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The Pontryagin Forms of Hessian Manifolds

J. Armstrong[1] and S.Amari[2]

[1] King's College London, London, WC2R 2LS, UK.,
john.1.armstrong@kcl.ac.uk,
[2] RIKEN Brain Science Institute, Saitama 351-0198, Japan,
amari@brain.riken.jp

Abstract. We show that Hessian manifolds of dimensions 4 and above must have vanishing Pontryagin forms. This gives a topological obstruction to the existence of Hessian metrics. We find an additional explicit curvature identity for Hessian 4-manifolds. By contrast, we show that all analytic Riemannian 2-manifolds are Hessian.

1 Introduction

At GSI2013, S. Amari asked the question of when a given Riemannian metric is a Hessian metric. In other words, for what metrics g do there exist local coordinates at every point such that g can be written as the Hessian of some convex potential function ϕ?

As a first result we will show that:

- All analytic metrics in 2 dimensions are Hessian metrics.
- Not all analytic metrics in 3 dimensions and higher are Hessian metrics.
- In dimensions 4 and above there are restrictions on the possible curvature tensors of Hessian metrics.

We will see that these results are quite simple to prove using Cartan–Kähler theory and were found independently by Robert Bryant [3].

A further question posed by Amari was to find conditions and invariants which characterize the Riemannian metrics which are Hessian. The ultimate goal would be to define a set of tensors such that the metric is Hessian if and only if these tensors vanish. We cannot achieve this goal in full. However, a partial answer that we can demonstrate is that the *Pontryagin forms* of the metric must vanish. By the *Pontryagin forms* we mean the differential forms defined in terms of the curvature that provide representatives of the Pontryagin classes.

We note that this provides a topological obstruction to the existence of a Hessian metric: a compact manifold that admits a Hessian metric must have vanishing Pontryagin classes.

To put this into context we recall that Hessian metrics are be locally equivalent to g-dually flat structures. That is g is Hessian if and only if one can locally find flat affine connections ∇ and ∇^* satisfying:

$$g(\nabla_Z X, Y) = g(X, \nabla^*_Z Y).$$

In [2] the question of when a manifold admits a global g-dually flat structure is considered and some topological obstructions are found. Our result is related, but distinct. We have found an obstruction to the existence of a metric which is required to be locally g-dually flat but which need not have globally defined connections ∇ and ∇^*.

It is trivial that a manifold that is globally g-dually flat must have vanishing Pontryagin classes: simply consider the Pontryagin forms defined by the flat connection. By the same token, the Euler characteristic must vanish on any manifold which is globally g-dually flat.

On the other hand as mentioned above, all 2-manifolds admit Hessian metrics including those with non-vanishing Euler characteristic. Thus in 2 dimensions there is a large difference between the set of manifolds which admit Hessian metrics and those which admit global g-dually flat structures. One can generalize this example to higher dimensions by considering quotients of hyperbolic space. It is well known that the hyperbolic metric is a Hessian metric, yet quotients of hyperbolic space may have non-vanishing Euler characterisic implying that they cannot admit a flat connection never mind a g-dually flat structure.

Thus this paper provides a first step towards answering the interesting question: which manifolds admit a Hessian metric?

This paper is a summary and update of a joint paper with S. Amari. Full details can be found in [1].

2 A counting argument

To define a Hessian metric locally near a point p an a manifold M^n we need to choose a set of coordinates $x : M^n \to \mathbb{R}$ defined in a neighbourhood of p and a strictly convex potential function ϕ. We can then write down a Hessian metric

$$g_{ij} = \frac{\partial^2 \phi}{\partial x_i \partial x_j}.$$

Speaking somewhat loosely we can say that a Hessian metric depends upon $n+1$ real valued functions of n variables: the n coordinate functions and the potential ϕ.

On the other hand to write down a general metric we need to choose the $\frac{n(n+1)}{2}$ tensor components g_{ij} in some neighbourhood of the point p. Thus a general Riemannian metric depends upon $\frac{n(n+1)}{2}$ real valued functions of n variables.

Since $\frac{n(n+1)}{2} > n+1$ when $n > 2$ this strongly suggests that there are many more Riemannian metrics than Hessian metrics in dimensions greater than 3.

This argument is suggestive but not rigorous. In particular it gives the wrong answer in dimension 1! Our formulae would suggest that there are more Hessian metrics than Riemannian metrics in dimension 1. The reason for this is that we haven't taken into account the diffeomorphism group when counting.

To make the argument rigorous we need to consider jet bundles. If the metric is Hessian we see that the k-jet of the metric depends upon the $k+2$-jet of the functions x and ϕ. The dimension of the space of $(k+2)$-jets of $(n+1)$ functions of n real variables is:

$$\dim J_{k+2}(x,\phi) := \sum_{i=0}^{k+2}(n+1)\dim(S^i T_p) = \sum_{i=0}^{k+2}(n+1)\binom{n+i-1}{i}.$$

Similarly the dimension of the space of k-jets of g_{ij} is:

$$\dim J_k(g) := \sum_{i=0}^{k}\frac{n(n+1)}{2}\dim(S^i T_p) = \sum_{i=0}^{k}\frac{n(n+1)}{2}\binom{n+i-1}{i}.$$

If we now compare the growth rate of $\dim J_k(g)$ and $\dim J_{k+2}(x,\phi)$ as k increases we see that so long as $n > 2$, $\dim J_k(g) > \dim J_{k+2}(x,\phi)$ for sufficiently large n. To see this note that we can write:

$$\dim J_k(g) - \dim J_{k+2}(x,\phi) = (n+1)(a_{k,n} - b_{k,n})$$

where

$$a_{k,n} := \left(\frac{n}{2}-1\right)\sum_{i=1}^{k}\binom{n+1-i}{i},$$
$$b_{k,n} := \binom{n+k}{k+1} + \binom{n+k+1}{k+2}.$$

We can now rigorously conclude that in dimensions greater than 2 there really are more Riemannian metrics than Hessian metrics. The growth rate of jet bundles provides a rigorous language for heuristic counting arguments.

3 Dimension 2

In dimension 2 our counting argument fails. It seems conceivable that every Riemannian metric is a Hessian metric. One can go further and explicitly identify the mapping from $(k+2)$ jets of (x,ϕ) to k-jets of metrics. It is not difficult to do so for low values of k. One discovers that the mapping is onto. It is easy to write a computer program that computes the mapping for a given value of k, in which case one will again discover that the mapping is onto.

One would like to be able to find a proof that this mapping is onto for all k and one would like to be able to deduce from this that all Riemannian metrics are Hessian metrics.

Fortunately a toolkit already exists for solving precisely this kind of problem. It is called Cartan–Kähler theory.

A general setting is to consider two vector bundles V and W over some n-manifold M^n. Let $D : \Gamma(V) \longrightarrow \Gamma(W)$ be an order k differential operator mapping sections of V to sections of W. In other words let D map k-jets of V at p to elements of W_p.

The top order term of this mapping is called the symbol σ_D of D.

$$\sigma_D : S^k T^*_p \otimes V_p \longrightarrow W_p$$

The reason that the top order term acts on a symmetric power of the tangent bundle simply comes from the fact that derivatives in different directions commute. The top order term only depends on the k-th derivatives of a section $v \in \Gamma(V)$. If we assume that D is quasilinear then σ will be a linear map.

If σ is onto then the differential equation $Dv = w$ can always be solved up to order k at p. Now consider differentiating the equation $Dv = w$. We will get a $k+1$-th order differential equation. We can associate a symbol σ_1 to this differential equation. If σ_1 is onto too then we can always solve the equation to $k+1$-st order. Continuing in this way we can define a sequence of symbols σ_i. If they are all onto then we can solve the differential equation to any desired order. Note that the σ_i can be easily computed directly from σ. Thus requiring that σ_i is onto for all i is just an algebraic condition on σ.

How can one prove that σ_i is onto for all i? The solution is to use *Cartan's test* which we will now describe. Given a basis $\{v_1, v_2, \dots v_n\}$ for T^*M, define the map:

$$\sigma_{i,m} : S^{k+i}\langle v_1, v_2, \dots v_m\rangle \otimes V_p \longrightarrow S^i T^*_p \otimes W_p$$

to be the restriction of σ_i. Define $g_{i,m} := \dim \ker \sigma_{i,m}$. If one can find a basis $\{v_1, v_2, \dots v_n\}$ and a number α such that σ_i is onto for all $i \leqslant \alpha$ and such that $g_{\alpha,n} = \sum_{\beta=0}^{k} g_{\alpha-1\beta}$ then the differential equation is said to be involutive. It can be shown that this implies that $\sigma_{\alpha+i}$ is onto for all i. Moreover, if one is working in the analytic category, one can then prove that solutions to the differential equation exist [5, 6, 4].

This gives a strategy for proving that all analytic 2-metrics are locally g-dually flat and hence Hessian. Given a metric g we can interpret the requirement that it is Hessian as requiring that we can locally find a g-dually flat connection. This gives rise to a 1-st order differential equation for a connection A.

To be precise, we define

$$\iota : T^* \otimes T^* \otimes T \longrightarrow T^* \otimes T^* \otimes T^*$$

to be raising the final index using the metric then to find a g-dually flat connection we seek a tensor $A \in \iota^{-1}(S^3 T^*)$ such that the connection $\nabla + A$ has curvature zero. It is well known that such a tensor is equivalent to a g-dually flat connection.

The details are not illuminating. The point is that we have expressed the problem as a differential equation and it is a simple algebraic exercise to check that this equation is involutive. It follows that all analytic 2-metrics are Hessian.

4 Dimensions > 4

Our aim in this section is to find more concrete obstructions to the existence of Hessian metrics. The key result is the following [8]:

Proposition 1. *Let (M, g) be a Riemannian manifold. Let ∇ denote the Levi–Civita connection and let $\overline{\nabla} = \nabla + A$ be a g-dually flat connection. Then*

*(i) The tensor A_{ijk} lies in S^3T^*. We shall call it the S^3-*tensor *of $\overline{\nabla}$.*
(ii) The S^3-tensor determines the Riemann curvature tensor as follows:

$$R_{ijkl} = -g^{ab}A_{ika}A_{jlb} + g^{ab}A_{ila}A_{jkb}. \tag{1}$$

Proof. $A \in T^* \otimes T^* \otimes T$. The condition that $\overline{\nabla}$ is torsion free is equivalent to requiring that $A \in S^2T^* \otimes T$. Using the metric to identify T and T^*, the condition that $\overline{\nabla}$ is dually torsion free can be written as $A \in S^3T^*$.

Expanding the formula $\overline{R}_{XY}Z = \overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X - \overline{\nabla}_{[X,Y]}Z$ in terms of ∇ and A, one obtains the following curvature identity:

$$\overline{R}_{XY}Z = R_{XY}Z + 2(\nabla_{[X}A)_{Y]}Z + 2A_{[X}A_{Y]}Z. \tag{2}$$

Here $\overline{R} = 0$ is the curvature of $\overline{\nabla}$ and the square brackets denote anti-symmetrization. Since $\overline{\nabla}$ is dually flat $\overline{R} = 0$.

Continuing to use the metric to identify T and T^*, the symmetries of the curvature tensor tell us that $R \in \Lambda^2T \otimes \Lambda^2T$. On the other hand, $(\nabla_{[.}A)_{.]} \in \Lambda^2T \otimes S^2T$. Thus if one projects equation (2) onto $\Lambda^2T \otimes \Lambda^2T$ one obtains the curvature identity (1).

We define a quadratic equivariant map ρ from $S^3T^* \longrightarrow \Lambda^2T^* \otimes \Lambda^2T^*$ by:

$$\rho(A_{ijk}) = -g^{ab}A_{ika}A_{jlb} + g^{ab}A_{ila}A_{jkb}$$

Corollary 1. *In dimensions > 4 the condition that R lies in the image of ρ gives a non-trivial necessary condition for a metric g to be a Hessian metric.*

Proof. $\dim S^3T = \binom{n+2}{n-1} = \frac{1}{6}n(1+n)(2+n)$. The dimension of the space of algebraic curvature tensors, $\mathcal{R}$, is $\dim \mathcal{R} = \frac{1}{12}n^2(n^2-1)$. So $\dim \mathcal{R} - \dim S^3T = \frac{1}{12}n(n-4)(1+n)^2$. This is strictly positive if $n > 4$.

5 Dimension 4

Surprisingly the condition that R lies in the image of ρ gives a non-trivial condition in dimension 4. In dimension 4, $\dim S^3T = \dim \mathcal{R} = 20$, yet the dimension of the image of ρ is only 18. The authors discovered this by computer experiment: we picked a random tensor $A \in S^3T^*$ and then computed the rank of the derivative ρ_* at A. By Sard's theorem we could be rather confident that ρ is not onto.

To prove this rigorously we wanted to identify the explicit conditions on an algebraic curvature tensor that were required for it to lie in the image of ρ. We found these conditions using a computer search. We assumed that the explicit conditions could be written as SO(4)-equivariant polynomials in the curvature R and catalogued the possibilities using the representation theory of

SO(4). This was feasible to program due to the simple representation theory of Spin(4) $\cong$ SU(2) $\times$ SU(2). We only had to examine up to cubic polynomials to find the 2-dimensions of curvature obstruction suggested by our numerical experiments.

Theorem 1. *The space of possible curvature tensors for a Hessian* 4*-manifold is* 18 *dimensional. In particular the curvature tensor must satisfy the identities:*

$$\alpha({R_{ija}}^b {R_{klb}}^a) = 0 \tag{3}$$

$$\alpha(R_{iajb} {R_k}^b{}_{cd} {R_l}^{dac} - 2R_{iajb} {R_{kc}}^a{}_d {R_l}^{dbc}) = 0 \tag{4}$$

where α denotes antisymmetrization of the i, j, k and l indices.

Proof. Using a symbolic algebra package, write the general tensor in S^3T^* with respect to an orthonormal basis in terms of its 20 components. Compute the curvature tensor using equation (1). One can then directly check the above identities.

The first of these equations is particularly interesting. The tensor defined in equation 3 is a closed 4-form. Its de Rham cohomology class is independent of the metric and hence defines a topological invariant of the manifold - the first Pontrjagin class. The integral of this form over the 4-manifold is the signature of the 4-manifold. We have proved that the signature must vanish on a Hessian 4-manifold.

6 Pontryagin classes of Hessian manifolds

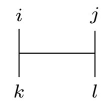
Let us generalize this last result to higher dimensions. To make the proof as vivid as possible, we introduce a graphical notation that simplifies manipulating symmetric powers of the S^3-tensor A (this is based on the notation given in the appendix of [7]). When using this notation we will always assume that our coordinates are orthonormal at the point where we perform the calculations so we can ignore the difference between upper and lower indices of ordinary tensor notation.

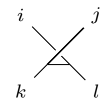
Given a tensor defined by taking the n-th tensor power of the S^3-tensor tensor A followed by a number of contractions we can define an associated graph by:

- Adding one vertex to the graph for each occurrence of A;
- Adding an edge connecting the vertices for each contraction between the vertices;
- Adding a vertex for each tensor index that is not contracted and labelling it with the same symbol used for the index. Join this vertex to the vertex representing the associated occurrence of A.

When two tensors written in the Einstein summation convention are juxtaposed in a formula, we will refer to this as "multiplying" the tensors. This multiplication corresponds graphically to connecting labelled vertices of the graphs according to the contractions that need to be performed when the tensors are juxtaposed. Since this multiplication is commutative, and since the S^3-tensor is symmetric, one sees that there is a one to one correspondence between isomorphism classes of such graphs and equivalently defined tensors.

We can use these graphs in formulae as an alternative notation for the tensor represented by the graph. For example, we can write the curvature identity (1) graphically as

$$R_{ijkl} = -$$ 

$+$ 

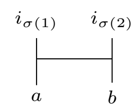
$.$ (5)

Theorem 2. *The tensor*

$$Q^p_{i_1 i_2 \dots i_{2p}} = \sum_{\sigma \in S_{2p}} \operatorname{sgn}(\sigma) R_{i_{\sigma(1)} i_{\sigma(2)} a_1}{}^{a_2} R_{i_{\sigma(3)} i_{\sigma(4)} a_2}{}^{a_3} R_{i_{\sigma(5)} i_{\sigma(6)} a_3}{}^{a_4} \dots R_{i_{\sigma(2p-1)} i_{\sigma(2p)} a_p}{}^{a_1}$$

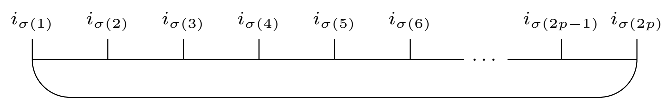
vanishes on a Hessian manifold. Hence all Pontryagin forms vanish on a Hessian manifold.

Proof. We can rewrite the curvature identity (1) as:

$$R_{i_1 i_2 ab} = \sum_{\sigma \in S_2} - \operatorname{sgn}(\sigma)$$ 

$.$

Thus we can replace each R in the formula for Q^p with an 'H'. The legs of adjacent H's are then connected. The result is:

$$Q^p_{i_1 i_2 \dots i_{2p}} = (-1)^p \sum_{\sigma \in S_{2p}} \operatorname{sgn}(\sigma)$$ 

$.$

Since the cycle $1 \to 2 \to 3 \dots \to 2p \to 1$ is an odd permutation, one sees that $Q^p = 0$.

The import of this result is that the Pontryagin forms of the manifold can be expressed in terms of the Q_p tensors. Thus it is a corollary that the Pontryagin forms, and hence the Pontryagin classes, vanish on a Hessian manifold. This result is an easy consequence of the standard definition of the Pontryagin forms combined with standard results on symmetric polynomials.

We have seen that equation (3) generalizes easily to higher dimensions. Equation (4) on the other hand does not hold in dimensions $\geqslant 5$. We list some interesting questions that this raises. Can one efficiently find all the explicit curvature conditions that must be satisfied by a Hessian metric in a fixed dimension $n \geqslant 5$? Can one find all the curvature conditions that hold for all n? For large enough n, is the condition that the curvature lies in the image of ρ a sufficient condition for a metric to be Hessian?

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