$\mathcal{PT}$-symmetric quantum field theory in D dimensions

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$\mathcal{PT}$-symmetric quantum mechanics began with a study of the Hamiltonian $H = p^2 + x^2(ix)^\varepsilon$. A surprising feature of this non-Hermitian Hamiltonian is that its eigenvalues are discrete, real, and positive when $\varepsilon \geq 0$. This paper examines the corresponding quantum-field-theoretic Hamiltonian $H = \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}\phi^2(i\phi)^\varepsilon$ in D-dimensional spacetime, where ϕ is a pseudoscalar field. It is shown how to calculate the Green's functions as series in powers of ε directly from the Euclidean partition function. Exact finite expressions for the vacuum energy density, all of the connected n-point Green's functions, and the renormalized mass to order ε are derived for $0 \leq D < 2$. For $D \geq 2$ the one-point Green's function and the renormalized mass are divergent, but perturbative renormalization can be performed. The remarkable spectral properties of $\mathcal{PT}$-symmetric quantum mechanics appear to persist in $\mathcal{PT}$-symmetric quantum field theory.

I. INTRODUCTION

The study of $\mathcal{PT}$-symmetric quantum theory may be traced back to a series of papers that proposed a new perturbative approach to scalar quantum field theory. Instead of a conventional perturbation expansion in powers of a coupling constant, it was proposed that a parameter δ that measures the nonlinearity of the theory could be used as a perturbation parameter [1, 2]. Thus, to solve a $g\phi^4$ field theory we studied a $g\phi^2(\phi^2)^\delta$ theory and treated the parameter δ as small. The procedure was to obtain a perturbation expansion in powers of δ and then to set $\delta = 1$ to obtain the results for the $g\phi^4$ theory. Detailed investigation showed that this perturbative calculation is numerically accurate and does not require the coupling constant g to be small [1, 2]. An important feature of this approach was that ϕ^2 and not ϕ had to be raised to the power δ in order to avoid raising a negative quantity to a noninteger power, thereby generating complex numbers as an artifact of the procedure.

This δ expansion was also used to solve nonlinear classical differential equations of physics [3]: the Thomas-Fermi equation (nuclear charge density) $y''(x) = [y(x)]^{3/2}/\sqrt{x}$ becomes $y''(x) = y(x)[y(x)/x]^\delta$; the Lane-Emdon equation (stellar structure) $y''(x) + 2y'(x)/x + [y(x)]^n = 0$ becomes $y''(x) + 2y'(x)/x + [y(x)]^{1+\delta}$; the Blasius equation (fluid dynamics) $y'''(x) + y''(x)y(x) = 0$ becomes $y'''(x) + y''(x)[y(x)]^\delta = 0$; the Korteweg-de Vries equation (nonlinear waves) $u_t + uu_x + u_{xxx} = 0$ becomes $u_t + u^\delta u_x + u_{xxx} = 0$. In each case the quantity raised to the power δ is positive and when $\delta = 0$ the equation becomes linear. Also, these δ expansions have a nonzero radius of convergence and are numerically accurate.

$\mathcal{PT}$-symmetric quantum mechanics began with the surprising discovery that spurious complex numbers do not appear if the quantity raised to the power δ is $\mathcal{PT}$ symmetric (invariant under combined space and time reflection) [4, 5]. This fact is highly nontrivial and was totally unexpected. Indeed, the eigenvalues of the non-Hermitian $\mathcal{PT}$-symmetric Hamiltonian

$$H = p^2 + x^2(ix)^\varepsilon \quad (\varepsilon \geq 0) \tag{1}$$

are entirely real, positive, and discrete when $\varepsilon \geq 0$ because ix is $\mathcal{PT}$ invariant. A proof that the spectrum is real when $\varepsilon > 0$ was given by Dorey, Dunning, and Tateo [6, 7]. Numerous $\mathcal{PT}$-symmetric model Hamiltonians have been studied at a theoretical level [8] and many laboratory experiments have been performed on $\mathcal{PT}$-symmetric physical systems [9–21].

The purpose of this paper is to introduce powerful new tools and techniques that can be used to investigate $\mathcal{PT}$-symmetric quantum field theories. We illustrate these tools by studying the quantum-field-theoretic analog of (1) whose D-dimensional Euclidean-space Lagrangian density is

$$\mathcal{L} = \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}\phi^2(i\phi)^\varepsilon \quad (\varepsilon \geq 0), \tag{2}$$

where ϕ is a pseudoscalar field so that $\mathcal{L}$ is $\mathcal{PT}$ invariant. We treat ε as small and show how to calculate the vacuum energy density E_0, the connected n-point Green's functions G_n, and the renormalized mass $M_{\rm R}$ as series in powers of ε. In this paper we assume that $0 \leq D < 2$ to avoid the appearance of renormalization infinities and then we comment briefly on the perturbative renormalization procedure for the case $D \geq 2$.

To first order in ε ($\varepsilon << 1$), the unusual Lagrangian density $\mathcal{L}$ in (1) has a *logarithmic* self-interaction term:

$$\mathcal{L} = \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}\phi^2 + \tfrac{1}{2}\varepsilon\phi^2\log(i\phi) + \mathrm{O}(\varepsilon^2). \tag{3}$$

For a quantum field theory having a complex logarithmic interaction term it is not obvious whether one can find

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Feynman rules for performing perturbative diagramatic calculations. We will show how to construct such Feynman rules. We begin by replacing the complex logarithm with a real logarithm and we do so in such a way as to preserve $\mathcal{PT}$ symmetry; to wit, we define

$$\log(i\phi) \equiv \tfrac{1}{2}i\pi + \log(\phi) \quad (\text{if } \phi > 0)$$

and we define

$$\log(i\phi) \equiv -\tfrac{1}{2}i\pi + \log(-\phi) \quad (\text{if } \phi < 0).$$

Combining these two equations, we make the replacement

$$\log(i\phi) = \tfrac{1}{2}i\pi\,|\phi|/\phi + \tfrac{1}{2}\log(\phi^2). \tag{4}$$

Note that in (4) the imaginary part is odd in ϕ and the real part is even in ϕ. Thus, (4) enforces the $\mathcal{PT}$ symmetry because the pseudoscalar field ϕ changes sign under parity $\mathcal{P}$ and i changes sign under time reversal $\mathcal{T}$. [To derive (4) we must assume that ϕ is real. The reality of ϕ is explained in Sec. II.]

Graphical techniques were developed in Ref. [1] to handle real logarithmic interaction terms. These techniques are generalizations of the replica trick [22], which has been used in the study of spin glasses. The idea of the replica trick is that a logarithmic term $\log A$ can be reformulated as the limit $\log A = \lim_{N\to 0}\frac{1}{N}\big(A^N - 1\big)$, or equivalently and slightly more simply as the limit

$$\log A = \lim_{N\to 0}\tfrac{d}{dN}A^N. \tag{5}$$

One then regards N as an integer and identifies A^N as an N-point vertex in a graphical expansion. Of course, this procedure is not rigorous because it requires taking the *continuous* limit $N \to 0$. The validity of this approach has not been proved, but when it is possible to compare with exactly known results in low-dimensional theories, the replica trick gives the correct answer. In this paper we verify our field-theoretic results by comparing them with the exact answers for $D = 0$ (where the functional integral becomes an ordinary integral) and for $D = 1$ (quantum mechanics).

The graphical calculations in this paper are done in coordinate space. Once the vertices have been identified, all that one needs is the free propagator in D-dimensional Euclidean space $\Delta(x-y)$, which satisfies the differential equation

$$\big(-\nabla_x^2 + 1\big)\Delta(x-y) = \delta^{(D)}(x-y). \tag{6}$$

Taking the Fourier transform of this equation gives the amplitude for the free propagator of a particle of mass 1 in momentum space:

$$\tilde{\Delta}(p) = 1/(p^2+1).$$

The D-dimensional inverse Fourier transform of this expression then gives the D-dimensional coordinate-space propagator in terms of an associated Bessel function:

$$\Delta(x_1 - x_2) = (2\pi)^{-\frac{D}{2}}\,|x_1 - x_2|^{1-\frac{D}{2}}\,\mathrm{K}_{1-\frac{D}{2}}(|x_1 - x_2|). \tag{7}$$

If we let $x_1 \to x_2$, we obtain the amplitude $\Delta(0)$ for a *self loop*, which is the amplitude for a line to originate from and return to the same point:

$$\Delta(0) = (4\pi)^{-D/2}\Gamma(1 - D/2). \tag{8}$$

This expression is finite and nonsingular for $0 \le D < 2$.

This paper is organized as follows. We calculate the ground-state energy density E_0 in Sec. II, the one-point Green's function G_1 in Sec. III, the two-point Green's function G_2 and the renormalized mass M_{R} in Sec. IV, and the general connected n-point Green's function G_n in Sec. V, all to first order in ε. These quantities are finite when $0 \le D < 2$, but the three quantities E_0, G_1, and M_{R} diverge when $D \ge 2$ so it is necessary to introduce a renormalization procedure. In Sec. VI we discuss the issues of renormalization. We show that a redefinition of the energy scale, an additive shift in the field, and a mass counterterm eliminate these infinities. In this section we also discuss our future calculational objectives, namely, calculating the Green's functions to higher order in ε.

II. FIRST-ORDER CALCULATION OF THE GROUND-STATE ENERGY DENSITY

If we expand the partition function

$$Z(\varepsilon) = \int \mathcal{D}\phi\, e^{-\int d^D x\,\mathcal{L}} \tag{9}$$

for the Lagrangian density $\mathcal{L}$ to first order in ε and use (4), the functional integral (9) becomes

$$Z(\varepsilon) \;=\; \int \mathcal{D}\phi\, e^{-\int d^D x\,\mathcal{L}_0}\bigg(1 - \frac{\varepsilon}{4}\int d^D y\,\big\{i\pi\phi(y)|\phi(y) + \phi^2(y)\log\big[\phi^2(y)\big]\big\} + \mathrm{O}(\varepsilon^2)\bigg), \tag{10}$$

where the free Lagrangian density

$$\mathcal{L}_0 = \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}\phi^2 \tag{11}$$

is obtained by setting $\varepsilon = 0$ in $\mathcal{L}$. Note that the imaginary part of the functional integrand in (10) is odd in ϕ, so $Z(\varepsilon)$ is *real*.

We emphasize that the functional integration in (10) is performed along the *real*-ϕ axis and not in the complex-ϕ domain. This justifies the use of (4). We are not concerned here with complex functional integration paths that terminate in complex Stokes sectors because the functional integral (10) converges term-by-term in powers of ε. In $\mathcal{PT}$-symmetric quantum mechanics the boundary conditions on the Schrödinger equation associated with the Hamiltonian (1) [see (42)] are imposed in complex Stokes sectors [5]. However, in the context of quantum field theory it would be hopelessly unwieldy to consider *functional* Stokes sectors. This is why we treat ε as small. This paper is concerned with calculating the coefficients

in the ε series and we do not consider here the mathematical issues involved with the summation of such a series for large values of ε.

In general, a partition function is the exponential of the ground-state energy density E_0 multiplied by the volume of spacetime V: $Z = e^{-E_0 V}$. Thus, the *shift* in the ground-state energy density ΔE to order ε is given by

$$\Delta E = \frac{\varepsilon}{4Z(0)V}\int \mathcal{D}\phi\, e^{-\int d^D x\, \mathcal{L}_0}\int d^D y\, \phi^2(y)\log\big[\phi^2(y)\big].$$

Hence, from (5) we obtain

$$\Delta E = \lim_{N\to 1}\frac{\varepsilon}{4Z(0)V}\,\frac{d}{dN}\int \mathcal{D}\phi\, e^{-\int d^D x\, \mathcal{L}_0}\int d^D y\, \phi^{2N}(y).$$

This expression has a graphical interpretation as the product of N self loops from the spacetime point y back to y. There are exactly $(2N-1)!!$ ways to construct these self loops, so the expression for ΔE simplifies to

$$\Delta E = \lim_{N\to 1}\frac{\varepsilon}{4V}\,\frac{d}{dN}\int d^D y\,[\Delta(0)]^N(2N-1)!!.$$

Next, we note that the integral $\int d^D y$ is the volume of spacetime V, so this formula simplifies further:

$$\Delta E = \lim_{N\to 1}\frac{\varepsilon}{4}\,\frac{d}{dN}[\Delta(0)]^N(2N-1)!!.$$

Finally, we use the duplication formula for the gamma function [23] to write

$$(2N-1)!! = 2^N\Gamma\big(N+\tfrac{1}{2}\big)/\sqrt{\pi}$$

and then take the derivative with respect to N to get

$$\Delta E = \tfrac{1}{4}\varepsilon\Delta(0)\left\{\log[2\Delta(0)] + \Gamma'\big(\tfrac{3}{2}\big)/\Gamma\big(\tfrac{3}{2}\big)\right\}, \qquad (12)$$

where $\Gamma'\big(\frac{3}{2}\big)/\Gamma\big(\frac{3}{2}\big) = 2-\gamma-2\log 2$. The result in (12) may be verified for the special cases of $D=0$ and $D=1$.

Special case $D=0$: For $D=0$ the normalized partition function becomes an ordinary integral, which we can expand to first order in ε:

$$\begin{aligned} Z &= \frac{1}{Z_0}\int_{-\infty}^{\infty} d\phi\, e^{-\frac{1}{2}\phi^2(i\phi)^\varepsilon} \\ &\sim \frac{1}{Z_0}\int_{-\infty}^{\infty} d\phi\, e^{-\frac{1}{2}\phi^2}\big[1-\tfrac{1}{2}\varepsilon\phi^2\log(i\phi)\big], \end{aligned} \qquad (13)$$

where $Z_0 = \sqrt{2\pi}$. If we take the negative logarithm of this result, we obtain the first-order shift in the ground-state energy density

$$\Delta E = \frac{\varepsilon}{2\sqrt{2\pi}}\int_{-\infty}^{\infty} d\phi\, e^{-\frac{1}{2}\phi^2}\phi^2\log(i\phi).$$

We then integrate separately from $-\infty$ to 0 and from 0 to ∞ and combine the two integrals to obtain a single real integral that we evaluate as follows:

$$\begin{aligned} \Delta E &= \frac{\varepsilon}{\sqrt{2\pi}}\int_0^{\infty} d\phi\, e^{-\frac{1}{2}\phi^2}\phi^2\log\phi \\ &= \tfrac{\varepsilon}{4}\big[\Gamma'\big(\tfrac{3}{2}\big)/\Gamma\big(\tfrac{3}{2}\big)+\log 2\big]. \end{aligned} \qquad (14)$$

Taking $D=0$ in (8) gives $\Delta(0)=1$, so (12) reduces exactly to (14) in zero-dimensional spacetime.

Special case $D=1$: In quantum mechanics, ΔE to leading order in ε is the expectation value of the interaction Hamiltonian $H_I = \frac{1}{2}\varepsilon x^2\log(ix)$ [see (3)] in the unperturbed ground-state eigenfunction $\psi_0(x) = \exp\big(-\frac{1}{2}x^2\big)$:

$$\begin{aligned} \Delta E &= \frac{\varepsilon}{2}\int_{-\infty}^{\infty} dx\, e^{-x^2}x^2\log(ix)\Big/\int_{-\infty}^{\infty} dx\, e^{-x^2} \\ &= \tfrac{1}{8}\varepsilon\Gamma'\big(\tfrac{3}{2}\big)/\Gamma\big(\tfrac{3}{2}\big). \end{aligned} \qquad (15)$$

Taking $D=1$ in (8) gives $\Delta(0)=\frac{1}{2}$, so the general result in (12) reduces exactly to (15) in one-dimensional spacetime.

Note that in $\mathcal{PT}$-symmetric quantum mechanics the calculation of expectation values requires the $\mathcal{C}$ operator [8]. However, the $\mathcal{C}$ operator is not needed for any of the calculations in this paper because a functional integral involves *vacuum* expectation values. The vacuum state is an eigenstate of the $\mathcal{C}$ operator with eigenvalue 1, $\mathcal{C}|0\rangle = |0\rangle$, so all reference to $\mathcal{C}$ disappears. This simplification was first pointed out in Ref. [24].

III. FIRST-ORDER CALCULATION OF THE ONE-POINT GREEN'S FUNCTION

The one-point Green's function G_1 is *nonperturbative* in character but it can be calculated by following the approach used above to calculate ΔE. Keeping terms that do not vanish under $\phi\to-\phi$, we evaluate directly the functional-integral representation

$$\begin{aligned} G_1(a) &= -\frac{\varepsilon}{4Z(0)}\int \mathcal{D}\phi\,\phi(a)e^{-\int d^D x\,\mathcal{L}_0} \\ &\quad\times\int d^D y\, i\pi\phi(y)|\phi(y)|, \end{aligned} \qquad (16)$$

where $\mathcal{L}_0$ is the free Euclidean Lagrangian in (11).

We use the integral identity

$$\phi|\phi| = \frac{2}{\pi}\phi^2\int_0^{\infty}\frac{dt}{t}\sin(t\phi) \qquad (17)$$

to replace $\phi(y)|\phi(y)|$ in the functional integral (16) and then we replace $\sin(t\phi)$ by its Taylor series:

$$\sin(t\phi) = \sum_{n=0}^{\infty}\frac{(-1)^n t^{2n+1}}{(2n+1)!}\phi^{2n+1}. \qquad (18)$$

This converts (16) to the product of an infinite sum in n, a one-dimensional integral in t, a D-dimensional integral in y, and a functional integral in ϕ:

$$\begin{aligned} G_1(a) &= -\frac{i\varepsilon}{2}\int_{t=0}^{\infty} dt\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{(2n+1)!}\int d^D y \\ &\quad\times\frac{1}{Z(0)}\int \mathcal{D}\phi\, e^{-\int d^D x\,\mathcal{L}_0}\phi(a)[\phi(y)]^{2n+3}. \end{aligned} \qquad (19)$$

The sum and multiple integrals in (19) may appear to be difficult, but like the calculation of the ground-state energy density, the functional integral in the second line of (19) also has a graphical interpretation; it is merely the product of the free propagator $\Delta(a-y)$ representing a line from a to y multiplied by $n+1$ self loops from y to y, and this product is accompanied by the combinatorial factor $(2n+3)!!$. Thus, the second line in (19) reduces to $(2n+3)!!\Delta(y-a)\Delta^{n+1}(0)$.

This result simplifies further because, as we can see from (6), the D-dimensional integral is trivial: $\int d^D y \Delta(y-a) = 1$. This establishes the translation invariance of G_1. The rest is straightforward:

$$\begin{aligned} G_1 &= -\frac{i\varepsilon}{2}\int_{t=0}^{\infty} dt \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}\Delta^{n+1}(0)(2n+3)!!}{(2n+1)!} \\ &= -\frac{i\varepsilon}{2}\int_{t=0}^{\infty} dt\, \Delta(0)\big[3-\Delta(0)t^2\big]e^{-\frac{1}{2}\Delta(0)t^2} \\ &= -i\varepsilon\sqrt{\pi\Delta(0)/2}. \end{aligned} \tag{20}$$

This expression for G_1 is exact to order ε.

Observe that the expression for G_1 is a *negative imaginary* number. This is precisely what we would expect based on previous studies of classical $\mathcal{PT}$-symmetric systems. The classical trajectories in complex coordinate space of a particle described by the Hamiltonian (1) are left-right symmetric but they lie mostly in the lower-half plane [8]. Thus, the average value of the classical orbits is a negative-imaginary number.

Special case $D = 0$: The one-point Green's function in $D = 0$ is given by the second line in (13) with an additional extra factor of ϕ:

$$G_1 \sim \frac{1}{Z(0)}\int_{-\infty}^{\infty} d\phi\, \phi e^{-\frac{1}{2}\phi^2}\big[1-\tfrac{1}{2}\varepsilon\phi^2\log(i\phi)\big],$$

where $Z(0) = \sqrt{2\pi}$. The integration over the first term in the square brackets vanishes by oddness. We evaluate the contribution of the second term by integrating first from $-\infty$ to 0 and then from 0 to ∞. Combining these two integrals, we obtain

$$G_1 = -i\varepsilon\sqrt{\pi/2}.$$

This result is in exact agreement with the general result in (20) because $\Delta(0) = 1$ when $D = 0$.

Special case $D = 1$: When $D = 1$, the expression for G_1 in (20) reduces to

$$G_1 = -\tfrac{1}{2}i\varepsilon\sqrt{\pi}. \tag{21}$$

In the Appendix we derive G_1 in quantum mechanics and verify (21).

IV. FIRST-ORDER CALCULATION OF THE TWO-POINT GREEN'S FUNCTION

To obtain the *connected* two-point Green's function $G_2(a,b)$, one must subtract G_1^2 from the vacuum expectation value of $\phi(a)\phi(b)$. However, we have seen that G_1 is of order ε. Therefore, to first order in ε we need only evaluate $Z(\varepsilon)$ in (10) with $\phi(a)\phi(b)$ inserted after $\mathcal{D}\phi$ and then divide this integral by $Z(\varepsilon)$. We may neglect the imaginary terms in these integrals because they are odd under $\phi \to -\phi$. Thus, the expression that we must evaluate for $G_2(a,b)$ is

$$\frac{\int \mathcal{D}\phi\, \phi(a)\phi(b)e^{-\int d^D x\, \mathcal{L}_0}\left\{1-\frac{\varepsilon}{4}\int d^D y\, \phi^2(y)\log\left[\phi^2(y)\right]\right\}}{\int \mathcal{D}\phi\, e^{-\int d^D x\, \mathcal{L}_0}\left\{1-\frac{\varepsilon}{4}\int d^D y\, \phi^2(y)\log\left[\phi^2(y)\right]\right\}}.$$

We expand this expression to first order in ε as a sum of three functional integrals:

$$G_2(a,b) = A + B + C, \tag{22}$$

where

$$\begin{aligned} A &= \frac{1}{Z(0)}\int \mathcal{D}\phi\, \phi(a)\phi(b)e^{-\int d^D x\, \mathcal{L}_0}, \\ B &= -\frac{\varepsilon}{4Z(0)}\int \mathcal{D}\phi\, \phi(a)\phi(b)e^{-\int d^D x\, \mathcal{L}_0} \\ &\quad\times \int d^D y\, \phi^2(y)\log\left[\phi^2(y)\right], \\ C &= \frac{\varepsilon}{4Z(0)}\int \mathcal{D}\phi\, \phi(a)\phi(b)e^{-\int d^D x\, \mathcal{L}_0} \\ &\quad\times\frac{1}{Z(0)}\int \mathcal{D}\phi\, e^{-\int d^D x\, \mathcal{L}_0}\int d^D y\, \phi^2(y)\log\left[\phi^2(y)\right]. \end{aligned} \tag{23}$$

We must now evaluate the three contributions A, B, and C. The functional integral A in (23) is simply the free propagator $\Delta(a-b)$. This result verifies that if we set $\varepsilon = 0$ in the Lagrangian (2), we obtain a free field theory; the two-point Green's function for such a field theory is $\Delta(a-b)$.

The double functional integral C presents a complication. The first line of C is proportional to A and evaluates to $\frac{1}{4}\varepsilon\Delta(a-b)$. However, as we showed in the calculation of the ground-state energy, the next two lines of C evaluate to $\frac{4}{\varepsilon}V\Delta E$, where ΔE is the first-order shift in the ground-state energy density (12) and V is the volume of Euclidean spacetime. Thus, $C = \Delta(a-b)V\Delta E$, and this quantity is *divergent* because V is infinite.

We resolve this divergence problem by calculating B. Using (5) we express the B integral as

$$\begin{aligned} B &= -\frac{\varepsilon}{4Z(0)}\lim_{N\to 1}\frac{d}{dN}\int \mathcal{D}\phi\, e^{-\int d^D x\, \mathcal{L}_0} \\ &\quad\times\int d^D y\, \phi(a)\phi(b)\phi^{2N}(y). \end{aligned} \tag{24}$$

This functional integral requires that we connect in a pairwise fashion the set of $2N+2$ points consisting of a, b, and the $2N$ points y with the free propagator Δ in (7).

There are two cases to consider. In the first case a is connected to b and the remaining $2N$ points at y are connected in pairs. Note that this reproduces the result above for C except with the opposite sign. Thus, the volume divergences exactly cancel.

In the second case a is not connected to b. Instead, a connects to a point y (there are $2N$ ways to do this) and b connects to one of the remaining $2N-1$ points y (there are $2N-1$ ways to do this). The rest of the $2N-2$ points y are joined in pairs [there are $(2N-3)!!$ ways to do this]. The amplitude for this case is

$$-\frac{\varepsilon}{2}\int d^D y\,\Delta(a-y)\Delta(y-b)\lim_{N\to 1}\frac{d}{dN}N(2N-1)!!\Delta^{N-1}(0),$$

which simplifies to $-\varepsilon K\int d^D y\,\Delta(a-y)\Delta(y-b)$, where

$$\begin{aligned} K &= \tfrac{1}{2}+\tfrac{1}{2}\Gamma'\big(\tfrac{3}{2}\big)/\Gamma\big(\tfrac{3}{2}\big)+\tfrac{1}{2}\log[2\Delta(0)] \\ &= \tfrac{3}{2}-\tfrac{1}{2}\gamma+\tfrac{1}{2}\log\big[\tfrac{1}{2}\Delta(0)\big]. \end{aligned} \tag{25}$$

Thus, our final result for the coordinate-space two-point Green's function to order ε is

$$G_2(a-b)=\Delta(a-b)-\varepsilon K\int d^D y\,\Delta(a-y)\Delta(y-b). \tag{26}$$

In momentum space this is

$$\tilde{G}_2(p)=\frac{1}{p^2+1}-\varepsilon\frac{K}{(p^2+1)^2}+\mathrm{O}\big(\varepsilon^2\big). \tag{27}$$

From (27) we construct the $(0,1)$ Padé approximant, which is just the geometric sum of a chain of bubbles:

$$\tilde{G}_2(p)=\frac{1}{p^2+1+\varepsilon K+\mathrm{O}\big(\varepsilon^2\big)}. \tag{28}$$

We then read off the square of the renormalized mass to first order in ε:

$$M_{\mathrm{R}}^2=1+K\varepsilon+\mathrm{O}\big(\varepsilon^2\big). \tag{29}$$

Special case $D=0$: To verify (28) and (29) for the case $D=0$ we evaluate the ordinary one-dimensional integrals in

$$\begin{aligned} G_2 &= \frac{\int dx\,x^2 e^{-x^2(ix)^\varepsilon/2}}{\int dx\,e^{-x^2(ix)^\varepsilon/2}} \\ &= \frac{\int_0^\infty dx\,x^2 e^{-x^2/2}\left[1-\frac{\varepsilon}{4}x^2\log(x^2)\right]}{\int_0^\infty dx\,e^{-x^2/2}\left[1-\frac{\varepsilon}{4}x^2\log(x^2)\right]}. \end{aligned}$$

These integrals are not difficult and to order ε we get

$$G_2=\frac{1}{1+\varepsilon\big(\frac{3}{2}-\frac{1}{2}\gamma-\frac{1}{2}\log 2\big)}. \tag{30}$$

This result agrees exactly with that in (28) for $D=0$.

Special case $D=1$: We verify (29) for the case $D=1$ by calculating the energy level of the *first* excited state of the Hamiltonian H in (1). When $\varepsilon=0$, H becomes the harmonic-oscillator Hamiltonian, so the first excited state eigenfunction is $xe^{-x^2/2}$ and the associated eigenvalue is $\frac{3}{2}$. We solve the time-independent Schrödinger equation $H\psi=E\psi$ perturbatively by substituting

$$\begin{aligned} \psi(x) &= xe^{-x^2/2}+\varepsilon\psi_1(x)+\mathrm{O}\big(\varepsilon^2\big), \\ E &= \tfrac{3}{2}+\varepsilon E_1+\mathrm{O}\big(\varepsilon^2\big), \end{aligned} \tag{31}$$

and collecting powers of ε. To first order in ε the function $\psi_1(x)$ satisfies

$$\begin{aligned} &-\tfrac{1}{2}\psi_1''(x)+\tfrac{1}{2}x^2\psi_1(x)-\tfrac{3}{2}\psi_1(x) \\ &\quad=\big[-\tfrac{1}{2}x^2\log(ix)+E_1\big]\,xe^{-x^2/2}. \end{aligned}$$

To solve this equation we use the technique of reduction of order and let $\psi_1(x)=xe^{-x^2/2}f(x)$. The equation for $f(x)$ is then

$$xf''(x)+\big(2-2x^2\big)f(x)=-2E_1x+x^3\log(ix).$$

Multiplying this equation by the integrating factor $x\exp\big(-x^2\big)$ gives

$$\frac{d}{dx}\left[x^2f'(x)e^{-x^2}\right]=\big[-2E_1x^2+x^4\log(ix)\big]\,e^{-x^2}.$$

Therefore, if we integrate this equation from $-\infty$ to ∞, we obtain an equation for E_1:

$$E_1=\frac{1}{4}\int_{-\infty}^{\infty}dx\,x^4\log\big(x^2\big)e^{-x^2}\Big/\int_{-\infty}^{\infty}dx\,x^2e^{-x^2},$$

where we have replaced $\log(ix)$ by $\frac{1}{2}\log\big(x^2\big)$. These integrals are easy to evaluate and we get

$$E_1=\tfrac{3}{8}\left(\tfrac{8}{3}-\gamma-2\log 2\right).$$

Thus, the first excited eigenvalue of H to order ε is

$$\tfrac{3}{2}+\tfrac{3}{8}\varepsilon\left(\tfrac{8}{3}-\gamma-2\log 2\right). \tag{32}$$

We have already calculated the ground-state energy in Sec. II:

$$\tfrac{1}{2}+\tfrac{1}{8}\varepsilon(2-\gamma-2\log 2). \tag{33}$$

The renormalized mass M_{R} is the first excitation above the ground state so M_{R} is the difference of these two energies:

$$M_{\mathrm{R}}=1+\tfrac{1}{4}\varepsilon(3-\gamma-2\log 2). \tag{34}$$

If we square this result and keep terms of order ε, we get

$$M_{\mathrm{R}}^2=1+\varepsilon\left(\tfrac{3}{2}-\tfrac{1}{2}\gamma-\log 2\right), \tag{35}$$

which exactly reproduces (29) for the case $D=1$.

V. HIGHER-ORDER GREEN'S FUNCTIONS

The connected three-point Green's function is given by the cumulant

$$\begin{aligned} G_3(x,y,z) &= \frac{\langle\phi(x)\phi(y)\phi(z)\rangle}{Z}-\frac{\langle\phi(x)\phi(y)\rangle\langle\phi(z)\rangle}{Z^2} \\ &\quad-\frac{\langle\phi(x)\phi(z)\rangle\langle\phi(y)\rangle}{Z^2}-\frac{\langle\phi(y)\phi(z)\rangle\langle\phi(x)\rangle}{Z^2} \\ &\quad+2\frac{\langle\phi(x)\rangle\langle\phi(y)\rangle\langle\phi(z)\rangle}{Z^3}. \end{aligned}$$

However, to order ε

$$G_3(x,y,z) = \frac{\langle\phi(x)\phi(y)\phi(z)\rangle}{Z} - \Delta(x-y)G_1 - \Delta(x-z)G_1 - \Delta(y-z)G_1.$$

The calculation of G_3 is somewhat tedious, but the procedure follows exactly the calculation of the two-point Green's function. The final result after the disconnected terms have canceled is

$$G_3(x,y,z) = -i\frac{\varepsilon\sqrt{\pi}}{\sqrt{2\Delta(0)}} \times \int d^D u\, \Delta(x-u)\Delta(y-u)\Delta(z-u). \quad (36)$$

The connected four-point function is defined by the cumulant

$$\begin{aligned} G_4(x,y,z,w) &= -\frac{\langle\phi(x)\phi(y)\phi(z)\phi(w)\rangle}{Z} \\ &-\frac{\langle\phi(x)\phi(y)\phi(z)\rangle\langle\phi(w)\rangle}{Z^2} - (\text{three permutations}) \\ &-\frac{\langle\phi(x)\phi(y)\rangle\langle\phi(z)\phi(w)\rangle}{Z^2} - (\text{two permutations}) \\ &+2\frac{\langle\phi(x)\phi(y)\rangle\langle\phi(z)\rangle\langle\phi(w)\rangle}{Z^3} + (\text{five permutations}) \\ &-6\frac{\langle\phi(x)\rangle\langle\phi(y)\rangle\langle\phi(z)\rangle\langle\phi(w)\rangle}{Z^4}. \end{aligned}$$

Again, calculating G_4 is tedious but the result is simply

$$G_4(x,y,z,w) = -\frac{\varepsilon}{\Delta(0)}\int d^D u\, \Delta(x-u) \times\Delta(y-u)\Delta(z-u)\Delta(w-u). \quad (37)$$

The pattern is now evident and with some effort we can calculate the connected n-point Green's function to order ε and obtain a general formula valid for all $n \neq 2$:

$$G_n(x_1,x_2,\ldots x_n) = -\tfrac{1}{2}\varepsilon(-i)^n\Gamma\big(\tfrac{1}{2}n-1\big) \times\big[\tfrac{1}{2}\Delta(0)\big]^{1-n/2}\int d^D u\, \prod_{k=1}^{n}\Delta\big(x_k-u\big). \quad (38)$$

Thus, the connected Green's functions are all of order ε except for G_2 in (26), which is of order 1. Observe that (38) reduces to (20), (36), and (37) for $n = 1$, 3, and 4.

We emphasize that (38) holds for both odd and even n. Our calculation of the odd-n Green's functions uses the *first* term on the right side of (4) and our calculation proceeds by introducing the integral representation in (17) followed by using the Taylor series in (18). On the other hand, our calculation of the even-n Green's functions uses the *second* term on the right side of (4) and the calculation proceeds by applying the derivative identity in (5). It is satisfying that these two strikingly different techniques lead to the single universal formula in (38) for G_n.

VI. DISCUSSION AND FUTURE WORK

The principal advance reported in this paper is that we have developed all of the machinery necessary to calculate the Green's functions of a $\mathcal{PT}$-symmetric quantum field theory in (2) to any order in ε. Thus, this paper opens a vast area for future study and investigation; one can investigate the masses of the theory (the poles of the Green's functions), scattering amplitudes, critical indices, and so on. The Green's-function calculations done in Secs. II-V are exact to first order in ε. However, the procedures presented here immediately generalize to all orders in ε.

Furthermore, as we show below, even in low orders the perturbation series in powers of ε is highly accurate and it continues to be accurate for large ε. To illustrate, we calculate the one-point Green's function G_1 in $D = 0$ to *second* order in ε. In $D = 0$ this Green's function is a ratio of two ordinary integrals:

$$G_1 = \frac{\int_{-\infty}^{\infty} dx\, x\, \exp\left[-\frac{1}{2}x^2\left(1+\varepsilon L+\frac{1}{2}\varepsilon^2 L^2\right)\right]}{\int_{-\infty}^{\infty} dx\, \exp\left[-\frac{1}{2}x^2\left(1+\varepsilon L\right)\right]}, \quad (39)$$

where $L = \log(ix) = \frac{1}{2}i\pi|x|/x + \frac{1}{2}\log\left(x^2\right)$. Evaluating these integrals is straightforward and the result is

$$\begin{aligned} G_1 &= -i\varepsilon\sqrt{\tfrac{\pi}{2}}\left[1+\tfrac{1}{4}\varepsilon\left(\gamma-2-3\log 2\right)+\mathrm{O}\left(\varepsilon^2\right)\right] \\ &= -i\varepsilon\sqrt{\tfrac{\pi}{2}}\left[1-0.8756\varepsilon+\mathrm{O}\left(\varepsilon^2\right)\right]. \end{aligned} \quad (40)$$

To check of the accuracy of (40) we calculate the one-point Green's function for a cubic theory ($\varepsilon = 1$). We convert the expansion in (40) to a $[0,1]$ Padé approximant,

$$G_1 = -i\varepsilon\sqrt{\tfrac{\pi}{2}}\frac{1}{1+0.8756\varepsilon},$$

and then set $\varepsilon = 1$ to obtain the approximate result $G_1 = -0.6682i$.

The exact value of G_1 for the zero-dimensional cubic theory $\varepsilon = 1$ is given by the ratio of integrals

$$G_1 = \frac{\int_{-\infty}^{\infty} dx\, x \exp\left(-\frac{1}{2}ix^3\right)}{\int_{-\infty}^{\infty} dx\, \exp\left(-\frac{1}{2}ix^3\right)} = -i2^{\frac{1}{3}}\frac{\Gamma(2/3)}{\Gamma(1/3)} = -0.6369i.$$

Thus, the *two*-term ε expansion (40) has an accuracy of 5%, which is impressive for such a large value of ε. This good result is consistent with the results found in previous studies of the accuracy of the ε expansion for various classical equations (see Ref. [3]).

The third-order version of (40) is

$$\begin{aligned} G_1 &= -i\varepsilon\sqrt{\tfrac{\pi}{2}}\left[1+\tfrac{1}{4}\varepsilon(\gamma-2-3\log 2)\right. \\ &\quad +\tfrac{1}{192}\varepsilon^2\left(54\log^2 2+144\log 2-36\gamma\log 2\right. \\ &\quad \left.\left.-\pi^2+6\gamma^2-48\gamma+48\right)+\mathrm{O}\left(\varepsilon^3\right)\right] \\ &= -i\varepsilon\sqrt{\tfrac{\pi}{2}}\left[1-0.8756\varepsilon+0.6447\varepsilon^2+\mathrm{O}\left(\varepsilon^3\right)\right]. \end{aligned}$$

Converting the expansion above to a $[1,1]$ Padé approximant and setting $\varepsilon = 1$, we obtain the result that $G_1 = -0.6213i$, which now differs from the exact result by only -2%. These numerical results strongly motivate us to extend our studies of the ε expansion of $\mathcal{PT}$-symmetric quantum field theories to higher order in ε. We will publish the higher-order calculations in a future paper.

A second issue that that needs to be examined in depth is that of renormalization. Because $\Delta(0)$ becomes singular when the dimension D of Euclidean spacetime reaches 2, the one-point Green's function G_1 and the renormalized mass $M_{\rm R}$ become singular. (To first order in ε the higher Green's functions do not become infinite when $D = 2$.) Thus, for $D \geq 2$ we must undertake a perturbative renormalization procedure.

For simplicity, in this paper we have worked entirely with dimensionless quantities. However, to carry out a perturbative renormalization one must work with the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}\mu^2\phi^2 + \tfrac{1}{2}g\mu_0^2\phi^2\bigl(i\mu_0^{1-D/2}\phi\bigr)^\varepsilon$$

for which the dimensional parameters are explicit: μ is the unrenormalized mass μ, μ_0 is a fixed parameter having dimensions of mass, and g is a dimensionless unrenormalized coupling constant. The mass renormalization procedure consists of expressing the renormalized mass $M_{\rm R}$ in terms of these Lagrangian parameters and absorbing the divergence that arises when $D \geq 2$ into parameter μ. The coupling-constant renormalization procedure is similar; we define the renormalized coupling constant $G_{\rm R}$ as the value of the three-point or four-point Green's functions at particular values of the external momentum and again absorb the divergence that arises into the Lagrangian parameter g. One must then verify that all higher Green's functions are finite when expressed in terms of $M_{\rm R}$ and $G_{\rm R}$. This program will be carried out explicitly in a future paper.

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VII. APPENDIX

In this Appendix we calculate the one-point Green's function G_1 for the case $D = 1$ (quantum mechanics). In quantum mechanics G_1 is the expectation value of the operator x in the ground state $|0\rangle$. Thus, if $\psi(x)$ is the (unnormalized) ground-state eigenfunction in coordinate space, we can express G_1 as the ratio of integrals

$$G_1 \equiv \frac{\langle 0|x|0\rangle}{\langle 0|0\rangle} = \int dx\, x\psi^2(x) \Big/ \int dx\, \psi^2(x). \tag{41}$$

For the quantum-mechanical Hamiltonian (1) the ground-state eigenfunction obeys the time-independent Schrödinger equation

$$-\tfrac{1}{2}\psi''(x) + \tfrac{1}{2}x^2(ix)^\varepsilon\psi(x) = E\psi(x), \tag{42}$$

where E is the ground-state energy. To first-order in ε this differential equation becomes

$$-\tfrac{1}{2}\psi''(x) + \tfrac{1}{2}x^2\psi(x) + \tfrac{1}{2}\varepsilon x^2\log(ix)\psi(x) = E\psi(x). \tag{43}$$

We solve this equation perturbatively by substituting

$$\begin{aligned}\psi(x) &= \psi_0(x) + \varepsilon\psi_1(x) + \mathrm{O}\bigl(\varepsilon^2\bigr),\\ E &= E_0 + \varepsilon E_1 + \mathrm{O}\bigl(\varepsilon^2\bigr),\end{aligned} \tag{44}$$

where $\psi_0(x) = \exp\bigl(-\tfrac{1}{2}x^2\bigr)$, $E_0 = \tfrac{1}{2}$, and as derived in (15), $E_1 = \frac{1}{8}\Gamma'(\frac{3}{2})/\Gamma\bigl(\frac{3}{2}\bigr)$. Collecting powers in ε, we see that $\psi_0(x)$ and $\psi_1(x)$ satisfy the differential equations

$$\begin{aligned}-\tfrac{1}{2}\psi_0''(x) + \tfrac{1}{2}x^2\psi_0(x) - \tfrac{1}{2}\psi_0(x) &= 0,\\ -\tfrac{1}{2}\psi_1''(x) + \tfrac{1}{2}x^2\psi_1(x) - \tfrac{1}{2}\psi_1(x)&\\ = -\tfrac{1}{2}x^2\log(ix)\psi_0(x) + E_1\psi_0(x).&\end{aligned} \tag{45}$$

The equation for $\psi_1(x)$ is an inhomogenous version of the equation for $\psi_0(x)$. Thus, we use reduction of order to solve the $\psi_1(x)$ equation; we substitute

$$\psi_1(x) = \psi_0 f(x) = e^{-\frac{1}{2}x^2} f(x) \tag{46}$$

and obtain

$$f''(x) - 2xf'(x) = x^2\log(ix) - 2E_1.$$

We then multiply this equation by the integrating factor $\exp(-x^2)$ and integrate from $-\infty$ to x:

$$f'(x)e^{-x^2} = C + \int_{-\infty}^{x} ds\,\bigl[s^2\log(is) - 2E_1\bigr]e^{-s^2}.$$

The integration constant C vanishes because

$$2E_1 = \int_{-\infty}^{\infty} ds\, s^2\log(is)e^{-s^2} \Big/ \int_{-\infty}^{\infty} ds\, e^{-s^2}.$$

Thus, we find that

$$\psi_1(x) = e^{-\frac{1}{2}x^2}\int_0^x dt\, e^{t^2}\int_{-\infty}^{t} ds\bigl[s^2\log(is) - 2E_1\bigr]e^{-s^2}. \tag{47}$$

We can now evaluate the integrals in (41) to first order in ε. This expression simplifies considerably because of the factor of x:

$$\begin{aligned}G_1 &= -\frac{2\varepsilon}{\sqrt{\pi}}\int_{-\infty}^{\infty} dx\, xe^{-x^2}\int_0^x dt\, e^{t^2}\\ &\quad\times\int_t^{\infty} ds\bigl[s^2\log(is) - 2E_1\bigr]e^{-s^2}.\end{aligned} \tag{48}$$

To evaluate this integral we integrate from $x = -\infty$ to 0 and then from $x = 0$ to ∞ and combine the two integrals. The resulting integral simplifies further because the logarithm in the integrand collapses to $\log(-1) = i\pi$:

$$G_1 = -2i\varepsilon\sqrt{\pi}\int_0^\infty dx\, xe^{-x^2}\int_0^x dt\, e^{t^2}\int_t^\infty ds\, s^2 e^{-s^2}. \quad (49)$$

Evaluating this triple integral is not trivial, but by using some tricks it can be calculated exactly and in closed form. We begin by interchanging the order of the s and t integrals:

$$G_1 = -2i\varepsilon\sqrt{\pi}\int_0^\infty dx\, xe^{-x^2}\int_0^\infty ds\, s^2 e^{-s^2}\int_0^{\min(s,x)} dt\, e^{t^2}. \quad (50)$$

Next, we introduce polar coordinates $x = r\cos\theta$ and $s = r\sin\theta$ and make the change of variable $t = rz$:

$$G_1 = -2i\varepsilon\sqrt{\pi}\int_0^\infty dr\, r^5\int_0^{\pi/2} d\theta\, \cos\theta\sin^2\theta \times\int_0^{\min(\sin\theta,\cos\theta)} dz\, e^{r^2z^2-r^2}. \quad (51)$$

The r integral can now be done, and we obtain a sum of two double integrals:

$$G_1 = -2i\varepsilon\sqrt{\pi}\left[\int_0^{\pi/4} d\theta\, \cos\theta\sin^2\theta\int_0^{\sin\theta} dz\frac{1}{(1-z^2)^3} + \int_{\pi/4}^{\pi/2} d\theta\, \cos\theta\sin^2\theta\int_0^{\cos\theta} dz\frac{1}{(1-z^2)^3}\right]. \quad (52)$$

These double integrals may be evaluated by using any algebraic manipulation code such as Mathematica. The final result is

$$G_1 = -\tfrac{1}{2}i\varepsilon\sqrt{\pi}. \quad (53)$$

This verifies the general result in (20) for the case $D = 1$.

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