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DOI:

[10.1093/teamat/hru017](https://doi.org/10.1093/teamat/hru017)

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Citation for published version (APA):

Healy, L., & De Carvalho, C. C. S. (2014). Evidence-based, theoretically informed design as a means to investigate and transform proof practices in school mathematics. *Teaching Mathematics and its Applications*, 33(3), 150-165. <https://doi.org/10.1093/teamat/hru017>

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Evidence-based, theoretically informed design as a means to investigate and transform proof practices in school mathematics

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The search to understand the complexities associated with the teaching and learning of argumentation and proof in mathematics has attracted a large number of researchers throughout the history of the field of mathematics education. Given the special place occupied by proofs and the processes associated with proving in mathematical practice, perhaps this is not altogether surprising. As Celia Hoyles put it in 1997:

“Proof lies at the heart of mathematics. It has traditionally separated mathematics from the empirical sciences as the indubitable method of testing knowledge which contrasts with natural induction from empirical pursuits. Deductive mathematical proof offers human beings the purest form of how to distinguish right from wrong [Wu, in press], a characteristic, some have argued, which has been responsible for the central role of mathematics in Western thought [Aleksandrov, 1963].” (Hoyles, 1997, p. 7).

Her research into school students’ understandings of proof and the proving process, which began in the mid 1990s, was not, though, motivated only by epistemological concerns. Rather it grew from a desire to bring a social dimension to the debate, which, at that time, had largely been dominated by epistemological and psychological concerns. In the already considerable body of mathematics education research that had been published, the tendency was to argue that students’ understanding of the proving process develops according to some kind of universal, cognitively determined, hierarchy of “proving competencies”, in which explorations of empirical examples are located on the lowest of levels, while rigorous, formally presented,

deductive arguments represent the pinnacle of proving achievements. Certainly, the evidence from published studies had pointed to a range of challenges associated with engaging with different aspects of the proving process and had shown, in particular, how many students find it difficult to construct or follow formally presented arguments and are unsure as to how such arguments differ from empirical evidence or can be used to derive further results (Balacheff, 1988; Chazan, 1993; Fischbein, 1982; Harel & Sowder, 1998; Porteous, 1994; Schoenfeld, 1989). According to the review of the literature presented by Reid and Knipping (2010), more recent research continues to confirm the prevalence amongst not only school students, but also university students and mathematics teachers, of the same difficulties. While in her work Celia Hoyles accepted local hierarchies in the development of mathematical knowledge, the question she raised, and explored in detail during two large-scale studies between 1995 and 2003¹, was whether it makes sense to seek interpretations for the tendencies in students' proving practices posed in purely cognitive terms.

Her focus on the social shaping of mathematical practices is characteristic of her whole research programme and not only the studies related to proof. In the field of mathematics education, she has made a major contribution to the development of theoretical explanations that throw light on how participation in mathematical practices is moulded by the cultural constraints and affordances of the context in which it occurs, and, together with Richard Noss, has expressed criticism and concern about the consequences of ignoring such factors:

if we restrict our terms of reference simply to the interaction of epistemology and psychology – ignoring the social dimension – then it is inevitable that mathematics learning will be perceived as the acquisition of context-dependent knowledge within a hierarchical framework. Starting from a position of epistemology/psychological locks research and its findings into a tautological loop. Moreover, we do not see it as accidental that these hierarchies mirror the various sequences of lessons in the school curriculum and bolster the assumption that mathematics is only appropriate for a small elite.” (Noss & Hoyles, 1996, p.48).

¹ The first of these two projects *Justifying and Proving in School Mathematics* from 1995 to 1998 under the direction of Celia Hoyles and the first author of this paper, Lulu Healy. The second *A Longitudinal Study of Mathematical Reasoning: Student Development and School Influences* ran from 1999 to 2003, with Celia Hoyles as director and Dietmar Küchemann as principal research officer. Both projects were funded by the ESRC.

In highlighting the shaping forces of the social dimension, her research has also been characterised by a commitment to change the status quo, to attempt to create mathematical cultures, tools, activities and teaching approaches that make mathematics an attractive and accessible option to the many rather than the few. This has also involved her in pioneering innovative methodologies which combine different research techniques, aimed to preserve scientific rigour while at the same time not losing contact with the authentic settings she researches.

In this paper, we intend to reflect upon how Celia's approaches, concentrating particularly on her attention to method, have impacted on our own research. Her influence on each of us has taken rather different forms. For one of us (Lulu), a member of Celia's team at the Institute of Education in London for a decade, before relocating to live and work in Brazil, this influence has been of a very direct nature. The project Justifying and Proving in School Mathematics (JPSM Project) was the final project developed during this collaboration. For the other (Claudia), a secondary school mathematics teacher based in the State of São Paulo, the relationship with what we might term as the "Hoyles legacy" began in earnest as she embarked on her post-graduate studies in mathematics education and in particular began to explore proof and proving in Brazilian schools.

In the rest of the paper, we concentrate on how the lessons learnt from the British project have and are impacting upon our own ongoing considerations of the teaching and learning of proof in Brazil. Although it is not our intention to describe the original project in detail (the results have been published in various articles, including Hoyles, 1997, Healy & Hoyles, 2000; Hoyles, 2001, Healy & Hoyles, 2001, Hoyles & Healy, 2007), we begin by briefly revising the JPSM Project to highlight those aspects which have been fundamental in fashioning our own research.

Revisiting the project Justifying and Proving in School Mathematics

This project began seven years after the first implementation of a National Curriculum for Mathematics in England and Wales. This curriculum prescribed an approach to teaching proof organised into a set of levels that largely corresponded to the hierarchy of proof competencies described above – the introduction of formal proof was delayed until after students had progressed through stages of reasoning empirically and

explaining their conjectures. Schools and teachers had a statutory requirement to follow the curriculum guidelines and it seemed timely to investigate how the approach was influencing students' conceptualisations of proving. In fact, the approach had already been the subject of some criticism, notably in the form of a report commissioned by the London Mathematical Society, which stated that

Most students entering higher education no longer understand that mathematics is a precise discipline in which exact, reliable calculation, logical exposition and proof play essential roles; yet it is these features which make mathematics important. (London Mathematical Society 1995, p.2)

Rather than seeking to simply confirm or refute this claim, or to emphasise only what students lack, the aim of the first phase of the JPSM Project was to provide a comprehensive picture of precisely how the new curriculum, and the ways in which it was being enacted in practice, was affecting students' views of, and approaches to, different aspects of the proving process. In this way, the project attempted to look beyond purely cognitive interpretations of students' performances and search for explanations that also took account of students' beliefs and motivations, teaching and curriculum factors, as well as school and societal influences. In the second phase of the project, experiments were devised in which activities were specifically crafted to exploit the students' strengths as identified in Phase 1 whilst also addressing areas identified as problematic. These activities incorporated the use of digital tools in and aimed to allow students to experience and express relationships between mathematical properties in a variety of ways – visually and dynamically, as well as symbolically.

From a methodological point of view, this project had a number of interesting, even pioneering, features. The statistical methods used in the first phase combined the use of descriptive statistics with the use of multilevel statistical models (Goldstein, 1995). Multilevel modelling is a technique that can be used to model how parameters vary at different levels and is used for analysing data with hierarchical structures (students in classes in schools, provide a case in point). At the time of the project, it was not a technique commonly applied in mathematics education research. It permitted a focus on how variance in students' proof performances occurred at the school as well as the individual level and enabled the identification of what were termed as “outlier schools”, in which particular aspects of the proving process varied significantly from the usual patterns. The experimental methods used in the second

phase would today be described as Design Experiments (Cobb *et al.* 2003). This methodology has become increasingly recognised and prevalent in the field of mathematics education in the 21st Century, but was less well known when Celia introduced its use in her comprehensive programme of research into the potential offered by digital tools to the doing and learning of mathematics that began with the Logo Mathematics Project in the early 1980s (Hoyles & Sutherland, 1989).

By combining the quantitative techniques of multilevel modelling with the essentially qualitative methods associated with conducting design experiments, the project aimed to provide new ways of looking at how student factors along with teaching and school factors contribute to variations in students' attitudes and practices.

To document students' conceptualisations of proof a nationwide survey was conducted, involving 2459 students (aged between 14 and 15 years) from 94 classes in 90 schools, spread across England and Wales. The students who participated were all in the top sets for mathematics or, in the very small number of school with mixed-ability mathematics classes, were identified as high attainers by their mathematics teachers. It is likely then that the sample included those who were most likely to continue studying mathematics beyond at advanced levels of study, perhaps the very group of students that those at the London Mathematical Society were concerned about.

Their worry related to the preparation of students for the activities of formal proving they encounter at the university level, something that needs to start during the elementary school level. Indeed, focussing on the conceptions of proof and proving developed by students of this age range might help clarify some of the issues faced by mathematics students at the higher secondary and tertiary levels as they grapple with the formal aspects of proving. As Campbell and Zazkis (2002) have argued in relation to number theory (the mathematical domain explored in all the studies considered in this paper), underpinning the axiomatic approaches towards elementary number as formally defined over the ring of integers adopted in post-secondary courses are ideas such as whole number patterns, multiples, factors, divisors, primes and prime decomposition and rules of divisibility. They see number theory as a conceptual field, permeating all educational levels and we hope that the focus on students nearing the end of the compulsory period of mathematics education on the UK will open windows onto the practices of those studying at the subsequent levels.

Returning to the JPSM survey, two instruments were implemented during the collection of quantitative data: A school questionnaire and a student proof questionnaire. The school questionnaire was designed to obtain data about the school and the mathematics teacher of the classes in which the students studied. The student proof questionnaire consisted of three kinds of items: a question to ascertain student views on the role of proof, followed by items in arithmetic/algebra and geometry presented in multiple-choice and open formats in which students were asked to construct their own proofs. The results of the students' responses to the questions from the arithmetic/algebra (number theory) and geometry items respectively can be found in Healy and Hoyles (2000) and Hoyles and Healy (2007), suffice to report here that the analyses indicated the prevalence of a particular strategy in students' proof constructions. This strategy involved testing empirically that the statement held for a number of cases (often presented in tabular form) followed by a reaffirmation of the original statement, this time in the format of some kind of "formula". In many cases, this formula was followed by a further final confirming example. This same routine characterised, on the whole, students' responses for the items involving number theoretic statements and geometry ones. It also essentially mirrored the presentation of proof and proving in the National Curriculum of the time.

Although the use of this strategy did not seem to lead the majority of students to produce logically deduced proofs, the picture obtained from the questionnaire analysis was not entirely negative. Students had much more success in evaluating the validity of the arguments presented in the multiple-choice questions than in constructing such arguments. Most seemed to know, for example, that arguments based on examples only proved the statement to be true for the number of cases given. When arguments based on mathematical structure were presented, they were more coherent when communicated in words, accompanied sometimes by visual representations, than in any formal mathematical representation – although students also believed that the formal form, and especially algebra, would be most highly valued by their teachers.

These results show that although students may have had some misgivings about the general validity of the arguments they were producing, the curriculum approach they had followed – based on the assumption that students should first explore empirical arguments and only later, once they reach the "higher" levels, attend to mathematical structure – may have actually had the result of confining many

to the use of examples – hence confirming the supposed hierarchy of proof competencies and locking students into the tautological loop mentioned above by Noss and Hoyles (1996). One possible way forward, explored in the teaching experiments in phase two, involved exploiting the potential of building generic mathematical constructions by defining their properties using digital tools, exploring these constructions dynamically to identify and explain the existence of dependent properties and relations, then explicitly introducing the formal ways of expressing the chains of reasoning by which one set of properties can be deduced from the other in conventional (paper and pencil) proofs.

These design experiments were carried out in only three schools, with the researchers rather than the class teachers assuming temporarily the teaching role. The results indicated that students responded positively to the challenge of attempting logical argument alongside informal argumentation, although there was considerable between-school variation in the three sites in which the experiments were conducted (Hoyles & Healy, 1999). Moreover, perhaps because of the distance of these activities from the requirements of the National Curriculum, it was only possible to implement them with small groups outside of regular classroom activities, which inevitably limited the authenticity of the settings and hence the generalisability of the results.

At the end of the project then, whilst answers to some of the initial questions had been gleaned, the questions of how to develop classroom cultures which support learners in all aspects of proving and of how such cultures are constrained and afforded by different characteristics of the contexts in which the associated practices are employed remained relatively open. In the rest of the paper, we cross the equator from London to São Paulo and consider how the methods and insights from the JPSM Project contributed to our further attempts to explore these questions. We focus on two rather different projects. The first represents an attempt to “replicate” aspects of the JPSM Project in the Brazilian context, only with teachers playing a more prominent and active role than in the first project. The second is more concerned with the challenge of developing digitally mediated possibilities to involve learners in the activities of conjecturing and proving.

Involving teachers in rethinking perspectives on proof in school mathematics

Earlier we stated that major motivation behind the JPSM Project was Celia's ongoing commitment to understand the myriad of factors that shape students' participation in practices related to proof, and the interrelations between student factors, factors associated with teachers and curriculum as well as factors at the level of school and society. This commitment continues to characterise our work. Inevitably, in any research design, we are constrained to focus our attention on some rather than all potentially shaping factors. In the first phase of JPSM, we could say that the lens was focussed particularly on the curriculum and in the second on the resources through which proof practices were to be brought into play. The project's conclusions recommended some changes to the existing curriculum structure and pointed to the potential of digital tools for enabling new ways of teaching and learning proof. Successful approaches to teaching proof do not, however, depend solely on innovative learning situations; they also require the acceptance and appropriation of the new contexts and tools on the part of the teachers. This was an issue that was not directly addressed in JPSM but became a central concern of its Brazilian successor *Argumentation and Proof in School Mathematics (AProvaME)*², a project involving a two-year collaboration between six university-based lecturers in mathematics education and 27 mathematics teachers. All the mathematics teachers were undertaking a professional Masters qualification in mathematics education and all taught mathematics in schools within the public school system of the state of São Paulo. Like its predecessor, the project was divided into two phases, with the first dedicated providing a picture of the proof performances of 14-15 year old students in São Paulo schools and the second concentrating on conducting design experiments to explore digitally-based learning scenarios for proof.

From a research point of view, the active participation of twenty-seven mathematics teachers, at first sight at least, could have offered an opportunity to attempt to "scale-up" the design experiments of the JPSM project – certainly the scaling up of successful innovations from a limited number of schools has been a growing concern in Celia's work (Clark-Wilson, Hoyles & Noss, 2013 report on an ongoing project with exactly this aim). However, since context profoundly shapes research outcomes, transporting projects from one country to another also implies

² The project *Argumentation and Proof in School Mathematics* was funded by CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico). Lulu Healy was also in receipt of a CNPq grant during the preparation of this paper.

fine-tuning research activities to fit the cultural differences between them. In Brazil, there is no National Mathematics Curriculum, prescribing a particular approach to proof (although a set of National Curriculum parameters offers guidelines, these are not statutory), neither had there been any large-scale quantitative studies to provide a picture of Brazilian students' proof practices.

To maximise the participation of the teachers, we decided to adopt a methodology in this project which combined quantitative explorations and methods associated with design experiments with a kind of participatory action research, co-generative investigation (Greenwood and Levin, 2000). In co-generative studies all participants contribute to the construction of knowledge through a process of collaborative communication. Constructed knowledge is intended to lead to social action, with reflections upon the research activities involving participants in re-signifying their practices. In the case of the project AProvaMe, the social action in question related to the challenges inherent in the teaching and learning of proof, and particularly, how this challenge manifested itself within the schools of the teachers involved.

In the first phase of the project, then, the focus was, once again, on mapping the current situation – both in terms of the proofs performances of the students of the teachers involved and the teachers' own views and experiences of proof and proving. The main instrument for mapping the practices and performances of students was a questionnaire, modified by the whole research team from the questionnaire originally used in the JPSM Project. Before the process of modification began, each teacher was asked to provide a set of data which was used to compose their initial profile (experiences during initial teacher education, description of a proof-related activity they had developed, responses with to items from the unmodified version of the proof questionnaire). These initial profiles indicated the teachers did not feel well prepared to work on argumentation and proof in their own classrooms. Their visions of the proving process was limited to their experiences during initial education courses (which emphasised only formal proof), along with, for a handful only, experiences within the classroom indicating formal proofs to be inaccessible to their students. Their initial visions valued proof as a product and many of them (16 of the 27) were unable to describe a single proof-related activity they had developed with their students. In contrast to the uniform curricula approach that characterised the British situation, these profiles suggested that in Brazil, or at least in São Paulo classrooms,

proofs and the process of proving were hardly present at all. Additional confirming evidence of the relative absence of activities related to constructing conjectures and proofs comes from studies of the textbooks in use in Brazilian classrooms. An analysis of 11 textbook collections recommended by MEC (the Brazilian Ministry for Education and Culture) for use in Brazilian schools the time of the AProvaME project showed that, even in the three collections that gave most attention to proof, in the context of Number Theory, for example, not one of the books included activities in which student proofs were solicited (Carvalho, 2007).

In the Brazilian version of the questionnaire, some of the items remained very similar to those used in the original project (Figure 1), while others were new (the complete version in Portuguese is available in Healy, Jahn & Pitta Coelho, 2007). In the multiple-choice question on Figure 1, four of the five options had appeared in the original questionnaire (Artur, Bia, Duda and Franklin), two options had been dropped (an incorrect algebraic argument and a correct argument similar to that of Artur's based expressed in words) and one new option introduced. The new argument (Hanna's) was an example of the use of a generic example – something discussed in the literature but quite new to, and increasingly valued by, the majority of the participating teachers. One notable difference between the two questionnaires was that while the British version had items related to proofs of familiar and unfamiliar properties, the Brazilian teachers decided that their version would focus exclusively on content they expected to be familiar to their students.

A1: Artur, Beth, Duda, Franklin and Hanna were trying to prove that the following statement is always true

When you add any two even numbers, the result is always even.

Artur's answer

a is any whole number
 b is any whole number
 $2a$ and $2b$ are even numbers
 $2a + 2b = 2(a + b)$

So Artur says the statement is true.

Beth's answer

$2 + 2 = 4$ $4 + 2 = 6$
 $2 + 4 = 6$ $4 + 4 = 8$
 $2 + 6 = 8$ $4 + 6 = 10$

So Beth says the statement is true.

Duda's answer

Even numbers end in 0, 2, 4, 6 or 8.
When you add two of these, the answer will still end in 0, 2, 4, 6 or 8.

So Duda says the statement is true..

Franklin's answer

So Franklin says the statement is true.

Hanna's answer

$8 + 6 = 14$

$8 = 2 \times 4$
 $6 = 2 \times 3$
 $14 = 2 \times (4 + 3)$

$8 + 6 = 2 \times 7$

So Hanna says the statement is true.

From the above answers, choose one that would be closest to what you would do if you were asked to answer this question.

From the above answers, chose the one to which your teacher would give the best mark.

Figure 1: Item A1 in the AprovaME version of the questionnaire

The final version of the AProvaME questionnaire was administered in 81 classes from 31 different schools, resulting in dataset consisting of 1998 protocols. Each teacher was responsible for the coding of the protocols of the students in the 3 classes for which he or she has collected data. This data was then analysed by a subgroup of the research team, composed of 6 teachers and 3 researchers. Despite the attempts to ensure that the content would be familiar to their students, the results of these analyses indicated that the students had experienced considerable difficulties in completing the questionnaire, with three groups of students identified (Healy, Jahn & Frant, 2009): those who were unable to offer any justification for mathematics statements (for many items, this was the largest group), those whose justifications were based on calculations with particular cases and a third, extremely small group, who managed to construct arguments based on mathematical properties – in relation

to the construction of a proof that the sum of two odd numbers is always even, for example, only 143 of the 1998 students produced arguments which included some attention to mathematical structure (and only 10 were considered complete proofs).

These results provide further evidence of students' tendency to produce empirical arguments more frequently than conceptual proofs. In the Brazilian context, however, a more elementary difficulty can also be identified: many students were unable to construct even a confirming or refuting example. Considering this result in isolation might suggest that Brazilian students have more difficulties in dealing with argumentation and proof than their contemporaries in other countries. But when the scarcity of proof practices in Brazilian school mathematics is taken in to account, it seems likely that these results reflect pedagogic as much as cognitive issues.

In the light of these results, the teachers were yet more convinced of the need to conceive learning scenarios which might contribute to the emergence of cultures of proving within their own classrooms. For the activities of Phase 2, dedicated to the design and evaluation of these scenarios, the research team divided into 5 groups in each of which the development of the learning situations occurred in a cyclical manner. Each group was responsible for the development of at least two technology-integrated activities (one in the domain of geometry and the other number/algebra). Two kinds of digital tools were selected, dynamic geometry and spreadsheets. During the design process, an initial preference for activities predominantly leading to particular proofs evolved into longer sequences of activities, in order that dynamic aspects of the software might be appropriated and explored. This in turn led the teachers to consider encouraging new strategies for treating examples – in order to facilitate the construction of empirical arguments, while also bringing elements with which students might begin to contemplate the difference between these and arguments based on properties.

Of the three main strategies for smoothing the passage between empirical and conceptual proofs, of particular note was the gradual inclusion into the learning scenarios of activities involving the exploration of generic examples (an idea that was completely unknown at the start of the project). The other two strategies were related to the form in which arguments were expressed. The teachers became increasingly convinced that the formal presentation was not the only form that should be encouraged, and in all the learning scenarios, the natural language register was valued. The teachers also came to recognise that when more formal presentation were

desired it was necessary to include activities specifically addressing the introduction of structures for the organizing of logical chains of the steps of the constructed proofs. When it came to experimenting with the final versions of the scenarios, individual teachers assumed the responsibility for implementing what then became “their scenario” (and the focus of the final coursework for their Masters).

A total of thirteen scenarios were designed and tested, nine involving content associated with geometry and only four content from the arithmetic/algebra domain. All the geometry scenarios were based on the use of the dynamic geometry software, as were two of the scenarios arithmetic/algebra domain (these involved proofs related to arithmetic and geometric progressions). In the other two scenarios, use of the spreadsheet Excel was incorporated. Full details are available in Jahn and Healy (2008).

On the one hand, the development of thirteen digitally mediated learning environments involving proof-related activities can be seen as a strength of the project. Since one of the aims of the project was for teachers to face the challenge of bringing proof and proving into their own classrooms, the co-generative methods that were employed privileged local change in authentic situations. On the other hand, each scenario was tested in only one classroom and there was a low degree of uniformity in the ways in which student data was collected and analysed by different teachers. Hence, while the aim of involving teachers in design-based research did contribute to the construction of new meanings for their teaching practices, one of the costs was the loss of opportunity to explore how the success of particular innovations varies across contexts.

Another issue was the software itself. The criteria for the selection of the digital tools to be used within the scenarios were availability and familiarity – and both tools necessitate a learning curve on the part of the teachers and students, an additional demand to the already taxing problem of creating proof activities. The added complexity associated with appropriating the digital tools meant also that the scenarios were based around extended learning sequences, which brought further complications for their integration into the existing curriculum. This factor, along with the problems of access to digital technology in many of the Brazilian schools, meant that the majority of the teachers in the project felt it would not be an easy task to use the scenarios they had created as part of regular curriculum activities.

What we felt we needed as a next step was to use the evidence from this project, as well as the results from the research into proof of the international community, to create a digital environment that might more easily be incorporated into the existing rhythms of Brazilian classrooms, while at the same time offering new possibilities for students to engage in the proving process. This brings us to a second Brazilian project and also back to the work of Celia. In the next section, we describe how her characterisation of the potential offered by digital tools to transform teaching and learning and reinvigorate involvement with mathematics contributed to the development of the digital environment *Consecutivo* (or *Consecutive* in English), designed to engage students with the properties of consecutive numbers.

Linking evidence and theory to provide fine-tuned digital support for proving

In 2008, during her plenary lecture at the International Congress on Mathematics Education, Celia pointed to six features of digital tools highlighted in her research since the early 1908s. She reiterated these characteristics in a paper presented at the research seminar of the Brazilian Society of Mathematics Education in Petrópolis in 2012, as follows:

- *dynamic & visual tools* that allow mathematics to be explored in a shared space - changing how mathematics is learned and taught;
- tools that *outsource processing power* that previously could only be undertaken by humans - changing the collective focus of attention during mathematics learning;
- *new representational infrastructures* for mathematics – changing what can be learned and for whom;
- *connectivity* – opening new opportunities for shared knowledge construction and for student autonomy over their mathematical work;
- *connections between school mathematics and learners' agendas and culture* – bridging the gap between school mathematics and problem solving 'in the real world';
- *intelligent support* for learners while engaged in an exploratory environment.

(Hoyles, 2012, p.1)

These potentialities can be exploited in different ways and explored under different theoretical lenses. All of Celia's work with digital technologies has been informed by

the constructionist vision originally proposed by Seymour Papert (Papert, 1980, 1991). Indeed, any consideration of Celia's work would be incomplete without reference to its constructionist underpinning, and especially the notion of a *microworld* which she describes as

both an epistemological and an emotional universe, a place where powerful (mathematical, but also scientific, musical or artistic) ideas can be explored; but explored 'in safety', acting as an incubator both in the sense of fostering conceptual growth, and a place where it is safe to make mistakes and show ignorance: And, of course, centrally these days, a place where ideas can be effortlessly shared, remixed and improved.

(Hoyles, 2012, p.2)

It is with these ideas in mind that the digital environment *Consecutivo*, whose objective is to foster the creation of conjectures and conceptual proofs related to the domain of the number theory in school mathematics, is being developed. We might term *Consecutivo* a third generation digital proof-related scenario, with some characteristics inherited from the scenarios developed in the previous two projects and others a consequence of the results emanating from them. Our approach to design is hence both evidence-based and theoretically informed. We see its attempts to incorporate lessons from empirical studies as well as theoretical stances as another of Celia's bequests – her untiring efforts to connect the worlds of research and teaching. To end this paper, we give a brief description of *Consecutivo*, which we hope might provide a small taste of this ongoing enterprise, as well as Celia's imprint upon it (a more detailed description is available in Carvalho, 2014).

Consecutivo was written using the *Imagine* version of the Logo programming language, this means that the environment can be used as is, or extended by others with some experience with this language. In common with its first generation ancestor, *Mathsticks* (Noss, Healy & Hoyles, 1997), the aim in creating *Consecutivo* was not to build a well-polished “finished” software, rather, in line with the constructionist agenda, it was to develop a digital tool that can be subsequently used and tinkered with by others (Papert, 1991).

The main page on opening *Consecutivo* is represented in Figure 2. It is composed of a blue panel (1) containing two slider bars, two textboxes and four representation buttons, a green panel (2) containing four task buttons and a number

line (3), with two consecutive numbers selected in a green rectangle. The sliders in the blue panel allow the user to determine the quantity of consecutive numbers to be operated with and the value of the first one. The text-boxes display the sum and product of the numbers. In Figure 2, for example, two consecutive numbers, starting with zero have been selected, so the sum displayed is 1 and the product 0. As the values are altered the sum and product are recalculated and the appropriate section of the number line displayed. In this way, students experience dynamically the results of changing the values of sliders and are able to obtain a multitude of examples easily and quickly – a feature intended to contribute to students' engagement in the construction of conjectures and an example of how the outsourcing of processing power was embedding in the tool. No lists or tables of particular cases are recorded by the tool itself, though, of course, students are free to do this for themselves if they choose.

Figure 2: Consecutivo's main page

Instead of building support for systemising examples into the environment, we chose to offer four different ways in which the structure of the examples might be expressed. These can be accessed by clicking on the representation buttons. Figure 3 presents the feedback associated with three of these buttons. The *Factorisation* button displays each number inside the green rectangle on the number line in terms of its prime factors (Figure 3). The *Remainder* button presents each selected number as an expression in relation to a chosen divisor (which can be altered using the divisor slider), with quotient and remainder also displayed. In the feedback associated with

the *Algebra* button, n becomes the value of the first number of the sequence, $n+1$ the second and so forth, showing the algebraic structure of the consecutive terms.

Figure 3: Feedback from the Factorisation, Remainder and Algebra buttons.

The three ways of representing structure shown in Figure 3 are all conventional representations that Brazilian students are likely to have encountered. Figure 4 shows the feedback associated with the fourth button, *Animal Sum*, a button inspired by the idea of using digital tools to provide (attractive) alternatives to the conventional. The feedback from *Animal Sum* combines aspects of mathematical structure also offered by the *Algebra* and *Remainder* tools, although it presents them in a rather different form. On the left of the *Animal Sum* panel, each number in the green rectangle on the number of line is presented visually as a number of elements (spiders, dogs or turtles). Each element can be physically dragged to a new position, or organised into a number of rows (determined by changing the value of the slider), with any remaining elements specifically highlighted, as shown on the right of the panel. In this way, the *Animal Sum* tool offers digital and visual means of interacting with mathematics structures, a step in the direction of the alternative mathematical infrastructures made possible by digital technology that Celia alludes to. In *Consecutivo*, though, users currently have the option of only limited interaction with this representation form, constraining its expressive power. Still, it is this tool, more than any of the others, to which *Consecutivo* users appear to become most emotionally attached (Carvalho, 2014).

Figure 4: Feedback from the Animal Sum button

Clicking on the icon “i” displayed in the *Animal Sum* panel opens a window with an explanation in text of the representation as well as a video which takes the user step-by-step through the resolution of a particular problem using this representation (Figure 5). Information is available for all four ways of expressing structure, offering a built-in reference for student support, although it does not correspond to the kind of intelligent support included in Celia’s list of digital potentials – in *Consecutivo*, students, rather than the system, decide whether to seek the offered support.

Figure 5: Student support related to the Animal Sum representation

Consecutivo does not exploit all the new forms of connectivity afforded by digital tools, largely because access to virtual connections in Brazilian classrooms are still relatively rare, but also because, in order to minimise the learning curve associated

with its appropriation, students' constructions – their conjectures and proofs – are to be presented in paper and pencil rather than digital form. This design decision was influenced by messages coming from the investigations of the second generation scenarios and the aim of producing something that might be incorporated more seamlessly into classroom practices.

Student activities with *Consecutivo* are divided into four groups: Explore, Conjecture, Organise, Prove. The activities opened when the Explore button is activated are intended to involve students in rather specific investigations of particular tools. For example, one activity asked them to use the *Animal Sum* button to help them locate four consecutive numbers whose sum is divisible by 9. The rest of the activities were expected to involve students in various aspects of the process of proving, including conjecturing, looking for properties that might explain constructed or given conjectures and organising their arguments into valid chains of reasoning. The difference was in the ways the activities were presented. For example, the conjecturing activities, highlight patterns to be located by the students, although they also ask for explanations of any regularities observed. In contrast, in a proving activity, the conjecture might be given, or students might be questioned about the possibility of constructing a particular result in addition to providing a proof of the validity of their answer. The organisation activities are the most highly structured. Their motivation was research indicating the importance of some explicit introduction to the formal structure of mathematical arguments, something stressed in the JPSM project as well as in other research (Duval & Egret, 1989, Almouloud, 2003), and represent another attempt at building student support into the tool. In these activities, students are given a set of scrambled arguments to be organised accorded to a given argument frame. The idea was to drag each element of the proof into the given frame so that it could be evaluated automatically. Figure 6 presents an example.

Figure 6: An Organise activity.

Work with *Consecutivo* is still ongoing. It has been trialled by small groups of students, by teachers and in the whole-class situation. The results suggest that users find the interface accessible and easy to navigate. Perhaps more important though are the student arguments it appears to motivate; arguments in which the use of empirical evidence continues to proliferate, but not in isolation as students also seem to be encouraged to include attention to aspects of the mathematical structures, as modelled by some of the *Consecutivo* tools, as they justify their empirically made conjectures with reference to the properties of the objects they result from. From the results we have achieved so far, it now seems realistic to attempt to scale-out, implementing this innovation across Brazil – although only in those schools adequately equipped with whole-class access to technology.

Mathematics education and technology: the tortoise and the hare?

Our aim in this paper has been to reflect upon the impact of Celia Hoyles' research into proof and proving in the mathematics classroom upon our own. In a way, this research has served as a generic example through which to view her contributions to the field of mathematics more widely. Her work has been characterised by an overarching desire to contribute to the development of a more learnable school mathematics and a more mathematically literate society. This has involved her in seeking to understand all the forces that shape existing practices of school

mathematics, while, at the same time, looking for more effective ways of teaching and developing more engaging media for exploring and expressing mathematical ideas. This is a hard balance to achieve – while technology changes at an exponential rate, educational institutions and structures do not. In this sense, it is important to stress that as Celia pursues her ground-breaking work with digital technologies, she does not lose sight of her commitment to understand, and to respect, the constraints faced by those at the chalk face (even in its electronic versions). We admire her commitment. We also share it.

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