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# A comparison of Landau-Ginzburg models for odd dimensional Quadrics

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## Abstract

In [Rie08], the second author defined a Landau-Ginzburg model for homogeneous spaces  $G/P$ , as a regular function on an affine subvariety of the Langlands dual group. In this paper, we reformulate this LG model  $(\check{X}, W_t)$  in the case of the odd-dimensional quadric  $Q_{2m-1}$  as a rational function on a Langlands dual projective space, in the spirit of work by R. Marsh and the second author [MR12] for type A Grassmannians and by both authors [PR13] for Lagrangian Grassmannians.

We also compare this LG model with the one obtained independently by Gorbounov and Smirnov in [GS13], and we use this comparison to deduce part of [Rie08, Conj. 8.1] for odd-dimensional quadrics.

## 1 Introduction

In 2000 Hori and Vafa wrote down a conjectured LG model for any hypersurface in a (weighted) complex projective space [HV00], [Prz07, Rmk. 19]. This is a Laurent polynomial associated to the hypersurface which plays the part of the B-model to the hypersurface in mirror symmetry, meaning its singularities are meant to encode various structures to do with Gromov-Witten theory of the hypersurface. In the case of the smooth quadric  $Q_3$  in  $\mathbb{P}^4$  the LG model is

$$Y_1 + Y_2 + \frac{(Y_3 + q)^2}{Y_1 Y_2 Y_3},$$

and in this special case it was written down earlier by Eguchi, Hori, and Xiong [EHX97]. For a quadric  $Q_{2m-1}$  the formula of Hori and Vafa reads

$$Y_1 + Y_2 + \dots + Y_{m-1} + \frac{(Y_m + q)^2}{Y_1 Y_2 \dots Y_m}.$$

One issue with these Laurent polynomial formulas is that they do not always have the expected number of critical points (at fixed generic value of  $q$ ) which should be equal to  $\dim(H^*(Q_{2m-1}))$ . This was already observed in [EHX97], where it was suggested to solve this problem using a partial compactification, and this was carried out for the first time albeit in an ad hoc fashion.

The quadratic hypersurfaces  $Q_{2m-1}$  have a large symmetry group. Indeed  $Q_{2m-1}$  is a cominuscule homogeneous space for the group  $\text{Spin}_{2m+1}(\mathbb{C})$ . Therefore there is already

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another LG model on an affine variety generally larger than a torus, which was defined by the second author using a Lie theoretic construction [Rie08]. Namely for any projective homogeneous space  $G/P$  of a simple complex algebraic group, [Rie08] constructs a conjectural LG model, which is a regular function on an affine subvariety of the Langlands dual group. It is shown in [Rie08] that this LG model recovers the Peterson variety presentation [Pet97] of the quantum cohomology of  $G/P$ . It therefore defines an LG model whose Jacobi ring has the correct dimension.

For odd-dimensional quadrics  $Q_{2m-1}$  a recent paper [GS13] of Gorbounov and Smirnov constructed directly a partial compactification of the Hori-Vafa mirrors, without making use of [Rie08]. Moreover they proved a version of mirror symmetry, which identifies the initial data of the Frobenius manifold associated to the LG model with that constructed out of the quantum cohomology of  $Q_{2m-1}$ .

The goal of this note is twofold. We first express the LG model from [Rie08] in the case of  $Q_{2m-1}$  in terms of natural coordinates on an affine subvariety of a ‘mirror homogeneous space’  $X^\vee = IG_1(2m) \cong \mathbb{P}^{2m-1}$ . For example in the case of  $Q_3$  we obtain

$$W_q = p_1 + \frac{p_2^2}{p_1 p_2 - p_3} + q \frac{p_1}{p_3}.$$

The first main result generalises this formula. Define

$$\check{X}^\circ := \check{X} \setminus D, \tag{1}$$

where  $D := D_0 + D_1 + \dots + D_{m-1} + D_m$ , the  $D_i$ ’s being given by

$$\begin{aligned} D_0 &:= \{p_0 = 0\}, \\ D_l &:= \left\{ p_l p_{2m-1-l} - p_{l-1} p_{2m-l} + \dots + (-1)^l p_0 p_{2m-1} = 0 \right\} \text{ for } 1 \leq l \leq m-1, \\ D_m &:= \{p_{2m-1} = 0\}. \end{aligned}$$

The divisor  $D$  is an anti-canonical divisor. Indeed, the index of  $\check{X} = \mathbb{P}(V^*)$  is  $2m$ .

**Theorem 1.** *The LG model  $\mathcal{F}_q : \mathcal{R} \rightarrow \mathbb{C}$  from [Rie08] is isomorphic to  $W_q : \check{X}^\circ \rightarrow \mathbb{C}$  defined by*

$$W_q = p_1 + \sum_{l=1}^{m-1} \frac{p_{l+1} p_{2m-1-l}}{p_l p_{2m-1-l} - p_{l-1} p_{2m-l} + \dots + (-1)^l p_{2m-1}} + q \frac{p_1}{p_{2m-1}}. \tag{2}$$

**Corollary 2.** *There is an isomorphism*

$$\mathbb{C}[\check{X}^\circ \times \mathbb{C}^*] / (\partial W_q) \rightarrow QH^*(X)[q^{-1}] \tag{3}$$

*defined by sending  $p_i$  to the Schubert class  $\sigma_i \in H^{2i}(X)$ .*

This follows from Thm. 1 together with [Rie08]. Indeed the isomorphism in Cor. 2 fits in well with the geometric Satake correspondence (see [Lus83], [Gin95], [MV07]), by which

$$H^*(Q_{2m-1}) = V_{\omega_1}^{\text{PSp}_{2m}}.$$

With this in mind it is natural to identify  $\check{X}$  with  $\mathbb{P}(H^*(Q_{2m-1})^*)$  and the coordinates  $\{p_i\}$  with the Schubert basis  $\{\sigma_i\}$  of  $H^*(Q_{2m-1})$ .

It is interesting to note that under the isomorphism from Cor. 2, the denominators of  $W_q$  actually map to something extremely simple inside the quantum cohomology of the quadric :

**Corollary 3.** For  $1 \leq l \leq m-1$ , the denominator  $p_l p_{2m-1-l} - p_{l-1} p_{2m-l} + \cdots + (-1)^l p_{2m-1}$  represents an element in the Jacobi ring of  $W_q$  which maps to

$$\sigma_l \sigma_{2m-1-l} - \sigma_{l-1} \sigma_{2m-l} + \cdots + (-1)^l \sigma_{2m-1} = q$$

inside  $QH^*(X)$  under the isomorphism (3).

This is an easy consequence of quantum Schubert calculus on the quadric (which can be deduced from the quantum Chevalley formula of [FW04]).

Finally, in Sec. 6 we recall a partial compactification of the Hori-Vafa mirror defined by Gorbounov and Smirnov. We then show the following corollary.

**Corollary 4.** The partially compactified LG model defined in Gorbounov and Smirnov is related to the formula (2) by a change of coordinates. In particular the Gorbounov and Smirnov LG model is isomorphic to the LG model defined in [Rie08].

Together with Cor. 4, the work of Gorbounov and Smirnov implies a part of the mirror conjecture stated in [Rie08, Conjecture 8.1] for the groups  $\mathrm{Spin}_{2m+1}(\mathbb{C})$  with maximal parabolic  $P = P_{\omega_1}$ , see Sec. 7.

## 2 Notations and Definitions

The LG model for  $Q_{2m-1} = \mathrm{Spin}_{2m+1}/P_{\omega_1}$  defined in [Rie08] takes place on an open Richardson variety inside the Langlands dual flag variety  $\mathrm{PSP}_{2m}/B_-$ . We let  $G = \mathrm{PSP}_{2m}(\mathbb{C})$ , since this is the group we will primarily be working with. Then  $G^\vee = \mathrm{Spin}_{2m+1}(\mathbb{C})$  and  $Q_{2m-1} = G^\vee/P^\vee$  for the parabolic subgroup  $P^\vee$  associated to the first node of the Dynkin diagram of type  $B_m$ :

$$\begin{array}{ccccccc} \circ & \text{---} & \circ & \text{---} & \cdots & \text{---} & \circ & \Rightarrow & \circ \\ 1 & & 2 & & & & m & & \end{array}$$

Let  $V = \mathbb{C}^{2m}$  with fixed symplectic form

$$J = \begin{pmatrix} & & & & -1 \\ & & & 1 & \\ & & \ddots & & \\ & -1 & & & \\ 1 & & & & \end{pmatrix}.$$

For  $G = \mathrm{PSP}(V, J)$  we fix Chevalley generators  $(e_i)_{1 \leq i \leq m}$  and  $(f_i)_{1 \leq i \leq m}$ . To be explicit we embed  $\mathfrak{sp}(V, J)$  into  $\mathfrak{gl}(V)$  and set

$$e_i = E_{i,i+1} + E_{2m-i,2m-i+1}, \quad \text{for } i = 1, \dots, m-1, \text{ and } e_m = E_{m,m+1}.$$

and  $f_i := e_i^T$ , the transpose matrix, for every  $i = 1, \dots, m$ . Here  $E_{i,j} = (\delta_{i,k} \delta_{l,j})_{k,l}$  is the standard basis of  $\mathfrak{gl}(V)$ . For elements of the group  $\mathrm{PSP}(V)$ , we will take matrices to represent their equivalence classes. We have Borel subgroups  $B_+ = TU_+$  and  $B_- = TU_-$  consisting of upper-triangular and lower-triangular matrices in  $\mathrm{PSP}(V)$ , respectively.  $T$  is the maximal torus of  $\mathrm{PSP}(V)$ , consisting of diagonal matrices  $(d_{ij})$  with non-zero entries  $d_{i,i} = d_{2m-i+1,2m-i+1}^{-1}$ .

The parabolic subgroup  $P$  we are interested in is the one whose Lie algebra  $\mathfrak{p}$  is generated by all of the  $e_i$  together with  $f_2, \dots, f_m$ , leaving out  $f_1$ . Let  $x_i(a) := \exp(ae_i)$  and  $y_i(a) =$

$\exp(af_i)$ . The Weyl group  $W$  of  $\mathrm{PSP}_{2m}$  is generated by simple reflections  $s_i$  for which we choose representatives

$$\dot{s}_i = y_i(-1)x_i(1)y_i(-1).$$

We let  $W_P$  denote the parabolic subgroup of the Weyl group  $W$ , namely  $W_P = \langle s_2, \dots, s_m \rangle$ . The length of a Weyl group element  $w$  is denoted by  $\ell(w)$ . The longest element in  $W_P$  is denoted by  $w_P$ . We also let  $w_0$  be the longest element in  $W$ . Next  $W^P$  is defined to be the set of minimal length coset representatives for  $W/W_P$ . The minimal length coset representative for  $w_0$  is denoted by  $w^P$ . Let  $\dot{w}$  denote the representative of  $w \in W$  in  $G$  obtained by setting  $\dot{w} = \dot{s}_{i_1} \cdots \dot{s}_{i_m}$ , where  $w = s_{i_1} \cdots s_{i_m}$  is a reduced expression.

We consider the open Richardson variety  $\mathcal{R} := R_{w_P, w_0} \subset G/B_-$ , namely

$$\mathcal{R} := R_{w_P, w_0} = (B_+ \dot{w}_P B_- \cap B_- \dot{w}_0 B_-) / B_-.$$

Let  $T^{W_P}$  be the  $W_P$ -fixed part of the maximal torus  $T$ , and fix  $d \in T^{W_P}$ . Then we also define

$$Z_d := B_- \dot{w}_0 \cap U_+ d \dot{w}_P U_-.$$

The map

$$\pi_R : Z_d \rightarrow \mathcal{R} : z \mapsto z B_-,$$

is an isomorphism from  $Z_d$  to the open Richardson variety.

Let  $q$  be the coordinate  $\alpha_1$  on the 1-dimensional torus  $T^{W_P}$ . The mirror LG model is a regular function on  $\mathcal{R}$  depending also on  $q$ , so a regular function on  $\mathcal{R} \times T^{W_P}$ . It is defined as follows [Rie08],

$$\mathcal{F} : (u_1 \dot{w}_P B_-, d) \mapsto z = u_1 \dot{w}_P d \bar{u}_2 \in Z_d \mapsto \sum e_i^*(u_1) + \sum f_i^*(\bar{u}_2). \quad (4)$$

The corresponding map from  $\mathcal{R}$ , when  $d$  is fixed, is denoted

$$\mathcal{F}_d : \mathcal{R} \rightarrow \mathbb{C} : u_1 \dot{w}_P B_- \mapsto \mathcal{F}(u_1 \dot{w}_P B_-, d).$$

We also define another embedding

$$\pi_L : Z_d \rightarrow P \backslash \mathrm{PSP}(V) : z \mapsto Pz,$$

which maps  $Z_d$  isomorphically to an open subvariety of a big cell in  $P \backslash \mathrm{PSP}(V)$ . Note that  $P \backslash \mathrm{PSP}(V)$  is canonically the isotropic Grassmannian of lines in  $V^*$ , when this Grassmannian is viewed as a homogeneous space via the action of  $\mathrm{PSP}(V)$  from the right. Moreover the isotropic Grassmannian of lines is just  $\mathbb{P}(V^*)$ , since any line is automatically isotropic. Therefore the second embedding  $\pi_L$  has an advantage, that it is just an embedding into a projective space.

**Definition 2.1** (Plücker coordinates). First we introduce notation for the elements of  $W^P$  :

$$w_k = \begin{cases} s_k s_{k-1} \cdots s_1 & \text{if } k \leq m, \\ s_{2m-k} \cdots s_{m-1} s_m s_{m-1} \cdots s_1 & \text{if } m+1 \leq k \leq 2m-1. \end{cases}$$

The associated Plücker coordinates  $p_k$  are defined by

$$p_k(g) = \langle v_{\omega_1}^-, g, w_k \cdot v_{\omega_1}^- \rangle.$$

Note that the Plücker coordinates are just the homogeneous coordinates on the projective space  $\mathbb{P}(V^*)$ . For a coset  $Pg$  they are given by the bottom row entries of  $g$  read from right to left. If  $g = u_1 \dot{w}_P d \bar{u}_2$  then

$$(p_0(g) : \dots : p_{2m-1}(g)) = (p_0(\bar{u}_2) : \dots : p_{2m-1}(\bar{u}_2)).$$

Our goal is to express  $\mathcal{F}$  as a rational function in the Plücker coordinates and  $q = \alpha_1(d)$ . We first illustrate our result in the smallest interesting example : that of the three-dimensional quadric  $Q_3$ .

### 3 The mirror to $Q_3$

A generic element of  $Z_d := B_- \dot{w}_0 \cap U_+ \dot{w}_P U_-$  can be written as  $u_1 d \dot{w}_P \bar{u}_2$ , where

$$\bar{u}_2 = y_1(a_1)y_2(c)y_1(b_1)$$

and  $a_1, c, b_1$  are non-zero. Hence

$$\bar{u}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_1 + b_1 & 1 & 0 & 0 \\ cb_1 & c & 1 & 0 \\ a_1 cb_1 & a_1 c & a_1 + b_1 & 1 \end{pmatrix}$$

The map  $\pi_L : Z_d \rightarrow P \backslash \text{PSp}(V) \cong \mathbb{P}(V^*)$  takes  $z = u_1 d \dot{w}_P \bar{u}_2$  to  $Pz = P\bar{u}_2$ . This may be interpreted as taking  $z$  to the span of the reverse row vector corresponding to the last row of  $\bar{u}_2$  after the identification  $P \backslash \text{PSp}(V) \cong \mathbb{P}(V^*)$ . The Plücker coordinates of  $\bar{u}_2$  are given by  $p_0 = 1, p_1 = a_1 + b_1, p_2 = a_1 c, p_3 = a_1 c b_1$ .

If we are interested in the image of  $Z_d$  in  $\mathbb{P}(V^*)$  then first of all we can observe that it is independent of  $d$ . So we may choose for  $d$  the identity element, and restrict our attention to  $B_- \dot{w}_0 \cap U_+ \dot{w}_P U_-$ . It turns out that the image of  $Z_d$  in  $\mathbb{P}(V^*)$  is obtained from  $\mathbb{P}(V^*)$  in coordinates

$$(p_0 : p_1 : p_2 : p_3) \in \mathbb{P}(V^*)$$

by removing  $\{p_0 = 0\} \cup \{p_3 p_0 - p_2 p_1 = 0\} \cup \{p_3 = 0\}$ . We call this variety  $\check{X}^\circ$ , and the isomorphism with  $Z_d$  in Prop. 9 shows that  $\check{X}^\circ$  is also isomorphic to the open Richardson variety  $\mathcal{R}$ .

Let us denote by  $W : \check{X}^\circ \times \mathbb{C}^* \rightarrow \mathbb{C}$  the map obtained from  $\mathcal{F}$ , see (4), after the identifications  $\mathcal{R} \cong \check{X}^\circ$  and  $(T)^{W_P} \cong \mathbb{C}^*$  via  $d \mapsto \alpha_1(d) = q$ . In this way we can compute the superpotential  $\mathcal{F}$  from [Rie08] in the coordinates on  $\mathbb{P}(V^*)$ :

$$W = \frac{p_1}{p_0} + \frac{p_2^2}{p_1 p_2 - p_0 p_3} + q \frac{p_1}{p_3}.$$

This is equivalent to the following Landau-Ginzburg model of [GS13]:

$$g = y + yz + q \frac{x^2}{(xy - 1)z}$$

via the change of coordinates:

$$x = \frac{p_0 p_2}{p_1 p_2 - p_0 p_3}; y = \frac{p_1}{p_0}; z = \frac{q p_0}{p_3}.$$

Note that in [GS13] the superpotential denoted  $\tilde{f}$  is  $g$  where  $z$  is replaced by  $z + 1$ .

### 4 The mirror to $Q_{2m-1}$

We now write down  $W_q = (\pi_L)_* \pi_R^* \mathcal{F}_d$  as a rational function on  $\check{X}$ , where  $d \in (T)^{W_P}$  is such that  $\alpha_1(d) = q$ . We will then prove in the next section that the locus  $\check{X}^\circ$  where it is defined is isomorphic to the open Richardson variety  $\mathcal{R}$ .

**Proposition 5.** *As a rational function on  $\check{X}$*

$$W_q = \frac{p_1}{p_0} + \sum_{l=1}^{m-1} \frac{p_{l+1} p_{2m-1-l}}{p_l p_{2m-1-l} - p_{l-1} p_{2m-l} + \cdots + (-1)^l p_0 p_{2m-1}} + q \frac{p_1}{p_{2m-1}}.$$

To prove the result, we first recall that

$$\pi_R^* \mathcal{F}_d : z = u_1 \dot{w}_P d \bar{u}_2 \in Z_d \mapsto \sum e_i^*(u_1) + \sum f_i^*(\bar{u}_2).$$

Now  $\bar{u}_2$  appearing in  $u_1 \dot{w}_P d \bar{u}_2 \in Z_d \dot{w}_0$  can be assumed to lie in  $U_- \cap B_+(\dot{w}^P)^{-1} B_+$ . This is because we have two birational maps

$$\begin{aligned} \Psi_1 : U_- \cap B_+(\dot{w}^P)^{-1} B_+ &\rightarrow P \backslash G : & \bar{u}_2 &\mapsto P \bar{u}_2, \\ \Psi_2 : B_- \cap U^+ d \dot{w}^P U_- &\rightarrow P \backslash G : & b_- &= u_1 d \dot{w}_P \bar{u}_2 \mapsto P b_-, \end{aligned}$$

which compose to give  $\Psi_1^{-1} \circ \Psi_2 : b_- \mapsto \bar{u}_2$ . This gives a birational map

$$\Psi_1^{-1} \circ \Psi_2 : Z_d \dot{w}_0 \rightarrow U_- \cap B_+(\dot{w}^P)^{-1} B_+.$$

Now a generic element  $\bar{u}_2$  in  $U_- \cap B_+(\dot{w}^P)^{-1} B_+$  can be assumed to have a particular factorisation. The smallest representative  $w^P$  in  $W$  of  $[w_0] \in W/W_P$  has the following reduced expression :

$$w^P = s_1 \dots s_{m-1} s_m s_{m-1} \dots s_m.$$

It follows that as a generic element of  $U_- \cap B_+(\dot{w}^P)^{-1} B_+$ , the element  $\bar{u}_2$  can be assumed to be written as

$$\bar{u}_2 = y_1(a_1) \dots y_{m-1}(a_{m-1}) y_m(c) y_{m-1}(b_{m-1}) \dots y_1(b_1), \quad (5)$$

where  $a_i, c, b_j \neq 0$ . We have the following standard expression for the  $p_k$  on factorized elements, which is a simple consequence of their definition.

**Lemma 6.** *Fix  $0 \leq k \leq 2m - 1$  an integer. Then if  $\bar{u}_2$  is of the form (5) we have*

$$p_k(\bar{u}_2) = \begin{cases} 1 & \text{if } k = 0, \\ a_1 \dots a_{k-1} (a_k + b_k) & \text{if } 1 \leq k \leq m - 1, \\ a_1 \dots a_{m-1} c b_{m-1} \dots b_{2m-k} & \text{otherwise.} \end{cases} \quad \square$$

We will also need the following :

**Lemma 7.** *If  $u_1$  and  $\bar{u}_2$  are as above then we have the following identities*

$$f_i^*(\bar{u}_2) = \begin{cases} a_i + b_i & \text{if } 1 \leq i \leq m, \\ c & \text{otherwise.} \end{cases} \quad (6)$$

$$e_i^*(u_1) = \begin{cases} 0 & \text{if } 2 \leq i \leq m, \\ e^t \frac{a_1 + b_1}{a_1 \dots a_{m-1} c b_{m-1} \dots b_1} & \text{if } i = 1. \end{cases} \quad (7)$$

*Proof.* Equation (6) is obtained immediately from the definition of  $\bar{u}_2$ . For Equation (7), notice that

$$\begin{aligned} e_i^*(u_1) &= \frac{\langle u_1^{-1} \cdot v_{\omega_i}^-, e_i \cdot v_{\omega_i}^- \rangle}{\langle u_1^{-1} \cdot v_{\omega_i}^-, v_{\omega_i}^- \rangle} \\ &= \frac{\langle e^h \dot{w}_P \bar{u}_2 \cdot v_{\omega_i}^+, e_i \cdot v_{\omega_i}^- \rangle}{\langle e^h \dot{w}_P \bar{u}_2 \cdot v_{\omega_i}^+, v_{\omega_i}^- \rangle}. \end{aligned}$$

Assume  $2 \leq i \leq m$ . Then  $e_i^*(u_1) = 0$  if and only if  $\langle \bar{u}_2 \cdot v_{\omega_i}^+, \dot{w}_P^{-1} e_i \cdot v_{\omega_i}^- \rangle = 0$ . Now the vector  $w_P^{-1} e_i \cdot v_{\omega_i}^-$  is in the  $\mu$ -weight space of the  $i$ -th fundamental representation, where  $\mu = w_P^{-1} s_i(-\omega_i)$ . Moreover,  $\bar{u}_2 \in B_+(\dot{w}^P)^{-1} B_+$ , hence  $\bar{u}_2 \cdot v_{\omega_i}^+$  can have non-zero components only down to the weight space of weight  $(w^P)^{-1}(\omega_i) = w_P^{-1}(-\omega_i)$ . Since  $l(w_P^{-1} s_i) > l(w_P^{-1})$  for  $2 \leq i \leq m$ , this is higher than  $\mu$ , which proves that  $e_i^*(u_1) = 0$ .

Now assume  $i = 1$ . We have

$$\begin{aligned} e_1^*(u_1) &= \frac{\langle e^h \dot{w}_P \bar{u}_2 \cdot v_{\omega_1}^+, e_1 \cdot v_{\omega_1}^- \rangle}{\langle e^h \dot{w}_P \bar{u}_2 \cdot v_{\omega_1}^+, v_{\omega_1}^- \rangle} \\ &= (\omega_1 + \alpha_1 - \omega_1)(e^h) \frac{\langle \bar{u}_2 \cdot v_{\omega_1}^+, \dot{w}_P^{-1} e_1 \cdot v_{\omega_1}^- \rangle}{\langle \bar{u}_2 \cdot v_{\omega_1}^+, \dot{w}_P v_{\omega_1}^- \rangle} \\ &= e^t \frac{\langle \bar{u}_2 \cdot v_{\omega_1}^+, \dot{w}_P^{-1} e_1 \cdot v_{\omega_1}^- \rangle}{\langle \bar{u}_2 \cdot v_{\omega_1}^+, v_{\omega_1}^- \rangle}. \end{aligned}$$

First look at the denominator. The only way to go from the highest weight vector  $v_{\omega_1}^+$  of the first fundamental representation to the lowest  $v_{\omega_1}^-$  is to apply  $g \in B_+ w B_+$  for  $w \geq (w^P)^{-1}$ . Since  $\bar{u}_2 \in B_+(\dot{w}^P)^{-1} B_+$ , it follows that we need to take all factors of  $\bar{u}_2$ , and normalising  $v_{\omega_1}^-$  appropriately, we get

$$\langle \bar{u}_2 \cdot v_{\omega_1}^+, v_{\omega_1}^- \rangle = a_1 \dots a_{m-1} c b_{m-1} \dots b_1.$$

Finally, we look at the numerator  $\langle \bar{u}_2 \cdot v_{\omega_1}^+, \dot{w}_P^{-1} e_1 \cdot v_{\omega_1}^- \rangle$ . The vector  $\dot{w}_P^{-1} e_1 \cdot v_{\omega_1}^-$  has weight

$$\mu' = \dot{w}_P^{-1} s_1(-\omega_1) = \dot{w}_P^{-1}(-\epsilon_2) = \epsilon_2.$$

Write  $w_P^{-1} s_1$  as a prefix  $w' = s_1 s_2 \dots s_{m-1} s_m s_{m-1} \dots s_2$  of  $(w^P)^{-1}$ . We have  $w' s_1 = (w^P)^{-1}$ , hence the way from  $v_{\omega_1}^+$  to  $w' \cdot v_{\omega_1}^-$  is through  $s_1$ . From the shape of  $\bar{u}_2$ , it follows that  $\langle \bar{u}_2 \cdot v_{\omega_1}^+, \dot{w}_P^{-1} e_1 \cdot v_{\omega_1}^- \rangle = a_1 + b_1$ .  $\square$

Using the expression (4) of the superpotential from [Rie08], we immediately deduce from Lem. 7 a intermediate expression for the Landau-Ginzburg model  $W_q$  of the odd-dimensional quadric as a Laurent polynomial :

**Proposition 8.**

$$W_q = a_1 + \dots + a_{m-1} + c + b_{m-1} + \dots + b_1 + q \frac{a_1 + b_1}{a_1 \dots a_{m-1} c b_{m-1} \dots b_1} \quad \square \quad (8)$$

Now with the help of Lem. 6 and Prop. 8, we prove the second expression of  $W_q$  :

*Proof of Prop. 5.* From Lem. 6, it follows that for  $\bar{u}_2$  as in (5)

$$p_{l+1} p_{2m-1-l}(\bar{u}_2) = \begin{cases} (a_{l+1} + b_{l+1})(a_1 \dots a_l)^2 a_{l+1} \dots a_{m-1} c b_{m-1} \dots b_{l+1} & \text{if } l \leq m-2, \\ (a_1 \dots a_{m-1} c)^2 & \text{if } l = m-1. \end{cases}$$

and

$$p_k p_{2m-1-k}(\bar{u}_2) = (a_k + b_k)(a_1 \dots a_{k-1})^2 a_k \dots a_{m-1} c b_{m-1} \dots b_{k+1}.$$

Hence most terms in  $\sum_{k=0}^l (-1)^k p_{l-k} p_{2m-1+k-l}(\bar{u}_2)$  cancel, and

$$\sum_{k=0}^l (-1)^k p_{l-k} p_{2m-1+k-l}(\bar{u}_2) = (a_1 \dots a_l)^2 a_{l+1} \dots a_{m-1} c b_{m-1} \dots b_{l+1}.$$



This proves that

$$\frac{p_{l+1}p_{2m-1-l}}{p_l p_{2m-1-l} - p_{l-1} p_{2m-l} + \cdots + (-1)^l p_0 p_{2m-1}}(\bar{u}_2) = \begin{cases} a_{l+1} + b_{l+1} & \text{if } l \leq m-2, \\ c & \text{if } l = m-1. \end{cases}$$

For the first and last terms, we obtain

$$\frac{p_1}{p_0}(\bar{u}_2) = a_1 + b_1$$

and

$$\frac{p_1}{p_{2m-1}}(\bar{u}_2) = \frac{a_1 + b_1}{a_1 \cdots a_{m-1} c b_{m-1} \cdots b_1}$$

as easy consequences of Lem. 6.  $\square$

## 5 The open Richardson variety

We now prove that the affine subvariety  $\check{X}^\circ$  defined in Equation (1) is isomorphic to the open Richardson variety  $\mathcal{R}$ .

Recall that  $\check{X}^\circ = \check{X} \setminus D$ , where

$$D := D_0 + D_1 + \cdots + D_{m-1} + D_m$$

and

$$D_0 := \{p_0 = 0\},$$

$$D_l := \left\{ p_l p_{2m-1-l} - p_{l-1} p_{2m-l} + \cdots + (-1)^l p_0 p_{2m-1} = 0 \right\} \text{ for } 1 \leq l \leq m-1,$$

$$D_m := \{p_{2m-1} = 0\}.$$

By definition,  $\check{X}^\circ$  is the locus where  $W_q$  is regular. Since  $p_0$  is non-zero on  $\check{X}^\circ$ , we may assume that  $p_0 = 1$ . Hence we have affine coordinates  $(p_1, \dots, p_{2m-1})$  on  $\check{X}^\circ$ . We also set, for  $1 \leq j \leq 2m-1$ :

$$r_j := \sum_{k=0}^j (-1)^k p_{j-k} p_{2m-1+k-j}.$$

**Proposition 9.** *The map  $\pi_L \circ \pi_R^{-1} : \mathcal{R} \rightarrow \check{X}^\circ$  is an isomorphism.*

We will prove the result by constructing the inverse map. But first, let us check that the image of this map is indeed inside  $\check{X}^\circ$  (and not just inside  $\check{X}$ ). Clearly,  $\mathcal{F}_d$  equals  $W_q \circ \pi_L \circ \pi_R^{-1}$  as a rational map. Since  $\mathcal{F}_d$  is regular on  $\mathcal{R}$ , it means that  $W_q \circ \pi_L \circ \pi_R^{-1}$  also is, hence that  $W_q$  is regular on the image of  $\pi_L \circ \pi_R^{-1}$ . This proves that this image is contained in  $\check{X}^\circ$ .

We now define a map  $\Phi : \check{X}^\circ \rightarrow B_-^{\text{PSL}} \dot{w}_0$ , where  $B_-^{\text{PSL}}$  is the Borel of lower triangular matrices in  $\text{PSL}_{2m}$ , so that  $\Phi(p_1, \dots, p_{2m-1}) \cdot v_j$  is equal to

$$\left\{ \begin{array}{l} p_{2m-1} v_{2m} \text{ if } j = 1, \\ (-1)^j \frac{r_{j-1}}{r_{j-2}} + p_{2m-j} \left( \sum_{k=1}^{j-2} (-1)^{l+1} \frac{p_l}{r_{l-1}} v_{2m-l} + v_{2m} \right) \text{ if } 2 \leq j \leq m, \\ (-1)^j \frac{r_{2m-1-j}}{r_{2m-j}} v_{2m+1-j} + p_{2m-j} \left( \sum_{k=m+1}^{j-1} (-1)^k \frac{p_{k-1}}{r_{2m-k}} v_{2m+1-k} + \sum_{k=1}^{m-1} (-1)^{k-1} \frac{p_k}{r_{k-1}} v_{2m-k} \right) \\ \quad \text{if } m+1 \leq j \leq 2m-1, \\ \frac{-1}{r_0} v_1 + \sum_{k=1}^{m-1} (-1)^k \frac{p_{2m-1-k}}{r_k} v_{k+1} + \sum_{k=1}^{m-1} (-1)^{k+1} \frac{p_k}{r_{k-1}} v_{2m-k} + v_{2m} \text{ if } j = 2m. \end{array} \right.$$

Let  $\Omega$  be the open dense subset of  $\check{X}^\circ$  where the coordinates  $p_m, p_{m-1}, \dots, p_{2m-2}$  do not vanish and define coordinates on  $\Omega$  (as follows from Lem. 10) by

$$\begin{aligned} a_i &= \frac{p_{2m-1} r_i}{p_{2m-1-i} r_{i-1}} \text{ for all } 1 \leq i \leq m-1; \\ b_i &= \frac{p_{2m-1}}{p_{2m-1-i}} \text{ for all } 1 \leq i \leq m-1; \\ c &= \frac{p_m^2}{r_{m-1}}. \end{aligned}$$

**Lemma 10.** For all  $(p_1, \dots, p_{2m-1}) \in \Omega$ ,  $\Phi(p_1, \dots, p_{2m-1})$  factorizes as  $u_1 \dot{w}_P \bar{u}_2$ , where

$$\bar{u}_2 = y_1(a_1) \dots y_{m-1}(a_{m-1}) y_m(c) y_{m-1}(b_{m-1}) \dots y_1(b_1)$$

and  $u_1$  equals

$$\begin{pmatrix} 1 & \frac{a_1+b_1}{a_1 \dots a_{m-1} c b_{m-1} \dots b_1} & \dots & \frac{a_{m-1}+b_{m-1}}{a_1 \dots a_{m-1} c b_{m-1}} & \frac{1}{a_1 \dots a_{m-1}} & \dots & \frac{1}{a_1} & \frac{-1}{a_1 \dots a_{m-1} c b_{m-1} \dots b_1} \\ & 1 & & & & & & \frac{-1}{a_1} \\ & & \ddots & & & & & \vdots \\ & & & 1 & & & & \frac{(-1)^m}{a_1 \dots a_{m-1}} \\ & & & & 1 & & & (-1)^{m-1} \frac{a_{m-1}+b_{m-1}}{a_1 \dots a_{m-1} c b_{m-1}} \\ & & & & & \ddots & & \vdots \\ & & & & & & 1 & \frac{a_1+b_1}{a_1 \dots a_{m-1} c b_{m-1} \dots b_1} \\ & & & & & & & 1 \end{pmatrix}$$

*Proof.* Using the definition of the  $y'_i$ 's, it is easy to check that  $\bar{u}_2 \cdot v_j$  is equal to

$$\begin{aligned} v_j + \sum_{k=0}^{m-1-j} (a_{j+k} + b_{j+k}) b_{j+k-1} \dots b_{j+1} b_j v_{j+k+1} + \sum_{k=0}^{m-1} a_{m-k} \dots a_{m-1} c b_{m-1} \dots b_j v_{m+1+k} \\ \text{if } 1 \leq j \leq m-1, \\ v_m + \sum_{k=0}^{m-1} a_{m-k} \dots a_{m-1} c \text{ if } j = m, \\ v_j + (a_{2m-j} + b_{2m-j}) \sum_{k=0}^{2m-1-j} a_{2m-k-j} \dots a_{2m-2-j} a_{2m-1-j} v_{j+1+k} \text{ if } m+1 \leq j \leq 2m, \end{aligned}$$

Now a straightforward, if slightly tedious, computation shows that  $\Phi(p_1, \dots, p_{2m-1}) = u_1 \dot{w}_P \bar{u}_2$ .  $\square$

We now need to prove that the entire image of  $\Phi$  is in fact contained in  $B_- \dot{w}_0 \cap U_+ \dot{w}_P U_-$  inside  $\text{PSp}_{2m}$ :

**Lemma 11.**

$$\Phi(\check{X}^\circ) \subset B_- \dot{w}_0 \cap U_+ \dot{w}_P U_-.$$

*Proof.* We first prove that  $\Phi(\Omega) \subset B_- \dot{w}_0 \cap U_+ \dot{w}_P U_-$  inside  $\text{PSp}_{2m}$ . Indeed, from Lem. 10, we know that for all  $(p_1, \dots, p_{2m-1}) \in \Omega$ ,  $\Phi(p_1, \dots, p_{2m-1})$  factorises as  $u_1 \dot{w}_P \bar{u}_2$ , where  $u_1$  and  $\bar{u}_2$  are defined in the statement of the lemma. The factorisation of  $\bar{u}_2$  means that

$\bar{u}_2$  is in  $U_-$  (hence in particular in  $\mathrm{PSp}_{2m}$ ). Now we prove that  $u_1$  is also in  $\mathrm{PSp}_{2m}$ , by showing directly that  ${}^t u_1 J u_1 = J$  using the formula from Lem. 10. This is the result of a straightforward computation. It follows that  $u_1 \in U_+$ , hence  $\Phi(p_1, \dots, p_{2m-1}) \in U_+ \dot{w}_P U_- \subset \mathrm{PSp}_{2m}$  in this case. Now also  $\Phi(p_1, \dots, p_{2m-1}) \in B_-^{\mathrm{PSL}} \dot{w}_0 \cap \mathrm{PSp}_{2m} = B_- \dot{w}_0$ . Therefore  $\Phi(\Omega) \subset B_- \dot{w}_0 \cap U_+ \dot{w}_P U_-$ .

Since  $\Omega$  is open dense in  $\check{X}^\circ$  we now have that  $\Phi(\check{X}^\circ) \subset B_- \dot{w}_0 \cap \overline{U_+ \dot{w}_P U_-}$ . Suppose there exists  $(p_1, \dots, p_{2m-1})$  in  $\check{X}^\circ$  such that  $\Phi(p_1, \dots, p_{2m-1}) \notin U_+ \dot{w}_P U_-$ . Then from Bruhat decomposition, we get  $\Phi(p_1, \dots, p_{2m-1}) \dot{w}_0^{-1} \in U_+ \dot{w} U_+$  with  $w < w_P w_0$ . It follows that we must have

$$\langle \Phi(p_1, \dots, p_{2m-1}) \dot{w}_0^{-1} v_{\omega_1}^+, v_{\omega_1}^- \rangle = \langle \Phi(p_1, \dots, p_{2m-1}) v_{\omega_1}^-, v_{\omega_1}^- \rangle = 0,$$

hence the lower-right corner of the matrix  $\Phi(p_1, \dots, p_{2m-1})$  has to be zero. But this coefficient is always 1, hence the result.  $\square$

We can now prove Prop. 9 :

*Proof of Prop. 9.* We have showed that the image of  $\pi_L$  is contained inside  $\check{X}^\circ$ . Moreover, we have defined a map  $\Phi : \check{X}^\circ \rightarrow Z_1$ , and a straightforward computation shows that it is the inverse of  $\pi_L$ . Hence  $\pi_L$  is an isomorphism. Since we saw in Sec. 2 that  $\pi_R$  is also an isomorphism, the proposition follows.  $\square$

The proof of Thm. 1 then follows from Prop. 5 and 9.

## 6 Comparison with the LG model of [GS13]

We now want to prove that our Landau-Ginzburg model (2) is isomorphic to the one stated in [GS13], which goes as follows

$$g = \sum_{i=1}^{m-1} y_i (1 + z_i) + q \frac{x^2}{(x y_1 y_2 \dots y_{m-1} - 1) z_1 z_2 \dots z_{m-1}}. \quad (9)$$

Note that as for  $Q_3$ , in [GS13] the superpotential denoted  $\tilde{f}$  is  $g$  where the  $z_i$  are replaced by  $z_i + 1$ .

Assume  $p_0 = 1$  and consider the change of variables :

$$\begin{aligned} y_1 &= p_1; & y_i &= \frac{p_i}{p_{i-1}} & \forall 2 \leq i \leq m-1; \\ z_1 &= \frac{q}{p_5}; & z_i &= \frac{\sum_{k=0}^{i-2} (-1)^k p_{i-2-k} p_{2m+1+k-i}}{\sum_{k=0}^{i-1} (-1)^k p_{i-1-k} p_{2m+k-i}} & \forall 2 \leq i \leq m-1; \\ x &= \frac{p_m}{\sum_{k=0}^{m-1} (-1)^k p_{m-1-k} p_{m+k}}. \end{aligned}$$

**Proposition 12.** *The above change of coordinates  $\{x, y_i, z_i\} \mapsto \{p_i\}$  defines an isomorphism between the Landau-Ginzburg model (9) and ours (2).*

*Proof.* We have  $y_1(1 + z_1) = p_1 + \frac{q}{p_{2m-1}}$ , and

$$y_i(1 + z_i) = \frac{p_i p_{2m-i}}{\sum_{k=0}^{i-1} (-1)^k p_{i-1-k} p_{2m+k-i}}.$$

Moreover

$$xy_1 \dots y_{m-1} = \frac{\sum_{k=0}^{m-2} (-1)^k p_{m-2-k} p_{m+1+k}}{\sum_{k=0}^{m-1} (-1)^k p_{m-1-k} p_{m+k}},$$

$$z_1 \dots z_{m-1} = \frac{q}{\sum_{k=0}^{m-2} (-1)^k p_{m-2-k} p_{m+1+k}},$$

and

$$x^2 = \frac{p_m^2}{\left( \sum_{k=0}^{m-1} (-1)^k p_{m-1-k} p_{m+k} \right)^2},$$

hence

$$q \frac{x^2}{(xy_1 y_2 \dots y_{m-1} - 1) z_1 z_2 \dots z_{m-1}} = \frac{p_m^2}{\sum_{k=0}^{m-1} (-1)^k p_{m-1-k} p_{m+k}}.$$

Hence the change of variables maps (9) to (2). Finally, it is clear that both domains of definition are the same.  $\square$

This proves Cor. 4.

## 7 Consequences

Let  $\mathcal{H}_A$  be the sheaf of regular functions of the trivial vector bundle with fiber  $H^*(X, \mathbb{C})$  over  $\mathbb{C}_\hbar^* \times \mathbb{C}_q^*$  the two-dimensional complex torus with coordinates  $\hbar$  and  $q$ . The *A-model connection* is defined on  $\mathcal{H}_A$  by

$${}^A\nabla_{q\partial_q} = q \frac{\partial}{\partial q} + \frac{1}{\hbar} p_1 \star_q \bullet$$

$${}^A\nabla_{\hbar\partial_\hbar} = \hbar \frac{\partial}{\partial \hbar} + \text{gr} - \frac{1}{\hbar} c_1(TX) \star_q \bullet,$$

where  $\text{gr}$  is a diagonal operator on  $H^*(X)$  given by  $\text{gr}(\alpha) = k$  for  $\alpha \in H^{2k}(X)$ . Here we are using the conventions of [Iri09]. Let  $\mathcal{H}_A^\vee$  be the vector bundle on  $\mathbb{C}_\hbar^* \times \mathbb{C}_q^*$  defined by  $\mathcal{H}_A^\vee = j^* \mathcal{H}_A$  for  $j : (\hbar, q) \mapsto (-\hbar, q)$ . This vector bundle with the pulled back connection  ${}^A\nabla^\vee = j^*({}^A\nabla)$  is dual to  $(\mathcal{H}_A, {}^A\nabla)$  via the flat non-degenerate pairing,

$$\langle \sigma_i, \sigma_j \rangle = (2\pi i \hbar)^N \int_{[X]} \sigma_i \cup \sigma_j = (2\pi i \hbar)^N \delta_{i+j, N},$$

where  $N = 2m - 1$  is the dimension of  $\check{X}^\circ$ . The dual A-model connection  ${}^A\nabla^\vee$  defines a system of differential equations called the (small) quantum differential equations

$${}^A\nabla_{q\partial_q}^\vee S = 0. \tag{10}$$

Define the  $\mathbb{C}[\hbar^{\pm 1}, q^{\pm 1}]$ -module

$$G = \Omega^N(\check{X}^\circ)[\hbar^{\pm 1}, q^{\pm 1}] / (d - \frac{1}{\hbar} dW_q \wedge \bullet) \Omega^{N-1}(\check{X}^\circ)[\hbar^{\pm 1}, q^{\pm 1}],$$

where  $\Omega^k(\check{X}^\circ)$  is the space of holomorphic  $k$ -forms on  $\check{X}^\circ$ . We denote by  $\mathcal{H}_B$  the sheaf with global sections  $G$ . Because  $W_q$  is cohomologically tame [GS13],  $G$  is a free  $\mathbb{C}[\hbar^{\pm 1}, q^{\pm 1}]$ -module of rank  $2m$  (cf. [Sab99]), and  $\mathcal{H}_B$  a trivial vector bundle of that dimension. It has

a (Gauss-Manin) connection given by

$$\begin{aligned} {}^B\nabla_{q\partial_q}[\eta] &= q\frac{\partial}{\partial q}[\eta] + \frac{1}{\hbar} \left[ q\frac{\partial W_q}{\partial q}\eta \right] \\ {}^B\nabla_{\hbar\partial_{\hbar}}[\eta] &= \hbar\frac{\partial}{\partial \hbar}[\eta] - \frac{1}{\hbar} [W_q\eta]. \end{aligned}$$

Let  $\omega$  be the canonical  $N$ -form on  $\check{X}^\circ$ .

**Corollary 13.** *The two bundles with connection  $(\mathcal{H}_A, {}^A\nabla)$  and  $(\mathcal{H}_B, {}^B\nabla)$  are isomorphic via  $\sigma_i \mapsto [p_i\omega]$ .*

*Proof.* The corollary is a consequence of the isomorphism of our LG model  $W_q$  and the one of [GS13] (see Cor. 4) together with the results of Gorbounov and Smirnov.  $\square$

Let  $\Gamma_0$  be a compact oriented real  $N$ -dimensional submanifold of  $\check{X}^\circ$  representing a cycle in  $H^N(\check{X}^\circ, \mathbb{Z})$  dual to  $\omega$ , in the sense that  $\frac{1}{(2i\pi)^N} \int_{\Gamma_0} \omega = 1$ . Then :

**Corollary 14.** *The integral*

$$S_0(z, q) = \frac{1}{(2i\pi z)^N} \int_{\Gamma_0} e^{\frac{W_q}{\hbar}} \omega$$

*is a solution to the quantum differential equation (10).*

This implies part of [Ric08, Conj. 8.1] for odd-dimensional quadrics.

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