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# Recent Advances of Palindromic Factorization

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**Abstract.** This paper provides an overview of six particular problems of palindromic factorization and recent algorithmic improvements in solving them.

## 1 Introduction

### 1.1 General Definitions

Let  $S = S[1]S[2] \cdots S[n]$  be a *string* of length  $|S| = n$  over an alphabet  $\Sigma$ . We consider the case of an integer alphabet; in this case each letter can be replaced by its rank so that the resulting string consists of integers in the range  $\{1, \dots, n\}$ . For two positions  $i$  and  $j$ , where  $1 \leq i \leq j \leq n$ , in  $S$ , we denote the *factor*  $S[i]S[i+1] \cdots S[j]$  of  $S$  by  $S[i..j]$ . We denote the reverse string of  $S$  by  $S^R$ , i.e.  $S^R = S[n]S[n-1] \cdots S[1]$ . The empty string (denoted by  $\varepsilon$ ) is the unique string over  $\Sigma$  of length 0. A string  $S$  is said to be a *palindrome* if and only if  $S = S^R$ . If  $S[i..j]$  is a palindrome, the number  $\frac{i+j}{2}$  is called the center of  $S[i..j]$ . Let  $S[i..j]$ , where  $1 \leq i \leq j \leq n$ , be a palindromic factor in  $S$ . It is said to be a *maximal palindrome* if there is no longer palindrome in  $S$  with center  $\frac{i+j}{2}$ . Note that a maximal palindrome can be a factor of another palindrome.

**Definition 1.** A (*maximal*) *palindromic decomposition* of  $S$  such that the number of (*maximal*) *palindromes* is minimal is called a (*maximal*) *palindromic factorization* of  $S$ .

Note that any single letter is a palindrome and, hence, every string can always be factorized into palindromes. However, not every string can be factorized into maximal palindromes; e.g. consider  $S = \text{abaca}$  [2].

In this paper we present a survey of five novel algorithms of palindromic factorization. We start with maximal palindromic factorization presented by [2] in section 2. Later, we explain palindromic factorization with gaps, maximal palindromic factorization with errors and maximal palindromic factorization with gaps and errors presented by [1] in sections 3, 4 and 5.

Finally, we show in section 6 an efficient algorithm of palindromes in weighted strings presented by [4]

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## 2 Maximal Palindromic Factorization

In this section we present an algorithm to compute the maximal palindromic factorization of a given string  $S$  presented by Alatabbi et al. [2]. They first present some notions required to present the algorithm. First of all, they use  $\mathcal{MP}(S)$  to denote the set of center distinct maximal palindromes of  $S$ . They further extend this notation as follows. They use  $\mathcal{MP}(S)[i]$ , where  $1 \leq i \leq n$  to denote the set of maximal palindromes with center  $i$ . Further, for the string  $S$ , they denote the set of all *prefix palindromes* (*suffix palindromes*) as  $\mathcal{PP}(S)$  ( $\mathcal{SP}(S)$ ).

**Proposition 1.** *The position  $i$  could be the center of at most two maximal palindromic factors, therefore;  $\mathcal{MP}(S)[i]$  contains at most two elements, where  $1 \leq i \leq n$ , hence; there are at most  $2n$  elements in  $\mathcal{MP}(S)$ .*

On the other hand, they use  $\mathcal{MPL}(S)[i]$  to denote the set of the lengths of all maximal palindromes ending at position  $i$ , where  $1 \leq i \leq n$  in  $S$ .

$$\begin{aligned} \mathcal{MPL}(s)[i] = \{ & 2\ell - 1 \mid s[i - \ell + 1 \dots i + \ell - 1] \in \mathcal{MP}(s) \} \\ & \cup \{ 2\ell' \mid s[i - \ell' \dots i + \ell' - 1] \in \mathcal{MP}(s) \} \end{aligned} \quad (1)$$

where  $1 \leq i \leq n$ , with  $2\ell$  and  $2\ell' + 1$  are the lengths of the odd and even palindromic factors respectively.

**Proposition 2.** *The set  $\mathcal{MPL}(S)$  (Equation 1) can be computed in linear time from the set  $\mathcal{MP}(s)$ .*

They define the list  $\mathcal{U}(S)$  such that for each  $1 \leq i \leq n$ ,

$\mathcal{U}(S)[i]$  stores the position  $j$  such that  $j + 1$  is the starting position of a maximal palindromic factors ending at  $i$  and  $j$  is the end of another maximal palindromic substring.

Clearly, this can be easily computed once  $\mathcal{MPL}(S)$  is computed.

$$\mathcal{U}[i][j] = i - \mathcal{MPL}(s)[i][j] \quad (2)$$

One can observe, from 1, that the sets  $\mathcal{MPL}(S)$  and  $\mathcal{U}(S)$  contain at most  $2n$  elements. Given the list  $\mathcal{U}(S)$  for a string  $S$ , they define a directed graph  $\mathcal{G}_s = (\mathcal{V}, \mathcal{E})$  as follows. There are  $\mathcal{V} = \{i \mid 1 \leq i \leq n\}$  and  $\mathcal{E} = \{(i, j) \mid j \in \mathcal{U}(S)[i]\}$ . Note that  $(i, j)$  is a directed edge where the direction is from  $i$  to  $j$ . The steps of the proposed algorithms are as follows.

### MPF Algorithm: Maximal Palindromic Factorization Algorithm

**Input:** A String  $S$  of length  $n$

**Output:** Maximal Palindromic Factorization of  $S$

1. Compute the set of maximal palindromes  $\mathcal{MP}(S)$  and identify the set of prefix palindromes  $\mathcal{PP}(S)$ .
2. Compute the list  $\mathcal{MPL}(S)$ .

3. Compute the list  $\mathcal{U}(S)$ .
4. Construct the graph  $\mathcal{G}_s = (\mathcal{V}, \mathcal{E})$ .
5. Do a breadth first search on  $\mathcal{G}_s$  assuming the vertex  $n$  as the source.
6. Identify the shortest path  $P \approx n \rightsquigarrow v$  such that  $v$  is the end position of a palindrome belonging to  $\mathcal{PP}(S)$ . Suppose  $P \approx \langle n = p_k p_{k-1} \dots p_2 p_1 = v \rangle$ .
7. Return  $S = S[1..p_1] S[p_1 + 1..p_2] \dots S[p_{k-1} + 1..p_k]$ .

**Theorem 1.** *Given a string  $S$  of length  $n$ , (Maximal Palindromic Factorization (MPF)) Algorithm correctly computes the maximal palindromic factorization of  $S$  in  $O(n)$  time.*

### 3 Palindromic Factorization with Gaps

In this section we present an efficient solution to the PALINDROMIC FACTORIZATION WITH GAPS problem has been introduced by Adamczyk et al. [1].

It is based on several transformations of the algorithm for computing a palindromic factorization by Fici et al. [6]. For a string  $S$  of length  $n$  this algorithm works in  $\mathcal{O}(n \log n)$  time. The algorithm consists of two steps:

1. Let  $P_j$  be the sorted list of starting positions of all palindromes ending at position  $j$  in  $S$ . This list may have size  $\mathcal{O}(j)$ . However, it follows from combinatorial properties of palindromes that the sequence of consecutive differences in  $P_j$  is non-increasing and contains at most  $\mathcal{O}(\log j)$  distinct values. Let  $P_{j,\Delta}$  be the maximal sublist of  $P_j$  containing elements whose predecessor in  $P_j$  is smaller by exactly  $\Delta$ . Then there are  $\mathcal{O}(\log j)$  such sublists in  $P_j$ . Hence,  $P_j$  can be represented by a set  $G_j$  of size  $\mathcal{O}(\log j)$  which consists of triples of the form  $(i, \Delta, k)$  that represent  $P_{j,\Delta} = \{i, i + \Delta, \dots, i + (k-1)\Delta\}$ . The triples are sorted according to decreasing values of  $\Delta$  and all starting positions in each triple are greater than in the previous one. Fici et al. show that  $G_j$  can be computed from  $G_{j-1}$  in  $\mathcal{O}(\log j)$  time.
2. Let  $PL[j]$  denote the number of palindromes in a palindromic factorization of  $S[1..j]$ . Fici et al. show that it can be computed via a dynamic programming approach, using all palindromes from  $G_j$  in  $\mathcal{O}(\log j)$  time. Their algorithm works as follows. Let  $PL_\Delta[j]$  be the minimum number of palindromes we can factorized  $S[1..j]$  in, provided that we use a palindrome from  $(i, \Delta, k) \in G_j$ . Then  $PL_\Delta[j]$  can be computed in constant time using  $PL_\Delta[j - \Delta]$  based on the fact that if  $(i, \Delta, k) \in G_j$  and  $k \geq 2$ , then  $(i, \Delta, k-1) \in G_{j-\Delta}$ . Exploiting this fact,  $PL_\Delta[j]$  can be computed by only considering  $PL_\Delta[j - \Delta]$  and the shortest palindrome in  $(i, \Delta, k)$ .

Finally,  $PL[j]$  can be computed from all such  $PL_\Delta[j]$  values.

To solve the PALINDROMIC FACTORIZATION WITH GAPS problem, Adamczyk et al. [1] algorithm firstly modify each of the triples in  $G_j$  to reflect the length constraint ( $m$ ). More precisely, due to the length constraint, in each  $G_j$  some triples will disappear completely, and at most one triple will get *trimmed* (i.e. the parameter  $k$  will be decreased).

The algorithm then computes an array  $MG[1..n][0..g]$  such that  $MG[j][q]$  is the minimum possible total length of gaps in a palindromic factorization of  $S[1..j]$ , provided that there are at most  $q$  gaps. Simultaneously, Adamczyk et al. [1] algorithm computes an auxiliary array  $MG'[1..n][0..g]$  such that  $MG'[j][q]$  is the minimum possible total length of gaps up to position  $j$  provided that this position belongs to a gap: at most the  $q$ -th one.

the following formula for  $j > 0$  and  $q \geq 0$  :

$$MG[j][q] = \min(MG'[j][q], \min_{\Delta} \{MG_{\Delta}[j][q]\})$$

where  $MG_{\Delta}[j][q]$  is the partial minimum computed only using palindromes from  $(i, \Delta, k) \in G_j$ . The formula means: either there is a gap at position  $j$ , or using a palindrome ending at position  $j$ . Also  $MG[0][q]$  is filled by zeros for any  $q \geq 0$ .

Adamczyk et al. [1] algorithm computes  $MG_{\Delta}[j][q]$  for  $(i, \Delta, k) \in G_j$  using the same approach as Fici et al. [6] used for  $PL_{\Delta}$ , ignoring the triples that disappear due to the length constraint. If there is a triple that got trimmed, then the corresponding triple at position  $j - \Delta$  (from which they reuse the values in the dynamic programming) must have got trimmed as well. More precisely, if the triple  $(i, \Delta, k)$  is trimmed to  $(i, \Delta, k')$  at position  $j$ , then at position  $j - \Delta$  there is a triple  $(i, \Delta, k - 1)$  which is trimmed to  $(i, \Delta, k' - 1)$ ; that is, by the same number of palindromes. Consequently, to compute  $MG_{\Delta}[j][q]$  from  $MG_{\Delta}[j - \Delta][q]$ , they need to include one additional palindrome (the shortest one in the triple) just as in Fici et al.'s approach.

Finally, for  $j > 0$  and  $q > 0$  they compute  $MG'$  using the following formula:

$$MG'[j][q] = \min(MG'[j - 1][q], MG[j - 1][q - 1]) + 1.$$

The first case corresponds to continuing the gap from position  $j$ , whereas the second to using a palindrome finishing at position  $j - 1$  or a gap finishing at position  $j - 1$  (the latter will be suboptimal). Here the border cases are  $MG'[j][0] = \infty$  for  $j \geq 0$  and  $MG'[0][q] = \infty$  for  $q > 0$ .

Thus Adamczyk et al. [1] arrive at the complete solution to the problem.

**Theorem 2.** *The PALINDROMIC FACTORIZATION WITH GAPS problem can be solved in  $\mathcal{O}(n \log n \cdot g)$  time and  $\mathcal{O}(n \cdot g)$  space.*

## 4 Computing Maximal Palindromes with Errors

We show an algorithm presented by Adamczyk et al. [1] to compute maximal  $\delta$ -palindromes under the edit distance within  $\mathcal{O}(n \cdot \delta)$ . If  $u$  is a  $\delta$ -palindrome under the edit distance, then there exists a palindrome  $v$  such that the minimal number of edit operations (insertion, deletion, substitution) required to transform  $u$  to  $v$  is at most  $\delta$ . The following simple observation shows that it can restrict edit operations to deletions and substitutions only, which Adamczyk et al. [1] call in what follows the *restricted edit operations*. Intuitively, instead of inserting at position  $i$  a character to match the character at position  $|u| - i + 1$ , the character can be deleted at position  $|u| - i + 1$ .

**Observation 3** Let  $u$  be a  $\delta$ -palindrome and  $v$  a palindrome such that the edit distance between  $u$  and  $v$  is minimal. Then there exists a palindrome  $v'$  such that the number of restricted edit operations needed to transform  $u$  to  $v'$  is equal to the edit distance between  $u$  and  $v$ .

**Definition 2.** A (LGPal-queries) is a maximal palindromes are computed using Gusfield's approach [7]

Adamczyk et al. [1] extend a maximal  $\delta$ -palindrome  $S[i..j]$  to a maximal  $(\delta+1)$ -palindrome in three ways; either ignore the letter  $S[i-1]$  and then perform an LGPal-query, or ignore the letter  $S[j+1]$  and then perform an LGPal-query, or ignore both and then perform the LGPal-query. More formally:

**Definition 3.** Assume that  $S[i..j]$  is a  $\delta$ -palindrome. Then it says that each of the factors  $S[i'..j']$  for:

- $i' = i - 1 - d, j' = j + d$ , where  $d = LGPal(i - 2, j + 1)$
- $i' = i - d, j' = j + 1 + d$ , where  $d = LGPal(i - 1, j + 2)$
- $i' = i - 1 - d, j' = j + 1 + d$ , where  $d = LGPal(i - 2, j + 2)$

is an extension of  $S[i..j]$ . If the index  $i'$  is smaller than 1 or the index  $j'$  is greater than  $|S|$ , the corresponding extension is not possible. They also say that  $S[i..j]$  can be extended to any of the three strings  $S[i'..j']$ .

Clearly, the extensions of a  $\delta$ -palindrome are always  $(\delta+1)$ -palindromes.

To facilitate the case of  $\delta$ -palindromes being prefixes or suffixes of the text, they also introduce the following *border-reductions* for  $S[i..j]$  being a  $\delta$ -palindrome:

- If  $i = 1$ , a border reduction leads to  $S[1..j-1]$ .
- If  $j = n$ , a border reduction leads to  $S[i+1..n]$ .

If any of the reductions is possible, they also say that  $S[i..j]$  can be border-reduced to the corresponding strings. As previously, border-reductions of a  $\delta$ -palindrome are always  $(\delta+1)$ -palindromes.

**Lemma 1.** Given a maximal  $\delta$ -palindrome  $S[i'..j']$  with  $\delta > 0$ , there exists a maximal  $(\delta-1)$ -palindrome  $S[i..j]$  which can be extended or border-reduced to  $S[i'..j']$ .

The combinatorial characterization of Lemma 1 yields an algorithm for generating all maximal  $d$ -palindromes, for all centers and subsequent  $d = 0, \dots, \delta$ .

Recall maximal 0-palindromes are computed using Gusfield's approach (LGPal-queries). For a given  $d < \delta$ , they consider all the maximal  $d$ -palindromes and try to extend each of them in all three possible ways (and border-reduce, if possible). This way they obtain a number of  $(d+1)$ -palindromes amongst which, by Lemma 1, are all maximal  $(d+1)$ -palindromes. To exclude the non-maximal ones, they group the  $(d+1)$ -palindromes by their centers (in  $\mathcal{O}(n)$  time via bucket sort) and retain only the longest one for each center.

They arrive at the following intermediate result.

**Lemma 2.** Under the edit distance, all maximal  $\delta$ -palindromes in a string of length  $n$  can be computed in  $\mathcal{O}(n \cdot \delta)$  time and  $\mathcal{O}(n)$  space.

## 5 Maximal Palindromic Factorization with Gaps and Errors

We show an algorithm presented by Adamczyk et al. [1] to solve maximal palindromic factorization with gaps and errors problem in  $\mathcal{O}(n \cdot (g + \delta))$  time and  $\mathcal{O}(n \cdot g)$  space.

Let  $\mathcal{F}$  be a set of factors of the text  $S[1..n]$ . In this section they develop a general framework that allows to factorized  $S$  into factors from  $\mathcal{F}$ , allowing at most  $g$  gaps. They call such a factorization a  $(g, \mathcal{F})$ -factorization of  $S$ .

The goal is to find a  $(g, \mathcal{F})$ -factorization of  $S$  that minimizes the total length of gaps. The authors aim at the time complexity  $\mathcal{O}((n + |\mathcal{F}|) \cdot g)$  and space complexity  $\mathcal{O}(n \cdot g + |\mathcal{F}|)$ .

In the proposed solution Adamczyk et al. [1] use dynamic programming to compute two arrays, similar to the ones used in Section 3:

$MG[1..n][0..g]$ :  $MG[j][q]$  is the minimum total length of gaps in a  $(q, \mathcal{F})$ -factorization of  $S[1..j]$ .

$MG'[1..n][0..g]$ :  $MG'[j][q]$  is the minimum total length of gaps in a  $(q, \mathcal{F})$ -factorization of  $S[1..j]$  for which the position  $j$  belongs to a gap.

They use the following formulas, for  $j > 0$  and  $q > 0$ :

$$MG[j][q] = \min(MG'[j][q], \min_{S[a..j] \in \mathcal{F}} MG[a-1][q])$$

$$MG'[j][q] = \min(MG[j-1][q-1], MG'[j-1][q]) + 1$$

The border cases are exactly the same as in Section 3.

They apply this approach to maximal  $\delta$ -palindromes in each of the considered metrics (see the classic result from [7] for the Hamming distance and Lemma 2 for the edit distance) to obtain the following result.

**Theorem 4.** *The MAXIMAL  $\delta$ -PALINDROMIC FACTORIZATION WITH GAPS problem under the Hamming distance or the edit distance can be solved in  $\mathcal{O}(n \cdot (g + \delta))$  time and  $\mathcal{O}(n \cdot g)$  space.*

## 6 Maximal Palindromic Factorization of Weighted String

In this section, we show an algorithm to compute a smallest maximal  $z$ -palindromic factorization of a given weighted string  $X$  of length  $n$  for a given cumulative threshold  $1/z \in (0, 1]$  has been presented by alzamel et al [4]. Our algorithm follows the one of Alatabbi et al for computing a smallest maximal palindromic factorization of standard strings [3] with some crucial modifications. Recall by  $\mathcal{MP}(x)$ , we denote the set of center-distinct maximal palindromes of string  $x$ . We will use the below two facts related to palindromes:

**Fact 5 ([7])** *Given a string  $x$ ,  $\mathcal{MP}(x)$  can be computed in time  $\mathcal{O}(|x|)$ .*

**Fact 6 (Trivial)** Let  $x[i..j]$  be a palindrome of string  $x$  with center  $c$  and let  $u$ ,  $|u| < j - i + 1$ , be a factor of  $x$  with center  $c$ . Then  $u$  is also a palindrome.

Note that for clarity we use upper case letters for weighted strings, e.g.  $X$ , and lower case letters, e.g.  $x$ , for standard strings.

we start with some definitions related to weighted strings :

**Definition 4.** A weighted string  $X$  on an alphabet  $\Sigma$  is a finite sequence of  $n$  sets. Every  $X[i]$ , for all  $0 \leq i < n$ , is a set of ordered pairs  $(s_j, \pi_i(s_j))$ , where  $s_j \in \Sigma$  and  $\pi_i(s_j)$  is the probability of having letter  $s_j$  at position  $i$ . Formally,  $X[i] = \{(s_j, \pi_i(s_j)) \mid s_j \neq s_l \text{ for } j \neq l, \text{ and } \sum \pi_i(s_j) = 1\}$ . A letter  $s_j$  occurs at position  $i$  of  $X$  if and only if the occurrence probability of letter  $s_j$  at position  $i$ ,  $\pi_i(s_j)$ , is greater than 0.

**Definition 5.** A string  $u$  of length  $m$  is a factor of a weighted string  $X$  if and only if it occurs at starting position  $i$  with cumulative probability  $\prod_{j=0}^{m-1} \pi_{i+j}(u[j]) > 0$ . Given a cumulative weight threshold  $1/z \in (0, 1]$ , we say factor  $u$  is  $z$ -valid, if it occurs at position  $i$  with cumulative probability  $\prod_{j=0}^{m-1} \pi_{i+j}(u[j]) \geq 1/z$ .

**Definition 6.** Given a cumulative weight threshold  $1/z \in (0, 1]$ , a weighted string  $X$  of length  $m$  is a  $z$ -palindrome if and only if there exists at least one  $z$ -valid factor  $u$  of  $X$  of length  $m$  which is a palindrome.

If the weighted string  $X[i..j]$  is a  $z$ -palindrome, we analogously define the number  $\frac{i+j}{2}$  as the center of  $X[i..j]$  in  $X$  and  $\frac{j-i+1}{2}$  as the radius of  $X[i..j]$ .

**Definition 7.** Let  $X$  be a weighted string of length  $n$ ,  $1/z \in (0, 1]$  a cumulative weight threshold, and  $X[i..j]$ , where  $0 \leq i \leq j \leq n - 1$ , a  $z$ -palindrome. Then  $X[i..j]$  is a maximal  $z$ -palindrome if there is no other  $z$ -palindrome in  $X$  with center  $\frac{i+j}{2}$  and larger radius.

We proceed as follows: By  $\mathcal{MP}(X, z)$ , we denote the set of center-distinct maximal  $z$ -palindromes of our weighted string  $X$ . We present a  $z$ -palindrome with center  $c$  and radius  $r$  by  $(c, r)$ . For each position of  $X$  we define the *heaviest letter* as the letter with the maximum probability (breaking ties arbitrarily). We consider the string obtained from  $X$  by choosing at each position the heaviest letter. We call this the *heavy string* of  $X$ .

We define a collection  $\mathcal{Z}_X$  of  $\lfloor z \rfloor$  *special-weighted strings* of  $X$ , denoted by  $\mathcal{Z}_k$ ,  $0 \leq k < \lfloor z \rfloor$ . Each  $\mathcal{Z}_k$  is of length  $n$  and it has the following properties. Each position  $j$  in  $\mathcal{Z}_k$  contains at most one letter with positive probability and it corresponds to position  $j$  in  $X$ . If  $f$  is a  $z$ -valid factor occurring at position  $j$  of  $X$ , then  $f$  occurs at position  $j$  in some of the  $\mathcal{Z}_k$ 's. The combinatorial observation telling us that this is possible is due to Barton et al [5]. For clarity of presentation we write  $\mathcal{Z}_k$ 's as standard strings.

**Lemma 3 ([5]).** Given a weighted string  $X$  of length  $n$  and a cumulative weight threshold  $1/z \in (0, 1]$ , the  $\lfloor z \rfloor$  *special-weighted strings* of  $X$  can be constructed in time and space  $\mathcal{O}(nz)$ .



**Fact 7** *Given a weighted string  $X$  of length  $n$  and a cumulative weight threshold  $1/z \in (0, 1]$ , we have that  $\mathcal{MP}(X, z) \subseteq \mathcal{MP}(\mathcal{Z}_0, z) \cup \mathcal{MP}(\mathcal{Z}_1, z) \cup \dots \cup \mathcal{MP}(\mathcal{Z}_{\lfloor z \rfloor - 1}, z)$ .*

There are two steps for the correct computation of  $\mathcal{MP}(X, z)$ . First, we compute the set  $\mathcal{A}_k$  of all maximal palindromes of the heavy string of  $\mathcal{Z}_k$ , for all  $0 \leq k < \lfloor z \rfloor$ , using Fact 5. We then need to adjust the radius of each reported palindrome for  $\mathcal{Z}_k$  to ensure that it is  $z$ -valid in  $X$  (the center should not change). To achieve this, we compute an array  $\mathcal{R}_k$ , for each  $\mathcal{Z}_k$ , such that  $\mathcal{R}_k[2c]$  stores the radius of the longest factor at center  $c$  in  $\mathcal{Z}_k$  which is a  $z$ -valid factor of  $X$  at center  $c$ , e.g.  $\mathcal{R}_k[2c] = \frac{j-i+1}{2}$ ,  $c = (i+j)/2$ , if  $\mathcal{Z}_k[i..j]$  is a  $z$ -valid factor of  $X$  centered at  $c$ , and  $\mathcal{Z}_k[i-1..j+1]$  is not a  $z$ -valid factor of  $X$ . By Fact 7, we cannot guarantee that all  $(c, r)$  in  $\mathcal{MP}(\mathcal{Z}_k, z)$  are necessarily in  $\mathcal{MP}(X, z)$ . Hence, the second step is to compute  $\mathcal{MP}(X, z)$  from  $\mathcal{MP}(\mathcal{Z}_k, z)$  by taking the maximum radius per center and filtering out everything else.

**Lemma 4.** *Given a weighted string  $X$  of length  $n$ , a cumulative weight threshold  $1/z \in (0, 1]$ , and the special-weighted strings  $\mathcal{Z}_X$  of  $X$ , each  $\mathcal{R}_k$ ,  $0 \leq k < \lfloor z \rfloor$ , can be computed in time  $\mathcal{O}(n)$ .*

After computing  $\mathcal{A}_k$  and  $\mathcal{R}_k$ , we perform the following check for each palindrome  $(c, r) \in \mathcal{A}_k$ . If  $r > \mathcal{R}_k[2c]$ , the palindrome with radius  $r$  is not  $z$ -valid but the factor with radius  $\mathcal{R}_k[2c]$  is  $z$ -valid and maximal (by definition) and palindromic (by Fact 6); if  $r \leq \mathcal{R}_k[2c]$ , the palindrome with radius  $r$  must be  $z$ -valid and it is maximal. Therefore we set  $(c, r) \in \mathcal{MP}(\mathcal{Z}_k, z)$ , such that  $r = \min\{r, \mathcal{R}_k[2c]\}$ ,  $0 \leq 2c \leq 2n - 2$ , and  $r \geq 1/2$ .

To go from  $\mathcal{MP}(\mathcal{Z}_k, z)$  to  $\mathcal{MP}(X, z)$  we need to take the maximum radius for each center. Therefore for each center  $c/2$ ,  $0 \leq c \leq 2n - 2$ , we set  $(c/2, r) \in \mathcal{MP}(X, z)$ , such that  $r = \max\{r_k \mid (c/2, r_k) \in \mathcal{MP}(\mathcal{Z}_k, z), 0 \leq k < \lfloor z \rfloor\}$ . We thus arrive at the first result of this article.

**Theorem 8.** *Given a weighted string  $X$  of length  $n$  and a cumulative weight threshold  $1/z \in (0, 1]$ , all maximal  $z$ -palindromes in  $X$  can be computed in time and space  $\mathcal{O}(nz)$ .*

After the computation of  $\mathcal{MP}(X, z)$ , we are in a position to apply the algorithm by Alatabbi et al [3] to find the smallest maximal  $z$ -palindromic factorization. We define a list  $\mathcal{F}$  such that  $\mathcal{F}[i]$ ,  $0 \leq i \leq n - 1$ , stores the set of the lengths of all maximal  $z$ -palindromes ending at position  $i$  in  $X$ . We also define a list  $\mathcal{U}$  such that  $\mathcal{U}[i]$ ,  $0 \leq i \leq n - 1$ , stores the set of positions  $j$ , such that  $j + 1$  is the starting position of a maximal  $z$ -palindrome in  $X$  and  $i$  is the ending position of this  $z$ -palindrome. Thus for a given  $\mathcal{F}[i] = \{\ell_0, \ell_1, \dots, \ell_q\}$ , we have that  $\mathcal{U}[i] = \{i - \ell_0, i - \ell_1, \dots, i - \ell_q\}$ . Note that  $\mathcal{U}[i]$  can contain a “ $-1$ ” element if there exists a maximal  $z$ -palindrome starting at position 0 and ending at position  $i$ . Note that the number of elements in  $\mathcal{MP}(X, z)$  is at most  $2n - 1$ , and, hence,  $\mathcal{F}$  and  $\mathcal{U}$  can contain at most  $2n - 2$  elements. The lists  $\mathcal{F}$  and  $\mathcal{U}$  can be computed trivially from  $\mathcal{MP}(X, z)$ . Finally, we define a directed graph  $\mathcal{G}_X = (\mathcal{V}, \mathcal{E})$ , where

$\mathcal{V} = \{i \mid -1 \leq i \leq n-1\}$  and  $\mathcal{E} = \{(i, j) \mid j \in \mathcal{U}[i]\}$ . Note that  $(i, j)$  is a directed edge from  $i$  to  $j$ . We do a breath first search on  $\mathcal{G}_X$  assuming the vertex  $n-1$  as the source and identify the shortest path from  $n-1$  to  $-1$ , which gives the factorization. We formally present the above as Algorithm SMPF for computing a smallest maximal  $z$ -palindromic factorization and obtain the following result.

**Theorem 9.** *Given a weighted string  $X$  of length  $n$  and a cumulative weight threshold  $1/z \in (0, 1]$ , Algorithm SMPF correctly solves the problem SMALLEST MAXIMAL  $z$ -PALINDROMIC FACTORIZATION in time and space  $\mathcal{O}(nz)$ .*

```

1  Algorithm SMPF( $X, n, 1/z$ )
2      Construct the set  $\mathcal{Z}_X$  of special-weighted strings of  $X$ ;
3      foreach  $\mathcal{Z}_k \in \mathcal{Z}_X$  do
4           $\mathcal{A}_k \leftarrow$  maximal palindromes of the heavy string of  $\mathcal{Z}_k$ ;
5          Compute  $\mathcal{R}_k$  for  $\mathcal{Z}_k$ ;
6           $\mathcal{MP}(\mathcal{Z}_k, z) \leftarrow$  EMPTYLIST();
7          foreach  $(c, r) \in \mathcal{A}_k$  do
8               $r \leftarrow \min\{r, \mathcal{R}_k[2c]\}$ ;
9              if  $r \geq \frac{1}{2}$  Insert  $(c, r)$  in  $\mathcal{MP}(\mathcal{Z}_k, z)$ ;
10          $\mathcal{MP}(X, z) \leftarrow$  EMPTYLIST();
11         foreach  $c \in [0, 2n-2]$  do
12              $r \leftarrow \max\{r_k \mid (c/2, r_k) \in \mathcal{MP}(\mathcal{Z}_k, z), 0 \leq k < \lfloor z \rfloor\}$ ;
13             Insert  $(c/2, r)$  in  $\mathcal{MP}(X, z)$ ;
14          $\mathcal{F} \leftarrow$  EMPTYLIST();
15          $\mathcal{U} \leftarrow$  EMPTYLIST();
16         foreach  $(c, r) \in \mathcal{MP}(X, z)$  do
17              $j \leftarrow \lfloor c+r \rfloor$ ;
18             Insert  $2r$  in  $\mathcal{F}[j]$ ;
19             Insert  $j-2r$  in  $\mathcal{U}[j]$ ;
20         Construct directed graph  $\mathcal{G}_X = (\mathcal{V}, \mathcal{E})$ , where  $\mathcal{V} = \{i \mid -1 \leq i \leq n-1\}$ ,
            $\mathcal{E} = \{(i, j) \mid j \in \mathcal{U}[i]\}$  and  $(i, j)$  is a directed edge from  $i$  to  $j$ ;
21         Breadth first search on  $\mathcal{G}_X$  assuming the vertex  $n-1$  as the source;
22         Identify the shortest path  $P \approx \langle n-1 = p_\ell, p_{\ell-1}, \dots, p_2, p_1, p_0 = -1 \rangle$ ;
23         Return  $X[0..p_1], X[p_1+1..p_2], \dots, X[p_{\ell-1}+1..p_\ell]$ ;

```

## 7 Conclusion

In this paper we present a review of recent advances of palindromic factorization.

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