}^\pm$ by construction.

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Remark 3.2 Note that by construction, any rotational surface contained in $\Omega$ must intersect $\tilde{C}_1 \cup \tilde{C}_2$. In particular, $Y^+ \cup Y^-$ intersects all $H$-catenoids $\tilde{C}_d$ for $d \in (d_1, d_2)$ as the circles $C_d \cap \mathbb{H}^2_0$ intersect either the circle $\tau^+_t$ or the circle $\tau^-_t$ for some $t > 0$ since $\delta_1 + \delta_2 > \delta$.

In $\tilde{\Omega}$, let $B^+$ be the lift of $Y^+$ in $\tilde{\Omega}$ which intersects the slice $\{\tilde{\theta} = -10\pi\}$. Similarly, let $B^-$ be the lift of $Y^-$ in $\tilde{\Omega}$ which intersects the slice $\{\tilde{\theta} = 10\pi\}$. Note that each lift of $Y^+$ or $Y^-$ is contained in a region where the $\tilde{\theta}$ values of their points lie in ranges of the form $(\theta_0 - \pi, \theta_0 + \pi)$ and so $B^+ \cap B^- = \emptyset$. See Fig. 2.

The $H$-surfaces $B^\pm$ near the top and bottom of $\tilde{\Omega}$ will act as barriers (infinite bumps) in the next section, ensuring that the limit $H$-plane of a certain sequence of compact $H$-surfaces does not collapse to an $H$-lamination of $\tilde{\Omega}$ all of whose leaves are invariant under translations in the $\tilde{\theta}$-direction.

Next we modify $\tilde{\Omega}$ as follows. Consider the component of $\tilde{\Omega} - (B^+ \cup B^-)$ containing the slice $\{\tilde{\theta} = 0\}$. From now on we will call the closure of this region $\tilde{\Omega}^*$.

3.3 The compact exhaustion of $\tilde{\Omega}^*$

Consider the rotationally invariant $H$-planes $E_H, -E_H$ described in Sect. 2. Recall that $E_H$ is a graph over the horizontal slice $\mathbb{H}^2_0$ and it is also tangent to $\mathbb{H}^2_0$ at the origin. Given $t \in \mathbb{R}$, let $E_H^t = -E_H + (0, 0, t)$ and $-E_H^t = E_H - (0, 0, t)$. Both families $\{E_H^t\}_{t \in \mathbb{R}}$ and $\{-E_H^t\}_{t \in \mathbb{R}}$ foliate $\mathbb{H}^2 \times \mathbb{R}$. Moreover, there exists $n_0 \in \mathbb{N}$ such that for any $n > n_0$, $n \in \mathbb{N}$, the following holds. The highest (lowest) component of the intersection $S_n^+ := E_H^n \cap \Omega$ ($S_n^- := -E_H^n \cap \Omega$) is a rotationally invariant annulus with boundary components contained in $C_1$ and $C_2$. The annulus $S_n^+$ lies “above” $S_n^-$ and their intersection is empty. The region $\mathcal{U}_n$ in $\Omega$ between $S_n^+$ and $S_n^-$ is a solid torus, see Fig. 3-left, and the mean curvature vectors of $S_n^+$ and $S_n^-$ point into $\mathcal{U}_n$.

Let $\tilde{\mathcal{U}}_n \subset \tilde{\Omega}$ be the universal cover of $\mathcal{U}_n$, see Fig. 3-right. Then, $\partial \tilde{\mathcal{U}}_n - \partial \tilde{\Omega} = \tilde{S}_n^+ \cup \tilde{S}_n^-$ where can view $\tilde{S}_n^\pm$ as a lift to $\tilde{\mathcal{U}}_n$ of the universal cover of the annulus $S_n^\pm$. Hence,

![Fig. 3](image-url)
\[ \tilde{S}_n^\pm \] is an infinite \( H \)-strip in \( \tilde{\Omega} \), and the mean curvature vectors of the surfaces \( \tilde{S}_n^+, \tilde{S}_n^- \) point into \( \tilde{U}_n \) along \( \tilde{S}_n^\pm \). Note that each \( \tilde{U}_n \) has bounded \( t \)-coordinate. Furthermore, we can view \( \tilde{U}_n \) as \((\mathcal{U}_n \cap \mathcal{P}_0) \times \mathbb{R}\), where \( \mathcal{P}_0 \) is the half-plane \( \{\theta = 0\} \) and the second coordinate is \( \tilde{\theta} \). Abusing the notation, we redefine \( \tilde{U}_n \) to be \( \tilde{U}_n \cap \tilde{\Omega}^* \), that is we have removed the infinite bumps \( B^\pm \) from \( \tilde{U}_n \).

Now, we will perform a sequence of modifications of \( \tilde{U}_n \) so that for each of these modifications, the \( \theta \)-coordinate in \( \tilde{U}_n \) is bounded and so that we obtain a compact exhaustion of \( \tilde{\Omega}^* \). In order to do this, we will use arguments that are similar to those in Claim 3.1. Recall that the necksize of \( C_2 \) is \( \eta_2 = b_2(0) \). Let \( \tilde{C}_3 = \tilde{\varphi}_{\eta_2}(C_2) \), see equation (3) for the definition of \( \tilde{\varphi}_{\eta_2} \). Then, \( \tilde{C}_3 \) is a rotationally invariant catenoid whose rotational axis is the line \((\eta_2, 0) \times \mathbb{R}\) (Fig. 4-left).

**Lemma 3.3** The intersection \( \tilde{C}_3 \cap \Omega \) is a pair of infinite strips.

**Proof** It suffices to show that \( \tilde{C}_3 \cap C_1 \) and \( \tilde{C}_3 \cap C_2 \) each consists of a pair of infinite lines. Now, consider the horizontal circles \( \tau_1^1, \tau_1^2, \) and \( \tau_1^3 \) in the intersection of \( \mathbb{H}_t^2 \) and \( C_1, C_2, \) and \( \tilde{C}_3 \) respectively, where \( \mathbb{H}_t^2 = \mathbb{H}^2 \times \{t\} \). For any \( t \in \mathbb{R} \), \( \tau_t^1 \) is a circle of radius \( b_1(t) \) in \( \mathbb{H}_t^2 \) with center \((0, 0, t)\). Similarly, \( \tau_t^3 \) is a circle of radius \( b_2(t) \) in \( \mathbb{H}_t^2 \) with center \((\eta_2, 0, t)\), see Fig. 4-right. Hence, it suffices to show that for any \( t \in \mathbb{R} \) each of the intersection \( \tau_t^1 \cap \tilde{\tau}_t^3 \) and \( \tau_t^2 \cap \tilde{\tau}_t^3 \) consists of two points.

By construction, it is easy to see \( \tau_t^2 \cap \tilde{\tau}_t^3 \) consists of two points. This is because \( \tau_t^2 \) and \( \tilde{\tau}_t^3 \) have the same radius, \( b_2(t) \) and \( \eta_2 + b_2(t) > b_2(t) \) and \( \eta_2 - b_2(t) > -b_2(t) \). Therefore, it remains to show that \( \tau_t^1 \cap \tilde{\tau}_t^3 \) consists of two points. By construction, this would be the case if \( \eta_2 - b_2(t) < b_1(t) \) and \( \eta_2 - b_2(t) > -b_1(t) \). The first inequality follows because \( \eta_2 = \inf_{t \in \mathbb{R}} b_2(t) \). The second inequality follows from Lemma 2.1 because

\[
\eta_2 > \eta_2 - \eta_1 = \sup_{t \in \mathbb{R}} (b_2(t) - b_1(t))
\]
Now, let $\tilde{\mathcal{C}}_3 \cap \Omega = T^+ \cup T^-$, where $T^+$ is the infinite strip with $\theta \in (0, \pi)$, and $T^-$ is the infinite strip with $\theta \in (-\pi, 0)$. Note that $T^\pm$ is a $\theta$-graph over the infinite strip $\tilde{\mathcal{P}}_0 = \Omega \cap \mathcal{P}_0$ where $\mathcal{P}_0$ is the half plane $\{ \theta = 0 \}$. Let $\mathcal{V}$ be the component of $\Omega - \tilde{\mathcal{C}}_3$ containing $\tilde{\mathcal{P}}_0$. Notice that the mean curvature vector $\mathbf{H}$ of $\partial \mathcal{V}$ points into $\mathcal{V}$ on both $T^+$ and $T^-$. 

Consider the lifts of $T^+$ and $T^-$ in $\tilde{\Omega}$. For $n \in \mathbb{Z}$, let $\tilde{T}^+_n$ be the lift of $T^+$ which belongs to the region $\tilde{\theta} \in (2n\pi, (2n+1)\pi)$. Similarly, let $\tilde{T}^-_n$ be the lift of $T^-$ which belongs to the region $\tilde{\theta} \in ((2n-1)\pi, 2n\pi)$. Let $\mathcal{V}_n$ be the closed region in $\tilde{\Omega}$ between the infinite strips $\tilde{T}^+_n$ and $\tilde{T}^-_n$. Notice that for $n$ sufficiently large, $B^\pm \subset \mathcal{V}_n$.

Next we define the compact exhaustion $\Delta_n$ of $\tilde{\Omega}$ as follows: $\Delta_n := \tilde{U}_n \cap \mathcal{V}_n$. Furthermore, the absolute value of the mean curvature of $\partial \Delta_n$ is equal to $H$ and the mean curvature vector $\mathbf{H}$ of $\partial \Delta_n$ points into $\Delta_n$ on $\partial \Delta_n - [ (\partial \Delta_n \cap \tilde{\mathcal{C}}_1) \cup B^- ]$.

### 3.4 The sequence of $H$-surfaces

We next define a sequence of compact $H$-surfaces $\{ \Sigma_n \}_{n \in \mathbb{N}}$ where $\Sigma_n \subset \Delta_n$. For each $n$ sufficiently large, we define a simple closed curve $\Gamma_n$ in $\partial \Delta_n$, and then we solve the $H$-Plateau problem for $\Gamma_n$ in $\Delta_n$. This will provide an embedded $H$-surface $\Sigma_n$ in $\Delta_n$ with $\partial \Sigma_n = \Gamma_n$ for each $n$.

**The Construction of $\Gamma_n$ in $\partial \Delta_n$:**

First, consider the annulus $A_n = \partial \Delta_n - (\tilde{\mathcal{C}}_1 \cup \tilde{\mathcal{C}}_2 \cup B^+ \cup B^-)$ in $\partial \Delta_n$. Let $\tilde{l}^+_n = \tilde{C}^+_1 \cap \tilde{T}^+_n$, and $\tilde{l}^-_n = \tilde{C}^-_2 \cap \tilde{T}^-_n$ be the pair of infinite lines in $\tilde{\Omega}$. Let $l^+_n = l^+_n \cap A_n$. Let $\mu^+_n$ be an arc in $\tilde{S}^+_n \cap A_n$, whose $\tilde{\theta}$ and $\rho$ coordinates are strictly increasing as a function of the parameter and whose endpoints are $l^+_n \cap \tilde{S}^+_n$ and $l^-_n \cap \tilde{S}^-_n$ (Fig. 5-left). Similarly, define $\mu^-_n$ to be a monotone arc in $\tilde{S}^-_n \cap A_n$ whose endpoints are $l^+_n \cap \tilde{S}^-_n$ and $l^-_n \cap \tilde{S}^+_n$. Note that these arcs $\mu^+_n$ and $\mu^-_n$ are by construction disjoint from the infinite bumps $B^\pm$. Then, $\Gamma_n = \mu^+_n \cup l^+_n \cup \mu^-_n \cup l^-_n$ is a simple closed curve in $A_n \subset \partial \Delta_n$ (Fig. 5-right).

Next, consider the following variational problem ($H$-Plateau problem): Given the simple closed curve $\Gamma_n$ in $A_n$, let $M$ be a smooth compact embedded surface in $\Delta_n$ with $\partial M = \Gamma_n$. Since $\Delta_n$ is simply-connected, $M$ separates $\Delta_n$ into two regions. Let $Q$ be the region in $\Delta_n - \Sigma$ with $Q \cap \tilde{C}_2 \neq \emptyset$, the “upper” region. Then define the functional $I_H = \text{Area}(M) + 2H \text{ Volume}(Q)$.

![Fig. 5](image-url) In the left, $\mu^+_n$ is pictured in $\tilde{S}^+_n$. On the right, the curve $\Gamma_n$ is described in $\partial \Delta_n$. 
By working with integral currents, it is known that there exists a smooth (except at the 4 corners of $\Gamma_n$), compact, embedded $H$-surface $\Sigma_n \subset \Delta_n$ with $\text{Int}(\Sigma_n) \subset \text{Int}(\Delta_n)$ and $\partial \Sigma_n = \Gamma_n$. Note that in our setting, $\Delta_n$ is not $H$-mean convex along $\Delta_n \cap \vec{C}_1$. However, the mean curvature vector along $\Sigma_n$ points outside $Q$ because of the construction of the variational problem. Therefore $\Delta_n \cap \vec{C}_1$ is still a good barrier for solving the $H$-Plateau problem. In fact, $\Sigma_n$ can be chosen to be, and we will assume it is, a minimizer for this variational problem, i.e., $I(\Sigma_n) \leq I(M)$ for any $M \subset \Delta_n$ with $\partial M = \Gamma_n$; see for instance [12, Theorem 2.1] and [1, Theorem 1]. In particular, the fact that $\text{Int}(\Sigma_n) \subset \text{Int}(\Delta_n)$ is proven in Lemma 3 of [4]. Moreover, $\Sigma_n$ separates $\Delta_n$ into two regions.

Similarly to Lemma 4.1 in [3], in the following lemma we show that for any such $\Gamma_n$, the minimizer surface $\Sigma_n$ is a $\tilde{\theta}$-graph.

**Lemma 3.4** Let $E_n := A_n \cap T_n^+$. The minimizer surface $\Sigma_n$ is a $\tilde{\theta}$-graph over the compact disk $E_n$. In particular, the related Jacobi function $J_n$ on $\Sigma_n$ induced by the inner product of the unit normal field to $\Sigma_n$ with the Killing field $\partial_{\vec{O}}$ is positive in the interior of $\Sigma_n$.

**Proof** The proof is almost identical to the proof of Lemma 4.1 in [3], and for the sake of completeness, we give it here. Let $T_\alpha$ be the isometry of $\tilde{\Omega}$ which is a translation by $\alpha$ in the $\tilde{\theta}$ direction, i.e.,

$$T_\alpha(\rho, \tilde{\theta}, t) = (\rho, \tilde{\theta} + \alpha, t). \quad (4)$$

Let $T_\alpha(\Sigma_n) = \Sigma_n^\alpha$ and $T_\alpha(\Gamma_n) = \Gamma_n^\alpha$. We claim that $\Sigma_n^\alpha \cap \Sigma_n = \emptyset$ for any $\alpha \in \mathbb{R} \setminus \{0\}$ which implies that $\Sigma_n$ is a $\tilde{\theta}$-graph; we will use that $\Gamma_n^\alpha$ is disjoint from $\Sigma_n$ for any $\alpha \in \mathbb{R} \setminus \{0\}$.

Arguing by contradiction, suppose that $\Sigma_n^\alpha \cap \Sigma_n \neq \emptyset$ for a certain $\alpha \neq 0$. By compactness of $\Sigma_n$, there exists a largest positive number $\alpha'$ such that $\Sigma_n^{\alpha'} \cap \Sigma_n \neq \emptyset$. Let $p \in \Sigma_n^{\alpha'} \cap \Sigma_n$. Since $\partial \Sigma_n^{\alpha'} \cap \partial \Sigma_n = \emptyset$ and the interior of $\Sigma_n$, respectively $\Sigma_n^{\alpha'}$, lie in the interior of $\Delta_n$, respectively $T_{\alpha'}(\Delta_n)$, then $p \in \text{Int}(\Sigma_n^{\alpha'}) \cap \text{Int}(\Sigma_n)$. Since the surfaces $\text{Int}(\Sigma_n^{\alpha'})$, $\text{Int}(\Sigma_n)$ lie on one side of each other and intersect tangentially at the point $p$ with the same mean curvature vector, then we obtain a contradiction to the mean curvature comparison principle for constant mean curvature surfaces, see Proposition 2.2. This proves that $\Sigma_n$ is graphical over its $\tilde{\theta}$-projection to $E_n$.

Since by construction every integral curve, $(\bar{\rho}, s, \bar{t})$ with $\bar{\rho}, \bar{t}$ fixed and $(\bar{\rho}, s_0, \bar{t}) \in E_n$ for a certain $s_0$, of the Killing field $\partial_{\vec{O}}$ has non-zero intersection number with any compact surface bounded by $\Gamma_n$, we conclude that every such integral curve intersects both the disk $E_n$ and $\Sigma_n$ in single points. This means that $\Sigma_n$ is a $\tilde{\theta}$-graph over $E_n$ and thus the related Jacobi function $J_n$ on $\Sigma_n$ induced by the inner product of the unit normal field to $\Sigma_n$ with the Killing field $\partial_{\vec{O}}$ is non-negative in the interior of $\Sigma_n$. Since $J_n$ is a non-negative Jacobi function, then either $J_n \equiv 0$ or $J_n > 0$. Since by construction $J_n$ is positive somewhere in the interior, then $J_n$ is positive everywhere in the interior. This finishes the proof of the lemma. \qed

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4 The proof of Theorem 1.1

With \( \Gamma_n \) as previously described, we have so far constructed a sequence of compact stable \( H \)-disks \( \Sigma_n \) with \( \partial \Sigma_n = \Gamma_n \subset \partial \Delta_n \). Let \( J_n \) be the related non-negative Jacobi function described in Lemma 3.4.

By the curvature estimates for stable \( H \)-surfaces given in [11], the norms of the second fundamental forms of the \( \Sigma_n \) are uniformly bounded from above at points which are at intrinsic distance at least one from their boundaries. Since the boundaries of the \( \Sigma_n \) leave every compact subset of \( \tilde{\Omega}^* \), for each compact set of \( \tilde{\Omega}^* \), the norms of the second fundamental forms of the \( \Sigma_n \) are uniformly bounded for values \( n \) sufficiently large and such a bound does not depend on the chosen compact set. Standard compactness arguments give that, after passing to a subsequence, \( \Sigma_n \) converges to a (weak) \( H \)-lamination \( \tilde{\mathcal{L}} \) of \( \tilde{\Omega}^* \) and the leaves of \( \tilde{\mathcal{L}} \) are complete and have uniformly bounded norm of their second fundamental forms, see for instance [5].

Let \( \beta \) be a compact embedded arc contained in \( \tilde{\Omega}^* \) such that its end points \( p_+ \) and \( p_- \) are contained respectively in \( B^+ \) and \( B^- \), and such that these are the only points in the intersection \([B^+ \cup B^-] \cap \beta \). Then, for \( n \)-sufficiently large, the linking number between \( \Gamma_n \) and \( \beta \) is one, which gives that, for \( n \) sufficiently large, \( \Sigma_n \) intersects \( \beta \) in an odd number of points. In particular \( \Sigma_n \cap \beta \neq \emptyset \) which implies that the lamination \( \tilde{\mathcal{L}} \) is not empty.

**Remark 4.1** By Remark 3.2, a leaf of \( \tilde{\mathcal{L}} \) that is invariant with respect to \( \tilde{\theta} \)-translations cannot be contained in \( \tilde{\Omega}^* \). Therefore none of the leaves of \( \tilde{\mathcal{L}} \) are invariant with respect to \( \tilde{\theta} \)-translations.

Let \( \tilde{\mathcal{L}} \) be a leaf of \( \tilde{\mathcal{L}} \) and let \( J_{\tilde{L}} \) be the Jacobi function induced by taking the inner product of \( \partial_{\tilde{\mathcal{L}}} \) with the unit normal of \( \tilde{L} \). Then, by the nature of the convergence, \( J_{\tilde{L}} \geq 0 \) and therefore since it is a Jacobi field, it is either positive or identically zero. In the latter case, \( \tilde{\mathcal{L}} \) would be invariant with respect to \( \theta \)-translations, contradicting Remark 4.1. Thus, by Remark 4.1, we have that \( J_{\tilde{L}} \) is positive and therefore \( \tilde{L} \) is a Killing graph with respect to \( \partial_{\tilde{\mathcal{L}}} \).

**Claim 4.2** Each leaf \( \tilde{L} \) of \( \tilde{\mathcal{L}} \) is properly embedded in \( \tilde{\Omega}^* \).

**Proof** Arguing by contradiction, suppose there exists a leaf \( \tilde{L} \) of \( \tilde{\mathcal{L}} \) that is NOT proper in \( \tilde{\Omega}^* \). Then, since the leaf \( \tilde{L} \) has uniformly bounded norm of its second fundamental form, the closure of \( \tilde{L} \) in \( \tilde{\Omega}^* \) is a lamination of \( \tilde{\Omega}^* \) with a limit leaf \( \Lambda \), namely \( \Lambda \subset \overline{\tilde{L}} \). Let \( J_{\Lambda} \) be the Jacobi function induced by taking the inner product of \( \partial_{\Lambda} \) with the unit normal of \( \Lambda \).

Just like in the previous discussion, by the nature of the convergence, \( J_{\Lambda} \geq 0 \) and therefore, since it is a Jacobi field, it is either positive or identically zero. In the latter case, \( \Lambda \) would be invariant with respect to \( \tilde{\theta} \)-translations and thus, by Remark 4.1, \( \Lambda \) cannot be contained in \( \tilde{\Omega}^* \). However, since \( \Lambda \) is contained in the closure of \( \tilde{L} \), this would imply that \( \tilde{L} \) is not contained in \( \tilde{\Omega}^* \), giving a contradiction. Thus, \( J_{\Lambda} \) must be positive and therefore, \( \Lambda \) is a Killing graph with respect to \( \partial_{\Lambda} \). However, this implies that \( \tilde{L} \) cannot be a Killing graph with respect to \( \partial_{\tilde{\mathcal{L}}} \). This follows because if we fix a point \( p \) in \( \Lambda \) and let \( U_p \subset \Lambda \) be neighborhood of such point, then by the nature of
the convergence, $U_p$ is the limit of a sequence of disjoint domains $U_{p_n}$ in $\bar{\mathcal{L}}$ where $p_n \in \bar{\mathcal{L}}$ is a sequence of points converging to $p$ and $U_{p_n} \subset \bar{\mathcal{L}}$ is a neighborhood of $p_n$. While each domain $U_{p_n}$ is a Killing graph with respect to $\partial_\gamma$, the convergence to $U_p$ implies that their union is not. This gives a contradiction and proves that $\Lambda$ cannot be a Killing graph with respect to $\partial_\gamma$. Since we have already shown that $\Lambda$ must be a Killing graph with respect to $\partial_\gamma$, this gives a contradiction. Thus $\Lambda$ cannot exist and each leaf $\bar{\mathcal{L}}$ of $\bar{\mathcal{L}}$ is properly embedded in $\bar{\Omega}^*$.

Arguing similarly to the proof of the previous claim, it follows that a small perturbation of $\beta$, which we still denote by $\beta$ intersects $\Sigma_n$ and $\bar{\mathcal{L}}$ transversally in a finite number of points. Note that $\bar{\mathcal{L}}$ is obtained as the limit of $\Sigma_n$. Indeed, since $\Sigma_n$ separates $B^+$ and $B^-$ in $\bar{\Omega}^*$, the algebraic intersection number of $\beta$ and $\Sigma_n$ must be one, which implies that $\beta$ intersects $\Sigma_n$ in an odd number of points. Then $\beta$ intersects $\bar{\mathcal{L}}$ in an odd number of points and the claim below follows.

**Claim 4.3** The curve $\beta$ intersects $\bar{\mathcal{L}}$ in an odd number of points.

In particular $\beta$ intersects only a finite collection of leaves in $\bar{\mathcal{L}}$ and we let $\mathcal{F}$ denote the non-empty finite collection of leaves that intersect $\beta$.

**Definition 4.1** Let $(\rho_1, \tilde{\theta}_0, t_0)$ be a fixed point in $\bar{\mathcal{L}}_1$ and let $\rho_2(\tilde{\theta}_0, t_0) > \rho_1$ such that $(\rho_2(\tilde{\theta}_0, t_0), \tilde{\theta}_0, t_0)$ is in $\bar{\mathcal{L}}_2$. Then we call the arc in $\bar{\Omega}$ given by

$$\left(\rho_1 + s(\rho_2 - \rho_1), \tilde{\theta}_0, t_0\right), \quad s \in [0, 1].$$

the vertical line segment based at $(\rho_1, \tilde{\theta}_0, t_0)$.

**Claim 4.4** There exists at least one leaf $\bar{\mathcal{L}}_\beta$ in $\mathcal{F}$ that intersects $\beta$ in an odd number of points and the leaf $\bar{\mathcal{L}}_\beta$ must intersect each vertical line segment at least once.

**Proof** The existence of $\bar{\mathcal{L}}_\beta$ follows because otherwise, if all the leaves in $\mathcal{F}$ intersected $\beta$ in an even number of points, then the number of points in the intersection $\beta \cap \mathcal{F}$ would be even. Given $\bar{\mathcal{L}}_\beta$ a leaf in $\mathcal{F}$ that intersects $\beta$ in an odd number of points, suppose there exists a vertical line segment which does not intersect $\bar{\mathcal{L}}_\beta$. Then since by Claim 4.2 $\bar{\mathcal{L}}_\beta$ is properly embedded, using elementary separation arguments would give that the number of points of intersection in $\beta \cap \bar{\mathcal{L}}_\beta$ must be zero mod 2, that is even, contradicting the previous statement.

Let $\Pi$ be the covering map defined in equation (2) and let $\mathcal{P}_H := \Pi(\bar{\mathcal{L}}_\beta)$. The previous discussion and the fact that $\Pi$ is a local diffeomorphism, implies that $\mathcal{P}_H$ is a stable complete $H$-surface embedded in $\bar{\Omega}$. Indeed, $\mathcal{P}_H$ is a graph over its $\theta$-projection to $\text{Int}(\Omega) \cap \{(\rho, 0, t) \mid \rho > 0, t \in \mathbb{R}\}$, which we denote by $\theta(\mathcal{P}_H)$. Abusing the notation, let $J_{\mathcal{P}_H}$ be the Jacobi function induced by taking the inner product of $\partial_\theta$ with the unit normal of $\mathcal{P}_H$, then $J_{\mathcal{P}_H}$ is positive. Finally, since the norm of the second fundamental form of $\mathcal{P}_H$ is uniformly bounded, standard compactness arguments imply that its closure $\overline{\mathcal{P}_H}$ is an $H$-lamination $\mathcal{L}$ of $\Omega$, see for instance [5].

**Claim 4.5** The closure of $\mathcal{P}_H$ is an $H$-lamination of $\Omega$ consisting of itself and two $H$-catenoids $L_1, L_2 \subset \Omega$ that form the limit set of $\mathcal{P}_H$. 

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Remark 4.6 Note that these two $H$-catenoids are not necessarily the ones which determine $\partial \Omega$.

Proof Given $(\rho_1, \tilde{\theta}_0, t_0) \in \tilde{C}_1$, let $\tilde{\gamma}$ be the fixed vertical line segment in $\tilde{\Omega}$ based at $(\rho_1, \tilde{\theta}_0, t_0)$, let $\tilde{p}_0$ be a point in the intersection $\tilde{L}_\beta \cap \tilde{\gamma}$ (recall that by Claim 4.4 such intersection is not empty) and let $p_0 = \Pi(\tilde{p}_0) \in \Pi(\tilde{\gamma}) \cap \mathcal{P}_H$. Then, by Claim 4.4, for any $i \in \mathbb{N}$, the vertical line segment $T_{2\pi i}(\tilde{\gamma})$ intersects $\tilde{L}_\beta$ in at least a point $\tilde{p}_i$, and $\tilde{p}_{i+1}$ is above $\tilde{p}_i$, where $T$ is the translation defined in equation (4). Namely, $\tilde{p}_0 = (r_0, \tilde{\theta}_0, t_0)$, $\tilde{p}_i = (r_i, \tilde{\theta}_0 + 2\pi i, t_0)$ and $r_i < r_{i+1} < \rho_2(\tilde{\theta}_0, t_0)$. The point $\tilde{p}_i \in \tilde{L}_\beta$ corresponds to the point $p_i = \Pi(\tilde{p}_i) = (r_i, \tilde{\theta}_0 \mod 2\pi, t_0) \in \mathcal{P}_H$. Let $r(2) := \lim_{r \to \infty} r_i$ then $r(2) < \rho_2(\tilde{\theta}_0, t_0)$ and note that since $\lim_{i \to \infty} (r_{i+1} - r_i) = 0$, then the value of the Jacobi function $J_{\mathcal{P}_H}$ at $p_i$ must be going to zero as $i$ goes to infinity. Clearly, the point $Q := (r(2), \theta_0 \mod 2\pi, t_0) \in \Omega$ is in the closure of $\mathcal{P}_H$, that is $\mathcal{L}$. Let $L_2$ be the leaf of $\mathcal{L}$ containing $Q$. By the previous discussion $J_{L_2}(Q) = 0$. Since by the nature of the convergence, either $J_{L_2}$ is positive or $L_2$ is rotational, then $L_2$ is rotational, namely an $H$-catenoid.

Arguing similarly but considering the intersection of $\tilde{L}_\beta$ with the vertical line segments $T_{-2\pi i}(\tilde{\gamma})$, $i \in \mathbb{N}$, one obtains another $H$-catenoid $L_1$, different from $L_2$, in the lamination $\mathcal{L}$. This shows that the closure of $\mathcal{P}_H$ contains the two $H$-catenoids $L_1$ and $L_2$.

Let $\Omega_g$ be the rotationally invariant, connected region of $\Omega - [L_1 \cup L_2]$ whose boundary contains $L_1 \cup L_2$. Note that since $\mathcal{P}_H$ is connected and $L_1 \cup L_2$ is contained in its closure, then $\mathcal{P}_H \subset \Omega_g$. It remains to show that $\mathcal{L} = \mathcal{P}_H \cup L_1 \cup L_2$, i.e. $\overline{\mathcal{P}_H} - \mathcal{P}_H = L_1 \cup L_2$. If $\overline{\mathcal{P}_H} - \mathcal{P}_H \neq L_1 \cup L_2$ then there would be another leaf $L_3 \in \mathcal{L} \cap \Omega_g$ and by previous argument, $L_3$ would be an $H$-catenoid. Thus $L_3$ would separate $\Omega_g$ into two regions, contradicting that fact that $\mathcal{P}_H$ is connected and $L_1 \cup L_2$ are contained in its closure. This finishes the proof of the claim. □

Note that by the previous claim, $\mathcal{P}_H$ is properly embedded in $\Omega_g$.

Claim 4.7 The $H$-surface $\mathcal{P}_H$ is simply-connected and every integral curve of $\partial \Omega$ that lies in $\Omega_g$ intersects $\mathcal{P}_H$ in exactly one point.

Proof Let $D_g := \text{Int}(\Omega_g) \cap \{(\rho, 0, t) \mid \rho > 0, t \in \mathbb{R}\}$, then $\mathcal{P}_H$ is a graph over its $\theta$-projection to $D_g$, that is $\theta(\mathcal{P}_H)$. Since $\theta : \Omega_g \to D_g$ is a proper submersion and $\mathcal{P}_H$ is properly embedded in $\Omega_g$, then $\theta(\mathcal{P}_H) = D_g$, which implies that every integral curve of $\partial \theta$ that lies in $\Omega_g$ intersects $\mathcal{P}_H$ in exactly one point. Moreover, since $D_g$ is simply-connected, this gives that $\mathcal{P}_H$ is also simply-connected. This finishes the proof of the claim. □

From this claim, it clearly follows that $\Omega_g$ is foliated by $H$-surfaces, where the leaves of this foliation are $L_1$, $L_2$ and the rotated images $\mathcal{P}_H(\theta)$ of $\mathcal{P}_H$ around the $t$-axis by angles $\theta \in \{0, 2\pi\}$. The existence of the examples $\Sigma_H$ in the statement of Theorem 1.1 can easily be proven by using $\mathcal{P}_H$. We set $\Sigma_H = \mathcal{P}_H$, and $C_i = L_i$ for $i = 1, 2$. This finishes the proof of Theorem 1.1.
Appendix: Disjoint $H$-catenoids

In this section, we will show the existence of disjoint $H$-catenoids in $\mathbb{H}^2 \times \mathbb{R}$. In particular, we will prove Lemma 2.1. Given $H \in (0, \frac{1}{2})$ and $d \in [-2H, \infty)$, recall that $\eta_d = \cosh^{-1} \left( \frac{2dH + \sqrt{1 - 4H^2 + d^2}}{1 - 4H^2} \right)$ and that $\lambda_d : [\eta_d, \infty) \to [0, \infty)$ is the function defined as follows.

$$\lambda_d(\rho) = \int_{\eta_d}^{\rho} \frac{d + 2H \cosh r}{\sqrt{\sinh^2 r - (d + 2H \cosh r)^2}} \, dr. \quad (6)$$

Recall that $\lambda_d(\rho)$ is a monotone increasing function with $\lim_{\rho \to \infty} \lambda_d(\rho) = \infty$ and that $\lambda'_d(\eta_d) = \infty$ when $d \in (-2H, \infty)$. The $H$-catenoid $C^H_d$, $d \in (-2H, \infty)$, is obtained by rotating a generating curve $\hat{\lambda}_d(\rho)$ about the $t$-axis. The generating curve $\hat{\lambda}_d(\rho)$ is obtained by doubling the curve $(\rho, 0, \lambda_d(\rho))$, $\rho \in [\eta_d, \infty)$, with its reflection $(\rho, 0, -\lambda_d(\rho))$, $\rho \in [\eta_d, \infty)$.

Finally, recall that $b_d(t) := \lambda^{-1}_d(t)$ for $t \geq 0$, hence $b_d(0) = \eta_d$, and that abusing the notation $b_d(t) := b_d(-t)$ for $t \leq 0$.

Lemma 2.1 (Disjoint $H$-catenoids) Given $d_1 > 2$ there exist $d_0 > d_1$ and $\delta_0 > 0$ such that for any $d_2 \in [d_0, \infty)$ and $t > 0$ then

$$\inf_{t \in \mathbb{R}} (b_{d_2}(t) - b_{d_1}(t)) \geq \delta_0.$$  

In particular, the corresponding $H$-catenoids are disjoint, i.e., $C^H_{d_1} \cap C^H_{d_2} = \emptyset$.

Moreover, $b_{d_2}(t) - b_{d_1}(t)$ is decreasing for $t > 0$ and increasing for $t < 0$. In particular,

$$\sup_{t \in \mathbb{R}} (b_{d_2}(t) - b_{d_1}(t)) = b_{d_2}(0) - b_{d_1}(0) = \eta_{d_2} - \eta_{d_1}.$$  

Proof We begin by introducing the following notations that will be used for the computations in the proof of this lemma,

$$c := \cosh r = \frac{e^r + e^{-r}}{2}, \quad s := \sinh r = \frac{e^r - e^{-r}}{2}.$$  

Recall that $c^2 - s^2 = 1$ and $c - s = e^{-r}$. Using these notations,

$$\lambda_d(\rho) = \int_{\eta_d}^{\rho} \frac{d + 2H \cosh r}{\sqrt{\sinh^2 r - (d + 2H \cosh r)^2}} \, dr \quad (7)$$

can be rewritten as

$$\lambda_d(\rho) = \int_{\eta_d}^{\rho} \frac{d + 2H(s + e^{-r})}{\sqrt{s^2 - (d + 2Hc)^2}} \, dr = f_d(\rho) + J_d(\rho), \quad (8)$$

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where

\[ f_d(\rho) = \int_{\eta_d}^\rho \frac{2Hs}{\sqrt{s^2 - (d + 2Hc)^2}} \, dr \quad \text{and} \quad J_d(\rho) = \int_{\eta_d}^\rho \frac{d + 2He^{-r}}{\sqrt{s^2 - (d + 2Hc)^2}} \, dr \]

First, by using a series of substitutions, we will get an explicit description of \( f_d(\rho) \). Then, we will show that for \( d > 2 \), \( J_d(\rho) \) is bounded independently of \( \rho \) and \( d \).

Claim 4.8

\[ f_d(\rho) = \frac{2H}{\sqrt{1 - 4H^2}} \cosh^{-1}\left( \frac{(1 - 4H^2) \cosh \rho - 2dH}{\sqrt{d^2 + 1 - 4H^2}} \right). \tag{9} \]

Remark 4.9 After finding \( f_d(\rho) \), we used Wolfram Alpha to compute the derivative of \( f_d(\rho) \) and verify our claim. For the sake of completeness, we give a proof.

Proof of Claim 4.8 The proof is a computation with requires several integrations by substitution. Consider

\[ \int \frac{2Hs}{\sqrt{s^2 - (d + 2Hc)^2}} \, dr \]

By using the fact that \( s^2 = c^2 - 1 \) and applying the substitution \( \{ u = c, du = \frac{dc}{dr} dr = sdr \} \) we obtain that

\[ \int \frac{2Hs}{\sqrt{s^2 - (d + 2Hc)^2}} \, dr = \int \frac{2H}{\sqrt{u^2 - 1 - (d + 2Hu)^2}} \, du. \]

Note that

\[ u^2 - 1 - (d + 2Hu)^2 = u^2 - 1 - (d^2 + 4dHu + 4H^2u^2) \\
= (1 - 4H^2)u^2 - 4dHu - d^2 - 1 \\
= (1 - 4H^2) \left( u^2 - \frac{4dH}{1 - 4H^2} u + \frac{4d^2H^2}{(1 - 4H^2)^2} \right) - \frac{4d^2H^2}{1 - 4H^2} - d^2 - 1 \\
= (1 - 4H^2) \left( u - \frac{2dH}{(1 - 4H^2)} \right)^2 - \left( \frac{4d^2H^2}{(1 - 4H^2)^2} + \frac{d^2 + 1}{1 - 4H^2} \right) \\
= (1 - 4H^2) \left[ \left( u - \frac{2dH}{(1 - 4H^2)} \right)^2 - \frac{4d^2H^2 + (1 - 4H^2)(d^2 + 1)}{(1 - 4H^2)^2} \right] \\
= (1 - 4H^2) \left[ \left( u - \frac{2dH}{(1 - 4H^2)} \right)^2 - \frac{d^2 + 1 - 4H^2}{(1 - 4H^2)^2} \right]. \]
Therefore, by applying a second substitution, \( w = u - \frac{2dH}{(1-4H^2)} \), and letting \( a^2 = \left( \frac{d^2+1-4H^2}{(1-4H^2)^2} \right) \) we get that

\[
\int \frac{2H}{\sqrt{u^2 - 1 -(d + 2Hu)^2}} \, du = \int \frac{2H}{\sqrt{1-4H^2} \sqrt{w^2 - a^2}} \, dw
\]

By using the fact that \( \sec^2 x - 1 = \tan^2 x \) and applying a third substitution, \( w = a \sec t, \, dw = a \sec t \tan t \, dt \), we obtain that

\[
\int \frac{2H a \sec t \tan t}{\sqrt{1 - 4H^2} \sqrt{a^2 \sec^2 t - a^2}} \, dt = \int \frac{2H \sec t}{\sqrt{1 - 4H^2}} \, dt
\]

\[
= \frac{2H}{\sqrt{1 - 4H^2}} \ln | \sec t + \tan t |
\]

Therefore

\[
\int \frac{2H}{\sqrt{1 - 4H^2} \sqrt{w^2 - a^2}} \, dw = \frac{2H}{\sqrt{1 - 4H^2}} \ln \left( \frac{w}{a} + \sqrt{\frac{w^2}{a^2} - 1} \right)
\]

\[
= \frac{2H}{\sqrt{1 - 4H^2}} \cosh^{-1} \left( \frac{w}{a} \right)
\]

Since \( w = u - \frac{2dH}{(1-4H^2)} \) then

\[
\int \frac{2H}{\sqrt{u^2 - 1 -(d + 2Hu)^2}} \, du = \frac{2H}{\sqrt{1 - 4H^2}} \cosh^{-1} \left( \frac{u - \frac{2dH}{(1-4H^2)}}{a} \right)
\]

\[
= \frac{2H}{\sqrt{1 - 4H^2}} \cosh^{-1} \left( \frac{u - \frac{2dH}{(1-4H^2)}}{\sqrt{d^2+1-4H^2}} \right)
\]

\[
= \frac{2H}{\sqrt{1 - 4H^2}} \cosh^{-1} \left( \frac{(1 - 4H^2)u - 2dH}{\sqrt{d^2 + 1 - 4H^2}} \right).
\]

Finally, since \( u = \cosh \rho \)

\[
\int_{nd}^{\rho} \frac{2Hs}{\sqrt{s^2 - (d + 2Hc)^2}} = \frac{2H}{\sqrt{1 - 4H^2}} \cosh^{-1} \left( \frac{(1 - 4H^2) \cosh \rho - 2dH}{\sqrt{d^2 + 1 - 4H^2}} \right) \bigg|_{nd}^{\rho}
\]

\[
= \frac{2H}{\sqrt{1 - 4H^2}} \left( \cosh^{-1} \left( \frac{(1 - 4H^2) \cosh \rho - 2dH}{\sqrt{d^2 + 1 - 4H^2}} \right) - \cosh^{-1} \left( \frac{(1 - 4H^2) \cosh nd - 2dH}{\sqrt{d^2 + 1 - 4H^2}} \right) \right)
\]
Recall that \( \eta_d = \cosh^{-1} \left( \frac{2dH + \sqrt{1 - 4H^2 + d^2}}{1 - 4H^2} \right) \) and thus

\[
\frac{(1 - 4H^2) \cosh \eta_d - 2dH}{\sqrt{d^2 + 1 - 4H^2}} = \frac{(1 - 4H^2) \left( \frac{2dH + \sqrt{1 - 4H^2 + d^2}}{1 - 4H^2} \right) - 2dH}{\sqrt{d^2 + 1 - 4H^2}} = 1.
\]

This implies that

\[
f_d(\rho) = \frac{2H}{\sqrt{1 - 4H^2}} \cosh^{-1} \left( \frac{(1 - 4H^2) \cosh \rho - 2dH}{\sqrt{d^2 + 1 - 4H^2}} \right).
\]

By Claim 4.8 we have that

\[
f_d(\rho) = \frac{2H}{\sqrt{1 - 4H^2}} \left( \cosh^{-1} \left( \frac{(1 - 4H^2) \cosh \rho - 2dH}{\sqrt{d^2 + 1 - 4H^2}} \right) \right) = \frac{2H}{\sqrt{1 - 4H^2}} \left( \rho + \ln \frac{1 - 4H^2}{\sqrt{d^2 + 1 - 4H^2}} \right) + g_d(\rho),
\]

where \( \lim_{\rho \to \infty} g_d(\rho) = 0. \)

Recall that \( \lambda_d(\rho) = f_d(\rho) + J_d(\rho) \) where

\[
J_d(\rho) = \int_{\eta_d}^{\rho} \frac{d + 2He^{-r}}{\sqrt{s^2 - (d + 2Hc)^2}} \, dr = \int_{\eta_d}^{\rho} \frac{d + 2He^{-r}}{\sqrt{c^2 - 1 - (d + 2Hc)^2}} \, dr.
\]

**Claim 4.10**

\[
\sup_{d \in (2, \infty), \rho \in (\eta_d, \infty)} J_d(\rho) \leq \pi \sqrt{1 - 2H}.
\]

**Proof of Claim 4.10** Let

\[
\alpha = \frac{2dH + \sqrt{1 - 4H^2 + d^2}}{1 - 4H^2} \quad \text{and} \quad \beta = \frac{2dH - \sqrt{1 - 4H^2 + d^2}}{1 - 4H^2}
\]

be the roots of \( c^2 - 1 - (d + 2Hc)^2 \), i.e.

\[
c^2 - 1 - (d + 2Hc)^2 = (1 - 4H^2) \left( c^2 - \frac{4dH}{1 - 4H^2}c - \frac{1 + d^2}{1 - 4H^2} \right) = (1 - 4H^2)(c - \alpha)(c - \beta).
\]

\( \square \)
Note that $\alpha = \cosh \eta_d$ and that as $H \in (0, \frac{1}{2})$, $\beta < 0 < \alpha$. Furthermore, $2He^{-r} < 2H < 1 < d$. Thus we have,

$$J_d(\rho) = \int_{\eta_d}^{\rho} \frac{d + 2He^{-r}}{\sqrt{1 - 4H^2}(c - \alpha)(c - \beta)} dr < \frac{2d}{\sqrt{1 - 4H^2}} \frac{\int_{\eta_d}^{\infty} dr}{\frac{2d}{\sqrt{1 - 4H^2}} \int_{\eta_d}^{\infty} dr} < \frac{2d}{\sqrt{1 - 4H^2}} \frac{\sqrt{1 - 4H^2}}{\frac{2d}{\sqrt{1 - 4H^2}} \int_{\eta_d}^{\infty} \frac{dr}{\sqrt{c - \alpha}}}$$

where the last inequality holds because for $r > \eta_d$, $\cosh r > \alpha$ and thus $\sqrt{\alpha - \beta} < \sqrt{c - \alpha}$. Notice that $\alpha - \beta = \frac{2\sqrt{1 - 4H^2 + d^2}}{1 - 4H^2} > \frac{2d}{1 - 4H^2}$. Therefore

$$\frac{2d}{\sqrt{1 - 4H^2}} \frac{\sqrt{1 - 4H^2}}{\frac{2d}{\sqrt{1 - 4H^2}}} = \frac{2d}{\sqrt{1 - 4H^2}} \frac{\sqrt{1 - 4H^2}}{\frac{2d}{\sqrt{1 - 4H^2}}} = \sqrt{2d}$$

and

$$J_d(\rho) < \sqrt{2d} \int_{\eta_d}^{\infty} \frac{dr}{\sqrt{c - \alpha}}.$$

Applying the substitution $\{ u = c - \alpha, du = sdr = \sqrt{(u + \alpha)^2 - 1}dr \}$, we obtain that

$$\int_{\eta_d}^{\infty} \frac{dr}{\sqrt{c - \alpha}} = \int_{0}^{\infty} \frac{du}{\sqrt{u \sqrt{(u + \alpha)^2 - 1}}}$$

(10)

Let $\omega = \alpha - 1$. Note that since $d \geq 1$ then $\alpha > 1$ and we have that $(u + \alpha)^2 - 1 > (u + \omega)^2$ as $u > 0$. This gives that

$$\int_{0}^{\infty} \frac{du}{\sqrt{u \sqrt{(u + \alpha)^2 - 1}}} < \int_{0}^{\infty} \frac{du}{\sqrt{u(u + \omega)}}$$

Applying the substitution $\{ v = \sqrt{u}, dv = \frac{du}{2\sqrt{u}} \}$ we get

$$\int_{0}^{\infty} \frac{dv}{\sqrt{v(u + \omega)}} = \int_{0}^{\infty} \frac{2dv}{v^2 + \omega} = \frac{2}{\sqrt{\omega}} \text{arctan} \frac{w}{\sqrt{\omega}} \bigg|_{0}^{\infty} < \frac{\pi}{\sqrt{\omega}}$$

and thus

$$J_d(\rho) < \sqrt{\frac{2d}{\omega} \pi}.$$
Note that
\[ \omega = \alpha - 1 = \frac{2dH + \sqrt{1 - 4H^2} + d^2}{1 - 4H^2} - 1 \]
\[ > \frac{(1 + 2H)d}{1 - 4H^2} - 1 = \frac{d}{1 - 2H} - 1. \]

Since \( d > 2 \), we have \( 2\omega > \frac{d}{1 - 2H} \) and \( \frac{d}{\omega} < 2(1 - 2H) \). Then
\[ \sqrt{\frac{2d}{\omega}} < 2 \sqrt{1 - 2H}. \]

Finally, this gives that
\[ J_d(\rho) < 2\pi \sqrt{1 - 2H} \]

independently on \( d > 2 \) and \( \rho > \eta_d \). This finishes the proof of the claim. \( \Box \)

Using Claims 4.8 and 4.10, we can now prove the next claim.

**Claim 4.11** Given \( d_2 > d_1 > 2 \) there exists \( T \in \mathbb{R} \) such for any \( t > T \), we have that
\[
\frac{2H}{\sqrt{1 - 4H^2}}(\lambda^{-1}_{d_2}(t) - \lambda^{-1}_{d_1}(t)) > \frac{1}{2} \ln \left( \frac{d_2^2 + 1 - 4H^2}{d_1^2 + 1 - 4H^2} \right) - 2\pi \sqrt{1 - 2H}.
\]

**Proof of Claim 4.11** Recall that \( \lambda_d(\rho) = f_d(\rho) + J_d(\rho) \) and that by Claims 4.8 and 4.10 we have that
\[
f_d(\rho) = \frac{2H}{\sqrt{1 - 4H^2}} \left( \rho + \ln \frac{1 - 4H^2}{\sqrt{d^2 + 1 - 4H^2}} \right) + g_d(\rho), \tag{11}
\]
where \( \lim_{\rho \to \infty} g_d(\rho) = 0 \), and that
\[
\sup_{d \in (2, \infty), \rho \in (\eta_d, \infty)} J_d(\rho) \leq 2\pi \sqrt{1 - 2H}. \tag{12}
\]

Let \( \rho_i(t) := \lambda^{-1}_{d_i}(t), i = 1, 2 \). Using this notation, since \( t = \lambda_1(\rho_1(t)) = \lambda_2(\rho_2(t)) \) we obtain that
\[
0 = \lambda_2(\rho_2(t)) - \lambda_1(\rho_1(t)) \]
\[
= f_{d_2}(\rho_2(t)) + J_{d_2}(\rho_2(t)) - f_{d_1}(\rho_1(t)) - J_{d_1}(\rho_1(t)) \]
\[
= \frac{2H}{\sqrt{1 - 4H^2}} \left( \rho_2(t) + \ln \frac{1 - 4H^2}{\sqrt{d_2^2 + 1 - 4H^2}} \right) + g_{d_2}(\rho_2(t)) + J_{d_2}(\rho_2(t)) \]
\[
- \frac{2H}{\sqrt{1 - 4H^2}} \left( \rho_1(t) - \ln \frac{1 - 4H^2}{\sqrt{d_1^2 + 1 - 4H^2}} \right) - g_{d_1}(\rho_1(t)) - J_{d_1}(\rho_1(t)).
\]

\( \Box \) Springer
Recall that \( \lim_{t \to \infty} \rho_i(t) = \infty, i = 1, 2 \), therefore given \( \varepsilon > 0 \) there exists \( T_\varepsilon \in \mathbb{R} \) such that for any \( t > T_\varepsilon \), \(|g_{d_i}(\rho_i(t))| \leq \varepsilon \). Taking
\[
4\varepsilon < \ln \frac{d_2^2 + 1 - 4H^2}{d_1^2 + 1 - 4H^2}
\]
for \( t > T_\varepsilon \) we get that
\[
\frac{2H}{\sqrt{1 - 4H^2}} (\rho_2(t) - \rho_1(t)) > \ln \frac{d_2^2 + 1 - 4H^2}{d_1^2 + 1 - 4H^2} + J_{d_1}(\rho_1(t)) - J_{d_2}(\rho_2(t)) - 2\varepsilon
\]
\[
> \frac{1}{2} \ln \frac{d_2^2 + 1 - 4H^2}{d_1^2 + 1 - 4H^2} + J_{d_1}(\rho_1(t)) - J_{d_2}(\rho_2(t)).
\]
Notice that \( J_{d_1}(\rho_1(t)) > 0 \) and that Claim 4.10 gives that
\[
\sup_{\rho \in (\eta_{d_2}, \infty)} J_{d_2}(\rho) \leq 2\pi \sqrt{1 - 2H}.
\]
Therefore
\[
\frac{2H}{\sqrt{1 - 4H^2}} (\rho_2(t) - \rho_1(t)) > \frac{1}{2} \ln \frac{d_2^2 + 1 - 4H^2}{d_1^2 + 1 - 4H^2} - 2\pi \sqrt{1 - 2H}.
\]
This finishes the proof of the claim. \( \square \)

We can now use Claim 4.11 to finish the proof of the lemma. Given \( d_1 > 2 \) fix \( d_0 > d_1 \) such that
\[
\frac{\sqrt{1 - 4H^2}}{4H} \left( \ln \frac{d_0^2 + 1 - 4H^2}{d_1^2 + 1 - 4H^2} - 4\pi \sqrt{1 - 2H} \right) = 1.
\]
Then, by Claim 4.11, given \( d_2 \geq d_0 \) there exists \( T > 0 \) such that \( \lambda_{d_2}^{-1}(t) - \lambda_{d_1}^{-1}(t) > 1 \) for any \( t > T \). Notice that since for any \( \rho \in (\eta_2, \infty), \lambda_{d_2}'(\rho) > \lambda_{d_1}'(\rho) \), then there exists at most one \( t_0 > 0 \) such that \( \lambda_{d_2}^{-1}(t_0) - \lambda_{d_1}^{-1}(t_0) = 0 \). Therefore, since there exists \( T > 0 \) such that \( \lambda_{d_2}^{-1}(t) - \lambda_{d_1}^{-1}(t) > 1 \) for any \( t > T \) and \( \lambda_{d_2}^{-1}(0) - \lambda_{d_1}^{-1}(0) = \eta_{d_2} - \eta_{d_1} > 0 \), this implies that there exists a constant \( \delta(d_2) > 0 \) such that for any \( t > 0 \),
\[
\lambda_{d_2}^{-1}(t) - \lambda_{d_1}^{-1}(t) > \delta(d_2).
\]
A priori it could happen that \( \lim_{d_2 \to \infty} \delta(d_2) = 0 \). The fact that \( \lim_{d_2 \to \infty} \delta(d_2) > 0 \) follows easy by noticing that by applying Claim 4.11 and using the same arguments as in the previous paragraph there exists \( d_3 > d_0 \) such that for any \( d \geq d_3 \) and \( t > 0 \),

\[
\lambda_d^{-1}(t) - \lambda_{d_0}^{-1}(t) > 0.
\]

Therefore, for any \( d \geq d_3 \) and \( t > 0 \),

\[
\lambda_d^{-1}(t) - \lambda_{d_0}^{-1}(t) > \lambda_{d_1}^{-1}(t) - \lambda_{d_1}^{-1}(t) > \delta(d_0)
\]

which implies that

\[
\lim_{d_2 \to \infty} \delta(d_2) \geq \delta(d_0) > 0.
\]

Setting \( \delta_0 = \inf_{d \in [d_0, \infty)} \delta(d_2) > 0 \) gives that

\[
\inf_{t \in \mathbb{R}_{\geq 0}} (\lambda_{d_2}^{-1}(t) - \lambda_{d_1}^{-1}(t)) \geq \delta_0.
\]

By definition of \( b_d(t) \) then

\[
\inf_{t \in \mathbb{R}} (b_{d_2}(t) - b_{d_1}(t)) = \inf_{t \in \mathbb{R}_{\geq 0}} (\lambda_{d_2}^{-1}(t) - \lambda_{d_1}^{-1}(t)) \geq \delta_0.
\]

It remains to prove that \( b_2(t) - b_1(t) \) is decreasing for \( t > 0 \) and increasing for \( t < 0 \). By definition of \( b_d(t) \), it suffices to show that \( b_2(t) - b_1(t) \) is decreasing for \( t > 0 \). We are going to show \( \frac{d}{dt} (b_2(t) - b_1(t)) < 0 \) when \( t > 0 \).

By definition of \( b_1 \), for \( t > 0 \) we have that \( \lambda_1(b_1(t)) = t \) and thus \( b_1'(t) = \frac{1}{\lambda_1'(b_1(t))} \).

By definition of \( \lambda_d(t) \) for \( t > 0 \) the following holds,

\[
b_1'(t) = \frac{1}{\lambda_1'(b_1(t))} > \frac{1}{\lambda_1'(b_2(t))} > \frac{1}{\lambda_2'(b_2(t))} = b_2'(t).
\]

The first inequality is due to the convexity of the function \( \lambda_1(t) \) and the second inequality is due to the fact that \( \lambda_1'(\rho) < \lambda_2'(\rho) \) for any \( \rho > \eta_2 \). This proves that \( \frac{d}{dt} (b_2(t) - b_1(t)) = b_2'(t) - b_1'(t) < 0 \) for \( t > 0 \) and finishes the proof of the claim.

Note that if \( d \) is sufficiently close to \( -2H \) then \( C_d^H \) must be unstable. This follows because as \( d \) approaches \( -2H \), the norm of the second fundamental form of \( C_d^H \) becomes arbitrarily large at points that approach the “origin” of \( \mathbb{H}^2 \times \mathbb{R} \) and a simple rescaling argument gives that a sequence of subdomains of \( C_d^H \) converge to a catenoid, which is an unstable minimal surface. This observation, together with our previous lemma suggests the following conjecture.

**Conjecture:** Given \( H \in (0, \frac{1}{2}) \) there exists \( d_H > -2H \) such that the following holds. For any \( d > d' > d_H \), \( C_d^H \cap C_{d'}^H = \emptyset \), and the family \( \{C_d^H \mid d \in [d_H, \infty)\} \) gives a
foliation of the closure of the non-simply-connected component of $\mathbb{H}^2 \times \mathbb{R} - \mathcal{C}_{dH}^H$. The $H$-catenoid $\mathcal{C}^H_{dH}$ is unstable if $d \in (-2H, d_H)$ and stable if $d \in (d_H, \infty)$. The $H$-catenoid $\mathcal{C}^H_{dH}$ is a stable-unstable catenoid.

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References