Research Article

Optimal Bounds for the Variance of Self-Intersection Local Times

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1. Introduction and Main Results

Let \(X, X_1, X_2, \ldots\) be independent, identically distributed, \(\mathbb{Z}^d\)-valued random variables, and define the random walk \(S_0 = 0, S_n = \sum_{j=1}^{n} X_j\) for \(n \geq 1\). The special case with \(P(X_j = e) = 1/(2d)\), for all \(e \in \mathbb{Z}^d\) with \(|e| = 1\), is known as the simple random walk in \(\mathbb{Z}^d\) and will be denoted by \((\text{SRW}_n)_{n \in \mathbb{N}_0}\).

Let \(l(n, x)\) be the local time of \((S_n)_{n \in \mathbb{N}_0}\) at the site \(x \in \mathbb{Z}^d\), and define for a positive integer \(\alpha\) the \(\alpha\)-fold self-intersection local time

\[
L_n = L_n(\alpha) = \sum_{x \in \mathbb{Z}^d} l(n, x)^\alpha
\]

\[
= \sum_{i_1, \ldots, i_n \in \mathbb{N}_0} \mathbb{1}(S_{i_1} = \ldots = S_{i_n}).
\]

We will denote the corresponding quantities for simple random walk in \(\mathbb{Z}^d\) by \(L_n^{\text{SRW}}(\alpha, d)\) or simply \(L_n^{\text{SRW}}(\alpha)\) when the dimension is clear from the context.

Let \(R^+\) and \(R^-\) be, respectively, the semigroup and the group generated by the support of \(X\),

\[
R^+ = \left\{ x \in \mathbb{Z}^d \mid P(S_n = x) > 0 \text{ for some } n \geq 0 \right\},
\]

\[
R^- = \left\{ x \in \mathbb{Z}^d \mid x = y - z \text{ for some } x, y \in R^+ \right\}.
\]

Following Spitzer [1], we call the random variable \(X\) and the random walk it generates genuinely \(d\)-dimensional if the group \(R\) is \(d\)-dimensional.

The quantity \(L_n(\alpha)\) has received considerable attention in the literature due to its relation to self-avoiding walks and random walks in random scenery. In particular let the random scenery \(\xi_x, x \in \mathbb{Z}^d\) be a collection of i.i.d. random variables, independent of \((S_n)_{n \in \mathbb{N}_0}\), and define the process \(Z_0 = 0, Z_n = \sum_{j=1}^{n} \xi_{S_j}\). Then \((Z_n)_{n \in \mathbb{N}_0}\) is commonly referred to as \(\text{random walk in random scenery}\) and was introduced in Kesten and Spitzer [2], where functional limit theorems were obtained for \(Z_{[nt]}\) under appropriate normalization for the case \(d = 1\). The case \(d = 2\), with \(X_1\) centered with nonsingular covariance matrix, was treated in [3] where it...
was shown that $Z_{[n]}/\sqrt{n \log n}$ converges weakly to Brownian motion. As is obvious from the identities $Z_n = \sum_{x \in \mathbb{Z}^d} l(n, x)^{k} x$ and $\text{var}(Z_n) = \text{var}[L_n(2)]$ $\text{var}(z)$, limit theorems for $(Z_n)_n$ usually require asymptotic results for the local times of the random walk $(S_n)_n$.

Such asymptotic results are usually obtained from Fourier techniques applied to the characteristic function $f(t) = \mathbb{E}[\exp(i \cdot X)]$, under the additional assumption of a Taylor expansion of the form $f(t) = 1 + \mathcal{O}(|t|^2)$, where $\Sigma$ is a positive definite covariance matrix [3–7], which further requires that $\mathbb{E}|X|^2 < \infty$ and $\mathbb{E}X = 0$. Similar restrictions are also required for the application of local limit theorems such as in [8, 9].

In this paper, motivated by the results of Spitzer [1] for genuinely $d$-dimensional random walks and the approach of Becker and König [10], we will study the asymptotic behavior of $\text{var}(L_n(\alpha))$ without imposing any moment assumptions on the random walk. The central idea behind our approach is to compare the self-intersection local times $L_n(\alpha)$ of a general $d$-dimensional walk with those of its symmetrized version. In addition we will compare the self-intersection local times of a general $d$-dimensional random walk with those of the $d$-dimensional simple symmetric random walk, $(\text{SRW}_n)_{n \in \mathbb{N}_0}$. It is well known that, for some positive constants $K_{a,d}$, $\text{var}(f_{\text{SRW}}(\alpha, d)) \sim K_{a,d} v_{d,a}(n)$ as $n \to \infty$, for

$$
\begin{align*}
V_{1,a} (n) &= n^{1+\alpha}, \\
V_{2,a} (n) &= n^2 \log (n)^{2\alpha - 4}, \\
V_{3,a} (n) &= n \log (n), \\
V_{d,a} (n) &= n, \quad d \geq 4.
\end{align*}
$$

Several other cases have been treated in the literature, using a variety of methods.

A careful look at the literature reveals that the most difficult case in $d = 2$ is the near transient recurrent case, where $P(S_n = 0) \sim C/n$, which corresponds to genuinely 2-dimensional symmetric recurrent random walks, which will be referred to as a critical case. Surprisingly enough, the variance of the self-intersection local times in the critical case is asymptotically the largest.

**Theorem 1.** Let $X, X_1, X_2, \ldots$ be independent, identically distributed, and genuinely $d$-dimensional $\mathbb{Z}^d$-valued random variables, for any $d \geq 1$. Then, there exist positive constants $C_{a,X} > c_{a,X} > 0$, depending on $\alpha$ and the distribution of $X$, such that for all $n$ large enough

$$
\text{var} \left(L_n (\alpha)\right) \leq c_{a,X} \text{var} \left(L_{\text{SRW}} (\alpha, d)\right) \leq C_{a,X} v_{d,a} (n). \tag{4}
$$

The result was motivated by [1, 10] and improves related results of Becker and König for $d = 3$ and $d = 4$. Several cases treated in [3, 4, 10–13] can then be obtained as particular cases.

Moreover, we also show the surprising converse. More precisely, we show that the right asymptotic behaviour of $\text{var}(L_n)$ implies that the jumps must have zero mean and finite second moment.

**Theorem 2.** Let $X, X_1, X_2, \ldots$ be independent, identically distributed, and genuinely $d$-dimensional with $d \leq 3$. If

$$
\liminf_{n \to \infty} \frac{\text{var}(L_n (\alpha))}{\text{var}(L_{\text{SRW}} (\alpha))} > 0, \tag{5}
$$

then $E|X|^2 < \infty$ and $\mathbb{E}X = 0$.

As it follows from Theorem 3 given below for $d = 2, 3$ and from Theorem 5.2.3 in Chen [12] for $d = 1$, if $\mathbb{E}X = 0$ and $0 < E|X|^2 < \infty$, then $\liminf_n \text{var}(L_n(\alpha))/v_{d,a}(n) > 0$.

For any genuinely $d$-dimensional random walk with finite second moments and zero mean, the asymptotic behaviour of $\text{var}(L_n(\alpha))$ is similar to that of the $d$-dimensional simple symmetric random walk. Also, as it follows from our general bounds (see Proposition 4 and Corollary 7) that the asymptotic results for the genuinely $d$-dimensional random walk can be reproduced by those of the symmetric one-dimensional random walk with appropriately chosen heavy tails, as was indicated by Kesten and Spitzer [2]. The proofs are based on adapting the Tauberian approach developed in [13].

**Theorem 3.** Let $d = 1, 2, 3$, and suppose that for $t \in \Gamma := [-\pi, \pi]^d$ one has

$$
\begin{align*}
f (t) &= 1 - |t|^2 + R(t), \quad \text{for } d = 1, \\
&= 1 - \langle \Sigma, t \rangle + R(t), \quad \text{for } d = 2, 3,
\end{align*}
$$

where $\Sigma$ is a nonsingular covariance matrix and $R(t) = o(|t|)$ for $d = 1$ and $o(|t|^2)$ for $d = 2, 3$ as $t \to 0$. Then

$$
\begin{align*}
\text{var}(L_n (\alpha)) &= \begin{cases} 
\left(\frac{\pi^2 + 6}{12}\right) (\alpha t)^2 (\alpha - 1)^2 n^2 \log (n)^{2\alpha - 4}, & \text{for } d = 1, \\
\frac{4}{3} (\alpha t)^2 (\alpha - 1)^2 n^2 \log (n)^{2\alpha - 4} (\kappa + 1), & \text{for } d = 2, \\
(\kappa_1 + \kappa_2) n \log n, & \text{for } d = 3, \quad \alpha = 2,
\end{cases}
\end{align*}
$$

where

$$
\kappa = \int_0^\infty dr ds \left[ (1 + r) (1 + s) \sqrt{1 + r + s} \right]^{d - 1}.
$$

and $\kappa_1$ and $\kappa_2$ are defined in (58) and (63), respectively.

Moreover, if $L_n (\alpha)$ is the self-intersection local time of another random walk, independent of $(S_n)_n$, whose characteristic function also satisfies (6), then $\text{var}(L_n (\alpha)) = \text{var}(L_n (\alpha))(1 + o(1))$.

2. Proofs

2.1. General Bounds. We first develop a technique to treat random walks with independent but not necessarily identically distributed increments.
Proposition 4 (general upper bound). Assume that \(X_1, X_2, \ldots\) are independent \(\mathbb{Z}^d\)-valued random variables and let \(S_{u,v} = X_u + \cdots + X_{u+v}\). Suppose further that for all \(n \in \mathbb{N}\) and integers \(a, b, v \geq 0\), with \(a + u \leq b\) and any \(x \in \mathbb{Z}^d\), one has

\[
P\left(S_{a,u} \pm S_{b,v} = x\right) \leq \phi(u + v), \tag{A}
\]

\[
P\left(S_{a,u} = 0\right) - P\left(S_{a,u} + S_{b,v} = 0\right) \leq \psi(u, v), \tag{B}
\]

where \(\phi(u)\) is nonincreasing and \(\psi(u, v)\) is nonincreasing in \(u\) and is nondecreasing and subadditive in \(v\) in the sense that \(\psi(u, v + w) \leq A\psi(\psi(u, v) + \psi(u, w))\), for some constant \(A\) independent of \(u, v,\) and \(w\). Then, for some constant \(K = cA(1 + A)^{\alpha-2}\) depending only on \(\alpha\)

\[
\text{var} \left(L_n(\alpha)\right) \leq Kn\left(\sum_{i=0}^{n-1} \phi(i)\right)^{2\alpha-4}
\]

\[
\cdot \sum_{i,j=0}^{n-1} \left[\phi(j \land i) \phi(k \land i) + \phi(j) \psi(i + k, j)\right].
\]

Proof of Proposition 4. We first write out the variance as a sum

\[
\text{var} \left(L_n(\alpha)\right) = (\alpha!)^2 
\]

\[
I_n := \sum_{k_1 \leq \cdots \leq k_n} \prod_{1 \leq t \leq n} P\left(S_{k_t} = \cdots = S_{k_{t+1}} = \cdots = S_{k_n}\right)
\]

\[
= \sum_{x, y \in \mathbb{Z}^d} \sum_{1 \leq t \leq 2\alpha} \sum_{l_1 \leq \cdots \leq l_2\alpha} \sum_{v(\delta) \geq 3} \prod_{1 \leq t \leq n} P\left(S_{k_t} = x, S_{l_t} = x + \epsilon_2 y, \ldots, S_{p_2} = x + \epsilon_2 y\right)
\]

\[
= \sum_{x, y \in \mathbb{Z}^d} \sum_{m_0} \sum_{m_1 \leq \cdots \leq m_{2\alpha-1}} \sum_{\delta(\delta) \geq 3} \prod_{1 \leq t \leq n} P\left(S_{m_t} = \delta_1 y\right) \cdots \prod_{1 \leq t \leq n} P\left(S_{m_t} = \delta_{2\alpha-1} y\right).
\]

Summing over the free index \(m_t\), it is clear that

\[
I_n \leq (n + 1)
\]

\[
\cdot \sum_{m_1 \leq \cdots \leq m_{2\alpha-1}} \sum_{x, y \in \mathbb{Z}^d} \sum_{\delta(\delta) \geq 3} \prod_{t=1}^{2\alpha-1} \sup_{w} P\left(S_{w+m_t} = \delta_1 y\right).
\]

(12)

For any \(\delta = (\delta_1, \ldots, \delta_{2\alpha-1})\) with \(v(\delta) = v\), exactly \(u = 2\alpha-1-v\) elements are equal to 0, and therefore by Assumption (A) with \(x = 0\) we have

\[
I_n \leq C(n + 1) \sum_{i=0}^{n} \phi(i)^{2\alpha-1-v}
\]

\[
\cdot \sum_{j_1, \ldots, j_v \in \mathbb{Z}^d} \prod_{t=1}^{v} \sup_{u_t} P\left(S_{u_t+j_t} = \delta_{j_t} y\right).
\]

(13)

Letting \((S_n)_{n \in \mathbb{N}}\) denote an independent copy of the random walk \((S_n)_{n \in \mathbb{N}}\) and assuming without loss of generality that \(j_1 \leq \cdots \leq j_v\), we have that for any \(\delta \in \{-1, +1\}^v\)

\[
\sum_{u_t} \prod_{t=1}^{v} \sup_{w_t} P\left(S_{w_t+j_t} = \delta_{j_t} y\right)
\]

\[
\leq \left(\prod_{t=2}^{v-1} \sup_{y} P\left(S_{w_{t-1}} = y\right)\right)
\]

\[
\cdot \sup_{u_t, w_t} P\left(S_{u_t+j_t} - \delta_{j_t} S_{w_{t-1}} = 0\right) \leq \left(\prod_{t=2}^{v-1} \phi(j_t)\right)
\]

\[
\cdot \phi(j_1 + j_v) \leq \prod_{t=2}^{v} \phi(j_t, v, j_1).
\]

(14)
Let $G_n := \sum_{i=0}^{n} \phi(i)$. Since $\phi$ is nonincreasing we have that
\[
\Delta_{n,v} = \sum_{0 \leq j_1 < j_2 < \ldots < j_v \leq n} \prod_{i=2}^{v} \phi(j_i \lor j_{i-1}) \\
\leq \sum_{j=0}^{n} \phi(j) \sum_{0 \leq j_1 < j_2 < \ldots < j_v \leq n} \prod_{i=2}^{v} \phi(j_i \lor j_{i-1}) \\
= G_n \Delta_{n,v-1},
\]
and iterating this procedure, for $v \geq 3$, we have that $\Delta_{n,v} \leq \Delta_{n,3} C_{n,v-3}^2$. Combining the two bounds and summing over $v = 3, \ldots, 2\alpha - 1$, we have that
\[
I_n \leq \sum_{v=3}^{2\alpha - 1} c(\alpha) n^{2\alpha - 1 - v} \Delta_{n,v} \leq c(\alpha)n^{2\alpha - 1 - \alpha - 3} \Delta_{n,3} \\
= c(\alpha)n^{2\alpha - 4} \Delta_{n,3},
\]
where $c(\alpha)$ is a constant depending only on $\alpha$.

Terms with $v = 2$. Next we consider the sum $J_n$ over the terms with $v = 2$, which occurs when, for some $j$, the indices $1, \ldots, n$ all lie in $[k, k+1]$. Then it is easy to see that this sum $J_n$ is bounded above by
\[
J_n \leq C_n \sup_{w_{-1} = \ldots = w_{k-1} = 0} \sum_{m_{-1} = 0}^{n} \prod_{r = -1}^{\alpha - 2} \mathcal{P}(S_{w_r, m_r} = 0) \\
\cdot \sum_{m_{-1} = 0}^{n} \prod_{r = -1}^{\alpha - 2} \mathcal{P}(S_{w_r, m_r} = 0) \\
\cdot \sup_{w_{-1} = \ldots = w_{k-1} = 0} \sum_{m_{-1} = 0}^{n} \prod_{r = -1}^{\alpha - 2} \mathcal{P}(S_{w_r, m_r} = 0) \\
\cdot \sum_{m_{-1} = 0}^{n} \prod_{r = -1}^{\alpha - 2} \mathcal{P}(S_{w_r, m_r} = 0) \\
\leq C_n G_n^{\alpha - 2} \sum_{m_{-1} = 0}^{n} \prod_{r = -1}^{\alpha - 2} \phi(m_r) \psi(m_0 + m_1, m_1) \\
+ \cdots + \phi(m_{-1}) \leq C_n G_n^{\alpha - 2} A_\psi (1 + A_\psi)^{\alpha - 2} \\
\cdot \left( \sum_{m_{-1} = 0}^{n} \prod_{r = -1}^{\alpha - 2} \phi(m_r) \right) \sum_{m_{0}, m_{1}} \phi(m_1) \psi(m_0 + m_1, m_1) \\
\cdot \left( \sum_{m_{-1} = 0}^{n} \prod_{r = -1}^{\alpha - 2} \phi(m_r) \right) \sum_{m_{0}, m_{1}} \phi(m_1) \psi(m_0 + m_1, m_1) \\
\cdot \sum_{i,j,k=0}^{n} \phi(i) \psi(i + k, j).
\]

The following corollary provides explicit bounds in the cases that are usually considered in the literature.

**Corollary 5.** Assume that the conditions of Proposition 4 are satisfied with $\phi(m) = T/m$ and $\psi(m, k) = T/m^{r-1}(k \land m)$. Then,
\[
\var(L_n(\alpha)) \leq c_n T^{2\alpha - 2} \\
v_n \log(n), \quad if \ r = 3/2.
\]

It is straightforward to see that Corollary 5 includes random walks with mean zero and finite second moment; for example, $d = 2$ corresponds to $r = 1$ and $d = 3$ to $r = 3/2$. Therefore several relevant results in [3, 7–13] are obtained as a special case of Corollary 5 and extended to the case of independent but not necessarily identically distributed variables, for example, by applying the local limit theorem, as conducted in [8].

Also when the random walk increment $X$ is in the domain of attraction of the one-dimensional symmetric Cauchy law [13, 14] or in the case of planar random walk with second moments [3, 7–9, 11], it is well known that the conditions of Proposition 4 are satisfied with $\phi(m) = T/m$ and $\psi(m, k) = T/m^{r-1}(k \land m)$.

However, we can do better for symmetric variables and show that condition (A) implies (B), which together with the comparison technique motivates the following results. For a real number $x$, we write $[x]$ for the integer part of $x$.

**Proposition 6** (bounds via comparison with characteristic function of symmetric random variables). Let $X_1, X_2, \ldots$, be independent $\mathbb{Z}^d$-valued random variables and let $f_i(t) := E \exp(i t X_i)$. Assume that there exist a measurable function $f : \Gamma \to [0, 1]$ and a positive nonincreasing sequence $(\phi(m))_{m \in \mathbb{N}_0}$, such that
\[
|1 - f_i(t)| \leq T f(t), \\
|f_i(\pm t)| \leq f(t), \\
\int f(t)^m \, dt \leq \phi(m),
\]
for all integers $i, m \geq 0$, all $t \in \Gamma$, and some positive constant $T$. Then there exists another positive constant $K = c(\alpha, d, T)$ such that
\[
\var(L_n(\alpha)) \leq K n^{1 - \alpha} \sum_{i,j,k=0}^{n} \phi(i) \psi(i + k, j).
\]
Proof of Proposition 6. Using the notation of Proposition 4, for positive integers $a, u, b$, and $v$, with $a + u \leq b$, $e_j = \pm 1$, and any $x \in \mathbb{Z}^d$

\[ P(S_{a,u} + e \cdot S_{b,v} = x) \]

\[ \leq \frac{1}{(2\pi)^d} \int \prod_{j \in [a,u] \cup [b,v]} |f_j(e_jt)| \, dt \]

\[ \leq \frac{1}{(2\pi)^d} \int f(t)^{u+v} \, dt \leq \frac{1}{(2\pi)^d} \phi(u+v). \]

To find $\psi(u, v)$, notice that since $f(t) \geq 0$,

\[ \phi(u) \geq \int f(t)^u \, dt \]

\[ = \sum_{j=0}^{m-1} \int f(t)^{uj} \, dt \]

\[ \geq m \int f(t)^m \, dt = mQ(m) \]

whence $Q(m) \leq 2\phi([m/2])/m$. Therefore,

\[ ||P(S_{a,u} = 0) - P(S_{a,u} + S_{b,v} = 0)|| \]

\[ \leq \frac{1}{(2\pi)^d} \int \prod_{j \in [a,u] \cup [b,v]} |f_j(t)| \, dt \]

\[ \leq CT \int f(t)^m \, dt \leq \frac{CT\phi([u/2])}{u}. \]

A telescoping argument implies that

\[ ||P(S_{a,u} = 0) - P(S_{a,u} + S_{b,v} = 0)|| \leq CT\phi \left( \frac{u}{2} \right) \frac{v}{u}. \]

On the other hand for $u \leq v$ we can obtain a tighter bound through

\[ P(S_{a,u} = 0) - P(S_{a,u} + S_{b,v} = 0) \leq P(S_{a,u} = 0) \]

\[ \leq \phi(u). \]

Combining the two bounds above it follows that (B) is satisfied with $\psi(u, v) = \phi([u/2]) \min(u, v)/u$. Thus all conditions of Proposition 4 are satisfied and the result follows. \(\square\)

The following corollary allows for the case where $\phi(m)$ is regularly varying.

Corollary 7. Assume that the conditions of Proposition 6 are satisfied with $\phi(m) = h(m)m^{-r}$, $r \geq 1$, where $h(\cdot)$ is slowly varying at $\infty$. Then,

\[ \var (L_n(\alpha)) \leq K\Delta_n(\alpha, \phi) \]

\[ \leq c_\alpha T^{2r-2} \left\{ \begin{array}{ll}
    n^2 \sum_{k=1}^{n} h(k)^2, & \text{for } r = 1, \\
    n^{4-2r} h^2(n), & \text{for } 1 < r < \frac{3}{2}, \\
    n, & \text{for } r > \frac{3}{2}.
\end{array} \right. \]

Several results in [3, 7–13] are obtained as a special case of Corollary 7 and can be extended to dependent variables, for example, a random walk driven by a hidden Markov chain. In addition, following [2], we can construct a one-dimensional symmetric random walk with characteristic function $f(t) = 1 - c|t|^{1/2} + o(|t|^{1/2})$, where $r = 2/d$ for $d = 2, 3$ and $r = 1/2$ for $d \geq 4$, whose asymptotic behaviour is similar to that of genuinely $d$-dimensional random walk.

The following example of genuinely 2-dimensional recurrent walk with infinite variance was motivated by Spitzer [1, pp. 87].

Example 8. Let $X_1, X_2, \ldots$ be independent, identically distributed, $\mathbb{Z}^2$-valued random variables, such that $P(|X_1| = k) = c/(k^2 \log(k)^g)$, for $k \geq 4$ and $g \in [0, 1)$. Let $(S_n)_{n \in \mathbb{N}}$ be the corresponding random walk in $\mathbb{Z}^2$. Then we have

\[ \var (L_n(\alpha)) \]

\[ \leq cn^2 \max \left\{ \log n^g, \log \log n \right\}^{2\alpha-4} \log n^{2(1-g)}, \]

for $n \geq 10$. Under these assumptions we have that $P(S_n = 0) \leq c/n \log(n)^{1-g}$, which is in the critical range, where the random walk is recurrent, without second moment. To see why, we note that by a lengthy but straightforward calculation it can be shown that the characteristic function of $X$ satisfies (19) with

\[ \phi(n) = \frac{c}{n \log(e \vee n)^{1-g}}, \]

\[ f(t) = \exp \left[ -A|t|^2 h(|t|^2) \right], \]

where $h(r) = \left[ 1 + \log \left( \frac{1}{r} \right) \right]^{1-g}$. The sequence $\phi(m)$ is identified via Fourier inversion, polar coordinates, and a Laplace argument,

\[ \int f(t)^n \, dt \leq c \int_0^1 \exp \left( -nr h \left( \frac{1}{r} \right) \right) \]

\[ + O(e^{-n}) \leq \frac{c}{n \log(e \vee n)^{1-g}} = \phi(n). \]

2.2. Bounds for Identically Distributed Variables

Proposition 9 (general upper bound for i.i.d.). Let $X, X_1, X_2, \ldots$ be independent, identically distributed,
$\mathbb{Z}^d$-valued random variables. Suppose that for any $x \in \mathbb{Z}^d$ and all positive integers $a$, $b$, $u$, and $v$, with $a + u \leq b$, it holds that
\begin{equation}
\Pr(S_{a,u} + S_{k,v} = x) \leq \phi(u + v),
\end{equation}
where $\phi(m)$ is a nonincreasing sequence. Then for some constant $K = c(\alpha)$ we have that
\begin{equation}
\var(L_n(\alpha)) \leq Kn\left(\sum_{i=0}^{n-1} \phi(i)\right)^{2a-4} \sum_{j=0}^{n} j^2 \phi(j) \sum_{k=j}^{n} \phi\left(\left\lfloor \frac{k}{\alpha} \right\rfloor\right).
\end{equation}

**Proof of Proposition 9.** By inspecting the proof of Proposition 6, we notice that we only need to bound the term $(\ref{eq:var})$. Consider typical ordering
\begin{equation}
0 \leq i_1 \leq \cdots \leq i_k \leq \cdots \leq i_{k+1} \leq \cdots \leq i_n \leq n
\end{equation}
and let us change variables to $(m_0, \ldots, m_{2a})$ such that $m_0 + \cdots + m_{2a} = n$. Then the contribution to $J_n$ is given by
\begin{align}
\sum_{m_0, \ldots, m_{2a}} & \prod_{j \leq k \leq \alpha} \Pr(S_{m_j} = 0) \\
& \cdot \left[\Pr(S_{m_{k+1}} = 0) - \Pr(S_{m_{k+1}+m_{k+2}} = 0)\right].
\end{align}
We keep $m_j$ fixed for $j \neq \alpha, k + \alpha$ and we sum over $m = m_k + m_{k+1}$ from 0 to some $M = M(n, m_j)$. Then for given $m_{k+1}, \ldots, m_{k+\alpha-1}$, the term in the sum is
\begin{equation}
\sum_{m=0}^{M} (m+1) \Pr(S_m = 0) - \Pr(S_{m+q} = 0),
\end{equation}
where $q = m_{k+1} + \cdots + m_{k+\alpha-1}$. Then since $M \leq n - q$, it is an easy exercise to show that this sum is bounded above by
\begin{align}
\sum_{m=0}^{M} & (m+1) \Pr(S_m = 0) - \Pr(S_{m+q} = 0) \\
& \leq \sum_{m=0}^{q-1} (m+1) \Pr(S_m = 0) + q \Pr(S_{m+q} = 0) \\
& \leq \sum_{m=0}^{m} \Pr(S_m = 0) \leq \sum_{m=0}^{n} (m+1) \Pr(S_m = 0) \\
& + am^* \sum_{m=m^*}^{n} \Pr(S_m = 0),
\end{align}
where $m^* = \max(m_{k+1}, \ldots, m_{k+\alpha-1})$. The result follows by summing over all indices apart from $m^*$ and changing the order of summation. $\square$

### 2.3 Proofs of Main Results

**Proof of Theorem 1.** We apply a comparison argument found to be useful in many areas (e.g., Montgomery-Smith and Pruss [15], and Lefèvre and Utev [16]). More specifically we bound the quantity $\var(L_n)$ by the corresponding quantity for the symmetrised random walk.

Following Spitzer’s argument we notice that with $f(t) = E[\exp(it \cdot X_1)]$
\begin{align}
\Pr(S_{a,n} + \epsilon S_{b,r} = x) & \leq c \int f(t)^{a} |f(-t)|^{r} \, dt \\
& = c \int |f(t)|^{\alpha/2} |f(-t)|^{\gamma/2} \, dt.
\end{align}
Since $|f(t)|^2$ is the characteristic function of a symmetric random variable in $\mathbb{Z}^d$, for some positive $\lambda$, we have $1 - |f(t)|^2 \geq \lambda |t|^2$, and, hence,
\begin{equation}
\Pr(S_{a,n} + \epsilon S_{b,r} = x) \leq c \int \exp\left[-\frac{\lambda (u + v)}{2} |t|^2\right] \, dt \leq c \leq (u + v)^{-d/2}.
\end{equation}
The result follows from Proposition 9 applied with $\phi(m) = m^{-d/2}$. $\square$

The proof of Theorem 2 will be based on the following lemma.

**Lemma 10.** Assume $X, X_1, X_2, \ldots$ are independent, identically distributed, genuinely $d$-dimensional random variables such that $E|X|^2 = \infty$. Then there exists a monotone, slowly varying sequence $(h_n)_{n \in \mathbb{N}}$, such that $h_n \to 0$ as $n \to \infty$ and
\begin{equation}
\sup_{x \in \mathbb{Z}^d} \Pr(S_n = x) \leq c_d \int \left|E \exp(\beta X)^n \right| \, dt \leq h_n n^{-d/2}.
\end{equation}

**Proof of Lemma 10.** Without loss of generality we assume that $X$ is symmetric. Let $\sigma_{\ell} := E[e^{-\ell/2}|X| \leq L]$. Following Spitzer, since $X$ is genuinely $d$-dimensional, we may assume that there exist positive constants $c, \lambda$, such that for any unit vector $|e| = 1$ we have that $\sigma_{\ell} \geq c$ and $1 - f(t) \geq c|t|^2$ for all $\ell \in \Gamma$. Let $\lambda_d$ be the $d$-dimensional Lebesgue measure on $\mathbb{R}^d$, and $\mu_d$ the Lebesgue-Haar measure on $\mathbb{S}^{d-1} := \{e \in \mathbb{R}^d : |e| = 1\}$. Notice that since $E|X|^2 = \infty$, for any $K$, we have $\mu_d(e : \sigma_{e \ell} < K) = 0$.

Fix a small positive $\epsilon$ such that $\sqrt{c/\epsilon} \geq 2W$, and for any $\epsilon > 0$ let $K = K(\epsilon) = \epsilon^{-d/2}$. Then there exists $L = L(\epsilon) > 0$ small enough so that $\mu_d(e : \sigma_{\epsilon e} < K) \leq \epsilon^{-d/2}$. We partition $S^{d-1}$ in two sets
\begin{align}
A_{L,K} & = \{e \in S_d : \sigma_{e \ell} \geq K\}, \\
\overline{A}_{L,K} & = \{e \in S_d : \sigma_{e \ell} < K\},
\end{align}
so that, for any direction $e \in \overline{A}_{L,K}$,
\begin{align}
\{z \in \mathbb{R} : 1 - f(z e) \leq \epsilon \} & \subseteq \{z : ce^2 \leq z\} \\
& \subseteq \{z : |z| \leq \sqrt{\frac{x}{c}}\}.
\end{align}
Hence, using $d$-dimensional spherical coordinates,

$$
\lambda_d \left\{ (z,e) \in \mathbb{R} \times A_{L,K} : 1 - f(\varepsilon z) \leq x \right\} \leq \mu_d \left\{ A_{L,K} \right\} \left( \frac{x^c}{\varepsilon} \right)^{d/2} \leq \varepsilon^{d/2} \left( \frac{x^c}{\varepsilon} \right)^{d/2} \left( \frac{1}{d} \right).
$$

(41)

On the other hand, for any $t$,

$$
1 - f(t) = 2 \sum_{k \in \mathbb{Z}} \sin \left( \frac{|t - k|}{2} \right) P(X = k) \geq \left( \frac{1}{4} \right) E \left[ (t - X)^2 I \left( |t - X| \leq \frac{1}{2} \right) \right] \geq \left( \frac{|t|^2}{4} \right) \sigma_{e,1/2} \geq \left( \frac{z^2}{4} \right) \sigma_{e,L}.
$$

(42)

Now, assume that $\sqrt{c/x} \geq 2L$. Then for any direction $e \in A_{L,K}$, by choice of $x$ and since $\sigma_{e,L}$ is increasing in $L$, for $c z^2 \leq 1 - f(\varepsilon z) \leq x$ or $|z| \leq \sqrt{x/c}$, it must be the case that

$$
x \geq 1 - f(\varepsilon z) \geq \left( \frac{z^2}{4} \right) \sigma_{e,1/2} \geq \left( \frac{z^2}{4} \right) \sigma_{e,L}.
$$

(43)

implying that, on the set $A_{L,K}$, it must be that $|z| \leq 2 \sqrt{x/K}$. Changing to $d$-dimensional polar coordinates, we find that

$$
\lambda_d \left\{ (z,e) \in \mathbb{R} \times A_{L,K} : 1 - f(\varepsilon z) \leq x \right\} \leq \int_{A_{L,K}} \int_{0}^{\sqrt{x/K}} r^{d-1} dr \, d\varepsilon \leq C_d \varepsilon^{d/2} x^{d/2}.
$$

(44)

Overall, for $x \leq c/4L^2$, $\lambda_d \left\{ t : 1 - f(t) \leq x \right\} \leq c_d (xe)^{d/2}$, and hence $[t \in \Gamma : 1 - f(t) \leq x]$ has Lebesgue measure $o(x^{d/2})$.

Let $F(x)$ be the cumulative distribution function of the random variable log$(1/f(\cdot))$ defined on the probability space $\Gamma$ with normalised Lebesgue measure. Then $F$ is continuous at $x = 0$ and supported on $\mathbb{R}^+$. Moreover, we have that $F(x) = o(x^{d/2})$ as $x \downarrow 0$. Therefore, for some positive sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ with $\varepsilon_n \to 0$, we have that

$$
\frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} f(t)^n \, dt = \int_{0}^{\infty} e^{-nx} dF(x) = n \int_{0}^{\infty} e^{-nx} F(x) \, dx \leq n^{-d/2} \varepsilon_n.
$$

(45)

It remains to show that there exists a positive, monotone, slowly varying sequence $(h_n)_{n \in \mathbb{N}_0}$, such that $\varepsilon_n \leq h(n) \to 0$ as $n \to \infty$. Let $\delta_n = \sup_{j \geq n} \varepsilon_j$ and $\check{a}_0 = 0$ and for $n \geq 1$ define $a_n$ recursively by $a_n = \min(2a_{n-1}, 1/\delta_n)$, for $2^{-1} < n < 2^2$, so that $a_n \to \infty$ is monotone, $a_{2^2} \leq 2a_{2^1}$ implying that $a_{n+1} \leq 4a_n$, and $1/a_n \geq \delta_n \geq \varepsilon_n$. Finally, take $h_n = 1/\max(a_0, \log a_n)$.

Proof of Theorem 2. Assume that $\mathbb{E}|X|^2 = \infty$ and $d = 2$ or $d = 3$. Then, by Lemma 10 there exists a slowly varying sequence $h_n \to 0$ as $n \to \infty$ such that $\int_{R} |\mathbb{E}\exp(\varepsilon X)|^d \, d\varepsilon \leq h_n n^{-d/2}$. Applying Corollary 7 with $r = 1$ and $r = 3/2$ we, respectively, find that

$$
\text{var}(L_n(\alpha)) \leq \begin{cases} K_n \left( \sum_{k \geq 1} h(k) \frac{(n^d)}{k} \right)^{2a_n-4} = o(\mathbb{N}^{(\log n)^{2a_n-4}}), & \text{for } d = 2, \\ K_n \left( \sum_{k \geq 1} h(k) \frac{(n^d)}{k} \right)^{2a_n-4} = o(n \ln n), & \text{for } d = 3. \end{cases}
$$

(46)

Finally assume that $\mathbb{E}|X|^2 < \infty$ and $\mathbb{E}[X] = \mu \neq 0$. Then $\mathbb{P}(S_n = 0) = \mathbb{P}(\bar{S}_n = -n\mu)$ whence it follows that $\mathbb{P}(S_n = 0) = o(n^{d/2})$ (see, e.g., [17, Theorem 2.3.10]). Then inspecting the proof of Proposition 4, one can readily obtain the desired bound for the $I_n$ term, while with slight modification the bound for the $J_n$ term also follows.

Note that for $d = 1$ the situation is much simpler since then $\text{var}(L_n^{SRW}(\alpha)) \sim C\mathbb{E}[L_n^{SRW}(\alpha, d)]^2$ and if $\mathbb{E}|X|^2 = \infty$ or $\mathbb{E}[X] \neq 0, \mathbb{E}_n^{SRW}(\alpha, d) = o(n^{1+d/2})$.

Proof of Theorem 3. We first give the proof for the case $d = 1$. As in the proof of Proposition 4 we begin from expression (10) and define the sequences $j_i$ and $\delta_i$ for $i = 1, \ldots, 2a - 1$, and the quantity $\nu(\delta) = \sum_{i=1}^{\infty} \delta_i$. Recall that $\mathbb{P}(\delta)$ measures the interlacement of the two sequences $k_{i+1}, \ldots, k_{i+1}$ and $l_{i+1}$, in which case the contribution vanishes by the Markov property. On the other hand $\mathbb{P}(\delta) = 2$ when, for example, $l_{i+1}, \ldots, l_{i+1} \in [k_i, k_{i+1}]$ for some $i$. Finally $\mathbb{P}(\delta) = 3$ occurs when, for example,

$$
k_1 \leq \ldots \leq k_r \leq l_1 \leq \ldots \leq l_s \leq k_{r+1} \leq \ldots \leq k_{s+1} \leq l_{r+1} \leq \ldots \leq l_s \leq n.
$$

(47)

From the proof of Proposition 4, and using the bound $\mathbb{P}(S_n = 0) \leq c/n$, the terms of the sum are bounded above by $n^2 \log(n)^{2a-1-\nu(\delta)}$, and thus the leading term appears when either $\nu(\delta) = 2, 3$, with other terms giving strictly lower order. We will therefore analyze these two situations in detail in order to derive the exact asymptotic constants. When $\nu = 3$, the two terms in the difference individually give the correct order and will be treated by the classical Tauberian theory. However for $\nu = 2$, the two terms only give the correct order when considered together. This however forbids the use of Karamata’s Tauberian theorem since the monotonicity restriction would require roughly that $X_i$ is symmetric. Thus the complex Tauberian approach, as developed in [13], is required to justify the answer.

**Case 1 ($\nu(\delta) = 3$).** Assume that part of the sequence $I = \{l_1, \ldots, l_s\}$ lies between $k_r$ and $k_{r+1}$ and the rest between $k_s$ and $k_{s+1}$. Then using the change of variables
\[\begin{align*}
i_1 &= m_0, \\
i_2 &= m_0 + m_1, \\
&\vdots \\
\ell &= m_0 + \cdots + m_{r-1} \\
j_1 &= m_0 + \cdots + m_r, \\
j_2 &= m_0 + \cdots + m_{r+1}, \\
&\vdots \\
\ell &= m_0 + \cdots + m_{\ell-1} \\
j &= m_0 + \cdots + m_\omega, \\
k &= m_0 + \cdots + m_{\omega+1}, \\
k &= m_0 + \cdots + m_{\omega+2}, \\
&\vdots \\
k &= m_0 + \cdots + m_{\omega+s-1} \\
p &= m_0 + \cdots + m_{\omega+s} \\
p &= m_0 + \cdots + m_{\omega+s+1} \\
&\vdots \\
p &= m_0 + \cdots + m_{\omega+2s-1} \\
p &= m_0 + \cdots + m_{\omega+2s}.
\end{align*}\]

Then, by direct calculations and Fourier inversion formula
\[
\sum_{n=0}^\infty \lambda^n a(n) = c \left( 1 - \lambda \right) a(\lambda)
\]
\[
\cdot \sum_{x \in \mathbb{Z}, k_1 \geq 0} \lambda^{k_1 + k_2} \mathbb{P}(S_{k_1} = -x) \mathbb{P}(S_{k_2} = -x)
\]
\[
= \mathbb{P}\left(S_{m_0} = \cdots = S_{m_\omega} = 0 \right) \prod_{j=r+1}^{\omega+s-1} \mathbb{P}(S_{m_j} = 0).
\]

Next we consider the negative term in (10)
\[
b(n) = \sum_{m_0, \ldots, m_{\omega+s-1} + m_{\omega+s} + \cdots + m_{\omega+2s-1} + m_{\omega+2s} = 0} \mathbb{P}(S_{m_0} = \cdots = S_{m_\omega} = 0) \mathbb{P}(S_{m_0} = \cdots = S_{m_\omega} = 0) \mathbb{P}(S_{m_0} = \cdots = S_{m_\omega} = 0).
\]

By direct calculations and (6),
\[
\sum_{n=0}^\infty \lambda^n b(n) = \left( \frac{1}{\pi^2} \log \left( \frac{1}{1 - \lambda} \right) \right)^{\omega-\omega+2} \left( 1 - \lambda \right)^{-2}
\]
\[
\cdot \sum_{m_0, \ldots, m_{\omega+s} = 0}^\infty \lambda^{m_0 + \cdots + m_{\omega+s}}
\]
\[
\cdot \mathbb{P}\left(S_{m_0} = \cdots = S_{m_\omega} = 0 \right) \prod_{j=r+1}^{\omega+s-1} \mathbb{P}(S_{m_j} = 0) \mathbb{P}(S_{m_j} = \cdots = S_{m_\omega} = 0) \mathbb{P}(S_{m_j} = \cdots = S_{m_\omega} = 0).
\]

and using Fourier inversion and (6) the internal sum behaves as
\[
(2\pi)^{-\omega+r+s} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} (1 - \lambda \phi(x))^r (1 - \lambda \phi(y))^s (1 - \lambda \phi(y))^s (1 - \lambda \phi(y))^s
\]
\[
\cdot \left[ \prod_{j=r+1}^{\omega+s-1} (1 - \lambda \phi(x))^r (1 - \lambda \phi(y))^s \frac{1}{2\pi} \right] d\lambda d\phi
\]
\[
\cdot \left( \frac{1}{1 - \lambda} \right)^{r+s+2} \frac{1}{2\pi}.
\]
Then, we have \( \sum_n \lambda^n b(n) \sim (\pi^2/6)\pi^{2\alpha-2}) a(\lambda) \), whence the Tauberian theorem implies that \( a(n) = b(n) \sim n^2 \log(n)^{3\alpha-4}/24\pi^{3\alpha-4} \). Most importantly we see that the lengths and locations of the chains, \( r \) and \( s \), do not affect the asymptotic behaviour. Noting that if \( 1 \leq r, s \leq \alpha - 1 \), we can partition \( 2 \alpha = r + s + (\alpha - r) + (\alpha - s) \) in \( (\alpha - 1)^2 \) ways, and thus overall the total contribution from terms with \( v \geq 3 \) is
\[
\left( \frac{\alpha!}{12\pi^{2\alpha-2}} \right) n^2 \log(n)^{2\alpha-4}. \tag{55}
\]

**Case 2** \( (v(\delta) = 2) \). The typical term \( c(n) \) was introduced in (33) in the proof of Proposition 9. Now we let \( \lambda \in \mathbb{C} \), with \( |\lambda| < 1 \). By lengthy but direct calculations we can derive an expression of the form
\[
\sum_n \lambda^n c(n) = \frac{\alpha - 1}{(\gamma \pi)^2} a(\lambda) + o(a(\lambda)), \quad \lambda \to 1. \tag{56}
\]

The approach developed in [13] can then be used to bound the error terms and show that \( c(n) \sim (1 - 1/2)(\gamma \pi)^{-2} n^2 \log(n)^{2\alpha-4} \).

Finally taking into account the fact that \( l_1, \ldots, l_\alpha \) can be in any of the \( (\alpha - 1) \) intervals \([k_i, k_{i+1}]\), for \( i = 1, \ldots, \alpha - 1 \), the result follows the overall contribution with \( v(\delta) = 2 \)
\[
\frac{(\alpha - 1)^2}{2(\gamma \pi)^{2\alpha-2}} \log(n)^{2\alpha-4}. \tag{57}
\]

The case for \( d = 2 \) is very similar, so we move on to the case \( d = 3 \).

**Case 3** \( (d = 3 \text{ and } \alpha = 2) \). Using the same notation as before, we have three terms to consider \( a(n), b(n) \), and \( c(n) \). We first consider \( c(n) \). Letting \( K = e/\sqrt{1 - \lambda} \) and using the usual power series construction and spherical coordinates
\[
\sum_n \lambda^n c(n) \sim (1 - \lambda)^2 (2\pi)^{-6}
\]

\[
I_1(\lambda) \sim |\Sigma|^{-1} \int_{r,s=0}^{2\pi} \int_{\theta_1,\theta_2=0}^{\pi} r^2 s^2 \sin(\theta_1) \sin(\theta_2) d\theta_1 d\theta_2 d\phi_1 d\phi_2 d\phi_3 \frac{d r d s}{(1 - \lambda + \lambda r^2)(1 - \lambda + \lambda s^2)}
\]

\[
= |\Sigma|^{-1} \int_{\theta_1,\theta_2=0}^{\pi} \int_{\phi_1,\phi_2=0}^{2\pi} \int_{r,s=0}^{K} \sin(\theta_1) \sin(\theta_2) r^2 s^2 \sin(\theta_1) \sin(\theta_2) d\theta_1 d\theta_2 d\phi_1 d\phi_2 d\phi_3\frac{d r d s}{(1 + r^2)(1 + s^2)}
\]

\[
= |\Sigma|^{-1} \log(K) \int_{\theta_1,\theta_2=0}^{\pi} \int_{\phi_1,\phi_2=0}^{2\pi} \sin(\theta_1) \sin(\theta_2) \frac{\arcsin(\cos(\phi_1) \cos(\phi_2))}{1 - A(\theta_1,\theta_2,\phi_1,\phi_2)} d\phi_1 d\phi_2 d\theta_1 d\theta_2.
\]

The other integral is slightly easier
\[
I_2(\lambda) \sim |\Sigma|^{-1} \frac{\pi}{2} \log K
\]

\[
\cdot \int_{\theta_1,\theta_2=0}^{\pi} \int_{\phi_1,\phi_2=0}^{2\pi} \sin(\theta_1) \sin(\theta_2) d\phi_1 d\phi_2 d\theta_1 d\theta_2, \tag{62}
\]

and thus overall we must have that
\[
(I_1 - I_2)(\lambda) \sim \frac{1}{2} (2\pi)^{-6} |\Sigma|^{-1} (1 - \lambda)^{-2} \log \left( \frac{1}{1 - \lambda} \right)
\]

\[
\cdot \int_{\theta_1,\theta_2=0}^{\pi} \int_{\phi_1,\phi_2=0}^{2\pi} \frac{\arcsin(\lambda)}{\sqrt{1 - A^2}} \frac{\pi}{2} \sin(\theta_1)
\]
\[ \int \sin(\theta_2) \, d\phi_1 \, d\phi_2 \, d\theta_1 \, d\theta_2 = \kappa_2 \left( 1 - \lambda \right)^{-2} \log \left( \frac{1}{1 - \lambda} \right), \]

whence it follows that \( \text{var}(L_n(2)) \sim (\kappa_1 + \kappa_2)n \log n. \)

To prove the last claim let \( S'_n = X'_1 + \cdots + X'_n \) be another random walk, independent of \( S_n \), such that its characteristic function \( f'(t) = \mathbb{E} \left[ \exp(itX'_1) \right] \) also satisfies the expansion (6). Then using [13, Lemma 3.1], one can adapt the proof of [13, Theorem 2.1] to show that \( L'_n(\alpha) = L_n(\alpha) + o(L_n(\alpha)). \)

\[ \square \]

Competing Interests

The authors declare that they have no competing interests.

References


