This paper for the first time investigates a family of line-symmetric Bricard linkages by means of two generated toroids and reveals their intersection that leads to a set of special Bricard linkages with various branches of reconfiguration. The discovery is made in the concentric toroid-toroid intersection. By manipulating the construction parameters of the toroids all possible bifurcation points are explored. This leads to the common bi-tangent planes that present singularities in the intersection set.

The study reveals the presence of Villarceau and secondary circles in the toroids intersection. Therefore, a way to reconfigure the Bricard linkage to a pair of different types of Bennett linkage is uncovered. Further, a linkage with two Bricard and two Bennett motion branches is explored. In addition, the paper reveals the Altmann linkage as a member of the family of special line-symmetric Bricard linkage studied in this paper. The method is applied to the plane-symmetric case in a different paper submitted by the authors in parallel with this paper.

Nomenclature

- $a_{i,j}$: DH parameter of link length between axes $i$ and $j$ in the proximal convention.
- $d_i$: DH parameter of axial displacement through axis $i$ in the proximal convention.
- $S_i$: Instantaneous screw associated to kinematic pair $i$.
- $\mathbb{V}^n$: Cartesian product of $n$ fields: $K_1 \times \ldots \times K_n$.
- $T^n$: $n$-dimensional torus.
- $\mathbb{Z}$: The set of integer numbers, $\mathbb{Z}^+$ is the set of positive integers and $\mathbb{Z}^*$ the set of non-negative integers.
- $K[x_1, \ldots, x_n]$: The set of all polynomials in $x_1, \ldots, x_n$ with coefficients in the field $K$.
- $L(\hat{u}, P)$: The line containing point $P \in \mathbb{R}^3$ and parallel to $\hat{u} \in \mathbb{R}^3$.
- $R(\theta, \hat{u})$: Rotation matrix representing a rotation of $\theta \in \mathbb{T}$ radians about an axis that contains the origin and is aligned with $\hat{u} \in \mathbb{R}^3$.
- $T(R, t)$: Homogeneous transformation matrix obtained from a rotation represented by $R \in SO(3)$ and a translation $t \in \mathbb{R}^3$.
- $im$: Image of a map.
- $sing$: Singular part of a variety.

1 Introduction

The fascination for the six types of overconstrained 6R linkages discovered by Bricard \cite{1,2} has left almost a century of research on their geometry and mobility that even led...
to the discovery of more general forms of these loops [3], Hunt [4], Phillips [5], Baker [6] and Bricard himself [2], among others, thoroughly analyzed these closed loops obtaining striking findings on their geometry. Even though nowadays the mysteries related to the mobility of the Bricard linkages have been clarified, there is still room for research related to the application, optimization and particularly reconfigurability of these closed loops. This paper focuses on the latter.

Overconstrained linkages [7][8][9] present a mobility which could not be revealed by Kutzbach-Grübler criterion but can be obtained by using the modified mobility criterion [9][10]. The geometry of an overconstrained linkage presents symmetries that make it movable. Bricard reported six movable 6R closed kinematic chains, each presenting specific symmetries that allowed the mobility of the overconstrained linkage. Bricard proved that these symmetries allowed all the axes in the kinematic chains to belong to the same linear complex [11][12][4][13]. The six cases reported by Bricard are the line-symmetric case, the plane-symmetric case, the trihedral case, the line-symmetric octahedral case, the plane-symmetric octahedral case and the doubly collapsible octahedral case. The DH parameters for the first case are the following [6].

\[
\begin{align*}
    a_{1,2} &= a_{4,5},
    a_{2,3} &= a_{5,6},
    a_{3,4} &= a_{6,1},
    a_{3,2} &= a_{4,5},
    a_{2,3} &= a_{5,6},
    a_{3,4} &= a_{6,1},
    d_1 &= d_4,
    d_2 &= d_5,
    d_3 &= d_6.
\end{align*}
\]

The general line-symmetric case is depicted in figure 1. In this case each member of the linkage is symmetric to another member through a line \(L\) (figure 1), which, therein, perpendicularly bisects a line segment that joins the corresponding points of these members. It is possible to find a line reciprocal to every axis of the linkage, however, in general this will not intersect each axis, therefore the linear complex has a non-zero pitch. Refer to the Appendix to [14] to find the central axis of this linear complex, this axis turns out to intersect perpendicularly the line of symmetry.

Surface generation is a method of linkage analysis and design based on kinematic dyads joined by spherical pairs [15][16][17][18][19][20][21]. The method has been used in the design of parallel platforms [22] and to prove the existence of some overconstrained linkages [23]. The method has also been applied to describe the reconfigurability phenomenon [24]. It has been proved [24] that the nature of the intersection of the generated surfaces can be linked to the configuration space of the linkage to obtain linkages that can work in different motion branches. The method has allowed the design of linkages whose configuration space is composed of an infinity of manifolds [25], such linkages always present helical joints.

A reconfigurable linkage [26] is able to work in different motion branches and/or can change its topological structure [27]. In addition to this, the kinematotropic [28][29] linkages are able to change their finite mobility in different configurations. The introduction of link annexation leading to topology change or of the reconfigurable joints that change their functionality resulting in change of configuration space leads to metamorphic linkages [30][31][32], which have found an important number of applications [33]. Some recent advances in reconfigurable linkages include a platform with 15 motion branches [34], synthesis of kinematotropic platforms [35], an extension [36] of the classical theory for the synthesis of kinematotropic linkages developed in [29], a reconfigurable platform whose hybrid legs include a 4-bar diamond [17], the study of reconfigurability of single-DOF loops by means of the reciprocity of screws [38][39], the use of screw system approach [9] to analyze a 14-state reconfigurable linkage [40], the application of higher order kinematic analysis to prove the local mobility of kinematotropic linkages [41][42][43], the use of morphing systemization to analyze ways of reconfiguration [44] and the application of dual quaternions in the analysis of reconfigurability [45][54].

Some examples of reconfigurable Bricard linkages have been reported [38][45][47][38] and applications of the Bricard loops in large deployable structures have been presented [49][50][51]. However, the reconfigurability, particularly its intrinsic properties that cast light to design, have not been presented. In this paper, the reconfigurability of the line-symmetric type of Bricard linkages is obtained by means of the intersection of generated toroids, while the planesymmetric case is to be studied by the authors in the following paper. The design is made by manipulating the construction parameters of two concentric toroids. Any possibility of tangency between the two surfaces is considered to obtain various cases of reconfigurability. It will be proven that certain members of this family of Bricard linkages can evolve to a pair of different types of Bennett linkage.

Toroids play an important role in the theory of linkages as it is generated by a kinematic dyad conformed by two revolute joints with skew axes. It is known that the only subgroup of motion of the Euclidean group that includes the motion generated by such a dyad is the whole Euclidean
group, thus, the analysis of linkages that include such dyad may lead to complications. Using the toroid as a generated surface the existence of several overconstrained linkages can be explained: The Bennett 4R linkage \cite{53,54,55,56}, the Myard 5R linkage \cite{23} and the family of overconstrained 5R loops with two consecutive parallel axes presented by Baker \cite{56,24}. Toroids have also been used in the analysis of 5R spatial linkages obtained from the combination of 4R linkages designed from the geometry of toroids \cite{57}, the derivation of equivalent lower pairs linkage for linkages involving a higher pair based on tori surface contact \cite{58} and the analysis of RSSR linkages \cite{59}.

The contributions of this paper include: the design of linkages obtained from the intersection of surfaces by exploring all the possibilities of tangency between surfaces, the analysis of the line-symmetric case of Bricard linkage by means of the intersection of two conrector toroids, the design of the intersection set forcing it to include circles in order to obtain idle joints in the linkage and the explanation of the relationship between the Bennett linkage and the Bricard line-symmetric linkages by means of the toroids geometry. In addition, very few contributions, including this paper and probably only \cite{46}, not only present the analysis of examples of reconfigurable Bricard linkages, but also develop a method for the design of these linkages.

This paper is organized as follows: Firstly the relationship between the configuration space and the toroid is reviewed, then the Bricard linkage is separated in its two toroid generators. The conditions for the linkage to be able to move along the intersection are established. The intersection between the two toroids is obtained and then the points of tangency between the two surfaces are explored. With this aim, the mutual bi-tangent planes are analyzed, leading to the discovery of motion branches of the Bennett type for reconfiguration. An example of a linkage with two Bricard and two Bennett motion branches is presented. Finally, some remarks about the Altmann linkage, which is well-known to be a special case of line-symmetric Bricard linkage, are made considering it as a member of the family of reconfigurable Bricard linkages studied in this paper. A second part of this research is presented in a different manuscript \cite{60} submitted by the authors in parallel with the present paper, in such paper the method is applied to the plane-symmetric case of Bricard loop.

2 Toroids generated by kinematic dyads.

The configuration space \cite{61,62,42}, \( V \subseteq \mathbb{V}^n \), of any closed kinematic chain with \( n \in \mathbb{Z}^+ \) joints and \( m \in \mathbb{Z}^+ \) loops, is the variety containing every joint parameters vector \( q \in \mathbb{V}^n \) that satisfies the \( m \) smooth closure functions \( f_i, i = 1, \ldots, m \). Thus,

\[
V = \cap_{i=1}^m f_i^{-1} (\text{id}_{\mathbb{S}^1(3)}) \subseteq \mathbb{V}^n
\]

which is the set of zeros of the \( m \) closure functions. Let \( n_s \in \mathbb{Z}^+ \) represent the number of revolute joints, \( n_p \in \mathbb{Z}^+ \) the number of prismatic joints and \( n_R \in \mathbb{Z}^+ \) the number of helical joints in a kinematic chain, then \( \mathbb{V}^n = \mathbb{V}^{n_s} \times \mathbb{R}^{n_p+n_R} \). The configuration space \( V \subseteq \mathbb{V}^n \), since there is no restriction on the values of the joint parameters. However, if the kinematic chain is closed, \( V \) is no longer a vector space and is, in general, not smooth. It may feature singularities and it can be composed of several components which are usually smooth manifolds, however they can also present singularities, e.g. cusps. These components may or may not intersect, the intersection of two or more components represents a singularity in the configuration space, which is no longer a manifold in such points. Each of these components represents an assembly mode of the linkage. If \( V \) includes at least two components \( V_i \) and \( V_j \), such that \( i \neq j \) and \( V_i \cap V_j \neq \emptyset \), then the linkage is reconfigurable and \( V_i \) and \( V_j \) are called motion branches of the linkage.

A point \( p \in V_i \cap V_j \) is a singularity in \( V \) and the dimension of the tangent space \( T_p V \) increases since the actual tangents to \( V \) at \( p \) conform a cone rather than a vector space. \( p \) is usually called bifurcation point, Hunt used the term increased mobility configuration. \cite{12} for this concept.

For a kinematic chain with \( n = 2 \) kinematic pairs, the non-smooth map \( E : V \rightarrow \mathbb{R}^3 \), describes the position of a point \( E(q) \) attached to the end effector of the dyad in the configuration \( q \). The image of \( E \) under the whole configuration space is a surface \( S := \text{im}(E(V)) \subset \mathbb{R}^3 \) whose shape depends not only on the DH parameters of the dyad, but also on the choice of the point that defines \( E \), relative to the end effector link. Degenerated cases in which \( S \) is not a surface or it is a doubly covered plane are not of the interest of this paper.

Let \( \sigma \) be a parameterization of \( S \) defined as \( \sigma : U \rightarrow \mathbb{R}^3 \ni S = \text{im}(\sigma(U)) \), where \( U \subseteq \mathbb{R}^2 \) is given in variables \((u,v)\). A point \((p_u, p_v) \in U\) is regular in \( \sigma \) if it does not present self-intersection and \( \partial \sigma / \partial u |_{p_u,p_v} \times \partial \sigma / \partial v |_{p_u,p_v} \neq 0 \). If this product vanishes, the point is said to be singular in \( \sigma \). If, either \( \partial \sigma / \partial u |_{p_u,p_v} = 0 \) or \( \partial \sigma / \partial v |_{p_u,p_v} = 0 \), then the point is a conic singularity.

Consider an RR kinematic chain, if the axes of the revolute joints are skew, \( S \) is a toroid as shown in figure \ref{fig:toroid}. The toroid \( T_{i,r,g,s}(\hat{k},O) \) has a base circle \( C_1 \) of radius \( l \) and center \( Q \) and lies on a plane perpendicular to \( \hat{k} \), a secondary circle \( C_2(\mu) \) of radius \( r \) is swept through \( C_1 \) to generate the surface of revolution. The constant angle between the normals of the planes containing \( C_1 \) and \( C_2 \) is \( \gamma \) and the secondary offset, i.e. the distance from \( C_1 \) to the center of \( C_2 \) measured along the axis of \( C_2 \), is \( s \). It is the closed curve obtained by intersecting the toroid with any plane containing the \( Z \) axis. In figure \ref{fig:toroid} \( \hat{a} = \hat{k} \) and \( Q = O \), for the sake of shortening the notation we define this toroid as \( T_{i,r,g,s}(\hat{k},O) \). The generator open kinematic chain is defined by \( a_{1.2} = \gamma, a_{1,2} = l \) and \( d_2 = s \).

It has to be noted that the same surface is generated with the angles \( \gamma, \gamma + \pi, -\gamma, -\gamma - \pi \) and \( \pi - \gamma \). Thus, we define \( \gamma' \in [0, \frac{\pi}{2}] \), which, along with \( l, r \) and \( s \) defines uniquely a
toroid. \( \gamma \) can be obtained from any value of \( \gamma \) according to:

\[
\begin{align*}
\gamma \in [\frac{2\pi}{3}, \frac{2\pi}{2} \Rightarrow \gamma' = 2\pi - \gamma, \\
\frac{2\pi}{2} \leq \gamma < \pi \Rightarrow \gamma' = \pi - \gamma, \\
\pi \leq \gamma < \frac{3\pi}{2} \Rightarrow \gamma' = \gamma - \pi, \\
\frac{3\pi}{2} \leq \gamma < 2\pi \Rightarrow \gamma' = 2\pi - \gamma.
\end{align*}
\]

and \( T_{i,\gamma',\xi} = \text{Im} (\sigma(U)) \). The implicit form \( \phi \in \mathbb{R}[x,y,z] \) is a quartic on \( x,y,z \) such that \( T_{i,\gamma',\xi} = \{ \mathbf{r}_i \in \mathbb{R}^3 | \phi(\mathbf{r}_i) = 0 \} \). This implicit form is given by [23]:

\[
\phi(x,y,z) = \left( x^2 + y^2 + z^2 - l^2 - r^2 - s^2 \right)^2 - 4l^2 \left[ r^2 - \left( \frac{z - s \cos \gamma}{\sin^2 \gamma} \right)^2 \right]
\]  

(2)

Fig. 2. An RR dyad generating a general toroid.

Using the parameters shown in figure 2 (\( u, v \) \( U = U^2 \)) the following parameterization for the general form \( T_{i,\gamma',\xi} \) can be obtained:

\[
\begin{align*}
\sigma(u,v) &= R(u,\hat{k}) [ R(\gamma,\hat{i}) (r R(v,\hat{k}) \hat{i} + \hat{k}) + \hat{i} ] \\
&= ( (s \sin \gamma - r \cos \gamma \sin v) \sin u + (r \cos v + l) \cos u, \\
&(-s \sin \gamma + r \cos \gamma \sin v) \cos u + (r \cos v + l) \sin u, \\
r \sin \gamma \sin v + s \cos \gamma )
\end{align*}
\]  

(1)

3 The line-symmetric Bricard linkage generator of concentric toroids

In the line-symmetric case of Bricard linkages any adjacent pair of revolute joints generate a toroid if the two axes are skew lines, i.e., the length-link parameter between the two axes is different to zero. Let \( \alpha = \{ 1, \ldots, 6 \} \) be a cyclic sequence of numbers, if in the 6R Bricard linkage a link connecting the axes \( S_i \) and \( S_{i+1}, i \in \alpha \), has zero length, i.e., \( d_i = d_{i+1} = 0 \), then the point \( E \) common to \( S_i \) and \( S_{i+1} \) describes the intersection of two toroids with respect to the opposite link connecting axes \( S_{i-3} \) and \( S_{i+4} \). Thus the linkage can be analyzed using the method of intersection of surfaces. In the octahedral cases both toroids degenerate into points. These degenerate cases are not addressed in this contribution, neither are the cases in which one of the toroids is any other degenerate form.

Due to symmetry if \( a_{i,i+1} = 0 \) and \( d_i = d_{i+1} = 0 \), then \( a_{i-3,i+4} = 0 \) and \( d_{i-3} = d_{i+4} = 0 \). Therefore, the two generated toroids are concentric.

Fig. 3 shows a general example of line-symmetric Bricard linkage derived from the intersection of two concentric toroids. Torfason and Sharma [17] solved numerically the polynomial equations of the intersection of two toroids generated by a general RRSRR non-overconstrained linkage. The intersection of two right torus is also of interest in computer graphics [65]. In this paper,
closed-form solutions for the concentric toroid-toroid intersection are obtained since the aim of the paper is the analysis of bifurcations and the design for reconfigurability.

For the Bricard line-symmetric case, let the toroid $T_{A_i:A_2:A_3:A_4}(\mathbf{k}_A, O) = \{E_A(\mathbf{q}_A) | \mathbf{q}_A \in \mathbb{T}^2\}$ be generated by kinematic pairs $A_1$ and $A_2$, while $T_{B,\tau;\rho;\gamma}(\mathbf{k}_B, O) = \{E_B(\mathbf{q}_B) | \mathbf{q}_B \in \mathbb{T}^2\}$ is generated by kinematic pairs $B_1$ and $B_2$, where $\mathbf{q}_i = (u_i, v_i), i = A, B$ and the toroids are related to coordinate systems $A$ and $B$, whose origins are coincident with $O$ and $Z_i$ axis is coincident with the $S_{1}, i = A, B$. To simplify the notation, let $T_{A_i:A_2:A_3:A_4}(\mathbf{k}_A, O) = T_{B,\tau;\rho;\gamma}(\mathbf{k}_B, O)$.

Since both open kinematic chains are connected by link $C$ between coincident revolute joints $A_3$ and $B_3$, $E_A(\mathbf{q}_A) = E_B(\mathbf{q}_B) = E(\mathbf{q})$, where $\mathbf{q} := (q_{A_1}, q_{A_2}, q_{A_3}, q_{B_2}, q_{B_1}) \in V \subset \mathbb{T}^6$. The joint parameters can be measured as shown in figure 4, and then $q_{A1} = q_{A2} = v_A$, $q_{B1} = u_B$ and $q_{B2} = v_B$. $q_{A3}$ and $q_{B3}$ are in symmetric concordance with $u_B$ and $u_A$, respectively. Since $E(\mathbf{q}) \in T_{A_i:A_2:A_3:A_4}$ and $E(\mathbf{q}) \in T_{B,\tau;\rho;\gamma}$ it follows that $E(\mathbf{q}) \in T_{A_i:A_2:A_3:A_4} \cap T_{B,\tau;\rho;\gamma}$.

Let $C := \{E(\mathbf{q}) | E(\mathbf{q}) \in T_{A_i:A_2:A_3:A_4} \cap T_{B,\tau;\rho;\gamma}\}$, then $C$ may feature $n \in \mathbb{Z}^\ast$ components such that $\mathbf{C} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$. These components can be points, curves or, in the trivial case that both toroids are the same, a surface.

Both toroids are represented by their parameterizations $^A\mathbf{r}_A$ and $^B\mathbf{r}_B$ respectively referred to coordinate systems $A$ and $B$. The relationship between the coordinate systems is given by the transformation matrix $^A\mathbf{T} \in SE(3)$. Given $\mathbf{r}_A := ^A\mathbf{r}_A - ^A\mathbf{r}_B$, $i \in \{A, B\}$, the intersection $\mathbf{C}$ is computed by $^A\mathbf{C} = \{^A\mathbf{r}_A(i), \mathbf{v}_A, \mathbf{u}_B, \mathbf{v}_B) = \{0\} = \{^A\mathbf{r}_B(i), \mathbf{v}_A, \mathbf{u}_B, \mathbf{v}_B) = \{0\}, i \in \{A, B\}\}$. $\mathbf{C}$ may include isolated points and curves in $\mathbb{R}^3$, or it can be a surface, in the case that both toroids are the same surface.

Each component of $\mathbf{C}$ is related to a component of the configuration space $V$ of the linkage, and thus, to a motion branch or assembly mode. However, in the case of these 6R linkages it has to be proved that the symmetries allow $E$ to move throughout the intersection. It will be concluded that for the line-symmetric case an extra restriction is required. When a curve $\mathbf{C} \subset \mathbf{C}$ appears in the intersection, the equivalent component in $V$ is 1-dimensional and the linkage has 1 DOF; this leads to the typical motion of the overconstrained linkage. On the other hand, if $\mathbf{C}$ features an isolated point, this point is equivalent to a 0-dimensional component in $V$ and the linkage can be assembled as a structure. However, if, for the same linkage, $\mathbf{C}$ features a curve or a double-coincident conic singularity, the same linkage can be assembled to have 1 finite degree of freedom.

Refer to figure 5 which depicts the common perpendiculars $\mathbf{d}$ for the line-symmetric case with $s_A = s_B = 0$. The axis of symmetry is given by $^A\mathbf{d}(\mathbf{E} \times (\mathbf{p}_A - \mathbf{p}_B), \frac{1}{2}\mathbf{E})$. Observe that the common perpendiculars between the joint axes always assemble two equal triangles, $\mathbf{O}_A\mathbf{E}$ and $\mathbf{O}_B\mathbf{E}$, drawn on planes perpendicular to $\mathbf{S}_A$ and $\mathbf{S}_B$ respectively. Therefore, the inner angles $\beta$ (figure 6) are always the same in both triangles. The condition $\beta = |\mathbf{v}_A|/|\mathbf{v}_B|$ allows the different possibilities shown in figures 4a and 4b. The change in the sign of $\mathbf{v}_i, i = A, B$ does not affect the symmetry since the angles $\beta$ are always the same for both triangles. The screw axes can be reversed from one assembly to another to obtain the proper signs of symmetry. Therefore it can be concluded that the symmetry is not lost while $E$ moves through any component of $\mathbf{C}$. However, if $s_A = s_B \neq 0$ such a change in sign would break the symmetry. A change in the direction of the screw axes attempting to fix the symmetry would affect the topology of the linkage due to the change of sign in the DH parameter $d$. In fact, it has been proven [60] that the change in sign in the articulated variables, i.e., a negative relationship between them, is not a solution of the closure equations of the general line-symmetric case. The intersection of two concentric toroids with non-zero secondary offset would require a third revolute joint intersecting $E$ and non-coplanar with $\mathbf{S}_A$ and $\mathbf{S}_B$ to ensure that $E$ can move through any component of $\mathbf{C}$. Without this third revolute joint the linkage would be movable only in certain components of $\mathbf{C}$ that do not break the symmetry. Due to this situation, only cases with $s_A = s_B = 0$, i.e., flattened forms, are studied in this paper.

From the previous paragraph and the required conditions for the method of generated surfaces explained in Section 2, the special line-symmetric Bricard loops to be designed in this paper present the following restrictions:

- $d_i = 0, i = 1, \ldots, 6$, and
- $a_{i+1} = 0 \Rightarrow a_{i+1,i+2} \neq 0, i \in \{1, \ldots, 6\}$

where $\{1, \ldots, 6\}$ is a cyclic sequence. Hence, all the special Bricard linkages have two, and only two, opposite zero-length links and all the axial displacements are zero for all the joints.
4 Concentric toroid-toroid intersection.

The line-symmetry condition applied to the intersection of toroids $A$ and $B$ forces the radius of the base circle of one toroid to be the same as the radius of the secondary circle of the other toroid. Therefore, let $b_1 := l_A = r_B$ and $b_2 := l_B = r_A$. Let the relationship between coordinate systems $A$ and $B$ be given by $\mathbf{T} = T(R(\theta, \tilde{\gamma}, 0))$, such that the toroids are concentric and the axis of $B$ is obtained by rotating the axis of $A$ radians about the $Y := Y_A = Y_B$ axis.

The parameterizations of both surfaces referred to coordinate system $A$ are:

$$A\sigma_A(u_A, v_A) = \left( (b_1 + b_1 \cos \gamma_A) \cos u_A - b_1 \cos \gamma_A \sin u_A, (b_1 + b_1 \cos \gamma_A) \sin u_A + b_1 \cos \gamma_A \cos u_A, b_1 \sin \gamma_A \sin v_A \right),$$

$$A\sigma_B(u_B, v_B) = \left( (b_2 + b_2 \cos \gamma_B) \cos u_B - b_2 \cos \gamma_B \sin u_B, (b_2 + b_2 \cos \gamma_B) \sin u_B + b_2 \cos \gamma_B \cos u_B, b_2 \cos \gamma_B \sin v_B \right),$$

and the implicit forms are given by:

$$\phi_A(x, y) = \left( x^2 + y^2 + z^2 - b_1^2 - b_2^2 \right)^2 - 4b_2^2 \left( b_1^2 - \frac{z^2}{\sin^2 \gamma_A} \right),$$

$$\phi_B(x, y) = \frac{1}{\sin^2 \gamma_B} \left[ -\left( x^2 + y^2 + z^2 - b_1^2 - b_2^2 + 2b_1 b_2 - b_2^2 \right) \left( x^2 + y^2 + z^2 - b_1^2 - b_2^2 \right)^2 \cos^2 \gamma_A + \left( -4x^2 + 4z^2 \right) b_2^2 \cos^2 \theta + 8 \cos \theta \sin \tilde{\gamma} z b_2^2 + b_2^4 + \left( 2x^2 - 2y^2 - 2z^2 - 2b_1 b_2 \right) dz^2 + \left( x^2 + y^2 + z^2 - b_1^2 \right)^2 \right]$$

After solving $f(u_A, v_A, u_B, v_B) = 0$, the values of $u_A, v_A, u_B, v_B$ in terms of $v_B$ are obtained. This expressions are listed in Appendix A. Four sets of solutions are found. As expected, $v_A$ and $v_B$ bear the condition of symmetry as shown in figure 4. Taking $A\sigma_B(u_B(v_B), v_B)$, the parameterizations for four 1-dimensional curves are found:

$$A\varsigma_1(v_B) = \left( \frac{(b_2 \sin \gamma_B - b_1 \cos \theta \sin \gamma_A) \sin v_B}{\sin \theta}, \frac{-\sqrt{K_1}}{\sin \theta \cos \gamma_B} b_1 \sin \gamma_A \sin v_B \right),$$

$$A\varsigma_2(v_B) = \left( \frac{(b_2 \sin \gamma_B - b_1 \cos \theta \sin \gamma_A) \sin v_B}{\sin \theta}, \frac{-\sqrt{K_2}}{\sin \theta \cos \gamma_B} b_1 \sin \gamma_A \sin v_B \right),$$

$$A\varsigma_3(v_B) = \left( \frac{(b_2 \sin \gamma_B + b_1 \cos \theta \sin \gamma_A) \sin v_B}{\sin \theta}, \frac{-\sqrt{K_2}}{\sin \theta \cos \gamma_B} b_1 \sin \gamma_A \sin v_B \right),$$

$$A\varsigma_4(v_B) = \left( \frac{(b_2 \sin \gamma_B + b_1 \cos \theta \sin \gamma_A) \sin v_B}{\sin \theta}, \frac{-\sqrt{K_1}}{\sin \theta \cos \gamma_B} b_1 \sin \gamma_A \sin v_B \right),$$

where $K_1$ and $K_2$ are defined in Appendix A. Thus, $C = \cup_{i=1}^4 C_i$, where $C_i = \text{im}(\varsigma_i(W))$, where $W \subseteq \mathbb{T}$. However, the expressions in Eq. (5) for the general intersection may take discontinuous forms containing parts of different components depending each specific numerical case. By eliminating the discontinuities and rearranging the components, the correct parameterizations for each curve can be obtained.

5 The bitangent planes and singularities.

In the concentric toroid-toroid intersection there exist $i, j \ni i \neq j$ such that $C_i \cap C_j \neq \emptyset$ the linkage is re-configurable with at least 2 motion branches, which are connected through at least one configuration $q_{ij} \in V_i \cap V_j$. It can be proved [24] that for the 1-dimensional components of $V$, the toroids are tangent to each other in $E(q_{ij})$. The intersection is non-transverse in $E(q_{ij})$, and it follows that if $q_{ij} := (u_B^p, v_B^p, q_{ij}^A, q_{ij}^B, v_B^p, u_B^p)$, then $\mathbf{n}_i(E(q_{ij})) \times \mathbf{n}_j(E(q_{ij})) = 0$, where $\mathbf{n}_i : S_i \to \mathbb{R}^3$ is the normal vector to the surface in the given point. This vector can be found by means of either the partial derivatives of the parameterization of the surface or the gradient of the implicit form.

In this paper, the latter is used, then the intersection is non-transverse if $\nabla \phi_A(x^p, y^p, z^p) \times \nabla \phi_B(x^p, y^p, z^p) = 0$, where $(x^p, y^p, z^p) = E(q_{ij})$. The points in $V$ that map to points of tangency may be bifurcation configurations of the linkages. These points in $V$ may represent the intersection of two components of $V$, or may be the self-crossing of the same component. The surfaces are also tangent to each other when they touch in one point, which would lead to an isolated point in $V$. In addition, if a continuum of points of tangency is found, the surfaces are touching in a curve in $C$.

To find the points where the intersection becomes non-transverse, the real points $(x, y, z) \in \mathbb{R}^3$ that make $\nabla \phi_A(x, y, z) \times \nabla \phi_B(x, y, z) = 0$ and also satisfy $\phi_A(x, y, z) = \phi_B(x, y, z) = 0$, are explored. Four immediate results are found: $(0, b_1 + b_2, 0), (0, b_1 - b_2, 0), (0, -b_1 - b_2, 0)$ and $(0, -b_1 + b_2, 0)$. These four points are independent of any construction parameter of the toroids. The toroids are tangent to each other through these four points in the $Y$ axis. This is a consequence of the symmetry condition $l_A = r_B = l_B = r_A$. The flattened toroids are symmetric with respect to the plane that contains its base circle, the intersection of these planes with toroids $A$ and $B$ lead to the same pair of concentric circles of radii $b_1 + b_2$ and $|b_1 - b_2|$. As the axis of $B$ is rotated $\theta$ radians about the $Y$ axis these two circles intersect in the four found points and the surfaces are always tangent there. Replacing $v_B \in \{0, \pi\}$ in parameterizations $\varsigma_i, i = 1 \ldots 4$ it is found that these four points are the intersections between pairs of curves in $C$. From these results it can be concluded that any line-symmetric linkage of this family (featuring two opposite link-lengths equal to
zero and all axial offsets equal to zero) is reconfigurable with at least 4 bifurcation configurations \( q_i^p \in V, i = 1 \ldots 4 \), such that \( E(q_i^p) \in \{(0, b_1 + b_2, 0), (0, b_1 - b_2, 0), (0, -b_1 - b_2, 0), (0, -b_1 + b_2, 0)\} \). We call these four points the permanent points of tangency, since they are always present in any member of this family of line-symmetric Bricard linkages.

When \( E(q) \) reaches any of these points, the linkage is in a flattened configuration and \( v_A \) and \( v_B \) are either 0 or \( \pi \) (which is equal to \(-\pi \) in \( T \)). Therefore, the change in the sign of \( v_g \) explained in figure [4] will occur in these configurations. A degenerate case occurs when \( \theta = 0 \) or \( \pi \). In this case the 4 curves become the two circles in the \( X_A Y_A \) plane with radii \( b_1 + b_2 \) and \( |b_1 - b_2| \). Throughout these circles the intersection is transverse since the surfaces are touching each other in the circles. \( A_1 \) and \( B_1 \) become coaxial as well as \( A_3 \) and \( B_3 \), but \( A_2 \) and \( B_2 \) are idle. Thus, the linkage has partitioned mobility with 2 DOF.

From \( \mathcal{V}_A(x, y, z) \times \mathcal{V}_B(x, y, z) = 0 \) other solutions, all implying \( y = 0 \), are found. Therefore, any point of tangency lies on either the \( Y \) axis or the \( XZ \) plane. The solutions in the \( XZ \) plane are large expressions that involve not only the construction parameters of the toroids, but also the angle between axes, \( \theta \). Consider the intersections of both toroids with the \( XZ \) plane, which are the \( B_A \) and \( B_B \) curves. If \( B_A \) and \( B_B \) are not tangent to each other, the only bifurcation points of the linkage are the ones discussed in the previous paragraph. Now imagine that \( \theta \) starts increasing from zero until \( B_A \) and \( B_B \) are tangent to each other. The following proposition reveals the values of \( \theta \) for which this result holds:

\[
0 < l_A < r_A < l_B < r_B = b_1 \quad \text{and} \quad l_A < r_A < l_B - r_B
\]

\[
\tan \beta_A x = 0 \quad \text{and} \quad \tan \beta_B (x + \theta) x = 0
\]

Among the possible values of \( \theta \), since they are always present in any \( T \), it is easy to find the points \( \{P_{A\theta}, P_{B\theta}\} = B_A \cap \mathcal{L}_i \) and \( \{P_{B\theta}, P_{B\theta}\} = B_B \cap \mathcal{L}_i \). It will turn out that \( A_{P_{A\theta}} = A_{P_{B\theta}} = \pm \sqrt{b_1^2 - b_2^2} (\cos \gamma_A, 0, \sin \gamma_A) \) and \( A_{P_{B\theta}} = A_{P_{B\theta}} = \pm \sqrt{b_1^2 - b_2^2} (\cos \gamma_B, 0, -\sin \gamma_B) \) and the plane is bi-tangent to both toroids in the same two points.

Without the symmetry condition, the toroids sharing the same bi-tangent plane may intersect the plane in different points.

6 Villarceau circles in the toroids intersection and re-configurability to Bennett linkages.

Using this proposition it can be concluded that whenever the bi-tangent planes of the toroids coincide the surfaces will be tangent to each other and two potential bifurcation points will appear in the \( XZ \) plane. Since, in figure [5] \( B_A \) and \( B_B \) can appear in different sides of \( \mathcal{L}_i \), this can happen for eight different values of \( \theta \):

\[
\begin{align*}
\beta_A + \beta_B, \\
\beta_A - \beta_B, \\
\beta_A - \beta_B, \\
(\pi - \beta_A + \beta_B), \\
(\pi - \beta_A - \beta_B), \\
- (\beta_A + \beta_B), \\
- (\beta_A - \beta_B), \\
- (\pi - (\beta_A + \beta_B))
\end{align*}
\]

It is known that the bi-tangent planes to any toroid intersect the latter in two circles [23] symmetrically disposed with the \( XZ \) plane (in our case). Since the bi-tangent plane is shared by both toroids these two circles are components

![Fig. 5. Intersection of toroids A and B and their bi-tangent plane with the XZ plane.](image-url)
of $C$. If (w.l.o.g.) $r < l$ for toroid $B$, these circles are the famous Villarceau circles \cite{67}, of radii $l_B = b_1$, centers $(0, \pm(l_B - r_B), 0)$ and intersecting in $P_3$ and $P_4$, while for toroid $A \ l < r$ and the two circles are the secondary circles. One of these circles is generated by $E$ rotating around $S_{A2}$ while $\alpha_A = \frac{1}{2} \pi$, then revolute joint $A1$ is idle for this branch of motion. Due to the line-symmetry condition revolute joint $B3$ must be idle too, then the linkage evolves to a different Bennett linkage in this branch of motion, since it is the only spatial skew 4R linkage that is movable.

It is well known \cite{52,53,54} that a Bennett linkage can be explained as a generator of a toroid where $E$ is confined to move in a Villarceau circle by means of link between $E$ and a revolute joint that generates the circle. Thus the Bennett conditions \cite{68} must be present in this Bricard linkage: since $A1$ is idle, the DH parameter for the twist angle of now adjacent axes $S_{B1}$ and $S_{A2}$ is $\alpha_{B1,A2} = \gamma_A - \theta = \pm \arcsin((b_2/b_1) \sin \gamma_B)$ or, depending on the choice of $\theta$, $\alpha_{B1,A2} = \gamma_A - \theta = \pi \pm \arcsin((b_2/b_1) \sin \gamma_B)$, as expected from the Bennett conditions. In the opposite case $A1$ is active since it cannot be generated by $E$ rotating around $S_{A2}$. These two circles intersect the other two curves of $C$ in the permanent points of tangency in the $Y$ axis, through which reconfigurability is possible. Figure 6a shows this case in which the intersection includes the two Villarceau circles intersecting in points 6 and 5, which are points of tangency, the intersection also includes two other closed curves that are not circles.

Using Eq. (1) or (2) it can be proved that $T_{r,r,0} = T_{r,l,0}$, if $\sin \beta = \pm (r/l) \sin \gamma$. Therefore, whenever $\sin \gamma < l$, the Villarceau circles in the toroid can be used as secondary circles for the same toroid \cite{23}. This property was used in \cite{55} in the design of the different types of Bennett linkages. This different way of generating the same toroid can be used in the line-symmetric Bricard linkage, since, if $l = b_1$ and $r = b_2$, for the other generator $l = b_2$ and $r = b_1$, fulfilling the line-symmetry condition. Such a Bricard linkage would be obtained from the intersection of a toroid with $r \sin \gamma < l$ and a copy of itself rotating about the $Y$ axis.

This result is interesting because in such case it is possible to force the two toroids to share a secondary circle. For this aim refer to figure 7 where a toroid is intersecting a copy of itself which has been rotated $\theta$ degrees about the $Y_A$ axis. On any non-right toroid two families of secondary circles can be drawn. Let the two secondary circles crossing the $Y_A = Y_B$ axis of toroid $i \in \{1,2\}$ be $C_2i$ and $C_2'i$ (each belonging to each family). By manipulating the rotation angle $\theta$ it is possible to make coincident a pair of these circles. As illustrated in figure 7 from simple geometry it can be concluded that if $\theta = 2 \gamma$, then $C_{2A} = C_{2'B}$ and if $\theta = \pi - 2 \gamma'$, then $C_{2A'} = C_{2'B}$, where $\gamma' = \gamma_A = \gamma_B$.

If $\theta$ is any of the two values obtained in the previous paragraph, then $C$ includes two circles of radii $r = b_2$ and center at $(0, b_1, 0)$, again two revolute joints are idle and the Bricard linkage evolves to a different 4R Bennett linkage. The Bennett condition is present in the evolved linkage: if toroid $A$ is the one generated with $l > r$, $\gamma_A = \gamma$ and toroid $B$ is generated by a Villarceau circle $(r > l$, $\gamma_B = \beta_B = \pm \arcsin((r/l) \sin \gamma))$, then revolute joint $A1$ is idle in the branch of motion for which $E$ generates the circle about $S_{A2}$. Then, the DH parameter for the twist angle between now adjacent axes $S_{B1}$ and $S_{A2}$ is $\alpha_{B1,A2} = \gamma_A = \gamma$. On the other hand, $\alpha_{B1,B2} = \gamma_B = \beta_A = \pm \arcsin((r/l) \sin \gamma_A)$, which is the Bennett condition. Figure 6b shows this case in which the intersection includes two disjoint secondary circles, the intersection also includes two other closed curves that are not circles.

7 A linkage with two Bricard and two Bennett motion branches.

As an example consider the linkage shown in figure 8 which generates two concentric toroids for which $\gamma_A = -\frac{1}{3} \pi$, $\gamma_B = b$, $l_A = b/\sqrt{3}$, $r_B = b/\sqrt{3}$, $l_B = b$ and $\theta = \frac{3}{2} \pi$. Both toroids are the same surface since $\gamma_A' = \beta_B = \arcsin((r_B/l_B) \sin \gamma_B) = \frac{1}{2} \pi$, with toroid $A$ obtained from its generator with $r > l$ and $B$ from its generator with $r < l$. Also, note that $\theta = 2 \gamma_B = \frac{3}{2} \pi$, therefore the intersection features two secondary circles, $S_{A1}$ and $S_{B1}$, like the ones in figure 6b. In addition, $\theta = \pi - (\gamma_A + \beta_B) = \frac{5}{2} \pi$, therefore the inter-
section features two Villarceau circles, \(C_1\) and \(C_2\), like the ones in figure 6. Thus \(C = C_1 \cup C_2 \cup C_3 \cup C_4\) and the linkage evolves in two different types of Bennett linkage, one related to a secondary circle and another related to a Villarceau circle. In the other two circles all joints are active.

The bifurcation points between secondary and Villarceau circles are the four permanent points of tangency, present in every linkage of this family. The two Villarceau circles intersect in two bifurcation points in the \(XY\) plane.

Figure 10 shows the two concentric toroids and different configurations of the linkage. It can be observed that if \(V_i\) is the component of \(V\) related to the circle \(C_i\), \(i = 1 \ldots 4\), then in \(V_2\) and \(V_3\) the linkage behaves as a Bennett linkage, as joints \(A1\) and \(B3\) are idle in \(V_2\), while joints \(B1\) and \(A3\) are idle in \(V_3\). In \(V_1\) and \(V_4\) all joints are active. Since the same link-lengths appear in the two Bennett motion branches, but the angles between active adjacent joints change, the linkage is able to work in two different types of Bennett linkage, from the four well-known cases [54]. The change between the two Bennett motion branches happens in the flattened configuration for which \(E = P_2 = C_2 \cap C_3\).

The figure 9a shows a representation of \(\mathbb{T}^2\) with the curves of \(\gamma_A\) versus \(\gamma_B\) for each motion branch. A flat plot of a cycle of the curves is shown in figure 9b. Each bifurcation point in figure 8 is indicated in the plot as a corresponding intersection of curves. However, it has to be noted that even though the curves for \(V_1\) and \(V_4\) intersect in three points, only \(P_3\) located as \((\frac{1}{2}\pi, \frac{1}{2}\pi)\), is a bifurcation point. In the other two crossing points the other parameters, \(\nu_A\) and \(\nu_B\), should not coincide.

A similar linkage which is a special case of this family of reconfigurable linkages was presented in a previous paper [24] without any further explanation on how to obtain such linkage. Reconfigurable linkages that are able to switch between two types of Bennett linkage were obtained in [69], by adding two revolute joints, the necessary change of orientation of the four axes was obtained, however there is no Bricard branches in such linkages since there is no branch of motion with the 6 joints active. The reconfigurable linkage presented in [46] features two Bricard and one Bennett branch, such linkage was obtained applying the concept of spatial triangle.

8 The Altmann linkage as a member of this family of special Bricard linkages.

The Altmann linkage [70, 71, 5] is an example belonging to this family of reconfigurable Bricard linkages. For this very special case \(\gamma_A = \gamma_B = \theta = \frac{1}{2}\pi\). Both toroids are right torus of the common form. The torus for which \(r > l\) is singular with the two conic singularities lying on the axis of the torus. However, it is easy to prove that these singularities
Fig. 9. The four curves of $u_A$ versus $u_B$ obtained from the motion branches of the linkage. Bifurcation points are located in relation with figure 8. a) toroidal representation, b) cartesian plot.

are never part of the intersection of the tori in the Altmann linkage case. The intersection features four curves crossing the four permanent points of tangency as in the general cases explained in this paper.

Throughout the motion of the Altmann linkage all joint axes intersect in two points, thus all of them belong to the same special lineal complex for it is always possible to locate the central axis intersecting these two points. It is easy to present a configuration of a linkage belonging to the family presented in this paper, in which a line intersects all the joints axes. The intersection of these two planes containing axes $\{S_{A1}, S_{B1}\}$ and $\{S_{A3}, S_{B3}\}$ is always perpendicular to the line of symmetry, hence, through this line it is possible to draw two skew symmetric axes for $S_{A2}$ and $S_{B2}$ and the axis of the special linear complex would be the intersection of such planes. However, this setup would change in the next instant and the found line will no longer intersect all axes. In the Altmann linkage the two points where the axes intersect lie on the intersection of the two planes, this intersection is the central axis of the linear complex, which, as expected, intersects perpendicularly the axis of symmetry. In general, the joint axes of the family of linkages presented in this section belong to the same non-special linear complex whose axis can be found geometrically following the Appendix to [13]. Baker [72] was able to generalize the Altmann linkage keeping its peculiarity of axes permanently intersecting in two points. Cui and Dai [73] explore the 6R linkages with axes intersecting in two points but in adjacent groups of three joint axes.

Another member of this family of Bricard variations is the reconfigurable linkage with both line and plane symmetries reported in [47], if $R = 0$ in such paper, i.e. the axial displacement parameter for the joints with intersecting axes, the linkage becomes a special case of the family of Bricard variations presented in this paper with both toroids singular.

9 Conclusions.

A family of reconfigurable line-symmetric Bricard variations was obtained applying the method of generated surfaces. It was concluded that the generated toroids are always concentric and have symmetrical flattened forms. The general intersection of these toroids lead to a maximum of four curves. Any possibility of tangency between the surfaces was explored. The four permanent points of tangency in the line-symmetric case proved that this family of Bricard variations is always reconfigurable. The Bennett condition was found in the case of line-symmetric toroids that are also tangent to each other in the $X_A Z_A$ plane, allowing the linkage to evolve into a 4R linkage. Looking for other circles in the general intersection of these toroids, another type of Bennett branch of motion was found when the intersection includes the secondary circles.

An example in which the four curves in the intersection are circles was presented. Applying the theory developed, the computation of the intersection for this specific case is not required and the curves are recognized as two Villarceau circles and two secondary circles which alternate two Bricard and two Bennett branches of motion.

It is important to mention that the authors applied the same method to the plane-symmetric case in a following paper [60] published in parallel with this paper.

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Fig. 10. Different positions for an example of line-symmetric linkage where the generated toroids intersect in two circles.

References


Appendix A: Solutions for the toroid parameters.

In this Appendix the four solutions for the concentric toroid-toroid intersection with line symmetry are listed.

Solution 1:

\[
\begin{align*}
&u_1(v_b) = \arctan \left[ -\frac{1}{\sin B \cos \gamma_b} \left( b_2 (b_2 + b_1 \cos \gamma_b) \sqrt{K_1} \right) \\
&- b_1 b_2 \cos \gamma_b \cos \gamma_b \sin^2 B (b_1 \sin \gamma_b \cos \theta - b_2 \sin \gamma_b) \right], \\
&v_1(v_b) = v_b \\
&u_2(v_b) = \arctan \left[ \frac{b_2 \cos \gamma_b \sqrt{K_1} - b_1 b_2 \cos \gamma_b \cos \gamma_b \sin^2 B (b_1 \sin \gamma_b \cos \theta - b_2 \sin \gamma_b)}{b_2 \sin^2 B \cos^2 \gamma_b + (b_1 + b_2 \cos^2 \gamma_b) B} \right]. \\
&v_2(v_b) = v_b
\end{align*}
\]

Solution 2:

\[
\begin{align*}
&u_1(v_b) = \arctan \left[ -\frac{1}{\sin B \cos \gamma_b} \left( -b_2 (b_2 + b_1 \cos \gamma_b) \sqrt{K_1} \right) \\
&- b_1 b_2 \cos \gamma_b \cos \gamma_b \sin^2 B (b_1 \sin \gamma_b \cos \theta + b_2 \sin \gamma_b) \right], \\
&v_1(v_b) = v_b \\
&u_2(v_b) = \arctan \left[ -\frac{b_2 \cos \gamma_b \sqrt{K_1} - b_1 b_2 \cos \gamma_b \cos \gamma_b \sin^2 B (b_1 \sin \gamma_b \cos \theta + b_2 \sin \gamma_b)}{b_2 \sin^2 B \cos^2 \gamma_b + (b_1 + b_2 \cos^2 \gamma_b) B} \right]. \\
&v_2(v_b) = v_b
\end{align*}
\]

Solution 3:

\[
\begin{align*}
&u_1(v_b) = \arctan \left[ -\frac{1}{\sin \theta \cos \gamma_b} \left( b_2 (b_2 + b_1 \cos \gamma_b) \sqrt{K_2} \right) \\
&- b_1 b_2 \cos \gamma_b \cos \gamma_b \sin^2 B (b_1 \sin \gamma_b \cos \theta - b_2 \sin \gamma_b) \right], \\
&v_1(v_b) = -v_b \\
&u_2(v_b) = \arctan \left[ -\frac{b_2 \cos \gamma_b \sqrt{K_2} - b_1 b_2 \cos \gamma_b \cos \gamma_b \sin^2 B (b_1 \sin \gamma_b \cos \theta - b_2 \sin \gamma_b)}{b_2 \sin^2 B \cos^2 \gamma_b + (b_1 + b_2 \cos^2 \gamma_b) B} \right]. \\
&v_2(v_b) = -v_b
\end{align*}
\]

Solution 4:

\[
\begin{align*}
&u_1(v_b) = \arctan \left[ 1 \right] \left( b_2 (b_2 + b_1 \cos \gamma_b) \sqrt{K_2} \right), \\
&v_1(v_b) = v_b \\
&u_2(v_b) = \arctan \left[ -\frac{b_2 \cos \gamma_b \sqrt{K_2} - b_1 b_2 \cos \gamma_b \cos \gamma_b \sin^2 B (b_1 \sin \gamma_b \cos \theta - b_2 \sin \gamma_b)}{b_2 \sin^2 B \cos^2 \gamma_b + (b_1 + b_2 \cos^2 \gamma_b) B} \right]. \\
&v_2(v_b) = v_b
\end{align*}
\]

where:

\[
\begin{align*}
K_1 &= -\cos \gamma_b \left[ (2B_1 b_2 \sin \gamma_b \sin \gamma_b \cos \theta - b_2^2 \sin^2 \gamma_b \\
&- b_1^2 \sin^2 \gamma_b) \cos^2 \gamma_b - 2B_1 b_2 \sin \gamma_b \cos \gamma_b + (b_1^2 + b_2^2) \cos^2 \gamma_b \right] \\
K_2 &= -\cos \gamma_b \left[ -2B_1 b_2 \sin \gamma_b \sin \gamma_b \cos \theta - b_2^2 \sin^2 \gamma_b \\
&- b_1^2 \sin^2 \gamma_b) \cos^2 \gamma_b - 2B_1 b_2 \sin \gamma_b \cos \gamma_b + (b_1^2 + b_2^2) \cos^2 \gamma_b \right] \\
&- 2B_1 b_2 \sin \gamma_b \sin \gamma_b \cos \theta - b_2^2 \cos^2 \gamma_b \right]
\end{align*}
\]
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