A comparison of Landau-Ginzburg models for odd dimensional Quadrics

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Abstract

In [Rie08], the second author defined a Landau-Ginzburg model for homogeneous spaces \(G/P\), as a regular function on an affine subvariety of the Langlands dual group. In this paper, we reformulate this LG model \((\tilde{X}, W_t)\) in the case of the odd-dimensional quadric \(Q_{2m-1}\) as a rational function on a Langlands dual projective space, in the spirit of work by R. Marsh and the second author [MR12] for type A Grassmannians and by both authors [PR13] for Lagrangian Grassmannians.

We also compare this LG model with the one obtained independently by Gorbounov and Smirnov in [GS13], and we use this comparison to deduce part of [Rie08, Conj. 8.1] for odd-dimensional quadrics.

1 Introduction

In 2000 Hori and Vafa wrote down a conjectured LG model for any hypersurface in a (weighted) complex projective space [HV00], [Prz07, Rmk. 19]. This is a Laurent polynomial associated to the hypersurface which plays the part of the B-model to the hypersurface in mirror symmetry, meaning its singularities are meant to encode various structures to do with Gromov-Witten theory of the hypersurface. In the case of the smooth quadric \(Q_3\) in \(\mathbb{P}^4\) the LG model is

\[
Y_1 + Y_2 + \frac{(Y_3 + q)^2}{Y_1 Y_2 Y_3},
\]

and in this special case it was written down earlier by Eguchi, Hori, and Xiong [EHX97].

For a quadric \(Q_{2m-1}\) the formula of Hori and Vafa reads

\[
Y_1 + Y_2 + \ldots + Y_{m-1} + \frac{(Y_m + q)^2}{Y_1 Y_2 \ldots Y_m}.
\]

One issue with these Laurent polynomial formulas is that they do not always have the expected number of critical points (at fixed generic value of \(q\)) which should be equal to \(\dim(H^*(Q_{2m-1}))\). This was already observed in [EHX97], where it was suggested to solve this problem using a partial compactification, and this was carried out for the first time albeit in an ad hoc fashion.

The quadratic hypersurfaces \(Q_{2m-1}\) have a large symmetry group. Indeed \(Q_{2m-1}\) is a cominuscule homogeneous space for the group \(\text{Spin}_{2m+1}(\mathbb{C})\). Therefore there is already

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another LG model on an affine variety generally larger than a torus, which was defined by
the second author using a Lie theoretic construction \textsuperscript{[Rie08]}. Namely for any projective
homogeneous space \(G/P\) of a simple complex algebraic group, \textsuperscript{[Rie08]} constructs a conjectural
LG model, which is a regular function on an affine subvariety of the Langlands dual
group. It is shown in \textsuperscript{[Rie08]} that this LG model recovers the Peterson variety presentation
\textsuperscript{[Pet97]} of the quantum cohomology of \(G/P\). It therefore defines an LG model whose Jacobi
ring has the correct dimension.

For odd-dimensional quadrics \(Q_{2m-1}\) a recent paper \textsuperscript{[GS13]} of Gorbounov and Smirnov
constructed directly a partial compactification of the Hori-Vafa mirrors, without making
use of \textsuperscript{[Rie08]}. Moreover they proved a version of mirror symmetry, which identifies the
initial data of the Frobenius manifold associated to the LG model with that constructed
out of the quantum cohomology of \(Q_{2m-1}\).

The goal of this note is twofold. We first express the LG model from \textsuperscript{[Rie08]} in the case
of \(Q_{2m-1}\) in terms of natural coordinates on an affine subvariety of a ‘mirror homogeneous
space’ \(X^\vee = IG_1(2m) \cong \mathbb{P}^{2m-1}\). For example in the case of \(Q_3\) we obtain

\[
W_q = \frac{p_1 + p_2}{p_1 p_2 - p_3} + \frac{q}{p_3}.
\]

The first main result generalises this formula. Define

\[
\tilde{X}^\circ := \tilde{X} \setminus D,
\]

where \(D := D_0 + D_1 + \ldots + D_{m-1} + D_m\), the \(D_i\)’s being given by

\[
D_0 := \{p_0 = 0\},
\]

\[
D_i := \left\{ p_0 p_{2m-1-i} - p_{i-1} p_{2m-i} + \cdots + (-1)^i p_0 p_{2m-1} = 0 \right\} \text{ for } 1 \leq l \leq m-1,
\]

\[
D_m := \{p_{2m-1} = 0\}.
\]

The divisor \(D\) is an anti-canonical divisor. Indeed, the index of \(\tilde{X} = \mathbb{P}(V^*)\) is \(2m\).

**Theorem 1.** The LG model \(F_q : \mathcal{R} \to \mathbb{C}\) from \textsuperscript{[Rie08]} is isomorphic to \(W_q : \tilde{X}^\circ \to \mathbb{C}\)
defined by

\[
W_q = p_1 + \sum_{l=1}^{m-1} \frac{p_l p_{2m-1-l}}{p_0 p_{2m-1-i} - p_{i-1} p_{2m-i} + \cdots + (-1)^i p_{2m-1}} + \frac{q}{p_{2m-1}}.
\]

**Corollary 2.** There is an isomorphism

\[
\mathbb{C}[\tilde{X}^\circ \times \mathbb{C}^*]/(\partial W_q) \to QH^*(X)[q^{-1}]
\]

defined by sending \(p_i\) to the Schubert class \(\sigma_i \in H^{2i}(X)\).

This follows from Thm. \textsuperscript{[1]} together with \textsuperscript{[Rie08]}. Indeed the isomorphism in Cor. \textsuperscript{[2]} fits in
well with the geometric Satake correspondence (see \textsuperscript{[Lus83], Gin95, [MV07]}), by which

\[
H^*(Q_{2m-1}) = V_{\omega_1}^{PSp_{2m}}.
\]

With this in mind it is natural to identify \(\tilde{X}\) with \(\mathbb{P}(H^*(Q_{2m-1})^*)\) and the coordinates \(\{p_i\}\)
with the Schubert basis \(\{\sigma_i\}\) of \(H^*(Q_{2m-1})\).

In is interesting to note that under the isomorphism from Cor. \textsuperscript{[2]} the denominators of \(W_q\) actually map to something extremely simple inside the quantum cohomology of the quadric :
Corollary 3. For $1 \leq l \leq m - 1$, the denominator $p_l p_{2m-l-1} \cdots p_{2m-1} - (-1)^l p_{2m-1}$ represents an element in the Jacobi ring of $W_\mathcal{q}$ which maps to 

$$\sigma_l \sigma_{2m-1-l} + \cdots + (-1)^l \sigma_{2m-1} = \mathcal{q}$$

inside $QH^*(X)$ under the isomorphism $\mathfrak{3}$.

This is an easy consequence of quantum Schubert calculus on the quadric (which can be deduced from the quantum Chevalley formula of [FW04]).

Finally, in Sec. $\mathfrak{6}$ we recall a partial compactification of the Hori-Vafa mirror defined by Gorbounov and Smirnov. We then show the following corollary.

Corollary 4. The partially compactified LG model defined in Gorbounov and Smirnov is related to the formula $\mathfrak{2}$ by a change of coordinates. In particular the Gorbounov and Smirnov LG model is isomorphic to the LG model defined in [Rie08].

Together with Cor. $\mathfrak{4}$ the work of Gorbounov and Smirnov implies a part of the mirror conjecture stated in [Rie08, Conjecture 8.1] for the groups $\text{Spin}_{2m+1}(\mathbb{C})$ with maximal parabolic $P = P_{\omega_1}$, see Sec. $\mathfrak{7}$.

2 Notations and Definitions

The LG model for $Q_{2m-1} = \text{Spin}_{2m+1}/P_{\omega_1}$ defined in [Rie08] takes place on an open Richardson variety inside the Langlands dual flag variety $\text{PSp}_{2m}/B_\pm$. We let $G = \text{PSp}_{2m}(\mathbb{C})$, since this is the group we will primarily be working with. Then $G^\vee = \text{Spin}_{2m+1}(\mathbb{C})$ and $Q_{2m-1} = G^\vee/P^\vee$ for the parabolic subgroup $P^\vee$ associated to the first node of the Dynkin diagram of type $B_m$.

$$\circ \rightarrow \circ \rightarrow \cdots \rightarrow \circ \Rightarrow \circ$$

Let $V = \mathbb{C}^{2m}$ with fixed symplectic form

$$J = \begin{pmatrix}
  -1 \\
  1 & \ddots \\
  & 1 \\
  1 & & -1 \\
\end{pmatrix}.$$

For $G = \text{PSp}(V, J)$ we fix Chevalley generators $(e_i)_{1 \leq i \leq m}$ and $(f_i)_{1 \leq i \leq m}$. To be explicit we embed $\mathfrak{sp}(V, J)$ into $\mathfrak{gl}(V)$ and set

$$e_i = E_{i,i+1} + E_{2m-i,2m-i+1}, \text{ for } i = 1, \ldots, m-1, \text{ and } e_m = E_{m,m+1}.$$ 

and $f_i := e_i^T$, the transpose matrix, for every $i = 1, \ldots, m$. Here $E_{i,j} = (\delta_{i,k} \delta_{l,j})_{k,l}$ is the standard basis of $\mathfrak{gl}(V)$. For elements of the group $\text{PSp}(V)$, we will take matrices to represent their equivalence classes. We have Borel subgroups $B_+ = TU_+$ and $B_- = TU_-$ consisting of upper-triangular and lower-triangular matrices in $\text{PSp}(V)$, respectively. $T$ is the maximal torus of $\text{PSp}(V)$, consisting of diagonal matrices $(d_{i,i})$ with non-zero entries $d_{i,i} = d_{2m-i+1,2m-i+1}$. The parabolic subgroup $P$ we are interested in is the one whose Lie algebra $\mathfrak{p}$ is generated by all of the $e_i$ together with $f_2, \ldots, f_m$, leaving out $f_1$. Let $x_i(a) := \exp(ae_i)$ and $y_i (a) = \exp(-a)$. 


exp(af_i). The Weyl group W of PSp_{2m} is generated by simple reflections s_i for which we choose representatives
\[ \dot{s}_i = y_i(-1)x_i(1)y_i(-1). \]
We let \( W_P \) denote the parabolic subgroup of the Weyl group W, namely \( W_P = \langle s_2, \ldots, s_m \rangle \).
The length of a Weyl group element \( w \) is denoted by \( \ell(w) \). The longest element in \( W_P \) is denoted by \( w_P \). We also let \( w_0 \) be the longest element in \( W \). Next \( W_P \) is defined to be the set of minimal length coset representatives for \( W/W_P \). The minimal length coset representative for \( w_0 \) is denoted by \( w^P \). Let \( \dot{w} \) denote the representative of \( w \) in \( G \) obtained by setting \( \dot{w} = \dot{s}_{i_1} \cdots \dot{s}_{i_m} \), where \( w = s_{i_1} \cdots s_{i_m} \) is a reduced expression.

We consider the open Richardson variety \( \mathcal{R} := R_{w_P, w_0} \subset G/B, \) namely
\[ \mathcal{R} := R_{w_P, w_0} = (B_+ \dot{w}_P B_- \cap B_- \dot{w}_0 B_-)/B_-. \]
Let \( T^{W_P} \) be the \( W_P \)-fixed part of the maximal torus \( T \), and fix \( d \in T^{W_P} \). Then we also define
\[ Z_d := B_- \dot{w}_0 \cap U_+ d \dot{w}_P U_. \]
The map
\[ \pi_R : Z_d \to \mathcal{R} : z \mapsto zB_-, \]
is an isomorphism from \( Z_d \) to the open Richardson variety.

Let \( q \) be the coordinate \( \alpha_1 \) on the 1-dimensional torus \( T^{W_P} \). The mirror LG model is a regular function on \( \mathcal{R} \) depending also on \( q \), so a regular function on \( \mathcal{R} \times T^{W_P} \). It is defined as follows [Ric08],
\[ \mathcal{F} : (u_1 \dot{w}_P B_-, d) \mapsto z = u_1 \dot{w}_P d \dot{u}_2 \in Z_d \mapsto \sum e^*_i(u_1) + \sum f^*_i(\dot{u}_2). \]
The corresponding map from \( \mathcal{R} \), when \( d \) is fixed, is denoted
\[ \mathcal{F}_d : \mathcal{R} \to \mathbb{C} : u_1 \dot{w}_P B_- \mapsto \mathcal{F}(u_1 \dot{w}_P B_-, d). \]

We also define another embedding
\[ \pi_L : Z_d \to P \backslash \text{PSp}(V) : z \mapsto Pz, \]
which maps \( Z_d \) isomorphically to an open subvariety of a big cell in \( P \backslash \text{PSp}(V) \). Note that \( P \backslash \text{PSp}(V) \) is canonically the isotropic Grassmannian of lines in \( V^* \), when this Grassmannian is viewed as a homogeneous space via the action of \( \text{PSp}(V) \) from the right. Moreover the isotropic Grassmannian of lines is just \( \mathbb{F}(V^*) \), since any line is automatically isotropic. Therefore the second embedding \( \pi_L \) has an advantage, that it is just an embedding into a projective space.

**Definition 2.1 (Plücker coordinates).** First we introduce notation for the elements of \( W^P \):
\[ w_k = \begin{cases} s_k s_{k-1} \cdots s_1 & \text{if } k \leq m, \\ s_{2m-k} \cdots s_{m-1} s_m s_{m-1} \cdots s_1 & \text{if } m + 1 \leq k \leq 2m - 1. \end{cases} \]
The associated Plücker coordinates \( p_k \) are defined by
\[ p_k(g) = \langle v^-_{w_k} g, w_k \cdot v^-_{w_k} \rangle. \]

Note that the Plücker coordinates are just the homogeneous coordinates on the projective space \( \mathbb{P}(V^*) \). For a coset \( Pg \) they are given by the bottom row entries of \( g \) read from right to left. If \( g = u_1 \dot{w}_P d \dot{u}_2 \) then
\[ (p_0(g) : \ldots : p_{2m-1}(g)) = (p_0(\dot{u}_2) : \ldots : p_{2m-1}(\dot{u}_2)). \]
Our goal is to express \( \mathcal{F} \) as a rational function in the Plücker coordinates and \( q = \alpha_1(d) \). We first illustrate our result in the smallest interesting example: that of the three-dimensional quadric \( Q_3 \).
3 The mirror to $Q_3$

A generic element of $Z_d := B_\sim -\bar{w}_0 \cap U_+ d\bar{w}_p U_-$ can be written as $u_1 d\bar{w}_p \bar{u}_2$, where
\[
\bar{u}_2 = y_1(a_1)y_2(c)y_1(b_1)
\]
and $a_1, c, b_1$ are non-zero. Hence
\[
\bar{u}_2 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
a_1 + b_1 & 1 & 0 & 0 \\
cb & c & 1 & 0 \\
a_1 cb & a_1 c & a_1 + b_1 & 1
\end{pmatrix}
\]
The map $\pi_L : Z_d \to P \setminus \text{PSp}(V) \cong \mathbb{P}(V^*)$ takes $z = u_1 d\bar{w}_p \bar{u}_2$ to $Pz = P\bar{u}_2$. This may be interpreted as taking $z$ to the span of the reverse row vector corresponding to the last row of $\bar{u}_2$ after the identification $P \setminus \text{PSp}(V) \cong \mathbb{P}(V^*)$. The Plücker coordinates of $\bar{u}_2$ are given by $p_0 = 1, p_1 = a_1 + b_1, p_2 = a_1 c, p_3 = a_1 cb_1$.

If we are interested in the image of $Z_d$ in $\mathbb{P}(V^*)$ then first of all we can observe that it is independent of $d$. So we may choose for $d$ the identity element, and restrict our attention to $B_\sim -\bar{w}_0 \cap U_+ d\bar{w}_p U_-$. It turns out that the image of $Z_d$ in $\mathbb{P}(V^*)$ is obtained from $\mathbb{P}(V^*)$ in coordinates
\[
(p_0 : p_1 : p_2 : p_3) \in \mathbb{P}(V^*)
\]
by removing $\{p_0 = 0\} \cup \{p_3p_0 - p_2p_1 = 0\} \cup \{p_3 = 0\}$. We call this variety $\bar{X}^o$, and the isomorphism with $Z_d$ in Prop. 9 shows that $\bar{X}^o$ is also isomorphic to the open Richardson variety $\mathcal{R}$.

Let us denote by $W : \bar{X}^o \times \mathbb{C}^* \to \mathbb{C}$ the map obtained from $F$, see [44], after the identifications $\mathcal{R} \cong \bar{X}^o$ and $(T)^{W_p} \cong \mathbb{C}^*$ via $d \mapsto \alpha_1(d) = q$. In this way we can compute the superpotential $F$ from [Rie08] in the coordinates on $\mathbb{P}(V^*)$:
\[
W = \frac{p_1}{p_0} + \frac{p_2^2}{p_1p_2 - p_0p_3} + q \frac{p_1}{p_3}.
\]
This is equivalent to the following Landau-Ginzburg model of [GS13]:
\[
W = \frac{p_1}{p_0} + \frac{p_2}{p_1p_2 - p_0p_3} + q \frac{p_1}{p_3}.
\]
via the change of coordinates:
\[
x = \frac{p_0p_2}{p_1p_2 - p_0p_3}; y = \frac{p_1}{p_0}; z = \frac{q p_0}{p_3}.
\]
Note that in [GS13] the superpotential denoted $\tilde{f}$ is $g$ where $z$ is replaced by $z + 1$.

4 The mirror to $Q_{2m-1}$

We now write down $W_q = (\pi_L)_* \pi_R^* F_d$ as a rational function on $\bar{X}$, where $d \in (T)^{W_p}$ is such that $\alpha_1(d) = q$. We will then prove in the next section that the locus $\bar{X}^o$ where it is defined is isomorphic to the open Richardson variety $\mathcal{R}$.

Proposition 5. As a rational function on $\bar{X}$
\[
W_q = \frac{p_1}{p_0} + \sum_{l=1}^{m-1} \frac{p_{l+1}p_{2m-1-l}}{p_{l}p_{2m-1-l} - p_{l-1}p_{2m-l} + \cdots + (-1)^l p_0p_{2m-l}} + q \frac{p_1}{p_{2m-1}}.
\]
To prove the result, we first recall that
\[ \pi_R^* \mathcal{F}_d : z = u_1 \hat{w}_P d \bar{u}_2 \in \mathbb{Z}_d \mapsto \sum e_i^*(u_1) + \sum f_i^*(\bar{u}_2). \]

Now \( \bar{u}_2 \) appearing in \( u_1 \hat{w}_P d \bar{u}_2 \in \mathbb{Z}_d \bar{u}_0 \) can be assumed to lie in \( U_- \cap B_+ (\hat{w}_P)^{-1} B_+ \). This is because we have two birational maps
\[ \Psi_1 : U_- \cap B_+ (\hat{w}_P)^{-1} B_+ \to P \setminus G : \bar{u}_2 \mapsto P \bar{u}_2, \]
\[ \Psi_2 : B_- \cap U_+ \hat{w}_P U_- \to P \setminus G : b_- = u_1 \hat{w}_P \bar{u}_2 \mapsto P b_-, \]
which compose to give \( \Psi_1^{-1} \circ \Psi_2 : b_- \mapsto \bar{u}_2 \). This gives a birational map
\[ \Psi_1^{-1} \circ \Psi_2 : Z_d \bar{u}_0 \to U_- \cap B_+ (\hat{w}_P)^{-1} B_+. \]

Now a generic element \( \bar{u}_2 \) in \( U_- \cap B_+ (\hat{w}_P)^{-1} B_+ \) can be assumed to have a particular factorisation. The smallest representative \( w_P \) in \( W \) of \( [w_0] \in W/W_P \) has the following reduced expression:
\[ w_P = s_1 \ldots s_{m-1} s_m s_{m-1} \ldots s_m. \]

It follows that as a generic element of \( U_- \cap B_+ (\hat{w}_P)^{-1} B_+ \), the element \( \bar{u}_2 \) can be assumed to be written as
\[ \bar{u}_2 = y_1(a_1) \ldots y_{m-1}(a_{m-1}) y_m(c) y_{m-1}(b_{m-1}) \ldots y_1(b_1), \]
(5)
where \( a_i, c, b_j \neq 0 \). We have the following standard expression for the \( p_k \) on factorized elements, which is a simple consequence of their definition.

**Lemma 6.** Fix \( 0 \leq k \leq 2m-1 \) an integer. Then if \( \bar{u}_2 \) is of the form (5) we have
\[ p_k(\bar{u}_2) = \begin{cases} 1 & \text{if } k = 0, \\ a_1 \ldots a_k (a_k + b_k) & \text{if } 1 \leq k \leq m - 1, \\ a_1 \ldots a_{m-1} c b_{m-1} \ldots b_{2m-k} & \text{otherwise}. \end{cases} \]

We will also need the following:

**Lemma 7.** If \( u_1 \) and \( \bar{u}_2 \) are as above then we have the following identities
\[ f_i^*(\bar{u}_2) = \begin{cases} a_i + b_i & \text{if } 1 \leq i \leq m, \\ c & \text{otherwise}. \end{cases} \]
(6)
\[ e_i^*(u_1) = \begin{cases} 0 & \text{if } 2 \leq i \leq m, \\ e_{a_1 \ldots a_{i-1} a_i + b_i} & \text{if } i = 1. \end{cases} \]
(7)

*Proof.* Equation (6) is obtained immediately from the definition of \( \bar{u}_2 \). For Equation (7), notice that
\[ e_i^*(u_1) = \frac{\langle u_1^{-1} \cdot v_u, e_i \cdot v_u \rangle}{\langle u_1^{-1} \cdot v_u, v_u \rangle} = \frac{\langle e_{h \hat{w}_P \bar{u}_2} \cdot v_w, e_i \cdot v_w \rangle}{\langle e_{h \hat{w}_P \bar{u}_2} \cdot v_w, v_w \rangle} \]
Assume $2 \leq i \leq m$. Then $e_i^*(u_1) = 0$ if and only if $\langle \tilde{u}_2 \cdot v_{\omega_i}^+, w_P^{-1} e_i \cdot v_{\omega_i}^- \rangle = 0$. Now the vector $w_P^{-1} e_i \cdot v_{\omega_i}^+$ is in the $\mu$-weight space of the $i$-th fundamental representation, where $\mu = w_P^{-1} s_i(-\omega_i)$. Moreover, $\tilde{u}_2 \in B_+(w_P)^{-1} B_+$, hence $\tilde{u}_2 \cdot v_{\omega_i}^+$ can have non-zero components only down to the weight space of weight $(w_P)^{-1}(\omega_i) = w_P^{-1}(-\omega_i)$. Since $l(w_P^{-1}s_i) > l(w_P^{-1})$ for $2 \leq i \leq m$, this is higher than $\mu$, which proves that $e_i^*(u_1) = 0$.

Now assume $i = 1$. We have

$$e_1^*(u_1) = \frac{\langle e^h w_P \tilde{u}_2 \cdot v_{\omega_1}^+, e_1 \cdot v_{\omega_1}^- \rangle}{\langle e^h w_P \tilde{u}_2 \cdot v_{\omega_1}^+, v_{\omega_1}^- \rangle} = \frac{(\omega_1 + \alpha_1 - \omega_1)(e^h) \langle \tilde{u}_2 \cdot v_{\omega_1}^+, w_P^{-1} e_1 \cdot v_{\omega_1}^- \rangle}{\langle \tilde{u}_2 \cdot v_{\omega_1}^+, w_P e_{\omega_1}^- \rangle} = e_1^*(u_1) \frac{\langle \tilde{u}_2 \cdot v_{\omega_1}^+, w_P^{-1} e_1 \cdot v_{\omega_1}^- \rangle}{\langle \tilde{u}_2 \cdot v_{\omega_1}^+, v_{\omega_1}^- \rangle}.$$

First look at the denominator. The only way to go from the highest weight vector $v_{\omega_1}^+$ of the first fundamental representation to the lowest $v_{\omega_1}^-$ is to apply $g \in B_+ w B_+$ for $w \geq (w_P)^{-1}$. Since $\tilde{u}_2 \in B_+(w_P)^{-1} B_+$, it follows that we need to take all factors of $\tilde{u}_2$, and normalising $v_{\omega_1}^-$ appropriately, we get

$$\langle \tilde{u}_2 \cdot v_{\omega_1}^+, v_{\omega_1}^- \rangle = a_1 \ldots a_{m-1} c b_{m-1} \ldots b_1.$$

Finally, we look at the numerator $\langle \tilde{u}_2 \cdot v_{\omega_1}^+, w_P^{-1} e_1 \cdot v_{\omega_1}^- \rangle$. The vector $w_P^{-1} e_1 \cdot v_{\omega_1}^-$ has weight

$$\mu' = w_P^{-1} s_1(-\omega_1) = w_P^{-1}(-\epsilon_2) = \epsilon_2.$$

Write $w_P^{-1} s_1$ as a prefix $w' = s_1 s_2 \ldots s_{m-1} s_m s_{m-1} \ldots s_2$ of $(w_P)^{-1}$. We have $w' s_1 = (w_P)^{-1}$, hence the way from $v_{\omega_1}^+$ to $w' \cdot v_{\omega_1}^-$ is through $s_1$. From the shape of $\tilde{u}_2$, it follows that

$$\langle \tilde{u}_2 \cdot v_{\omega_1}^+, w_P^{-1} e_1 \cdot v_{\omega_1}^- \rangle = a_1 + b_1.$$

Using the expression (4) of the superpotential from [Rie08], we immediately deduce from Lem.7 an intermediate expression for the Landau-Ginzburg model $W_q$ of the odd-dimensional quadric as a Laurent polynomial:

**Proposition 8.**

$$W_q = a_1 + \ldots + a_{m-1} + c + b_{m-1} + \ldots + b_1 + q \frac{a_1 + b_1}{a_1 \ldots a_{m-1} c b_{m-1} \ldots b_1} \quad (8)$$

Now with the help of Lem.6 and Prop.8 we prove the second expression of $W_q$:

**Proof of Prop.** From Lem.6 it follows that for $\tilde{u}_2$ as in (5)

$$p_{l+1} p_{2m-1-l}(\tilde{u}_2) = \begin{cases} (a_{l+1} + b_{l+1})(a_1 \ldots a_l)^2 a_{l+1} \ldots a_{m-1} c b_{m-1} \ldots b_{l+1} & \text{if } l \leq m - 2, \\ (a_1 \ldots a_{m-1} c)^2 & \text{if } l = m - 1. \end{cases}$$

and

$$p_k p_{2m-1-k}(\tilde{u}_2) = (a_k + b_k)(a_1 \ldots a_{k-1})^2 a_k \ldots a_{m-1} c b_1 \ldots b_{k+1}.$$

Hence most terms in $\sum_{k=0}^{l} (-1)^k p_{l-k} p_{2m-1-k-l}(\tilde{u}_2)$ cancel, and

$$\sum_{k=0}^{l} (-1)^k p_{l-k} p_{2m-1-k-l}(\tilde{u}_2) = (a_1 \ldots a_l)^2 a_{l+1} \ldots a_{m-1} c b_{m-1} \ldots b_{l+1}.$$
This proves that
\[
\frac{p_{l+1}p_{2m-1-l}}{p_lp_{2m-1-l} - p_{l-1}p_{2m-l} + \cdots + (-1)^lp_0p_{2m-1}}(\bar{u}_2) = \begin{cases} 
  a_{l+1} + b_{l+1} & \text{if } l \leq m - 2, \\
  c & \text{if } l = m - 1.
\end{cases}
\]

For the first and last terms, we obtain
\[
\frac{p_1}{p_0}(\bar{u}_2) = a_1 + b_1
\]
and
\[
\frac{p_1}{p_{2m-1}}(\bar{u}_2) = \frac{a_1 + b_1}{a_1 \ldots a_{m-1}b_{m-1} \ldots b_1}
\]
as easy consequences of Lem. [6].

\section{The open Richardson variety}

We now prove that the affine subvariety $\tilde{X}^0$ defined in Equation (1) is isomorphic to the open Richardson variety $\mathcal{R}$.

Recall that $\tilde{X}^0 = \tilde{X} \setminus D$, where
\[
D := D_0 + D_1 + \ldots + D_{m-1} + D_m
\]
and
\[
D_0 := \{ p_0 = 0 \}, \\
D_l := \{ p_l p_{2m-1-l} - p_{l-1} p_{2m-l} + \cdots + (-1)^l p_0 p_{2m-1} = 0 \} \quad \text{for } 1 \leq l \leq m - 1, \\
D_m := \{ p_{2m-1} = 0 \}.
\]

By definition, $\tilde{X}^0$ is the locus where $W_q$ is regular. Since $p_0$ is non-zero on $\tilde{X}^0$, we may assume that $p_0 = 1$. Hence we have affine coordinates $(p_1, \ldots, p_{2m-1})$ on $\tilde{X}^0$. We also set, for $1 \leq j \leq 2m - 1$:
\[
r_j := \sum_{k=0}^{j} (-1)^k p_j k p_{m-1+k-j}.
\]

\begin{proposition}
The map $\pi_L \circ \pi_R^{-1} : \mathcal{R} \to \tilde{X}^0$ is an isomorphism.
\end{proposition}

We will prove the result by constructing the inverse map. But first, let us check that the image of this map is indeed inside $\tilde{X}^0$ (and not just inside $\tilde{X}$). Clearly, $\mathcal{F}_d$ equals $W_q \circ \pi_L \circ \pi_R^{-1}$ as a rational map. Since $\mathcal{F}_d$ is regular on $\mathcal{R}$, it means that $W_q \circ \pi_L \circ \pi_R^{-1}$ also is, hence that $W_q$ is regular on the image of $\pi_L \circ \pi_R^{-1}$. This proves that this image is contained in $\tilde{X}^0$.

We now define a map $\Phi : \tilde{X}^0 \to B^\text{PSL}_m \bar{w}_0$, where $B^\text{PSL}_m$ is the Borel of lower triangular matrices in $\text{PSL}_{2m}$, so that $\Phi(p_1, \ldots, p_{2m-1}) \cdot v_j$ is equal to
\[
\begin{cases}
  p_{2m-1}v_{2m} & \text{if } j = 1, \\
  (-1)^j \sum_{k=2}^{m-j} (-1)^{j+k-1} v_{2m-l} + v_{2m} & \text{if } 2 \leq j \leq m, \\
  (-1)^j \sum_{k=m+1}^{j} (-1)^{k-1} p_{k-1} v_{2m+1-k} + \sum_{k=1}^{m-k} (-1)^{k-1} p_{k-1} v_{2m-k} & \text{if } m+1 \leq j \leq 2m - 1, \\
  \sum_{k=1}^{m-1} (-1)^k p_{2m-k} v_{k+1} + \sum_{k=1}^{m-k} (-1)^k p_{k-1} v_{2m-k} + v_{2m} & \text{if } j = 2m.
\end{cases}
\]
Let $\Omega$ be the open dense subset of $X^{\circ}$ where the coordinates $p_{m}, p_{m-1}, \ldots, p_{2m-2}$ do not vanish and define coordinates on $\Omega$ (as follows from Lem. 10) by

$$a_{i} = \frac{p_{2m-1} r_{i}}{p_{2m-1} - r_{i}} \quad \text{for all } 1 \leq i \leq m - 1;$$

$$b_{i} = \frac{p_{2m-1}}{p_{2m-1} - i} \quad \text{for all } 1 \leq i \leq m - 1;$$

$$c = \frac{p_{2m}}{r_{m-1}}.$$

**Lemma 10.** For all $(p_{1}, \ldots, p_{2m-1}) \in \Omega$, $\Phi(p_{1}, \ldots, p_{2m-1})$ factorizes as $u_{1} \hat{w}_{p} u_{2}$, where

$$\bar{u}_{2} = y_{1}(a_{1}) \cdots y_{m-1}(a_{m-1}) y_{m}(c) y_{m-1}(b_{m-1}) \cdots y_{1}(b_{1})$$

and $u_{1}$ equals

$$
\begin{pmatrix}
1 & a_{1} & \cdots & a_{1} + b_{1} & \cdots & a_{1} + b_{m-1} & \cdots & a_{1} + \sum_{i=1}^{m} b_{i} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
1 & \cdots & \cdots & 1 & \cdots & \cdots & \cdots & \cdots \\
1 & \cdots & \cdots & 1 & \cdots & \cdots & \cdots & \cdots \\
1 & \cdots & \cdots & 1 & \cdots & \cdots & \cdots & \cdots \\
1 & \cdots & \cdots & 1 & \cdots & \cdots & \cdots & \cdots \\
1 & \cdots & \cdots & 1 & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
$$

and $u_{2}$ equals

$$\begin{pmatrix}
(-1)^{m} & \cdots & (-1)^{m-1} b_{m-1} \\
\vdots & \ddots & \vdots \\
(-1)^{m-1} b_{1} & \cdots & \cdots & \cdots \\
\vdots & \ddots & \vdots & \vdots \\
1 & \cdots & \cdots & \cdots \\
\end{pmatrix}
$$

**Proof.** Using the definition of the $y_{i}$'s, it is easy to check that $\bar{u}_{2} \cdot v_{j}$ is equal to

$$v_{j} + \sum_{k=0}^{m-1-j} (a_{j+k} + b_{j+k}) b_{j+k-1} \cdots b_{j+1} b_{j} v_{j+k+1} + \sum_{k=0}^{m-1} a_{m-k} \cdots a_{m-1} c b_{m-1} \cdots b_{j} v_{m+1+k}$$

if $1 \leq j \leq m - 1,$

$$v_{m} + \sum_{k=0}^{m-1} a_{m-k} \cdots a_{m-1} c$$

if $j = m,$

$$v_{j} + (a_{2m-j} + b_{2m-j}) \sum_{k=0}^{2m-1-j} a_{2m-j-k} \cdots a_{2m-2-j} a_{2m-1-j} v_{j+1+k}$$

if $m + 1 \leq j \leq 2m,$

Now a straightforward, if slightly tedious, computation shows that $\Phi(p_{1}, \ldots, p_{2m-1}) = u_{1} \hat{w}_{p} u_{2}$. ∎

We now need to prove that the entire image of $\Phi$ is in fact contained in $B_{-} \hat{w}_{0} \cap U_{+} \hat{w}_{p} U_{-}$ inside $\text{PSp}_{2m}$.

**Lemma 11.**

$$\Phi(X^{\circ}) \subset B_{-} \hat{w}_{0} \cap U_{+} \hat{w}_{p} U_{-}.$$

**Proof.** We first prove that $\Phi(\Omega) \subset B_{-} \hat{w}_{0} \cap U_{+} \hat{w}_{p} U_{-}$ inside $\text{PSp}_{2m}$. Indeed, from Lem. 11 we know that for all $(p_{1}, \ldots, p_{2m-1}) \in \Omega$, $\Phi(p_{1}, \ldots, p_{2m-1})$ factorizes as $u_{1} \hat{w}_{p} u_{2}$, where $u_{1}$ and $u_{2}$ are defined in the statement of the lemma. The factorisation of $u_{2}$ means that
\( \tilde{u}_2 \) is in \( U_\omega \) (hence in particular in \( \text{PSp}_{2m} \)). Now we prove that \( u_1 \) is also in \( \text{PSp}_{2m} \), by showing directly that \( ^t u_1 J u_1 = J \) using the formula from Lem. 10. This is the result of a straightforward computation. It follows that \( u_1 \in U_\omega \), hence \( \Phi(p_1, \ldots, p_{2m-1}) \in U_\omega \wedge U_{\omega} \subset \text{PSp}_{2m} \) in this case. Now also \( \Phi(p_1, \ldots, p_{2m-1}) \in B_{\omega} \wedge \text{PSp}_{2m} = B_{-\omega} \). Therefore \( \Phi(\Omega) \subset B_{-\omega} \wedge U_{\omega} \).

Since \( \Omega \) is open dense in \( \hat{X}^\circ \) we now have that \( \Phi(\hat{X}^\circ) \subset B_{-\omega} \wedge U_{\omega} \). Suppose there exists \( (p_1, \ldots, p_{2m-1}) \) in \( \hat{X}^\circ \) such that \( \Phi(p_1, \ldots, p_{2m-1}) \notin U_{\omega} \wedge U_{\omega} \). Then from Bruhat decomposition, we get \( \Phi(p_1, \ldots, p_{2m-1}) \tilde{u}_0^{-1} \in U_{\omega} \wedge u_+ U_+ \) with \( w < w_+ \). It follows that we must have

\[
\langle \Phi(p_1, \ldots, p_{2m-1}) \tilde{u}_0^{-1} v_1^+, v_{-1} \rangle = \langle \Phi(p_1, \ldots, p_{2m-1}) v_1^-, v_{-1} \rangle = 0,
\]
hence the lower-right corner of the matrix \( \Phi(p_1, \ldots, p_{2m-1}) \) has to be zero. But this coefficient is always 1, hence the result.

We can now prove Prop. 9:

**Proof of Prop. 9** We have showed that the image of \( \pi_L \) is contained inside \( \hat{X}^\circ \). Moreover, we have defined a map \( \Phi : \hat{X}^\circ \rightarrow Z_1 \), and a straightforward computation shows that it is the inverse of \( \pi_L \). Hence \( \pi_L \) is an isomorphism. Since we saw in Sec. 2 that \( \pi_R \) is also an isomorphism, the proposition follows.

The proof of Thm. 1 then follows from Prop. 5 and 9.

### 6 Comparison with the LG model of [GS13]

We now want to prove that our Landau-Ginzburg model (2) is isomorphic to the one stated in [GS13], which goes as follows

\[
g = \sum_{i=1}^{m-1} y_i (1 + z_i) + q \frac{x^2 (y_1 y_2 \cdots y_{m-1} - 1) z_1 z_2 \ldots z_{m-1}}{z_1 \cdots z_{m-1}}. \tag{9}
\]

Note that as for \( Q_3 \), in [GS13] the superpotential denoted \( \tilde{f} \) is \( g \) where the \( z_i \) are replaced by \( z_i + 1 \).

Assume \( p_0 = 1 \) and consider the change of variables:

\[
y_1 = p_1; \quad y_i = \frac{p_i}{p_{i-1}} \quad \forall \ 2 \leq i \leq m - 1;
\]

\[
z_1 = \frac{q}{p_5}; \quad z_i = \frac{\sum_{k=0}^{i-2} (-1)^k p_{i-2-k} p_{2m+1+k-i}}{\sum_{k=0}^{i-1} (-1)^k p_{i-1-k} p_{2m+k-i}} \quad \forall \ 2 \leq i \leq m - 1;
\]

\[
x = \frac{\sum_{k=0}^{m-1} (-1)^k p_{m-1-k} p_{m+k}}{p_{m}}.
\]

**Proposition 12.** *The above change of coordinates \( \{x, y_i, z_i\} \rightarrow \{p_i\} \) defines an isomorphism between the Landau-Ginzburg model \( 4 \) and ours \( 2 \).*

**Proof.** We have \( y_1 (1 + z_1) = p_1 + \frac{q}{p_{2m-1}} \), and

\[
y_i (1 + z_i) = \frac{p_i p_{2m-i}}{\sum_{k=0}^{i-1} (-1)^k p_{i-1-k} p_{2m+k-i}}.
\]
Moreover
\[
x y_1 \cdots y_{m-1} = \frac{\sum_{k=0}^{m-2} (-1)^k p_{m-2-k} p_{m+1+k}}{\sum_{k=0}^{m-1} (-1)^k p_{m-1-k} p_{m+k}}.
\]
\[
z_1 \cdots z_{m-1} = \frac{q}{\sum_{k=0}^{m-2} (-1)^k p_{m-2-k} p_{m+1+k}},
\]
and
\[
x^2 = \frac{p_m^2}{\left(\sum_{k=0}^{m-1} (-1)^k p_{m-1-k} p_{m+k}\right)^2},
\]
hence
\[
q(x y_1 y_2 \cdots y_{m-1} - 1) z_1 z_2 \cdots z_{m-1} = \frac{p_m^2}{\sum_{k=0}^{m-1} (-1)^k p_{m-1-k} p_{m+k}}.
\]
Hence the change of variables maps (9) to (2). Finally, it is clear that both domains of definition are the same.

This proves Cor. 4.

7 Consequences

Let \( H_A \) be the sheaf of regular functions of the trivial vector bundle with fiber \( H^*(X, \mathbb{C}) \) over \( \mathbb{C}_h^* \times \mathbb{C}_q^* \) the two-dimensional complex torus with coordinates \( h \) and \( q \). The \textit{A-model connection} is defined on \( H_A \) by
\[
A\nabla_{q \partial_q} = q \frac{\partial}{\partial q} + \frac{1}{h} p_1 * q \bullet
\]
\[
A\nabla_{h \partial_h} = h \frac{\partial}{\partial h} + \text{gr} - \frac{1}{h} c_1(TX) * q \bullet,
\]
where \( \text{gr} \) is a diagonal operator on \( H^*(X) \) given by \( \text{gr}(\alpha) = k \) for \( \alpha \in H^{2k}(X) \). Here we are using the conventions of [Iri09]. Let \( H_A^j \) be the vector bundle on \( \mathbb{C}_h^* \times \mathbb{C}_q^* \) defined by \( H_A^j = j^*H_A \) for \( j : (h, q) \mapsto (-h, q) \). This vector bundle with the pulled back connection \( A\nabla^\vee = j^*(A\nabla) \) is dual to \( (H_A, A\nabla) \) via the flat non-degenerate pairing,
\[
\langle \sigma_i, \sigma_j \rangle = (2\pi i h)^N \int_{[X]} \sigma_i \cup \sigma_j = (2\pi i h)^N \delta_{i+j,N},
\]
where \( N = 2m - 1 \) is the dimension of \( \hat{X}^o \). The dual A-model connection \( A\nabla^\vee \) defines a system of differential equations called the (small) quantum differential equations
\[
A\nabla^\vee_{q \partial_q} S = 0. \tag{10}
\]
Define the \( \mathbb{C}[h^{\pm 1}, q^{\pm 1}] \)-module
\[
G = \Omega^N(\hat{X}^o)[h^{\pm 1}, q^{\pm 1}] / (d - \frac{1}{h} dW_q \wedge \bullet) \Omega^{N-1}(\hat{X}^o)[h^{\pm 1}, q^{\pm 1}],
\]
where \( \Omega^k(\hat{X}^o) \) is the space of holomorphic \( k \)-forms on \( \hat{X}^o \). We denote by \( H_B \) the sheaf with global sections \( G \). Because \( W_q \) is cohomologically tame [GS13], \( G \) is a free \( \mathbb{C}[h^{\pm 1}, q^{\pm 1}] \)-module of rank \( 2m \) (cf. [Sab99]), and \( H_B \) a trivial vector bundle of that dimension. It has
a (Gauss-Manin) connection given by

\[ B\nabla_{q_0, \eta} = q \frac{\partial}{\partial q} [\eta] + \frac{1}{\hbar} \left[ q \frac{\partial W_q}{\partial q} \eta \right] \]

\[ B\nabla_{\hbar_0, \eta} = \hbar \frac{\partial}{\partial \hbar} [\eta] - \frac{1}{\hbar} [W_q \eta]. \]

Let \( \omega \) be the canonical \( N \)-form on \( \tilde{X}^0 \).

**Corollary 13.** The two bundles with connection \( (\mathcal{H}_A, A\nabla) \) and \( (\mathcal{H}_B, B\nabla) \) are isomorphic via \( \sigma_i \mapsto [p_i \omega] \).

**Proof.** The corollary is a consequence of the isomorphism of our LG model \( W_q \) and the one of [GS13] (see Cor. [3]) together with the results of Gorbounov and Smirnov. \( \square \)

Let \( \Gamma_0 \) be a compact oriented real \( N \)-dimensional submanifold of \( \tilde{X}^0 \) representing a cycle in \( H^N(X^0, \mathbb{Z}) \) dual to \( \omega \), in the sense that \( \frac{1}{(2\pi)^N} \int_{\Gamma_0} \omega = 1 \). Then:

**Corollary 14.** The integral

\[ S_0(z, q) = \frac{1}{(2i\pi)^N} \int_{\Gamma_0} e^{\frac{W_q}{\hbar}} \omega \]

is a solution to the quantum differential equation (10).

This implies part of [Rie08] Conj. 8.1 for odd-dimensional quadrics.

**References**


