On Strictly Positive Modal Logics with $\mathsf{S4.3}$ Frames

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Abstract

We investigate the lattice of strictly positive (SP) modal logics that contain the SP-fragment of the propositional modal logic $\mathsf{S4.3}$ of linear quasi-orders. We are interested in Kripke (in)completeness of these logics, their computational complexity, as well as the definability of Kripke frames by means of SP-implications. We compare the lattice of these SP-logics with the lattice of normal modal logics above $\mathsf{S4.3}$. We also consider global consequence relations for SP-logics, focusing on definability and Kripke completeness.

Keywords: strictly positive modal logic, meet-semilattices with monotone operators, Kripke completeness, definability, decidability.

1 Introduction

The lattice NExt$\mathsf{S4.3}$ of propositional normal modal logics containing $\mathsf{S4.3}$ is a rare example of a non-trivial class of 'well-behaved' modal logics. Indeed, all of them are finitely axiomatisable, have the finite model property [6,10], and are decidable in coNP [22]. Although Fine [10] complained that "the full lattice [of these logics] is one of great complexity", its structure is perfectly understandable (though difficult to depict) if one recalls the fact [7] that every $L \in \mathrm{NExt}\mathsf{S4.3}$ can be axiomatised by Yankov (aka characteristic or frame) formulas for finite linear quasi-ordered frames.

In this paper, our concern is the fragment of the modal language that comprises implications $\sigma \to \tau$, where σ and τ are *strictly positive* modal formulas [3,23] constructed from variables using $\wedge$, $\Diamond$, and the constant $\top$. We call such implications *SP-implications*. A natural algebraic semantics for SP-implications is given by meet-semilattices with monotone operators (SLOs, for short) [21,2,16]. We denote the corresponding syntactic consequence relation by $\vdash_{\mathsf{SLO}}$ and call any $\vdash_{\mathsf{SLO}}$-closed set of SP-implications an *SP-logic*. The lattice of varieties of SLOs validating the SP-implicational S4 axioms $p \to \Diamond p$ and $\Diamond\Diamond p \to \Diamond p$ (*closure* SLOs) was studied by Jackson [15] (see also [9]). Jackson showed, in particular, that the SP-fragment P_{S4} of S4 is *complex* [12] in the sense that each closure SLO can be embedded into the full complex algebra of some Kripke frame for S4. It follows that the SP-logic axiomatised by $\{p \to \Diamond p, \Diamond\Diamond p \to \Diamond p\}$ is (*Kripke*) *complete* in the sense that its SP-implicational $\vdash_{\mathsf{SLO}}$-consequences coincide with those over Kripke frames, that is, with P_{S4}. On the other hand, it is shown in [16] that the SP-extension of P_{S4} with

$$\Diamond(p \wedge q) \wedge \Diamond(p \wedge r) \to \Diamond(p \wedge \Diamond q \wedge \Diamond r) \qquad (wcon)$$

is not complex; yet, it is complete and axiomatises the SP-fragment $P_{\mathsf{S4.3}}$ of S4.3. Both P_{S4} and $P_{\mathsf{S4.3}}$ are decidable in polynomial time [1,21,16].

Our aim here is to investigate SP-logics above $P_{\mathsf{S4.3}}$—focusing on completeness, definability (of Kripke frames) and computational complexity—and compare them with classical modal logics above S4.3. Our first important observation is that, unlike classical modal formulas, a randomly chosen SP-implication will most probably axiomatise an *incomplete* SP-extension of $P_{\mathsf{S4.3}}$. Such are, for example, $\Diamond p \to \Diamond q$ and $\Diamond p \wedge \Diamond q \to \Diamond(p \wedge q)$ [16]. In Section 3, we construct infinite sequences of SP-logics sharing the same two-point frames. But the main result of this paper is that we do identify *all complete* SP-logics above $P_{\mathsf{S4.3}}$.

Over S4.3 frames, modal formulas define exactly those frame classes that are closed under cofinal subframes [10,24]. On the other hand, over P_{S4}, SP-implications are capable of defining only FO-definable classes closed under subframes [16] (which implies that frames for any proper extension of $P_{\mathsf{S4.3}}$ are of bounded depth). In Section 4, we show that *all* FO-definable subframe-closed classes of S4.3-frames are SP-*definable*. We also give *finite axiomatisations* for the SP-logics of these SP-definable classes by means of SP-analogues of subframe formulas [11,24], and prove that each of the resulting SP-logics is decidable in *polynomial time*.

In Section 5, we investigate the lattice $\mathrm{Ext}^{+}P_{\mathsf{S4.3}}$ of SP-logics containing $P_{\mathsf{S4.3}}$ by comparing it with the lattice NExtS4.3. Following [4], we consider two maps: one associates with every modal logic L its SP-fragment $\pi(L)$, the other one associates with every SP-logic P the modal logic $\mu(P) = \mathsf{S4.3} \oplus P$. We give a frame-theoretic characterisation of π and show that each $\pi^{-1}(P)$ has a greatest element. On the other hand, as follows from Section 3, $\mu^{-1}(L)$ often contains infinitely many incomplete logics.

In Section 6, we consider rules $\varrho = \frac{\iota_1,\dots,\iota_n}{\iota}$ of SP-implications. In modal logics above $\mathsf{S4}$, ϱ can be expressed by the (non-SP) formula $\Box(\iota_1 \wedge\cdots\wedge \iota_n) \to \iota$. We show that a class of $\mathsf{S4.3}$-frames is modally definable iff it is definable by SP-rules. In general, an SP-logic is complete with respect to SP-rules iff it is complex [16]. We show that the only SP-logics in $\mathrm{Ext}^{+}P_{\mathsf{S4.3}}$ complete with respect to SP-rules are P_{S5} and the logic of the singleton cluster. These are also the only complete SP-logics for which deciding valid SP-rules is in polynomial time [17]. We make a step towards axiomatising SP-rules by giving axiomatisations for the SP-rule logics of n-element clusters, $n > 1$.

2 Preliminaries

We assume that the reader is familiar with basic notions of modal logic and Kripke semantics [7]. In particular, $\mathfrak{M}, w \models \varphi$ means that φ holds at world w in Kripke model $\mathfrak{M}$, and $\mathfrak{F} \models \varphi$ says that φ is valid in Kripke frame $\mathfrak{F}$. We write $\mathcal{C} \models \varphi$, for a class $\mathcal{C}$ of frames, if φ is valid in every $\mathfrak{F} \in \mathcal{C}$.

We denote SP-implications by $\iota = (\sigma \to \tau)$ and refer to σ, τ as *terms*. By regarding ι as the equality $\sigma \wedge \tau = \sigma$, we can naturally evaluate it in SLOs, i.e., structures $\mathfrak{A} = (A, \wedge, \top, \Diamond)$, where $(A, \wedge, \top)$ is a semilattice with top element $\top$ and $\Diamond(a \wedge b) \wedge \Diamond b = \Diamond(a \wedge b)$, for any $a, b \in A$. We write $\mathfrak{A} \models \iota$ to say that $\sigma \wedge \tau = \sigma$ holds in $\mathfrak{A}$ under any valuation. For a set $P \cup \{\iota\}$ of SP-implications, we write $P \models_{\mathsf{SLO}} \iota$ if $\mathfrak{A} \models \iota$ for every SLO $\mathfrak{A}$ validating all implications in P. We denote by $\boldsymbol{P}[\mathfrak{A}]$ the set of SP-implications ι for which $\mathfrak{A} \models \iota$.

Since equational consequence can be characterised syntactically by Birkhoff's equational calculus [5,13], it is readily seen that

$$P \models_{\mathsf{SLO}} \iota \quad \text{iff} \quad P \vdash_{\mathsf{SLO}} \iota,$$

where $P \vdash_{\mathsf{SLO}} \iota$ means that there is a sequence (derivation) $\iota_0, \dots, \iota_n = \iota$, with each ι_i being a substitution instance of a member in P or one of the axioms

$$p \to p, \qquad p \to \top, \qquad p \wedge q \to q \wedge p, \qquad p \wedge q \to p,$$

or obtained from earlier members of the sequence using one of the rules

$$\frac{\sigma \to \tau \quad \tau \to \varrho}{\sigma \to \varrho}, \qquad \frac{\sigma \to \tau \quad \sigma \to \varrho}{\sigma \to \tau \wedge \varrho}, \qquad \frac{\sigma \to \tau}{\Diamond\sigma \to \Diamond\tau}$$

(see also the Reflection Calculus RC of [2,8]). We say that SP-implications ι and ι' are *P-equivalent*, if $P \cup \{\iota\} \vdash_{\mathsf{SLO}} \iota'$ and $P \cup \{\iota'\} \vdash_{\mathsf{SLO}} \iota$. We say that terms σ and τ are *P-equivalent* if $P \vdash_{\mathsf{SLO}} \sigma \to \tau$ and $P \vdash_{\mathsf{SLO}} \tau \to \sigma$.

For any class $\mathcal{C}$ of Kripke frames, the set of modal formulas (with full Booleans) validated by the frames in $\mathcal{C}$ is called the *modal logic* of $\mathcal{C}$ and denoted by $\boldsymbol{L}[\mathcal{C}]$; the restriction of $\boldsymbol{L}[\mathcal{C}]$ to SP-implications is called the *SP-logic* of $\mathcal{C}$ and denoted by $\boldsymbol{P}[\mathcal{C}]$. For finite $\mathcal{C} = \{\mathfrak{F}_1, \dots, \mathfrak{F}_n\}$, we write $\boldsymbol{L}[\mathfrak{F}_1 \dots, \mathfrak{F}_n]$ and $\boldsymbol{P}[\mathfrak{F}_1, \dots, \mathfrak{F}_n]$, respectively. For any set Φ of modal formulas, $\mathsf{Fr}(\Phi)$ is the class of frames validating Φ. Every Kripke frame $\mathfrak{F} = (W, R)$ gives rise to a SLO $\mathfrak{F}^\star = (2^W, \cap, W, \Diamond^+)$ where $\Diamond^+ X = \{w \in W \mid R(w, v) \text{ for some } v \in X\}$, for $X \subseteq W$ (that is, $\mathfrak{F}^\star$ is the $(\wedge, \Diamond, \top)$-type reduct of the *full complex algebra of* $\mathfrak{F}$ [12]). As Kripke models over $\mathfrak{F}$ and valuations in the algebra $\mathfrak{F}^\star$ are the same thing, we always have $\boldsymbol{P}[\mathfrak{F}] = \boldsymbol{P}[\mathfrak{F}^\star]$.

$\mathsf{S4.3}$ is the modal logic whose rooted Kripke frames are linear quasi-orders $\mathfrak{F} = (W, R)$, i.e., R is a reflexive and transitive relation on W with xRy or yRx, for any $x, y \in W$. From now on, we refer to rooted frames for $\mathsf{S4.3}$ as simply *frames*. A *cluster* in $\mathfrak{F}$ is any set of the form $C(x) = \{y \in W \mid xRy \;\&\; yRx\}$. If $|W| \leq \omega$ and the number of clusters in $\mathfrak{F}$ is finite, we write $\mathfrak{F} = (m_1, \dots, m_n)$, $1 \leq m_i \leq \omega$, to say that the ith cluster in $\mathfrak{F}$ (starting from the root) has m_i points. In this case, we also say that $\mathfrak{F}$ is of *depth* n and write $d(\mathfrak{F}) = n$. A linear order with n points is denoted by $\mathfrak{L}_n$.

We call frame $\mathfrak{F} = (W, R)$ a *subframe* of a frame $\mathfrak{F}' = (W', R')$ and write $\mathfrak{F} \subseteq \mathfrak{F}'$ if $W \subseteq W'$ and R is the restriction of R' to W. A subframe $\mathfrak{F}$ of $\mathfrak{F}'$ is *proper* if W is a proper subset of W'. We say that $\mathfrak{F}$ is a *cofinal subframe* of $\mathfrak{F}'$ and write $\mathfrak{F} \Subset \mathfrak{F}'$ if $\mathfrak{F} \subseteq \mathfrak{F}'$ and, for any $x \in W$ and $y \in W'$ with $xR'y$, there is $z \in W$ with $yR'z$. (As we only deal with frames for $\mathsf{S4.3}$, $\mathfrak{F} \Subset \mathfrak{F}'$ iff $\mathfrak{F}$ is a p-morphic image of $\mathfrak{F}'$.)

The normal extension L of $\mathsf{S4.3}$ with a set Φ of modal formulas is denoted by $L = \mathsf{S4.3} \oplus \Phi$; $\mathrm{NExt}\mathsf{S4.3} = \{\mathsf{S4.3} \oplus \Phi \mid \Phi \text{ is a set of modal formulas}\}$. The minimal SP-logic P containing $P_{\mathsf{S4.3}}$ and a set Φ of SP-implications is denoted by $P = P_{\mathsf{S4.3}} + \Phi$; $\mathrm{Ext}^+ P_{\mathsf{S4.3}} = \{P_{\mathsf{S4.3}} + \Phi \mid \Phi \text{ is a set of SP-implications}\}$.

Given P and $\boldsymbol{\iota}$, we write $P \models_{\mathsf{Kr}} \boldsymbol{\iota}$ iff $\boldsymbol{\iota} \in \boldsymbol{P}[\mathsf{Fr}(P)]$. It is easy to see that

$$P \vdash_{\mathsf{SLO}} \boldsymbol{\iota} \text{ always implies } P \models_{\mathsf{Kr}} \boldsymbol{\iota}. \tag{1}$$

We call P *complete* if the converse also holds for every $\boldsymbol{\iota}$. As shown in [16], $P_{\mathsf{S4.3}}$ is complete and $\mathsf{Fr}(P_{\mathsf{S4.3}}) = \mathsf{Fr}(\mathsf{S4.3})$.

Throughout (see Figs. 1, 3 and 6), we describe examples of SLOs for $P_{\mathsf{S4.3}}$ by Hasse diagrams where $\circ$ vertices represent (closed) elements a with $\Diamond a = a$ and $\bullet$ vertices represent elements a with $\Diamond a > a$. Note that, for each $\bullet$ element a, $\Diamond a$ is the (unique) smallest $\circ$ element b with $b > a$.

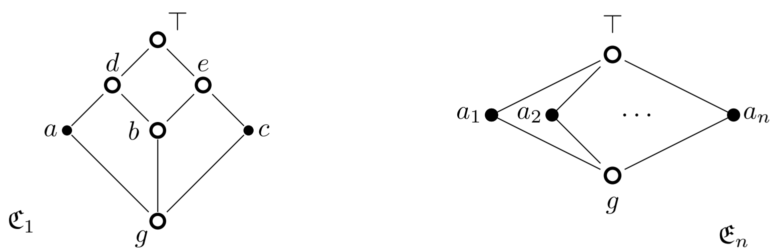

Fig. 1. SLOs for $P_{\mathsf{S4.3}}$.

3 Incomplete SP-logics

Axiomatising by SP-implications the SP-logics of even very simple frame classes turns out to be a challenging problem because many natural and innocuously looking axiom-candidates will most probably give incomplete SP-logics. For instance, let

$$\sigma_n(p,q) \;=\; \underbrace{\Diamond\big(p \wedge \Diamond\big(q \wedge \Diamond(p \wedge \dots)\dots\big)}_{\Diamond \text{ used } n \text{ times, } 1 \le n < \omega}.$$

Consider the following SP-implications (the reader may find it useful to compare them with the SP-implications defined in the next section):

$$\begin{aligned}
\boldsymbol{\iota}^2_{\mathit{fun}} &= \Diamond(p \wedge q) \wedge \Diamond(p \wedge r) \wedge \Diamond(q \wedge r) \to \Diamond(p \wedge q \wedge r), && (2)\\
\boldsymbol{\epsilon}_1 &= \Diamond p \wedge \Diamond q \to \Diamond\big(p \wedge \Diamond q\big), && (3)\\
\boldsymbol{\epsilon}_2 &= \Diamond(p \wedge \Diamond q) \wedge \Diamond(p \wedge \Diamond r) \wedge \Diamond(q \wedge \Diamond r) \to \Diamond\big(p \wedge \Diamond(q \wedge \Diamond r)\big), && (4)\\
\boldsymbol{\varphi} &= \Diamond(p \wedge \Diamond q) \wedge \Diamond(p \wedge \Diamond r) \wedge \Diamond(q \wedge r) \to \Diamond\big(p \wedge \Diamond(q \wedge r)\big), && (5)\\
\boldsymbol{\beta}_n &= \sigma_n(p,q) \wedge \sigma_n(q,p) \;\to\; \Diamond(p \wedge q), && (6)\\
\boldsymbol{\gamma}_n &= \sigma_{n+1}(p,q) \;\to\; \Diamond(p \wedge q), &&\\
\boldsymbol{\delta}_n &= \sigma_{n+1}(p,q) \;\to\; \sigma_{n+1}(q,p). &&
\end{aligned}$$

Figure 2 describes inclusions between the SP-logics above $P^2_{\mathit{fun}} = P_{\mathsf{S4.3}} + \boldsymbol{\iota}^2_{\mathit{fun}}$ axiomatised by these SP-implications. Using results from [15] and Claim 3.1 below, it is not hard to see that all of the depicted inclusions are proper.

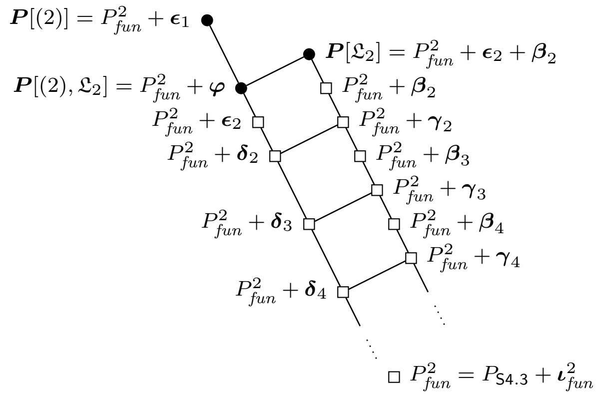

Fig. 2. Complete and incomplete SP-logics above $P^2_{\mathit{fun}} = P_{\mathsf{S4.3}} + \boldsymbol{\iota}^2_{\mathit{fun}}$.

The only complete SP-logics in the picture are shown by $\bullet$ (that they are axiomatised by the indicated SP-implications follows from [15]; see also Theorem 4.4 below). On the other hand, it is not hard to check that, for each SP-logic P in Fig. 2, if P is below $\boldsymbol{P}[(2), \mathfrak{L}_2]$ then the rooted frames for P are $\mathfrak{L}_1, (2), \mathfrak{L}_2$, and if P is below $\boldsymbol{P}[\mathfrak{L}_2]$ but not below $\boldsymbol{P}[(2), \mathfrak{L}_2]$ then the rooted frames for P are $\mathfrak{L}_1, \mathfrak{L}_2$. Therefore, all SP-logics shown by $\square$ in the picture are incomplete.

Claim 3.1 *(i)* $P^2_{\mathit{fun}}+\boldsymbol{\epsilon}_2 \not\models_{\mathsf{SLO}} \boldsymbol{\varphi}$ *and* $P^2_{\mathit{fun}}+\boldsymbol{\epsilon}_2 \not\models_{\mathsf{SLO}} \boldsymbol{\beta}_2$, *(ii)* $P^2_{\mathit{fun}}+\boldsymbol{\beta}_2 \not\models_{\mathsf{SLO}} \boldsymbol{\epsilon}_2$, *(iii)* $P^2_{\mathit{fun}}+\boldsymbol{\gamma}_n \not\models_{\mathsf{SLO}} \boldsymbol{\beta}_n$, *(iv)* $P^2_{\mathit{fun}}+\boldsymbol{\beta}_{n+1} \not\models_{\mathsf{SLO}} \boldsymbol{\delta}_n$, *(v)* $P^2_{\mathit{fun}}+\boldsymbol{\delta}_n \not\models_{\mathsf{SLO}} \boldsymbol{\beta}_{n+1}$.

Proof. We use the SLOs in Fig. 3. Claim (i) can be shown by $\mathfrak{A}_1$, (ii) by $\mathfrak{A}_2$, (iii) by $\mathfrak{C}_n$, (iv) by $\mathfrak{B}_n$, and (v) by $\mathfrak{D}_n$. □

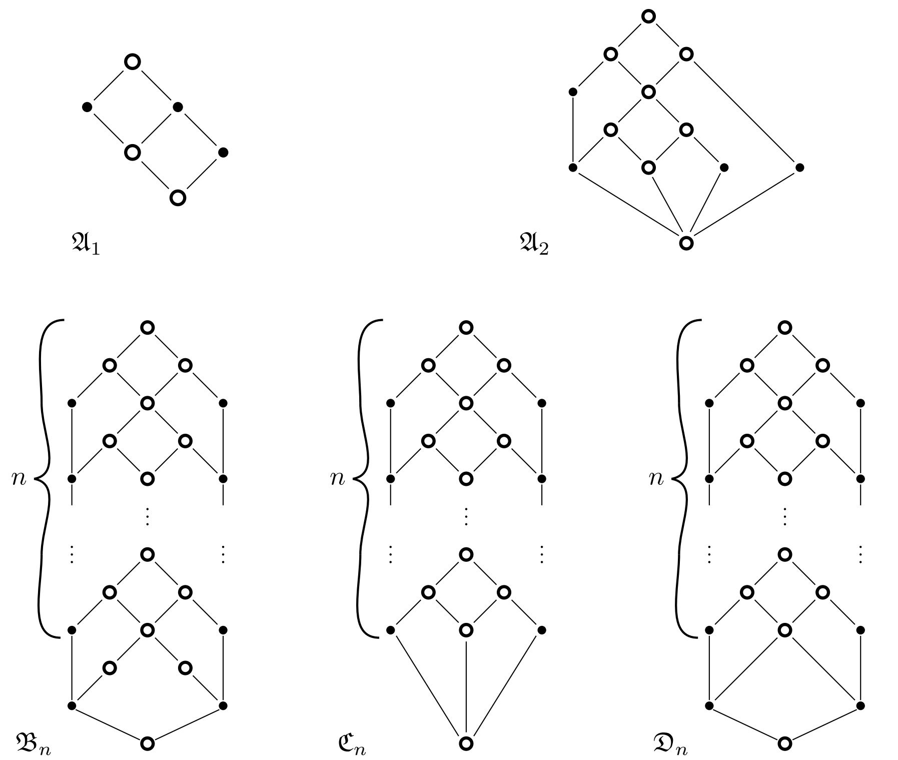

Fig. 3. SLOs validating P^2_{fun}.

The frames for the incomplete SP-logics above have at most 2 points. Similar incomplete SP-logics can clearly be defined for more complex frames, which makes the lattice $\mathrm{Ext}^+P_{\mathsf{S4.3}}$ much more involved compared to $\mathrm{NExt}\mathsf{S4.3}$.

4 Axiomatising and defining frame classes by SP-implications

As any SP-implication $\boldsymbol{\iota}$ is a Sahlqvist formula, $\mathsf{Fr}(\boldsymbol{\iota})$ is FO-definable [20]. As shown in [16], the class of *reflexive and transitive* frames validating $\boldsymbol{\iota}$ is closed under subframes. Using the results from [11,24] on FO-definable subframe logics, we obtain the following theorem (which can be readily proved directly):

Theorem 4.1 *For every* $\boldsymbol{\iota} \notin P_{\mathsf{S4.3}}$*, there exists* $n < \omega$ *such that* $\mathsf{Fr}(\boldsymbol{\iota})$ *is closed under subframes and any frame in* $\mathsf{Fr}(\boldsymbol{\iota})$ *is of depth* $< n$*. Thus, for any SP-logic* $P \in \mathrm{Ext}^+P_{\mathsf{S4.3}}$*, either* $P = P_{\mathsf{S4.3}}$ *or* $\mathsf{Fr}(P)$ *is of bounded depth.*

Our aim now is to axiomatise the SP-logic of every FO-definable subframe-closed class of $\mathsf{S4.3}$-frames. To this end, recall from [11,24] that, for any finite frame $\mathfrak{F}$, there are modal formulas $\alpha(\mathfrak{F})$ and $\alpha^\sharp(\mathfrak{F})$, called the *subframe* and *cofinal subframe formulas for* $\mathfrak{F}$, respectively, such that, for any frame $\mathfrak{G}$,

$$\mathfrak{G} \not\models \alpha(\mathfrak{F}) \quad \text{iff} \quad \mathfrak{F} \subseteq \mathfrak{G}, \qquad\qquad \mathfrak{G} \not\models \alpha^\sharp(\mathfrak{F}) \quad \text{iff} \quad \mathfrak{F} \Subset \mathfrak{G}. \tag{7}$$

In fact, every $L \in \mathrm{NExt}\mathsf{S4.3}$ can be represented in the form

$$L = \mathsf{S4.3} \oplus \alpha^\sharp(\mathfrak{F}_1) \oplus \cdots \oplus \alpha^\sharp(\mathfrak{F}_m),$$

for some $\mathfrak{F}_i$. If $\mathcal{C}$ is a nonempty FO-definable class of frames closed under subframes and $N < \omega$ is the maximal depth of frames in $\mathcal{C}$, then there exists a finite set $F_\mathcal{C}$ of frames of depth $\leq N$ such that

$$\boldsymbol{L}[\mathcal{C}] = \mathsf{S4.3} \oplus \alpha(\mathfrak{L}_{N+1}) \oplus \{\alpha(\mathfrak{F}) \mid \mathfrak{F} \in F_\mathcal{C}\}. \tag{8}$$

Note that $\mathsf{S4.3} \oplus \alpha(\mathfrak{F}) = \mathsf{S4.3} \oplus \alpha^\sharp(\mathfrak{F}) \oplus \alpha^\sharp(\mathfrak{F}^\circ)$, where $\mathfrak{F}^\circ$ is obtained from $\mathfrak{F}$ by adding a single-point cluster on top of $\mathfrak{F}$, and that $\mathsf{S4.3} \oplus \alpha(\mathfrak{L}_n) = \mathsf{S4.3} \oplus \alpha^\sharp(\mathfrak{L}_n)$.

In the remainder of this section we (i) construct SP-analogues $\boldsymbol{\kappa}^N(\mathfrak{F})$ of the subframe formulas $\alpha(\mathfrak{F})$ such that $\mathfrak{G} \models \boldsymbol{\kappa}^N(\mathfrak{F})$ iff $d(\mathfrak{G}) \leq N$ and $\mathfrak{G} \models \alpha(\mathfrak{F})$; and (ii) prove that, using some of the $\boldsymbol{\kappa}^N(\mathfrak{F})$ formulas, one can axiomatise all but two complete SP-logics properly extending $P_{\mathsf{S4.3}}$.

Let $\mathfrak{F} = (n_1, \dots, n_f)$ be a finite frame with $f \leq N < \omega$. We begin by defining an SP-implication $\boldsymbol{\iota}^N(\mathfrak{F})$ as follows. First, we take the terms

$$\tau(\mathfrak{F}) \;=\; \Diamond\big(\bigwedge P_1 \wedge \Diamond\big(\bigwedge P_2 \wedge \Diamond(\dots \Diamond \bigwedge P_f) \dots)\big), \tag{9}$$

where the P_i are pairwise disjoint sets of variables with $|P_i| = n_i$, for $i \leq f$. Next, denote by $\Sigma^N(\mathfrak{F})$ the set of all terms of the form

$$\Diamond\big(\bigwedge Q_1 \wedge \Diamond\big(\bigwedge Q_2 \wedge \Diamond(\dots \Diamond \bigwedge Q_N) \dots)\big),$$

where there exist $1 = x_1 < x_2 < \cdots < x_{f+1} = N + 1$ such that:

$$\text{for any } i \text{ and } j, \text{ if } x_i \leq j < x_{i+1} \text{ then } Q_j \subseteq P_i \text{ and } |Q_j| \geq |P_i| - 1, \tag{10}$$

$$\text{there is } i \text{ such that } |Q_j| = |P_i| - 1 \text{ for any } j \text{ with } x_i \leq j < x_{i+1}. \tag{11}$$

(Note that $\bigwedge \emptyset = \top$, so when some $Q_j = \emptyset$, the corresponding term in $\Sigma^N(\mathfrak{F})$ is P_{S4}-equivalent to a term of modal depth $< N$.) It is not hard to see that $|\Sigma^N(\mathfrak{F})| \leq (\max_i |P_i| + 1)^N$, and so

$$|\Sigma^N(\mathfrak{F})| \text{ is polynomial in } |\mathfrak{F}|. \tag{12}$$

Finally, we set

$$\boldsymbol{\iota}^N(\mathfrak{F}) \;=\; \bigwedge_{\sigma \in \Sigma^N(\mathfrak{F})} \sigma \;\to\; \tau(\mathfrak{F}). \tag{13}$$

For example, $\boldsymbol{\iota}^1((n))$ is

$$\bigwedge_{\substack{Q\subseteq\{p_1,\dots,p_n\}\\|Q|=n-1}} \Diamond \bigwedge Q \ \to\ \Diamond(p_1 \wedge \dots \wedge p_n)$$

defining $(n-1)$-functionality (in particular, $\boldsymbol{\iota}^1((3))$ is $\boldsymbol{\iota}^2_{fun}$ in (2)). For every $m \geq n$, the SP-implication $\boldsymbol{\iota}^m(\mathfrak{L}_n)$ is (P_{S4}-equivalent to)

$$\bigwedge_{i=1}^{n} \Diamond\Big[p_1 \wedge \Diamond\Big(p_2 \wedge \Diamond\big[\dots \Diamond\big(p_{i-1} \wedge \Diamond[p_{i+1} \wedge \Diamond(\dots \Diamond p_n)\dots]\big)\dots\big]\Big)\Big] \ \to \\ \Diamond\Big(p_1 \wedge \Diamond\big(p_2 \wedge \Diamond(\dots \Diamond p_n)\dots\big)\Big)$$

defining the property of having depth $< n$. In particular, $\boldsymbol{\iota}^n(\mathfrak{L}_n)$ is $\boldsymbol{\epsilon}_{n-1}$ in (3)–(4), for $n = 2, 3$; $\boldsymbol{\iota}^2((1,2))$ is $\boldsymbol{\varphi}$ in (5), and $\boldsymbol{\iota}^2((2))$ is $\boldsymbol{\beta}_2$ in (6).

In order to define $\boldsymbol{\kappa}^N(\mathfrak{F})$, we require the following:

Claim 4.2 [15, Lemma 7.7] *For any finite set Φ of SP-implications, there is a single SP-implication $\boldsymbol{\iota}_\Phi$ such that $P_{\mathsf{S4}} + \Phi = P_{\mathsf{S4}} + \boldsymbol{\iota}_\Phi$.*

Now, we set

$$\boldsymbol{\kappa}^N(\mathfrak{F}) \ = \ \boldsymbol{\iota}_{\{\boldsymbol{\iota}^{N+1}(\mathfrak{L}_{N+1}),\boldsymbol{\iota}^N(\mathfrak{F})\}}.$$

Observe that $P_{\mathsf{S4}} + \boldsymbol{\iota}^{n+1}(\mathfrak{L}_{n+1}) + \boldsymbol{\iota}^n(\mathfrak{L}_n) \ = \ P_{\mathsf{S4}} + \boldsymbol{\iota}^n(\mathfrak{L}_n)$, and so $\boldsymbol{\kappa}^n(\mathfrak{L}_n)$ is P_{S4}-equivalent to $\boldsymbol{\iota}^n(\mathfrak{L}_n)$.

Theorem 4.3 *For any finite $\mathsf{S4.3}$-frame $\mathfrak{F}$, any N with $d(\mathfrak{F}) \leq N < \omega$, and any $\mathsf{S4.3}$-frame $\mathfrak{G}$, we have $\mathfrak{G} \models \boldsymbol{\kappa}^N(\mathfrak{F})$ iff $d(\mathfrak{G}) \leq N$ and $\mathfrak{G} \models \alpha(\mathfrak{F})$.*

Proof. By (7), it suffices to show that (i) if $\mathfrak{F} \subseteq \mathfrak{G}$ then $\mathfrak{G} \not\models \boldsymbol{\iota}^N(\mathfrak{F})$; and (ii) if $d(\mathfrak{G}) \leq N$ and $\mathfrak{F} \not\subseteq \mathfrak{G}$ then $\mathfrak{G} \models \boldsymbol{\iota}^N(\mathfrak{F})$.

(i) Let $\mathfrak{F} = (n_1, \dots, n_f)$ and $\mathfrak{F} \subseteq \mathfrak{G}$. Then there exist clusters $(\zeta_1), \dots, (\zeta_f)$ in $\mathfrak{G}$ such that $(\zeta_1, \dots, \zeta_f)$ is a subframe of $\mathfrak{G}$ and $\zeta_i \geq n_i$, for $i \leq f$. Take the term $\tau(\mathfrak{F})$ from (9). We define a model $\mathfrak{M}$ based on $\mathfrak{G}$ in the following way. For each $i \leq f$, we

- take n_i distinct points from (ζ_i) and let each of them validate a different $n_i - 1$-element subset Q of P_i;
- take some point from (ζ_i) and let it validate $\bigcup_{j=1, j\neq i}^{f} P_j$.

All other points in $\mathfrak{G}$ validate no variables. Take any point r in the root cluster of $\mathfrak{G}$. Since for every $i \leq f$ we have $\mathfrak{M}, x \not\models \bigwedge P_i$ for any $x \in (\zeta_i)$, we clearly have $\mathfrak{M}, r \not\models \tau(\mathfrak{F})$. On the other hand, take some $\sigma \in \Sigma^N(\mathfrak{F})$ of the form $\Diamond\big(\bigwedge Q_1 \wedge \Diamond\big(\bigwedge Q_2 \wedge \Diamond(\dots \Diamond \bigwedge Q_N)\dots\big)\big)$. By (11), there is $i \leq f$ such that $|Q_j| = n_i - 1$ for all those Q_j for which $Q_j \subseteq P_i$. Therefore, by (10) for every $j \leq N$ we have $\mathfrak{M}, x_j \models \bigwedge Q_j$ for some $x_j \in (\zeta_i)$, and so $\mathfrak{M}, r \models \sigma$ as required.

(ii) is proved by induction on $d(\mathfrak{F})$. To begin with, we show that if $(n) \not\subseteq \mathfrak{G}$ for some $n < \omega$, then $\mathfrak{G} \models \boldsymbol{\iota}^N((n))$ for any $N \geq d(\mathfrak{G})$. To this end, suppose $\tau((n)) = \Diamond \bigwedge P_1$ for some set P_1 of variables with $|P_1| = n$. Let $\mathfrak{M}$ be a model based on $\mathfrak{G} = (\zeta_1, \dots, \zeta_g)$ such that the left-hand side $\bigwedge \Sigma^N((n))$ of $\boldsymbol{\iota}^N((n))$ holds at some point r in $\mathfrak{M}$. (We may assume that $r \in (\zeta_1)$.) We claim that

$$\text{there is } j^* \leq g \text{ such that, for every } Q \subseteq P_1,\ |Q| = n-1, \\ \text{we have } \mathfrak{M}, y_Q \models \bigwedge Q \text{ for some } y_Q \in (\zeta_{j^*}). \quad (14)$$

Indeed, suppose otherwise. Then, for every $j \leq g$, there is an $n-1$-element subset Q_j of P_1 such that $\mathfrak{M}, y \not\models \bigwedge Q_j$ for any $y \in (\zeta_j)$. Take the term

$$\delta \;=\; \Diamond\big(\bigwedge Q_1 \wedge \Diamond(\dots \Diamond \bigwedge Q_g)\dots\big).$$

By the choice of the Q_j, it is easy to see that $\mathfrak{M}, r \not\models \delta$. On the other hand, as $g \leq N$, there is $\sigma \in \Sigma^N((n))$ with $P_{\mathsf{S4}} \vdash_{\mathsf{SLO}} \sigma \to \delta$, and so $\mathfrak{M}, r \models \delta$, which is a contradiction proving (14). Now as $(n) \not\subseteq \mathfrak{G}$, it follows that $\zeta_{j^*} < n$. Therefore, by the pigeonhole principle and (14), there is $x \in (\zeta_{j^*})$ with $\mathfrak{M}, x \models \bigwedge P_1$, and so $\mathfrak{M}, r \models \tau((n))$, as required.

Now let $d(\mathfrak{F}) > 1$ and suppose inductively that, for all $\mathfrak{F}'$, if $d(\mathfrak{F}') < d(\mathfrak{F})$ then $\mathfrak{G} \models \boldsymbol{\iota}^N(\mathfrak{F}')$ for all N and $\mathfrak{G}$ such that $N \geq \max\big(d(\mathfrak{G}), d(\mathfrak{F}')\big)$ and $\mathfrak{F}' \not\subseteq \mathfrak{G}$. Take $\mathfrak{F} = (n_1, \dots, n_f)$, $\mathfrak{G} = (\zeta_1, \dots, \zeta_g)$, $N \geq \max(f, g)$ and $\mathfrak{F} \not\subseteq \mathfrak{G}$. Take $\boldsymbol{\iota}^N(\mathfrak{F})$ from (13), and suppose $\mathfrak{M}$ is a model based on $\mathfrak{G}$ such that

$$\mathfrak{M}, r \models \bigwedge_{\sigma \in \Sigma^N(\mathfrak{F})} \sigma, \quad \text{for some } r \in (\zeta_1). \tag{15}$$

Since $P_{\mathsf{S4}} \vdash_{\mathsf{SLO}} \sigma \to \Diamond \bigwedge P_f$ for some $\sigma \in \Sigma^N(\mathfrak{F})$, by (15), there is x with $\mathfrak{M}, x \models \bigwedge P_f$. Let

$$m = \max\{j \mid \mathfrak{M}, x \models \bigwedge P_f \text{ for some } x \in (\zeta_j)\} \tag{16}$$

and let $\mathfrak{M}^-$ be the restriction of $\mathfrak{M}$ to $(\zeta_1, \dots, \zeta_m)$. Let $\mathfrak{F}^- = (n_1, \dots, n_{f-1})$. Then $\tau(\mathfrak{F}^-) \;=\; \Diamond\big(\bigwedge P_1 \wedge \Diamond\big(\bigwedge P_2 \wedge \Diamond(\dots \Diamond \bigwedge P_{f-1})\dots\big)\big)$. We show that

$$\mathfrak{M}^-, r \models \tau(\mathfrak{F}^-), \tag{17}$$

from which $\mathfrak{M}, r \models \tau(\mathfrak{F})$ would clearly follow by (16).

To begin with, if $m = g$ then $\mathfrak{M}^- = \mathfrak{M}$. As $f - 1 \leq N$, there is $\sigma \in \Sigma^N(\mathfrak{F})$ such that $P_{\mathsf{S4}} \vdash_{\mathsf{SLO}} \sigma \to \tau(\mathfrak{F}^-)$, and so we have $\mathfrak{M}^-, r \models \tau(\mathfrak{F}^-)$ by (15). Now suppose that $m < g$. Two cases are possible:

Case 1: $\mathfrak{F}^- \not\subseteq (\zeta_1, \dots, \zeta_m)$. Let $k = \max(m, f-1)$. By IH, we then have

$$(\zeta_1, \dots, \zeta_m) \models \boldsymbol{\iota}^k(\mathfrak{F}^-). \tag{18}$$

Also, as $m < g \leq N$ and $f - 1 \leq N - 1 < N$, we have $k < N$. Thus, for every $\delta \in \Sigma^k(\mathfrak{F}^-)$ of the form $\Diamond\big(\bigwedge Q_1 \wedge \Diamond\big(\bigwedge Q_2 \wedge \Diamond(\dots \Diamond \bigwedge Q_k)\dots\big)\big)$, there is some $\sigma \in \Sigma^N(\mathfrak{F})$ with $P_{\mathsf{S4}} \vdash_{\mathsf{SLO}} \sigma \to \delta^+$ for the term

$$\delta^+ \;=\; \Diamond\Big(\bigwedge Q_1 \wedge \Diamond\big(\dots \Diamond\big(\bigwedge Q_k \wedge \Diamond\big(\bigwedge P_f\big)\big)\big)\Big).$$

Thus, $\mathfrak{M}, r \models \delta^+$ by (15), and so $\mathfrak{M}^-, r \models \delta$ follows by (16). By (18), we have $\mathfrak{M}^-, r \models \tau(\mathfrak{F}^-)$, and so (17) holds as required.

Case 2: $\mathfrak{F}^- \subseteq (\zeta_1, \dots, \zeta_m)$. Then $f-1 \leq m$ and $(f-1)+(g-m) \leq g \leq N$. Thus, for any sequence $\bar{Q} = (Q_{m+1}, \dots, Q_g)$ of (n_f-1)-element subsets of P_f, there is $\sigma \in \Sigma^N(\mathfrak{F})$ such that

$$P_{\mathsf{S4}} \vdash_{\mathsf{SLO}} \sigma \to \Diamond\big(\bigwedge P_1 \wedge \Diamond\big(\dots \Diamond\big(\bigwedge P_{f-1} \wedge \delta_{\bar{Q}}\big) \dots\big)\big)$$

for the term

$$\delta_{\bar{Q}} \ = \ \Diamond\big(\bigwedge Q_{m+1} \wedge \Diamond\big(\dots \Diamond \bigwedge Q_g\big) \dots\big).$$

So we have $\mathfrak{M}, r \models \Diamond\big(\bigwedge P_1 \wedge \Diamond\big(\dots \Diamond\big(\bigwedge P_{f-1} \wedge \delta_{\bar{Q}}\big) \dots\big)\big)$ by (15). Now suppose (17) does not hold. Then, for every such $\bar{Q}$, we have $\mathfrak{M}, x_{\bar{Q}} \models \delta_{\bar{Q}}$ for some $x_{\bar{Q}} \in (\zeta_{m+1})$. Thus, it is easy to see that there is j^* with $m < j^* \leq g$ such that, for each of the n_f-many (n_f-1)-element subsets Q of P_f, we have $\mathfrak{M}, y_Q \models \bigwedge Q$ for some $y_Q \in (\zeta_{j^*})$. (Otherwise, for every j with $m < j \leq g$, take some Q_j such that $\mathfrak{M}, y \not\models \bigwedge Q_j$ for any $y \in (\zeta_j)$, and then consider the sequence $\bar{Q} = (Q_{m+1}, \dots, Q_g)$.) As $\mathfrak{F}^- \subseteq (\zeta_1, \dots, \zeta_m)$ and $\mathfrak{F} \not\subseteq \mathfrak{G}$, it follows that $\zeta_{j^*} < n_f$. Therefore, by the pigeonhole principle $\mathfrak{M}, x \models \bigwedge P_f$ for some $x \in (\zeta_{j^*})$, contrary to $j^* > m$ and (16), proving that (17) holds as required. □

We now prove our main theorem on definability, completeness (axiomatisability) and computational complexity. We show that, using some of the $\boldsymbol{\kappa}^N(\mathfrak{F})$ formulas, we can axiomatise all but two complete SP-logics properly extending $P_{\mathsf{S4.3}}$. We begin with the two exceptions. First, it is straightforward to see that $\boldsymbol{P}[\emptyset] = P_{\mathsf{S4.3}} + (p \to q)$. Second, it is shown in [16, Theorem 29] that $\boldsymbol{P}[\mathcal{C}_{id}] = P_{\mathsf{S4.3}} + (\Diamond p \to p)$, where

$$\mathcal{C}_{id} = \big\{(W,R) \mid \forall x, y \in W \ \big(R(x,y) \leftrightarrow x = y\big)\big\}.$$

Now, suppose that $\mathcal{C}$ is a nonempty FO-definable class of S4.3-frames that is closed under subframes and different from $\mathsf{Fr}(P_{\mathsf{S4.3}})$, and let $N < \omega$ be the maximal depth of frames in $\mathcal{C}$. Take some finite set $F_{\mathcal{C}}$ of depth $\leq N$ frames such that (8) holds. This $F_{\mathcal{C}}$ consists of 'forbidden subframes' in the sense that, for any frame $\mathfrak{G}$ of depth $\leq N$,

$$\mathfrak{G} \in \mathcal{C} \quad \text{iff} \quad \mathfrak{F} \not\subseteq \mathfrak{G} \text{ for any } \mathfrak{F} \in F_{\mathcal{C}}. \tag{19}$$

Note that such an $F_{\mathcal{C}}$ is not unique. In (8), this is not a problem since $P_{\mathsf{S4}} + \alpha(\mathfrak{F}_1) \models_{\mathsf{Kr}} \alpha(\mathfrak{F}_2)$ whenever $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$ by (7). However, it is not always the case that $P_{\mathsf{S4.3}} + \boldsymbol{\kappa}^N(\mathfrak{F}_1) \vdash_{\mathsf{SLO}} \boldsymbol{\kappa}^N(\mathfrak{F}_2)$ whenever $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$. (For example, it follows from Claim 3.1 that $P_{\mathsf{S4.3}} + \boldsymbol{\kappa}^2((2)) \not\vdash_{\mathsf{SLO}} \boldsymbol{\kappa}^2((1,2))$.) Therefore, in what follows, we assume that $F_{\mathcal{C}}$ has the following closure property:

For every $\mathfrak{G}$ with $d(\mathfrak{G}) \leq N$,

$$\text{if } \mathfrak{G} \notin \mathcal{C} \text{ then there is } \mathfrak{F} \in F_{\mathcal{C}} \text{ such that } \mathfrak{F} \subseteq \mathfrak{G} \text{ and } d(\mathfrak{F}) = d(\mathfrak{G}). \tag{20}$$

Now take the minimal set $F_\mathcal{C}$ of depth $\leq N$ frames such that (19) and (20) hold. It is easy to see that $F_\mathcal{C}$ is always finite and unique (up to isomorphism of its frames). (For example, if $\mathcal{C}$ consists of all frames isomorphic to some subframe of $(2, 1)$, then $F_\mathcal{C} = \{(1, 2), (3), (3, 1)\}$.) Let

$$P_\mathcal{C} = \begin{cases} P_{\mathsf{S4.3}} + \{\boldsymbol{\kappa}^N(\mathfrak{F}) \mid \mathfrak{F} \in F_\mathcal{C}\}, & \text{if } F_\mathcal{C} \neq \emptyset, \\ P_{\mathsf{S4.3}} + \boldsymbol{\kappa}^{N+1}(\mathfrak{L}_{N+1}), & \text{otherwise.} \end{cases}$$

Theorem 4.4 *Let $\mathcal{C}$ be any FO-definable class of $\mathsf{S4.3}$-frames that is closed under subframes and different from $\emptyset$, $\mathcal{C}_{id}$ and $\mathsf{Fr}(P_{\mathsf{S4.3}})$. Then the following hold:*

(i) $\mathcal{C} = \mathsf{Fr}(P_\mathcal{C})$.

(ii) *$P_\mathcal{C}$ is Kripke complete, and so $\boldsymbol{P}[\mathcal{C}] = P_\mathcal{C}$.*

(iii) *$P_\mathcal{C}$ is decidable in* PTime.

Proof. *(i)* is a straightforward consequence of Claim 4.2 and Theorem 4.3.

(ii) By an *$F_\mathcal{C}$-normal form* we mean any term of the following form: $\Diamond\big(\bigwedge P^1 \wedge \Diamond\big(\bigwedge P^2 \wedge \Diamond(\dots \Diamond \bigwedge P^k)\dots\big)\big)$, where $k \leq N$ and each P^i is a finite nonempty set of variables for which there is no $\mathfrak{F} \in F_\mathcal{C}$ such that $\mathfrak{F} \subseteq \big(|P^1|, |P^2|, \dots, |P^k|\big)$. We do not require that the P^i are disjoint.

Claim **4.4.1** *For every term σ, there is a set N_σ of $F_\mathcal{C}$-normal forms such that $|N_\sigma|$ is polynomial in the size of σ, and $\Diamond\sigma$ is $P_\mathcal{C}$-equivalent to $\bigwedge_{\varrho \in N_\sigma} \varrho$.*

Proof. We proceed via a series of steps (a)–(c).

(a) As shown in [16, Claim 48.1]), $\Diamond\sigma$ is $P_{\mathsf{S4.3}}$-equivalent to a conjunction of terms of the form $\Diamond\big(\bigwedge Q^1 \wedge \Diamond\big(\bigwedge Q^2 \wedge \Diamond(\dots \Diamond \bigwedge Q^k)\dots\big)\big)$, where $k < \omega$ and each Q^i is a finite set of variables. Each such term describes a full linear branch in the 'term tree' of $\Diamond\sigma$, and so

(p1) each k is polynomial in the size $|\sigma|$ of σ;

(p2) each $|Q^i|$ of each term is polynomial in $|\sigma|$; and

(p3) the overall number of terms we obtain this way is polynomial in $|\sigma|$.

(b) By $\boldsymbol{\iota}^{N+1}(\mathfrak{L}_{N+1})$, we can take $k \leq N$ in (a). By (12) and **(p1)**–**(p3)**, the number of terms we thus obtain is polynomial in $|\sigma|$, so we are done if $F_\mathcal{C} = \emptyset$.

(c) Let $F_\mathcal{C} \neq \emptyset$. We claim that each term in (b) is $P_{\mathsf{S4}} + \{\boldsymbol{\iota}^N(\mathfrak{F}) \mid \mathfrak{F} \in F_\mathcal{C}\}$-equivalent to a conjunction of $F_\mathcal{C}$-normal forms. Indeed, suppose

$$\chi = \Diamond\big(\bigwedge Q^1 \wedge \Diamond\big(\bigwedge Q^2 \wedge \Diamond(\dots \Diamond \bigwedge Q^k)\dots\big)\big)$$

is a term in (b), for some $k \leq N$ and finite sets Q^i of variables. By P_{S4}, we may assume that each Q^i is nonempty. If such a term is not an $F_\mathcal{C}$-normal form, then $\mathfrak{F} \subseteq \big(|Q^1|, |Q^2|, \dots, |Q^k|\big)$ for some $\mathfrak{F} \in F_\mathcal{C}$. By (19) and (20), we may assume that $d(\mathfrak{F}) = k$. Let $\mathfrak{F} = (n_1, \dots, n_k)$ and $n_i \leq |Q^i|$ for $i \leq k$. For $i \leq k$, let $P_i = \{p^i_1, \dots, p^i_{n_i}\}$ be the variables in $\boldsymbol{\iota}^N(\mathfrak{F})$ and $Q^i = \{q^i_1, \dots, q^i_{|Q_i|}\}$. We define a substitution S for the variables in $\bigcup_{i \leq k} P_i$ as follows: for any $i \leq k$ and any $j \leq n_i - 1$, we substitute q^i_j for p^i_j, and $q^i_{n_i} \wedge \dots \wedge q^i_{|Q_i|}$ for $p^i_{n_i}$. Then the S-instance of $\tau(\mathfrak{F})$ is χ. It is easy to see that $P_{\mathsf{S4}} \vdash_{\mathsf{SLO}} \tau(\mathfrak{F}) \to \bigwedge \Sigma^N(\mathfrak{F})$ always holds, and so $P_{\mathsf{S4}} \vdash_{\mathsf{SLO}} \chi \to S(\sigma)$ for the S-instance $S(\sigma)$ of every $\sigma \in \Sigma^N(\mathfrak{F})$. On the other hand, we have $\boldsymbol{\iota}^N(\mathfrak{F}) \vdash_{\mathsf{SLO}} \bigwedge_{\sigma \in \Sigma^N(\mathfrak{F})} S(\sigma) \to \chi$. By (10) and (11), each $S(\sigma)$ is of the form $\Diamond\big(\bigwedge R^1 \wedge \Diamond(\bigwedge R^2 \wedge \Diamond(\dots \Diamond \bigwedge R^N)\dots)\big)$ such that

(n1) for every $i \leq N$, there is some $j_i \leq k$ with $R^i \subseteq Q^{j_i}$;

(n2) there is $j \leq k$ such that $|R^i| < |Q^j|$, for every i with $j_i = j$.

Take those $R^{i_1}, \dots, R^{i_\ell}$ that are nonempty. Then $S(\sigma)$ is P_{S4}-equivalent to

$$\vartheta = \Diamond\Big(\bigwedge R^{i_1} \wedge \Diamond\big(\bigwedge R^{i_2} \wedge \Diamond(\dots \Diamond \bigwedge R^{i_\ell}) \dots\big)\Big).$$

If such a ϑ is not an $F_{\mathcal{C}}$-normal form, then again there is some $\mathfrak{F}' \in F_{\mathcal{C}}$ such that $d(\mathfrak{F}') = \ell$ and $\mathfrak{F}' \subseteq \big(|R^{i_1}|, |R^{i_2}|, \dots, |R^{i_\ell}|\big)$. Thus, we can continue by 'applying' $\boldsymbol{\iota}^N(\mathfrak{F}')$ to ϑ. By **(n1)** and **(n2)**, sooner or later the procedure stops and we obtain a set of $F_{\mathcal{C}}$-normal forms. In fact, by (12) and **(p1)**–**(p3)**, both the number of required steps and the number of terms we obtain in each step are polynomial in $|\sigma|$. ☐

For a set Σ of terms, let $var(\Sigma)$ be the set of propositional variables in Σ. We write $var(\sigma)$ for $var\big(\{\sigma\}\big)$.

Claim **4.4.2** *For every finite set $\Sigma \cup \{\xi\}$ of $F_{\mathcal{C}}$-normal forms such that $var(\Sigma) \subseteq var(\xi)$, if $P_{\mathsf{S4}} \not\vdash_{\mathsf{SLO}} \bigwedge \Sigma \to \xi$ then $\mathfrak{G} \not\models \bigwedge \Sigma \to \xi$, for some $\mathfrak{G} \in \mathcal{C}$.*

Proof. Suppose $\xi = \Diamond\big(\bigwedge P^1 \wedge \Diamond(\bigwedge P^2 \wedge \Diamond(\dots \Diamond \bigwedge P^k)\dots)\big)$ where $k \leq N$, and each P^i is a finite set of variables such that there is no $\mathfrak{F} \in F_{\mathcal{C}}$ with $\mathfrak{F} \subseteq \big(|P^1|, |P^2|, \dots, |P^k|\big)$. Then $\big(|P^1|, |P^2|, \dots, |P^k|\big) \in \mathcal{C}$ by (19). By P_{S4}, we may assume that neither $P^i \subseteq P^{i+1}$ nor $P^{i+1} \subseteq P^i$ for any $i \leq k$. Let $\mathfrak{M}$ be the following model over $\big(|P^1|, |P^2|, \dots, |P^k|\big)$: for every $i \leq k$, let each point in $\big(|P^i|\big)$ validate $\bigcup_{j=1}^k P^j \setminus \{p\}$ for each different $p \in P^i$. Then clearly $\mathfrak{M}, r \not\models \xi$ for any $r \in \big(|P^i|\big)$. On the other hand, take any $\chi \in \Sigma$. As $var(\chi) \subseteq var(\xi)$, χ is of the form $\Diamond\big(\bigwedge R^1 \wedge \Diamond(\bigwedge R^2 \wedge \Diamond(\dots \Diamond \bigwedge R^\ell)\dots)\big)$ such that $\ell \leq N$ and $R^i \subseteq \bigcup_{j=1}^k P^j$ for $i \leq \ell$. As $P_{\mathsf{S4}} \not\vdash_{\mathsf{SLO}} \chi \to \xi$, there is no subsequence $(R^{i_1}, \dots, R^{i_k})$ of $(R^1, \dots, R^\ell)$ with $P^j \subseteq R^{i_j}$ for $j \leq k$. So it is not hard to check that $\mathfrak{M}, r \models \chi$. Therefore, $\mathfrak{M}, r \models \bigwedge \Sigma$, and so $\big(|P^1|, |P^2|, \dots, |P^k|\big) \not\models \bigwedge \Sigma \to \xi$. ☐

Claim **4.4.3** [15, Lemma 5.1] *Suppose σ and τ are terms.*

(a) $\sigma \to \tau$ is $\emptyset$-equivalent to $\sigma_\tau[\top] \to \tau$, where $\sigma_\tau[\top]$ is obtained from σ by substituting $\top$ for all variables not in $var(\tau)$.

(b) If $P_{\mathsf{S4}} + (\sigma \to \tau) \not\vdash_{\mathsf{SLO}} \Diamond p \to p$, then $\sigma \to \tau$ is P_{S4}-equivalent to $\Diamond\sigma \to \Diamond\tau$.

Proof. (a) is a straightforward.

(b) We clearly have $\sigma \to \tau \vdash_{\mathsf{SLO}} \Diamond\sigma \to \Diamond\tau$. For the other direction, suppose

$$\sigma = \bigwedge_{p \in P_\sigma} p \wedge \bigwedge_{\chi \in T_\sigma} \Diamond\chi \qquad \text{and} \qquad \tau = \bigwedge_{p \in P_\tau} p \wedge \bigwedge_{\vartheta \in T_\tau} \Diamond\vartheta,$$

for some sets P_σ, P_τ of variables, and sets T_σ, T_τ of terms. We claim that

$$P_\tau \subseteq P_\sigma. \tag{21}$$

Suppose otherwise. In this case, we take some $p \in P_\tau \setminus P_\sigma$, and let σ^- be obtained from σ by replacing all variables different from p with $\top$. Then we have $P_{\mathsf{S4}} + (\sigma \to \tau) \vdash_{\mathsf{SLO}} \sigma^- \to p$. As $P_{\mathsf{S4}} + (\sigma \to \tau) \not\vdash_{\mathsf{SLO}} \top \to p$, we may assume that $var(\tau) \subseteq var(\sigma)$, and so $p \in var(\sigma)$. It follows that $P_{\mathsf{S4}} \vdash_{\mathsf{SLO}} \Diamond p \to \sigma^-$, and so $P_{\mathsf{S4}} + (\sigma \to \tau) \vdash_{\mathsf{SLO}} \Diamond p \to p$, which is a contradiction proving (21). As we clearly have $P_{\mathsf{S4}} \vdash_{\mathsf{SLO}} \sigma \to \bigwedge P_\sigma \wedge \Diamond\sigma$, and $P_{\mathsf{S4}} \vdash_{\mathsf{SLO}} \Diamond\tau \to \bigwedge_{\vartheta \in T_\tau} \Diamond\vartheta$, by (21) we obtain $P_{\mathsf{S4}} + (\Diamond\sigma \to \Diamond\tau) \vdash_{\mathsf{SLO}} \sigma \to \tau$, as required. ☐

Now the proof of (ii) can be completed as follows. Suppose $P_{\mathcal{C}} \models_{\mathsf{Kr}} \sigma \to \tau$. As $\mathcal{C}$ is different from $\emptyset$ and $\mathcal{C}_{id}$, it follows that $P_{\mathsf{S4}} + (\sigma \to \tau) \not\vdash_{\mathsf{SLO}} \Diamond p \to p$, and so $\sigma \to \tau$ is P_{S4}-equivalent to $\Diamond\sigma \to \Diamond\tau$ by Claim 4.4.3 (b). Thus, by Claim 4.4.1 and (1), we have

$$P_{\mathcal{C}} \models_{\mathsf{Kr}} \bigwedge_{\chi \in N_\sigma} \chi \to \bigwedge_{\vartheta \in N_\tau} \vartheta,$$

and so $P_{\mathcal{C}} \models_{\mathsf{Kr}} \bigwedge_{\chi \in N_\sigma} \chi \to \vartheta$, for every $\vartheta \in N_\tau$. Take any $\vartheta \in N_\tau$. It is straightforward to see that by substituting $\top$ for some variables in an $F_{\mathcal{C}}$-normal form we obtain a term that is P_{S4}-equivalent to an $F_{\mathcal{C}}$-normal form. So by (i) and Claims 4.4.3 (a), 4.4.2, we have $P_{\mathsf{S4}} \vdash_{\mathsf{SLO}} \bigwedge_{\chi \in N_\sigma} \chi \to \vartheta$. Therefore,

$$P_{\mathsf{S4}} \vdash_{\mathsf{SLO}} \bigwedge_{\chi \in N_\sigma} \chi \to \bigwedge_{\vartheta \in N_\tau} \vartheta,$$

and so $P_{\mathcal{C}} \vdash_{\mathsf{SLO}} \sigma \to \tau$ follows by Claim 4.4.1.

(iii) follows from Claims 4.4.1 and 4.4.2, and the tractability of P_{S4}. ☐

It is to be noted that Theorem 4.4 does not hold for $\mathcal{C} = \mathcal{C}_{id}$. In this case $N = 1$ and $F_{\mathcal{C}_{id}} = \{(2)\}$. It is easy to see that

$$P_{\mathsf{S4.3}} + \boldsymbol{\kappa}^1((2)) = P_{\mathsf{S4.3}} + \boldsymbol{\iota}^1((2)) = P_{\mathsf{S4.3}} + \big(\Diamond p \wedge \Diamond q \to \Diamond(p \wedge q)\big).$$

As shown in [16, Theorem 29], this SP-logic is incomplete.

Note also that the axiomatisations given by Theorem 4.4 are not necessarily independent and often can be simplified. For instance, it is not hard to show the following:

– $P_{\mathsf{S4}} + \boldsymbol{\kappa}^N(\mathfrak{F}_1) \vdash_{\mathsf{SLO}} \boldsymbol{\kappa}^N(\mathfrak{F}_2)$ whenever $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$ and $d(\mathfrak{F}_1) = d(\mathfrak{F}_2) \leq N$.
– $P_{\mathsf{S4.3}} + \boldsymbol{\kappa}^N(\mathfrak{F}) \vdash_{\mathsf{SLO}} \boldsymbol{\kappa}^N(\mathfrak{F}^\circ)$ whenever $d(\mathfrak{F}) < N$, where $\mathfrak{F}^\circ$ is obtained from $\mathfrak{F}$ by adding a single-point cluster on top of $\mathfrak{F}$.

5 Modal logics vs. SP-logics

The set NExt$\mathsf{S4.3}$ ordered by $\subseteq$ forms a distributive lattice, a pseudo-Boolean algebra, to be more precise [19]. On the other hand, as follows from [15, Proposition 5.11], the lattice $\mathrm{Ext}^+P_{\mathsf{S4.3}}$ is not even modular because it contains the pentagon N_5 lattice; see Fig. 2.

Following [4], we compare these two lattices of logics using the map $\pi\colon \mathrm{NExt}\mathsf{S4.3} \to \mathrm{Ext}^+P_{\mathsf{S4.3}}$, which gives the *SP-fragment* $\pi(L)$ of any modal logic L, and the map $\mu\colon \mathrm{Ext}^+P_{\mathsf{S4.3}} \to \mathrm{NExt}\mathsf{S4.3}$, which gives the *modal extension* $\mu(P) = \mathsf{S4.3} \oplus P$ of any SP-logic P. We call L a *modal companion* of $\pi(L)$, and P an *SP-companion* of $\mu(P)$. It was observed in [4] that μ and π are monotone and form a Galois connection: $\mu(P) \subseteq L$ iff $P \subseteq \pi(L)$. In this section, we obtain answers to some open questions from [4] in the context of NExt$\mathsf{S4.3}$ and $\mathrm{Ext}^+P_{\mathsf{S4.3}}$. First, we give a characterisation of those SP-logics that have modal companions (which actually holds for all multi-modal SP-logics).

Theorem 5.1 *An SP-logic P has a modal companion iff P is complete.*

Proof. ($\Rightarrow$) If $\pi(L) = P$ and $\boldsymbol{\iota} \notin P$, then $\boldsymbol{\iota} \notin \mathsf{S4.3} \oplus P$, and so, by Sahlqvist completeness, there is a Kripke frame for P refuting $\boldsymbol{\iota}$, as required. ($\Leftarrow$) It is readily seen that, if P is complete, then $\mathsf{S4.3} \oplus P$ is its modal companion. $\square$

Next, we give a frame-theoretic characterisation of modal companions of any complete SP-logic in $\mathrm{Ext}^+P_{\mathsf{S4.3}}$, which consists of two parts. The first part describes the set $\pi^{-1}(P_{\mathsf{S4.3}})$ of modal companions of $P_{\mathsf{S4.3}}$ and shows that it comprises exactly the logics in NExt$\mathsf{S4.3}$ that have frames of unbounded depth. We say that such logics are of *slice* ω.

Theorem 5.2 *For any modal logic $L \in$ NExt$\mathsf{S4.3}$, we have $\pi(L) = P_{\mathsf{S4.3}}$ iff $L \subseteq \mathsf{Grz.3} = \mathsf{S4.3} \oplus \alpha((2))$. Thus, $\mathsf{Grz.3}$ is the greatest companion of $P_{\mathsf{S4.3}}$.*

Proof. ($\Rightarrow$) If $L \not\subseteq \mathsf{Grz.3}$, then $\alpha(\mathfrak{L}_n) \in L$, for some $n < \omega$. By Theorem 4.3, it follows that $\boldsymbol{\kappa}^n(\mathfrak{L}_n) \in L$, contrary to $\pi(L) = P_{\mathsf{S4.3}}$. ($\Leftarrow$) Suppose there is $\boldsymbol{\iota} \in \pi(L) \setminus P_{\mathsf{S4.3}}$. By Theorem 4.1, there is $n < \omega$ such that $\mathfrak{L}_n \not\models \boldsymbol{\iota}$, which is impossible since $\mathfrak{L}_n \models \mathsf{Grz.3}$. $\square$

In other words, $\pi^{-1}(P_{\mathsf{S4.3}})$ comprises $\mathsf{S4.3}$ and all those (infinitely many) logics in NExt$\mathsf{S4.3}$ whose classes of frames are *not FO-definable.* By a standard Löwenheim–Skolem–Tarski argument, if $\mathcal{C}$ is FO-definable, then both $\boldsymbol{L}[\mathcal{C}]$ and $\boldsymbol{P}[\mathcal{C}]$ are determined by the countable frames in $\mathcal{C}$. So the remaining logics can be classified according to the depth of their countable frames. (This classification was suggested in [14,18].) We say that a modal logic L is *of slice* n $(0 < n < \omega)$ if $\mathfrak{L}_n \models L$ but $\mathfrak{L}_{n+1} \not\models L$. Within NExt$\mathsf{S4.3}$, the logics L of slice n form the infinite interval

$$\mathsf{S4.3} \oplus \alpha(\mathfrak{L}_{n+1}) = \boldsymbol{L}[(\underbrace{\omega, \dots, \omega}_{n})] \subseteq L \subseteq \boldsymbol{L}[\mathfrak{L}_n] = \mathsf{S4.3} \oplus \alpha(\mathfrak{L}_{n+1}) \oplus \alpha\big((2)\big).$$

In particular, the logics L of slice 1 form the (infinite) interval $\mathsf{S5} \subseteq L \subseteq \boldsymbol{L}[\mathfrak{L}_1]$. (It is shown in [18] that all logics of finite slices above $\mathsf{S4}$ are locally finite.)

We now characterise the modal companions of complete SP-logics properly extending $P_{\mathsf{S4.3}}$. Given the class $\mathsf{Fr}^\omega(P)$ of all countable rooted frames for P, denote by $\downarrow\mathsf{Fr}^\omega(P)$ its smallest subclass whose closure under subframes gives $\mathsf{Fr}^\omega(P)$. (It is not hard to see that $\downarrow\mathsf{Fr}^\omega(P)$ always exists.)

Theorem 5.3 *For any complete SP-logic* $P \supsetneqq P_{\mathsf{S4.3}}$*,*

$$\pi^{-1}(P) = \{L \in \mathrm{NExt}\mathsf{S4.3} \mid \downarrow\mathsf{Fr}^\omega(P) \subseteq \mathsf{Fr}^\omega(L) \subseteq \mathsf{Fr}^\omega(P)\}.$$

Thus, $\boldsymbol{L}[\downarrow\mathsf{Fr}^\omega(P)]$ *is the greatest modal companion of* P*.*

Proof. Suppose $\downarrow\mathsf{Fr}^\omega(P) \subseteq \mathsf{Fr}^\omega(L) \subseteq \mathsf{Fr}^\omega(P)$ and show that $\pi(L) = P$. If $\boldsymbol{\iota} \in L$, then $\mathsf{Fr}^\omega(L) \models \boldsymbol{\iota}$, and so $\downarrow\mathsf{Fr}^\omega(P) \models \boldsymbol{\iota}$. By Theorem 4.1, it follows that $\mathsf{Fr}^\omega(P) \models \boldsymbol{\iota}$ and $\boldsymbol{\iota} \in P$. The implication $\boldsymbol{\iota} \in P \Rightarrow \boldsymbol{\iota} \in L$ is trivial.

Next, we have to prove that every modal companion L of P belongs to the specified interval. It suffices to show that if $L = \mathsf{S4.3} \oplus P \oplus \alpha^\sharp(\mathfrak{F})$ is a modal companion of P, then $\downarrow\mathsf{Fr}^\omega(P) \models \alpha^\sharp(\mathfrak{F})$. Suppose otherwise and take some $\mathfrak{G} \in \downarrow\mathsf{Fr}^\omega(P)$ such that $\mathfrak{F} \Subset \mathfrak{G}$. By the construction of $\downarrow\mathsf{Fr}^\omega(P)$, we can always find a *finite* frame $\mathfrak{G}'$ such that $\mathfrak{F} \Subset \mathfrak{G}' \subseteq \mathfrak{G}$ and $\mathfrak{G}' \not\subseteq \mathfrak{H}$, for any $\mathfrak{H}$ in $\downarrow\mathsf{Fr}^\omega(P)$ different from $\mathfrak{G}$. But then $\alpha(\mathfrak{G}') \in L$, and so $\boldsymbol{\kappa}^n(\mathfrak{G}') \in P$ by Theorem 4.3, where n is the slice of P, which is impossible because $\mathfrak{G}' \in \mathsf{Fr}^\omega(P)$. □

Intuitively, $\boldsymbol{L}[\downarrow\mathsf{Fr}^\omega(P)]$ saturates $\mathsf{S4.3} \oplus P$ with all those formulas $\alpha^\sharp(\mathfrak{F})$ that do not derive any $\alpha(\mathfrak{G}) \notin \mathsf{S4.3} \oplus P$. We illustrate this by a few examples.

Example 5.4 (1) The SP-logic $\boldsymbol{P}[(\omega, \omega)] = P_{\mathsf{S4.3}} + \boldsymbol{\kappa}^3(\mathfrak{L}_3)$ has only one modal companion $\mathsf{S4.3} \oplus \alpha(\mathfrak{L}_3)$ since $(\omega, \omega) \not\models \alpha^\sharp(\mathfrak{F})$, for any $\mathfrak{F}$ of depth ≤ 2.

(2) The SP-logic $\boldsymbol{P}[(2, 1)] = P_{\mathsf{S4.3}} + \boldsymbol{\kappa}^2((1, 2)) + \boldsymbol{\kappa}^2((3))$ has two modal companions: $\mathsf{S4.3} \oplus \alpha(\mathfrak{L}_3) \oplus \alpha\big((1, 2)\big) \oplus \alpha\big((3)\big) = \boldsymbol{L}[(2), (2, 1)]$ and its extension with $\alpha^\sharp\big((2)\big)$, i.e., $\boldsymbol{L}[(2, 1)]$. Note that $\mathsf{S4} \oplus \alpha^\sharp\big((2)\big) = \mathsf{S4} \oplus \Box\Diamond p \to \Diamond\Box p = \mathsf{S4.1}$.

(3) The companions L of $\boldsymbol{P}[(\omega, 1)] = P_{\mathsf{S4.3}} + \boldsymbol{\kappa}^2((1, 2))$ form the infinite interval

$$\boldsymbol{L}[(\omega), (\omega, 1)] = \mathsf{S4.3} \oplus \alpha(\mathfrak{L}_3) \oplus \alpha\big((1, 2)\big) \subseteq L \subseteq \\ \mathsf{S4.3} \oplus \alpha(\mathfrak{L}_3) \oplus \alpha^\sharp\big((2)\big) = \boldsymbol{L}[(\omega, 1)].$$

As follows from our results above, π maps $\mathrm{NExt}\mathsf{S4.3}$ onto the complete SP-logics in $\mathrm{Ext}^+P_{\mathsf{S4.3}}$. On the other hand, for a complete SP-logic P, there may exist Kripke incomplete logics P' such that $\mathsf{Fr}(P) = \mathsf{Fr}(P')$. All these logics form the set $\mu^{-1}(\mathsf{S4.3} \oplus P)$. Thus, for every $L \in \mathrm{NExt}\mathsf{S4.3}$, the set $\mu^{-1}(L)$ has $\pi(L)$ as its greatest element; see Fig. 4. We do not know whether $\mu^{-1}(L)$ always has a least element, whether it has non-finitely axiomatisable SP-logics, and whether there are a continuum of them.

As we saw above, any SP-logic different from $P_{\mathsf{S4.3}}$ belongs to some finite slice. In Fig. 5, we show slice 1 and a part of slice 2 (to minimise clutter, we only give the frames or SLOs determining modal and SP-logics; as before, □ indicates incomplete SP-logics). The structure of slice 1 was detailed in [15]. As shown in [16], it has two incomplete SP-logics: $\boldsymbol{P}[\mathfrak{S}]$ and $\boldsymbol{P}[\mathfrak{C}_1]$, where $\mathfrak{S}$ is

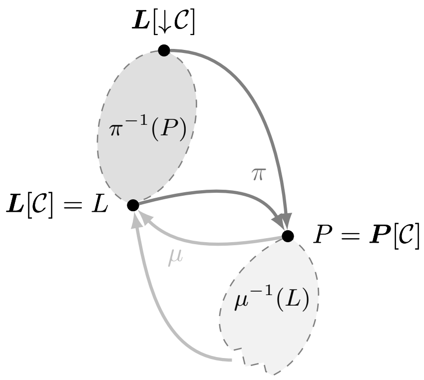

Fig. 4. Maps π and μ.

a SLO with two elements $a \leq \top$ such that $\Diamond a = \Diamond \top = \top$, and $\mathfrak{E}_1$ is in Fig. 1. Note that $\mu\big(\boldsymbol{P}[\mathfrak{S}]\big)$ is inconsistent, while $\mu^{-1}\big(\boldsymbol{L}[(1)]\big) = \big\{\boldsymbol{P}[(1)], \boldsymbol{P}[\mathfrak{E}_1]\big\}$. Slice 2 is much more involved. Its sublattice of Kripke complete logics comprises the SP-logics of the form $\boldsymbol{P}[\mathcal{C}]$, where $\mathcal{C}$ is a finite set of frames of depth ≤ 2 at least one of which is of depth 2. As shown in Section 3, modal logics $\boldsymbol{L}[\mathcal{C}]$ of slice 2 typically have infinitely many SP-companions in $\mu^{-1}(L)$; see Fig. 2. On the other hand, Example 5.4 shows SP-logics with multiple modal companions.

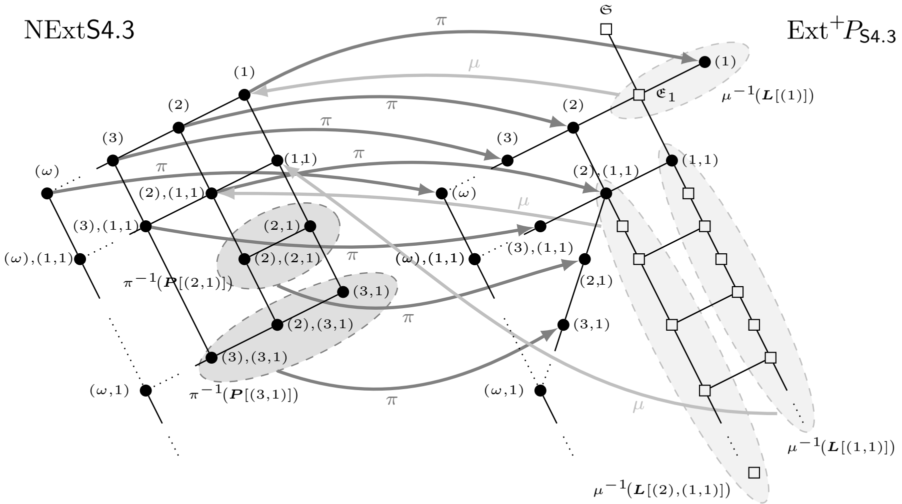

Fig. 5. Modal and SP-logics of slices 1 and 2.

6 SP-rules

An SP-rule, ϱ, takes the form $\frac{\boldsymbol{\iota}_1,\dots,\boldsymbol{\iota}_n}{\boldsymbol{\iota}}$, where $\boldsymbol{\iota}_1, \dots, \boldsymbol{\iota}_n, \boldsymbol{\iota}$ are SP-implications. We say that ϱ is *valid* in a SLO $\mathfrak{A}$ and write $\mathfrak{A} \models \varrho$ if $\mathfrak{A}$ satisfies the *quasi-*

equation $(\boldsymbol{\iota}_1^* \wedge \ldots \wedge \boldsymbol{\iota}_n^*) \to \boldsymbol{\iota}^*$, where $(\sigma \to \tau)^* = (\sigma \wedge \tau = \sigma)$. We identify the rule $\frac{\emptyset}{\boldsymbol{\iota}}$ with $\boldsymbol{\iota}$. Given a set R of SP-rules and an SP-rule ϱ, we set $R \models_{\mathsf{SLO}} \varrho$ if $\mathfrak{A} \models \varrho$ for any SLO $\mathfrak{A}$ validating every rule in R. We denote by $\boldsymbol{R}[\mathfrak{A}]$ the set of SP-rules that are valid in $\mathfrak{A}$ and, for a frame $\mathfrak{F}$, set $\boldsymbol{R}[\mathfrak{F}] = \boldsymbol{R}[\mathfrak{F}^*]$. For a class $\mathcal{C}$ of SLOs (or frames), we define the *SP-rule logic* $\boldsymbol{R}[\mathcal{C}]$ of $\mathcal{C}$ as the intersection of all $\boldsymbol{R}[\mathfrak{A}]$ with $\mathfrak{A} \in \mathcal{C}$. Clearly, there is a one-to-one correspondence between SP-rule logics and quasi-varieties of SLOs. The minimal SP-rule logic R containing $P_{\mathsf{S4.3}}$ and a set Φ of SP-rules is denoted by $P_{\mathsf{S4.3}}+\Phi$. We call R *globally complete* in case $\varrho \in R$ iff $\varrho \in \boldsymbol{R}[\mathfrak{F}]$ for any frame $\mathfrak{F}$ validating all rules in R. If Φ is a set of SP-implications, then $P_{\mathsf{S4.3}} + \Phi$ is complete whenever it is globally complete. However, the converse only holds for two non-trivial SP-logics containing $P_{\mathsf{S4.3}}$:

Theorem 6.1 *A non-trivial $P \in \mathrm{Ext}^+ P_{\mathsf{S4.3}}$ is globally complete iff P is complex iff $P \in \{P_{\mathsf{S5}}, \boldsymbol{P}[(1)]\}$.*

Proof. The first equivalence is shown in [16]. For the second one, it is known from [16] that P_{S5} and $\boldsymbol{P}[(1)]$ are complex. Conversely, let $P \notin \{P_{\mathsf{S5}}, \boldsymbol{P}[(1)]\}$. Suppose first that $(1,1) \in \mathsf{Fr}(P)$. Then the SLO $\mathfrak{C}_1$ in Fig. 1 validates P but there is no frame $\mathfrak{F} \in \mathsf{Fr}(P_{\mathsf{S4.3}})$ such that $\mathfrak{C}_1$ is embeddable into $\mathfrak{F}^*$. Thus, P is not complex. Now suppose $(1,1) \notin \mathsf{Fr}(P)$. Then $P \supset P_{\mathsf{S5}}$ and there exists a minimal $n > 2$ such that $(n) \notin \mathsf{Fr}(P)$. Then the SLO $\mathfrak{E}_n$ in Fig. 1 validates P but there is no frame $\mathfrak{F} \in \mathsf{Fr}(P)$ such that $\mathfrak{E}_n$ is embeddable into $\mathfrak{F}^*$. ☐

It follows that the axiomatisations of SP-logics given above do not provide axiomatisations of the corresponding SP-rule logics, except for two cases. This is in sharp contrast to normal modal logics containing $\mathsf{S4.3}$ where the introduction of rules does not extend the expressive power of formulas as any rule $\varrho = \frac{\boldsymbol{\iota}_1,\ldots,\boldsymbol{\iota}_n}{\boldsymbol{\iota}}$ can be equivalently expressed by the formula $\Box(\boldsymbol{\iota}_1 \wedge \cdots \wedge \boldsymbol{\iota}_n) \to \boldsymbol{\iota}$. We next observe that SP-rules have sufficient expressive power to define any modally definable class of $\mathsf{S4.3}$-frames:

Theorem 6.2 *For every finite $\mathsf{S4.3}$-frame $\mathfrak{F}$, there is an SP-rule $\varrho(\mathfrak{F})$ such that, for any $\mathsf{S4.3}$-frame $\mathfrak{G}$, we have $\mathfrak{G} \not\models \varrho(\mathfrak{F})$ iff $\mathfrak{F}$ is a p-morphic image of $\mathfrak{G}$.*

Proof. Let $\mathfrak{F} = (W, R)$ be a finite $\mathsf{S4.3}$-frame with root r. For each $x \in W$, take a variable p_x and the following set $\Delta_{\mathfrak{F}}$ of SP-implications: $p_x \to \Diamond p_y$ if $R(x,y)$, $p_x \wedge \Diamond p_y \to q$ if $\neg R(x,y)$, $p_x \wedge p_y \to q$ for $x \neq y$, $\top \to \Diamond p_y$ for y in the final cluster of $\mathfrak{F}$. Let

$$\varrho(\mathfrak{F}) = \frac{\Delta_{\mathfrak{F}}}{p_r \to \Diamond q}.$$

It is straightforward to show that $\varrho(\mathfrak{F})$ is as required. ☐

Corollary 6.3 *A class of frames is modally definable iff it is SP-rule definable.*

We now return to the axiomatisation problem for SP-rule logics. The SLOs in Fig. 1 show that the rules from Theorem 6.2 do not axiomatise globally complete SP-rule logics. Here, we give an axiomatisation of $\boldsymbol{R}[(2)]$, which can

Fig. 6. The SLOs $\mathfrak{A}_3$ and $\mathfrak{A}_4$.

be easily generalised to any $\boldsymbol{R}[(n)]$ with $n > 2$. Consider the rules

$$\varrho^1 = \frac{(a \to \Diamond c_i), \quad (a \wedge \Diamond(c_i \wedge c_j) \to b),\ 1 \le i \ne j \le 3}{a \to b}$$

$$\varrho^2 = \frac{(c_1 \to b), (c_2 \to b), (a \to \Diamond c_1), (a \to \Diamond c_2), (a \wedge \Diamond(c_1 \wedge c_2) \to b)}{a \to b}$$

$$\varrho^3 = \frac{(a \to \Diamond c_1), (a \to \Diamond c_2), (c_1 \to b), (a \wedge \Diamond(c_1 \wedge c_2) \to b), (a \wedge \Diamond(a \wedge c_2) \to b)}{a \to b}$$

It is easy to see that a cluster (n) validates all the ϱ^i iff $n \le 2$. Thus, restricted to the set of clusters, each rule ϱ^i defines the intended class of frames. Let $R_{(2)} = P_{\mathsf{S5}} + \{\varrho^1, \varrho^2, \varrho^3\}$.

Theorem 6.4 (i) $R_{(2)} = \boldsymbol{R}[(2)]$. (ii) $P_{\mathsf{S5}} + \Phi \ne \boldsymbol{R}[(2)]$, *for any proper subset* Φ *of* $\{\varrho^1, \varrho^2, \varrho^3\}$.

Proof. For the proof of (ii), take $\mathfrak{E}_3$ in Fig. 1, and $\mathfrak{A}_3$ and $\mathfrak{A}_4$ in Fig. 6. Then $\mathfrak{E}_3, \mathfrak{A}_3, \mathfrak{A}_4$ all validate P_{S5} and

- $\mathfrak{E}_3 \not\models \varrho^1$, but $\mathfrak{E}_3 \models \varrho^2, \varrho^3$;
- $\mathfrak{A}_3 \not\models \varrho^2$, but $\mathfrak{A}_3 \models \varrho^1, \varrho^3$;
- $\mathfrak{A}_4 \not\models \varrho^3$, but $\mathfrak{A}_4 \models \varrho^1, \varrho^2$.

For (i), it suffices to provide an embedding of any SLO $\mathfrak{A}$ with $\mathfrak{A} \models R_{(2)}$ into $\mathfrak{F}^*$, for a union $\mathfrak{F}$ of two-element clusters. Recall that a *filter* F in $\mathfrak{A}$ is a subset of A containing $\top$ and such that $a \in F$ and $a \le b$ imply $b \in F$ and $a, b \in F$ imply $a \wedge b \in F$. For a filter F in $\mathfrak{A}$, we set $\Diamond F = \{\Diamond a \mid a \in F\}$ and define $\mathfrak{F} = (W, R)$ using a set $\mathcal{X}$ of pairs of filters in $\mathfrak{A}$. Let $(F_1, F_2) \in \mathcal{X}$ if F_1, F_2 are filters in $\mathfrak{A}$ such that

- $\Diamond F_1 \subseteq F_2$ and $\Diamond F_2 \subseteq F_1$;
- if $F_1 \subseteq \Diamond F'$ or $F_2 \subseteq \Diamond F'$ for a filter F', then $F_1 \supseteq F'$ or $F_2 \supseteq F'$.

For any $w = (F_1, F_2) \in \mathcal{X}$, take fresh 1_w and 2_w and set

$$W = \{1_w, 2_w \mid w \in \mathcal{X}\}, \quad R = \{(1_w, 2_w), (1_w, 1_w), (2_w, 2_w), (2_w, 1_w) \mid w \in \mathcal{X}\}.$$

By definition, $\mathfrak{F} = (W, R)$ is a union of two-element clusters. We show that $f(a) = \{1_{F_1,F_2} \mid a \in F_1\} \cup \{2_{F_1,F_2} \mid a \in F_2\}$ is an embedding of $\mathfrak{A}$ into $\mathfrak{F}^*$. We first show that if $a \neq b$, then $f(a) \neq f(b)$. We may assume that $a \not\leq b$. Let F_0 be a maximal filter containing a such that $b \notin F_0$. We show that there exists a pair $(F_1, F_2) \in \mathcal{X}$ with $a \in F_1$ and $b \notin F_1$.

Let $\mathcal{M}$ be the set of all maximal filters G in $\mathfrak{A}$ such that $\Diamond G \subseteq F_0$ and $\Diamond F_0 \subseteq G$. Observe that there exists a filter $G \in \mathcal{M}$ containing a set $X \subseteq A$ iff $\Diamond(a_1 \wedge \cdots \wedge a_n) \in F_0$ for all $a_1, \ldots, a_n \in X$. It follows that, for every filter F' in $\mathfrak{A}$ with $F_0 \subseteq \Diamond F'$, there exists $G \in \mathcal{M}$ with $F' \subseteq G$. We now make a case distinction according to the cardinality of $\mathcal{M}$.

- $|\mathcal{M}| = 1$. Let $\mathcal{M} = \{G\}$. Then $G \supseteq F_0$ and $(F_0, G) \in \mathcal{X}$. We have $a \in F_0$ and $b \notin F_0$, as required.
- $|\mathcal{M}| = 2$. Let $\mathcal{M} = \{G_0, G_1\}$. We distinguish between two cases:
  Case 1: $G_0, G_1 \supseteq F_0$. Since neither $G_0 \subseteq G_1$ nor $G_1 \subseteq G_0$, we obtain $G_0 \neq F_0$ and $G_1 \neq F_0$. Thus, $b \in G_0 \cap G_1$. Then we find $c_1' \in G_0$ and $c_2' \in G_1$ and $a' \in F_0$ such that $a' \wedge \Diamond(c_1' \wedge c_2') \leq b$. Then, using the condition that filters are closed under $\wedge$, we find $c_1 \in G_0$, $c_2 \in G_1$, and $a \in F_0$ such that $c_1 \leq b$, $c_2 \leq b$, $a \leq \Diamond c_1$, $a \leq \Diamond c_2$, and $a \wedge \Diamond(c_1 \wedge c_2) \leq b$. By rule ϱ^2, $a \leq b$, and we have derived a contradiction as $b \in F_0$ follows.
  Case 2: $G_1 \not\supseteq F_0$. Then $G_0 \supseteq F_0$ as there exists at least one filter $G \in \mathcal{M}$ with $G \supseteq F_0$. Assume first that $G_0 \neq F_0$. Then $b \in G_0$. Similarly to Case 1 we thus find $c_1 \in G_0$, $c_2 \in G_1$, and $a \in F_0$ such that $a \leq \Diamond c_1$, $a \leq \Diamond c_2$, $c_1 \leq b$, $a \wedge \Diamond(c_1 \wedge c_2) \leq b$, and $a \wedge \Diamond(a \wedge c_2) \leq b$. But then, by rule ϱ^3, $a \leq b$, and we have derived a contradiction. Assume now that $G_0 = F_0$. Then $(F_0, G_1) \in \mathcal{X}$. We are done as $a \in F_0$ and $b \notin F_0$.
- $|\mathcal{M}| \geq 3$. Let $G_0, G_1, G_2 \in \mathcal{M}$. Then we find $a \in F_0$, $c_1 \in G_0$, $c_2 \in G_1$, and $c_3 \in G_2$ such that $a \leq \Diamond c_i$ for $1 \leq i \leq 3$ and $a \wedge \Diamond(c_i \wedge c_j) \leq b$ for $1 \leq i \neq j \leq 3$. But then $a \leq b$, by rule ϱ^1, and we have derived a contradiction.

We next show that $f(\Diamond a) = \Diamond f(a)$ for all $a \in A$. Suppose first $1_{F_1,F_2} \in \Diamond f(a)$. Then $a \in F_1$ or $a \in F_2$. Then $\Diamond a \in F_1$. Then $1_{F_1,F_2} \in f(\Diamond a)$, as required. Conversely, suppose $1_{F_1,F_2} \in f(\Diamond a)$. Then $\Diamond a \in F_1$. Then there exists a filter F' with $a \in F'$ and $F_1 \subseteq \Diamond F'$ and $F' \subseteq \Diamond F_1$. By definition, $F_1 \supseteq F'$ or $F_2 \supseteq F'$. Thus $1_{F_1,F_2} \in \Diamond f(a)$. Finally, $f(a_1 \wedge a_2) = f(a_1) \cap f(a_2)$ can we proved in a straightforward way. $\square$

The computational complexity of deciding $\boldsymbol{R}[\mathcal{C}]$ has been analysed in [17]. In contrast to SP-implications, deciding SP-rules is often coNP-hard. In fact, if $\mathcal{C}$ is a nonempty class of $\mathsf{S4.3}$-frames of the form $\mathsf{Fr}(P)$ for some SP-logic $P \in \mathrm{Ext}^+ P_{\mathsf{S4.3}}$, then $\boldsymbol{R}[\mathcal{C}]$ is in PTime iff $\mathcal{C}$ is the class of all clusters or a singleton cluster. Otherwise, $\boldsymbol{R}[\mathcal{C}]$ is coNP-complete.

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