Relaxed Stability Analysis of Fuzzy-Model-Based Control Systems

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Relaxed Stability Analysis of Fuzzy-Model-Based Control Systems

Thesis by
Yanbin Zhao

In Partial Fulfilment of the Requirements for the Degree of
Doctor of Philosophy in Robotics
at
King’s College London

Department of Informatics

May 2018
To
Li Shusen
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Abstract

This thesis presents and extrapolates on the research works concerning the stability analysis of fuzzy-model-based (FMB) control systems. In this study, two types of FMB control systems are considered: 1) Takagi-Sugeno (T-S) FMB control systems; and 2) polynomial fuzzy-model-based (PFMB) control systems. The control scheme illustrated in this thesis has great design flexibility because it allows the number and/or shape of membership functions of fuzzy controllers to be designed independently from the fuzzy models. However, in wake of the imperfectly matched membership functions, the stability conditions of the FMB control systems are typically very conservative given the fact that they are congruent with traditional stability analysis methods. In this thesis, based on Lyapunov stability theory, membership-function-dependent (MFD) stability analysis methods are proposed to relax the stability conditions. Firstly, piecewise membership functions (PMFs) are utilised as approximate membership functions to carry out a relaxed stability analysis of T-S FMB control system. Subsequently, PMF-based stability analysis is improved with the consideration of membership
function boundary information. Based on the PMF method, we propose a lower-upper-PFM-based stability analysis method. Relaxed stability conditions are obtained in the form of linear matrix inequalities (LMIs) in consideration of the approximation accuracy of the membership function.

For the purpose of stability analysis of PFMB control system, the other MFD method proposed is to extract the regional membership function information via operating domain partition. Two types of membership information are considered in each sub-domain: 1) the numerical relationship between all membership function overlap terms; and 2) the bounds of every single membership function overlap term. Thereafter, relaxed sum of squares (SOS)-based stability conditions are derived. In conjunction with these proposed MFD methods, sub-domain fuzzy controllers are utilised to enhance the capability of feedback compensation. In this thesis, all the LMI/SOS-based stability conditions obtained can be solved numerically using existing computational tools. Furthermore, simulation examples are provided to illustrate the validity and applicability of the proposed methods.
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# Acronyms

<table>
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<th>Description</th>
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<tbody>
<tr>
<td>FLS</td>
<td>Fuzzy logic system</td>
</tr>
<tr>
<td>FMB</td>
<td>Fuzzy-model-based</td>
</tr>
<tr>
<td>GA</td>
<td>Genetic algorithm</td>
</tr>
<tr>
<td>LMI</td>
<td>Linear matrix inequality</td>
</tr>
<tr>
<td>LQR</td>
<td>Linear quadratic regulator</td>
</tr>
<tr>
<td>MFD</td>
<td>Membership-function-dependent</td>
</tr>
<tr>
<td>MFI</td>
<td>Membership-function-independent</td>
</tr>
<tr>
<td>PDC</td>
<td>Parallel distributed compensation</td>
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<tr>
<td>PFMB</td>
<td>Polynomial fuzzy-model-based</td>
</tr>
<tr>
<td>PMF</td>
<td>Piecewise membership function</td>
</tr>
<tr>
<td>PID</td>
<td>Proportional-integral-derivative</td>
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<tr>
<td>SOS</td>
<td>Sum of squares</td>
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<tr>
<td>T-S</td>
<td>Takagi-Sugeno</td>
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<tr>
<td>T-S-K</td>
<td>Takagi-Sugeno-Kang</td>
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Chapter 1

Introduction

1.1 Background

Fuzzy control has witnessed substantial development over the past four decades [1, 2]. In the pioneering work of Mamdani based on fuzzy set theory [3, 4], the heuristic rule-based controller was designed to mimic human operator actions in the control loop in order to form a fuzzy logic system (FLS) [5, 6]. As compared with the crisp control (conventional control), this control approach is a model-free design scheme. Put simply, the controller synthesis can be performed without involving the mathematical model of the plant. Therefore, it serves as a powerful alternative control approach for complex and ill-defined nonlinear plants. Furthermore, it is an expert system-like control scheme that can involve “expert knowledge” into the task-oriented controller synthesis at a very early design stage. For conventional feedback control, which is characterized by set point-oriented, most of the operator’s experience can only be considered at the system tuning stage [7]. In the wake of aforementioned advantages, model-free fuzzy control has been extensively investigated as a promising control approach. It has also been successfully applied to many applications, ranging from consumer electronics to industrial applications, which mostly involves highly nonlinear systems. More details of these applications can be referred to as series of surveys [1, 8]. Although model-free fuzzy control entails the notion that “human-in-loop” offers a number
of advantages and is popular in industry, it still suffers from several drawbacks, such as the limitation of the “expert knowledge” for noise/disturbances handling [7]. More importantly, the design process of the FLS is heuristic, but not in a theoretical way, which makes it lacks a general method necessary for the stability analysis [9]. This drawback implies that the control system designed does not have any guarantee of stability and faces the risk of an unsatisfied performance.

In the late 1980s, a new type of fuzzy model named Takagi-Sugeno (T-S)/Takagi-Sugeno-Kang (T-S-K) fuzzy model was introduced [10, 11]. It is a systematic mathematical tool that utilizes “IF-THEN” form of fuzzy rule with a linear function in the consequence to fuzzily describe a highly nonlinear system. This type of fuzzy model offers a nice theoretical analysis platform for the stability analysis and fuzzy controller synthesis. T-S fuzzy-model-based (FMB) control system was constructed based on the T-S fuzzy model, and linear matrix inequality (LMI)-based approach [12] was posited as a systematic design methodology for the fuzzy control of a class of nonlinear systems. Through the application of Lyapunov stability theory, the problem of stability analysis as well as controller synthesis for T-S FMB control system can be cast to LMI problems [13]. The obtained LMI problems can be resolved numerically through convex optimisation methods such as the interior point method [14]. Within the framework of nonlinear system control, in general, FMB control methodology requires less complicated mathematical knowledge; at the same time, the controller obtained has a simple structure when compared to the existing nonlinear control techniques such as feedback linearization [15]. Given the fact that the consequence of each fuzzy rule is a linear sub-system, many mature linear control techniques such as $H_\infty$ design, linear quadratic regulator (LQR) can be involved within the design process [16, 17]. With the development of FMB control techniques, in order to solve the design problems that cannot be represented in terms of LMIs, polynomial fuzzy-model-based (PFMB) control was introduced as an extension to T-S FMB control [18, 19]. Notably, a sum of squares (SOS)-based approach [20] was
proposed to carry out the stability analysis of PFMB control systems [18], such that the stability conditions can be represented in terms of SOS instead of LMIs. In contrast to the LMI-based approach for T-S FMB control, this SOS-based approach for PFMB control can yield more extensive and/or relaxed results [21].

As illustrated in Fig. 1.1, there are generally two areas for fuzzy control. Model-free fuzzy control is a heuristic, knowledge-based approach. FMB control was developed in wake of the growing interest of stability analysis for fuzzy control systems. Comprising of T-S FMB control approach and PFMB control approach, FMB control is a class of fuzzy control approaches representing the nonlinear dynamical system as fuzzy models with specific mathematical form, i.e., T-S/polynomial fuzzy model, and can be applied to achieve rigorous stability conditions whilst performing systematical systematical fuzzy controller design. In this thesis, theoretical research is conducted on the stability analysis and controller synthesis of continuous-time FMB control systems. Both T-S FMB control systems and PFMB control systems are investigated. Relaxed stability conditions are achieved and fuzzy controllers with more design flexibility are synthesized to stabilise the nonlinear plants. Simulation examples are given to demonstrate the viability of the proposed methods. More details of the FMB control are presented in the following sections.
1.2 Literature Review

FMB control has been developed as a systematic framework for fuzzy control of a class of uncertain nonlinear systems in the past decades. There exists a large body of literature on fuzzy modelling, fuzzy controller design, stability analysis, and applications. In Section 1.2.1, an introduction to FMB control system is given. Various fuzzy models and fuzzy controllers are presented. In Section 1.2.2, stability analysis of FMB control system is discussed including the basic design procedure. The sources of conservativeness are introduced and techniques used to relax the stability conditions are presented. In Section 1.2.3, other control problems rather than stabilisation are introduced briefly. The application areas so far for FMB control are presented.

1.2.1 FMB Control System

FMB control is a powerful nonlinear control approach. In the operating domain, the nonlinear dynamic system is transferred to a fuzzy model, which is the fuzzy blending of sub-systems. The fuzzy feedback controller is applied to control the fuzzy model and form the closed-loop FMB control system.

1.2.1.1 Fuzzy Models

In model-free fuzzy control, the control system is modelled by representing the human operation into a set of “IF-THEN” fuzzy rules by fuzzy logic. The membership functions of each fuzzy term are obtained by experts experience or the data of experiments [22]. The advantage of this fuzzy control method is that the “linguist control rule” can be converted into fuzzy control rules directly and naturally, which is quite easy to understand by controller designer. However, it suffers the disadvantages that it is more or less tailored for every particular case, that it lacks rigorous mathematics tools such as pole placement applied in the conventional control approaches.
In FMB control approach, the plant with strong nonlinearity can be represented as the fuzzy model with a special form, i.e., T-S/polynomial fuzzy model. T-S fuzzy model was proposed in [10], of which the consequence in the fuzzy rule is not the fuzzy number as that in the FLS, but a linear function. It belongs to a type of fuzzy systems which are expressed by fuzzy rules with singleton consequence. In [9], a T-S fuzzy system was classified as the type-III fuzzy system (it is worth noting here that the fuzzy model in FLS with singleton consequence in the fuzzy rule was classified as type-I and its simplified version with a real number in the consequence of fuzzy rule was classified as type-II). There are in general two methods to obtain the T-S fuzzy model for a nonlinear plant. One method is applying system identification algorithms [10, 11, 23] to construct the T-S fuzzy model from input-output data pairs. The other method is to construct the T-S fuzzy model based on the mathematical model of the nonlinear plant using the sector nonlinearity technique [12].

As an extension to T-S fuzzy model, polynomial fuzzy model is proposed to represent the nonlinear dynamic system with a polynomial sub-system as its consequence. One advantage of polynomial fuzzy model is that less number of fuzzy rules is needed than T-S fuzzy model [18]. Furthermore, as T-S fuzzy model is only the special form of polynomial fuzzy model, polynomial fuzzy model can be used to represent more complicated nonlinear dynamic systems. Polynomial fuzzy model also can be constructed using sector nonlinearity technique [18]. An alternative way to convert the nonlinear system into polynomial fuzzy model is Taylor series approach which is an extension to the sector nonlinearity technique [24].

1.2.1.2 Fuzzy Controllers

The basic FMB control system is constructed by full-state feedback control scheme. Under the so-called parallel distributed compensation (PDC) [12, 25] design, i.e., for each sub-system in the fuzzy model, a linear sub-controller is designed to com-
pensate the sub-system. In this case, fuzzy controller shares the same premise membership functions and the number of fuzzy rules with the fuzzy model. The closed-loop fuzzy system is the fuzzy blending of all the sub-closed-loop systems. The main advantage of the PDC design is its functional simplicity. However, it highly limits the design flexibility of the fuzzy controller. As the number and shape of the membership functions of fuzzy model are determined in the fuzzy modelling process, it means which could be a part of the design parameters, the membership functions of fuzzy controller are determined under PDC design. Furthermore, in the case of fuzzy model has a huge number of complicated membership functions, the fuzzy controller designed under PDC would be complicated which leads a high implementation cost. To remove these drawbacks, the fuzzy controller designed under imperfect premise matching was proposed [26,27]. The fuzzy controller does not share the premise membership functions and the number of fuzzy rules as those of the fuzzy model, which offers a greater design flexibility to the fuzzy controller. As an extension, sub-domain fuzzy controllers designed under imperfect premise matching are proposed [28,29] to obtain more relaxed stability conditions. Various control schemes are also proposed such as a nonlinear feedback control law is applied in local models [30], control Lyapunov function design [31], switching fuzzy controller [32] and nonquadratic stabilisation design [33,34].

1.2.2 Relaxation of Stability Conditions

Stability is the fundamental of the performance and reliability of the control system. This research is concerned with the stability of equilibrium points for nonlinear dynamic system represented as FMB control system. The procedure of FMB control system design is shown in Fig. 1.2. For FMB control, the main tool is Lyapunov stability theory, some other techniques such as invariant set method [35] can also be found in the literature. The stability analysis framework for T-S FMB control system is LMI-based method [12]. Based on Lyapunov
1.2. LITERATURE REVIEW

stability theory, the stability conditions are recast into LMIs and solved numerically by convex optimisation tools. The basic stability conditions obtained for T-S FMB control system were given in [12] and more relaxed stability conditions were derived in [36] and [37]. As an extension to LMI-based approach, SOS-based method [18] is proposed for PFMB control system analysis. The stability conditions are cast into SOS problem and can be solved numerically. It should be mentioned that the LMI-based approach is only a special case of the SOS-based method. The stability conditions obtained using the SOS-based method are more relaxed than using the LMI-based approach [18].

There are several sources of conservativeness of T-S FMB fuzzy system and PFMB fuzzy system within the above stability analysis framework [38]. During the FMB control system design, the conservativeness leads to infeasible solutions which means the feedback gains cannot be obtained. Therefore, the applicability of FMB control scheme is restricted. Three main sources of conservativeness are discussed in the following paragraphs.
1.2. LITERATURE REVIEW

1.2.2.1 Form of Lyapunov Functions

One of the sources of conservativeness is the form of Lyapunov function candidates. The quadratic Lyapunov function and its first order derivative are simple, so they are extensively utilised in the stability analysis [12, 36, 37, 39, 40]. The main drawback of these quadratic stability analysis methods is that it leads to potentially conservative results, compared with the more sophisticated Lyapunov function candidates. In the literature, various methods using more complicated Lyapunov functions are proposed to release the conservativeness, such as piecewise Lyapunov function [32, 41–43], switching Lyapunov function [44], switching polynomial Lyapunov function [45], fuzzy Lyapunov function [46–51].

1.2.2.2 Positivity Conditions for Fuzzy Summations

As Lyapunov stability theory is used in the analysis and design of FMB control system, the stability conditions are derived from the positivity conditions for fuzzy summations, which is the other source of conservativeness. The quadratic stability conditions obtained in [36, 37] are more relaxed than those in [12] by improving the positivity conditions for fuzzy summations mathematically. In [52], a method is proposed to extend the idea from [37] by using the instinct feature of membership functions (the sum of all membership functions equals to one) to further relax the stability conditions. The asymptotically necessary and sufficient conditions for stability is derived by using the Pólya’s theorem [53, 54].

1.2.2.3 Membership Function Shape Information

FMB control also suffers conservativeness from the partial use of the membership function shape information. Taking LMI-based method in [16] as the example, stability conditions of T-S FMB control system are derived based on proving positiveness (or negativeness) of a so-called double fuzzy summations. Simplest sufficient conditions for positivity of this double fuzzy summations are the positiveness of the matrix parts, which can be recast as an LMI problem. Further
relaxed stability conditions are introduced in [25] by considering membership functions using the property of positiveness of sum of squares. However, these results are still very conservative due to the lack of consideration of the shape of the membership functions. Without taking membership function shape information into stability analysis, the stability conditions derived are for a family of models not for the original nonlinear model designed specifically. Therefore, by using membership function shape information to “narrow” the family of models under design, less conservative stability conditions can be obtained. Following with the above idea, various techniques which are classed as membership-function-dependent (MFD) analysis, using membership function shape information to relax stability conditions are proposed. As an extension to [40] which handles the uncertainty in the membership functions of the fuzzy control systems, the shape of membership functions for both fuzzy model and controller is considered and this information is represented as a group of affine inequalities in [39]. For the purpose of relaxing positivity conditions for fuzzy summations, the membership function shape information is considered in terms of the bounds of each single membership function and overlap term (products of two membership functions) in [55]. A more general form of these aforementioned constraints are proposed in [56], with the form of a group of inequalities with second-order polynomial in the terms of membership functions in the left-hand side. All the inequalities representing membership function shape information are involved in stability analysis using $S$-procedure [13]. For the purpose of extracting more membership function shape information, membership functions of FMB control systems are approximated as piecewise linear membership functions [57], polynomial membership functions [58] and staircase membership functions [59,60]. MFD methods also applied combining with various nonquadratic analysis such as in [42], the operation domain was partitioned to extract regional membership function shape information. Based on the partitions, piecewise Lyapunov functions were proposed for stability analysis.
1.2.3 Other Control Problems and Application Areas

FMB control has been applied to solve various control problems such as regulation [61,62], tracking control [63–65], output feedback control [66–69], robustness design [70–72], fuzzy observer-based control [73–75]. In addition, FMB control has been extended to various control approaches such as adaptive fuzzy control [1, 76–80], sampled-data fuzzy control [81–86], time-delay fuzzy control [87–89], etc. Furthermore, as an extension to the FMB control systems introduced above, Type-2 FMB control systems have been proposed to tackle the uncertainty of the conventional FMB control systems [90–94].

For the application of FMB control, variety of control systems have been introduced in publications, such as inverted pendulum system [95], bolt-tightening tool [91], DC-DC Converters [96], micro helicopter control [97] and vehicle suspension systems [98].

1.3 Objectives and Organization

The main objective of this research study is to improve the applicability of FMB control technique by relaxing the stability conditions whilst considering the membership function information. The FMB control systems investigated in this thesis have greater flexibility than the ones designed under the PDC method, and the fuzzy controller applied has more compensation capability based on the operating domain partition. The details include:

1) Relax stability conditions by approximated membership functions for T-S FMB control systems while taking computational burden into consideration.

2) Extracting regional and global membership function information to perform the stability analysis for PFMB control system. A numerical method is introduced for the purpose of regional information search.

3) All the T-S FMB control systems and PFMB control systems studies are
designed under imperfect premise matching. In addition, the sub-domain fuzzy controllers are utilised to enhance the feedback compensation capability of the control scheme.

Based on these objectives, this thesis comprises of two main chapters as shown in Fig. 1.3. The relaxation of stability conditions for T-S FMB control systems as well as PFMB control systems are discussed in these chapters. Under Section 3.3, the T-S FMB control system is constructed using sub-domain T-S fuzzy controllers. They are further generalised into sub-domain polynomial fuzzy controllers in Chapter 4. The remainder of this thesis is organized in the following manner:

- In Chapter 2, the fuzzy modelling method named sector nonlinearity is first introduced. This is followed by the description of the basic forms of T-S FMB control systems and PFMB control systems. Next, Lyapunov stability theory and LMI/SOS-based stability conditions are introduced. Finally, some definitions and lemmas are mentioned that will be extrapolated on in the subsequent chapters.

- In Chapter 3, the relaxation of stability conditions of T-S FMB control systems is proposed. The MFD stability analysis methods, including PMF-based and lower-upper-PMF-based methods, are explained. The latter part
of this chapter includes the introduction of the sub-domain T-S fuzzy controllers. The T-S FMB control systems studied are formed with this type of fuzzy controller, and the PMF-based stability analysis is then undertaken.

• In Chapter 4, the stability of PFMB control system is investigated in the consideration of the regional membership function shape information. Based on operating domain partition, the boundaries of each single membership function overlap term. Meanwhile the numerical relationship existing between all the membership function overlap terms are extracted and incorporated into stability analysis. A search algorithm based on genetic algorithm (GA) is proposed for implementation during the last part of this chapter.

• Chapter 5 consists of conclusion in addition to the prospects of future research works.
Chapter 2

Preliminary

2.1 Introduction

The subject of fuzzy control in this thesis deals with the stability analysis and fuzzy controller synthesis of nonlinear control systems, i.e., control systems containing at least one nonlinear component such as sinusoid term. These control systems appear in many real-world applications that the nonlinear terms are intrinsic to the nature of the system. Controlling this class of nonlinear systems is challenging, that many techniques developed such as feedback linearization [99] are rather involved. With the emergence of a powerful mathematical tool named T-S fuzzy model [10, 11], FMB control approach has been extensively investigated as an easy-to-understood nonlinear control methodology [100]. The feature of T-S fuzzy model is that it can be utilised to represent the nonlinear plants as an average weighted sum (fuzzy blending) of several linear sub-systems. As the nonlinearity of the plant is extracted and involved into the membership functions, it leaves the linearity in the linear sub-systems which can be handled by many mature linear control techniques for stability analysis and controller synthesis [38]. Various fuzzy control schemes (e.g., [66, 67, 101]) can be employed to form the closed loop system. In this research, we only focus on two state-feedback fuzzy control schemes: PDC design and imperfect premise matching. Based on Lyapunov stability theory, the stability analysis and control design problems are
2.2. NOTATION AND CONVENTIONS

reduced into LMI problems for T-S FMB control and SOS problems for PFMB control. The feasible solutions of these LMI/SOS problems can be solved by the corresponding convex optimisation methods [14,102].

In this chapter, we will provide the basic definitions and concepts as well as mathematical tools, which support the research works in this thesis. In Section 2.2, the notations and conventions are given. In Section 2.3, sector nonlinearity as a basic mathematical tool for constructing fuzzy model based on the mathematical model of the nonlinear plant is introduced briefly. In Section 2.4 and Section 2.5, T-S FMB control system and PFMB control system, of both, are designed under PDC scheme are presented respectively. In Section 2.6, Lyapunov stability theory is given with geometric interpretation. In Section 2.7, LMI/SOS-based stability conditions are presented to briefly illustrate the rough outline of the basic LMI/SOS-based methods. To make this thesis to be self-contained, Section 2.8 gives several definitions, propositions and useful lemmas. It is followed by concluding remarks in Section 2.9.

2.2 Notation and Conventions

The notation in this thesis is fairly standard. The expressions of $M > 0$, $M \geq 0$, $M < 0$ and $M \leq 0$ denote the positive, semi-positive, negative and semi-negative definite matrices $M$, respectively. $M > N$ means that $M - N > 0$ and $M = 0$ means $M$ is a zero matrix. The superscript “$T$" represents the transpose. The superscript “$-$” represents the inverse. “$\forall$” is to mean “for any”, “$\exists$” is for “there exists”, “$\in$” is for “in the set” and “$\Rightarrow$” is for “implies that”.

The following notations are adopted [20]. A monomial in $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$ is a function of the form $x_1^{d_1}(t)x_2^{d_2}(t) \cdots x_n^{d_n}(t)$, where $d_i \geq 0$, $i = 1, 2, \ldots, n$, are integers. The degree of a monomial is $d = \sum_{i=1}^{n} d_i$. A polynomial $p(x(t))$ is a finite linear combination of monomials with real coefficients. A polynomial $p(x(t))$ is an SOS if it can be written as $p(x(t)) = \sum_{j=1}^{m} q_j(x(t))^2$, where $q_j(x(t))$ is a polynomial and $m$ is a nonnegative integer. It can be concluded that if $p(x(t))$
2.3 Sector Nonlinearity Technique

Consider the following nonlinear dynamic system to be controlled,

\[ \dot{x}(t) = f(x(t), u(t)) \] (2.1)

where \( f(\cdot) \) is a smooth nonlinear function. \( x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \) is the state vector, and \( u(t) = [u_1(t), u_2(t), \ldots, u_m(t)]^T \) is the input vector. \((x(t) = 0, u(t) = 0)\) is an equilibrium point. \( f(x(t), u(t)) \) fulfils the required Lipschitz conditions for existence and uniqueness of the solution.

This nonlinear dynamic system can be represented as an FMB control system by using the sector-nonlinearity technique [16]. The idea is illustrated as follows.

For a scalar nonlinear dynamic system \( \dot{x}(t) = f(x(t)) \), a local sector can be found for the nonlinearity noted as \( \psi(\cdot) \) in \( \dot{x}(t) = f(x(t)) \) such that

\[ \alpha x(t)^2 \leq x(t) \psi(x(t)) \leq \beta x(t)^2, \quad \forall x(t) \in [a, b] \] (2.2)

in which \( \alpha, \beta, a \) and \( b \) are constants with \( \beta > \alpha \) and \( a < 0 < b \). If (2.2) holds for all \( x(t) \in (-\infty, +\infty) \), (2.2) obtained is called global sector condition [15]. Fig 2.1 shows the global sector nonlinearity and local sector nonlinearity.

After finding the sector (time \( t \) is omitted in the following), the nonlinearity \( \psi(x) \) considered can be represented as a fuzzy model in the form of “IF-THEN” such as

Rule 1: IF \( \psi(x) \) is around \( \alpha x \),

THEN \( \psi(x) = \alpha x \),
2.3. SECTOR NONLINEARITY TECHNIQUE

Rule 2 : IF $\psi(x)$ is around $\beta x$,

THEN $\psi(x) = \beta x$.

It should be mentioned that in this case as $\beta > \alpha$, the premise of the first rule can also be represented as “IF $\psi(x)$ is SMALL” and the premise of the second rule as “IF $\psi(x)$ is BIG”. The nonlinearity $\psi(x)$ can be represented as

$$\psi(x) = \mu_{M_1^1}(x)\alpha x + \mu_{M_1^2}(x)\beta x,$$

$$\mu_{M_1^1}(x) + \mu_{M_1^2}(x) = 1,$$

where $\mu_{M_1^1}(x)$ and $\mu_{M_1^2}(x)$ are the membership functions corresponding to the fuzzy terms $M_1^1$ and $M_1^2$, respectively. In this case, the fuzzy terms $M_1^1$ and $M_1^2$ are “around $\alpha x$ ” (or “SMALL”) and “around $\beta x$” (or “BIG”), respectively.
Therefore, we can obtain
\[
\mu_{M_1}(x) = \frac{\beta x - \psi(x)}{\beta x - \alpha x}, \quad \mu_{M_2}(x) = 1 - \mu_{M_1}(x).
\]

Using sector-nonlinearity technique for every nonlinearity in the nonlinear control system (2.1), a fuzzy model can be constructed. The number of fuzzy rules is \(2^l\), in which \(l\) is the number of the nonlinearities in (2.1). It should be mentioned that the way of choosing nonlinearities is not unique, that means the fuzzy model is not unique [24]. Some numerical examples of sector-nonlinearity technique can be referred in [15,16]. An extension to sector-nonlinearity technique is proposed in [24] using Taylor series expansion where the detail of this technique is omitted here.

### 2.4 T-S FMB Control System

Feedback control is a powerful tool to handle the uncertainties in dynamic systems and to compensate their dynamic behaviour. FMB control system is a state variable feedback system. As it is shown in Fig 2.2, it comprises a fuzzy model representing a nonlinear plant and a fuzzy controller connected in a closed loop, where \(x(t)\) is the state vector and \(u(t)\) is the control input. The objective of this thesis is to investigate its system stability. The fuzzy controller designed based on the stability conditions, is able to drive the system state \(x(t)\) to zero when time \(t\) tends to \(\infty\) (asymptotic stability).

![Figure 2.2: A simplified block diagram of FMB control systems.](image-url)
2.4. T-S FMB CONTROL SYSTEM

2.4.1 T-S Fuzzy Model

T-S FMB control system [12] is one of the most promising research topics in fuzzy control community over the past decades [100]. T-S fuzzy model has very strong expression capability in representing the nonlinear dynamic system. The consequence of each IF-THEN fuzzy rule is a linear control system. The overall fuzzy model is a fuzzy blending of the linear control systems.

It is assumed that the nonlinear plant (2.1) can be represented by a \( p \)-rule T-S fuzzy model [12] where the \( i \)-th rule is shown below:

\[
\text{Rule } i: \text{ IF } f_1(x(t)) \text{ is } M_1^i \text{ AND } \cdots \text{ AND } f_\Psi(x(t)) \text{ is } M_\Psi^i, \]

\[
\text{THEN } \dot{x}(t) = A_i x(t) + B_i u(t)
\]

where \( M_\alpha^i \) is a fuzzy term of rule \( i \) corresponding to the function \( f_\alpha(x(t)), \alpha = 1, 2, \ldots, \Psi, \Psi \) is a positive integer; \( i = 1, 2, \ldots, p; x(t) \in \mathbb{R}^n \) is the system state vector; \( u(t) \in \mathbb{R}^m \) is the input vector; \( A_i \in \mathbb{R}^{n \times n} \) and \( B_i \in \mathbb{R}^{n \times m} \) are the known system and input matrices, respectively. The system dynamics is defined as follows:

\[
\dot{x}(t) = \sum_{i=1}^{p} w_i(x(t))(A_i x(t) + B_i u(t)), \quad (2.3)
\]

where

\[
w_i(x(t)) \geq 0 \quad \forall \; i, \quad \sum_{i=1}^{p} w_i(x(t)) = 1, \quad (2.4)
\]

\[
w_i(x(t)) = \frac{\prod_{l=1}^{\Psi} \mu_{M_l^i}(f_l(x(t)))}{\sum_{k=1}^{p} \prod_{l=1}^{\Psi} \mu_{M_l^k}(f_l(x(t)))} \quad \forall \; i, \quad (2.5)
\]

\( w_i(x(t)), \; i = 1, 2, \ldots, p, \) are the normalised grades of membership and \( \mu_{M_\alpha^i}(f_\alpha(x(t))), \alpha = 1, 2, \ldots, \Psi, \) are the membership functions corresponding to the fuzzy term \( M_\alpha^i. \)
2.4.2 T-S Fuzzy Controller

Under PDC design scheme [12], a $p$-rule fuzzy controller of the following format is employed to stabilise the nonlinear plant represented by the T-S fuzzy model (2.3). The $j$-th rule of the fuzzy controller is shown below:

Rule $j$: IF $f_1(x(t))$ is $M_1^j$ AND $\cdots$ AND $f_p(x(t))$ is $M_p^j$,

THEN $u(t) = G_j x(t)$

where $G_j \in \mathbb{R}^{m \times n}$, $j = 1, 2, \ldots, p$, is the constant feedback gain in rule $j$.

The fuzzy controller is defined as follows:

$$u(t) = \sum_{j=1}^{p} w_j(x(t)) G_j x(t),$$

(2.6)

where $w_j(x(t))$, $j = 1, 2, \ldots, p$, are the membership functions.

Remark 1 It should be noted that, T-S fuzzy controller presented here is designed under PDC scheme. It shares the same number of fuzzy rules $p$ and the same membership functions $w_j(x(t))$, $j = 1, 2, \ldots, p$, with the T-S fuzzy model (2.3).

2.4.3 T-S FMB Control System

From (2.3) and (2.6), the T-S FMB control system is obtained as follows:

$$\dot{x}(t) = \sum_{i=1}^{p} w_i(x(t))(A_i x(t) + B_i \sum_{j=1}^{p} w_j(x(t)) G_j x(t))$$

$$= \sum_{i=1}^{p} \sum_{j=1}^{p} w_i(x(t)) w_j(x(t))(A_i + B_i G_j)x(t).$$

(2.7)

2.5 PFMB Control System

As an extension to T-S FMB control system, PFMB control system is also a state feedback control system but composing with a polynomial fuzzy model representing the nonlinear plant and a polynomial fuzzy controller [18].
2.5.1 Polynomial Fuzzy Model

As an extension to T-S fuzzy model, polynomial fuzzy model has polynomial sub-system as consequence part in every IF-THEN fuzzy rule.

It is assumed that the nonlinear plant (2.1) can be represented by a \( p \)-rule polynomial fuzzy model where the \( i \)-th rule is shown below:

Rule \( i \): IF \( f_1(x(t)) \) is \( M_1^i \) AND \( \ldots \) AND \( f_\Psi(x(t)) \) is \( M_\Psi^i \)

\[
\dot{x}(t) = A_i(x(t))\dot{x}(x(t)) + B_i(x(t))u(t) \tag{2.8}
\]

where \( M_\alpha^i \) is a fuzzy term of rule \( i \) corresponding to the function \( f_\alpha(x(t)) \), \( \alpha = 1, 2, \ldots, \Psi \); \( i = 1, 2, \ldots, p \); \( x(t) \in \mathbb{R}^n \) is the system state vector; \( \dot{x}(x(t)) \in \mathbb{R}^n \) is a vector of monomials in \( x(t) \), in which a monomial in \( x(t) \) is in the form \( x_1^{d_1}x_2^{d_2}\cdots x_n^{d_n} \), \( d_1, d_2, \ldots, d_n \) are nonnegative integers, the degree of this monomial is \( d = \sum_{i=1}^n d_i \); \( u(t) \in \mathbb{R}^m \) is the input vector; \( A_i(x(t)) \in \mathbb{R}^{n\times N} \) and \( B_i(x(t)) \in \mathbb{R}^{n\times m} \) are the known polynomial system and input matrices, respectively. It is assumed that \( \dot{x}(x(t)) = 0 \) iff \( x(t) = 0 \). The system dynamics is described as

\[
\dot{x}(t) = \sum_{i=1}^p w_i(x(t))(A_i(x(t))\dot{x}(x(t)) + B_i(x(t))u(t)), \tag{2.9}
\]

where \( w_i(x(t)) \), \( i = 1, 2, \ldots, p \), are the membership functions.

2.5.2 Polynomial Fuzzy Controller

Under PDC design scheme [18], a \( p \)-rule fuzzy controller of the following format is employed to stabilise the nonlinear plant represented by the polynomial fuzzy model (2.9). The \( j \)-th rule of the fuzzy controller is shown below:

Rule \( j \): IF \( f_1(x(t)) \) is \( M_1^j \) AND \( \ldots \) AND \( f_\Psi(x(t)) \) is \( M_\Psi^j \)

\[
u(t) = G_j(x(t))\dot{x}(x(t)),
\]
where $G_j(x(t)) \in \mathbb{R}^{m \times N}$, $j = 1, 2, \ldots, p$, is the polynomial feedback gain in rule $j$. The fuzzy controller is defined as follows:

$$u(t) = \sum_{j=1}^{p} w_j(x(t))G_j(x(t))\dot{x}(x(t)).$$

(2.10)

**Remark 2** It should be noted that, polynomial fuzzy controller presented here is designed under PDC scheme. It shares the same number of fuzzy rules $p$ and the same membership functions $w_j(x(t))$, $j = 1, 2, \ldots, p$, with the polynomial fuzzy model (2.9).

**Remark 3** The polynomial fuzzy controller (2.10) is reduced to the T-S fuzzy controller (2.6) when the feedback gains $G_j(x(t))$ become constant matrices, i.e., $G_j$ for all $j$, $j = 1, 2, \ldots, p$.

### 2.5.3 PFMB Control System

The PFMB control system formed by the polynomial fuzzy model (2.9) and the polynomial fuzzy controller (2.10) is obtained as follows:

$$\dot{x}(t) = \sum_{i=1}^{p} w_i(x(t))(A_i(x(t))\dot{x}(x(t)) + B_i(x(t))\sum_{j=1}^{p} w_j(x(t))G_j(x(t))\dot{x}(x(t)))$$

$$= \sum_{i=1}^{p} \sum_{j=1}^{p} w_i(x(t))w_j(x(t))(A_i(x(t)) + B_i(x(t))G_j(x(t))))\dot{x}(x(t)).$$

(2.11)

### 2.6 Lyapunov Stability Theory

Stability analysis plays a central role in control system design. The stability of an equilibrium point is a qualitative characteristic of a system. A system is described as stable if starting the system somewhere near its equilibrium point implies that it will stay around the point ever after [103]. The formal definition of stability is given as follows.
Consider the autonomous system
\[
\dot{x}(t) = f(x(t))
\]  
(2.12)

where \( f(\cdot) \) is locally Lipschitz over a domain \( D \subset \mathbb{R}^n \). Suppose \( \bar{x} \in \mathbb{R}^n \) is an equilibrium point of (2.12); that is, \( f(\bar{x}) = 0 \).

Without loss of generality, we assume that \( f(x(t)) \) satisfies \( f(0) = 0 \). The definition of stability is given as follows:

**Definition 1** [15] The equilibrium point \( x = 0 \) of (2.12) is stable if for each \( \varepsilon > 0 \), there is \( \delta > 0 \) (dependent on \( \varepsilon \)) such that \(|x(0)| < \delta \Rightarrow |x(t)| < \varepsilon\), \( \forall t \geq 0 \); unstable if it is not stable; asymptotically stable if it is stable and \( \delta \) can be chosen such that \(|x(0)| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0\).

Stability of equilibrium point is usually handled in the sense of Lyapunov. The Lyapunov stability theory has been a cornerstone of nonlinear system analysis for the past half-century [104]. It gives sufficient conditions for stability and asymptotic stability of equilibrium point. Lyapunov’s method based on Lyapunov stability theory can be applied to facilitate stability analysis of a nonlinear system by finding a scalar “energy like” function and examining the function’s time variation. The Lyapunov stability theory is given next:

**Theorem 1** [15] Let \( x = 0 \) be an equilibrium point for (2.12) and \( D \subset \mathbb{R}^n \) be a domain containing \( x = 0 \). Let \( V: D \to \mathbb{R} \) be a continuously differentiable function such that

\[
V(0) = 0;
\]  
(2.13)

\[
V(x) > 0 \quad \text{for all } x \in D \text{ with } x \neq 0; \]  
(2.14)

\[
\dot{V}(x) \leq 0 \quad \text{for all } x \in D; \]  
(2.15)
Then, $x = \mathbf{0}$ is a stable equilibrium point of (2.12). Moreover, if

$$\dot{V}(x) < 0 \quad \text{for all } x \in D \text{ with } x \neq \mathbf{0};$$

Then, $x = \mathbf{0}$ is a asymptotically stable equilibrium point.

The history of Lyapunov stability theory was far from the research activities to the connection between energy and stability of dynamic system several centuries ago [105]. From the “energy” point of view, the Lyapunov function is a “valid energy function” in the vicinity of the equilibrium point, and it decreases along the trajectories of the dynamic system, then the dynamic system is stable. Geometric interpretation of Lyapunov stability theory is given in Fig 2.3, in which, the black line indicates the trajectory, the blue lines indicate the energy levels of the system. In Fig. 2.3, it shows the energy decreases from top to bottom. The origin, i.e., the equilibrium point, is the minimum energy point which is the bottom of the bowl.
2.7 LMI/SOS-based Stability Conditions

As the systematic methods for FMB control, both of the LMI/SOS-based methods are based on Lyapunov stability theory. In this section, the processes of derivation of the basic LMI/SOS stability conditions are given for the purpose of showing a rough outline of the basic LMI-based method [12] and SOS-based method [18]. It should be mentioned that the part of LMI/SOS-based methods here are only for the analysis work, the design part is omitted here. More details about these two methods will be given in following chapters.

2.7.1 LMI-based Stability Conditions

Consider T-S FMB control system (2.3) with \( u(t) = 0 \),

\[
\dot{x}(t) = \sum_{i=1}^{p} w_i(x(t)) A_i x(t). \quad (2.17)
\]

Applying Lyapunov stability theory with a quadratic Lyapunov candidate function \( V(x(t)) = x(t)^T P x(t) \), where \( P \) is real symmetric positive definition matrix. The derivate of \( V \) along the trajectory of system (2.17) is given by

\[
\dot{V}(x(t)) = x(t)^T P \dot{x}(t) + \dot{x}(t)^T P x(t) = \sum_{i=1}^{p} w_i(x(t)) x(t)^T (PA_i + A_i^T P) x(t). \quad (2.18)
\]

The sufficient stability conditions for ensuring stability of (2.17) are given as follows:

**Theorem 2** [12] The equilibrium of the fuzzy system (2.17) is asymptotically stable if there exists a common matrix \( P \) such that

\[
P > 0; \quad PA_i + A_i^T P < 0 \quad i = 1, 2, \ldots, p; \quad (2.19)
\]

that is, a common \( P \) has to exist for all sub-systems.
The stability conditions in Theorem 2 are also called *simultaneous Lyapunov stability conditions* [13]. The problem of finding common $P$ is used to check the stability of fuzzy system (2.17) which is an LMI problem. It can be solved efficiently via convex optimisation techniques numerically [13]. Computational tool such as MATLAB® LMI Control Toolbox can be used.

### 2.7.2 SOS-based Stability Conditions

Consider PFMB control system (2.9) with $u(t) = 0$,

$$
\dot{x}(t) = \sum_{i=1}^{P} w_i(x(t))A_i(x(t))\dot{x}(t).
$$

(2.20)

The stability is investigated in using a polynomial Lyapunov candidate function based on Lyapunov stability theory. Sufficient stability conditions in terms of SOS for ensuring stability of (2.20) are given as follows:

**Theorem 3** [18] The equilibrium of the polynomial fuzzy system (2.20) is asymptotically stable if there exists a polynomial matrix $P(x(t)) \in \mathbb{R}^{N \times N}$, such that the following SOS-based conditions are satisfied:

$$
v^T(P(x(t)) - \varepsilon_1(x(t))I)v \text{ is SOS};
$$

(2.21)

$$
-v^T(Q_i(x(t)) + \varepsilon_2(x(t))I)v \text{ is SOS } \forall i;
$$

(2.22)

where $v \in \mathbb{R}^N$ is an arbitrary vector independent of $x(t)$, $\varepsilon_1(x(t)) > 0$ and $\varepsilon_2(x(t)) > 0$ are predefined scalar polynomials. $Q_i(x(t)) = P(x(t))T(x(t))A_i(x(t))+A_i(x(t))^TT(x(t))^TP(x(t))+\sum_{i=1}^{n} \frac{\partial P(x)}{\partial x_k}A_i^k(x(t))\dot{x}(x(t))$.

in which $T(x(t)) \in \mathbb{R}^{N \times n}$ is a polynomial matrix with $(i,j)$th entry given by $T_{ij}(x(t)) = \frac{\partial x_j}{\partial x_i}(x(t))$, $A_i^k(x(t)) \in \mathbb{R}^N$ denotes the $k^{th}$ row of $A_i(x(t))$.

The problem of finding common $P(x(t))$ is used to check the stability of fuzzy system (2.20). It is an SOS problem which can solved numerically using sum of squares programs by casting them as semidefinite programs [20]. It can
be converted into semidefinite programs (SDPs) by SOSTOOLS [102] and then solved by SPD solvers such as SeDuMi [106].

2.8 Several Definitions and Useful Lemmas

The formal definitions of LMI, LMI problem, and SOS are given in this section. The following lemmas are employed in the following chapters.

Definition 2 (LMI) [13] An LMI is a matrix inequality of the form

\[ F(x) \triangleq F_0 + \sum_{i=1}^{m} x_i F_i > 0, \]

where \( x = [x_1, x_2, \ldots, x_m]^T \) is the variable and the symmetric matrices \( F_i = F_i^T \in \mathbb{R}^{n \times n}, i = 0, 1, \ldots, m, \) are given.

Definition 3 (LMI problem) [16] Given an LMI as \( F(x) > 0, \) the LMI problem is to find \( x^{\text{feas}} \) such that \( F(x^{\text{feas}}) > 0 \) or determine that the LMI is infeasible. This is a convex feasibility problem.

Definition 4 (SOS) [20] A multivariate polynomial \( p(x_1, x_2, \ldots, x_n) \triangleq p(x) \) is a sum of squares (SOS), if there exist polynomials \( f_1(x), f_2(x), \ldots, f_m(x) \) such that

\[ p(x) = \sum_{i=1}^{m} f_i^2(x). \]

Proposition 1 [107] Let \( f(x(t)) \) be a polynomial \( x(t) \in \mathbb{R}^n \) of degree \( 2d. \) Let \( \dot{x}(x(t)) \) be a column vector whose entries are all monomials in \( x(t) \) with degree no greater than \( d. \) Then, \( f(x(t)) \) is an SOS iff there exists a positive semidefinite matrix \( P \) such that

\[ f(x(t)) = \dot{x}^T(x(t))P\dot{x}(x(t)). \]
Proposition 2 \[20\] Let \( F(x(t)) \) be an \( N \times N \) symmetric polynomial matrix of degree \( 2d \) in \( x(t) \in \mathbb{R}^n \). Let \( \dot{x}(x(t)) \) be a column vector whose entries are all monomials in \( x(t) \) with degree no greater than \( d \), and consider the following conditions.

1) \( F(x(t)) \geq 0 \) for all \( x(t) \in \mathbb{R}^n \).

2) \( \nu^T(t)F(x(t))\nu(t) \) is an SOS, where \( \nu(t) \in \mathbb{R}^N \).

3) There exists a positive semidefinite matrix \( Q \) such that \( \nu^T(t)F(x(t))\nu(t) = (\nu \otimes \dot{x}(x(t)))^TQ(\nu(t) \otimes \dot{x}(x(t))) \), where \( \otimes \) denotes the Kronecker product.

Then, \((1) \iff (2) \) and \((2) \iff (3) \).

Lemma 1 (S-procedure) \[13\] Let \( T_0, \ldots, T_p \) be symmetric matrices and vector \( v \) of appropriate dimensions, the following relation holds:

There exist \( \tau_1 \geq 0, \ldots, \tau_p \geq 0 \) such that \( T_0 - \sum_{i=1}^{p} \tau_i T_i > 0 \).

\[ \iff \] the following condition on \( T_0, \ldots, T_p \) holds

\[ v^TT_0v > 0 \] holds for all \( v \neq 0 \) that satisfy \( v^TT_i v > 0, \ i = 1, 2, \ldots, p. \)

Lemma 2 \[18, 20\] For any invertible polynomial matrix \( X(y) \) where \( y = [y_1, y_2, \ldots, y_n]^T \), the following equation is true.

\[ \frac{\partial X(y)^{-1}}{\partial y_j} = -X(y)^{-1}\frac{\partial X(y)}{\partial y_j}X(y)^{-1} \forall j. \]

2.9 Conclusion

In this chapter, sector nonlinearity technique is introduced. With the calculation of upper and lower bounds of the chosen nonlinear term, a fuzzy model described by “IF-THEN” rules can be constructed. The preliminaries of T-S FMB control system and PFMB control system have been reviewed. The technical details of T-S/polynomial fuzzy models, T-S/polynomial fuzzy controllers, and T-S
FMB/PFMB control systems are provided. It should be mentioned that the T-S/ polynomial fuzzy controllers are designed under PDC scheme, which is also referred as the traditional fuzzy control scheme in this thesis. As its extension, imperfect premise matching control scheme will be introduced in Section 3.2.1. Lyapunov stability theory and LMI/SOS stability conditions are reviewed briefly. Moreover, some definitions, propositions and useful lemmas are given.
Chapter 3

Relaxed Stability Analysis of T-S FMB Control Systems

3.1 Introduction

T-S FMB control is a systematic design methodology for the nonlinear control system. With the success of algorithms which are for the LMI problems solving in the 1990s, LMI-based stability analysis of T-S FMB control systems was proposed early in [108, 109] and has been developed into a very influential research framework for the stability analysis and controller synthesis in the past two decades [100]. In [16], basic stability conditions in terms of LMIs were provided using the quadratic Lyapunov function based on Lyapunov stability theory. The stability conditions obtained were recast as LMI problem. For the stabilisation design problem, based on the T-S fuzzy model, a fuzzy controller is designed under PDC scheme which can quadratically stabilise the closed-loop system. The design variables in this problem were feedback gains of the fuzzy controller. The stability conditions of T-S FMB control system designed were sufficient or the existence of such a T-S fuzzy controller. Once the feasible solution of the LMI problem was found, the feedback gains were obtained.

There are two features of the basic LMI-based method [12]. First, the scheme
used for T-S fuzzy controller design is PDC design, as it is introduced in Section 2.4.2. As mentioned in Section 1.2, PDC design lacks design flexibility that it requires T-S fuzzy controller shares the same membership functions with the fuzzy model. Second, it is a membership-function-independent (MFI) stability analysis method that the stability analysis only considers the local sub-systems of T-S FMB control system without involving the information of the membership functions. The results of MFI stability analysis are potentially conservative due to the absence of membership functions information in the stability conditions.

For the purpose of improving the applicability of T-S FMB control system, in this chapter, we employ imperfect premise matching design scheme for T-S fuzzy controller design to facilitate MFD stability analysis vis PMFs.

This chapter contains two main sections. Section 3.2 presents two MFD stability analysis methods using PMFs. In Section 3.3, T-S FMB control system with sub-domain fuzzy controllers is considered that relaxed stability conditions are achieved by MFD stability analysis using PMFs and membership function boundary information.

3.2 Relaxed Stability Analysis via PMFs

In the framework of T-S FMB control, given nonlinear system is represented as a T-S fuzzy model using fuzzy modelling techniques such as sector nonlinearity [16]. Nonlinearity of the dynamic system at hand is extracted into the membership functions. Therefore, stability analysis ignoring membership function shape information is a source which leads to conservative results. In the issue of this problem, stability analysis approaches are classified into two types: MFI stability analysis and MFD stability analysis. The stability conditions proposed in the literature on stability analysis of T-S fuzzy control systems in [12, 36, 37, 39, 40, 55] are all results obtained by MFI stability analysis. These conditions can be applied to a family of T-S FMB control systems with arbitrary membership functions, not only to the one at hand with specified membership functions.
For the purpose of stability conditions relaxation, various MFD stability analysis methods are proposed to investigate the stability of FMB control systems [39, 40, 42, 55, 56, 59, 60, 110–114]. In this section, MFD methods using the concept of PMF [57] are presented to utilise the information of membership functions to further relax the stability conditions for T-S FMB control system. The PMFs are employed as approximation membership functions to facilitate the stability analysis. The relation between conservativeness and approximation accuracy will be investigated. Lower and upper PMFs are introduced for the purpose of providing an MFD method which can be applied for relaxation of stability conditions using the PMFs with lower approximation accuracy. The T-S FMB control system considered in this section is designed under imperfect premise matching scheme. Comparison study results are provided to demonstrate the viability of the proposed MFD stability methods under different scenarios.

This section is organised as follows. In Section 3.2.1, imperfect premise matching is introduced briefly. T-S fuzzy controller designed under imperfect premise matching is given. In Sections 3.2.2 and 3.2.3, based on Lyapunov stability theorem, relaxed LMI-based stability conditions derived via PMF-based stability analysis and lower-upper-PMF-based stability analysis are presented respectively. A simulation example is provided in Section 3.2.4 to illustrate the merits of proposed MFD stability analysis methods.

3.2.1 Imperfect Premise Matching Design

FMB control system investigated in this thesis is a full-state feedback control system formed by a fuzzy model which represents the given nonlinear system and a fuzzy controller. In the basic LMI-based stability analysis procedures [16], fuzzy controller was designed under PDC scheme. As represented in Section 2.4.1, T-S fuzzy controller (2.6) adopts the premises of T-S fuzzy model (2.3): the same number of fuzzy rules \( p \) and the same membership function \( w_j(x(t)) \), \( j = 1, 2, \ldots, p \). Since the premises of fuzzy controller and fuzzy model are perfectly matched,
PDC design is also called perfectly premise matching design. T-S FMB control system (3.5) designed using this scheme is able to offer more relaxed stability analysis conditions by grouping the cross terms of the membership functions [115]. However, PDC design scheme has two main drawbacks. First, as it requires the fuzzy controller to share the premise membership functions with the fuzzy model. Therefore, the design flexibility of the controller membership functions is heavily limited. In the case that fuzzy model has very complicated membership functions and/or the number of rules is large, the control implementation cost will increase correspondingly. The second drawback is the inherent robustness property of the fuzzy controller vanishes under perfectly premises matching [21].

For the purpose of tackling these drawbacks, the design scheme called imperfect premise matching is employed in the research activities on FMB control systems [26, 39, 40, 110, 116]. In imperfect premise matching, membership functions of fuzzy controller can be designed independently with fuzzy model, more specifically, it is not necessary to share the same membership functions and the same number of fuzzy rules with the fuzzy model. In this thesis, FMB control systems investigated are designed under imperfect premise matching design.

A fuzzy controller with \( c \) rules of the following format is proposed to stabilise the nonlinear plant represented by the T-S fuzzy model (2.3).

\[
\text{Rule } j: \quad \text{IF } g_1(x(t)) \text{ is } N^j_1 \text{ AND } \cdots \text{ AND } g_\Omega(x(t)) \text{ is } N^j_\Omega \quad \text{THEN } u(t) = G_j x(t) \quad (3.1)
\]

where \( N^j_\beta \) is the fuzzy term of rule \( j \) corresponding to the function \( g_\beta(x(t)) \), \( \beta = 1, 2, \ldots, \Omega; \ j = 1, 2, \ldots, c; \ \Omega \) is a positive integer; \( G_j \in \mathbb{R}^{m \times n}, \ j = 1, 2, \ldots, c, \) is the constant feedback gain to be determined. The fuzzy controller is defined as follows:

\[
u(t) = \sum_{j=1}^{c} m_j(x(t))G_j x(t), \quad (3.2)\]
where
\[
m_j(x(t)) \geq 0 \quad \forall j, \quad \sum_{j=1}^{c} m_j(x(t)) = 1, \tag{3.3}
\]
and
\[
m_j(x(t)) = \frac{\prod_{l=1}^{\Omega} \mu_{N_j^\beta}(g_l(x(t)))}{\sum_{k=1}^{c} \prod_{l=1}^{\Omega} \mu_{N_k^\beta}(g_l(x(t)))} \quad \forall j, \tag{3.4}
\]
m_j(x(t)), j = 1, 2, \ldots, c, are the normalised grades of membership, \(\mu_{N_j^\beta}(g_\beta(x(t)))\), \(\beta = 1, 2, \cdots, \Omega\), are the membership functions corresponding to the fuzzy term \(N_j^\beta\).

**Remark 4** It is noted that in imperfect premise matching design, the number of fuzzy rules of the fuzzy controller is different from fuzzy model (2.3): \(c \neq p\); The membership functions are not shared with fuzzy model (2.3): \(m_i(x(t)) \neq w_i(x(t))\) for any \(i\).

### 3.2.2 PMF-based Stability Analysis

The stability of the T-S FMB control system is investigated in this section. From (2.3) and (3.2), the FMB control system is obtained as follows:

\[
\dot{x}(t) = \sum_{i=1}^{p} w_i(x(t))(A_i x(t) + B_i \sum_{j=1}^{c} m_j(x(t)) G_j x(t))
\]
\[
= \sum_{i=1}^{p} \sum_{j=1}^{c} w_i(x(t)) m_j(x(t))(A_i + B_i G_j) x(t). \tag{3.5}
\]

The control objective is to determine the feedback gain \(G_j\) such that the FMB system (3.5) is asymptotically stable i.e., \(x(t) \to 0\) as time \(t \to \infty\).

As the T-S fuzzy model (2.3) and the fuzzy controller (3.2) do not share the same number of fuzzy rules and the membership functions, PDC-based analysis approach cannot be applied for relaxation of stability analysis. The following theorem offers the LMI-based stability conditions for the class of FMB control systems (3.5).
Theorem 4 [12, 25] The FMB control system (3.5) formed by a nonlinear plant represented by the T-S fuzzy model in the form of (2.3) and the fuzzy controller (3.2) connected in a closed loop, is asymptotically stable if there exist matrices $N_j \in \mathbb{R}^{m \times n}, j = 1, 2, \ldots, c$ and $X = X^T \in \mathbb{R}^{n \times n}$, such that the following LMIs are satisfied.

$$X > 0;$$

$$A_i X + X A_i^T + B_i N_j + N_j^T B_i^T < 0 \quad \forall \ i, j;$$

where the feedback gains are defined as $G_j = N_j X^{-1}$ for all $j$.

Remark 5 It should be noted that the LMI-based stability conditions in Theorem 4 are MFI stability conditions, of which the information of membership functions is not considered. Consequently, the solution is valid for the FMB based control systems with any shape of membership functions resulting in a conservative stability analysis result.

In this section, an MFD stability analysis method is proposed to relax the stability analysis result with the consideration of the information of membership functions. For brevity, the time $t$ associated with the variables is dropped in the following analysis for the situation without ambiguity, e.g., $x(t)$ and $u(t)$ are denoted as $x$ and $u$, respectively. Also, the property of membership functions leading to $\sum_{i=1}^{p} w_i = \sum_{j=1}^{c} m_j = \sum_{i=1}^{p} \sum_{j=1}^{c} w_i m_j = 1$ will be used.

3.2.2.1 Piecewise Membership Functions

Consider an operating domain $\Phi$ being divided into $q$ connected sub-domains $\Phi_k$ such that $\Phi = \bigcup_{k=1}^{q} \Phi_k$. Defining a vector $z = \left[ z_1 \ z_2 \ \cdots \ z_\eta \right]$ and considering $z \in \Phi_k, k = 1, 2, \ldots, q$, we have the lower and upper bounds of $z_r, r = 1, 2, \ldots, \eta$
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denoted as $z_{ri,k}$ and $z_{ri,k}$, respectively, $i_r = 1, 2$, such that $z_{ri,k} \leq z_r \leq z_{ri,k}$. Denoting the vertices of the sub-domains $\Phi_k$ as $z_{i_1 \ldots i_{\eta_k}} = \begin{bmatrix} z_{i_1,k} & z_{2i_2,k} & \cdots & z_{\eta_{\eta_k}} \end{bmatrix}$, the PMF $\hat{h}(z)$ [51,57] is defined as

$$\hat{h}(z) = \sum_{k=1}^{q} \sum_{i_1=1}^{2} \cdots \sum_{i_{\eta}=1}^{2} \prod_{r=1}^{n} v_{ri,k}(z_r) \hat{h}(z_{i_1 \ldots i_{\eta_k}}), \quad (3.6)$$

where $v_{ri,k}(z_r)$ is a function to be determined, which exhibits the properties that $0 \leq v_{ri,k}(z_r) \leq 1$ and $\sum_{i_r=1}^{2} v_{ri,k}(z_r) = 1$ for all $r$, $k$ and $z \in \Phi_k$, otherwise, $v_{ri,k}(z_r) = 0$; $\hat{h}(z_{i_1 \ldots i_{\eta_k}}) \geq 0$ is a constant scalar and its value is to be determined. Thus, it leads to the property that

$$\sum_{k=1}^{q} \sum_{i_1=1}^{2} \cdots \sum_{i_{\eta}=1}^{2} \prod_{r=1}^{n} v_{ri,k}(z_r) = 1. \quad (3.7)$$

An example is presented to illustrate the concept of PMFs. Considering $z = x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$, an example PMF $\hat{h}(x)$ is shown in Fig. 3.1. Corresponding to every single coordinate characterized by $x_1$ and $x_2$, it is associated with a grade of membership. The black dots in Fig. 3.1a denote the vertices. Each sub-region is a quadragon which is characterized by four vertices. In this example, 6 sub-regions are shown. By extending the ranges of $x_1$ and $x_2$, it can be imaged that we will have a number of sub-regions. The $k^{th}$ sub-region is shown in Fig. 3.1b. It can be shown that the grades of membership within the sub-region can be expressed as (3.6). The four vertices are denoted as $\hat{h}(x_{11k})$, $\hat{h}(x_{12k})$, $\hat{h}(x_{21k})$ and $\hat{h}(x_{22k})$ corresponding to the grades of membership at different coordinates $(x_1, x_2)$. The general form denoting a vertex is $\hat{h}(z_{i_1 \ldots i_{\eta_k}})$ where $i_r \in \{1, 2\}$, $r = 1, 2, \ldots, n$, indicates the two end points of $x_{i_r}$ and $k$ denotes the $k^{th}$ sub-region. For example, $\hat{h}(x_{12k})$ is the grade of membership corresponding to the lower value of $x_1$ and upper value of $x_2$ in the $k^{th}$ sub-region. The functions $v_{11k}(x_1)$, $v_{12k}(x_1)$, $v_{21k}(x_2)$ and $v_{22k}(x_2)$ are defined as triangular shape shown in Fig.3.1b (their values are 0 outside the $k^{th}$ sub-region) which are defined such that $v_{11k}(x_1) + v_{12k}(x_1) = 1$ and $v_{21k}(x_2) + v_{22k}(x_2) = 1$ ensuring that the property

The functions $v_{11k}(x_1)$, $v_{12k}(x_1)$, $v_{21k}(x_2)$ and $v_{22k}(x_2)$ are defined as triangular shape shown in Fig.3.1b (their values are 0 outside the $k^{th}$ sub-region) which are defined such that $v_{11k}(x_1) + v_{12k}(x_1) = 1$ and $v_{21k}(x_2) + v_{22k}(x_2) = 1$ ensuring that the property
From (5) and (10), recalling that $h_{ij}(x) \equiv w_i(x)m_j(x)$, we have
\[
\dot{V}(x) = x^T P \dot{x} + x^T P \dot{x} = \sum_{i=1}^{c} \sum_{j=1}^{c} \hat{h}_{ij}(x) z^T Q_{ij} z
\]
(11)

Define $X = P^{-1}, G_j = N_j X^{-1}, z = X - 1$ and $Q_{ij} = A_i X + X A_i^T + B_i N_j + N_j^T B_i^T$.
From (9) and (11), we have
\[
\dot{V}(x) = \sum_{i=1}^{c} \sum_{j=1}^{c} \hat{h}_{ij}(x) z^T Q_{ij} z + \sum_{i=1}^{c} \sum_{j=1}^{c} \hat{\hat{h}}_{ij}(x) z^T Q_{ij} z + \sum_{i=1}^{c} \sum_{j=1}^{c} \Delta t \hat{\hat{h}}_{ij}(x) z^T Q_{ij} z
\]

Figure 3.1: Example PMF: (a) PMF; (b) $k^{th}$ sub-region.

(3.7) holds. The grade of membership in the $k^{th}$ sub-region can be expressed as
\[
\hat{h}(x) = \sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \prod_{r=1}^{2} v_{r,i_1,k}(x_r) \hat{h}(x_{i_1 i_2 k})
\]
and the overall PMF is defined as in (3.6), i.e., $\tilde{h}(x) = \sum_{k=1}^{q} \sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \prod_{r=1}^{2} v_{r,i_1,k}(x_r) \hat{h}(x_{i_1 i_2 k})$. It should be noted in this example that the grade of membership of $\hat{h}(x)$ in each sub-region is a linear plane. By choosing different form of $v_{r,i,k}(x_r)$, say, a nonlinear function, a nonlinear surface of $\hat{h}(x)$ can be achieved.
3.2.2 Stability Conditions

The PMFs are proposed to approximate the original membership functions to facilitate the stability analysis. Based on the concept of PMFs in Section 3.2.2.1, from (3.6), we defined a PMF \( \hat{h}_{ij}(x) \) to approximate the multiplication of membership functions \( h_{ij}(x) \equiv w_i(x)m_j(x) \) as follows:

\[
\hat{h}_{ij}(x) = \sum_{k=1}^{2} \sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} \prod_{r=1}^{n} v_{ri,k}(x_r) \hat{h}_{ij}(x_{i_1 \cdots i_n k}). \tag{3.8}
\]

The difference between \( h_{ij}(x) \) and \( \hat{h}_{ij}(x) \) is defined as follows:

\[
\Delta h_{ij}(x) = h_{ij}(x) - \hat{h}_{ij}(x), \tag{3.9}
\]

which is bounded by the lower and upper bounds \( \Delta \underline{h}_{ij} \) and \( \Delta \bar{h}_{ij} \), respectively, such that \( \Delta \underline{h}_{ij} \leq \Delta h_{ij}(x) \leq \Delta \bar{h}_{ij} \).

**Remark 6** The PMF \( \hat{h}_{ij}(x) \) considers a general case that all system states \( x_1, \ldots, x_n \), are involved. The system states not involved in real situation can be removed from the expression (3.8).

To investigate the stability of the FMB control system (3.5), we consider the following quadratic Lyapunov function candidate:

\[
V(x) = x^T P x, \tag{3.10}
\]

where \( 0 < P = P^T \in \mathbb{R}^{n \times n} \).

From (3.5) and (3.10), recalling that \( h_{ij}(x) \equiv w_i(x)m_j(x) \), we have,

\[
\dot{V}(x) = x^T P x + x^T P \dot{x} = \sum_{i=1}^{p} \sum_{j=1}^{q} h_{ij}(x)x^T ((A_i + B_i G_j)^T P + P (A_i + B_i G_j)) x. \tag{3.11}
\]
Define $X = P^{-1}$, $G_j = N_jX^{-1}$, $z = X^{-1}x$ and $Q_{ij} = A_iX + XA_i^T + B_iN_j + N_j^TB_i^T$. From (3.9) and (3.11), we have,

$$
\dot{V}(x) = \sum_{i=1}^{p} \sum_{j=1}^{c} h_{ij}(x)z^TQ_{ij}z
$$

$$
= \sum_{i=1}^{p} \sum_{j=1}^{c} \hat{h}_{ij}(x)z^TQ_{ij}z + \sum_{i=1}^{p} \sum_{j=1}^{c} \Delta h_{ij}(x)z^TQ_{ij}z
$$

$$
= \sum_{i=1}^{p} \sum_{j=1}^{c} \hat{h}_{ij}(x)z^TQ_{ij}z + \sum_{i=1}^{p} \sum_{j=1}^{c} \Delta h_{ij}(x)z^TQ_{ij}z
$$

Define the slack matrix $Y_{ij} = Y^T_{ij} \in \mathbb{R}^{n \times n}$ satisfying $Y_{ij} \geq 0$ and $Y_{ij} \geq Q_{ij}$.

With the fact that $\Delta \bar{h}_{ij} - \Delta h_{ij} \geq \hat{h}_{ij}(x) - \Delta h_{ij} \geq 0$, it follows from (3.12) that

$$
\dot{V}(x) \leq \sum_{i=1}^{p} \sum_{j=1}^{c} (\hat{h}_{ij}(x) + \Delta h_{ij})z^TQ_{ij}z + \sum_{i=1}^{p} \sum_{j=1}^{c} (\Delta \bar{h}_{ij} - \Delta h_{ij})z^TY_{ij}z
$$

$$
= \sum_{i=1}^{p} \sum_{j=1}^{c} z^T((\hat{h}_{ij}(x) + \Delta h_{ij})Q_{ij} + (\Delta \bar{h}_{ij} - \Delta h_{ij})Y_{ij})z. \quad (3.13)
$$

Expanding $\hat{h}_{ij}(x)$ as in (3.8) and using the property (3.7), we have

$$
\dot{V}(x) = \sum_{i=1}^{p} \sum_{j=1}^{c} \sum_{k=1}^{q} \sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} \prod_{r=1}^{n} v_{ri,k}(x_r)\hat{h}_{ij}(x_{i_1 \cdots i_n}) + \Delta h_{ij}Q_{ij} +
$$

$$
(\Delta \bar{h}_{ij} - \Delta h_{ij})Y_{ij})z
$$

$$
= \sum_{k=1}^{q} \sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} \prod_{r=1}^{n} v_{ri,k}(x_r)z^T \sum_{i=1}^{p} \sum_{j=1}^{c} ((\hat{h}_{ij}(x_{i_1 \cdots i_n}) + \Delta h_{ij})Q_{ij} +
$$

$$
(\Delta \bar{h}_{ij} - \Delta h_{ij})Y_{ij})z. \quad (3.14)
$$

It can be seen from (3.14) that $\dot{V}(x) \leq 0$ (equality holds for $x = 0$)

if $\sum_{i=1}^{p} \sum_{j=1}^{c} ((\hat{h}_{ij}(x_{i_1 \cdots i_n}) + \Delta h_{ij})Q_{ij} + (\Delta \bar{h}_{ij} - \Delta h_{ij})Y_{ij}) < 0$ for all $i_1$, $i_2$, 


\[ \cdots, i_n \text{ and } k. \] Based on Lyapunov stability theory, the stability analysis result is summarised in the follow theorem.

**Theorem 5** The FMB control system (3.5) formed by a nonlinear plant represented by the T-S fuzzy model in the form of (2.3) and the fuzzy controller (3.2) connected in a closed loop, is asymptotically stable if there exist matrices \( N_j \in \mathbb{R}^{m \times n} \), \( Y_{ij} = Y_{ij}^T \in \mathbb{R}^{n \times n} \), \( i = 1, 2, \ldots, p \), \( j = 1, 2, \ldots, c \) and \( X = X^T \in \mathbb{R}^{n \times n} \), such that the following LMIs are satisfied.

\[
X > 0;
\]

\[
Y_{ij} \geq 0 \quad \forall \ i, j;
\]

\[
Y_{ij} \geq Q_{ij} \quad \forall \ i, j;
\]

\[
\sum_{i=1}^{p} \sum_{j=1}^{c} ((\hat{h}_{ij}(x_{i_1 \cdots i_n k}) + \Delta h_{ij}) Q_{ij} + (\Delta \bar{h}_{ij} - \Delta h_{ij}) Y_{ij}) < 0 \quad \forall \ i_1, \ldots, i_n, k;
\]

where \( \Delta h_{ij} \leq \Delta h_{ij}(x) \leq \Delta \bar{h}_{ij} \), \( Q_{ij} = A_i X + X A_i^T + B_i N_j + N_j^T B_i^T \) and the feedback gains are defined as \( G_j = N_j X^{-1} \) for all \( j \).

### 3.2.3 Lower-upper-PMF-based Stability Analysis

An alternative stability analysis approach is proposed in this section to investigate the stability of the FMB control system (3.5) using the PMFs. We introduce the lower and upper PMFs, \( \hat{h}_{ij}^L(x) \) and \( \hat{h}_{ij}^U(x) \), as follows:

\[
\hat{h}_{ij}^L(x) = \sum_{k=1}^{a} \sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} \prod_{r=1}^{n} v_{r \downarrow, k}(x_r) \hat{h}_{ij}^L(x_{i_1 \cdots i_n k}), \tag{3.15}
\]

\[
\hat{h}_{ij}^U(x) = \sum_{k=1}^{a} \sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} \prod_{r=1}^{n} v_{r \downarrow, k}(x_r) \hat{h}_{ij}^U(x_{i_1 \cdots i_n k}), \tag{3.16}
\]
where $\hat{h}_{ij}(x_{i_1\ldots i_n}) \geq 0$ and $\hat{h}^U_{ij}(x_{i_1\ldots i_n}) \geq 0$ are constant scalars to be determined.

The lower and upper PMFs $\hat{h}^L_{ij}(x)$ and $\hat{h}^U_{ij}(x)$ are chosen such that they satisfy the following inequalities:

\[
\hat{h}^L_{ij}(x) - h_{ij}(x) \geq 0, \quad (3.17)
\]

\[
\hat{h}^U_{ij}(x) - h_{ij}(x) \geq 0. \quad (3.18)
\]

Also, we consider that the lower and upper bounds of $h_{ij}(x)$ are known and satisfy

\[
h_{ij} \leq h_{ij}(x) \leq \bar{h}_{ij}, \quad (3.19)
\]

where $h_{ij}$ and $\bar{h}_{ij}$ are constant scalars to be determined.

Define some slack matrices $0 \leq H^L_{ij} = H^T_{ij} \in \mathbb{R}^{n \times n}$, $0 \leq H^U_{ij} = H^{T^T}_{ij} \in \mathbb{R}^{n \times n}$, $0 \leq H_{ij} = H^T_{ij} \in \mathbb{R}^{n \times n}$ and $0 \leq \bar{H}_{ij} = \bar{H}^T_{ij} \in \mathbb{R}^{n \times n}$. From (3.17)–(3.19), we have

\[
\sum_{i=1}^p \sum_{j=1}^c (h_{ij}(x) - \hat{h}^L_{ij}(x))H^L_{ij} \geq 0, \quad (3.20)
\]

\[
\sum_{i=1}^p \sum_{j=1}^c (\hat{h}^U_{ij}(x) - h_{ij}(x))H^U_{ij} \geq 0, \quad (3.21)
\]

\[
\sum_{i=1}^p \sum_{j=1}^c (h_{ij}(x) - \bar{h}_{ij})\bar{H}_{ij} \geq 0, \quad (3.22)
\]

\[
\sum_{i=1}^p \sum_{j=1}^c (\bar{h}_{ij} - h_{ij}(x))H_{ij} \geq 0. \quad (3.23)
\]
Adding (3.20)–(3.23) to (3.11), we have,

\[
\dot{V}(x) \leq \sum_{i=1}^{p} \sum_{j=1}^{c} h_{ij}(x)z^T Q_{ij} z + \sum_{i=1}^{p} \sum_{j=1}^{c} (h_{ij}(x) - \hat{h}_{ij}^L(x))z^T H_{ij}^L z + \sum_{i=1}^{p} \sum_{j=1}^{c} \left(\hat{h}_{ij}^U(x) - h_{ij}(x)\right)z^T H_{ij}^U z + \sum_{i=1}^{p} \sum_{j=1}^{c} (\bar{h}_{ij} - h_{ij}(x))z^T \bar{H}_{ij} z
\]

\[
= \sum_{i=1}^{p} \sum_{j=1}^{c} h_{ij}(x)z^T \left(Q_{ij} + H_{ij}^L - H_{ij}^U + \bar{H}_{ij} - \bar{H}_{ij}\right) + \sum_{r=1}^{p} \sum_{s=1}^{c} \left(-\hat{h}_{rs}^L(x)H_{rs}^L + \hat{h}_{rs}^U(x)H_{rs}^U - \tilde{h}_{rs}(x)\tilde{H}_{rs} + \tilde{h}_{rs}(x)\tilde{H}_{rs}\right) z.
\]

(3.24)

Expanding \(\hat{h}_{ij}^L(x)\) and \(\hat{h}_{ij}^U(x)\) as in (3.15) and (3.16), we have

\[
\dot{V}(x) = \sum_{k=1}^{q} \sum_{j_1=1}^{2} \cdots \sum_{i_n=1}^{2} \prod_{r=1}^{n} v_{r,i}(x_r) \sum_{i=1}^{p} \sum_{j=1}^{c} h_{ij}(x)z^T \left(Q_{ij} + H_{ij}^L - H_{ij}^U + \bar{H}_{ij} - \bar{H}_{ij}\right) + \sum_{r=1}^{p} \sum_{s=1}^{c} \left(-\hat{h}_{rs}^L(x_{i_1 \cdots i_n})H_{rs}^L + \hat{h}_{rs}^U(x_{i_1 \cdots i_n})H_{rs}^U - \tilde{h}_{rs}(x_{i_1 \cdots i_n})\tilde{H}_{rs} + \tilde{h}_{rs}(x_{i_1 \cdots i_n})\tilde{H}_{rs}\right) z.
\]

(3.25)

It can be seen from (3.25) that \(\dot{V}(x) \leq 0\) (equality holds for \(x = 0\)) if

\[Q_{ij} + H_{ij}^L - H_{ij}^U + \bar{H}_{ij} - \bar{H}_{ij} + \sum_{r=1}^{p} \sum_{s=1}^{c} \left(-\hat{h}_{rs}^L(x_{i_1 \cdots i_n})H_{rs}^L + \hat{h}_{rs}^U(x_{i_1 \cdots i_n})H_{rs}^U - \tilde{h}_{rs}(x_{i_1 \cdots i_n})\tilde{H}_{rs} + \tilde{h}_{rs}(x_{i_1 \cdots i_n})\tilde{H}_{rs}\right) < 0\] for all \(i, j, i_1, i_2, \cdots, i_n\) and \(k\). Based on Lyapunov stability theory, the stability analysis results are summarised in the follow theorem.

**Theorem 6** The FMB control system (3.5) formed by a nonlinear plant represented by the T-S fuzzy model in the form of (2.3) and the fuzzy controller (3.2) connected in a closed loop, is asymptotically stable if there exist matrices \(H_{ij}^L = H_{ij}^T \in \mathbb{R}^{n \times n}\), \(H_{ij}^U = H_{ij}^T \in \mathbb{R}^{n \times n}\), \(H_{ij} = H_{ij}^T \in \mathbb{R}^{n \times n}\), \(\bar{H}_{ij} = \bar{H}_{ij}^T \in \mathbb{R}^{n \times n}\), \(N_{ij} \in \mathbb{R}^{m \times n}, i = 1, 2, \ldots, p, j = 1, 2, \ldots, c\) and \(X = X^T \in \mathbb{R}^{n \times n}\), such that the following LMIs are satisfied.

\[
X > 0;
\]
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\[ H_{ij}^L \geq 0 \quad \forall \ i, j; \]

\[ H_{ij}^U \geq 0 \quad \forall \ i, j; \]

\[ H_{ij} \geq 0 \quad \forall \ i, j; \]

\[ \bar{H}_{ij} \geq 0 \quad \forall \ i, j; \]

\[ Q_{ij} + H_{ij}^L - H_{ij}^U + H_{ij} - \bar{H}_{ij} + \]

\[ \sum_{r=1}^{p} \sum_{s=1}^{c} (-\hat{h}_{rs}^L(x_{i_1 \cdots i_n k})H_{rs}^L + \]

\[ \hat{h}_{rs}^U(x_{i_1 \cdots i_n k})H_{rs}^U - h_{rs} \bar{H}_{rs} + \bar{h}_{rs} \bar{H}_{rs}) < 0 \]

\[ \forall \ i, j, i_1, i_2, \ldots, i_n, k; \]

where \( h_{ij}(x) - \hat{h}_{ij}^L(x) \geq 0, \hat{h}_{ij}^U(x) - h_{ij}(x) \geq 0, h_{ij} \leq h_{ij}(x) \leq \bar{h}_{ij}, \]

\( Q_{ij} = A_i X + X A_i^T + B_i N_j + N_j^T B_i^T \) and the feedback gains are defined as \( G_j = N_j X^{-1} \)

for all \( j. \)

\textbf{Remark 7} The purpose of the PMFs is to bring the information of membership functions, e.g., the shape and position, to the stability analysis resulting in MFD LMI-based stability conditions. With the consideration of the information of membership functions, more relaxed stability analysis result can be achieved, which is applied to the FMB control system with the specified membership functions rather than any shapes and positions.

\textbf{Remark 8} It can be shown that a feasible solution of the LMI-based stability conditions without membership function information in Theorem 4 is also that of Theorem 5 and Theorem 6 but may not the other way round. When there exists a feasible solution to Theorem 4, we have \( X > 0 \) and \( Q_{ij} < 0 \) for all \( i \) and \( j \), which implies that \( \sum_{i=1}^{p} \sum_{j=1}^{c} h_{ij}(x)Q_{ij} < 0. \) Referring to Theorem 5, setting \( Y_{ij} = 0 \) will satisfy \( Y_{ij} \geq 0 \) and \( Y_{ij} \geq Q_{ij}, \) the last LMI in Theorem
5 becomes \( \sum_{i=1}^{p} \sum_{j=1}^{c} (\hat{h}_{ij}(x_{i1\cdots i_nk}) + \Delta h_{ij})Q_{ij} < 0 \). By choosing \( \hat{h}_{ij}(x_{i1\cdots i_nk}) = h_{ij}(x_{i1\cdots i_nk}) \) (sample points of \( h_{ij}(x) \)), we have \( \sum_{i=1}^{p} \sum_{j=1}^{c} \hat{h}_{ij}(x_{i1\cdots i_nk})Q_{ij} < 0 \). Consequently, by choosing sufficiently dense sample points \( \hat{h}_{ij}(x_{i1\cdots i_nk}) \) resulting in a sufficiently small value of \( \Delta h_{ij} \), \( \sum_{i=1}^{p} \sum_{j=1}^{c} (\hat{h}_{ij}(x_{i1\cdots i_nk}) + \Delta h_{ij})Q_{ij} < 0 \) can be achieved. Referring to Theorem 6, setting \( H_{ij}^L = H_{ij}^U = H_{ij} = \overline{H}_{ij} = 0 \), the last LMI becomes \( Q_{ij} < 0 \) which is satisfied by the solution of Theorem 4. Thus, it can be concluded that the MFD LMI-based stability conditions in Theorem 4 and Theorem 6 are more relaxed than the LMI-based stability conditions without membership function information in Theorem 4.

**Remark 9** It can be shown that a feasible solution of Theorem 6 is also that of Theorem 5 but may not the other way round. When there exists a feasible solution to Theorem 6, we have \( \sum_{i=1}^{p} \sum_{j=1}^{c} h_{ij}(x)Q_{ij} < 0 \) leading to
\[
\sum_{i=1}^{p} \sum_{j=1}^{c} \hat{h}_{ij}(x_{i1\cdots i_nk})Q_{ij} < 0 \text{ with } \hat{h}_{ij}(x_{i1\cdots i_nk}) = h_{ij}(x_{i1\cdots i_nk}).
\]
Consequently, by choosing sufficiently dense sample points \( \hat{h}_{ij}(x_{i1\cdots i_nk}) \) resulting in a sufficiently small value of \( \Delta h_{ij} \) and \( \Delta \overline{h}_{ij} - \Delta h_{ij} \), the last LMI in Theorem 5, i.e.,
\[
\sum_{i=1}^{p} \sum_{j=1}^{c} ((\hat{h}_{ij}(x_{i1\cdots i_nk}) + \Delta h_{ij})Q_{ij} + (\Delta \overline{h}_{ij} - \Delta h_{ij})Y_{ij}) < 0
\]
can be satisfied. Similarly, by satisfying the LMI-based stability conditions in Theorem 5, we have \( \sum_{i=1}^{p} \sum_{j=1}^{c} h_{ij}(x)Q_{ij} < 0 \). Referring to the last LMI in Theorem 6, we can achieve
\[
\sum_{i=1}^{p} \sum_{j=1}^{c} h_{ij}(x)(Q_{ij} + H_{ij}^L - H_{ij}^U + \overline{H}_{ij} - H_{ij} + \sum_{r=1}^{p} \sum_{s=1}^{c} (-h_{rs}^L(x_{i1\cdots i_nk})H_{rs}^L + h_{rs}^U(x_{i1\cdots i_nk})H_{rs}^U - h_{rs}^L\overline{H}_{rs} + h_{rs}^U\overline{H}_{rs})) < 0
\]
by some sufficiently small values of matrices \( H_{rs}^L, H_{rs}^U, \overline{H}_{rs} \) and \( \overline{H}_{rs} \). However, it is not guaranteed that \( Q_{ij} + H_{ij}^L - H_{ij}^U + \overline{H}_{ij} - H_{ij} + \sum_{r=1}^{p} \sum_{s=1}^{c} (-h_{rs}^L(x_{i1\cdots i_nk})H_{rs}^L + h_{rs}^U(x_{i1\cdots i_nk})H_{rs}^U - h_{rs}^L\overline{H}_{rs} + h_{rs}^U\overline{H}_{rs}) < 0 \) for all \( i \) and \( j \) is satisfied. Thus, it can be concluded that the LMI-based stability conditions in Theorem 5 are more relaxed than the LMI-based stability conditions in Theorem 6.

**Remark 10** The numbers of decision variables and LMIs for Theorem 4 to Theorem 6 are given in Table 3.1. Among Theorem 4 to Theorem 6, Theorem 6 has the largest number of decision variables while Theorem 4 has the smallest. As the LMI-based stability conditions in Theorem 4 are MDI, the number of LMIs is fixed.
when the number of rules of T-S fuzzy model, \( p \), and the number of rules of fuzzy controller, \( c \), are determined. The number of LMIs in Theorem 5 and Theorem 6 is a variable depending on the number of sample points (\( q \)) employed to construct the PMFs and the number of state variables involved. It should be noted in (3.8), (3.15) and (3.16) that state variables \( x_1, \ldots, x_n \) are considered in general case to construct the PMFs. For the case that not all state variables are involved, the number of LMIs will be reduced. The numbers of decision variables and LMIs of Theorem 6 are the largest resulting in the highest computational demand when the same number of involved state variables are considered in both Theorem 5 and Theorem 6. However, it will be shown by a simulation example in the following section that the LMI-based stability conditions in Theorem 6 are feasible when a small value of \( q \) (less number of sample points) is used while those in Theorem 5 are not.

Table 3.1: Numbers of decision variables and LMIs

<table>
<thead>
<tr>
<th>Theorem</th>
<th>Number of Decision Variables</th>
<th>Number of LMIs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theorem 4</td>
<td>( 1 + c )</td>
<td>( 1 + pc )</td>
</tr>
<tr>
<td>Theorem 5</td>
<td>( 1 + (1 + c)p )</td>
<td>( 1 + 2pc + 2\eta q )</td>
</tr>
<tr>
<td>Theorem 6</td>
<td>( 1 + (1 + 4c)p )</td>
<td>( 1 + pc(4 + 2\eta q) )</td>
</tr>
</tbody>
</table>

Note: \( \eta \leq n \) denotes the number of state variables involved.

3.2.4 Simulation Example

In this simulation example, the conservativeness of LMI-based stability conditions from Theorem 4 to Theorem 6 are investigated.

In this thesis, the way for conservativeness comparison is to compare the size of the stability regions in terms of two scalars \( a \) and \( b \) [16, 115], in which, \( a \) and \( b \) are system parameters of the chosen fuzzy model. Applying the computational tool, the stability regions (also called feasible areas) satisfying the investigated
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stability conditions for different values of $a$ and $b$ can be found and plotted. An empty point means that no feasible solution is found. The stability conditions which are less conservative offers the larger size of the stability region.

In this section, the stability region produced by each theorem is investigated by an FMB control system in the form of (3.5), which is formed by a 3-rule T-S fuzzy model in the form of (2.3) and a 2-rule fuzzy controller in the form of (3.2) connected in a closed loop.

In this example, the chosen T-S fuzzy model is with the following system and input matrices [52]:

$$A_1 = \begin{bmatrix} 1.59 & -7.29 \\ 0.01 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0.02 & -4.64 \\ 0.35 & 0.21 \end{bmatrix}, A_3 = \begin{bmatrix} -a & -4.33 \\ 0 & 0.05 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} 8 \\ 0 \end{bmatrix}, B_3 = \begin{bmatrix} -b + 6 \\ 0 \end{bmatrix},$$

where $a$ and $b$ are scalar parameters.

The membership functions of the T-S fuzzy model are chosen as $w_1(x_1) = \mu_{M_1}(x_1) = 1 - \frac{1}{1+e^{-(x_1+\eta)}}$, $w_2(x_1) = \mu_{M_2}(x_1) = 1 - w_1(x_1) - w_3(x_1)$, $w_3(x_1) = \mu_{M_3}(x_1) = \frac{1}{1+e^{-(x_1-\xi)}}$.

A fuzzy controller with 2 rules in the form of (3.2) is employed to close the feedback loop of the T-S fuzzy model to form an FMB control system in the form of (3.5). The membership functions of the fuzzy controller are chosen as $m_1(x_1) = \mu_{N_1}(x_1) = \begin{cases} 1 & \text{for } x_1 < -10 \\ \frac{-x_1+10}{20} & \text{for } -10 \leq x_1 \leq 10 \\ 0 & \text{for } x_1 > 10 \end{cases}$ and $m_2(x_1) = \mu_{N_2}(x_1) = 1 - m_1(x_1)$.

As $h_{ij}(x_1) \equiv w_i(x_1)m_j(x_1)$ depends on the system state $x_1$ only in this example, it is logical to choose the PMFs in the form of (3.8) depending on $x_1$ as $\hat{h}_{ij}(x_1) = \sum_{k=1}^{q} \sum_{i_k=1}^{2} v_{i_1i_2}^{i_1i_2} h_{i_1i_2}(x_1i_2k)$. We choose $v_{i_1i_2}(x_1) = 1 - \frac{x_1-x_{i_1}^k}{x_{i_1}^k-x_{i_2}^k}$ and
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\[ v_{12k}(x_1) = 1 - v_{11k}(x_1) \text{ where } x_{1k} = (k - (\frac{q}{2} + 1))\delta, x_{2k} = (k - \frac{q}{2})\delta, k = 1, 2, \ldots, q \]

and \( \delta \) is a scalar to be determined. The value of \( \delta \) can be interpreted as the sampling interval of \( x_1 \).

Considering \( 0 \leq a \leq 10 \) at the interval of 1 and \( -10 \leq b \leq 50 \) at the interval of 2, various values of \( \delta \) are chosen, i.e., \( \delta = 5, 2, 0.5 \) and 0.25; \( q = \frac{20}{\delta} \), to investigate how they influence the size of stability region of the FMB control system given by the LMI-based stability conditions in Theorem 5. The values of \( \Delta h_{ij} \) and \( \Delta \hat{h}_{ij} \) corresponding to different values of \( \delta \) satisfying \( \Delta h_{ij} \leq h_{ij}(x_1) - \hat{h}_{ij}(x_1) \leq \Delta \hat{h}_{ij} \) are found numerically and shown in Table 3.2. The stability regions corresponding to \( \delta = 0.5 \) and 0.25 are shown in Fig. 3.3 and Fig. 3.4 indicated by “o”. However, there is no stability region found for \( \delta = 5 \) and \( \delta = 2 \). It can be seen from Fig. 3.3 and Fig. 3.4 that a larger stability region can be obtained with a smaller value of \( \delta \).

The stability regions given by the LMI-based stability conditions in Theorem 6 are investigated in the following. With the chosen membership functions, it is found numerically that \( \bar{h}_{11} = \bar{h}_{32} = 1.0000 \times 10^{-1}, \bar{h}_{12} = \bar{h}_{31} = 1.8467 \times 10^{-1}; \)

Figure 3.2: Membership functions of T-S fuzzy model (solid lines) and fuzzy controller (dotted lines).
Table 3.2: Lower and upper bounds of $\Delta h_{ij}$

<table>
<thead>
<tr>
<th>Theorem 5 with $\delta = 5$</th>
<th>$\Delta h_{11} = \Delta h_{32} = -1.6483 \times 10^{-1}$, $\Delta h_{12} = \Delta h_{31} = -3.0859 \times 10^{-2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\Delta h_{21} = \Delta h_{32} = -8.1893 \times 10^{-2}$, $\Delta h_{11} = \Delta h_{32} = 8.1935 \times 10^{-2}$</td>
</tr>
<tr>
<td></td>
<td>$\Delta h_{12} = \Delta h_{31} = 3.3996 \times 10^{-2}$, $\Delta h_{21} = \Delta h_{32} = 1.6837 \times 10^{-1}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Theorem 5 with $\delta = 2$</th>
<th>$\Delta h_{11} = \Delta h_{32} = -3.5974 \times 10^{-3}$, $\Delta h_{12} = \Delta h_{31} = -7.0632 \times 10^{-3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\Delta h_{21} = \Delta h_{32} = -2.1381 \times 10^{-2}$, $\Delta h_{11} = \Delta h_{32} = 2.1422 \times 10^{-2}$</td>
</tr>
<tr>
<td></td>
<td>$\Delta h_{12} = \Delta h_{31} = 1.9688 \times 10^{-2}$, $\Delta h_{21} = \Delta h_{32} = 3.6243 \times 10^{-2}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Theorem 5 with $\delta = 0.5$</th>
<th>$\Delta h_{11} = \Delta h_{32} = -2.4426 \times 10^{-3}$, $\Delta h_{12} = \Delta h_{31} = -6.7708 \times 10^{-4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\Delta h_{21} = \Delta h_{32} = -1.7826 \times 10^{-3}$, $\Delta h_{11} = \Delta h_{32} = 1.7839 \times 10^{-3}$</td>
</tr>
<tr>
<td></td>
<td>$\Delta h_{12} = \Delta h_{31} = 1.3139 \times 10^{-3}$, $\Delta h_{21} = \Delta h_{32} = 2.4622 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Theorem 5 with $\delta = 0.25$</th>
<th>$\Delta h_{11} = \Delta h_{32} = -6.1718 \times 10^{-4}$, $\Delta h_{12} = \Delta h_{31} = -6.9800 \times 10^{-4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\Delta h_{21} = \Delta h_{32} = -4.5397 \times 10^{-4}$, $\Delta h_{11} = \Delta h_{32} = 4.5432 \times 10^{-4}$</td>
</tr>
<tr>
<td></td>
<td>$\Delta h_{12} = \Delta h_{31} = 3.3236 \times 10^{-4}$, $\Delta h_{21} = \Delta h_{32} = 6.2154 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

$\bar{h}_{21} = \bar{h}_{22} = 5.2976 \times 10^{-1}$; $h_{ij} = 0$ for all $i$ and $j$.

We define $\hat{h}_{ij}^L(x_1) = \sum_{k=1}^{q} \sum_{i_1=1}^{q} v_{1i_1k}(x_1) \hat{h}_{ij}^L(x_{i_1k})$ and $\hat{h}_{ij}^U(x_1) = \sum_{k=1}^{q} \sum_{i_1=1}^{q} v_{1i_1k}(x_1) \hat{h}_{ij}^U(x_{i_1k})$ where $\hat{h}_{ij}^L(x_{i_1k}) = \max(0, w_i(x_{i_1k}) - \varepsilon) m_j(x_{i_1k})$, $\hat{h}_{ij}^U(x_{i_1k}) = \min(1, w_i(x_{i_1k}) + \varepsilon) m_j(x_{i_1k})$, $\min(\cdot)$ and $\max(\cdot)$ are the minimum and maximum operators which return respectively the minimum and maximum values among the arguments; $\varepsilon$ is a constant scalar which is chosen to be 0.55, 0.09, 0.01 and 0.002 corresponding to $\delta = 5$, 2, 0.5 and 0.25, respectively, such that $h_{ij}(x_1) - \hat{h}_{ij}^L(x_1) \geq 0$ and $\hat{h}_{ij}^U(x_1) - h_{ij}(x_1) \geq 0$ for all $i$ and $j$. The functions $v_{11k}(x_1)$ and $v_{12k}(x_1)$, and $x_{i_1k}$ are defined as the same as the above. The stability regions corresponding to $\delta = 5$, 2, 0.5 and 0.25 are shown in Fig. 3.3–3.5. It can be seen from these figures that the smaller the value of $\delta$, the larger the size of stability region can be obtained.

Considering the cases of smaller value of $\delta$ ($\delta = 0.5$ and $\delta = 0.25$), the LMI-based stability conditions in Theorem 5 are more relaxed compared with those in Theorem 6 as evident by larger stability regions as shown in Fig. 3.3 and Fig. 3.4, which is supported by Remark 9. When the cases of larger value of $\delta$ ($\delta = 5$ and $\delta = 2$) are considered, the LMI-based stability conditions in Theorem 6 is still able to offer a stability region but those in Theorem 5 cannot.

Remark 11 Based on these findings, to lower the computational demand, it is
Figure 3.3: Stability regions given by Theorem 5 ("o") and Theorem 6 ("x") with $\delta = 0.5$.

It is recommended that Theorem 6 can be employed in the beginning with a larger value of $\delta$. If a feasible solution cannot be found, Theorem 5 can be applied with a smaller value of $\delta$.

Table 3.3: Comparison between low-upper-PMFs method and the exiting methods

<table>
<thead>
<tr>
<th>Method</th>
<th>Stability region (with the setting $\delta = 2$ or $\delta = 5$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proposed Theorem 6</td>
<td>Can be found, as shown in Fig. 3.5</td>
</tr>
<tr>
<td>Theorem 4</td>
<td>Cannot be found</td>
</tr>
<tr>
<td>Theorem 1 in [57] (Piecewise linear membership functions)</td>
<td>Cannot be found</td>
</tr>
<tr>
<td>Theorem 1 in [60] (Staircase membership functions)</td>
<td>Cannot be found</td>
</tr>
</tbody>
</table>

The comparison between the proposed method and exiting methods is summarised in Table 3.3. As shown in the table, the LMI-based stability conditions without membership shape information in Theorem 4 are employed to check the
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Figure 3.4: Stability regions given by Theorem 5 and ("o") and Theorem 6 ("x") with $\delta = 0.25$.

system stability. However, no stability region can be found. Also, by applying the stability analysis results in [57] and [60] with the chosen values of $\delta$, stability region cannot be found. As the FMB based control system considered in this example that the fuzzy controller does not share the same set of premise membership functions and the same number of premise rules as those of the T-S fuzzy controller, the LMI-based stability conditions in [12,25,36,37,52–56,111,112,117,118] cannot be applied.
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Figure 3.5: Stability regions given by Theorem 6 with \( \delta = 2 \) (“○”), Theorem 6 (“□”) with \( \delta = 5 \) and Theorem 4 and (“×”).

Table 3.4: \( X \) and feedback gains \( G_j \) given by Theorem 5

<table>
<thead>
<tr>
<th>( \delta = 0.5 )</th>
<th>( X = \begin{bmatrix} 1.1687 \times 10^{-3} &amp; -1.3899 \times 10^{-4} \ -1.3899 \times 10^{-4} &amp; 5.0587 \times 10^{-5} \end{bmatrix} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G_1 = \begin{bmatrix} -12.6154 &amp; -27.3351 \end{bmatrix} )</td>
<td></td>
</tr>
<tr>
<td>( G_2 = \begin{bmatrix} 7.0691 &amp; 20.2116 \end{bmatrix} )</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \delta = 0.25 )</th>
<th>( X = \begin{bmatrix} 1.2136 \times 10^{-4} &amp; -1.5074 \times 10^{-5} \ -1.5074 \times 10^{-5} &amp; 2.7603 \times 10^{-6} \end{bmatrix} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G_1 = \begin{bmatrix} -12.2050 &amp; -55.5202 \end{bmatrix} )</td>
<td></td>
</tr>
<tr>
<td>( G_2 = \begin{bmatrix} 5.7518 &amp; 32.8570 \end{bmatrix} )</td>
<td></td>
</tr>
</tbody>
</table>
Table 3.5: $X$ and feedback gains $G_j$ given by Theorem 6

$$ \delta = 5 \quad X = \begin{bmatrix} 0.2203 & 0.8867 \\ 0.8867 & 7.9934 \end{bmatrix} $$

$$ G_1 = \begin{bmatrix} -8.8615 & -25.2316 \\ -4.0305 & -9.4055 \end{bmatrix} $$

$$ G_2 = \begin{bmatrix} -0.2203 & 0.8867 \\ 0.8867 & 7.9934 \end{bmatrix} $$

$$ \delta = 2 \quad X = \begin{bmatrix} 6058.6454 & 2084.6378 \times 10^1 \\ 2084.6378 \times 10^1 & 1123.8177 \times 10^2 \end{bmatrix} $$

$$ G_1 = \begin{bmatrix} -21.4911 & -64.1054 \\ -1.4363 & -2.3908 \end{bmatrix} $$

$$ G_2 = \begin{bmatrix} -0.214911 & -0.6094 \\ -1.4363 & -2.3908 \end{bmatrix} $$

$$ \delta = 0.5 \quad X = \begin{bmatrix} 1061.3989 \times 10^4 & 3615.0298 \times 10^2 \\ 3615.0298 \times 10^2 & 3897.9899 \times 10^3 \end{bmatrix} $$

$$ G_1 = \begin{bmatrix} -6.6094 & -12.8242 \\ 1.8450 & 6.5742 \end{bmatrix} $$

$$ G_2 = \begin{bmatrix} -0.66094 & -1.28242 \\ 1.8450 & 6.5742 \end{bmatrix} $$

$$ \delta = 0.25 \quad X = \begin{bmatrix} 5860.1523 \times 10^4 & 2532.5827 \times 10^5 \\ 2532.5827 \times 10^5 & 2084.6378 \times 10^6 \end{bmatrix} $$

$$ G_1 = \begin{bmatrix} -9.3553 & -30.5099 \\ 4.1911 & 18.9780 \end{bmatrix} $$

$$ G_2 = \begin{bmatrix} -0.93553 & -3.05099 \\ 4.1911 & 18.9780 \end{bmatrix} $$

Figure 3.6: Behaviours in $x_1$-$x_2$ plane for $a = 10$ and $b = 28$ with $\delta = 0.5$ in Theorem 5, where the initial conditions are indicated by “o”. 
Figure 3.7: Behaviours in $x_1$-$x_2$ plane for $a = 10$ and $b = 48$ with $\delta = 0.5$ in Theorem 5, where the initial conditions are indicated by “o”.

Figure 3.8: Behaviours in $x_1$-$x_2$ plane for $a = 1$ and $b = -10$ with $\delta = 5$ in Theorem 6, where the initial conditions are indicated by “o”.
Table 7 X and feedback gains given by Theorem 3

\[
\begin{bmatrix}
0.2203 & 0.8867 \\
0.8867 & 7.9934 \\
\end{bmatrix}
\]

Figure 3.9: Behaviours in $x_1$-$x_2$ plane for $a = 1$ and $b = -4$ with $\delta = 2$ in Theorem 6, where the initial conditions are indicated by "o".

Figure 3.10: Behaviours in $x_1$-$x_2$ plane for $a = 10$ and $b = 16$ with $\delta = 0.5$ in Theorem 6, where the initial conditions are indicated by "o".
3.3 MFD Stability Analysis of T-S FMB Control Systems Designed with Sub-Domain Fuzzy Controllers

In the previous section, relaxation of LMI-based stability conditions is achieved by utilising PMF-based and lower-upper-PMF-based stability analysis methods. The T-S FMB control system investigated is designed under imperfect premise...
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matching scheme, for which, a single fuzzy controller is employed to stabilise the nonlinear plant represented by the fuzzy model in the whole operating domain. In this section, as an extension to imperfect premise matching control scheme, we utilise sub-domain fuzzy controllers to stabilise the nonlinear plant. Furthermore, the MFD method using PMFs with the consideration of approximation error is proposed.

For the purpose to enhance the capacity of compensation, sub-domain fuzzy controllers [28, 115] were proposed to handle nonlinear control system with strong nonlinearity. The operating domain of the nonlinear plant is partitioned into several sub-domains. Corresponding to each sub-domain, a local fuzzy controller is designed to stabilise the fuzzy model. As the nonlinearity of the plant for each sub-domain is weaker than that for the whole operating domain, it offers a favour to perform more stable design in each sub-domain. When the FMB control system is working in one of the sub-domains, the corresponding sub-domain fuzzy controller is employed to stabilise the plant. Further relaxed stability conditions can be achieved potentially by utilizing this control scheme. The other motivation for applying sub-domain controller is that the design work can be facilitated combining with the proposed MFD method. As the operating domain partition is one of the analysis procedures in the MFD method proposed in this section, sub-domain controllers can be constructed naturally on this existing operating domain partition without extra effort.

In this section, T-S FMB control system with sub-domain fuzzy controllers is investigated. In Section 3.3.1, sub-domain fuzzy controllers are presented. Section 3.3.2 gives relaxed LMI-based stability conditions. Section 3.3.3 entails a numerical example to demonstrate the validity of the proposed design method.

### 3.3.1 Sub-domain T-S Fuzzy Controllers

In this section, we divide the operating domain of the FMB control system into $D$ sub-domains firstly. Denote the system state vector as $\mathbf{x}(t) = [x_1(t), x_2(t), \ldots, x_n(t)]$. 
The sub-domains $s_d$, $d = 1, 2, \ldots, D$, are characterized by the state variables $\underline{x}_{sd} \leq x_\varsigma(t) \leq \overline{x}_{sd}$, $\varsigma = 1, 2, \ldots, n$, where $\underline{x}_{sd}$ and $\overline{x}_{sd}$ denote the lower and upper bounds of the state variable $x_\varsigma(t)$ in the sub-domain $s_d$, respectively. Corresponding to the sub-domain $s_d$, we proposed a sub-domain fuzzy controller described by the following $c$ rules to stabilise the nonlinear plant represented by the fuzzy model (2.3).

Rule $j$: IF $g_\beta(x(t))$ is $N^j_\beta$ AND $\cdots$ AND $g_\Omega(x(t))$ is $N^j_\Omega$ THEN $u(t) = G_{jd}x(t)$  \hspace{1cm} (3.26)

where $N^j_\beta$ is a fuzzy term of rule $j$ corresponding to the function $g_\beta(x(t))$, $\beta = 1, 2, \ldots, \Omega$, $j = 1, 2, \ldots, c$, $\Omega$ is a positive integer and $G_{jd} \in \mathbb{R}^{m \times n}$, $j = 1, 2, \ldots, c$, $d = 1, 2, \ldots, D$, are the constant feedback gains to be determined. The fuzzy controller in the sub-domain $s_d$ is defined as follows:

$$u(t) = \sum_{j=1}^{c} m_j(x(t))G_{jd}x(t),$$  \hspace{1cm} (3.27)

where $m_j(x(t))$, $j = 1, 2, \ldots, c$, are the membership functions. It should be noted that in the boundary of the connected sub-domains, either one of the sub-domain fuzzy controllers can be applied for the control process. The sub-domain fuzzy controller is reduced to the traditional one [12, 25] when it shares a common set of feedback gains, i.e., $G_j = G_{jd}$ for all $s_d$.

### 3.3.2 Stability Analysis

The stability of the T-S FMB control system is investigated in this section. In the following analysis, for brevity, the time $t$ associated with the variables is dropped for the situation without ambiguity, e.g., $x(t)$ is denoted as $x$. 
From (2.3) and (3.27), considering the property of (3.3), we have,

\[
\dot{x}(t) = \sum_{i=1}^{p} \sum_{j=1}^{c} w_i(x(t))m_j(x(t))(A_i + B_iG_{jd})x(t) \quad \forall \, d. \quad (3.28)
\]

The control objective is to determine the feedback gains \(G_{jd}\), such that the fuzzy controller is able to drive the system states towards the origin, i.e., \(x(t) \to 0\) as time \(t \to \infty\).

In this section, we attempt to investigate the stability of FMB control systems under imperfect premise matching and bring the information of the membership functions to the stability conditions for relaxation. As the current form of the membership functions, \(w_i(x)\) and \(m_j(x)\), is not favourable for this purpose, we propose the following representation for the membership functions. Considering the sub-domain \(s_d\) characterized by \(x_{\varsigma d} \leq x_\varsigma(t) \leq x_{\varsigma d}\), the multiplication of the membership functions, \(w_i(x)m_j(x)\), is approximated by the following expression [57]:

\[
h_{ijd}(x) = \sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \cdots \sum_{i_n=1}^{2} v_{1i_1d}(x_1) \cdots v_{ni_nd}(x_n)
\]

\[
\times w_i(s_{i_1i_2\cdots i_n d})m_j(s_{i_1i_2\cdots i_n d}) \quad \forall \, i, j, d, \quad (3.29)
\]

where \(s_{i_1i_2\cdots i_n d} = [x_{1i_1}, x_{1i_2}, \ldots, x_{ni_n}]\), \(x_\varsigma = \begin{cases} x_{\varsigma d} & \text{for } i_\varsigma = 1 \\ x_{\varsigma d} & \text{for } i_\varsigma = 2 \end{cases}\), \(\varsigma = 1, 2, \ldots, n\), \(d = 1, 2, \ldots, D\). The functions \(v_{\varsigma i_\varsigma d}(x_\varsigma)\) are defined such that \(\sum_{i_\varsigma=1}^{2} v_{\varsigma i_\varsigma d}(x_\varsigma) = 1\) and \(\sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \cdots \sum_{i_n=1}^{2} v_{1i_1d}(x_1)v_{2i_2d}(x_2) \cdots v_{ni_nd}(x_n) = 1\) for all \(\varsigma\) and \(d\). As a result, with the property of the membership functions stated in (2.4) and (3.3), we have \(\sum_{i=1}^{p} \sum_{j=1}^{c} h_{ijd}(x) = 1\) that will be utilised in the following analysis.

The multiplication of the membership functions in the sub-domain \(s_d\) can be represented as follows:

\[
w_i(x)m_j(x) = h_{ijd}(x) + \Delta_{ijd}(x) \quad \forall \, i, j, d, \quad (3.30)
\]
where $\Delta_{ijd}(x)$ is the approximation error.

We consider the following quadratic Lyapunov function candidate to investigate the stability of the FMB control system (3.28):

$$V = x^T P x,$$  

(3.31)

where $0 < P = P^T \in \mathbb{R}^{n \times n}$.

From (3.28) and (3.31), we have,

$$\dot{V} = \sum_{i=1}^{p} \sum_{j=1}^{c} w_i(x) m_j(x) x^T ((A_i + B_j G_{jd})^T P + P (A_i + B_j G_{jd})) x \quad \forall \; d,$$

(3.32)

Denote $z = P x$ and $X = P^{-1}$. Define the feedback gains $G_{jd} = N_{jd} X^{-1}$, where $N_{jd} \in \mathbb{R}^{m \times n}$, $j = 1, 2, \ldots, c$, $d = 1, 2, \ldots, D$, are matrices to be determined.

From (3.32), with the representation in (3.30), we have,

$$\dot{V} = \sum_{i=1}^{p} \sum_{j=1}^{c} w_i(x) m_j(x) z^T Q_{ijd} z$$

$$= \sum_{i=1}^{p} \sum_{j=1}^{c} (h_{ijd}(x) + \Delta_{ijd}(x)) z^T Q_{ijd} z \quad \forall \; d,$$

(3.33)

where $Q_{ijd} = A_i X + X A_i^T + B_i N_{jd} + N_{jd}^T B_i^T$.

It can be seen from (3.33) that the FMB control system (3.28) is asymptotically stable in the sense of Lyapunov if the basic stability conditions $Q_{ijd} < 0$ for all $i$, $j$ and $d$ are satisfied. However, the membership functions are not considered, the stability conditions can be applied to any shape of membership functions and are thus very conservative. For relaxing the stability conditions, it makes more sense to consider $\sum_{i=1}^{p} \sum_{j=1}^{c} w_i(x) m_j(x) Q_{ijd} < 0$ for all $d$. However, as the membership functions are involved, the number of stability conditions will become infinity. It is thus impractical to find the solution numerically using convex programming techniques. Instead, to circumvent the difficulty, the original multipli-
cation of membership functions, \( w_i(x)m_j(x) \), is approximated by \( h_{ijd}(x) \) defined in (3.29). It has a nice property that the grades of membership in the sub-domain are characterized by some sample points. The infinite number of stability conditions can thus be approximated by the finite ones. Furthermore, through \( h_{ijd}(x) \), the information of membership functions at \( s_{i_1i_2...i_n} \), i.e., \( w_i(s_{i_1i_2...i_n})m_j(s_{i_1i_2...i_n}) \), can be brought to the stability conditions. Consequently, the stability conditions only need to be satisfied for membership functions approximated by \( h_{ijd}(x) \) but not for any shapes and relaxed stability conditions can be achieved.

The regional information of \( w_i(x)m_j(x) \) and \( \Delta_{ijd}(x) \) in each sub-domain will be carried by some slack matrices [110, 119] to the stability analysis through the S-procedure [13]. Considering \( \rho_{ijd} \leq w_i(x)m_j(x) \leq \bar{\rho}_{ijd} \) where \( \rho_{ijd} \) and \( \bar{\rho}_{ijd} \) are constants, and introducing slack matrices \( 0 \leq R_{ijd} = R_{ijd}^T \in \mathbb{R}^{n \times n} \) and \( 0 \leq S_{ijd} = S_{ijd}^T \in \mathbb{R}^{n \times n} \), we have the following inequalities for the regional information of \( w_i(x)m_j(x) \):

\[
\sum_{i=1}^{p} \sum_{j=1}^{c} (\rho_{ijd} - w_i(x)m_j(x))R_{ijd} \geq 0 \quad \forall \ d, \tag{3.34}
\]

\[
\sum_{i=1}^{p} \sum_{j=1}^{c} (w_i(x)m_j(x) - \rho_{ijd})S_{ijd} \geq 0 \quad \forall \ d. \tag{3.35}
\]

With the expression (3.29), (3.30) and the property of the membership functions (2.4) and (3.3), it leads to \( \sum_{i=1}^{p} \sum_{j=1}^{c} \Delta_{ijd}(x) = 0 \). Consequently, we have the following equation that brings free matrices \( M_d = M_d^T \in \mathbb{R}^{n \times n} \) to (3.33):

\[
\sum_{i=1}^{p} \sum_{j=1}^{c} \Delta_{ijd}(x)M_d = 0 \quad \forall \ d. \tag{3.36}
\]

Adding (3.34)–(3.36) to (3.33), recalling that \( \Delta_{ijd}(x) = w_i(x)m_j(x) - h_{ijd}(x) \)
(from (3.30)), we have

\[
\dot{V} \leq \sum_{i=1}^{p} \sum_{j=1}^{c} z^T (h_{ijd}(x)(Q_{ijd} - R_{ijd} + S_{ijd} + W_{ijd})) \\
+ \overline{r}_{ijd} R_{ijd} - \overline{r}_{ijd} S_{ijd} + z^T (Q_{ijd} - R_{ijd} + S_{ijd}) z \\
+ \sum_{i=1}^{p} \sum_{j=1}^{c} \Delta_{ijd}(x) z^T \left( Q_{ijd} - R_{ijd} + S_{ijd} + M_d + W_{ijd} \right) z \\
- \sum_{i=1}^{p} \sum_{j=1}^{c} w_i(x)m_j(x) z^T W_{ijd} z \quad \forall \, d,
\]

(3.37)

where \( W_{ijd} = W_{ijd}^T \in \mathbb{R}^{n \times n} \).

Based on the property of the membership functions, the approximation error in the sub-domain \( s_d \) is bounded that it satisfies \( \gamma_{ijd} \leq \Delta_{ijd}(x) \leq \overline{\gamma}_{ijd} \) where \( \gamma_{ijd} \) and \( \overline{\gamma}_{ijd} \) are constants. The upper bound information of \( \Delta_{ijd}(x) \) in sub-domain \( s_d \) is carried by the following inequalities to the stability analysis through the slack matrices \( 0 \leq T_{ijd} = T_{ijd}^T \in \mathbb{R}^{n \times n} \):

\[
\sum_{i=1}^{p} \sum_{j=1}^{c} (\gamma_{ijd} - \Delta_{ijd}(x)) T_{ijd} \geq 0 \quad \forall \, d.
\]

(3.38)

From (3.37) and (3.38), we have

\[
\dot{V} \leq \sum_{i=1}^{p} \sum_{j=1}^{c} z^T (h_{ijd}(x)(Q_{ijd} - R_{ijd} + S_{ijd} + W_{ijd})) \\
+ \overline{r}_{ijd} R_{ijd} - \overline{r}_{ijd} S_{ijd} + \overline{\gamma}_{ijd} T_{ijd} z \\
+ \sum_{i=1}^{p} \sum_{j=1}^{c} \Delta_{ijd}(x) z^T (Q_{ijd} - R_{ijd} + S_{ijd}) z \\
+ W_{ijd} - T_{ijd}) z - \sum_{i=1}^{p} \sum_{j=1}^{c} w_i(x)m_j(x) z^T W_{ijd} z \\
= \sum_{i=1}^{p} \sum_{j=1}^{c} z^T (h_{ijd}(x)(Q_{ijd} - R_{ijd} + S_{ijd} + W_{ijd})) \\
+ \overline{r}_{ijd} R_{ijd} - \overline{r}_{ijd} S_{ijd} + \overline{\gamma}_{ijd} T_{ijd} + \gamma_{ijd}(Q_{ijd} - R_{ijd} + S_{ijd}) + M_d + W_{ijd} - T_{ijd} \right) z
\]
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\[ + \sum_{i=1}^{p} \sum_{j=1}^{c} (\Delta_{ijd}(x) - \gamma_{ijd})z^T(Q_{ijd} - R_{ijd} + S_{ijd}) + M_d + W_{ijd} - T_{ijd})z \]

\[ - \sum_{i=1}^{p} \sum_{j=1}^{c} w_i(x)m_j(x)z^T W_{ijd}z \quad \forall \ d. \quad (3.39) \]

Similar to the inequalities (3.34) and (3.35), introducing the slack matrices \(0 \leq \underline{U}_{ijd} = \underline{U}_{ijd}^T \in \mathbb{R}^{n \times n}\) and \(0 \leq \overline{U}_{ijd} = \overline{U}_{ijd}^T \in \mathbb{R}^{n \times n}\), we have \(\sum_{i=1}^{p} \sum_{j=1}^{c} (\rho_{ijd} - w_i(x)m_j(x))\overline{U}_{ijd} \geq 0\) and \(\sum_{i=1}^{p} \sum_{j=1}^{c} (\rho_{ijd} - \overline{U}_{ijd})\underline{U}_{ijd} \geq 0\) for all \(d\).

Adding these two inequalities to (3.39) and using expression (3.29), we have

\[ V \leq \sum_{i=1}^{p} \sum_{j=1}^{c} z^T(h_{ijd}(x)(Q_{ijd} - R_{ijd} + S_{ijd} + W_{ijd}) \]

\[ + \rho_{ijd}(R_{ijd} + \overline{U}_{ijd}) - \rho_{ijd}(S_{ijd} + \underline{U}_{ijd}) \]

\[ + \gamma_{ijd} T_{ijd} + \gamma_{ijd} (Q_{ijd} - R_{ijd} + S_{ijd} + M_d + W_{ijd} - T_{ijd})z \]

\[ + \sum_{i=1}^{p} \sum_{j=1}^{c} w_i(x)m_j(x)z^T(-W_{ijd} + \overline{U}_{ijd} - \underline{U}_{ijd})z \]

\[ = \sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \cdots \sum_{i_n=1}^{2} v_{1i_1d}(x_1)v_{2i_2d}(x_2) \cdots v_{ni_nd}(x_n) \]

\[ \times z^T \sum_{i=1}^{p} \sum_{j=1}^{c} (w_i(s_{i_1i_2\cdots i_n})m_j(s_{i_1i_2\cdots i_n})) \]

\[ \times (Q_{ijd} - R_{ijd} + S_{ijd} + W_{ijd}) \]

\[ + \rho_{ijd}(R_{ijd} + \overline{U}_{ijd}) - \rho_{ijd}(S_{ijd} + \underline{U}_{ijd}) \]

\[ + \gamma_{ijd} T_{ijd} + \gamma_{ijd} (Q_{ijd} - R_{ijd} + S_{ijd} + M_d + W_{ijd} - T_{ijd})z \]

\[ + \sum_{i=1}^{p} \sum_{j=1}^{c} w_i(x)m_j(x)z^T(-W_{ijd} + \overline{U}_{ijd} - \underline{U}_{ijd})z \quad \forall \ d. \quad (3.40) \]
Based on Lyapunov stability theory, the FMB control system (3.28) is asymptotically stable when both $V > 0$ and $\dot{V} < 0$ (excluding $z = x = 0$), which can be achieved if the LMI-based stability conditions summarised in the following theorem are satisfied.

**Theorem 7** Considering the operating domain partitioned into $D$ sub-domains of which each is characterized by $\underline{x}_d \leq x(t) \leq \overline{x}_d$, $d = 1, 2, \ldots, D$, the FMB system (3.28) formed by a nonlinear plant (represented by the fuzzy model (2.3)) and a fuzzy controller (3.27) connected in a closed loop, is guaranteed to be asymptotically stable if there exist predefined scalars $\underline{\rho}_{ijd}$, $\overline{\rho}_{ijd}$, $\underline{\gamma}_{ijd}$ and $\overline{\gamma}_{ijd}$, satisfying $\underline{\rho}_{ijd} \leq w_i(x)m_j(x) \leq \overline{\rho}_{ijd}$ and $\underline{\gamma}_{ijd} \leq \Delta_{ijd}(x) \leq \overline{\gamma}_{ijd}$, respectively, $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, c$, $d = 1, 2, \ldots, D$, and there exist matrices $M_d = M_d^T \in \mathbb{R}^{n \times n}$, $N_j \in \mathbb{R}^{m \times n}$, $R_{ijd} = R_{ijd}^T \in \mathbb{R}^{n \times n}$, $S_{ijd} = S_{ijd}^T \in \mathbb{R}^{n \times n}$, $T_{ijd} = T_{ijd}^T \in \mathbb{R}^{n \times n}$, $U_{ijd} = U_{ijd}^T \in \mathbb{R}^{n \times n}$, $\mathbf{U}_{ijd} = \mathbf{U}_{ijd}^T \in \mathbb{R}^{n \times n}$, $\mathbf{X} = \mathbf{X}^T \in \mathbb{R}^{n \times n}$, $\mathbf{W}_{ijd} = \mathbf{W}_{ijd}^T \in \mathbb{R}^{n \times n}$ such that the following LMIs are satisfied.

$$\mathbf{X} > 0; \quad (3.41)$$

$$R_{ijd} \geq 0 \quad \forall \ i, j, d; \quad (3.42)$$

$$S_{ijd} \geq 0 \quad \forall \ i, j, d; \quad (3.43)$$

$$T_{ijd} \geq 0 \quad \forall \ i, j, d; \quad (3.44)$$

$$\mathbf{U}_{ijd} \geq 0 \quad \forall \ i, j, d; \quad (3.45)$$

$$\mathbf{U}_{ijd} \geq 0 \quad \forall \ i, j, d; \quad (3.46)$$

$$-\mathbf{W}_{ijd} + \mathbf{U}_{ijd} - \mathbf{U}_{ijd} \leq 0 \quad \forall \ i, j, d; \quad (3.47)$$

$$\sum_{i=1}^{p} \sum_{j=1}^{c} (w_i(s_{i1i2\ldots i_d})m_j(s_{i1i2\ldots i_d})(Q_{ijd} - R_{ijd} + S_{ijd})$$
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\[
\begin{align*}
\gamma_{ijd}T_{ijd} + \gamma_{ijd}(Q_{ijd} - R_{ijd} + S_{ijd} + M_d) \\
+ W_{ijd} - T_{ijd}) < 0 & \quad \forall i_1, i_2, \ldots, i_n, d; \quad (3.48)
\end{align*}
\]

\[
\begin{align*}
Q_{ijd} - R_{ijd} + S_{ijd} + M_d + W_{ijd} - T_{ijd} < 0 & \quad \forall i, j, d; \quad (3.49)
\end{align*}
\]

where \( Q_{ijd} = A_iX + XA_i^T + B_iN_{jd} + N_{jd}^TB_i^T, s_{i_1i_2\cdots i_n} = [x_{i_1}, x_{i_2}, \ldots, x_{i_n}] \).

\[
x_{i_1_1} = \begin{cases} 
  x_{i_1} & \text{for } i_1 = 1 \\
  x_{i_2} & \text{for } i_2 = 1 \\
  x_{i_3} & \text{for } i_3 = 2 \\
\end{cases}, \quad i = 1, 2, \ldots, p, \quad j = 1, 2, \ldots, c, \quad d = 1, 2, \ldots, D, \\
\zeta = 1, 2, \ldots, n, \text{ and the feedback gains are defined as } G_{jd} = N_{jd}X^{-1}.
\]

3.3.3 Simulation Example

In this simulation example, the effectiveness of proposed MFD stability analysis method and sub-domain fuzzy control scheme, is verified by investigating the LMI-based stability conditions given by Theorem 7. The way used is the same as in Section 3.2.4. The stability conditions which are less conservative offers the larger size of stability region.

In this section, the stability region produced by each theorem is investigated by an FMB control system which is formed by a 3-rule T-S fuzzy model in the form of (2.3) and a 2-rule sub-domain fuzzy controller in the form of (3.27) connected in a closed loop. In this example, we choose the same T-S fuzzy model used in Section 3.2.4 [52].

A 2-rule fuzzy controller in the form of (3.27) is employed to stabilise the nonlinear plant. The membership functions of the fuzzy controller are chosen as \( m_1(x_1) = \mu_{N_1^1}(x_1) = 1 - \frac{1}{1 + e^{-x_1}} \) and \( m_2(x_1) = \mu_{N_1^2}(x_1) = 1 - m_1(x_1) \). Both of the membership functions of fuzzy model and fuzzy controller is shown in Fig. 3.12.

The FMB control system is obtained as \( \dot{x} = \sum_{i=1}^{3} \sum_{j=1}^{2} w_i(x_1)m_j(x_1)(A_i + \)
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Considering the operating domain $-10 \leq x_1 \leq 10$, the number of sub-domains is chosen to be 4. The $s_d$ sub-domain is characterized by $5(d-3) \leq x_1 \leq 5(d-2)$, $d = 1, 2, 3, 4$. Define $v_{11d}(x_1) = 1 - \frac{x_1 - \xi_{1d}}{\xi_{1d} - \eta_{1d}}$ and $v_{12d}(x_1) = 1 - v_{11d}(x_1)$ where $\xi_{1d} = 5(d-3)$ and $\eta_{1d} = 5(d-2)$. From (3.29), we have $h_{i,j,d}(x_1) = \sum_{i_1=1}^{2} v_{1i_1d}(x_1)w_i(x_{1i_1})m_j(x_{1i_1})$ for all $i, j, d$ where $x_{1i_1} = \begin{cases} \xi_{1d} & \text{for } i_1 = 1 \\ \eta_{1d} & \text{for } i_1 = 2 \end{cases}$. The feedback gains are considered to be the same for all sub-domains, i.e., $G_j = G_{jd}$ for all $d$. The LMI-based stability conditions in Theorem 7 are employed to check the stability region of the FMB control system, using the MATLAB® LMI toolbox, with $20 \leq a \leq 30$ at the interval of 1 and $30 \leq b \leq 54$ at the interval of 2. The stability region is shown in Fig. 3.13 indicated by the symbol “×”.

Considering the same settings, we increase the number of sub-domains to 10 to investigate how it influences the stability region. In this case, the $s_d$ sub-domain is characterized by $2(d-6) \leq x_1 \leq 2(d-5)$, $d = 1, 2, \cdots, 10$. The stability region with 10 sub-domains is shown in Fig. 3.13 indicated by “◦”. Considering the feedback gains to be $G_{jd}$, the stability regions corresponding to 4 and 10
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sub-domains, respectively, are shown in Fig. 3.14. It is observed from the figures that the stability regions corresponding to 10 sub-domains are larger due to the smaller approximation error when the number of sub-regions increases. Moreover, by employing the feedback gains $G_{jd}$ for different sub-domains, it is possible to produce a much larger stability region. It should be noted that Fig. 3.13 and Fig. 3.14 are not in the same scale.

The stability conditions under perfect premise matching [12, 25, 36, 37, 52, 53, 55, 56, 111, 112, 117, 118] cannot be applied in this example as the premise membership functions between the membership functions of the fuzzy model and fuzzy controller do not match. Furthermore, the stability conditions under imperfect premise matching [39, 40, 110] cannot be applied as it is required that both fuzzy model and fuzzy controller share the same number of rules. For comparison purposes, the stability conditions under imperfect premise matching in [12] (i.e., $X > 0$ and $Q_{ij} < 0$ for all $i$ and $j$) and [60] (with the same interval as the above) are employed to check the stability region of the FMB control system. However, no feasible solution can be found.
Figure 3.13: Stability regions given by Theorem 7 using the same set of feedback gains for 4 sub-domains (“x”) and 10 sub-domains (“o”).

Figure 3.14: Stability regions given by Theorem 7 using different sets of feedback gains for 4 sub-domains (“x”) and 10 sub-domains (“o”).
3.4 Conclusion

In this chapter, the stability of T-S FMB control systems designed under imperfect premise matching, and stabilisation using sub-domain fuzzy controllers have been investigated. PMFs-based analysis is employed to facilitate the stability analysis. The utilisation of PMFs can effectively bring the membership function shape information into the stability conditions. With the consideration of computational burden, lower-upper-PMF-based analysis as an alternative stability analysis method, is proposed and relaxed stability conditions can be achieved in the case that PMFs employed have low accuracy.

Sub-domain fuzzy controllers are proposed to control the nonlinear system. They can be designed based on the operating domain partition which is performed for extracting membership function shape information. Therefore, controller design procedures can be combined with the proposed MFD stability analysis. LMI-based stability conditions have been derived to check the stability of T-S FMB control system and perform fuzzy controller synthesis. Two simulation examples are presented to demonstrate the viability of the proposed methods.
Chapter 4

Relaxed Stability Analysis of PFMB Control Systems

4.1 Introduction

In the previous chapter, the stability analysis of T-S FMB control systems under imperfect premise matching design was investigated. Two MFD stability analysis methods: PMF-based analysis and lower-upper-PMFs-based analysis were introduced for relaxing the stability conditions. The sub-domain fuzzy controllers were utilised for stabilisation of T-S FMB control system. In this chapter, PFMB control system designed with sub-domain polynomial fuzzy controllers is investigated with the consideration of regional membership function shape information.

As introduced in Section 2.5, PFMB control system [18] is a closed-loop system which consists of polynomial fuzzy model and polynomial fuzzy controller. With the consideration of polynomials, the capabilities of system modelling and feedback compensation are enhanced effectively. However, the stability conditions of PFMB can hardly recast as LMI problem due to the existence of the polynomial variables. To tackle this issue, SOS-based approach [18] was employed to facilitate the stability analysis of PFMB control system. The stability conditions derived are in the form of SOS and can be solved numerically using semidefinite
programming [107]. The SOS-based approach also has a feature that polynomial Lyapunov functions can be applied naturally in stability analysis, that the basic quadratic stability analysis for T-S FMB control system is its special case. Therefore, more relaxed stability conditions can be achieved in this general framework. PFMB control using SOS-based approach has attracted a lot of attention in fuzzy control community (e.g., [1, 21, 26, 43, 72, 75, 93, 116, 120]).

Although SOS-based stability analysis can lead to more relaxed stability conditions comparing with the LMI-based method. The lack of membership function shape information in stability analysis is still a source of conservativeness. For the purpose of extracting more membership function information, various MFD methods are proposed [57, 60, 120–122]. Membership functions were also handled as symbolic variables with the consideration of their properties and boundary information in the stability analysis relaxation [27]. For the MFD methods using approximated membership functions, the shape information of memberships function can be extracted effectively. However, there is a drawback of heavy computational burden due to a large number of the variables and conditions. In [42], the stability analysis method based on the operating domain partition was proposed. In this method, the operating domain partition heavily depends on the membership function shape and order relation. In [27, 57], the global/local boundary information of the membership functions are considered, but the numerical relationship between all the membership function terms is ignored.

In this section, an MFD stability analysis method using regional membership function shape information is proposed. The operating domain of the PFMB control system is partitioned into several sub-domains uniformly which makes the partition scheme to be more flexible compared with the method proposed in [42] as mentioned above. Based on operating domain partition, the regional information of the membership functions is extracted in the sub-domains. In each sub-domain, two types of membership function shape information will be considered: 1) the bounds of the single membership function overlap term; and 2) the numerical
relationship between all the membership function overlap terms. Both of them are represented in the form of inequalities and are brought into the stability analysis through the $S$-procedure. Combining with the proposed MFD method naturally, as an extension to the sub-domain T-S fuzzy controllers introduced in Section 3.2.1, sub-domain polynomial fuzzy controllers are employed to work in every sub-domain to stabilise the polynomial FMB control system.

The PMFB control system in [18] is constructed with PDC scheme, same as the case in basic LMI-based method [12] for T-S FMB control system, the polynomial fuzzy controller is required to share the premise membership functions with the polynomial fuzzy model. The drawbacks discussed in Section 3.2.1 also exist in the SOS-based method proposed in [18]. Therefore, the PFMB control system designed under imperfectly premise matching is considered in this chapter. The SOS-based stability conditions proposed in this section can be used for polynomial fuzzy controller synthesis with the design flexibility for fuzzy controller membership function design.

The rest of the section is organised as follows. In Section 4.2, the sub-domain polynomial fuzzy controllers are presented. Section 4.3 presents the MFI stability conditions for PFMB control system. In Section 4.4, two types of regional membership function shape information are introduced and the relaxed SOS-based stability conditions are given. Section 4.5 entails a simulation to verify the viability of the proposed method. A search approach applied in the given simulation example for searching the numerical relationship between all overlap terms is proposed in Section 4.6. It is followed by concluding remarks.

4.2 Sub-domain Polynomial Fuzzy Controllers

Combining with the MFD stability analysis method represented in following sections, sub-domain polynomial fuzzy controllers are employed to enhance the feedback compensation capability. They are designed to stabilise the nonlinear plant represented by a polynomial fuzzy model (2.9).
First, the system operating domain is partitioned into \( D \) connected subdomains, \( D \) is a positive integer. Recalling that the system state vector is denoted as \( x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \), the sub-domain denoted as \( s_d, d = 1, 2, \ldots, D \), is characterized by the state variables \( \underline{x}_{sd} \leq x_\varsigma(t) \leq \overline{x}_{sd}, \varsigma = 1, 2, \ldots, n \), where \( \underline{x}_{sd} \) and \( \overline{x}_{sd} \) are predefined constants used to denote the lower and upper bounds of the state variable \( x_\varsigma \) in the sub-domain \( s_d \), respectively. A set of sub-domain polynomial fuzzy controllers is proposed according to the above operating domain partition. As an extension to the controller proposed in [29] which has constant feedback gains working in each sub-domain, the controllers employed here have polynomial feedback gains to stabilise the polynomial fuzzy model (2.9).

Corresponding to the sub-domain \( s_d \), the sub-domain polynomial fuzzy controller is described by the following \( c \)-rules.

Rule \( j \): IF \( g_1(x(t)) \) is \( N^j_1 \) AND \( \cdots \) AND \( g_\Omega(x(t)) \) is \( N^j_\Omega \)

THEN \( u(t) = G_{jd}(x(t))\hat{x}(x(t)) \) \hspace{1cm} (4.1)

where \( N^j_\beta \) is a fuzzy term of rule \( j \) corresponding to the function \( g_\beta(x(t)), \beta = 1, 2, \ldots, \Omega, \Omega \) is a positive integer, \( j = 1, 2, \ldots, c \) and \( G_{jd}(x) \in \mathbb{R}^{m \times N}, j = 1, 2, \ldots, c, d = 1, 2, \ldots, D \), are the polynomial feedback gains to be determined.

The polynomial fuzzy controller in sub-domain \( s_d \) is defined as

\[
\mathbf{u}(t) = \sum_{j=1}^{c} m_j(x(t))G_{jd}(x(t))\hat{x}(x(t)),
\]

where \( m_j(x(t)), j = 1, 2, \ldots, c, \) are the membership functions.

In the control operating operation, the \( d^{th} \) sub-domain polynomial fuzzy controller is employed when the system state falls into the sub-domain \( s_d \). In the boundary of the connected sub-domains, either one of the sub-domain fuzzy controllers can be applied for the control process.

**Remark 12** In practice, if the switching control scheme is not suitable, when physical constraints are considered such as the issues of actuator, performance,
controller structure, implementation costs, etc, by choosing all sets of feedback gains from all sub-domains to be the same, i.e., $G_j(x(t)) = G_{jd}(x(t))$ for all $d$, the polynomial fuzzy controller (4.2) will be reduced to the traditional fuzzy controller without switching at the boundary of sub-domains.

**Remark 13** The number of rules $c$ and membership function $m_j(x(t))$ can be different from the fuzzy model in each sub-domain which can enhance the design flexibility of the polynomial fuzzy controller.

### 4.3 MFI Stability Analysis

The PFMB control system formed by connecting the polynomial fuzzy model (2.9) and the sub-domain polynomial fuzzy controller (4.2) in a closed loop is described as follows:

$$\dot{x}(t) = \sum_{i=1}^{p} \sum_{j=1}^{c} w_i(x(t)) m_j(x(t))(A_i(x(t)) + B_i(x(t))G_{jd}(x(t)))\hat{x} \quad \forall \, d. \quad (4.3)$$

The control objective is to determine the polynomial feedback gains $G_{jd}(x(t))$ such that the PFMB control system (4.3) is asymptotically stable, i.e., $x(t) \to 0$ as time $t \to \infty$.

The stability of the PFMB control system (4.3) is investigated in this section. In the following, to lighten the notation, the time $t$ associated with the variables is dropped for the situation without ambiguity, e.g., $x(t)$ and $u(t)$ are denoted as $x$ and $u$, respectively. Furthermore, we denote $\dot{x}(x(t))$, $w_i(x(t))$ and $m_j(x(t))$ as $\dot{x}$, $w_i$ and $m_j$, respectively.

From (4.3), denoting $\dot{x} = [\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_N]^T$,

$$\dot{\dot{x}} = \frac{\partial \dot{x}}{\partial x} \frac{dx}{dt} = T(x)\dot{x}$$

$$= \sum_{i=1}^{p} \sum_{j=1}^{c} w_i m_j (\tilde{A}_i(x) + \tilde{B}_i(x)G_{jd}(x))\dot{x} \quad \forall \, d. \quad (4.4)$$
4.3. MFI STABILITY ANALYSIS

where \( \hat{A}_i(x) = T(x)A_i(x), \hat{B}_i(x) = T(x)B_i(x) \), and \( T(x) \in \mathbb{R}^{N \times n} \) is polynomial matrix defined as

\[
T(x) = \begin{bmatrix}
\frac{\partial \hat{x}_1(x)}{\partial x_1} & \cdots & \frac{\partial \hat{x}_1(x)}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial \hat{x}_N(x)}{\partial x_1} & \cdots & \frac{\partial \hat{x}_N(x)}{\partial x_n}
\end{bmatrix}.
\]

(4.5)

Defining \( K = \{k_1, k_2, \cdots, k_q\} \) as a set of row number that the entries of the entire row of \( B_i(x) \) are all zero for all \( i \), and \( \hat{x} = (x_{k_1}, x_{k_2}, \cdots, x_{k_q}) \). As \( \hat{x} = 0 \) iff \( x = 0 \), we consider the following polynomial Lyapunov function candidate to investigate the system stability of (4.3):

\[
V(x) = \hat{x}^T X(\hat{x})^{-1} \hat{x},
\]

(4.6)

where \( 0 < X(\hat{x}) = X(\hat{x})^T \in \mathbb{R}^{N \times N} \) is a polynomial matrix. Note that all the sub-domains share the common Lyapunov function candidate (4.6).

In the following, we shall show that \( \dot{V} < 0 \) (excluding for \( \hat{x} = 0 \)) subject to \( V > 0 \) (excluding for \( \hat{x} = 0 \)), it is guaranteed by the Lyapunov stability theory that \( \hat{x} \to 0 \) when time \( t \to \infty \). As it is required that \( \hat{x}(x(t)) = 0 \) iff \( x(t) = 0 \), the stability of the PFMB control system (4.3) can thus be achieved.

From (4.3) and (4.6), we have

\[
\dot{V} = \dot{\hat{x}}^T X(\hat{x})^{-1} \dot{\hat{x}} + \dot{\hat{x}}^T X(\hat{x})^{-1} \dot{\hat{x}} + \hat{x}^T \frac{dX(\hat{x})^{-1}}{dt} \hat{x}
\]

\[
= \sum_{d=1}^{D} \sum_{i=1}^{p} \sum_{j=1}^{c} \xi_d w_i m_j \hat{x}^T \left( (\hat{A}_i(x) + \hat{B}_i(x)G_{jd}(x)) \right)^T X(\hat{x})^{-1}
\]

\[
+ X(\hat{x})^{-1} (\hat{A}_i(x) + \hat{B}_i(x)G_{jd}(x))) \hat{x} + \hat{x}^T \frac{dX(\hat{x})^{-1}}{dt} \hat{x}.
\]

(4.7)

Define \( z = X(\hat{x})^{-1} \hat{x} \) and \( G_{jd}(x) = N_{jd}(x)X(\hat{x})^{-1} \), where \( N_{jd}(x) \in \mathbb{R}^{m \times N} \), \( j = 1, 2, \ldots, c \), \( d = 1, 2, \ldots, D \), are polynomial matrices to be determined. And \( \xi_d \) is defined as

\[
\xi_d = \begin{cases} 
1 & \text{for } x \in \text{sub-domain } d \\
0 & \text{otherwise}
\end{cases}, \quad d = 1, 2, \ldots, D.
\]
From (4.7) we have
\[
\dot{V} = \sum_{d=1}^{D} \sum_{i=1}^{p} \sum_{j=1}^{c} \xi_d w_i m_j z^T Q_{ijd}(x) z,
\] (4.8)

where
\[
Q_{ijd}(x) = \tilde{A}_i(x) X(\tilde{x}) + X(\tilde{x}) \tilde{A}_i(x)^T + \tilde{B}_i(x) N_{jd}(x) + N_{jd}(x)^T \tilde{B}_i(x)^T - \sum_{k \in K} \frac{\partial X(\tilde{x})}{\partial x_k} A^k_i(x) \tilde{x},
\]

for \( i = 1, 2, \ldots, p, \ j = 1, 2, \ldots, c, \ d = 1, 2, \ldots, D \). \( A^k_i(x) \in \mathbb{R}^N \) denotes the \( k \)th row of \( A_i(x) \). It is required that \( \sum_{d=1}^{D} \xi_d = 1 \), which implies that there is only one active sub-domain at any instance.

The basic stability conditions based on Lyapunov stability theory to guarantee the stability of the PFMB control system (4.3) are summarised in the following theorem.

**Theorem 8** Considering the operating domain partitioned into \( D \) sub-domains of which each sub-domain is characterized by \( x_{\underline{\varsigma}d} \leq x_\varsigma \leq x_{\overline{\varsigma}d}, \varsigma = 1, 2, \ldots, n, \ d = 1, 2, \ldots, D \), the PFMB control system (4.3) is asymptotically stable if there exist polynomial matrices \( N_{jd}(x) \in \mathbb{R}^{m \times N}, \ j = 1, 2, \ldots, c, \ d = 1, 2, \ldots, D \) and \( X(\tilde{x}) = X(\tilde{x})^T \in \mathbb{R}^{N \times N} \), such that the following SOS-based conditions are satisfied:
\[
\begin{align*}
\upsilon^T (X(\tilde{x}) - \varepsilon_1(\tilde{x}) I) \upsilon &\text{ is SOS}; \\
-\upsilon^T (Q_{ijd}(x) + \varepsilon_2(x) I) \upsilon &\text{ is SOS} \ \forall \ i,j,d;
\end{align*}
\] (4.9) (4.10)

where \( \upsilon \in \mathbb{R}^N \) is an arbitrary vector independent of \( x \), \( \varepsilon_1(\tilde{x}) > 0 \) and \( \varepsilon_2(x) > 0 \) are predefined scalar polynomials. The polynomial feedback gains are defined as
\[
G_{jd}(x) = N_{jd}(x) X(\tilde{x})^{-1}, \ j = 1, 2, \ldots, c, \ d = 1, 2, \ldots, D.
\]

**Remark 14** The information of the membership functions is not taken into consideration in Theorem 8 during stability analysis. Theorem 8 works on the PFMB
control system with the same \( Q_{ijd}(x) \) but with any membership functions. It is the reason that the above theorem is conservative.

**Remark 15** The PFMB control system (4.3) is asymptotic stable in the sense of the Lyapunov stability theory that it requires \( V > 0 \) and \( \dot{V} < 0 \) for \( \dot{x} \neq 0 \). In Theorem 8, the condition (4.9) is to make sure that the Lyapunov function candidate (4.6) satisfying \( V > 0 \) and the condition (4.10) is to make sure that \( \dot{V} < 0 \) according to (4.8).

### 4.4 MFD Stability Analysis

In Section 4.3, it reveals that the membership function shape information plays an important role in the stability analysis. The membership functions of the fuzzy model can be obtained through certain fuzzy modelling methods such as sector nonlinearity technique [18]. For the imperfect premise matching, the membership functions of the polynomial fuzzy controller, which can be different from those of the polynomial fuzzy model, are designed by the designer. Therefore, the membership functions of both polynomial fuzzy model and polynomial fuzzy controller are well defined before conducting stability analysis for the PFMB control system (4.3). As the membership functions are known, it can be found numerically that these membership functions can satisfy some inequalities in the forms appear in [39] and [55]. Through considering these numerical relations, it effectively makes the family of systems represented by the FMB system become smaller in the stability analysis, therefore the stability conditions obtained are more relax [56]. However, taking the information of membership functions into account in the stability analysis is an open research topic so far as there is no systematic way.

In this section, two general forms used to represent the information of membership functions are introduced. Subsequently, an MFD stability analysis method is proposed to relax the stability conditions with the membership function infor-
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mation obtained.

4.4.1 Regional Information of Membership Functions

The information of the membership functions is relatively limited when the whole operating domain is considered. To extract more information from the membership functions, the system operating domain can be partitioned into several sub-domains. In each sub-domain $s_d$, the following regional membership function information is considered.

4.4.1.1 Regional Boundary Information

The following boundary information of membership functions can be found and utilised in stability analysis. In each sub-domain $s_d$, the constant lower and upper bound of each membership function overlap term $w_i m_j$, denoted as $\gamma_{ijd}$ and $\bar{\gamma}_{ijd}$ respectively, can be found to satisfy the following inequalities as

$$\gamma_{ijd} \leq w_i m_j \leq \bar{\gamma}_{ijd} \quad \forall \ i, j, d$$  \hspace{1cm} (4.11)

where $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, c$ and $d = 1, 2, \ldots, D$.

4.4.1.2 Regional Relationship Between Overlap Terms

The boundary information represented above is only the information of each single $w_i m_j$, which does not show the relationship with other membership functions for different values of $i$ and $j$. As a result, the lower and upper bound $\gamma_{ijd}$ and $\bar{\gamma}_{ijd}$ contain limited information of membership functions. The relation between all the membership functions will contain more information, which can help further relax the stability analysis results.

In each sub-domain $s_d$, we represent the relationship between all the membership functions using $R$ inequalities as follows,
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\[ \sum_{i=1}^{p} \sum_{j=1}^{c} k_{ijdr} w_{i} m_{j} + \zeta_{dr} \geq 0 \quad \forall \, d, r \] \hspace{1cm} (4.12)

where \( k_{ijdr} \) and \( \zeta_{dr} \), \( d = 1, 2, \ldots, D \), \( r = 1, 2, \ldots, R \), are the real constant scalars which can be found numerically.

For example, for a PFMB control system (4.3) in which \( i = 1, 2 \) and \( j = 1, 2 \), in a chosen operating sub-domain say \( s_3 \) i.e., \( d = 3 \), we find the membership functions satisfy the following two inequalities: \( w_{1} m_{1} - 0.3 w_{2} m_{1} \geq 0 \) and \( 0.9 w_{1} m_{1} - 0.7 w_{2} m_{2} + 0.1 \geq 0 \). The number of inequalities is two therefore \( R = 2 \).

For the first inequality i.e., \( r = 1 \), referring to (4.12), we denote \( k_{1131} = 1 \), \( k_{2131} = -0.3 \) and all the other overlap terms \( k_{1231} = k_{2231} = 0 \) and \( \zeta_{31} = 0 \) in this case. For the second inequality i.e., \( r = 2 \), we denote \( k_{1132} = 0.9 \), \( k_{2232} = -0.7 \) and all other overlap terms \( k_{1232} = k_{2132} = 0 \) and \( \zeta_{32} = 0.1 \).

The information of membership functions in (4.11) and (4.12) will be introduced to the stability analysis through some slack matrices using the \( S \)-procedure.

### 4.4.2 Stability Analysis

We shall investigate the stability of the PFMB control system (4.3) using Lyapunov stability theory subject to the control objective that is to determine the polynomial feedback gains \( G_{jd}(x) \) such that \( x(t) \to 0 \) as time \( t \to \infty \).

The regional information of the membership functions in Section 4.4.1 is then brought into the stability analysis using some slack matrices. Introducing the slack matrices \( 0 \leq R_{ijd}(x) = R_{ijd}(x)^T \in \mathbb{R}^{N \times N} \), \( 0 \leq S_{ijd}(x) = S_{ijd}(x)^T \in \mathbb{R}^{N \times N} \) and \( 0 \leq T_{dr}(x) = T_{dr}(x)^T \in \mathbb{R}^{N \times N} \), from (4.11) and (4.12), we have the following inequalities:

\[ \sum_{i=1}^{p} \sum_{j=1}^{c} (w_{i} m_{j} - \gamma_{ijd}) R_{ijd}(x) \geq 0 \quad \forall \, d, \] \hspace{1cm} (4.13)

\[ \sum_{i=1}^{p} \sum_{j=1}^{c} (\gamma_{ijd} - w_{i} m_{j}) S_{ijd}(x) \geq 0 \quad \forall \, d, \] \hspace{1cm} (4.14)
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\[ \sum_{r=1}^{R} \left( \sum_{i=1}^{p} \sum_{j=1}^{c} k_{ijd} w_i m_j + \omega_{dr} \right) T_{dr}(x) \geq 0 \quad \forall \ d. \quad (4.15) \]

Adding (4.13)-(4.15) to (4.8), we have

\[
\dot{V} \leq \sum_{d=1}^{D} \sum_{i=1}^{p} \sum_{j=1}^{c} \xi_d z^T \left( \tau_{ijd} S_{ijd}(x) - \gamma_{ijd} R_{ijd}(x) \right) + \sum_{r=1}^{R} \omega_{dr} T_{dr}(x) z + \sum_{d=1}^{D} \sum_{i=1}^{p} \sum_{j=1}^{c} \xi_d w_i m_j z^T (Q_{ijd}(x)) + R_{ijd}(x) - S_{ijd}(x) + \sum_{r=1}^{R} k_{ijd} T_{dr}(x) z. \quad (4.16)
\]

Introducing the slack matrices \(0 \leq Y_{ijd}(x) = Y_{ijd}(x)^T \in \mathbb{R}^{N \times N}\) which satisfies

\[
Y_{ijd}(x) \geq Q_{ijd}(x) + R_{ijd}(x) - S_{ijd}(x) + \sum_{r=1}^{R} k_{ijd} T_{dr}(x) \quad \forall \ i, j, d. \quad (4.17)
\]

Then, from (4.16) and (4.17), we have

\[
\dot{V} \leq \sum_{d=1}^{D} \sum_{i=1}^{p} \sum_{j=1}^{c} \xi_d z^T \left( \tau_{ijd} S_{ijd}(x) - \gamma_{ijd} R_{ijd}(x) \right) + \sum_{r=1}^{R} \omega_{dr} T_{dr}(x) z + \sum_{d=1}^{D} \sum_{i=1}^{p} \sum_{j=1}^{c} \xi_d w_i m_j z^T Y_{ijd}(x) z \\
\leq \sum_{d=1}^{D} \sum_{i=1}^{p} \sum_{j=1}^{c} \xi_d z^T \left( \tau_{ijd} S_{ijd}(x) - \gamma_{ijd} R_{ijd}(x) \right) + \sum_{r=1}^{R} \omega_{dr} T_{dr}(x) z + \tau_{ijd} Y_{ijd}(x) z. \quad (4.18)
\]

Based on Lyapunov stability theory, the PFMB control system (4.3) is asymptotically stable when both \(V(t) > 0\) and \(\dot{V}(t) < 0\) (excluding \(z = x = 0\)) hold, which can be achieved if the SOS-based stability conditions summarised in the following theorem are satisfied.
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Theorem 9 Considering the operating domain partitioned into $D$ sub-domains of which each is characterized by $\underline{\omega}_{cd} \leq x_{\varsigma} \leq \overline{\omega}_{cd}$, $\varsigma = 1, 2, \ldots, n$, $d = 1, 2, \ldots, D$, the PFMB control system (4.3) is asymptotically stable if there exist polynomial matrices $R_{ijd}(x) = R_{ijd}(x)^T \in \mathbb{R}^{N \times N}$, $S_{ijd}(x) = S_{ijd}(x)^T \in \mathbb{R}^{N \times N}$, $T_{dr}(x) = T_{dr}(x)^T \in \mathbb{R}^{N \times N}$, $Y_{ijd}(x) = Y_{ijd}(x)^T \in \mathbb{R}^{N \times N}$, $N_{jd}(x) \in \mathbb{R}^{m \times N}$, $i = 1, 2, \ldots p$, $j = 1, 2, \ldots c$, $r = 1, 2, \ldots, R$ and $X(\hat{x}) = X(\hat{x})^T \in \mathbb{R}^{N \times N}$, such that the following SOS-based conditions are satisfied.

\[ v^T(X(\hat{x}) - \varepsilon_1(\hat{x}) I) v \text{ is SOS}; \quad (4.19) \]

\[ v^T(R_{ijd}(x) - \varepsilon_2(x) I) v \text{ is SOS } \forall i,j,d; \quad (4.20) \]

\[ v^T(S_{ijd}(x) - \varepsilon_3(x) I) v \text{ is SOS } \forall i,j,d; \quad (4.21) \]

\[ v^T(Y_{ijd}(x) - \varepsilon_4(x) I) v \text{ is SOS } \forall i,j,d; \quad (4.22) \]

\[ v^T(T_{dr}(x) - \varepsilon_5(x) I) v \text{ is SOS } \forall d,r; \quad (4.23) \]

\[ v^T(Y_{ijd}(x) - Q_{ijd}(x) - R_{ijd}(x) + S_{ijd}(x)) \]

\[ - \sum_{r=1}^{R} k_{ijd} T_{dr} - \varepsilon_6(x) I) v \text{ is SOS } \forall i,j,d; \quad (4.24) \]

\[ -v^T(\sum_{i=1}^{p} \sum_{j=1}^{c}(\gamma_{ij} S_{ijd}(x) - \gamma_{ij} R_{ijd}(x)) + \sum_{r=1}^{R} \zeta_{dr} T_{dr}(x) \]

\[ + \gamma_{ijd} Y_{ijd}(x)) + \varepsilon_7(x) I) v \text{ is SOS } \forall d; \quad (4.25) \]

where $\nu \in \mathbb{R}^N$ is an arbitrary vector independent of $x$, $\varepsilon_1(\hat{x}) > 0$ and $\varepsilon_l(x) > 0$, $l = 2, 3, \ldots, 7$ are predefined scalar polynomials. $\gamma_{ij}$, $\gamma_{ij}$, $k_{ijd}$ and $\zeta_{dr}$, $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, c$, $d = 1, 2, \ldots, D$, $r = 1, 2, \ldots, R$, are predefined constant scalars satisfying (4.11) and (4.12), respectively. The polynomial feedback gains are defined as $G_{jd}(x) = N_{jd}(x) X(\hat{x})^{-1}$, $j = 1, 2, \ldots, c$, $d = 1, 2, \ldots, D$. 
Remark 16 The predefined scalars $\gamma_{ijd}$ and $\gamma_{ijd}$, i.e., the bounds of the membership function overlap term, can be computed numerically as the extrema of the $w_i(x)m_j(x)$ function in the corresponding sub-domain. The predefined scalars $k_{ijd}$ and $c_{dr}$ can be obtained numerically by some searching algorithms, e.g., GA.

Remark 17 It can be seen from Theorem 9 that the information of membership functions in the form of (4.11) and (4.12) are included in the stability conditions. The stability conditions are thus more dedicated to the nonlinear plant to be controlled. Consequently, more relaxed stability conditions are achieved compared with the basic stability conditions in Theorem 8.

Remark 18 The PFMB control system (4.3) is asymptotic stable in the sense of the Lyapunov stability theory that it requires $V > 0$ and $\dot{V} < 0$ for $\hat{x} \neq 0$. In Theorem 9, the condition (4.19) is to make sure that the Lyapunov function candidate (4.6) satisfying $V > 0$ and the condition (4.25) is to make sure that $\dot{V} < 0$ according to (4.8), while the conditions (4.20) to (4.24) are to make sure that the slack matrices satisfying the prescribed requirement.

Remark 19 Comparing Theorem 8 and Theorem 9 in terms of computational complexity, as the number of stability conditions and decision variables to address the relationship among membership functions, the computational demand required to solve a feasible solution to the stability conditions in Theorem 9 will be increased. When more and more information/sub-domains is considered, although more relaxed stability analysis results can be achieved, the computational complexity will increase accordingly. To save computational power, it is thus suggested that the stability conditions in Theorem 8 can be tried first. If no feasible solution is obtained from Theorem 8, the stability conditions in Theorem 9 are considered that the number of relations/sub-domains is increased gradually to look for a feasible solution.
4.5 Simulation Example

In this simulation example, the SOS-based stability conditions from Theorem 8 and Theorem 9 are investigated to verify the proposed MFD stability analysis method. The stability regions given by each theorem will be investigated under the way introduced in Section 3.2.4.

In this section, the stability region produced by each theorem is investigated by an FMB control system in the form of (4.3), which is formed by a 3-rule polynomial fuzzy model in the form of (2.9) and a 2-rule sub-domain polynomial fuzzy controller in the form of (4.2) connected in a closed loop.

In this example, the polynomial fuzzy model chosen is with the following system and input matrices:

\[
A_1(x_2) = \begin{bmatrix}
1.59 - 1.66x_2 & -7.29 + 0.22x_2 - 0.25x_2^2 \\
0 & -0.36
\end{bmatrix},
\]

\[
A_2(x_2) = \begin{bmatrix}
0.02 + 2.72x_2 & 5.46 - 1.25x_2 - 0.25x_2^2 \\
0 & -0.21
\end{bmatrix},
\]

\[
A_3(x_2) = \begin{bmatrix}
-a + 1.17x_2 & -4.33 - 3.36x_2 - 0.25x_2^2 \\
0 & -0.05
\end{bmatrix},
\]

\[
B_1(x_2) = \begin{bmatrix}
1 + x_2^2 \\
0
\end{bmatrix},
\]

\[
B_2(x_2) = \begin{bmatrix}
8 + x_2 \\
0
\end{bmatrix},
\]

\[
B_3(x_2) = \begin{bmatrix}
-b + 6 + 2.36x_2^2 \\
0
\end{bmatrix},
\]

\[
\hat{x} = x = [x_1, x_2]^T,
\]

where \(a\) and \(b\) are scalar parameters. It is assumed that \(x_2 \in [-10, 10]\).

The membership functions are chosen as \(w_1(x_2) = \mu_M^1(x_2) = 1 - \frac{1}{1+e^{-(x_2+a)}}\), \(w_2(x_2) = \mu_M^2(x_2) = 1 - w_1(x_2) - w_3(x_2)\), \(w_3(x_2) = \mu_M^3(x_2) = \frac{1}{1+e^{-(x_2-a)}}\) for \(x_2 \in [-10, 10]\), which are shown in Fig. 4.1.

A polynomial fuzzy controller with 2 rules in the form of (4.2) is proposed to control the nonlinear plant. The membership functions are chosen as \(m_1(x_2) = \)
4.5. SIMULATION EXAMPLE

\[ \mu_{N_1}(x_2) = 1 - \frac{1}{1 + e^{-\frac{x_2}{2}}}, \quad m_2(x_2) = \mu_{N_2}(x_2) = 1 - m_1(x_2), \]
which are also shown in Fig. 4.1.

Remark 20 It should be noted that the membership functions of both polynomial fuzzy model and polynomial fuzzy controller are not the same in this example. And the entries of the second row of the input matrices \( B_i(x) \) are all zero for all \( i \), therefore, \( \tilde{x} = (x_2) \).

The stability conditions of Theorem 9 are employed to check the stability region of the PFMB control system with \( 50 \leq a \leq 55 \) and \( 32 \leq b \leq 94 \) (both at the interval of 1). In this example, the solution is found numerically using SOSTOOLS [102] where the degrees of \( R_{ijd}(x) \), \( S_{ijd}(x) \), \( Y_{ijd}(x) \) and \( T_{dr} \) are chosen as 0 in \( x_2 \), the degrees of \( X(\tilde{x}) \) and \( N_{jd}(x) \) as 2 in \( x_2 \), \( \varepsilon_l = 0.001 \), \( l = 1, 2, \ldots, 7 \).

We partition the operating domain \( x_2 \in [-10, 10] \) into 4 sub-domains, i.e., \( D = 4 \). The \( s_d \) sub-domain is characterized by \( 5(d-3) \leq x_2 \leq 5(d-2) \), \( d = 1, 2, 3, 4 \). With the chosen membership functions, the regional boundary information

Figure 4.1: Membership functions of polynomial fuzzy model (solid lines) and polynomial fuzzy controller (dotted lines).
$\gamma_{ijd}$ and $\tau_{ijd}$ found numerically are shown in Table A.1 in the Appendix. The regional relation between overlap terms $\mathcal{L}_{dr}$ and $k_{ijdr}$ found numerically by GA are shown in Tables A.2–A.6 in the Appendix. All the parameters $\gamma_{ijd}$, $\tau_{ijd}$, $\mathcal{L}_{dr}$ and $k_{ijdr}$ in this example are computed using MATLAB®. Employing the sub-domain controllers in Theorem 9, consider two cases: Case 1 is with both regional boundary information and regional relation between overlap terms and Case 2 is with regional boundary information only, without regional relation between overlap terms. The stability regions obtained are shown in Fig. 4.2 indicated by “$\triangle$” (Case 1) and “$\times$” (Case 2). It shows that larger stability region can be obtained with the regional relation between overlap terms as the membership function information. To highlight the result and show the boundary of the stability region, we choose the range of $b$ in $88 \leq b \leq 94$ to show. It should be mentioned that the region for $b$ in the range of $32$ to $87$ at the interval of $1$ is stable for both Case 1 and Case 2.

To demonstrate the effect of the sub-domain controller, under the same operating domain partition as in Case 1 and Case 2, we employ the common controller instead of the sub-domain controller in the above cases. The common controller is that $G_j(x) = G_{jd}(x)$ for all $d = 1, 2, 3, 4$. We consider two cases: Case 3 is with both regional boundary information and regional relation between overlap terms and Case 4 is with regional boundary information only, without regional relation between overlap terms. As the operating domain partition are remain the same, the $\gamma_{ijd}$, $\tau_{ijd}$, $\mathcal{L}_{dr}$ and $k_{ijdr}$ are the same as in Case 1 and Case 2. The stability regions obtained are shown in Fig. 4.2 indicated by “$\square$” (Case 3) and “$\bullet$” (Case 4). We can find the same effect of regional relation between overlap terms, i.e., relaxing the stability conditions indicated by larger stability region. Same as the above, to highlight the boundary of the stability region, the range of $b$ shown is chosen as $52 \leq b \leq 56$ although the stability region can be found for $b$ in the range of $32$ to $55$ at the interval of $1$ for both Case 3 and Case 4. Comparing the stability regions shown in Fig. 4.2a and Fig. 4.2b, it can be seen
that Case 1 offers a larger stability region than Case 3 which demonstrates that the sub-domain polynomial fuzzy controller has a better feedback compensation capability than the common one. The same observation is applied to Case 2 and Case 4.

For the purpose of verifying the effect of the reducing of membership function information to the stability analysis results, we reduce the number of sub-domains from 4 to 1, i.e., \( D = 1 \). In this scenario, we only consider one single sub-domain which is actually the overall operating domain and the controller employed is common controller trivially. A group of new \( \gamma_{ijd} \) and \( \gamma_{ijd} \) which represent the global boundary information and a group of new \( \omega_{dr} \) and \( k_{ijdr} \) taken as the global relation between overlap terms are obtained numerically and shown in Table A.7 and Table A.8 in the Appendix, respectively. We consider two cases: Case 5 is with both global boundary information and global relation between overlap terms and Case 6 is without any membership function information. The stability regions obtained are shown in Fig. 4.2c indicated by “o” (Case 5) and “+” (Case 6). It can be seen in Fig. 4.2c, comparing with Case 5 and Case 6, the membership function information can relax the stability conditions resulting in offering a larger stability region. The range of \( b \) is chosen as \( 32 \leq b \leq 36 \) to highlight the boundary of the stability region. Since the region below \( b = 52 \) in Fig. 4.2b is stable for both Case 3 and Case 4, by comparing with Fig. 4.2b and Fig. 4.2c, it can be concluded that the more number of sub-domains is considered, the more the regional membership function information is extracted which will lead to more relaxed stability conditions.

It should be noted that, the stability conditions used for Case 6 are the conditions in Theorem 8. It can be considered as the basic stability conditions derived from [18] for the case of imperfect premise matching. For Case 4, with common fuzzy controller and without regional overlap term information, it is the case of the stability conditions of polynomial version for the case of imperfect premise matching derived from [112]. It can be seen from Fig. 4.2 that when regional
relation between overlap terms is considered and apply the regional fuzzy control strategy, more relaxed results can be achieved.

For verification, we show the phase plots of $x_1$ and $x_2$ of some stable points.
for the six cases subject to various initial conditions are shown in Figs. 4.3-4.8. It can be seen that the PFMB control systems are all stable, and the polynomial fuzzy controllers designed are able to drive the system states to the origin. The matrices $X(\tilde{x})$ and $N_{jd}(\tilde{x})$ of each case obtained are shown in Tables A.9–A.11 in Appendix A.

Figure 4.3: Behaviours in $x_1$-$x_2$ plane for Case 1 with $a = 55$ and $b = 94$, where the initial conditions are indicated by “o”.
Figure 4.4: Behaviours in $x_1$-$x_2$ plane for Case 2 with $a = 51$ and $b = 88$, where the initial conditions are indicated by “◦”.

Figure 4.5: Behaviours in $x_1$-$x_2$ plane for Case 3 with $a = 54$ and $b = 55$, where the initial conditions are indicated by “◦”.
4.5. SIMULATION EXAMPLE

Figure 4.6: Behaviours in $x_1$-$x_2$ plane for Case 4 with $a = 51$, $b = 52$, where the initial conditions are indicated by “◦”.

Figure 4.7: Behaviours in $x_1$-$x_2$ plane for Case 5 with $a = 55$ and $b = 35$, where the initial conditions are indicated by “○”.
4.6 Searching Numerical Relationship Between Overlap Terms via GA

As it was stated in Remark 16, the scalars $k_{ijdr}$ and $c_{dr}$ in (4.12) can be obtained numerically by using some searching algorithms. The search method via GA is what we use in the above example. We provide it in this section for user’s reference. For clarity, we keep using the above example to illustrate this method.

As it is shown in Fig. 4.9, the search method we propose consists of five steps. Consider the example given in Section 4.5, and equation (4.12) is restated as follows:

$$\sum_{i=1}^{p} \sum_{j=1}^{c} k_{ijdr} w_i m_j + c_{dr} \geq 0 \quad \forall \, d, r$$

where $k_{ijdr}$ and $c_{dr}$, $d = 1, 2, \ldots, D$, $r = 1, 2, \ldots, R$, are the real constant scalars. The $k_{ijdr}$ and $c_{dr}$ searched should satisfy this inequality.
4.6. SEARCHING NUMERICAL RELATIONSHIP BETWEEN OVERLAP TERMS VIA GA

Figure 4.9: Procedure of searching numerical relationship between overlap terms.

**Step 1:** For each sub-domain, that is, for each \(d\) and \(r\), the number of the undetermined \(k_{ijdr}\) are \(pc\), in which \(p\) and \(c\) are the numbers of the fuzzy rules of the model and designed controller respectively. In the above simulation example, \(p = 3\) and \(c = 2\). For the purpose of limiting the search range and reducing the number of variables, we pick one undetermined parameter \(k_{ijdr}\) to be 1, take the other undetermined parameters \(k_{ijdr}\) as decision variables and represented as a vector denoted as \(k_{dr}\). For each \(d\) and \(r\), there are six \(k_{ijdr}\) parameters as \(k_{11dr}, k_{12dr}, k_{21dr}, k_{22dr}, k_{31dr}\) and \(k_{32dr}\). We pick one undetermined parameter to be 1 by setting the \(l\)th entry in the row \([k_{11dr}, k_{12dr}, k_{21dr}, k_{22dr}, k_{31dr}, k_{32dr}]\) to be 1 for the \(l\)th inequality in each sub-domain. Take the other five undetermined parameters as variables and represented as a vector denoted as \(k_{dr}\). For Case 1 in the simulation example, we want to find the first inequality \((r = 1)\) in the first
sub-domain $s_1$ ($d = 1$), then $l = 1$ in this case, so we fix $k_{1111} = 1$, set other five parameters $k_{1211}, k_{2111}, k_{2211}, k_{3111}$ and $k_{3211}$ as decision variables for GA to search, that is $\mathbf{k}_{11} = [k_{1211}, k_{2111}, k_{2211}, k_{3111}, k_{3211}]^T$. The lower bound and upper bound of the decision variables are set as $-30$ and $-0.001$ respectively for all cases which are obtained by trial and error.

**Step 2:** We define the first term of the right hand of the (4.12) as

$$z_{dr} = \sum_{i=1}^{p} \sum_{j=1}^{c} k_{ijdr} w_i m_j \forall d, r.$$  \hfill (4.26)

In the simulation example Case 1, for the first inequality in the first sub-domain ($d = 1$ and $r = 1$), we find the values of $z_{11}(x_2)$ in the form of equation (4.26) at $-10 \leq x_2 \leq -5$ at the interval of 0.01, and denote the $s$th value as $z_{11s}, s = 1, 2, \ldots, 501$. For example, the value $z_{11}(-9.999)$ and $z_{11}(-9.998)$ are denoted as $z_{112}$ and $z_{113}$ receptively in this case. We define $y_{11} = \sum_{s=1}^{501} z_{11s}$ and find the decision variable $\mathbf{k}_{11}$ which can minimise $y_{11}$. This is the way we use to make $\mathbf{k}_{11}$ regionally carry the shape information between the overlap terms of the membership functions, i.e., $w_i m_j$ in sub-domain $s_1$. In the consideration of the constraint, $\mathbf{k}_{11}$ must satisfy $z_{11}(x_2) > 0$ for all $x_2$ in the first sub-domain. Namely, all the $z_{11s}, s = 1, 2, \ldots, 501$ must be positive. To construct this constraint, we count the number of the case that $z_{11s} < 0$ and denote it as $n_{11}$. As $z_{11s} < 0$ for some $s$ (that is $n_{11} > 0$) implies $\mathbf{k}_{11}$ obtained is infeasible, we penalize this case by multiply a very large positive scaler with $n_{11}$ and add it to the fitness function. Therefore, the fitness function is $f_{11} = y_{11} + \lambda n_{11}$ in this case. That is in general form, the fitness function is defined as

$$f_{dr} = y_{dr} + \lambda n_{dr} \forall d, r$$ \hfill (4.27)

in which $y_{dr} = \sum_{s=1}^{501} z_{drs}, n_{dr}$ is the number of the cases that $z_{drs} < 0, s = 1, 2, \ldots, 501$ and $\lambda$ is a predefined positive scalar. The fitness function defined in equation (4.27) is the function of the decision variable $\mathbf{k}_{dr}$.
Step 3: Search the decision variables by GA using MATLAB®. Solve the
minimisation problem

$$\min_{k_{dr}} f_{dr},$$

in which $f_{dr}$ is defined in equation (4.27), $k_{dr}$ defined in Step 1 represents the
decision variables. As mentioned above, for Case 1 in the simulation example
find the first inequality in the first sub-domain, the minimisation problem is as
follows:

$$\min_{k_{11}} f_{11},$$

in which $k_{11} = [k_{1211}, k_{2111}, k_{2211}, k_{3111}, k_{3211}]^T$. For Case 1 in the simulation exam-
ple find the second inequality in the first sub-domain, the minimisation problem is as
follows:

$$\min_{k_{12}} f_{12},$$

in which $k_{12} = [k_{1112}, k_{2112}, k_{2212}, k_{3112}, k_{3212}]^T$.

Step 4: Check the value of $n_{dr}$ by substituting $k_{ijdr}$ obtained from Step 3 to
equation (4.26), in the corresponding sub-domain $s_d$, with the interval 0.01 to
check whether all the $z_{drs}$, $s = 1, 2, \ldots, 501$, are all positive. As mentioned in
Step 2, the number of the cases that $z_{drs} < 0$ is denoted as $n_{dr}$. The value of $n_{dr}$
may be zero or a positive integer.

Step 5 ($n_{dr} = 0$ case): For the case $n_{dr} = 0$, it means $k_{ijdr}$ make $z_{dr}(x_2) > 0,$
in which $z_{dr}(x_2)$ is defined in the equation (4.26), be satisfied for all $x_2$ in the
corresponding sub-domain. In this case, we simply set $\zeta_{dr} = 0$. $k_{ijdr}$ and $\zeta_{dr}$
obtained can satisfy equation (4.12).

Step 5 ($n_{dr} > 0$ case): For the case $n_{dr} > 0$, it means $k_{ijdr}$ obtained can not
make $z_{dr}(x_2) > 0$ be satisfied for all $x_2$ in the corresponding sub-domain. We
add $\zeta_{dr}$ as bias to the equation (4.26) to make $k_{ijdr}$ and $\zeta_{dr}$ obtained can satisfy
equation (4.12). The way to find $\zeta_{dr}$ is to substitute $k_{ijdr}$ obtained from Step 3
to equation (2), handle all $z_{drs}$, $s = 1, 2, \ldots, 501$ as a set denoted as $Z_{dr}$. We can
obtain the $\inf \mathbf{Z}_{dr}$ which is a negative scalar and find $\mathcal{L}_{dr}$ as follows:

\[
\mathcal{L}_{dr} = 0.005 - \inf \mathbf{Z}_{dr} \quad \forall \, d, r
\]

in which 0.005 is a predefined bias to make equation (4.12) to be satisfied in the whole corresponding sub-domain $s_d$.

The $k_{ijdr}$ and $\mathcal{L}_{dr}$ for the other sub-domains can be obtained following the same steps.

As the membership functions are predefined, the membership function shape information can be obtained following the above procedure and brought into the relaxed stability analysis by the MFD method proposed.

**4.7 Conclusion**

In this chapter, the stability of PFMB control systems under imperfect premise matching have been investigated. The polynomial fuzzy controller employed is not required to share the same premise membership functions and the same number of fuzzy rules with the polynomial fuzzy model. The MFD stability analysis method using regional membership function information is proposed. The operating domain is partitioned into several sub-domains. The membership function shape information including boundary information of each membership function overlap terms and the numerical relationship between all the overlap terms are taken account into the stability analysis. Based on the operating domain partition, a set of sub-domain polynomial fuzzy controllers is designed to enhance the feedback compensation capability. Relaxed SOS-based stability conditions for PFMB control system are obtained and solved numerically. The search method via GA for the numerical relationship between all the overlap terms is provided for user’s reference. A simulation example is given to demonstrate the merit of proposed method.
Chapter 5

Conclusion

5.1 Summary

In this thesis, the stability of FMB control systems designed under imperfectly premise matching has been investigated. The stability conditions are relaxed by considering the membership function shape information, whereas the proposed techniques are summarised in Fig. 5.1.

For T-S FMB control system, the PMFs are employed to reduce the conservativeness of stability conditions. By approximating the original membership functions with the PMFs, the membership function shape information is integrated into the LMI-based stability conditions in consideration of the approximation er-
ror. The system stability is then determined by ascertaining the stability of the system at the sample points through which, the stability conditions achieved only pertain to the T-S FMB control system with specified PMFs as opposed to any shapes. Thus, these LMI-based stability conditions are more relaxed than the ones that are obtained without the involvement of membership function shape information. The relationship between conservativeness and membership function approximation accuracy is also investigated. Stability conditions are found to be more relaxed in case the PFMs used have higher approximation accuracy. However, the application of more accurate PMFs, i.e., more sample points considered in stability analysis, leads to an increase in the number of LMIs, thus increasing the computational demand.

In consideration of the computational demand, an alternative PMF-based technique is proposed to improve the applicability of the aforementioned PMF-based method for T-S FMB control system analysis. By introducing the concept of lower and upper PMFs, the proposed lower-upper-PMF-based analysis method can still lead to relaxed stability conditions, whereas only the lower accurate PMFs are applied. Based on the simulation result, a suggestion regarding the order of use of the proposed PMF-based method and lower-upper-PMF-based method for MFD stability analysis is included in Section 3.2.4. In addition, boundary-related information of membership functions also has been considered with the lower and upper PMFs to further enhance the effectiveness of MFD stability for releasing conservativeness.

For PFMB control system, an MFD stability analysis method using the regional membership function shape information is proposed. This method is based on the operating domain partition scheme. Two types of membership function shape information have been posited: 1) regional boundary information; 2) regional relationship between all the overlap terms. This information has been extracted and introduced into SOS-based stability conditions. As a result, the SOS-based stability conditions obtained are only related to the PMFB control sys-
tem which includes membership functions that specifically satisfy the numerical relations on each operating sub-domain instead of any shapes. When compared to the global information, regional information has a greater potential to yield results in more relaxed stability conditions. Through an example, it is illustrated that the proposed MFD method involving membership function regional information can effectively release the conservativeness. Moreover, in the given example, the regional relationship is explored between all the overlap terms by GA. In addition, the details about this procedure have been provided in order to facilitate stability analysis via this MFD method.

The applicability of FMB control is also improved upon by applying the fuzzy control scheme, which has a stronger compensation capacity for the highly non-linear control system. In this thesis, the sub-domain fuzzy control scheme is applied to both T-S FMB control system and PFMB control system design. In each system operating sub-domain, one sub-domain T-S/PFMB fuzzy controller is designed in order to stabilise the local fuzzy model. The simulation results reveal that the FMB control system with sub-domain fuzzy controller has a larger stability region than those that only have a common fuzzy controller (e.g. for sub-domain polynomial fuzzy controller $G_j(x(t)) = G_{jd}(x(t)$, for all $d$). In the form of LMI/SOS, the stability conditions are effectively relaxed by the proposed scheme. It must be noted that in order to facilitate the stability analysis more effectively, the sub-domain fuzzy control scheme shares the operating domain partition procedure with the MFD methods proposed within this thesis.
5.2 Future Works

Our work provides the following future directions that can be further explored.

1) In this thesis, PFMs employed to approximate the original membership functions are piecewise-linear functions. As an extension, it would be interesting to investigate the use of higher order functions as approximated membership functions in order to release conservativeness of T-S FMB control system.

2) In the current computation environment, the heavy computational burden curtails the applicability of FMB control. In this thesis, the stability conditions are recast as LMI/SOS problems that can be solved numerically using convex optimisation techniques. Computational burden is known to increase with an increase in the number of variables and/or constraints. To that end, further research works are necessary in order to develop the MFD stability analysis techniques which perform better both in terms of conservativeness and computational burden. One possible direction is to expand the idea of the proposed lower-upper-PMFs method that can adequately capture the information of membership functions with a small number of the sample points.

3) In Section 4.6, GA is applied to search the coefficients of inequalities used to denote the numerical relationship between all the membership function overlap terms. Since GA is a metaheuristics (random search algorithm), it gives different result each time used. Therefore, the extra result ascertaining step proposed in Section 4.6 (Step 4) cannot be omitted. This limitation of GA affects the applicability of the proposed MFD method. In future works, this problem can be overcome by applying more reliable and efficient optimization algorithms to undertake membership function information research.
4) In this thesis, imperfect premise matching is applied to fuzzy controller design. The feedback gains are found systematically using the proposed MFD analysis methods. However, the membership functions of fuzzy controllers have been predefined by the designers. Therefore, take the membership functions as design variables, perform the optimisation of membership functions for a specified cost function, such as performance criteria, in the process of fuzzy controller synthesis is one of the next research directions which can be explored.

5) The form of Lyapunov function affects the conservativeness of stability conditions. Only quadratic Lyapunov function and polynomial Lyapunov function are employed for the proposed methods in this thesis. Therefore, MFD stability analysis methods, in conjunction with non-quadratic analysis techniques, can be further investigated.
Appendix A

Membership Function Shape
Information and Feedback Gains
for Simulation Example in
Section 4.5
Table A.1: Lower and upper bounds of $\gamma_{ijd}$ for four-sub-domain case in Section 4.5

<table>
<thead>
<tr>
<th>Sub-domain $s_d$</th>
<th>$\gamma_{ijd}$</th>
<th>$\tilde{\gamma}_{ijd}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sub-domain $s_1$</td>
<td>$\gamma_{111} = 6.7560 \times 10^{-1}$</td>
<td>$\tilde{\gamma}_{111} = 9.9085 \times 10^{-1}$</td>
</tr>
<tr>
<td></td>
<td>$\gamma_{121} = 6.6763 \times 10^{-3}$</td>
<td>$\tilde{\gamma}_{121} = 5.5457 \times 10^{-2}$</td>
</tr>
<tr>
<td></td>
<td>$\gamma_{211} = 2.4552 \times 10^{-3}$</td>
<td>$\tilde{\gamma}_{211} = 2.4843 \times 10^{-1}$</td>
</tr>
<tr>
<td></td>
<td>$\gamma_{211} = 1.6543 \times 10^{-5}$</td>
<td>$\tilde{\gamma}_{221} = 2.0392 \times 10^{-2}$</td>
</tr>
<tr>
<td></td>
<td>$\gamma_{211} = 8.2596 \times 10^{-7}$</td>
<td>$\tilde{\gamma}_{311} = 1.1403 \times 10^{-4}$</td>
</tr>
<tr>
<td></td>
<td>$\gamma_{211} = 5.5653 \times 10^{-9}$</td>
<td>$\tilde{\gamma}_{321} = 9.3605 \times 10^{-6}$</td>
</tr>
<tr>
<td>Sub-domain $s_2$</td>
<td>$\gamma_{112} = 8.9931 \times 10^{-3}$</td>
<td>$\tilde{\gamma}_{112} = 6.7560 \times 10^{-1}$</td>
</tr>
<tr>
<td></td>
<td>$\gamma_{122} = 8.9931 \times 10^{-3}$</td>
<td>$\tilde{\gamma}_{122} = 5.9988 \times 10^{-2}$</td>
</tr>
<tr>
<td></td>
<td>$\gamma_{212} = 2.4843 \times 10^{-1}$</td>
<td>$\tilde{\gamma}_{212} = 6.4274 \times 10^{-1}$</td>
</tr>
<tr>
<td></td>
<td>$\gamma_{222} = 2.0392 \times 10^{-2}$</td>
<td>$\tilde{\gamma}_{222} = 4.8201 \times 10^{-1}$</td>
</tr>
<tr>
<td></td>
<td>$\gamma_{312} = 1.1403 \times 10^{-4}$</td>
<td>$\tilde{\gamma}_{312} = 8.9931 \times 10^{-3}$</td>
</tr>
<tr>
<td></td>
<td>$\gamma_{322} = 9.3605 \times 10^{-6}$</td>
<td>$\tilde{\gamma}_{322} = 8.9931 \times 10^{-3}$</td>
</tr>
<tr>
<td>Sub-domain $s_3$</td>
<td>$\gamma_{113} = 9.3605 \times 10^{-6}$</td>
<td>$\tilde{\gamma}_{113} = 8.9931 \times 10^{-3}$</td>
</tr>
<tr>
<td></td>
<td>$\gamma_{123} = 1.1403 \times 10^{-4}$</td>
<td>$\tilde{\gamma}_{123} = 8.9931 \times 10^{-3}$</td>
</tr>
<tr>
<td></td>
<td>$\gamma_{213} = 2.0392 \times 10^{-2}$</td>
<td>$\tilde{\gamma}_{213} = 4.8201 \times 10^{-1}$</td>
</tr>
<tr>
<td></td>
<td>$\gamma_{223} = 2.4843 \times 10^{-1}$</td>
<td>$\tilde{\gamma}_{223} = 6.4274 \times 10^{-1}$</td>
</tr>
<tr>
<td></td>
<td>$\gamma_{313} = 8.9931 \times 10^{-3}$</td>
<td>$\tilde{\gamma}_{313} = 5.9988 \times 10^{-2}$</td>
</tr>
<tr>
<td></td>
<td>$\gamma_{323} = 8.9931 \times 10^{-3}$</td>
<td>$\tilde{\gamma}_{323} = 6.7560 \times 10^{-1}$</td>
</tr>
<tr>
<td>Sub-domain $s_4$</td>
<td>$\gamma_{114} = 5.5653 \times 10^{-9}$</td>
<td>$\tilde{\gamma}_{114} = 9.3605 \times 10^{-6}$</td>
</tr>
<tr>
<td></td>
<td>$\gamma_{124} = 8.2596 \times 10^{-7}$</td>
<td>$\tilde{\gamma}_{124} = 1.1403 \times 10^{-4}$</td>
</tr>
<tr>
<td></td>
<td>$\gamma_{214} = 1.6543 \times 10^{-5}$</td>
<td>$\tilde{\gamma}_{214} = 2.0392 \times 10^{-2}$</td>
</tr>
<tr>
<td></td>
<td>$\gamma_{224} = 2.4552 \times 10^{-3}$</td>
<td>$\tilde{\gamma}_{224} = 2.4843 \times 10^{-1}$</td>
</tr>
<tr>
<td></td>
<td>$\gamma_{314} = 6.6763 \times 10^{-3}$</td>
<td>$\tilde{\gamma}_{314} = 5.5457 \times 10^{-2}$</td>
</tr>
<tr>
<td></td>
<td>$\gamma_{324} = 6.7560 \times 10^{-1}$</td>
<td>$\tilde{\gamma}_{324} = 9.9085 \times 10^{-1}$</td>
</tr>
</tbody>
</table>
Table A.2: $c_{dr}$ for four-sub-domain case in Section 4.5

<table>
<thead>
<tr>
<th>Sub-domain $s_d$</th>
<th>$c_{dr}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sub-domain $s_1$</td>
<td>$c_{11} = 0.0000$, $c_{12} = 3.5580 \times 10^{-1}$, $c_{13} = 0.0000$, $c_{14} = 3.6960 \times 10^{-1}$, $c_{15} = 1.1464$, $c_{16} = 2.0600 \times 10^{-2}$</td>
</tr>
<tr>
<td>Sub-domain $s_2$</td>
<td>$c_{21} = 0.0000$, $c_{22} = 0.0000$, $c_{23} = 0.0000$, $c_{24} = 0.0000$, $c_{25} = 8.0640 \times 10^{-1}$, $c_{26} = 6.6380 \times 10^{-1}$</td>
</tr>
<tr>
<td>Sub-domain $s_3$</td>
<td>$c_{31} = 7.1310 \times 10^{-1}$, $c_{32} = 8.2850 \times 10^{-1}$, $c_{33} = 0.0000$, $c_{34} = 0.0000$, $c_{35} = 5.6440 \times 10^{-1}$, $c_{36} = 0.0000$</td>
</tr>
<tr>
<td>Sub-domain $s_4$</td>
<td>$c_{41} = 1.7400 \times 10^{-2}$, $c_{42} = 9.3600 \times 10^{-2}$, $c_{43} = 2.5770 \times 10^{-1}$, $c_{44} = 0.0000$, $c_{45} = 0.0000$, $c_{46} = 0.0000$</td>
</tr>
</tbody>
</table>

Table A.3: $k_{ijdr}$ in sub-domain $s_1$ for four-sub-domain case in Section 4.5

<table>
<thead>
<tr>
<th>Sub-domain $s_d$</th>
<th>$k_{ijdr}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sub-domain $s_1$</td>
<td>$k_{1111} = 1.0000$, $k_{1112} = -3.6080 \times 10^{-1}$, $k_{1113} = -2.2000 \times 10^{-3}$, $k_{1114} = -3.6790 \times 10^{-1}$, $k_{1115} = -1.1519$, $k_{1116} = -1.0000 \times 10^{-3}$, $k_{1211} = -1.7500$, $k_{1212} = 1.0000$, $k_{1213} = -2.0000 \times 10^{-3}$, $k_{1214} = -1.5700 \times 10^{-2}$, $k_{1215} = -1.0000 \times 10^{-3}$, $k_{1216} = -1.0000 \times 10^{-3}$, $k_{2111} = -2.0411$, $k_{2112} = -1.0000 \times 10^{-3}$, $k_{2113} = 1.0000$, $k_{2114} = -1.0000 \times 10^{-3}$, $k_{2115} = -1.0000 \times 10^{-3}$, $k_{2116} = -5.9700 \times 10^{-2}$, $k_{3111} = -6.9110 \times 10^{-1}$, $k_{3112} = -1.0000 \times 10^{-3}$, $k_{3113} = -2.2842$, $k_{3114} = -1.0000 \times 10^{-3}$, $k_{3115} = 1.0000$, $k_{3116} = -1.0000 \times 10^{-3}$, $k_{3211} = -5.5390 \times 10^{-1}$, $k_{3212} = -8.5110 \times 10^{-1}$, $k_{3213} = -5.4030 \times 10^{-1}$, $k_{3214} = -1.0000 \times 10^{-3}$, $k_{3215} = -6.9830 \times 10^{-1}$, $k_{3216} = 1.0000$</td>
</tr>
</tbody>
</table>
Table A.4: $k_{ijdr}$ in sub-domain $s_2$ for four-sub-domain case in Section 4.5

<table>
<thead>
<tr>
<th>Sub-domain $s_d$</th>
<th>$k_{ijdr}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_{1121} = 0.0000$, $k_{1122} = 0.0000$, $k_{1123} = 0.0000$, $k_{1124} = 0.0000$, $k_{1125} = -1.1638$, $k_{1126} = -1.9540 \times 10^{-1}$, $k_{1221} = 0.0000$, $k_{1222} = 0.0000$, $k_{1223} = 0.0000$, $k_{1224} = 0.0000$, $k_{1225} = -1.0000 \times 10^{-3}$, $k_{1226} = -1.0000 \times 10^{-3}$, $k_{2121} = 0.0000$, $k_{2122} = 0.0000$, $k_{2123} = 0.0000$, $k_{2124} = 0.0000$, $k_{2125} = -6.0900 \times 10^{-2}$, $k_{2126} = -9.8940 \times 10^{-1}$, $k_{2221} = 0.0000$, $k_{2222} = 0.0000$, $k_{2223} = 0.0000$, $k_{2224} = 0.0000$, $k_{2225} = -1.0000 \times 10^{-3}$, $k_{2226} = -1.0000 \times 10^{-3}$, $k_{3121} = 0.0000$, $k_{3122} = 0.0000$, $k_{3123} = 0.0000$, $k_{3124} = 0.0000$, $k_{3125} = 1.0000$, $k_{3126} = -1.4857$, $k_{3221} = 0.0000$, $k_{3222} = 0.0000$, $k_{3223} = 0.0000$, $k_{3224} = 0.0000$, $k_{3225} = -1.0007$, $k_{3226} = 1.0000$</td>
<td></td>
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</tbody>
</table>

Table A.5: $k_{ijdr}$ in sub-domain $s_3$ for four-sub-domain case in Section 4.5

<table>
<thead>
<tr>
<th>Sub-domain $s_d$</th>
<th>$k_{ijdr}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_{1131} = 1.0000$, $k_{1132} = 0.0000$, $k_{1133} = -8.8135$, $k_{1134} = -3.9795$, $k_{1135} = -1.0000 \times 10^{-3}$, $k_{1136} = -9.4800 \times 10^{-2}$, $k_{1231} = -9.6050 \times 10^{-1}$, $k_{1232} = 1.0000$, $k_{1233} = -5.0030$, $k_{1234} = -6.0010$, $k_{1235} = -9.5110 \times 10^{-1}$, $k_{1236} = -1.0000 \times 10^{-3}$, $k_{2131} = -9.3030 \times 10^{-1}$, $c_{2132} = 0.0000$, $k_{2133} = 1.0000$, $k_{2134} = -7.5100 \times 10^{-1}$, $k_{2135} = -1.0000 \times 10^{-3}$, $k_{2136} = -1.0000 \times 10^{-3}$, $k_{2231} = -5.0050 \times 10^{-1}$, $k_{2232} = -2.0000 \times 10^{-1}$, $k_{2233} = -1.0000 \times 10^{-3}$, $k_{2234} = 1.0000$, $k_{2235} = -9.1750 \times 10^{-1}$, $k_{2236} = -6.2000 \times 10^{-3}$, $k_{3131} = -1.5007$, $k_{3132} = 0.0000$, $k_{3133} = -2.5100 \times 10^{-1}$, $k_{3134} = -2.9733$, $k_{3135} = 1.0000$, $k_{3136} = -5.1060 \times 10^{-1}$, $k_{3231} = -5.8660 \times 10^{-1}$, $k_{3232} = -1.1455$, $k_{3233} = -8.2000 \times 10^{-3}$, $k_{3234} = -9.9400 \times 10^{-2}$, $k_{3235} = -1.0000 \times 10^{-3}$, $k_{3236} = 1.0000$</td>
<td></td>
</tr>
</tbody>
</table>
Table A.6: $k_{ijdr}$ in sub-domain $s_4$ for four-sub-domain case in Section 4.5

<table>
<thead>
<tr>
<th>Sub-domain $s_d$</th>
<th>$k_{ijdr}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$k_{1141} = 1.0000, k_{1142} = 0.0000, $ $k_{1143} = -1.8171, k_{1144} = -1.0954, $ $k_{1145} = 0.0000, k_{1146} = 0.0000, $ $k_{1241} = -1.0000 \times 10^{-3}, k_{1242} = 1.0000, $ $k_{1243} = -8.5840 \times 10^{-1}, k_{1244} = -3.9385, $ $k_{1245} = 0.0000, k_{1246} = 0.0000, $ $k_{2141} = -5.6170 \times 10^{-1}, k_{2142} = -3.7530 \times 10^{-1}, $ $k_{2143} = 1.0000, k_{2144} = -1.1945 \times 10^{1}, $ $k_{2145} = 0.0000, k_{2146} = 0.0000, $ $k_{2241} = -1.0000 \times 10^{-3}, k_{2242} = -3.2620 \times 10^{-1}, $ $k_{2243} = -1.0960, k_{2244} = 1.0000, $ $k_{2245} = 0.0000, k_{2246} = 0.0000, $ $k_{3141} = -1.0000 \times 10^{-3}, k_{3142} = 0.0000, $ $k_{3143} = -1.0000 \times 10^{-3}, k_{3144} = -5.7700 \times 10^{-2}, $ $k_{3145} = 0.0000, k_{3146} = 0.0000, $ $k_{3241} = -1.0000 \times 10^{-3}, k_{3242} = 0.0000, $ $k_{3243} = -1.0000 \times 10^{-3}, k_{3244} = -1.0000 \times 10^{-3}, $ $k_{3245} = 0.0000, k_{3246} = 0.0000</td>
</tr>
</tbody>
</table>

Table A.7: Lower and upper bounds of $\gamma_{ijd}$ for one-sub-domain case in Section 4.5

<table>
<thead>
<tr>
<th>Sub-domain $s_d$</th>
<th>$\underline{\gamma}_{ijd}$</th>
<th>$\overline{\gamma}_{ijd}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\gamma_{111} = 5.5653 \times 10^{-9}$</td>
<td>$\overline{\gamma}_{111} = 9.9085 \times 10^{-1}$</td>
</tr>
<tr>
<td></td>
<td>$\gamma_{121} = 8.2596 \times 10^{-7}$</td>
<td>$\overline{\gamma}_{121} = 6.4274 \times 10^{-1}$</td>
</tr>
<tr>
<td></td>
<td>$\gamma_{211} = 1.6543 \times 10^{-5}$</td>
<td>$\overline{\gamma}_{211} = 5.9988 \times 10^{-2}$</td>
</tr>
<tr>
<td></td>
<td>$\gamma_{221} = 1.6543 \times 10^{-5}$</td>
<td>$\overline{\gamma}_{221} = 6.4274 \times 10^{-1}$</td>
</tr>
<tr>
<td></td>
<td>$\gamma_{311} = 8.2596 \times 10^{-7}$</td>
<td>$\overline{\gamma}_{311} = 5.9988 \times 10^{-2}$</td>
</tr>
<tr>
<td></td>
<td>$\gamma_{321} = 5.5653 \times 10^{-9}$</td>
<td>$\overline{\gamma}_{321} = 9.9085 \times 10^{-1}$</td>
</tr>
</tbody>
</table>

Sub-domain $s_1$
Table A.8: $c_{dr}$ and $k_{ijdr}$ for one-sub-domain case in Section 4.5

<table>
<thead>
<tr>
<th>Sub-domain $s_d$</th>
<th>$c_{dr}$</th>
<th>$k_{ijdr}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sub-domain $s_1$</td>
<td>$c_{i1} = 0.0000$</td>
<td>$k_{i1111} = 1.0000, k_{i1112} = -1.3000 \times 10^{-4}$</td>
</tr>
<tr>
<td></td>
<td>$c_{i2} = 0.0000$</td>
<td>$k_{i1113} = -1.0000 \times 10^{-3}, k_{i1211} = -2.7000 \times 10^{-3}$</td>
</tr>
<tr>
<td></td>
<td>$c_{i3} = 0.0000$</td>
<td>$k_{i1212} = -3.2500 \times 10^{-2}, k_{i1213} = -1.1000 \times 10^{-3}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$k_{i2111} = -1.0000 \times 10^{-3}, k_{i2112} = 1.0000$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$k_{i2113} = -1.0000 \times 10^{-3}, k_{i2211} = -1.0000 \times 10^{-3}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$k_{i2212} = -1.0000 \times 10^{-3}, k_{i2213} = 1.0000$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$k_{i3111} = -1.2000 \times 10^{-3}, k_{i3112} = -1.0000 \times 10^{-3}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$k_{i3113} = -8.8600 \times 10^{-2}, k_{i3211} = -1.0000 \times 10^{-3}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$k_{i3212} = -1.0000 \times 10^{-3}, k_{i3213} = -1.0000 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

Table A.9: Feedback gains $X(x_2)$ and $N_{jd}(x_2)$ for Case 1 with $a = 55$ and $b = 94$ in Section 4.5

<table>
<thead>
<tr>
<th>$s_d$</th>
<th>$N_{jd}(x_2) = [N_{jd1}(x_2) \ N_{jd2}(x_2)]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1$</td>
<td>$N_{i111}(x_2) = -5.9890 \times 10^{-3} x_2^3 - 2.3934 \times 10^{-2} x_2 - 2.6829 \times 10^{-1}$</td>
</tr>
<tr>
<td></td>
<td>$N_{i112}(x_2) = -2.0124 \times 10^{-5} x_2^2 - 2.6215 \times 10^{-4} x_2 + 3.2901 \times 10^{-3}$</td>
</tr>
<tr>
<td></td>
<td>$N_{i211}(x_2) = -5.7650 \times 10^{-3} x_2^3 - 3.6044 \times 10^{-2} x_2 - 7.0685 \times 10^{-2}$</td>
</tr>
<tr>
<td></td>
<td>$N_{i212}(x_2) = -9.5058 \times 10^{-6} x_2^2 - 1.0412 \times 10^{-4} x_2 + 2.4881 \times 10^{-3}$</td>
</tr>
<tr>
<td>$s_2$</td>
<td>$N_{i121}(x_2) = -5.8849 \times 10^{-3} x_2^3 - 4.2751 \times 10^{-2} x_2 - 1.5270 \times 10^{-1}$</td>
</tr>
<tr>
<td></td>
<td>$N_{i122}(x_2) = -4.3962 \times 10^{-6} x_2^2 - 8.7609 \times 10^{-7} x_2 + 2.1779 \times 10^{-4}$</td>
</tr>
<tr>
<td></td>
<td>$N_{i221}(x_2) = -5.7924 \times 10^{-3} x_2^3 - 4.3266 \times 10^{-2} x_2 - 6.9215 \times 10^{-2}$</td>
</tr>
<tr>
<td></td>
<td>$N_{i222}(x_2) = -6.7048 \times 10^{-6} x_2^2 - 2.7756 \times 10^{-5} x_2 - 2.4929 \times 10^{-4}$</td>
</tr>
<tr>
<td>$s_3$</td>
<td>$N_{i131}(x_2) = -6.0313 \times 10^{-3} x_2^3 - 4.9202 \times 10^{-2} x_2 - 4.8730 \times 10^{-2}$</td>
</tr>
<tr>
<td></td>
<td>$N_{i132}(x_2) = -6.4740 \times 10^{-6} x_2^2 - 1.7660 \times 10^{-5} x_2 - 6.2832 \times 10^{-4}$</td>
</tr>
<tr>
<td></td>
<td>$N_{i231}(x_2) = -6.1337 \times 10^{-3} x_2^3 - 4.9391 \times 10^{-2} x_2 - 4.7349 \times 10^{-2}$</td>
</tr>
<tr>
<td></td>
<td>$N_{i232}(x_2) = -1.6135 \times 10^{-5} x_2^2 - 2.6247 \times 10^{-5} x_2 - 5.0413 \times 10^{-4}$</td>
</tr>
<tr>
<td>$s_4$</td>
<td>$N_{i141}(x_2) = -5.7780 \times 10^{-3} x_2^3 - 3.9018 \times 10^{-2} x_2 - 5.8616 \times 10^{-2}$</td>
</tr>
<tr>
<td></td>
<td>$N_{i142}(x_2) = -1.7569 \times 10^{-5} x_2^2 - 2.7354 \times 10^{-5} x_2 - 4.4848 \times 10^{-4}$</td>
</tr>
<tr>
<td></td>
<td>$N_{i241}(x_2) = -5.7676 \times 10^{-3} x_2^3 - 4.3167 \times 10^{-2} x_2 - 3.0887 \times 10^{-2}$</td>
</tr>
<tr>
<td></td>
<td>$N_{i242}(x_2) = -2.9687 \times 10^{-5} x_2^2 - 3.4835 \times 10^{-5} x_2 - 3.1681 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

$X(x_2) = [X_{11}(x_2) \ X_{12}(x_2) \ X_{21}(x_2) \ X_{22}(x_2)]$

- $X_{11}(x_2) = 2.2790 \times 10^{-3} x_2^2 + 7.1140 \times 10^{-3} x_2 + 1.1080 \times 10^{-1}$
- $X_{12}(x_2) = X_{21}^T(x_2) = 2.0050 \times 10^{-5} x_2^2 - 9.8560 \times 10^{-6} x_2 + 4.9070 \times 10^{-4}$
- $X_{22}(x_2) = 3.9630 \times 10^{-6} x_2^2 - 1.4000 \times 10^{-4} x_2 + 1.2340 \times 10^{-3}$
Table A.10: Feedback gains $X(x_2)$ and $N_{jd}(x_2)$ for Case 2 with $a = 51$ and $b = 88$ in Section 4.5

$$X(x_2) = \begin{bmatrix} X_{11}(x_2) & X_{12}(x_2) \\ X_{21}(x_2) & X_{22}(x_2) \end{bmatrix}$$

$X_{11}(x_2) = 3.5910 \times 10^{-3}x_2^2 + 1.1820 \times 10^{-2}x_2 + 1.5640 \times 10^{-1}$

$X_{12}(x_2) = X_{21}(x_2) = 2.9150 \times 10^{-5}x_2^2 - 4.8560 \times 10^{-5}x_2 + 5.7200 \times 10^{-4}$

$X_{22}(x_2) = 5.1490 \times 10^{-6}x_2^2 - 2.2750 \times 10^{-4}x_2 + 2.4680 \times 10^{-3}$

$s_d$ $N_{jd}(x_2) = \begin{bmatrix} N_{j11}(x_2) & N_{j12}(x_2) \\ N_{j21}(x_2) & N_{j22}(x_2) \end{bmatrix}$

$s_1$

$N_{111}(x_2) = -9.5099 \times 10^{-3}x_2^2 - 4.2653 \times 10^{-4}x_2 - 3.6991 \times 10^{-1}$

$N_{112}(x_2) = -3.3067 \times 10^{-5}x_2^2 - 2.7368 \times 10^{-4}x_2 + 6.4091 \times 10^{-3}$

$N_{211}(x_2) = -9.1014 \times 10^{-3}x_2^2 - 6.5546 \times 10^{-3}x_2 - 1.0229 \times 10^{-1}$

$N_{212}(x_2) = -1.4738 \times 10^{-5}x_2^2 - 1.2790 \times 10^{-4}x_2 + 5.2481 \times 10^{-3}$

$s_2$

$N_{121}(x_2) = -9.1915 \times 10^{-3}x_2^2 - 6.7481 \times 10^{-4}x_2 - 2.0012 \times 10^{-1}$

$N_{122}(x_2) = 8.4487 \times 10^{-6}x_2^2 + 3.6431 \times 10^{-5}x_2 + 6.0266 \times 10^{-4}$

$N_{221}(x_2) = -9.0924 \times 10^{-3}x_2^2 - 7.1165 \times 10^{-3}x_2 - 9.2380 \times 10^{-2}$

$N_{222}(x_2) = 2.4669 \times 10^{-6}x_2^2 - 3.5544 \times 10^{-5}x_2 - 2.4507 \times 10^{-4}$

$s_3$

$N_{131}(x_2) = -9.3374 \times 10^{-3}x_2^2 - 7.9303 \times 10^{-4}x_2 - 6.4856 \times 10^{-2}$

$N_{132}(x_2) = -1.5222 \times 10^{-6}x_2^2 - 3.5818 \times 10^{-5}x_2 - 9.6682 \times 10^{-4}$

$N_{231}(x_2) = -9.6093 \times 10^{-3}x_2^2 - 8.1046 \times 10^{-3}x_2 - 6.4100 \times 10^{-2}$

$N_{232}(x_2) = -1.0876 \times 10^{-5}x_2^2 - 4.2272 \times 10^{-5}x_2 - 8.4239 \times 10^{-4}$

$s_4$

$N_{141}(x_2) = -9.1041 \times 10^{-3}x_2^2 - 6.8544 \times 10^{-4}x_2 - 8.2304 \times 10^{-2}$

$N_{142}(x_2) = -1.4327 \times 10^{-5}x_2^2 - 4.9133 \times 10^{-5}x_2 - 6.3889 \times 10^{-4}$

$N_{241}(x_2) = -9.1474 \times 10^{-3}x_2^2 - 7.2984 \times 10^{-3}x_2 - 5.0482 \times 10^{-2}$

$N_{242}(x_2) = -4.5419 \times 10^{-5}x_2^2 - 6.1082 \times 10^{-5}x_2 - 4.4539 \times 10^{-4}$
Table A.11: Feedback gains $X(x_2)$ and $N_{j_2}(x_2)$ for Case 3-6 in Section 4.5

<table>
<thead>
<tr>
<th>Case</th>
<th>$X(x_2)$</th>
<th>$N_j(x_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$X_{11}(x_2) = 8.1690 \times 10^{-3}x_2^2 + 6.4310 \times 10^{-4}x_2 + 3.0110 \times 10^{-1}$</td>
<td>$N_{j_1}(x_2)$, $N_{j_2}(x_2)$</td>
</tr>
<tr>
<td></td>
<td>$X_{12}(x_2) = X_{21}(x_2) = 1.4740 \times 10^{-5}x_2^2 + 5.3990 \times 10^{-5}x_2 + 3.1750 \times 10^{-4}$</td>
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<td></td>
<td>$X_{22}(x_2) = 6.8440 \times 10^{-7}x_2^2 + 1.8730 \times 10^{-5}x_2 + 4.3380 \times 10^{-4}$</td>
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<tr>
<td></td>
<td>$N_{11}(x_2) = -2.2295 \times 10^{-2}x_2^2 - 5.4350 \times 10^{-2}x_2 - 5.7727 \times 10^{-1}$</td>
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<tr>
<td></td>
<td>$N_{12}(x_2) = 1.0664 \times 10^{-6}x_2^2 - 5.8404 \times 10^{-5}x_2 - 2.7992 \times 10^{-4}$</td>
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<td></td>
<td>$N_{21}(x_2) = -2.1715 \times 10^{-2}x_2^2 - 6.6221 \times 10^{-2}x_2 - 3.0713 \times 10^{-1}$</td>
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<tr>
<td></td>
<td>$N_{22}(x_2) = -2.9942 \times 10^{-5}x_2^2 - 7.5285 \times 10^{-5}x_2 - 3.9447 \times 10^{-4}$</td>
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</tr>
<tr>
<td>4</td>
<td>$X_{11}(x_2) = 7.7430 \times 10^{-4}x_2^2 + 5.5730 \times 10^{-2}x_2 + 2.5890 \times 10^{-1}$</td>
<td>$N_{j_1}(x_2)$, $N_{j_2}(x_2)$</td>
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<td></td>
<td>$X_{12}(x_2) = X_{21}(x_2) = 1.1420 \times 10^{-5}x_2^2 + 1.3590 \times 10^{-5}x_2 + 7.3510 \times 10^{-5}$</td>
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<tr>
<td></td>
<td>$X_{22}(x_2) = 3.4420 \times 10^{-7}x_2^2 + 1.4890 \times 10^{-5}x_2 + 5.6580 \times 10^{-4}$</td>
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<tr>
<td></td>
<td>$N_{11}(x_2) = -2.1083 \times 10^{-2}x_2^2 - 4.6793 \times 10^{-2}x_2 - 5.0298 \times 10^{-1}$</td>
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<tr>
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<td>$N_{12}(x_2) = -1.4954 \times 10^{-6}x_2^2 - 1.6145 \times 10^{-5}x_2 + 2.8551 \times 10^{-4}$</td>
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<tr>
<td></td>
<td>$N_{21}(x_2) = -2.0674 \times 10^{-2}x_2^2 - 5.7567 \times 10^{-2}x_2 - 2.6635 \times 10^{-1}$</td>
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<tr>
<td></td>
<td>$N_{22}(x_2) = -8.8375 \times 10^{-6}x_2^2 - 3.8180 \times 10^{-5}x_2 - 1.7115 \times 10^{-4}$</td>
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<tr>
<td>5</td>
<td>$X_{11}(x_2) = 4.7360 \times 10^{-4}x_2^2 + 1.2380 \times 10^{-2}x_2 + 4.0220 \times 10^{-1}$</td>
<td>$N_{j_1}(x_2)$, $N_{j_2}(x_2)$</td>
</tr>
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<td></td>
<td>$X_{12}(x_2) = X_{21}(x_2) = 1.6930 \times 10^{-6}x_2^2 + 3.3500 \times 10^{-5}x_2 + 5.6760 \times 10^{-4}$</td>
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<td></td>
<td>$X_{22}(x_2) = 9.1070 \times 10^{-9}x_2^2 + 6.8950 \times 10^{-7}x_2 + 6.0320 \times 10^{-4}$</td>
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<tr>
<td></td>
<td>$N_{11}(x_2) = -1.2838 \times 10^{-1}x_2^2 - 2.0800 \times 10^{-1}x_2 - 7.7226 \times 10^{-1}$</td>
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<td></td>
<td>$N_{12}(x_2) = 5.1177 \times 10^{-7}x_2^2 - 1.5836 \times 10^{-6}x_2 - 1.1238 \times 10^{-3}$</td>
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<td></td>
<td>$N_{21}(x_2) = -1.2836 \times 10^{-1}x_2^2 - 2.1003 \times 10^{-1}x_2 - 7.4651 \times 10^{-1}$</td>
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<td></td>
<td>$N_{22}(x_2) = 5.5884 \times 10^{-7}x_2^2 - 1.8026 \times 10^{-6}x_2 - 1.1322 \times 10^{-3}$</td>
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<td>6</td>
<td>$X_{11}(x_2) = 9.2420 \times 10^{3}x_2^2 + 2.3810 \times 10^{1}x_2 + 6.9010 \times 10^{1}$</td>
<td>$N_{j_1}(x_2)$, $N_{j_2}(x_2)$</td>
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<tr>
<td></td>
<td>$X_{12}(x_2) = X_{21}(x_2) = 1.0400 \times 10^{-4}x_2^2 + 3.4520 \times 10^{-3}x_2 + 5.4280 \times 10^{-2}$</td>
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<tr>
<td></td>
<td>$X_{22}(x_2) = 2.9410 \times 10^{-7}x_2^2 + 6.8980 \times 10^{-5}x_2 + 5.3610 \times 10^{-2}$</td>
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</tr>
<tr>
<td></td>
<td>$N_{11}(x_2) = -2.5095 \times 10^{1}x_2^2 - 4.0912 \times 10^{1}x_2 - 1.3200 \times 10^{2}$</td>
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</tr>
<tr>
<td></td>
<td>$N_{12}(x_2) = 7.2167 \times 10^{-6}x_2^2 + 1.5115 \times 10^{-5}x_2 - 1.1037 \times 10^{-1}$</td>
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</tr>
<tr>
<td></td>
<td>$N_{21}(x_2) = -2.5095 \times 10^{1}x_2^2 - 4.0912 \times 10^{1}x_2 - 1.3200 \times 10^{2}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$N_{22}(x_2) = 7.2167 \times 10^{-6}x_2^2 + 1.5115 \times 10^{-5}x_2 - 1.1037 \times 10^{-1}$</td>
<td></td>
</tr>
</tbody>
</table>
Bibliography


[84] H. K. Lam and L. D. Seneviratne, “Chaotic Synchronization Using Sampled-Data Fuzzy Controller Based on Fuzzy-Model-Based Approach,”


