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A geometric dual of c -extremization

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ABSTRACT: We consider supersymmetric $\text{AdS}_3 \times Y_7$ and $\text{AdS}_2 \times Y_9$ solutions of type IIB and $D = 11$ supergravity, respectively, that are holographically dual to SCFTs with $(0, 2)$ supersymmetry in two dimensions and $\mathcal{N} = 2$ supersymmetry in one dimension. The geometry of Y_{2n+1} , which can be defined for $n \geq 3$, shares many similarities with Sasaki-Einstein geometry, including the existence of a canonical R-symmetry Killing vector, but there are also some crucial differences. We show that the R-symmetry Killing vector may be determined by extremizing a function that depends only on certain global, topological data. In particular, assuming it exists, for $n = 3$ one can compute the central charge of an $\text{AdS}_3 \times Y_7$ solution without knowing its explicit form. We interpret this as a geometric dual of c -extremization in $(0, 2)$ SCFTs. For the case of $\text{AdS}_2 \times Y_9$ solutions we show that the extremal problem can be used to obtain properties of the dual quantum mechanics, including obtaining the entropy of a class of supersymmetric black holes in AdS_4 . We also study many specific examples of the type $\text{AdS}_3 \times T^2 \times Y_5$, including a new family of explicit supergravity solutions. In addition we discuss the possibility that the $(0, 2)$ SCFTs dual to these solutions can arise from the compactification on T^2 of certain $d = 4$ quiver gauge theories associated with five-dimensional Sasaki-Einstein metrics and, surprisingly, come to a negative conclusion.

KEYWORDS: AdS-CFT Correspondence, Differential and Algebraic Geometry, Supersymmetric Gauge Theory, Black Holes in String Theory

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1 Introduction

The class of supersymmetric conformal field theories (SCFTs) in two spacetime dimensions that preserve $(0, 2)$ supersymmetry share several similarities with the class of $\mathcal{N} = 1$ SCFTs in four spacetime dimensions. For example, in both cases the SCFTs possess a continuous abelian R-symmetry which determines exact results about the spectrum of operators. Furthermore, the R-symmetry can be obtained, in rather general circumstances, by solving a variational problem whereby one extremizes a certain functional over the space of possible R-symmetries. In $d = 2$ this procedure, known as c -extremization [1], also yields the right moving central charge, c_R , of the SCFT. This is the direct analogue of the procedure in $d = 4$, known as a -maximization [2], which yields the a central charge.

A principal aim of this paper is to present a geometric dual¹ of c -extremization for the class of $d = 2$, $(0, 2)$ SCFTs with holographic duals given by $\text{AdS}_3 \times Y_7$ solutions of type IIB supergravity with only five-form flux [5]. This result can be viewed as the analogue of the geometric dual of a -maximization that was presented in [6, 7], for the general class of $d = 4$, $\mathcal{N} = 1$ SCFTs which are holographically dual to $\text{AdS}_5 \times SE_5$ solutions of type IIB supergravity, where SE_5 is a five-dimensional Sasaki-Einstein manifold.

Before discussing the $\text{AdS}_3 \times Y_7$ solutions of interest further, we first recall some salient aspects of the story involving $\text{AdS}_5 \times SE_5$ solutions. These solutions, which also have only five-form flux, arise from D3-branes sitting at the apex of the Calabi-Yau cone whose link (i.e. cross section) is the SE_5 manifold. The R-symmetry of the dual field theory is geometrically realized as a nowhere vanishing “Reeb” Killing vector on SE_5 . In the geometric dual of a -maximization discussed in [6, 7], one first goes “off-shell” by fixing the complex structure of the cone and then considering the more general class of compatible Sasaki metrics on the link of this cone. It is only when the Sasaki metric is also Einstein that the type IIB equations of motion are satisfied. It was shown that the Reeb vector field for the Sasaki-Einstein metric can be obtained by minimizing the normalized volume of the Sasaki manifold as a functional on the space of possible Reeb vector fields. An interesting corollary of this extremal problem is that the normalized volumes of Sasaki-Einstein manifolds, and hence the a central charges of the dual SCFTs in $d = 4$, are necessarily algebraic numbers.

There is also a parallel story that involves $\text{AdS}_4 \times SE_7$ solutions of eleven-dimensional supergravity, where SE_7 is a $d = 7$ Sasaki-Einstein manifold, and the four-form flux is purely electric. In this case the dual field theories are three-dimensional $\mathcal{N} = 2$ SCFTs that arise from membranes lying at the apex of the Calabi-Yau cone with link SE_7 . These SCFTs again have an abelian R-symmetry, which is now determined by extremizing the supersymmetric free energy, namely minus the logarithm of a supersymmetric partition function [8]. The volume minimization of [6, 7], which is valid for Sasaki-Einstein manifolds of arbitrary odd dimension greater than three, also provides a geometric description of this field theory variational problem.

We now return to the general class of supersymmetric $\text{AdS}_3 \times Y_7$ solutions of type IIB supergravity that will be the main focus of this paper. This class of solutions was first discussed in [5] and the geometry of Y_7 was further elucidated in [9]. As one might expect, the compact manifolds Y_7 share several similarities with Sasaki-Einstein manifolds. For example, they also have a non-vanishing R-symmetry Killing vector field and the cone over Y_7 is again complex. However, there are some crucial differences; for example, we will show here that the metric on Y_7 can never be Sasakian (in particular the R-symmetry Killing vector is never a Reeb vector). Furthermore, ensuring that the five-form flux is suitably quantized is more involved for the $\text{AdS}_3 \times Y_7$ solutions, as compared to their $\text{AdS}_5 \times SE_5$ cousins.

In presenting the geometric dual of c -extremization for the $\text{AdS}_3 \times Y_7$ solutions of [5], we need to find an appropriate way of going off-shell and, *a priori*, there is not a canonical procedure to do this. The approach we pursue here is to consider a specific class of geometries on a complex cone and admitting certain Killing spinors of the type discussed

¹We note that a connection between c -extremization and three-dimensional gauged supergravity was made in [3], generalizing a similar connection between a -maximization and five-dimensional gauged supergravity in [4].

in [9], but relaxing the equation of motion for the five-form. A key point is that we also need to impose a natural topological constraint in order to ensure that the five-form flux is properly quantized. With this set-up, the link of the cone still has an R-symmetry vector which, moreover, foliates the link with a transversely conformal Kähler metric. A main result of this paper is to show that the central charge c_R of the dual SCFT can be obtained by extremizing a specific functional that depends on the space of R-symmetry vectors as well as the basic cohomology class of the transverse Kähler form.

As we shall see, these complex cone geometries and the related extremal problem can be formulated for all the geometries Y_{2n+1} with $n \geq 3$ introduced in [9]. In particular, the results are also applicable to a class of supersymmetric $\text{AdS}_2 \times Y_9$ solutions of eleven-dimensional supergravity, with only electric four-form flux, introduced in [10], which are holographically dual to superconformal quantum mechanics with two supercharges. We will show that a naturally defined two-dimensional Newton constant, G_2 , is obtained from the extremization principle. Although determining the precise holographic dictionary for AdS_2 is still a work in progress (for some recent discussion and references, see e.g. [11]), one can expect that $1/(4G_2)$ determines the logarithm of the partition function of the dual superconformal quantum mechanics.

For a certain sub-class of $\text{AdS}_2 \times Y_9$ solutions we can also make a connection with recent work on microstate counting of supersymmetric AdS_4 black holes [12–17]. Specifically, we can consider the class of solutions of the form $\text{AdS}_2 \times \Sigma_g \times SE_7$, where Σ_g is a Riemann surface with genus $g > 1$, SE_7 is a Sasaki-Einstein manifold and the SE_7 is fibred over the Σ_g just in the direction of the Reeb vector of the SE_7 . These solutions arise as the near horizon limit of black holes that asymptotically approach $\text{AdS}_4 \times SE_7$ and hence have a clear dual interpretation. Specifically, after compactifying $D = 11$ supergravity on SE_7 one gets an $\mathcal{N} = 2$ SCFT in $d = 3$. One can then further compactify this on the Riemann surface Σ_g , with the addition of R-symmetry magnetic flux (only) on Σ_g ; this is the so-called “universal twist”. It has been shown in [16] that the entropy of the black holes, S_{BH} , is precisely equal to the logarithm of the twisted topological index [18–20] and furthermore, that this is equal to minus the on-shell action of the full AdS_4 black hole solution. A point that we make here is that the entropy is simply related to G_2 via $S_{BH} = 1/(4G_2)$, and hence, for this class of black hole solutions, we can obtain S_{BH} , as well as the twisted topological index, via a novel variational principle.

In section 2 of this paper we will present several new results concerning the general class of odd-dimensional “GK geometries” Y_{2n+1} of [9]. In particular, after significant preparation in earlier subsections, the extremal problem is presented in section 2.5. We will then restrict our attention, in the remainder of the paper, to the special class of geometries with $Y_7 = T^2 \times Y_5$, for which we make additional progress. These are associated with $\text{AdS}_3 \times T^2 \times Y_5$ solutions of type IIB supergravity, our main focus, or $\text{AdS}_2 \times T^4 \times Y_5$ solutions of $D = 11$ supergravity. When $Y_7 = T^2 \times Y_5$ we can show that the central charge c_R of the $\text{AdS}_3 \times T^2 \times Y_5$ solutions can be obtained by resolving a complex cone over Y_5 and then using localization, somewhat analogous to what was achieved in the Sasaki-Einstein case [6, 7]. In addition, we will also prove an interesting obstruction theorem: if the cone over Y_5 is Calabi-Yau, then a supersymmetric solution of the form $\text{AdS}_3 \times T^2 \times Y_5$ with the given complex structure on the cone does not exist.

It is interesting to point out that several infinite classes of explicit $\text{AdS}_3 \times Y_7$ solutions of the type we are considering (and also $\text{AdS}_2 \times Y_9$ solutions) have been known for some time [21–24], but the dual field theories for most of them have not yet been identified. Since the type IIB solutions have only non-trivial five-form flux, it is natural to consider them as arising from configurations involving a large number of D3-branes wrapping a complex submanifold inside a Calabi-Yau four-fold. For certain classes of solutions where Y_7 is the total space of a Y_5 fibration over Σ_g , where Σ_g is a Riemann surface, one can also anticipate that the dual $d = 2$ field theories might arise as the low energy limit of some four-dimensional “parent” SCFTs that have been appropriately compactified on Σ_g . Indeed some interesting progress in this direction has been made in [25, 26].

Here we will critically re-examine the possibility that specific examples of $\text{AdS}_3 \times T^2 \times Y_5$ solutions have $d = 2$ SCFTs duals which arise from compactifications of certain $d = 4$ quiver gauge theories dual to specific $\text{AdS}_5 \times SE_5$ solutions, as discussed in [26]. More precisely, the idea is to consider the quiver gauge theory compactified on T^2 with vanishing flavour flux and non-vanishing baryonic flux on the T^2 . In [26] it was suggested that this possibility is realized for an explicit class of solutions first found in [24], which we label here as $\text{AdS}_3 \times T^2 \times \mathcal{Y}^{p,q}$, and the quiver gauge theories [27] that are dual to the $\text{AdS}_5 \times Y^{p,q}$ solutions [28]. Although some evidence for this was provided in [26], including what seemed to be a remarkable matching of central charges as functions of p, q , here we will show, surprisingly, that this possibility is in fact not realized. It remains an interesting open problem to identify the SCFTs dual to the $\text{AdS}_3 \times T^2 \times \mathcal{Y}^{p,q}$ solutions as well as to determine the fate of the $d = 4$ $Y^{p,q}$ quiver gauge theories after they have been compactified on T^2 , with only baryon flux on the T^2 .

This analysis will be carried out in section 4, where we will also discuss some other examples of $\text{AdS}_3 \times T^2 \times Y_5$ solutions. In particular, we carry out the regularity analysis and also flux quantization for the local explicit solutions presented in [29], obtaining a new class of solutions which we label $\text{AdS}_3 \times T^2 \times \mathcal{L}^{a,b,c}$. The $\mathcal{L}^{a,b,c}$ metrics share several similarities with the $L^{a,b,c}$ Sasaki-Einstein metrics of [30]. However, as in the previous paragraph, there is no direct connection between the $L^{a,b,c}$ quiver gauge theory [31–33] compactified on T^2 and the $\text{AdS}_3 \times T^2 \times \mathcal{L}^{a,b,c}$ solutions.

The plan of the rest of the paper is as follows. In section 2 we set up the general geometric formalism, reviewing and extending the results of [9], and then discuss the variational problem. In section 3 we specialize to $Y_7 = T^2 \times Y_5$ and show, in particular, that the central charge can be obtained via localization. Section 4 discusses several specific classes of solutions with $Y_7 = T^2 \times Y_5$, including $Y_5 = \mathcal{Y}^{p,q}$ and $\mathcal{L}^{a,b,c}$. Section 4 also contains an analysis of quiver gauge theories, dual to certain Sasaki-Einstein spaces, after they have been reduced on T^2 . We conclude in section 5 with some discussion of our results and open questions.

2 General formalism

In this section we study a general class of supersymmetric $\text{AdS}_3 \times Y_7$ solutions of type IIB supergravity and $\text{AdS}_2 \times Y_9$ solutions of $D = 11$ supergravity. The geometric structure of

Y_7 and Y_9 , which share many similarities, was first described in [5] and [10], respectively, and then further clarified and also generalized to higher odd dimensions, Y_{2n+1} , in [9]. Here we will summarize some of these results and also significantly extend them. We introduce the relevant geometry in section 2.1 and then discuss the complex geometry on the cones over Y_{2n+1} in section 2.2. In sections 2.3 and 2.4 we show that both the action and flux quantization conditions for a natural class of off-shell geometries may be written in terms of certain global, topological data. A supersymmetric solution is necessarily a critical point of this action. In section 2.5 we summarize this extremal problem, and show for $n = 3$ that at a critical point the action coincides with the supergravity formula for the central charge of the dual (0, 2) SCFT. For $n = 4$ the critical point determines the value of the two-dimensional Newton constant, G_2 , which is related to the partition function of the dual quantum mechanics and also gives, for a subset of AdS_2 solutions, the entropy of a class of AdS_4 black hole solutions.

2.1 Geometric backgrounds

We consider AdS_3 solutions of type IIB supergravity where the ten-dimensional metric and Ramond-Ramond self-dual five-form F_5 are given by²

$$\begin{aligned} ds_{10}^2 &= L^2 e^{-B/2} (ds_{\text{AdS}_3}^2 + ds_7^2) , \\ F_5 &= -L^4 (\text{vol}_{\text{AdS}_3} \wedge F + *_7 F) . \end{aligned} \tag{2.1}$$

We also consider AdS_2 solutions of $D = 11$ supergravity where the eleven-dimensional metric and four-form G are given by

$$\begin{aligned} ds_{11}^2 &= L^2 e^{-2B/3} (ds_{\text{AdS}_2}^2 + ds_9^2) , \\ G &= L^3 \text{vol}_{\text{AdS}_2} \wedge F . \end{aligned} \tag{2.2}$$

In these expressions L is an overall dimensionful length scale, with $ds_{\text{AdS}_3}^2$ and $ds_{\text{AdS}_2}^2$ being the metrics on a unit radius AdS_3 and AdS_2 , respectively, with corresponding volume forms $\text{vol}_{\text{AdS}_3}$ and $\text{vol}_{\text{AdS}_2}$. In each case the warp factor B is a function on the smooth, compact Riemannian internal space (Y_7, ds_7^2) or (Y_9, ds_9^2) while F is a closed two-form on Y_7 or Y_9 with, in the former case, Hodge dual $*_7 F$.

The ansatz (2.1) is the most general AdS_3 background one can write down that has only metric and five-form turned on. Due to the presence of only five-form flux, they are in some sense D3-brane backgrounds. In particular, if one wraps a large number of D3-branes over a Riemann surface Σ_g , assuming the resulting two-dimensional low-energy theory flows to a CFT in the IR one would expect this to have a holographic dual of the form (2.1). The ansatz (2.1) is the natural analogue of the Freund-Rubin $\text{AdS}_5 \times SE_5$ ansatz, which describes holographic duals of unwrapped D3-brane worldvolume theories in flat space sitting at the apex of the cone over SE_5 . In the latter case, supersymmetry implies that

²This is in agreement with the type IIB conventions in appendix A of [24], and we note that we have a slightly different expression here for the five-form than in equation (3.8) of [9]. We also note that the two-form F_2 appearing in appendix A of [24] (when $n = 1$) for type IIB/ $D = 11$ supergravity is minus/plus that appearing in [9].

SE_5 is a Sasaki-Einstein manifold [34] and is realized when the cone over SE_5 is Calabi-Yau. On the other hand, supersymmetric $AdS_3 \times Y_7$ solutions of the form (2.1), which are dual to SCFTs preserving³ $(0, 2)$ supersymmetry [5], can arise when D3-branes wrap holomorphic cycles in Calabi-Yau four-folds. As we will review below, Y_7 in the supersymmetric $AdS_3 \times Y_7$ solutions share a number of features with Sasaki-Einstein geometry [9].

Similar comments apply to the ansatz (2.2). It is the most general AdS_2 background with purely electric four-form flux and is associated with M2-brane backgrounds. If one wraps a large number of M2-branes over a Riemann surface Σ_g , assuming the resulting one-dimensional low-energy theory flows to a conformal quantum mechanics in the IR one would expect this to have a holographic dual of the form (2.2). It is a natural analogue of the Freund-Rubin $AdS_4 \times SE_7$ ansatz. If we demand supersymmetry, SE_7 is again a Sasaki-Einstein manifold, while supersymmetric $AdS_2 \times Y_9$ solutions are dual to supersymmetric quantum mechanics preserving 2 supercharges⁴ and can arise from membranes wrapping holomorphic curves in a Calabi-Yau five-fold. It is also worth noting that $AdS_2 \times Y_9$ (supersymmetric) solutions with $Y_9 = T^2 \times Y_7$ can be dimensionally reduced to type IIA and then T-dualized to type IIB, and we find precisely the (supersymmetric) $AdS_3 \times Y_7$ solutions in (2.1) [23].

Substituting the ansatz (2.1) and (2.2) into the type IIB and $D = 11$ equations of motion, respectively, gives corresponding equations of motion for the metric ds_{2n+1}^2 on Y_{2n+1} , function B , and local one-form A with curvature $F \equiv dA$. Here $n = 3$ is the type IIB AdS_3 case and $n = 4$ is the $D = 11$ AdS_2 case. As shown in [9], these equations of motion can in turn be derived from the action

$$S = \int_{Y_{2n+1}} e^{(1-n)B} \left[R_{2n+1} - \frac{2n}{(n-2)^2} + \frac{n(2n-3)}{2} (dB)^2 + \frac{1}{4} e^{2B} F^2 \right] \text{vol}_{2n+1}. \quad (2.3)$$

Here we have written an action in general dimension $2n + 1$, with Riemannian manifold Y_{2n+1} having Ricci scalar R_{2n+1} and Riemannian volume form vol_{2n+1} , and $F^2 \equiv F_{ab}F^{ab}$.

Demanding that the $AdS_3 \times Y_7$ and $AdS_2 \times Y_9$ backgrounds preserve supersymmetry leads to the existence of specific Killing spinors on Y_{2n+1} , for $n = 3, 4$, and this implies additional geometric structure. It was shown in [9] that these properties can also be generalized to $n > 4$, so we continue with arbitrary $n \geq 3$. Supersymmetry, by which we now mean the existence of the Killing spinors given in [9], implies that the metric on Y_{2n+1} is equipped with a unit norm Killing vector field ξ , which we call the *R-symmetry vector field*. In local coordinates we write

$$\xi = \frac{1}{c} \partial_z, \quad \text{where } c \equiv \frac{1}{2}(n-2). \quad (2.4)$$

Since ξ is nowhere zero, it defines a foliation \mathcal{F}_ξ of Y_{2n+1} . The metric takes the form

$$ds_{2n+1}^2 = c^2 (dz + P)^2 + e^B ds^2, \quad (2.5)$$

³If one wants to obtain AdS_3 solutions that preserve $(0, 1)$ supersymmetry, one should allow for three-form flux. A concrete brane realization is D3-branes wrapping a holomorphic curve in a Calabi-Yau four-fold, combined with five-branes wrapping a SLAG four-cycle.

⁴If one wants AdS_2 solutions dual to quantum mechanics preserving one supercharge one should add magnetic four-form flux. A concrete brane realization is M5-branes wrapping a SLAG five-cycle inside a Calabi-Yau five-fold, with the option of also having M2-branes wrapping a holomorphic curve [35].

where ds^2 is a Kähler metric, transverse to the foliation \mathcal{F}_ξ . In fact this Kähler metric in real dimension $2n$ determines all of the remaining fields. The function B is fixed via

$$e^B = \frac{c^2}{2}R, \tag{2.6}$$

where R is the Ricci scalar of the transverse Kähler metric. Notice here that we need positive scalar curvature, $R > 0$, in order that the metric (2.5) is well-defined and positive definite.⁵ The local one-form P in (2.5) is the Ricci one-form of the transverse Kähler metric, so that $dP = \rho$ is the Ricci two-form. Finally, the closed two-form is given by

$$F = -\frac{1}{c}J + c d [e^{-B}(dz + P)] , \tag{2.7}$$

where J is the transverse Kähler form.

The above geometric conditions on Y_{2n+1} are equivalent to the existence of a non-trivial solution to the Killing spinor equations given in [9]. Moreover, imposing also the equation of motion for the two-form,

$$d [e^{(3-n)B} *_2 F] = 0 , \tag{2.8}$$

the supersymmetric backgrounds given by (2.5)–(2.7) automatically obey the equations of motion derived from the action (2.3) [9]. On the other hand, it is straightforward to show that (2.8) is equivalent to the PDE

$$\square R = \frac{1}{2}R^2 - R_{ij}R^{ij} , \tag{2.9}$$

where R_{ij} denotes the transverse Ricci tensor, and everything in (2.9) is computed using the transverse Kähler metric. We shall refer to equation (2.9) as the equation of motion in what follows. Moreover, to be clear, we refer to geometries satisfying (2.5)–(2.7) as *supersymmetric geometries* (since they solve the Killing spinor equations), and if the equation of motion (2.9) also holds these are then *supersymmetric solutions*. Thus, the supersymmetric geometries are “off-shell” in a precise sense which we utilize below.

The above supersymmetric geometry is at first glance very similar to Sasakian geometry. In both cases there is a unit norm Killing vector field ξ , which foliates the manifold Y_{2n+1} with a transversely conformal Kähler metric. However, in the present setting the dual one-form

$$\eta \equiv c(dz + P) , \tag{2.10}$$

satisfies $d\eta = c\rho$, where recall that ρ is the transverse Ricci form. On the other hand, in Sasakian geometry instead $d\eta$ is proportional to the transverse Kähler form J , implying that Y_{2n+1} is a contact manifold. In fact in section 2.4 we will see that Sasakian manifolds never solve the equation of motion (2.9), and in this sense the supersymmetric geometry we have is orthogonal to Sasakian geometry.

⁵When $R < 0$ one can obtain [23], after a double wick rotation, supersymmetric solutions of type IIB and $D = 11$ with S^3 and S^2 factors, respectively, that are also of interest, but we will not discuss them further here.

2.2 Complex geometry

Another feature in common with Sasakian geometry is that the real cone over Y_{2n+1} is a complex cone [9]. Here one introduces the $(2n + 2)$ -dimensional cone $C(Y_{2n+1}) \equiv \mathbb{R}_{>0} \times Y_{2n+1}$, equipped with the conical metric

$$ds_{2n+2}^2 = dr^2 + r^2 ds_{2n+1}^2, \tag{2.11}$$

where $r > 0$ is a coordinate on $\mathbb{R}_{>0}$. Notice that we exclude the tip of the cone $r = 0$, so that $C(Y_{2n+1})$ is a smooth manifold. There is a natural compatible $SU(n + 1)$ structure on this cone, with fundamental two-form \mathcal{J} and holomorphic volume form $\Omega_{(n+1,0)}$ given by

$$\begin{aligned} \mathcal{J} &= -c r dr \wedge (dz + P) + r^2 e^B J, \\ \Omega_{(n+1,0)} &= e^{iz} \left(e^{B/2} r \right)^n [dr - i r c (dz + P)] \wedge \Omega. \end{aligned} \tag{2.12}$$

Here Ω is a local $(n, 0)$ -form for the transverse Kähler metric, satisfying

$$d\Omega = iP \wedge \Omega, \tag{2.13}$$

where recall that $dP = \rho$. The real two-form \mathcal{J} is not closed, nor conformally closed, and so there is no (natural) symplectic structure on $C(Y_{2n+1})$. However, $\Omega_{(n+1,0)}$ is both globally defined and conformally closed:

$$d\Psi = 0, \quad \text{where } \Psi \equiv e^{-nB/2} r^{-\frac{n(n-1)}{(n-2)}} \Omega_{(n+1,0)}. \tag{2.14}$$

This implies that $C(Y_{2n+1})$ has an integrable complex structure, with zero first Chern class. We discuss this in more detail below. We note that $\Omega_{(n+1,0)}$, or equivalently Ψ , has charge $1/c$ under the R-symmetry vector field $\xi = \frac{1}{c} \partial_z$:

$$\mathcal{L}_\xi \Psi = \frac{i}{c} \Psi. \tag{2.15}$$

This implies that ξ is a holomorphic vector field. It pairs with the radial vector field $r\partial_r = -\mathcal{I}(\xi)$ under the complex structure tensor⁶ \mathcal{I} . The foliation generated by ξ is then also *transversely holomorphic*, meaning that there are local coordinate patches $\mathbb{R} \times \mathbb{C}^n$, with corresponding local coordinates z, z_1, \dots, z_n , where (z_1, \dots, z_n) are complex coordinates on \mathbb{C}^n . In particular the latter transform holomorphically between coordinate patches. For further details on this, and some of the other material in this subsection, see section 1 of [36] and references therein.

It will be convenient for what follows to introduce the *basic cohomology* of the foliation \mathcal{F}_ξ . A form α on Y_{2n+1} is called *basic* if

$$\xi \lrcorner \alpha = 0, \quad \mathcal{L}_\xi \alpha = 0. \tag{2.16}$$

The basic cohomology $H_B^*(\mathcal{F}_\xi)$ is by definition simply the cohomology defined by the exterior derivative d restricted to basic forms on Y_{2n+1} . For example, the transverse Kähler

⁶We use conventions where \mathcal{I} is obtained by raising an index on *minus* the Kähler two-form.

form J is basic and closed, and so defines a class $[J] \in H_B^2(\mathcal{F}_\xi)$. The one-form η is not basic, but its exterior derivative $d\eta = c\rho$ is, and in particular $[\rho] \in H_B^2(\mathcal{F}_\xi)$ also defines a basic cohomology class. Since basic cohomology classes are by definition represented by closed (basic) forms on Y_{2n+1} , there is a natural map

$$H_B^*(\mathcal{F}_\xi) \rightarrow H^*(Y_{2n+1}, \mathbb{R}). \tag{2.17}$$

Note that $[\rho]$ lies in the kernel of this map, since $\rho = \frac{1}{c}d\eta$. Since the foliation is also transversely holomorphic, we may further grade the basic cohomology by Hodge type. In particular the transverse Ricci form ρ has Hodge type $(1,1)$, and $[\rho/2\pi]$ represents the basic cohomology class $c_1^B \in H_B^{1,1}(\mathcal{F}_\xi)$. This is independent of the choice of transverse Kähler metric, and depends only on the transversely holomorphic foliation \mathcal{F}_ξ . Similarly, the Kähler form is Hodge type $(1,1)$, so we may more accurately write $[J] \in H_B^{1,1}(\mathcal{F}_\xi)$.

Consider now a general class of supersymmetric geometries of the form (2.5)–(2.7). We fix both the choice of manifold Y_{2n+1} and its complex cone $C(Y_{2n+1})$. However, we are still free to choose a nowhere zero holomorphic vector field ξ , together with a transverse Kähler metric. In fact any choice of such vector field and Kähler metric defines a supersymmetric geometry via (2.5)–(2.7), provided the scalar curvature $R > 0$. This class of off-shell geometries may be described as follows. We fix a choice of complex manifold $C(Y_{2n+1}) \cong \mathbb{R}_{>0} \times Y_{2n+1}$, with closed holomorphic volume form Ψ proportional to $\Omega_{(n+1,0)}$ (2.14). Since Ψ has fixed charge $1/c$ under ξ , we may write a general choice of R-symmetry vector field as

$$\xi = \sum_{i=1}^s b_i \partial_{\varphi_i} = \xi_0 + \sum_{i=2}^s c_i \partial_{\varphi_i}. \tag{2.18}$$

Here in the first expression ∂_{φ_i} , $i = 1, \dots, s \geq 1$, are real holomorphic vector fields generating a $U(1)^s$ action on $C(Y_{2n+1})$. We choose this basis so that the holomorphic volume form has unit charge under ∂_{φ_1} , but is uncharged under ∂_{φ_i} , $i = 2, \dots, s$. Physically, the latter will be dual to non-R global symmetries. Notice this fixes the coefficient $b_1 = 1/c$. For some considerations also the second expression in (2.18) will be useful. Here ξ_0 is a particular fiducial choice of R-symmetry vector, meaning that Ψ has charge $1/c$ under ξ_0 . We then have a family of R-symmetry vector fields in (2.18), parametrized either by $(b_1 = 1/c, b_2, \dots, b_s)$, or equivalently by (c_1, \dots, c_{s-1}) .

Note that contracting the complex holomorphic vector field $\xi - i\mathcal{L}(\xi)$ into Ψ gives a global $(n,0)$ -form that is transverse to the corresponding foliation \mathcal{F}_ξ . This is proportional to the form $e^{iz} \Omega$, where the phase-dependence is fixed by the fixed charge of Ψ under $\xi = \frac{1}{c} \partial_z$, and Ω is a local transverse $(n,0)$ -form. In fact neither factor in $e^{iz} \Omega$ is globally defined separately, with the transition functions between patches for each factor precisely cancelling. Indeed, Ω is a section of the basic canonical line bundle $\Lambda_B^{n,0}$, whose basic first Chern class is $-c_1^B \in H_B^{1,1}(\mathcal{F}_\xi)$ — see equation (2.13). Since $dz + P$ is a global one-form on Y_{2n+1} , where recall that $[\rho/2\pi] = c_1^B \in H_B^{1,1}(\mathcal{F}_\xi)$, this means that e^{iz} is a section of the dual line bundle. Having chosen an R-symmetry vector field and hence transversely holomorphic foliation, we are then free to pick a compatible transverse Kähler metric. In particular, this involves specifying a choice of the basic Kähler class $[J] \in H_B^2(\mathcal{F}_\xi)$.

Finally, as in Sasakian geometry we may classify these geometries according to whether the orbits of ξ all close or not. If ξ has a non-closed orbit we call the structure *irregular*, while if all orbits close, and are hence circles, then we call the structure *quasi-regular*. In the latter case ξ defines a $U(1)$ action on Y_{2n+1} . If this action is free then the structure is called *regular*, and in that case Y_{2n+1} is the total space of a circle bundle over a compact Kähler manifold (V, J) . In this case $H_B^*(\mathcal{F}_\xi) \cong H^*(V, \mathbb{R})$ is naturally isomorphic to the cohomology of the base V , and moreover $[\rho/2\pi] = c_1 \in H^2(V, \mathbb{Z})$ represents an integral cohomology class, the first Chern class. The largest positive integer I_V such that $c_1/I_V \in H^2(V, \mathbb{Z})$ is called the *Fano index* of V . In general the coordinate z may then have period $2\pi I_V/m$, where m is a positive integer that divides I_V . This latter condition is required in order that the Killing spinors, or equivalently the holomorphic volume form $\Omega_{(n+1,0)}$ in (2.12), are appropriately single-valued. Then Y_{2n+1} is the total space of the circle bundle over V associated to the line bundle K^{m/I_V} , where $K = \Lambda^{n,0}$ is the canonical line bundle of V . In fact all of these statements go through also in the quasi-regular case, when V is an orbifold, provided one replaces $H^2(V, \mathbb{Z})$ by the appropriate notion of orbifold cohomology group $H_{\text{orb}}^2(V, \mathbb{Z})$.

2.3 The action

Consider the general class of supersymmetric geometries of the form (2.5)–(2.7) described in the previous subsections. In particular we have a choice of nowhere zero holomorphic vector field ξ , of the form (2.18), and having fixed ξ then a choice of compatible transverse Kähler metric. One can then restrict the original action (2.3) to this off-shell class of supersymmetric geometries. They are off-shell because we do not (yet) impose the equation of motion (2.9). A straightforward computation leads to the remarkably simple formula

$$S_{\text{SUSY}} = \int_{Y_{2n+1}} \eta \wedge \rho \wedge \frac{J^{n-1}}{(n-1)!}. \tag{2.19}$$

Here we have dropped a total derivative term using Stokes' Theorem and the fact that Y_{2n+1} is compact without boundary. For future use, we note that this can also be written in the form⁷

$$S_{\text{SUSY}} = \frac{1}{c^2} \int_{Y_{2n+1}} e^{(1-n)B} \text{vol}_{2n+1}. \tag{2.20}$$

Notice that S_{SUSY} must be positive.

The action (2.19) is a functional of both the choice of R-symmetry vector field ξ , and also the transverse Kähler metric. However, an immediate observation is that S_{SUSY} only depends on this Kähler metric via its Kähler class $[J] \in H_B^2(\mathcal{F}_\xi)$. To be precise, suppose that J_1, J_2 are two Kähler forms related via

$$J_2 = J_1 + d\alpha, \tag{2.21}$$

⁷We can also write $\frac{(n-1)}{2} S_{\text{SUSY}} = \frac{1}{4} \int_{Y_{2n+1}} e^{(3-n)B} F^2 \text{vol}_{2n+1}$, where the right hand side is the last term appearing in (2.3). In addition $S_{\text{SUSY}} = c \int_{Y_{2n+1}} [(\nabla B)^2 + \frac{1}{2} c^2 e^{-B} \rho^2] \text{vol}_{2n}$, where in this particular expression indices are raised using the $2n$ -dimensional transverse Kähler metric ds^2 appearing in (2.5) and vol_{2n} is the corresponding volume form.

where α is a basic one-form on Y_{2n+1} , thus satisfying (2.16). The corresponding Ricci forms are then similarly related by $\rho_2 = \rho_1 + d\beta$, where β is also a basic one-form. An integration by parts then shows that the actions for J_1 and J_2 are equal. To see this, note that if Φ is any closed form with $\xi \lrcorner \Phi = 0$, then

$$\int_{Y_{2n+1}} \eta \wedge d\alpha \wedge \Phi = \int_{Y_{2n+1}} d\eta \wedge \alpha \wedge \Phi = 0. \tag{2.22}$$

The first equality follows from Stokes' Theorem (and integration by parts), while the second follows since the integrand has zero contraction with ξ and is hence identically zero. We have thus shown that S_{SUSY} depends on just the basic classes $[J]$ and $[\rho]$ both in $H_B^{1,1}(\mathcal{F}_\xi)$. Since $[\rho] \in H_B^{1,1}(\mathcal{F}_\xi)$ is an invariant of the transversely holomorphic foliation \mathcal{F}_ξ , independent of the choice of Kähler metric one uses to compute ρ , we deduce that S_{SUSY} depends on ξ and this basic Kähler class $[J] \in H_B^{1,1}(\mathcal{F}_\xi)$, and we have

$$S_{\text{SUSY}} = S_{\text{SUSY}}(\xi; [J]). \tag{2.23}$$

We conclude this subsection by noting that for quasi-regular structures the action (2.19) may be written as

$$S_{\text{SUSY}} = \frac{(2\pi)^2 c I_V}{\mathfrak{m}} \int_V c_1 \wedge \frac{J^{n-1}}{(n-1)!} = \frac{\pi c I_V}{\mathfrak{m}} \int_V R \frac{J^n}{n!}. \tag{2.24}$$

Here the notation is the same as that at the end of section 2.2, with I_V being the (orbifold) Fano index of $V = Y_{2n+1}/U(1)$. For fixed quasi-regular vector field ξ , the middle expression in (2.24) is more manifestly a function of the Kähler class $[J] \in H^2(V, \mathbb{R}) \cong H_B^2(\mathcal{F}_\xi)$.

2.4 Flux quantization

In order that solutions of the form (2.1) and (2.2) define consistent backgrounds of type IIB string theory and M-theory, respectively, we have to impose flux quantization. There are a number of subtleties that we need to discuss, including the fact that we want to impose a version of flux quantization for the off-shell supersymmetric geometries.

We first discuss flux quantization for the type IIB supersymmetric AdS₃ solutions. In order to define a consistent string theory background, the five-form must satisfy an appropriate Dirac quantization condition over all five-cycles $\Sigma_A \subset Y_7$, where $A = 1, \dots, \text{rank } H_5(Y_7, \mathbb{Z})$ runs over an integral basis for the free part of $H_5(Y_7, \mathbb{Z})$. Specifically, the condition is

$$\frac{1}{(2\pi\ell_s)^4 g_s} \int_{\Sigma_A} F_5 = N_A \in \mathbb{Z}, \tag{2.25}$$

where ℓ_s is the dimensionful string length, and g_s is the constant string coupling. Now, for the supersymmetric geometries the relevant part of the five-form is the piece on Y_7 which is given by

$$F_5|_{Y_7} = \frac{L^4}{4} \left[(dz + P) \wedge \rho \wedge J + \frac{1}{2} * dR \right]. \tag{2.26}$$

It is important to note that in the supersymmetric solutions $F_5|_{Y_7}$ is closed and hence (2.25) only depends on the homology class of Σ_A . For supersymmetric geometries $F_5|_{Y_7}$ is not closed and hence more care is required in imposing flux quantization, as we discuss below.

Before doing that, we next discuss flux quantization for the $D = 11$ supersymmetric AdS_2 solutions, which has a few distinctive subtleties. We first note that despite the fact that $G|_{Y_9} = 0$ there is still a non-trivial condition concerning flux quantization of G . Specifically, from [37] we require that the first Pontryagin class of Y_9 , $p_1(Y_9)$, is divisible by four, and so we will implicitly assume that this is the case. Since Y_9 is a spin manifold we necessarily have $p_1(Y_9)$ is divisible by two. The next subtlety concerns the quantization of the electric part of the four-form flux. The equation of motion for the four-form in $D = 11$ supergravity is given by

$$d *_{11} G + \frac{1}{2} G \wedge G = 0. \tag{2.27}$$

In general this requires that a suitably defined ‘‘Page charge’’ is appropriately quantized on Y_9 . However, for the ansatz (2.2) we have $G \wedge G = 0$, so we just need to impose

$$\frac{1}{(2\pi\ell_p)^6} \int_{\Sigma_A} *_{11} G = N_A \in \mathbb{Z}, \tag{2.28}$$

over all seven-cycles $\Sigma_A \subset Y_9$, where $A = 1, \dots, \text{rank } H_7(Y_9, \mathbb{Z})$ runs over an integral basis for the free part of $H_7(Y_9, \mathbb{Z})$, and ℓ_p denotes the eleven-dimensional Planck length. Now for the supersymmetric geometries we have, on Y_9 ,

$$*_{11} G = L^6 \left[(dz + P) \wedge \rho \wedge \frac{J^2}{2} + \frac{1}{2} * dR \right]. \tag{2.29}$$

For the supersymmetric solutions $*_{11} G$ is closed, being equivalent to (2.9), and hence (2.28) only depends on the homology class of Σ_A . For supersymmetric geometries $*_{11} G$ is not closed and again care is required in imposing flux quantization. A final subtlety for the $D = 11$ case is that the next order correction to the supergravity equations of motion gives rise to a contribution of $-\frac{(2\pi\ell_p)^6}{192} (p_1(Y_9)^2 - 4p_2(Y_9))$ appearing on the right hand side of (2.27), where $p_2(Y_9)$ is the second Pontryagin form [38, 39]. This term, which arises from anomaly considerations, is certainly important in properly imposing flux quantization, but it gives rise to corrections to the fluxes that are sub-leading in the large N_A limit, and hence for simplicity we will not consider them further in this paper. We note that when $Y_9 = T^2 \times Y_7$, all of the above quantum subtleties involving Pontryagin classes are absent, and indeed in this case the solutions may be reduced and T-dualized to type IIB solutions where these corrections are indeed not present.⁸

We are now in a position to discuss flux quantization for the off-shell supersymmetric geometries. We first observe that for the $\text{AdS}_3 \times Y_7$ and the $\text{AdS}_2 \times Y_9$ cases, imposing closure of $F_5|_{Y_7}$ and $*_{11} G$ is equivalent to imposing the equation of motion (2.9), which would put us on-shell. However, we may instead impose the weaker condition that the *integral* of (2.9) over Y_{2n+1} holds. A short computation reveals that (2.9) may be rewritten as

$$\square R = (J \wedge J) \lrcorner (\rho \wedge \rho). \tag{2.30}$$

⁸To see this, note first that for M any spin manifold $p_1(M)/2$ is congruent modulo 2 to the fourth Stiefel-Whitney class $w_4(M)$. On the other hand, for M a spin seven-manifold $w_4(M) = 0$ on dimensional grounds (for example, see [40]), and $w_4(T^2 \times Y_7)$ is simply a pull-back of $w_4(Y_7) = 0$. Similarly, the curvature forms representing Pontryagin classes p_1^2 and p_2 of each of T^2 and Y_7 are both identically zero on dimensional grounds, which implies they vanish for the product.

Since the integral of $\square R$ over Y_{2n+1} , using the measure $\eta \wedge J^n/n!$, vanishes using Stokes' Theorem, a necessary condition to solve the equation of motion is

$$\int_{Y_{2n+1}} \eta \wedge \rho^2 \wedge \frac{J^{n-2}}{(n-2)!} = 0. \tag{2.31}$$

Compare this expression to the supersymmetric action (2.19): the left hand side is again only a function of the choice of Killing vector ξ and basic Kähler class for a supersymmetric geometry. As we shall see in this subsection, (2.31) is also a sufficient condition in order to be able to impose flux quantization when $n = 3$ and $n = 4$ for a supersymmetric geometry of type IIB or $D = 11$ supergravity, respectively. We note in passing that Sasakian metrics with metric cones admitting a holomorphic $(n + 1, 0)$ -form have $[\rho]$ being a positive multiple of $[J]$, and hence the left hand side of (2.31) is then a positive multiple of the Riemannian volume of Y_{2n+1} . Thus Sasakian metrics can never be used for supersymmetric solutions of the type we are discussing. In particular the R-symmetry Killing vector is never a Reeb vector.

A sufficient topological condition to interpret (2.25), (2.28) for our supersymmetric geometries is that

$$H^2(Y_{2n+1}, \mathbb{R}) \cong H_B^2(\mathcal{F}_\xi)/[\rho]. \tag{2.32}$$

Note that $\rho = \frac{1}{c}d\eta$ is automatically exact in $H^2(Y_{2n+1}, \mathbb{R})$, as noted after equation (2.17). In fact the Gysin long exact sequence for the foliation, discussed for example in [41, 42], implies that $[\rho]$ spans the kernel of the map in (2.17). The content of (2.32) is hence that the map in (2.17) is *surjective* i.e. *all* closed two-form classes on Y_{2n+1} can be represented by *basic* closed two-forms. This holds in all examples of which we are aware. For example, it follows immediately from the Gysin long exact sequence for the foliation if $H_B^1(\mathcal{F}_\xi) = 0$. In fact this latter condition does *not* hold for the class of examples in section 3, but as discussed in that section (2.32) does hold.

In the following discussion we briefly restrict to the quasi-regular case for simplicity, although notice that an irregular R-symmetry vector may be viewed as a limit of a sequence of quasi-regular vector fields (since irrational numbers are limits of sequences of rationals). Since the quantities of interest are all continuous, the equations we deduce will hold also in the irregular case. For quasi-regular geometries we have $H_B^2(\mathcal{F}_\xi) \cong H^2(V, \mathbb{R})$, where $V = Y_{2n+1}/U(1)$ is the Kähler orbifold base. The dual homology statement to (2.32) implies that all $(2n - 1)$ -cycles in $H_{2n-1}(Y_{2n+1}, \mathbb{R}) \cong H^2(Y_{2n+1}, \mathbb{R})$ may be represented as circle fibrations over $(2n - 2)$ -cycles in V . Because of this, the second $*dR$ term in the flux (2.26) or (2.29) does not contribute to the integral, since it manifestly has zero contraction with ξ which generates the circle action. This will be true for any submanifold representing Σ_A , that is everywhere tangent to ξ . In this case only the first term in (2.26) contributes to the integral, leading to

$$\int_{\Sigma_A} \eta \wedge \rho \wedge \frac{J^{n-2}}{(n-2)!} = \begin{cases} \frac{2(2\pi\ell_s)^4 g_s}{L^4} N_A, & n = 3, \\ \frac{(2\pi\ell_p)^6}{L^6} N_A, & n = 4. \end{cases} \tag{2.33}$$

Again, compare the left hand side of (2.33) to the key formulae (2.19) and (2.31). However, *a priori* the left hand side of (2.33) is still not well-defined. We must require Σ_A to be tangent to ξ , as already discussed, but consider two such submanifolds $\Sigma_A^{(1)}, \Sigma_A^{(2)}$ that represent the same homology class in $H_{2n-1}(Y_{2n+1}, \mathbb{R})$. In the quasi-regular case these are circle bundles over $(2n-2)$ -dimensional subspaces $C_A^{(1)}, C_A^{(2)}$ of V representing $(2n-2)$ -cycles in $H_{2n-2}(V, \mathbb{R}) \cong H^2(V, \mathbb{R}) \cong H_B^2(\mathcal{F}_\xi)$. However, two such $(2n-2)$ -cycles with Poincaré duals differing by a multiple of $[\rho]$ both lift to the *same* $(2n-1)$ -cycle, due to (2.32). Writing $C_A^{(2)} - C_A^{(1)} = \lambda[\rho]_{\text{Poincaré dual}} \in H_{2n-2}(V, \mathbb{R})$, we then compute

$$\int_{\Sigma_A^{(2)}} \eta \wedge \rho \wedge J^{n-2} - \int_{\Sigma_A^{(1)}} \eta \wedge \rho \wedge J^{n-2} = \lambda \int_{Y_{2n+1}} \eta \wedge \rho \wedge \rho \wedge J^{n-2}. \quad (2.34)$$

Here $\lambda \in \mathbb{R}$ is arbitrary. Thus in order that (2.33) depends only on the homology class of $\Sigma_A \in H_{2n-1}(Y, \mathbb{R})$, the right hand side of (2.34) must be zero for all λ . But this is precisely the condition (2.31). This is perhaps not surprising: the constraint (2.31) is necessary (but not sufficient) for $F_5|_{Y_7}$ and $*_{11}G$ to be closed.

Provided we only use representatives of $(2n-1)$ -cycles that are tangent to ξ , and that (2.31) also holds, equation (2.33) makes sense as a topological flux quantization condition for our supersymmetric geometries. With this understanding, the left hand side depends only on the homology class of Σ_A , and the choice of vector field ξ and basic Kähler class. We will see many of the above general features exemplified in more detail in sections 3 and 4.

2.5 Extremal problem and the central charge

With all of the above background now in place, we can finally summarize the extremal problem of interest. For the case of $n = 3$ we will relate this to the central charge of the dual SCFT, hence giving a geometric dual of c -extremization. When $n = 4$ we will show that the extremization problem allows one to determine the two-dimensional Newton constant, G_2 , which is related to the partition function of the dual quantum mechanics and also with the entropy of certain black hole solutions in AdS₄.

Much as in [7], we fix a complex cone $C(Y_{2n+1}) = \mathbb{R}_{>0} \times Y_{2n+1}$ with holomorphic volume form, and holomorphic $U(1)^s$ action. A general choice of R-symmetry vector may then be written as in (2.18), under which the holomorphic volume form has fixed charge $1/c = 2/(n-2)$. For a particular choice of ξ and hence foliation \mathcal{F}_ξ we may then choose a transverse Kähler metric with basic class $[J] \in H_B^{1,1}(\mathcal{F}_\xi)$. Finally, we should also impose flux quantization, which requires us to first impose the constraint (2.31), and then (2.33) for the cases of $n = 3, 4$. For $n > 4$, one could impose (2.33) with an arbitrary constant factor multiplying N_A on the right hand side. These latter conditions will, in general, further constrain the choice of ξ and $[J]$. By construction, a solution to the equations of motion will be a critical point of the action (2.19), where we vary over the remaining unconstrained variables in ξ and $[J]$. We will see how to impose all of this concretely in a class of examples in section 3.

Geometrically we have set up a very analogous problem to volume minimization in Sasakian geometry [6, 7], which for Sasaki-Einstein five-manifolds is a geometric dual of

a -maximization in the dual four-dimensional SCFTs. It is thus natural to interpret the above, when $n = 3$, as a geometric dual to c -extremization, which is a precise analogue for two-dimensional $(0, 2)$ SCFTs [1]. However, for this analogy to hold our extremal function (2.19) should play the role of a trial central charge function, and in particular be equal to the central charge of the solution at a critical point.

To see that this is indeed the case, we begin by recalling the general formula for the central charge of the $d = 2$ SCFT,

$$c_{\text{sugra}} = \frac{3L}{2G_3}, \tag{2.35}$$

where G_3 denotes the effective Newton constant in three dimensions. This is easily computed via dimensional reduction for the class of IIB backgrounds we are considering when $n = 3$ (e.g. see appendix B of [26]), and one finds

$$\frac{1}{G_3} = \frac{L^7}{G_{10}} \int_{Y_7} e^{-2B} \text{vol}_7, \tag{2.36}$$

where the ten-dimensional Newton constant is

$$G_{10} = \frac{(2\pi)^7 g_s^2 \ell_s^8}{16\pi}. \tag{2.37}$$

Evaluating the general expression (2.36) for our class of off-shell supersymmetric geometries, we can define what we will call the “trial central charge”, \mathcal{Z} , via

$$\mathcal{Z} \equiv \frac{3L^8}{(2\pi)^6 g_s^2 \ell_s^8} S_{\text{SUSY}}, \tag{2.38}$$

where S_{SUSY} is the supersymmetric action (2.19) with $n = 3$. Then for an on-shell supersymmetric solution we get

$$\mathcal{Z}|_{\text{on-shell}} = c_{\text{sugra}}. \tag{2.39}$$

This completes our identification of a geometric version of c -extremization.

We can also consider the extremization problem for AdS_2 solutions of $D = 11$ supergravity by setting $n = 4$. We define the two-dimensional Newton constant G_2 by

$$\frac{1}{G_2} = \frac{L^9}{G_{11}} \int_{Y_9} e^{-3B} \text{vol}_9, \tag{2.40}$$

where the eleven-dimensional Newton constant is

$$G_{11} = \frac{(2\pi)^8 \ell_p^9}{16\pi}. \tag{2.41}$$

Evaluating (2.40) for our class of off-shell supersymmetric geometries, we have

$$\frac{1}{G_2} = \frac{16\pi L^9}{(2\pi)^8 \ell_p^9} \int_{Y_9} e^B \eta \wedge \frac{J^4}{4!} = \frac{16\pi L^9}{(2\pi)^8 \ell_p^9} S_{\text{SUSY}}. \tag{2.42}$$

Thus, our extremal problem allows us to determine G_2 .

While AdS₂ holography is still being developed (e.g. [11] and references therein), it is clear that G_2 encodes important information of the dual superconformal quantum mechanics. A simple dimensional reduction of $D = 11$ supergravity action to $D = 2$, leads to an action of the form $\frac{1}{16\pi G_2} \int d^2x \sqrt{-g_2} [R_2 + \dots]$. This action does not have any gravitational dynamics and in particular does not give rise to an AdS₂ vacuum, so we should therefore include additional minimal degrees of freedom in the reduction as discussed in [43, 44], for example. In any event we note that the renormalized action $\frac{1}{16\pi G_2} \int_M d^2x \sqrt{-g_2} R_2 + \frac{1}{8\pi G_2} \int_{\partial M} K$, where K is the trace of the extrinsic curvature, evaluates on the unit radius Euclidean AdS₂ vacuum, i.e. the hyperbolic disc, to give $-1/4G_2$, and it is therefore natural to identify this as minus the logarithm of the partition function for a one-dimensional dual superconformal quantum mechanics.

There is a special subclass of AdS₂ × Y₉ solutions where we can make a more precise statement and also make an interesting connection with black hole entropy computations for certain AdS₄ black hole solutions. To see this we first recall that there is a consistent Kaluza-Klein reduction of $D = 11$ supergravity on an arbitrary SE₇ to obtain minimal gauged supergravity theory in $D = 4$ [45]. This latter theory admits supersymmetric AdS₄ black hole solutions with black hole horizon given by AdS₂ × Σ_g, where Σ_g is a Riemann surface with genus $g > 1$ [46]. After uplifting the latter on SE₇ we obtain precisely a special example of the $D = 11$ solutions we are considering in this paper with eight-dimensional Kähler base given by Σ_g × KE₆, where KE₆ is the transverse Kähler-Einstein metric associated with the SE₇ metric. A short calculation, that we have included in appendix A, shows that the Bekenstein-Hawking entropy, S_{BH} , of this class of black hole solutions is directly related to G_2 via

$$S_{BH} = \frac{1}{4G_2}. \tag{2.43}$$

Thus, for this class of black hole solutions our variational problem gives rise to the black hole entropy. Furthermore, this is also the logarithm of the twisted topological index for the $\mathcal{N} = 2$ SCFT in $d = 3$, dual to AdS₄ × SE₇, after compactification on the Riemann surface Σ_g, with the addition of R-symmetry magnetic flux (only) on Σ_g, the “universal twist” [16].

Finally, recall that the AdS₂ × Y₉ solutions with $Y_9 = T^2 \times Y_7$ can be dimensionally reduced and then T-dualized to obtain AdS₃ × Y₇ solutions of type IIB [23]. In appendix B we also derive the relationship between c_{sugra} and G_2 for this class of solutions.

3 A class of examples: $Y_7 = T^2 \times Y_5$

In this section we use the general formalism described in section 2 to discuss a class of type IIB AdS₃ × Y₇ examples in which $Y_7 = T^2 \times Y_5$ is a product of a flat two-torus T^2 with a compact five-manifold Y_5 . As noted earlier, after T-duality and uplifting to $D = 11$ these give AdS₂ × T⁴ × Y₅ examples (see appendix B). We will focus on the type IIB perspective in the remainder of the paper.

We begin in section 3.1 by specializing the formulae of section 2 to this case. In particular, the main simplifying feature is that the dependence of S_{SUSY} on the Kähler class parameters can be entirely eliminated using flux quantization, so that the resulting extremal

function is a function only of the R-symmetry vector ξ . In section 3.2 we describe a general localization formula that allows one to compute this extremal function quite explicitly in terms of fixed point data of ξ . In section 3.3 we show that complex cones $C(Y_5) \equiv \mathbb{R}_{>0} \times Y_5$ which admit a compatible Kähler cone metric *never* have a corresponding $\text{AdS}_3 \times T^2 \times Y_5$ solution. This implies that the complex geometry of $C(Y_5)$ for $\text{AdS}_3 \times T^2 \times Y_5$ solutions is necessarily somewhat exotic, and we describe this in further detail in section 3.4, and in the explicit examples in sections 4.1 and 4.2. The result of section 3.3 also implies that compactifying four-dimensional SCFTs, dual to $\text{AdS}_5 \times Y_5$ Sasaki-Einstein solutions, on T^2 with no geometric twist cannot flow to two-dimensional $(0, 2)$ SCFTs with AdS_3 duals of the type studied in this paper, if the complex structure on the cones is preserved in the RG flow.

3.1 General formulas

Throughout this section we will assume that $Y_7 = T^2 \times Y_5$, where in addition $b_1(Y_5) \equiv \dim H_1(Y_5, \mathbb{R}) = 0$ and the R-symmetry vector ξ is taken to be tangent to Y_5 . We then write the transverse Kähler form as⁹

$$J = A \text{vol}_2 + \omega, \tag{3.1}$$

where the volume form vol_2 on T^2 is normalized so that $\int_{T^2} \text{vol}_2 = 1$, and $A > 0$ is a constant parametrizing the Kähler class of T^2 . The two-form ω is a transverse Kähler form on Y_5 . Notice that because the T^2 is flat, the transverse Kähler metric on Y_5 solves (2.9). Notice also that $\rho = dP$ is then similarly a transverse Ricci form on Y_5 , again since the torus is flat. For this class the supersymmetric action (2.19) is easily computed, giving

$$S_{\text{SUSY}} = A \int_{Y_5} \eta \wedge \rho \wedge \omega. \tag{3.2}$$

Cancelling an overall factor of $A \neq 0$, the constraint (2.31) similarly reads

$$\int_{Y_5} \eta \wedge \rho \wedge \rho = 0. \tag{3.3}$$

We next turn to flux quantization. Recall that we assume $b_1(Y_5) = 0$. The Künneth formula then implies that the five-cycles on Y_7 are spanned by a copy of Y_5 at a fixed point on T^2 , and $\Sigma_I = T^2 \times \sigma_I$, where $\sigma_I \subset Y_5$ form a basis of three-cycles in Y_5 , $I = 1, \dots, b_3(Y_5) \equiv \dim H_3(Y_5, \mathbb{R})$. Noting that ξ is tangent to Y_5 , (2.33) reads

$$\int_{Y_5} \eta \wedge \rho \wedge \omega = \frac{2(2\pi\ell_s)^4 g_s}{L^4} N, \tag{3.4}$$

where we have denoted the flux number for this distinguished five-cycle by $N \in \mathbb{N}$. Substituting into the off-shell supersymmetric action (3.2) immediately gives

$$S_{\text{SUSY}} = A \frac{2(2\pi\ell_s)^4 g_s}{L^4} N. \tag{3.5}$$

⁹In fact we only need this equation to hold in basic cohomology.

We next claim that there do not exist solutions with $b_3(Y_5) = 0$, for example ruling out $Y_5 = S^5$ topology. Recall that ξ leads to a foliation \mathcal{F}_ξ of Y_5 , and there is an associated long exact Gysin sequence. The relevant part of this for our purposes reads

$$0 \cong H^3(Y_5, \mathbb{R}) \longrightarrow H_B^2(\mathcal{F}_\xi) \longrightarrow H_B^4(\mathcal{F}_\xi) \longrightarrow H^4(Y_5, \mathbb{R}) \cong 0. \quad (3.6)$$

Here we have used $b_1(Y_5) = 0 = b_3(Y_5)$. This sequence implies $H_B^2(\mathcal{F}_\xi) \cong H_B^4(\mathcal{F}_\xi) \cong \mathbb{R}$, the latter being generated by the transverse volume form. On the other hand the foliation is transversely Kähler, so the transverse Kähler class must generate the former group. It follows that $[\rho] = \lambda[\omega] \in H_B^2(\mathcal{F}_\xi)$ for some constant $\lambda \in \mathbb{R}$. But the constraint (3.3) then implies

$$0 = \int_{Y_5} \eta \wedge \rho \wedge \rho = \lambda^2 \int_{Y_5} \eta \wedge \omega \wedge \omega = 2\lambda^2 \int_{Y_5} \eta \wedge \text{vol}_4, \quad (3.7)$$

which implies $\lambda = 0$, a contradiction. Indeed, note that the action (3.2) is then zero. Notice that when $b_3(Y_5) = 0$ there is by definition no baryonic $U(1)$ symmetry in the dual field theory.

We next look at the case $b_3(Y_5) = 1$, so there is a single three-cycle generated by $\sigma \subset Y_5$. Under our above topological assumptions one can similarly show that the Gysin long exact sequence implies

$$H^2(Y_5, \mathbb{R}) \cong H_B^2(\mathcal{F}_\xi)/[\rho]. \quad (3.8)$$

This in turn implies (2.32). It follows that σ may be taken to be tangent to the R-symmetry vector. The quantization condition (2.33) then reads¹⁰

$$A \int_\sigma \eta \wedge \rho = \frac{2(2\pi\ell_s)^4 g_s}{L^4} M, \quad (3.9)$$

where we have denoted the flux quantum number of the five-cycle $T^2 \times \sigma$ as $M \in \mathbb{Z}$. Putting everything together, we get the following very simple expression for what we will call the “trial” central charge

$$\mathcal{Z} \equiv \frac{12(2\pi)^2 MN}{\int_\sigma \eta \wedge \rho}. \quad (3.10)$$

The numerator is of course quantized. The flux quantization conditions have effectively allowed us to eliminate the dependence on the Kähler class in terms of the integers N and M , and the only dependence on the R-symmetry vector ξ is now purely in the denominator. We should thus now extremize (3.10) as a function of ξ , subject to the constraint (3.3).

Finally, for general $b_3(Y_5) \geq 1$ the formula (3.10) of course still holds, where we pick one of the generating three-cycles to be $\sigma \equiv \sigma_1$, with corresponding flux quantum number

¹⁰Much of the above analysis also applies, *mutatis mutandis*, for $D = 11$ solutions with $Y_9 = T^2 \times Y_7$, including the conclusion that there do not exist solutions with $b_5(Y_7) = 0$. However, the analogue of the integral in (3.9) will involve $\eta \wedge \rho \wedge \omega$, where ω is the transverse Kähler form on Y_7 and hence the flux quantization conditions still depend on the transverse Kähler class when $Y_9 = T^2 \times Y_7$.

$M \equiv M_1$. However, we must in addition impose flux quantization through each σ_I , $I = 2, \dots, b_3(Y_5)$, which is equivalent to imposing

$$\frac{\int_{\sigma_I} \eta \wedge \rho}{\int_{\sigma} \eta \wedge \rho} = \frac{M_I}{M}, \quad I = 2, \dots, b_3(Y_5). \quad (3.11)$$

The flux quantum numbers $\{M = M_1, M_2, \dots, M_{b_3(Y_5)}\}$ are part of the fixed global, topological data.

3.2 Fixed point theorem

In practice we would like to obtain more explicit expressions for quantities such as (3.3) and (3.10). To do this we may use similar techniques to those in [7]. For generality we again return to general complex dimension n , with $n = 3$ being the case relevant for type IIB supergravity solutions.¹¹ Thus associated to Y_{2n-1} we have a complex cone $C(Y_{2n-1}) = \mathbb{R}_{>0} \times Y_{2n-1}$, with coordinate $r > 0$ on the first factor. On $C(Y_{2n-1})$ we then introduce the two-form

$$\gamma \equiv \frac{1}{2} d(r^2 \eta) = r dr \wedge \eta + \frac{1}{2} r^2 d\eta. \quad (3.12)$$

It is straightforward to show that

$$\int_{C(Y_{2n-1})} e^{-\frac{1}{2} r^2 + \gamma} = \int_{Y_{2n-1}} \eta \wedge (d\eta)^{n-1}. \quad (3.13)$$

This equality follows simply by explicitly performing the integral over $r \in (0, \infty)$ on the cone. Since

$$d\left(-\frac{1}{2} r^2\right) = \xi \lrcorner \gamma, \quad (3.14)$$

it follows that $-\frac{r^2}{2} + \gamma$ is an equivariantly closed form under the derivative $d - \xi \lrcorner$. The left hand side of (3.13) may hence be computed using the Berline-Vergne localization formula. However, as in the similar application of the Duistermaat-Heckman formula to Sasakian geometry in [7], where γ is a symplectic form, the fixed point set of ξ is formally the origin $r = 0$. This is generically a singular point if we add it to compactify $C(Y_{2n-1})$ around $r = 0$, and the Berline-Vergne formula cannot be applied.

We may obtain a meaningful formula by instead *resolving* the singularity at the origin, again as in [7]. There is no unique way to do this, and the resulting formulas will take different forms for different resolutions. By a (partial) resolution here we mean a manifold (or respectively orbifold) $\hat{C}(Y_{2n-1})$ together with a map $\pi : \hat{C}(Y_{2n-1}) \rightarrow C(Y_{2n-1}) \cup \{r = 0\}$. This map should be equivariant under the holomorphic $U(1)^s$ action, and be a diffeomorphism when restricted to $C(Y_{2n-1})$. We refer to $\pi^{-1}(\{r = 0\})$ as the *exceptional set*. Since by definition ξ is nowhere zero on $C(Y_{2n-1})$, its fixed point set is a subset of the exceptional set. There is a canonical way to construct such a partial resolution: simply pick a quasi-regular \mathbb{R} -symmetry vector field ξ_0 , and take $\hat{C}(Y_{2n-1})$ to be the total space of the

¹¹It is worth noting that for the $D = 11$ $\text{AdS}_2 \times Y_9$ solutions with $Y_9 = T^2 \times Y_7$, or equivalently, $\text{AdS}_3 \times Y_7$ type IIB solutions, one needs to calculate different integrals, involving the transverse Kähler class of Y_7 , as noted in footnote 10.

orbifold line bundle K^m/I_{V_0} , described at the end of section 2.2. In this case the exceptional set is a copy of the Kähler orbifold V_0 , which maps to $\{r = 0\}$ under the map π . The fixed point set of ξ_0 is precisely V_0 , but for a more general R-symmetry vector (2.18) there will be an induced action of $U(1)^{s-1}$ on the exceptional set, where for generic ξ its fixed points will coincide with the fixed points of $U(1)^{s-1}$. On this particular partial resolution $\hat{C}(Y_{2n-1})$ we may identify r with a radial distance function on the complex line fibre. Notice that although η is not defined on the exceptional set, where ξ has fixed points, nevertheless the two-form γ defined in (3.12) is well-defined everywhere when pulled back to $\hat{C}(Y_{2n-1})$ — it is simply zero on the exceptional set.

With this notation in hand, we may then apply the Berline-Vergne fixed point theorem on $\hat{C}(Y_{2n-1})$, which gives

$$\int_{Y_{2n-1}} \eta \wedge (d\eta)^{n-1} = (2\pi)^n \sum_{\{\mathcal{F}\}} \prod_{\lambda=1}^R \frac{1}{(\vec{b}, \vec{u}_\lambda)^{n_\lambda}} \int_{\mathcal{F}} \frac{1}{d_{\mathcal{F}}} \prod_{\lambda=1}^R \left[\sum_{a \geq 0} \frac{c_a(\mathcal{E}_\lambda)}{(\vec{b}, \vec{u}_\lambda)^a} \right]^{-1}. \quad (3.15)$$

The notation here is exactly the same as in [7]. The sum is over connected components \mathcal{F} of the fixed point set of ξ on $\hat{C}(Y_{2n-1})$. As described above, each such \mathcal{F} is a subset of the exceptional set $\pi^{-1}(\{r = 0\})$. For each connected component \mathcal{F} of fixed points the $\vec{u}_\lambda \in \mathbb{Z}^s \subset \mathfrak{t}_s^*$ denote weights of the resulting linear action of ξ on the normal bundle, with multiplicities $n_\lambda \in \mathbb{N}$, so that the linear action by ξ on a given weight space \mathcal{E}_λ is $(\vec{b}, \vec{u}_\lambda)$. Here we use the basis for the Lie algebra \mathfrak{t}_s of $U(1)^s$ in (2.18), so that $\vec{b} = (b_1, \dots, b_s)$ parametrizes the choice of R-symmetry vector. Finally $c_a(\mathcal{E}_\lambda)$ denote Chern classes of the corresponding weight space bundles, and the positive integer $d_{\mathcal{F}}$ is the *order* of \mathcal{F} as an orbifold, and is required only for resolutions with orbifold singularities.

Although the general formula (3.15) is a little cumbersome, the point is that the right hand side is manifestly only a function of the trial R-symmetry vector \vec{b} , together with certain global topological data — namely Chern numbers and weights of the $U(1)^s$ action. If $\hat{C}(Y_{2n-1})$ is a smooth manifold with only isolated fixed points of $U(1)^n$ (a maximal torus action), the right hand side significantly simplifies to

$$\int_{Y_{2n-1}} \eta \wedge (d\eta)^{n-1} = (2\pi)^n \sum_{\{\text{fixed points}\}} \prod_{\lambda=1}^n \frac{1}{(\vec{b}, \vec{u}_\lambda)}, \quad (3.16)$$

where the n , possibly indistinct, weights are \vec{u}_λ , $\lambda = 1, \dots, n$. Recalling that $d\eta = c\rho = \frac{1}{2}\rho$ when $n = 3$, the formula (3.15) similarly implies that the constraint (3.3) and central charge (3.10) depend only on the trial R-symmetry vector \vec{b} and global, topological data. In particular we may apply a similar localization formula (3.15), simply replacing Y_{2n-1} by σ . Here we take $\sigma \subset Y_5$ to be a three-submanifold, invariant under $U(1)^s$. The resolution $\hat{C}(Y_5)$ of $C(Y_5) \cup \{r = 0\}$ will induce a resolution of the cone over σ , although again the point is that we may use any choice of resolution.

We shall make use of the general formulae in this section in the examples of section 4.

3.3 An obstruction for Kähler cones

Let Y_5 be a five-manifold with complex cone $C(Y_5) = \mathbb{R}_{>0} \times Y_5$ that is of Calabi-Yau type. By this we mean that the complex manifold $C(Y_5) \cong \mathbb{R}_{>0} \times Y_5$ admits a Kähler cone metric that is compatible with the given complex structure, and has a global holomorphic volume form of positive charge under the Reeb vector. In this section we show that there is no supersymmetric $\text{AdS}_3 \times T^2 \times Y_5$ solution with the given complex structure on $C(Y_5)$. In fact as we shall see, the problem is quite simple: an R-symmetry vector ξ satisfying the constraint (3.3) necessarily lies outside the *Reeb cone* (defined below) and in that case the putative radial vector $r\partial_r \equiv -\mathcal{I}(\xi)$ has no compatible radial coordinate $r > 0$. Thus, in a sense the complex geometry is not compatible with the radial slicing into $C(Y_5) = \mathbb{R}_{>0} \times Y_5$. For simplicity we shall prove this for toric Kähler cones, using some of the formalism of [6], although the proof can be generalized.

Thus let $C(Y_5)$ be a toric complex cone of Calabi-Yau type. Fix a choice of any compatible Kähler cone metric. Following reference [6] we may then introduce symplectic-toric coordinates $(y_1, y_2, y_3; \varphi_1, \varphi_2, \varphi_3)$, where the Killing vectors ∂_{φ_i} , $i = 1, 2, 3$, generate the effectively acting $U(1)^3$ action. The coordinates y_i arise as moment maps, and lie inside a convex polyhedral cone $\vec{y} \in \mathcal{C} \subset \mathbb{R}^3 \cong \mathfrak{t}_3^*$, where \mathfrak{t}_3 is the Lie algebra of the torus $U(1)^3$. The Kähler cone metric may be written

$$ds_{\text{Kähler}}^2 = \sum_{i,j=1}^3 G_{ij} dy_i dy_j + G^{ij} d\varphi_i d\varphi_j, \tag{3.17}$$

where $G_{ij} = G_{ij}(\vec{y})$ is homogeneous degree -1 , and is positive definite and smooth in the interior of \mathcal{C} , with a certain pole behaviour on the bounding facets of \mathcal{C} , required in order that the metric compactifies smoothly there. Here G^{ij} is simply the inverse matrix to G_{ij} , which correspondingly has reduced rank on the boundary components of \mathcal{C} . This Kähler cone metric will have an associated Reeb vector field

$$\text{Reeb} = \sum_{i,j=1}^3 2G_{ij} y_j \partial_{\varphi_i}. \tag{3.18}$$

However, this will *not* be a putative R-symmetry vector in the application to AdS_3 solutions, as we describe below. Instead we wish to use the above coordinates simply to describe the complex geometry of $C(Y_5)$, rather than the Kähler cone geometry that is also present in the above description.

Let

$$\xi = \sum_{i=1}^3 b_i \partial_{\varphi_i} \tag{3.19}$$

be a putative R-symmetry vector for a supersymmetric $\text{AdS}_3 \times T^2 \times Y_5$ solution. Recall that the *Reeb cone* $\mathcal{C}^* \subset \mathbb{R}^3 \cong \mathfrak{t}_3$ is defined as $\mathcal{C}^* = \{\vec{b} \in \mathbb{R}^3 \mid (\vec{b}, \vec{u}_\alpha) > 0\}$, where

$$\mathcal{C} = \left\{ \sum_{\alpha} t_{\alpha} \vec{u}_{\alpha} \in \mathbb{R}^3 \mid t_{\alpha} \geq 0 \right\}. \tag{3.20}$$

The $\vec{u}_\alpha \in \mathbb{Z}^3$ are the outward-pointing generating edges of the moment map polyhedral cone \mathcal{C} , and for the Reeb vector of a Kähler cone metric one necessarily has $\xi \in \mathcal{C}^*$ [6]. Moreover, for such a vector

$$\int_{Y_5} \eta \wedge \rho \wedge \rho = 32 \text{Vol}(\xi) > 0, \tag{3.21}$$

where $\text{Vol}(\xi)$ denotes the Riemannian volume of the corresponding Sasakian metric. Thus for $\xi \in \mathcal{C}^*$ the constraint (3.3) cannot hold, since this requires the left hand side of (3.21) to be zero. Thus $\xi \notin \mathcal{C}^*$, meaning there is at least one edge vector u_α with

$$(\vec{b}, \vec{u}_\alpha) < 0. \tag{3.22}$$

Geometrically, the edge vector \vec{u}_α corresponds to a one-dimensional torus-invariant complex submanifold $V_\alpha \cong \mathbb{C}^* = \text{U}(1) \times \mathbb{R}_{>0}$ of $C(Y_5)$. Here the $\text{U}(1)$ is the single non-vanishing circle over that edge in $\partial\mathcal{C}$. Being torus-invariant, the R-symmetry vector (3.19) is tangent to V_α , and so too is it's complex partner

$$- \mathcal{I}(\xi) = \sum_{i,j=1}^3 G^{ij} b_j \frac{\partial}{\partial y_i} = r \partial_r. \tag{3.23}$$

In the second equality we have used the formula for the complex structure in symplectic-toric coordinates (equation (2.16) of [6]), while the last equality follows from the relationship between the radial vector and Reeb vector described in section 2.2. Note that when restricted to an edge vector $\vec{y} \in \{t \vec{u}_\alpha \mid t \geq 0\} \subset \partial\mathcal{C}$, the matrix $G^{ij}(\vec{y})$ has rank 1, corresponding to the single $\text{U}(1)$ that is non-vanishing along the pre-image of that edge under the moment map. On the other hand, the outward-pointing directional derivative along the edge corresponding to \vec{u}_α is by definition

$$\nu_\alpha \equiv \sum_{i=1}^3 u_\alpha^i \frac{\partial}{\partial y_i}. \tag{3.24}$$

We thus see that

$$\sum_{j=1}^3 G^{ij} b_j \propto u_\alpha^i \tag{3.25}$$

holds along the edge generated by the vector \vec{u}_α . We may determine the proportionality factor by dotting this with \vec{b} . Note immediately that the right hand side is negative due to (3.22), while the left hand side is $\sum_{i,j=1}^3 G^{ij} b_i b_j = |\xi|^2 > 0$, the square length of the R-symmetry vector. Thus along the edge vector generated by \vec{u}_α , namely $\vec{y} \in \{t \vec{u}_\alpha \mid t \geq 0\} \subset \partial\mathcal{C}$, we have shown that

$$r \partial_r = \sum_{i,j=1}^3 G^{ij} b_j \frac{\partial}{\partial y_i} = \frac{|\xi|^2}{(\vec{b}, \vec{u}_\alpha)} \nu_\alpha. \tag{3.26}$$

Thus when the R-symmetry vector ξ lies outside the Reeb cone, so that (3.22) holds for some α , there is a corresponding submanifold $V_\alpha \subset C(Y_5)$ along which $r \partial_r$ points *towards*

the origin, rather than away from it. This is an immediate problem given the definition of the radial coordinate r . By definition we have

$$r\partial_r r^2 = 2r^2 > 0, \tag{3.27}$$

away from the origin $r = 0$. On the other hand, combining this with (3.26) says that r^2 is monotonic *decreasing* as one moves out along the edge, i.e. as t increases from zero in $\{t\vec{u}_\alpha \mid t \geq 0\}$. This is a contradiction, since $r^2 = 0$ at the tip of the cone $t = 0$, and should clearly be non-negative.

This concludes our proof, but since it is rather general (and abstract), it is perhaps helpful to give a simple example where the details can be seen more explicitly. Thus consider $C(Y_5) = \mathbb{C}^3 \setminus \{0\}$. In this case we may introduce polar coordinates (r_i, φ_i) , $i = 1, \dots, 3$, for each copy of \mathbb{C} in $\mathbb{C}^3 = \oplus_{i=1}^3 \mathbb{C}$. The moment map coordinates are $y_i = \frac{1}{2}r_i^2 \geq 0$, so that the polyhedral cone is $\mathcal{C} = (\mathbb{R}_{\geq 0})^3 \cong \mathcal{C}^*$. The matrix G_{ij} and its inverse G^{ij} for the flat Kähler metric on \mathbb{C}^3 are

$$G_{ij} = \text{diag}\left(\frac{1}{r_1^2}, \frac{1}{r_2^2}, \frac{1}{r_3^2}\right), \quad G^{ij} = \text{diag}(r_1^2, r_2^2, r_3^2). \tag{3.28}$$

The 3 bounding facets of \mathcal{C} are at $\{r_i = 0\}$, $i = 1, 2, 3$, where notice that G^{ij} has a zero. The generating edge vectors are $\vec{u}_1 = (1, 0, 0)$, $\vec{u}_2 = (0, 1, 0)$, $\vec{u}_3 = (0, 0, 1)$, where the edge corresponding to \vec{u}_1 is $\{r_2 = 0, r_3 = 0\}$, etc. Suppose the R-symmetry vector lies outside the Reeb cone, which means that $b_i < 0$ for at least one $i = 1, 2, 3$. Without loss of generality, let us suppose that $b_1 < 0$. Then along $V_1 \equiv C(Y_5) \cap \{r_2 = r_3 = 0\} \cong \mathbb{C}^*$ the radial vector is

$$r\partial_r = \sum_{i=1}^3 b_i r_i \partial_{r_i} = b_1 r_1 \partial_{r_1}. \tag{3.29}$$

But since $b_1 < 0$, this says that r^2 is a monotonic decreasing function of the radius r_1 along V_1 . For example, the expression $r^2 = \sum_{i=1}^3 r_i^{2/b_i}$ satisfies the homogeneity equation (3.27), and indeed notice that for $b_1 < 0$ it is a monotonic decreasing function of r_1 along $\{r_2 = 0, r_3 = 0\}$, as we have argued in general. However, $r_1 = 0$ maps to $r = \infty$ for this choice of “radial” coordinate!

3.4 Toric formulas and non-convex cones

In the previous subsection we have ruled out toric complex cones $C(Y_5)$ of Calabi-Yau type as giving rise to $\text{AdS}_3 \times T^2 \times Y_5$ solutions. However, one can nevertheless write down formulas for the physical quantities of interest in section 3.1 in this case, that we will use in the next section. Furthermore, they will lead us to some formulae, which we conjecture to hold for “non-convex” toric geometries (defined below), that are associated with some explicitly known $\text{AdS}_3 \times T^2 \times Y_5$ solutions.

We begin with the toric complex cones $C(Y_5)$ with a compatible Kähler cone metric, as in section 3.3. Denoting $d \geq 3$ as the number of facets of the associated polyhedral cone \mathcal{C} , we may rewrite the presentation of \mathcal{C} in (3.20) via a dual description as

$$\mathcal{C} = \{\vec{y} \in \mathbb{R}^3 \mid (\vec{y}, \vec{v}_a) \geq 0, \ a = 1, \dots, d\}. \tag{3.30}$$

Here $\vec{v}_a \in \mathbb{Z}^3$ are the *inward*-pointing normal vectors to the d facets of the cone. Each facet of \mathcal{C} , namely $\{(\vec{y}, \vec{v}_a) = 0\}$, corresponds to a complex codimension one submanifold, which is a cone over a three-manifold $S_a \subset Y_5$. We then have the formula [6]

$$\int_{S_a} \eta \wedge \rho = 2(2\pi)^2 \frac{(\vec{v}_{a-1}, \vec{v}_a, \vec{v}_{a+1})}{(\vec{b}, \vec{v}_{a-1}, \vec{v}_a)(\vec{b}, \vec{v}_a, \vec{v}_{a+1})}. \tag{3.31}$$

Similarly, it is a standard result of toric geometry (e.g. see eq. (17) of [47]) that the transverse basic first Chern class

$$c_1^B = \frac{[\rho]}{2\pi} = \sum_{a=1}^d [S_a]_{\text{Poincaré dual}} \in H_B^2(\mathcal{F}_\xi). \tag{3.32}$$

Using (3.32) and (3.31) we may then derive the formula

$$\int_{Y_5} \eta \wedge \rho \wedge \rho = \frac{4(2\pi)^3}{b_1} \sum_{a=1}^d \frac{(\vec{v}_{a-1}, \vec{v}_a, \vec{v}_{a+1})}{(\vec{b}, \vec{v}_{a-1}, \vec{v}_a)(\vec{b}, \vec{v}_a, \vec{v}_{a+1})}, \tag{3.33}$$

where recall that $b_1 = 2$ is fixed by the charge of the holomorphic volume form. Note that equation (3.33) was derived differently in [6].

Strictly speaking, (3.31) and (3.33) were proven in [6] only for \vec{b} lying in the Reeb cone, $\vec{b} \in \mathcal{C}^*$, for which there is a compatible Kähler cone metric. However, toric Kähler cones in dimension $n = 3$ can always be fully resolved, for which one can also apply the fixed point formula (3.16). On the other hand, this latter formula uses the general Berline-Vergne theorem, which does not require any Kähler cone structure, and in particular applies to any R-symmetry vector \vec{b} . Notice here that the radial coordinate r in section 3.2 can be *any* choice of radial coordinate on the toric Kähler cone $C(Y_5)$, and does not have to be paired with the R-symmetry vector via a complex structure. Since the fixed point formulas and (3.31), (3.33) necessarily agree for $\vec{b} \in \mathcal{C}^*$, being rational functions of \vec{b} it follows that they agree for all \vec{b} . We shall make use of (3.31) and (3.33) in the examples in section 4.

In the next section, associated with $\text{AdS}_3 \times T^2 \times Y_5$ solutions, we will meet examples of complex cones $C(Y_5) = \mathbb{R}_{>0} \times Y_5$ that have a holomorphic volume form and a holomorphic $U(1)^3$ action, but do *not* admit any compatible Kähler cone metric, consistent with the obstruction theorem in the last subsection. As for toric Kähler cones, one can still define vectors $\vec{v}_a \in \mathbb{Z}^3$ for such geometries: the index a labels the components of torus-invariant complex codimension one submanifolds, with the vector \vec{v}_a specifying the $U(1) \subset U(1)^3$ that fixes a given component. The vectors are normalized to be primitive, so that they define an effective $U(1)$ action on the normal space to the fixed point set. The signs are fixed so that minus the complex structure pairs each vector with a radial vector that points inwards from the fixed point set (rather than outwards). However, for the examples that we shall discuss the set of these \vec{v}_a 's no longer define a convex polyhedral cone \mathcal{C} , as in (3.30). We thus refer to them as “non-convex” toric cones. Notice that both sides of the equations (3.31) and (3.33) still make sense for this class of geometries. In fact we conjecture that (3.31) and (3.32) continue to hold for non-convex/non-Kähler cones, although we currently have

no general proof of this.¹² We shall instead find by explicit computation in examples that (3.31) and (3.33) do always hold, where recall that (3.33) follows from (3.31), (3.32).

Finally, assuming (3.32) holds, we may derive another interesting formula, that is also valid for non-convex toric cones. Similarly to (3.33) we may immediately deduce

$$\int_{Y_5} \eta \wedge \rho \wedge \omega = 2\pi \sum_{a=1}^d \int_{S_a} \eta \wedge \omega, \tag{3.34}$$

where recall that ω is the transverse Kähler form on Y_5 . On the other hand, combining this with (3.4) then gives

$$2N = \frac{2\pi L^4}{(2\pi\ell_s)^4 g_s} \sum_{a=1}^d \int_{S_a} \eta \wedge \omega. \tag{3.35}$$

One can calculate R-charges of baryonic operators in the dual SCFT that are associated with D3-branes wrapping supersymmetric three-cycles of $Y_5 \subset T^2 \times Y_5$. As discussed for the explicit examples in [48], that we re-examine in section 4.1, one can associate such a baryonic operator to each $S_a \subset Y_5$, $a = 1, \dots, d$. Moreover, we can recast the expression for the R-charges of these operators as

$$R[S_a] = \frac{L^4}{(2\pi)^3 \ell_s^4 g_s} \int_{S_a} e^{-B} \text{vol}(S_a) = \frac{L^4}{(2\pi)^3 \ell_s^4 g_s} \int_{S_a} \eta \wedge \omega, \tag{3.36}$$

where $\text{vol}(S_a)$ is the volume form with respect to the metric on Y_5 pulled back to the three-cycle, and similarly the integrand in the second expression is understood to be pulled back to the cycle. Combining with (3.35) then gives

$$2N = \sum_{a=1}^d R[S_a]. \tag{3.37}$$

We shall see via explicit computation that this indeed holds in all our examples.

4 Examples

In this section we present and discuss various examples of Y_5 geometries that we discussed in section 3. Each example has quite different features, both geometrically and physically. In all of the examples the complex cones $C(Y_5)$ are toric cones, which, as we discussed in section 3.4, are necessarily non-convex toric for the $\text{AdS}_3 \times T^2 \times Y_5$ solutions.

¹²Actually one can prove (3.31) by directly computing both sides, as sketched in this footnote, although we believe there should be a better approach. Note first, for example by choosing a quasi-regular R-symmetry vector, that the S_a are total spaces of orbifold circle bundles over a toric orbifold Riemann surface. The latter is necessarily a weighted projective space. One can then evaluate the left hand side using the localization formula in section 3.2, partially resolving the cone $C(S_a) = \mathbb{R}_{>0} \times S_a$ to the total space of the associated complex line orbundle over the weighted projective space. This gives a completely explicit formula that can be compared to the right hand side of (3.31), to see that they agree. Notice that when the degree of the line bundle is negative these are toric and Kähler, for which the methods of [6, 7] may instead be used to deduce (3.31), but for positive degree we only have the approach sketched here.

4.1 $Y_5 = \mathcal{Y}^{p,q}$

In this section we apply the above formalism to a family of examples where there is also an explicitly known supergravity solution, first constructed in [24]. A proposal for the dual field theory was made in [26], and also subsequently discussed in [48]. Here we first extend the discussion of the geometry of these solutions and demonstrate that the proposal of [26] is not correct.

From section 4 of [24], the metric on the internal space Y_7 is¹³

$$ds_7^2 = \frac{dy^2}{4\beta^2 y^2 U(y)} + \frac{U(y)}{4(\beta^2 - 1 + 2y)} (d\psi - \cos\theta d\phi)^2 + \frac{1}{4\beta^2} (d\theta^2 + \sin^2\theta d\phi^2) + \frac{\beta^2 - 1 + 2y}{4\beta^2} D\gamma^2 + \frac{y}{\beta^2} ds_{T^2}^2, \quad (4.1)$$

with $D\gamma \equiv d\gamma - g(y)(d\psi - \cos\theta d\phi)$, and where

$$U(y) = 1 - \frac{1}{\beta^2} (1 - y)^2, \quad g(y) = \frac{y}{\beta^2 - 1 + 2y}. \quad (4.2)$$

The parameter $\beta = p/q$, where p, q are relatively prime integers satisfying

$$q > p > 0. \quad (4.3)$$

The ranges of coordinates are $0 \leq \theta \leq \pi/2$, $0 \leq \phi < 2\pi$, $0 \leq \psi < 2\pi$, $1 - \beta \leq y \leq 1 + \beta$, $0 \leq \gamma < 2\pi l$, with $l = 2q/(q^2 - p^2)$. In these coordinates, the R-symmetry vector is

$$\xi = 2(\partial_\psi + \partial_\gamma). \quad (4.4)$$

Topologically $Y_7 = T^2 \times Y_5$, where $Y_5 = \mathcal{Y}^{p,q} \cong S^2 \times S^3$. The local metric on $\mathcal{Y}^{p,q}$, and how it extends globally to a smooth metric on $S^2 \times S^3$, is very analogous to the construction of the $Y^{p,q}$ Sasaki-Einstein manifolds [28]. In particular the first line of (4.1) is a smooth metric on a base four-manifold, realized geometrically as an S^2 bundle over S^2 . Here the y and ψ coordinates are polar and azimuthal coordinates on the fibre S^2 , respectively. There is then a circle bundle fibred over this base, with circle coordinate γ/l , with the integers p and q being Chern numbers describing the twisting. One may then equivalently view $\mathcal{Y}^{p,q}$ as the total space of a Lens space S^3/\mathbb{Z}_p fibred over S^2 , where the integer q determines the twisting.

The $\mathcal{Y}^{p,q}$ metrics have three commuting Killing vectors, namely $\partial_\phi, \partial_\psi, \partial_\gamma$. Appropriately normalized, these generate a $U(1)^3$ isometry, and following the discussion in [49] one can check that

$$\partial_{\varphi_1} \equiv \partial_\psi + \frac{q-p}{2} l \partial_\gamma, \quad \partial_{\varphi_2} \equiv -\partial_\phi - \frac{q-p}{2} l \partial_\gamma, \quad \partial_{\varphi_3} \equiv \partial_\phi + \left(\frac{q-p}{2} + 1\right) l \partial_\gamma, \quad (4.5)$$

generate an effective action of this torus. Thus each φ_i , $i = 1, 2, 3$, has canonical period 2π . Of course, this basis is unique only up to the action of $SL(3, \mathbb{Z})$. We have chosen

¹³Note that we follow the notation of [24], but we have relabelled the coordinate $\gamma = z^{there}$ and also $p = q_{there}$, $q = p_{there} + q_{there}$.

the basis in (4.5) analogously to that in [49] for the Sasaki-Einstein $Y^{p,q}$ metrics. The complex cone $C(\mathcal{Y}^{p,q})$ is then toric, which means there is a holomorphic action of $(\mathbb{C}^*)^3$, with $U(1)^3 \subset (\mathbb{C}^*)^3$ being the above isometry, and where the complex structure pairs each Killing vector with a corresponding holomorphic vector field. The holomorphic volume form Ψ on the cone, defined in (2.14), has charge 1 under ∂_ψ , but is not charged under ∂_ϕ or ∂_γ . The basis (4.5) is thus the same as in (2.18), with $\partial_{\varphi_2}, \partial_{\varphi_3}$ generating flavour isometries.

The torus action generated by (4.5) has fixed points, and it is straightforward to determine the location of the complex codimension one fixed point sets, which have maximal dimension. There are four, each fixed by a particular $U(1) \subset U(1)^3$ determined by the vectors

$$\vec{v}_1 = (1, 0, 0), \quad \vec{v}_2 = (1, 1, 0), \quad \vec{v}_3 = (1, p, p), \quad \vec{v}_4 = (1, p - q - 1, p - q). \quad (4.6)$$

Using (4.5) and the metric (4.1), one can check that the corresponding four Killing vectors vanish at $\{y = 1 + \beta\}$, $\{\theta = \pi\}$, $\{y = 1 - \beta\}$, $\{\theta = 0\}$, respectively. We denote the corresponding torus-invariant three-submanifolds of $\mathcal{Y}^{p,q}$ by S_a , $a = 1, 2, 3, 4$, respectively. As discussed at the end of section 3.3, the vectors in (4.6) are normalized to be primitive, with signs fixed so that the minus complex structure pairs each vector with a radial vector that points inwards from the fixed point set. This same toric data was derived via a different route in appendix E of [48]. In this reference the authors noted that if one writes $\vec{v}_a = (1, \vec{w}_a)$, with $\vec{w}_a \in \mathbb{Z}^2$, and plots the vectors \vec{w}_a in the plane $\mathbb{R}^2 \supset \mathbb{Z}^2$, the resulting so-called toric diagram is not convex. Correspondingly, for $q > p > 0$ the vectors (4.6) no longer define a convex polyhedral cone \mathcal{C} , via (3.30). As discussed in section 3.4, this non-convexity is a *necessary* condition to admit a solution. Notice that when $p > q > 0$ instead (4.6) gives the convex toric data [49] for the Calabi-Yau cones $C(Y^{p,q})$ over the Sasaki-Einstein five-manifolds $Y^{p,q}$ of [28], which of course do admit a compatible Kähler cone metric.

To understand the global structure more clearly, it is helpful to introduce a partial resolution $\hat{C}(\mathcal{Y}^{p,q})$ of the cone $C(\mathcal{Y}^{p,q}) \cong \mathbb{R}_{>0} \times \mathcal{Y}^{p,q}$. Since $\mathcal{Y}^{p,q}$ may be viewed as a Lens space S^3/\mathbb{Z}_p bundle over S^2 , there is a natural choice: namely we take $\hat{C}(\mathcal{Y}^{p,q})$ to be the total space of the associated $\mathbb{C}^2/\mathbb{Z}_p$ fibration over S^2 , where S^3/\mathbb{Z}_p may be viewed as the boundary of $\mathbb{C}^2/\mathbb{Z}_p$. When $p > q > 0$ explicit Calabi-Yau metrics were constructed on these spaces in [50], and we may describe the fibration structure following that reference. Denote standard complex coordinates on \mathbb{C}^2 by (z_1, z_2) , and let $U(1)_1$ rotate the z_1 coordinate with charge 1, and $U(1)_A$ be the anti-diagonal action in which z_1, z_2 have charges -1 and $+1$, respectively. Here the \mathbb{Z}_p action on \mathbb{C}^2 is via $\mathbb{Z}_p \subset U(1)_A$. Then we twist the $\mathbb{C}^2/\mathbb{Z}_p$ fibre over S^2 using the canonical bundle $O(-2)$ of S^2 for the $U(1)_1$ action, and using $O(-p - q)$ for the action of $U(1)_A/\mathbb{Z}_p$. The resulting space has a family of \mathbb{Z}_p orbifold singularities along the S^2 zero section. In particular notice that setting $z_1 = 0$ and $z_2 = 0$ both give \mathbb{C}/\mathbb{Z}_p fibrations over S^2 . The corresponding circle bundles $S^1 \subset \mathbb{C}/\mathbb{Z}_p$ are easily computed using the above description: setting $z_2 = 0$ gives the circle bundle corresponding to $O(-p - q)$, while $z_1 = 0$ gives $O(-2p + (p + q)) = O(-p + q)$. In fact this is precisely how the Chern numbers p and q are defined in the first place.

Since $q > p > 0$ are arbitrary, it is convenient to temporarily set $p = 1$ in what follows, so that the above partial resolution is in fact a smooth manifold. The above description

then identifies

$$\hat{C}(\mathcal{Y}^{1,q}) = \text{total space of } O(-1+q) \oplus O(-1-q) \rightarrow S^2. \quad (4.7)$$

Although this is a perfectly good smooth complex manifold, with an obvious holomorphic $U(1)^3$ action, it is *not* a toric manifold in the usual sense. The problem can be seen by looking at the zero section of the second $O(-1-q)$ factor, which gives an embedded copy of

$$M_q \equiv \text{total space of } O(-1+q) \rightarrow S^2. \quad (4.8)$$

For $q > 1$ this manifold is complex, but it has no regular non-constant holomorphic functions. Indeed, one can describe the holomorphic functions on M_q by Fourier decomposing along the fibre direction, or equivalently via charges under the $U(1)_1$ that acts on the fibre. A holomorphic function of charge $k \in \mathbb{Z}$ corresponds, in its dependence on the S^2 coordinates, to a section of $O(-k(-1+q))$. But for $k > 0$ there are no such holomorphic sections, only meromorphic sections, and for $k < 0$ the dependence of the function on the fibre coordinate z_1 is proportional to z_1^k , which has a pole singularity at the zero section $\{z_1 = 0\} \cong S^2$. When $k = 0$ we have just the constant function. On the other hand, when $q \leq 0$ instead M_q is a toric variety in the usual sense, admitting a compatible Kähler metric, and with a convex toric diagram. The spaces $\hat{C}(\mathcal{Y}^{1,q})$, and more generally $\hat{C}(\mathcal{Y}^{p,q})$ for $p \geq 1$, inherit this non-convexity and lack of holomorphic functions. In particular if one adds a point to the tip of the cones $C(\mathcal{Y}^{p,q}) = \mathbb{R}_{>0} \times \mathcal{Y}^{p,q}$, the resulting spaces are not affine varieties, since they are not generated by their ring of holomorphic functions. This is in contrast to the Calabi-Yau cones $C(Y^{p,q}) = \mathbb{R}_{>0} \times Y^{p,q}$ for $p > q$.

The final geometric data that we need is the weights of the $U(1)^3$ action generated by (4.5) on the fixed points of the partial resolution $\hat{C}(\mathcal{Y}^{p,q})$. There are two points which are fixed by the entire $U(1)^3$ action, namely the south and north poles of the S^2 zero section. Denoting the weights as $\vec{u}_i^{(1)}, \vec{u}_i^{(2)}$, $i = 1, 2, 3$, respectively, one finds

$$\begin{aligned} p\vec{u}_1^{(1)} &= (0, p-q, -p+q+1), & p\vec{u}_2^{(1)} &= (p, q, -1-q), & p\vec{u}_3^{(1)} &= (0, -p, p), \\ p\vec{u}_1^{(2)} &= (p, -p, p-1), & p\vec{u}_2^{(2)} &= (0, 0, 1), & p\vec{u}_3^{(2)} &= (0, p, -p). \end{aligned} \quad (4.9)$$

The normalizations here ensure that we have the correct corresponding weights for the torus action that enter the orbifold localization formula (3.15). When $p > q > 0$ the weights in (4.9) are easily derived using toric geometry methods. In particular, $\vec{u}_1^{(1)}, \vec{u}_2^{(1)}, \vec{u}_1^{(2)}, \vec{u}_2^{(2)}$ are the outward-pointing generating edges of the polyhedral cone \mathcal{C} , and thus each has zero dot products with a pair of vectors \vec{v}_a in (4.6), and positive dot products with the remaining pair. On the other hand, $\vec{u}_3^{(1)}, \vec{u}_3^{(2)}$ are the weights on the tangent space of the blown up S^2 , and these immediately follow from the basis (4.5), given that ∂_ϕ rotates this S^2 with weights ± 1 at the poles, while the other Killing vectors act trivially. This determines the weights at the fixed points in the basis of Killing vectors (4.5) for $p > q$, but since the weights are linear in p and q , in fact this then determines the weights for general p and q .

Of course since the explicit supergravity solution is known in this case, one can impose the flux quantization conditions and compute the central charge in gravity directly, and this

was done in [24]. However, let us see that we can instead recover this using *only* the global complex geometry above, together with the general formulae in section 3. Using (4.9) and the localization formula (3.15) allows us to compute

$$\begin{aligned} \int_{\mathcal{Y}^{p,q}} \eta \wedge \rho \wedge \rho &= 4(2\pi)^3 \sum_{\{\text{fixed points } p_A, A=1,2\}} \frac{1}{d_{p_A}} \prod_{i=1}^3 \frac{1}{(\vec{b}, \vec{u}_i^{(A)})} \\ &= \frac{4(2\pi)^3 p [b_3(p-q-2)q + b_1(q-p)p + b_2(q-p)q]}{b_3 [(b_3(p-1) + b_1p - b_2p) [b_3(q+1) - b_1p - b_2q] [b_3(p-q-1) + b_2(q-p)]]} \end{aligned} \quad (4.10)$$

Here $d_{p_A} = p$ for both $A = 1, 2$, since these are the orders of the orbifold singularities at the two fixed points. We may then impose the constraint (3.3), which sets (4.10) to zero. Solving this for b_3 gives

$$b_3 = \frac{(q-p)(b_1p + b_2q)}{q(q-p+2)}, \quad (4.11)$$

while the holomorphic volume form Ψ has charge 2 under the R-symmetry vector when

$$b_1 = 2. \quad (4.12)$$

This follows from (4.5), together with the fact that Ψ is only charged under ∂_ψ , with charge 1.

To compute the central charge in (3.10) we may again use the localization formula (3.15). There are two natural choices of torus-invariant three-submanifolds, namely the copies of the Lens space fibres S^3/\mathbb{Z}_p over the south and north poles of the base S^2 , where $\theta = \pi$ and $\theta = 0$, respectively. Recall we denoted these by S_2, S_4 , respectively. For p and q relatively prime, as in [28] one can show $\mathcal{Y}^{p,q} \cong S^2 \times S^3$, and $[S_2] = [S_4] = p \in H_3(\mathcal{Y}^{p,q}, \mathbb{Z}) \cong \mathbb{Z}$. On the other hand, using (3.15) we compute

$$\begin{aligned} \int_{S_2} \eta \wedge \rho &= 2(2\pi)^2 \frac{1}{d_{p_1}} \prod_{i=1}^2 \frac{1}{(\vec{b}, \vec{u}_i^{(1)})} = \frac{2(2\pi)^2 p}{b_3 [b_3(p-1) + b_1p - b_2p]}, \\ \int_{S_4} \eta \wedge \rho &= 2(2\pi)^2 \frac{1}{d_{p_2}} \prod_{i=1}^2 \frac{1}{(\vec{b}, \vec{u}_i^{(2)})} \\ &= \frac{2(2\pi)^2 p}{[b_3(q+1) - b_1p - b_2q] [b_3(p-q-1) + b_2(q-p)]}. \end{aligned} \quad (4.13)$$

Notice that for general $\vec{b} = (b_1, b_2, b_3)$ these last two expressions are *not* equal, as expected since although S_2, S_4 are in the same homology class, the forms that are being integrated are not closed. However, as shown in general in section 2.4, once we impose the constraint (3.3) these integrals become invariants of the homology class. We can see this very explicitly in this example: once we fix b_3 as in (4.11) so that the constraint (3.3) holds, we indeed find

$$\int_{S_2} \eta \wedge \rho = \int_{S_4} \eta \wedge \rho = \frac{2(2\pi)^2 p q^2 (q-p+2)^2}{(q-p)(p+q)(b_1p + b_2q) [b_1(q-p+1) - b_2q]}. \quad (4.14)$$

We note that (4.10) may also be computed using the inward-pointing normals \vec{v}_a in (4.6) together with (3.33). For $p > q$, when we have a toric Kähler cone, these are

guaranteed to be the same, as discussed in section 3.4. For $q > p$ the fact these expressions agree supports our conjecture in section 3.4 that (3.33) also holds for non-convex toric geometries.¹⁴ Similarly, the expressions (4.13) agree with (3.31) using (4.6) and setting the index a in (3.31) to $a = 2$ and $a = 4$, respectively.

We now have everything that we need. The trial central charge (3.10) is

$$\mathcal{Z} = \frac{12(2\pi)^2}{\int_{\sigma} \eta \wedge \rho} MN = \frac{12(2\pi)^2}{\frac{1}{p} \int_{S_2} \eta \wedge \rho} MN, \quad (4.15)$$

where $[\sigma] = 1 \in H_3(\mathcal{Y}^{p,q}, \mathbb{Z}) \cong \mathbb{Z}$ is the generating three-cycle. Setting $b_1 = 2$ and extremizing this function over b_2 , we find the critical R-symmetry vector

$$\vec{b} = (b_1, b_2, b_3) = \left(2, \frac{p(q-p)}{q}, \frac{p(q-p)}{q} \right), \quad (4.16)$$

and central charge

$$\mathcal{Z}|_{\text{on-shell}} = c_{\text{sugra}} = \frac{6p^2(q-p)(p+q)}{q^2} MN. \quad (4.17)$$

This agrees with the result for the explicit supergravity solution [24]. Moreover, using (4.16) and the definition of the basis (4.5), one easily checks that

$$\sum_{i=1}^3 b_i \partial_{\varphi_i} = 2(\partial_{\psi} + \partial_z) = \xi, \quad (4.18)$$

agreeing with the R-symmetry vector (4.4) of the original supergravity solution.

Notice that we have computed the central charge of an AdS₃ solution using only the complex geometry of the cone as an input! Of course, this follows from the general prescription of section 2. The supergravity solutions exist only for $q > p > 0$ [24]. Note that $N > 0$ follows from the general formula (3.5), since $A > 0$ is the Kähler class of the T^2 and $S_{\text{SUSY}} > 0$ from (2.20). Provided also $M > 0$ then the central charge (4.17) is positive only if $q > p$. In supergravity recall that c_{sugra} is a positive multiple of the positive supersymmetric action S_{SUSY} (2.38) and hence for consistency one must then take $M > 0$. However, just looking at (4.17) one could potentially have $q < p$ and $M < 0$, and still have a positive central charge; the problem with this is that the $q < p$ complex cones are Kähler, and hence section 3.3 implies such a solution cannot exist.

In the full supergravity solution it is also possible to calculate the R-charges of baryonic operators in the SCFT that are associated with D3-branes wrapping supersymmetric three-cycles of $\mathcal{Y}^{p,q} \subset T^2 \times \mathcal{Y}^{p,q}$. As discussed at the end of section 3.4, we may associate such an operator to each of the three-cycles $S_a \subset \mathcal{Y}^{p,q}$, $a = 1, 2, 3, 4$, introduced above. The corresponding R-charges $R[S_a]$ in (3.36) can be directly evaluated in this case [48] to give

$$R[S_2] = R[S_4] = \frac{p^2}{q^2} N, \quad R[S_1] = R[S_3] = \frac{q^2 - p^2}{q^2} N. \quad (4.19)$$

¹⁴In later examples, discussed in sections 4.3, 4.4, where we do not have explicit metrics or a simple explicit resolution of the relevant complex cones to use (4.10), we will utilize (3.33) to obtain some results which are then predicated on the validity of this conjecture.

Field	Multiplicity	R_0 -charge	$U(1)_B$	$U(1)_{F_1}$	$U(1)_{F_2}$
Y	$(p+q)N^2$	0	$p-q$	0	-1
Z	$(p-q)N^2$	0	$p+q$	0	1
U_1	pN^2	1	$-p$	1	0
U_2	pN^2	1	$-p$	-1	0
V_1	qN^2	1	q	1	1
V_2	qN^2	1	q	-1	1
λ	$2p(N^2-1)$	1	0	0	0

Table 1. The field content of the $Y^{p,q}$ quiver theories.

In particular, notice that (3.37) indeed holds, which supports the conjecture that (3.32) holds in general. Since $N > 0$, here one certainly needs $q > p$ in order that the R-charges of chiral primary operators are positive. It is interesting to note that in this case $R[S_2] + R[S_1] = N$.

We now switch gears and discuss the implications of the above results concerning the $AdS_3 \times T^2 \times \mathcal{Y}^{p,q}$ solutions in the context of the dual field theory proposed in [26]. Specifically, [26] proposed that these supergravity solutions are holographically dual to the four-dimensional $Y^{p,q}$ quiver gauge theories [27] compactified on T^2 , with a baryonic twist. There are some reasons to hope for such an identification. For example, the fact that the $\mathcal{Y}^{p,q}$ is trivially fibred over the T^2 is consistent with the $Y^{p,q}$ quiver theory on T^2 with vanishing twist with respect to the flavour symmetry. Furthermore, the fact that $\mathcal{Y}^{p,q}$ has the same topology as $Y^{p,q}$, with a single three-cycle, and that there is non-vanishing five-form flux on the product of this three-cycle with T^2 is consistent with the presence of non-vanishing baryonic twist. Also, both geometries are specified by a pair of relatively prime integers p, q and most strikingly, it was shown in [26] that the central charge as a function of p, q obtained by c -extremization agrees with that for $\mathcal{Y}^{p,q}$.

However, as already noted in [48] there is an immediate problem with this identification: for the $Y^{p,q}$ field theories necessarily $p \geq q$ (for example, $p - q$ is the multiplicity of the Z fields in the table below), while in the supergravity solutions based on $\mathcal{Y}^{p,q}$ instead $q > p$. Moreover, in this paper we have shown that if the complex cone $C(Y_5)$ is of Calabi-Yau type, then there is *not* a corresponding $AdS_3 \times T^2 \times Y_5$ solution, with the same complex structure on $C(Y_5)$. Thus there cannot be a supergravity solution describing an RG flow across dimensions from $AdS_5 \times Y^{p,q}$ in the UV to $AdS_3 \times T^2 \times \mathcal{Y}^{p,q}$ in the IR, where the complex structure is preserved.¹⁵

Nevertheless, it is interesting to re-examine the c -extremization calculation of [26] in this case as it exhibits a number of features that will recur in later examples. The field content of the $Y^{p,q}$ quiver theories [27] is summarized in the table 1. The gauge group is $SU(N)^{2p}$, the λ are the gauginos and the remaining fields are bifundamental matter fields. The $U(1)_B$ corresponds to the baryonic symmetry associated to the single three-

¹⁵A flow from $AdS_5 \times Y^{p,q}$ to $AdS_3 \times T^2 \times \mathcal{Y}^{q,p}$ is ruled out because the central charges would not agree.

cycle $\sigma = 1 \in H_3(Y^{p,q}, \mathbb{Z}) \cong \mathbb{Z}$, while $U(1)_{F_i}$, $i = 1, 2$, are flavour symmetries corresponding to $U(1)$ isometries under which the holomorphic volume form is uncharged. In particular $U(1)_1 \subset SU(2)$ is the Cartan of the $SU(2)$ isometry that acts on the round S^2 in the metric. Note that R_0 is *not* the R-charge of the dual SCFT in $d = 4$ (which can be found in [27]). Instead R_0 is a simple assignment of a fiducial R-charge that can be used in the c -extremization procedure for the putative $d = 2$ SCFT; we can use any assignment for R_0 compatible with the usual requirements of an R-symmetry (every term in the superpotential has R_0 -charge 2 and the gauginos, λ , have R_0 -charge 1).

We consider these $d = 4$ SCFTs theories wrapped on T^2 , introducing only baryonic flux in the topological twist:

$$T_{\text{top}} = BT_B. \tag{4.20}$$

Geometrically, if these were to be dual to solutions of the form $\text{AdS}_3 \times Y_7$, then this twisting would be associated with a product $Y_7 = T^2 \times Y_5$, as noted above. The trial R-charge is a linear combination

$$T_{\text{trial}} = T_{R_0} + \zeta T_B + \epsilon_1 T_{F_1} + \epsilon_2 T_{F_2}, \tag{4.21}$$

where ζ , ϵ_i are parameters. We now substitute this data into the trial central charge given in [1] and extremize over the parameters η , ϵ_i . The extremal point has

$$\zeta = \frac{q^2 - p^2}{pq^2}, \quad \epsilon_1 = 0, \quad \epsilon_2 = \frac{p^2 - q^2}{pq}. \tag{4.22}$$

and one finds [26]

$$c_{\text{c-ext}} = 6 \frac{p^2(q-p)(p+q)}{q^2} BN^2. \tag{4.23}$$

This formally agrees with the geometric result (4.17) on making the identification

$$M = NB, \tag{4.24}$$

of geometric and field theory baryonic flux parameters M , B , respectively. Physically, (4.24) is consistent with the fact that M is the number of units of F_5 flux through $T^2 \times \sigma$, where σ is the three-cycle in $Y_5 = Y^{p,q} \cong S^2 \times S^3$ that generates the $U(1)_B$ symmetry.

Of course, the immediate problem with identifying the field theory result (4.23) with the gravity result (4.17) is that the ranges of p and q are complimentary. However, as discussed above, the central charge (4.23) can be positive if $p > q$ and also $B < 0$. However, this may be ruled out in *field theory* by examining the R-charges of fields. These are easily computed using (4.22) and we get

$$R_{\text{c-ext}}[U_1] = R_{\text{c-ext}}[U_2] = \frac{p^2}{q^2} N, \quad R_{\text{c-ext}}[Z] = R_{\text{c-ext}}[Y] = \frac{q^2 - p^2}{q^2} N. \tag{4.25}$$

The gauge-invariant baryonic operators constructed from the fields Z, U_1, Y, U_2 precisely correspond to D3-branes wrapped on the three-cycles S_1, S_2, S_3, S_4 , respectively [27]. Thus (4.25) also match the supergravity results (4.19), except again the ranges of p and q are complimentary. Moreover, from the field theory results (4.25) where $p > q$ one sees that

$R[Z]$ and $R[Y]$ are *negative*, which is a contradiction for a chiral operator. The conclusion must be that such a superconformal fixed point, with such R-charges, does not exist. We discuss possible refinements of the c -extremization procedure in section 5.

We shall give some more elaborate examples of a similar “matching” in the remainder of this section. From these examples it seems clear that *mathematically* the central charges and R-charges being computed on both sides will *always agree*. Notice that the quiver theory has encoded in it the complex cone geometry, which arises as the mesonic moduli space, and thus this same complex data is present in both descriptions. It must then be the case that the supergravity formulae of section 2 are computing the *same* geometric objects as c -extremization in the field theory. This is currently far from obvious, but we state it here as a conjecture, based on examples. However, one must be careful when interpreting the results physically, as we have seen. There can be obstructions to the existence of supergravity solutions/superconformal fixed points with the assumed properties. The current status is that the field theory dual to the $\text{AdS}_3 \times T^2 \times \mathcal{Y}^{p,q}$ solutions are not known! On the other hand, the fate of the $Y^{p,q}$ quiver gauge theories on T^2 , with only baryonic twist, is also unknown!

To conclude this subsection we briefly consider the analogous results for the case of $T^{1,1}$. From the field theory perspective we are interested in the reduction of the Klebanov-Witten theory [51] on T^2 with baryon flux only. The four-dimensional theory has four bifundamental fields transforming under an $\text{SU}(N)^2$ gauge group with two abelian non-R flavour symmetries and one baryonic symmetry. The charge assignments of the fields may be obtained by setting $p = 1$ and $q = 0$ in table 1. After carrying out the c -extremization, it is easy to see that the trial central charge does not admit a critical point. The situation is therefore, in some sense, worse than that of $Y^{p,q}$ field theories as the putative theory does not admit a critical point regardless of the positivity of the R-charges. On the gravity side, we have already shown that there are no $\text{AdS}_3 \times T^2 \times T^{1,1}$ type solutions with the same complex structure on the conifold. Again, the situation is worse than the case of $Y^{p,q}$ since we can't even satisfy the constraint equation (3.3). Indeed, after inputting the $d = 4$ toric vectors for the conifold into the constraint equation (3.33), which gives the same result as (4.10) after setting $p = 1, q = 0$, we find that the only solution is $b_1 = 0$, contradicting $b_1 = 2$.

4.2 $Y_5 = \mathcal{L}^{a,b,c}$

In this section we analyse a class of supersymmetric AdS_3 solutions, labelled $Y_5 = \mathcal{L}^{a,b,c}$, that are specified by three positive integers, $\mathbf{a}, \mathbf{b}, \mathbf{c}$, satisfying some additional conditions given below. Just as the $\mathcal{Y}^{p,q}$ solutions are analogues of the $Y^{p,q}$ Sasaki-Einstein metrics [28], the $\mathcal{L}^{a,b,c}$ solutions are analogues¹⁶ of the $L^{a,b,c}$ Sasaki-Einstein metrics constructed in [30]. Moreover, the family of $\mathcal{L}^{a,b,c}$ solutions include the $\mathcal{Y}^{p,q}$ solutions discussed in the last subsection as a special case.

The local metrics for $Y_5 = \mathcal{L}^{a,b,c}$ were first constructed in appendix C of [29]. Here we present sufficient conditions for these to describe regular geometries with suitably quantized flux. Following this analysis we are able to calculate the central charge and R-charges for certain baryonic operators directly. Using coordinates similar to [52], the six-dimensional

¹⁶The reason for using different fonts for the integers, e.g. \mathbf{a} versus a , will become clear later.

local Kähler metric is given by¹⁷

$$ds_6^2 = \frac{\eta - \xi}{\mathcal{F}(\xi)} d\xi^2 + \frac{\mathcal{F}(\xi)}{\eta - \xi} (d\phi + \eta d\psi)^2 + \frac{\eta - \xi}{\mathcal{G}(\eta)} d\eta^2 + \frac{\mathcal{G}(\eta)}{\eta - \xi} (d\phi + \xi d\psi)^2 + ds^2(T^2). \quad (4.26)$$

This is fixed by two functions, which are explicitly given by

$$\mathcal{F}(\xi) = -\mathcal{G}(\xi) - (1 + \alpha)\xi^2, \quad \mathcal{G}(\eta) = -(\eta - 1)(\eta - \beta), \quad (4.27)$$

where α, β are constants. The type IIB AdS₃ solution with $Y_7 = T^2 \times Y_5$ can be obtained from (2.1), (2.5) using the data

$$e^B = \frac{1 + \alpha}{4(\eta - \xi)}, \quad P = \frac{(1 + \alpha)\xi}{\eta - \xi} (d\phi + \eta d\psi) + d\phi + \frac{\beta + 1}{2} d\psi, \\ F = -\frac{2}{(1 + \alpha)} \left(dz + d\phi + \frac{\beta + 1}{2} d\psi \right) \wedge (d\eta - d\xi) + 2d\phi \wedge d\eta - 2\text{vol}(T^2). \quad (4.28)$$

In order to get a positive definite metric we take $\alpha > 0$ and $\beta > 1$ and the ranges of the coordinates η and ξ are taken to be $1 \leq \eta \leq \beta$ and $\xi_- \leq \xi \leq \xi_+$, where ξ_{\pm} are the roots of the quadratic \mathcal{F} . Note that we have $\xi_- < 0 < \xi_+ < 1$ and therefore $\eta - \xi > 0$ everywhere.

We next analyse the additional conditions that are required in order to have a well behaved metric on a regular manifold $Y_7 = T^2 \times Y_5$. The T^2 factor is not relevant and so we can concentrate on Y_5 , closely following the analysis for the Sasaki-Einstein metrics $L^{a,b,c}$ given in [30]. The metric on Y_5 has three commuting Killing vectors given by ∂_ψ , ∂_ϕ and the R-symmetry Killing vector ∂_z . The complete metrics will be cohomogeneity two, with $U(1)^3$ principal orbits. These orbits degenerate at the roots of the two quadratic functions $\mathcal{F}(\xi)$ and $\mathcal{G}(\eta)$, and there are four such degeneration surfaces. Specifically, we find that the degenerating Killing vectors at the surfaces $\eta = 1, \beta$ and $\xi = \xi_{\pm}$ are given by

$$k_1 = -\frac{2}{\beta - 1} \left(\frac{\partial}{\partial\psi} - \frac{\partial}{\partial\phi} - \frac{\beta - 1}{2} \frac{\partial}{\partial z} \right), \\ k_\beta = \frac{2}{\beta - 1} \left(\frac{\partial}{\partial\psi} - \beta \frac{\partial}{\partial\phi} + \frac{\beta - 1}{2} \frac{\partial}{\partial z} \right), \\ l_{\pm} = \mp \frac{2}{\alpha(\xi_+ - \xi_-)} \left(\frac{\partial}{\partial\psi} - \xi_{\pm} \frac{\partial}{\partial\phi} - \left(\alpha\xi_{\pm} + \frac{\beta + 1}{2} \right) \frac{\partial}{\partial z} \right), \quad (4.29)$$

respectively. These vectors have been normalized to have unit surface gravity and the overall signs have been chosen for convenience.

To ensure that the collapsing orbits extend smoothly onto the degeneration surfaces without generating conical singularities, we need to impose suitable conditions on the parameters α, β . After some further analysis we find that the regular solutions, labelled

¹⁷Start from (C.4), (C.5) and (C.27) of [29], with $Q = 0$ (i.e. vanishing three-form flux) and then write $\mu_1 = \cos \frac{\theta}{2}$, $\mu_2 = \sin \frac{\theta}{2}$. After identifying $w = -4\xi$, $\cos \theta = \frac{1}{q_2 - q_1} (8\eta - (q_1 + q_2))$, $\phi_1 = \frac{1}{8} (4\phi + q_2\psi)$, $\phi_2 = \frac{1}{8} (4\phi + q_1\psi)$, $q_1 = 4$, $q_2 = 4\beta$, and $\lambda = -(1 + \alpha)$ one obtains precisely (4.26) and (4.27).

$Y_5 = \mathcal{L}^{a,b,c}$, are fixed by four positive integers a, b, c and d , satisfying the following conditions

$$\begin{aligned} -a + b &= c + d, \\ b > c \geq d > 0, \quad b > a > 0. \end{aligned} \tag{4.30}$$

In particular, only three of these integers are independent and the solutions can be labelled by a, b, c . We further demand that $\text{hcf}(a, b, c, d) = 1$, which implies that any three integers are coprime. In addition we demand that a, b are each coprime to each of c, d . In terms of these integers, the parameters α, β are given by¹⁸

$$\alpha = \frac{ab}{cd}, \quad \beta = \frac{c}{d}. \tag{4.31}$$

The roots of $\mathcal{F}(\xi)$ are then explicitly given by

$$\xi_+ = \frac{c}{b}, \quad \xi_- = -\frac{c}{a}. \tag{4.32}$$

Furthermore, in terms of a, b, c, d the linear relation between the four degenerating Killing vectors (4.29) is given by

$$ck_1 + dk_\beta + al_- - bl_+ = 0. \tag{4.33}$$

Now, since $\text{hcf}(b, c) = 1$, Bézout's identity implies there exist non-unique $k, l \in \mathbb{Z}$, satisfying

$$bl + ck = 1. \tag{4.34}$$

Let us fix such a pair k, l . Using this we can replace the coordinates (ϕ, ψ, z) parametrizing the three-torus with a new set of coordinates, $(\varphi_1, \varphi_2, \varphi_3)$, via

$$\begin{aligned} \phi &= -\frac{2c}{c-d}\varphi_1 + \frac{2c[a(a+b) + d(c-d)]}{a(a+b)(c-d)}\varphi_2 \\ &\quad + \frac{2[cd(c-d)k + (a+b)b(dl + c(k+l))]}{b(a+b)(c-d)}\varphi_3, \\ \psi &= \frac{2d}{c-d}\varphi_1 - \frac{4d(a+d)}{(a+b)(c-d)}\varphi_2 - \frac{4d[a+b + ak(b-c)]}{b(a+b)(c-d)}\varphi_3, \\ z &= \varphi_1. \end{aligned} \tag{4.35}$$

The Killing vectors ∂_{φ_i} generate an effective torus action, and moreover the range of these coordinates is $0 \leq \varphi_i \leq 2\pi$. Specifically, we have

$$\begin{pmatrix} k_\beta \\ k_1 \\ l_+ \\ l_- \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -ak & b \\ 1 & al & c \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \partial_{\varphi_1} \\ \partial_{\varphi_2} \\ \partial_{\varphi_3} \end{pmatrix}. \tag{4.36}$$

¹⁸Note that when $c = d$, the two roots of η coincide. This case, which leads back to the $\mathcal{Y}^{p,q}$ examples, needs to be treated with different coordinates; for example the change of coordinates described in footnote 17 is no longer valid.

As in the discussion around (4.6) we can extract the toric data on the cone over $\mathcal{L}^{a,b,c}$. Specifically, the four vectors that define vanishing $U(1)$ s along complex codimension one submanifolds are given by

$$\vec{v}_1 = (1, 1, 0), \quad \vec{v}_2 = (1, -\mathbf{ak}, \mathbf{b}), \quad \vec{v}_3 = (1, \mathbf{a1}, \mathbf{c}), \quad \vec{v}_4 = (1, 0, 0), \quad (4.37)$$

with the corresponding submanifolds being defined by $\xi = \xi_-, \eta = 1, \xi = \xi_+, \eta = \beta$, respectively. We denote the corresponding torus-invariant three-submanifolds of $\mathcal{L}^{a,b,c}$ by S_a , $a = 1, 2, 3, 4$, respectively. One can compare this with the toric data for the $L^{a,b,c}$ Sasaki-Einstein metrics given in eq. (3.2) of [33], and we observe that the sign of the second entry in \vec{v}_2 and \vec{v}_3 differs. This implies that the putative toric diagram that we can construct for the complex geometry on the cone associated with the $\mathcal{L}^{a,b,c}$ solutions is not convex (in contrast to that for $L^{a,b,c}$).

We can recover the $\mathcal{Y}^{p,q}$ solutions as a special case of the $\mathcal{L}^{a,b,c}$ solutions. Specifically, we should set

$$\mathbf{b} = p + q, \quad \mathbf{a} = q - p, \quad \mathbf{c} = \mathbf{d} = p, \quad (4.38)$$

which solves $-\mathbf{a} + \mathbf{b} = \mathbf{c} + \mathbf{d}$ (and we recall the comment in footnote 18). In particular, if one substitutes this into (4.37), and carries out a suitable $SL(3, \mathbb{Z})$ transformation, then we recover (4.6).

We now discuss the quantization of the five-form flux for the $\mathcal{L}^{a,b,c}$ solutions. The relevant part of the five-form in (2.1), tangent to $Y_7 = T^2 \times \mathcal{L}^{a,b,c}$, is given by

$$\begin{aligned} L^{-4}F_5|_{Y_7} = & -\frac{1+\alpha}{4(\eta-\xi)^2} \left(\text{vol}(T^2) \wedge (\mathcal{F}(\xi)d\eta + \mathcal{G}(\eta)d\xi) \wedge d\phi \wedge d\psi \right. \\ & + \text{vol}(T^2) \wedge (dz + P) \wedge [\xi d\eta \wedge (d\phi + \xi d\psi) - \eta d\xi \wedge (d\phi + \eta d\psi)] \\ & \left. + (\eta - \xi)^2 (dz + P) \wedge d\eta \wedge d\xi \wedge d\phi \wedge d\psi \right). \end{aligned} \quad (4.39)$$

The topology of $\mathcal{L}^{a,b,c}$ is $S^2 \times S^3$ and can be established as in [30, 33]. Thus, there is a single generator, σ , of $H_3(\mathcal{L}^{a,b,c}, \mathbb{Z}) \cong \mathbb{Z}$. With respect to the torus invariant three manifolds S_a , defined above, we have the homology relations in $H_3(\mathcal{L}^{a,b,c}, \mathbb{Z})$:

$$[S_1] = \mathbf{a}\sigma, \quad [S_2] = \mathbf{c}\sigma, \quad [S_3] = \mathbf{b}\sigma, \quad [S_4] = \mathbf{d}\sigma. \quad (4.40)$$

We find that it is sufficient if we impose the following quantization conditions

$$\frac{L^4}{\pi l_s^4 g_s} = \frac{\mathbf{a}^2 \mathbf{b}^2 \mathbf{d}}{(\mathbf{ab} + \mathbf{cd})^2} N, \quad \frac{1}{4\pi} \text{vol}(T^2) = \frac{(\mathbf{ab} + \mathbf{cd})\mathbf{c}}{\mathbf{ab}} \frac{M}{N}, \quad (4.41)$$

where N, M are positive integers. Indeed we then find

$$\frac{1}{(2\pi l_s)^4 g_s} \int_{Y_5} F_5 = N, \quad \frac{1}{(2\pi l_s)^4 g_s} \int_{T^2 \times \sigma} F_5 = M. \quad (4.42)$$

It is now straightforward to calculate the central charge for the dual two-dimensional CFT using (2.36), and we obtain

$$c_{\text{sugra}} = 6 \frac{\mathbf{abcd}}{\mathbf{ab} + \mathbf{cd}} NM. \quad (4.43)$$

It is also possible to calculate the R-charges of baryonic operators in the CFT that are associated with D3-branes wrapping supersymmetric three-cycles of $T^2 \times \mathcal{L}^{a,b,c}$. As discussed in [48], these are precisely the three-cycles S_a discussed above. Using the general expression (3.36) and the first expression in (4.42) we find¹⁹

$$\begin{aligned} R[S_2] &= R[S_4] = \frac{cd}{ab + cd} N, \\ R[S_1] &= R[S_3] = \frac{ab}{ab + cd} N. \end{aligned} \tag{4.44}$$

We observe that once again (3.37) indeed holds which provides further support for the conjecture that (3.32) holds in general. It is interesting to note that in this case we also have $R[S_2] + R[S_1] = N$. We also notice that upon substituting the $\mathcal{D}^{p,q}$ values for a, b, c, d , given in (4.38), into (4.43) and (4.44) we recover (4.17) and (4.19), respectively.

In principle, we can also obtain the central charge using the results of section 3. To use the formula (3.15) we would need to find a suitable resolution of the cone over $\mathcal{L}^{a,b,c}$. We will not pursue that here, but instead we point out that we obtain the correct central charge, $\mathcal{Z}|_{\text{on-shell}} = c_{\text{sugra}}$, if we again use (3.31) and (3.33), each of which may be checked explicitly in this case. In other words, this provides further evidence for the conjecture we made in section 3.4.

We now consider the $d = 4$ quiver gauge theories dual to the $\text{AdS}_5 \times L^{a,b,c}$ type IIB solutions, and reduce them on a T^2 with non-vanishing baryonic flux. Assuming that these field theories flow in the far IR to a $d = 2$ $(0, 2)$ SCFT we can attempt to calculate the putative central charge, $c_{c\text{-ext}}$, as well as the R-charges of certain operators, using c -extremization [1, 25, 26].

The field content of the $L^{a,b,c}$ quiver theories, which have gauge group $\text{SU}(N)^{a+b}$ [31–33] is summarized in the table 2. Once again the λ are the gauginos and the remaining fields are bifundamental or adjoint matter fields.

Here, a, b, c, d satisfy

$$\begin{aligned} a + b &= c + d, \\ b &\geq a, c, d > 0, \end{aligned} \tag{4.45}$$

and $k, l \in \mathbb{Z}$ are chosen to satisfy

$$bl + ck = 1. \tag{4.46}$$

Once again R_0 is a fiducial choice of R-charge to use in c -extremization. Note that the integers k, l are not unique: for a given (k, l) satisfying (4.46) we also have that $(k + nb, l - nc)$ satisfies (4.46) for arbitrary $n \in \mathbb{Z}$. In the quiver gauge theory, this ambiguity is associated with the following redefinition of the flavour symmetry: $\text{U}(1)_{F_2} \rightarrow \text{U}(1)_{F_2} + n \text{U}(1)_B - na \text{U}(1)_{F_1}$.

¹⁹In carrying out the explicit integrals, one needs to use a good set of coordinates on the three-cycle. The coordinates introduced in (4.35) have the feature that on the surface $\eta = \beta$, it is the Killing vector ∂_{φ_1} that is degenerating. Thus, we can use the coordinates $\xi, \varphi_2, \varphi_3$ to parametrize S_β and the integral is straightforward to carry out. For each of the other three degenerating surfaces we should use another set of coordinates, obtained by using a suitable $\text{SL}(3, \mathbb{Z})$ transformation on the ∂_{φ_i} in order to have a similar feature.

Field	Multiplicity	R_0 -charge	$U(1)_B$	$U(1)_{F_1}$	$U(1)_{F_2}$
Y	bN^2	0	a	1	0
Z	aN^2	0	b	0	k
U_1	dN^2	1	$-c$	0	l
U_2	cN^2	1	$-d$	-1	$-l - k$
V_1	$(b - c)N^2$	1	$b - d$	-1	$-l$
V_2	$(b - d)N^2$	1	$b - c$	0	$l + k$
λ	$(a + b)(N^2 - 1)$	1	0	0	0

Table 2. The field content of the $L^{a,b,c}$ quiver theories.

We now wrap these quiver theories on a T^2 and topologically twist with baryonic flux only, as in (4.20). The trial R-symmetry is constructed as in (4.21). After substituting this into the trial central charge given in [1] and extremizing over the parameters η, ϵ_i we find the central charge

$$c_{c\text{-ext}}(L^{a,b,c}) = 6 \frac{abcd}{(b-c)(b-d)} BN^2 = 6 \frac{abcd}{ab-cd} BN^2. \quad (4.47)$$

The middle expression is manifestly positive for $B > 0$ by (4.45). Furthermore,²⁰ the R-charges are

$$\begin{aligned} R_{c\text{-ext}}[Y] &= R_{c\text{-ext}}[Z] = -\frac{ab}{(b-d)(b-c)}N = \frac{ab}{ab-cd}N, \\ R_{c\text{-ext}}[U_1] &= R_{c\text{-ext}}[U_2] = \frac{cd}{(b-d)(c-b)}N = -\frac{cd}{ab-cd}N. \end{aligned} \quad (4.48)$$

From the R-charges it is clear that it is not possible to suitably tune the integers to simultaneously make all R-charges positive as required for chiral operators. The conclusion is the same as that of the previous section: there is no such superconformal fixed point. We make some additional comments on this point in section 5.

Nevertheless, it is interesting to compare the result of $c_{c\text{-ext}}(L^{a,b,c})$ and $R_{c\text{-ext}}$ with the results obtained for $\mathcal{L}^{a,b,c}$ in (4.43). A naïve matching as in the $Y^{p,q}$ versus $\mathcal{Y}^{p,q}$ scenario does not work. However, if we make the formal identification

$$a = -\mathbf{a}. \quad (4.49)$$

then they do precisely agree, as do the defining relations for the integers. Of course this matching is only formal, since both a and \mathbf{a} must be positive.

It remains an open question to identify the CFTs dual to the explicit $\text{AdS}_3 \times T^2 \times \mathcal{L}^{a,b,c}$ solutions. It also remains an open question to determine the IR behaviour of the $L^{a,b,c}$ quiver gauge theories reduced on T^2 , twisted with baryonic flux.

²⁰Observe that setting $c = d = p$, $a = p - q$, $b = p + q$, which takes $L^{a,b,c}$ to $Y^{p,q}$, we precisely recover the results in (4.23) and (4.25).

4.3 $Y_5 = X^{p,q}$

The examples in this subsection and the next are somewhat different. We have already argued in section 3.3 that if the complex manifold $C(Y_5) = \mathbb{R}_{>0} \times Y_5$ is of Calabi-Yau type, then there is no corresponding $\text{AdS}_3 \times T^2 \times Y_5$ solution. On the other hand, we have also seen in section 4.1 that if one formally applies the geometric formulae (3.31), (3.33) to the Calabi-Yau cones $C(Y^{p,q})$ with $p > q$, the resulting central charge agrees with that computed using c -extremization for the four-dimensional $Y^{p,q}$ quiver gauge theories [27] compactified on T^2 . Similar remarks apply to the $L^{a,b,c}$ quiver gauge theories [31–33] and section 4.2. In the next two subsections we show that this formal matching continues to hold for other complex cones $C(Y_5)$ of Calabi-Yau type. In particular for the examples in this subsection $b_3(Y_5) = 2$, for which we must also impose (3.11). In this case there is no extremization to perform in the geometric computation, since flux quantization determines uniquely the R-symmetry vector. The result perfectly matches the c -extremization result for the $X^{p,q}$ quiver gauge theories [53] compactified on T^2 . In this subsection we will first discuss these points before concluding with a conjecture concerning the existence of a new family of $\text{AdS}_3 \times T^2 \times \mathcal{X}^{p,q}$ solutions, with $q > p > 0$.

The $X^{p,q}$ Calabi-Yau cones are toric, with inward-pointing normal vectors

$$\begin{aligned} \vec{v}_1 &= (1, 1, 0), & \vec{v}_2 &= (1, 2, 0), & \vec{v}_3 &= (1, 1, p), \\ \vec{v}_4 &= (1, 0, p - q + 1), & \vec{v}_5 &= (1, 0, p - q). \end{aligned} \quad (4.50)$$

Here $p > q > 0$ are integers. Being Calabi-Yau cones, these give rise to supersymmetric $\text{AdS}_5 \times X^{p,q}$ solutions of type IIB which are holographically dual to corresponding four-dimensional $\mathcal{N} = 1$ quiver gauge theories. These gauge theories were presented in [53], and by construction may be Higgsed to the $Y^{p,q}$ theories. In particular for $p = 2$, $q = 1$ the complex cone is the canonical complex cone over the second del Pezzo surface, dP_2 . The five-manifolds $Y_5 = X^{p,q}$ have $b_3(X^{p,q}) = 2$.

Using this data we may apply our general formulae from section 3. Inserting (4.50) into the constraint equation (3.3) using (3.33) immediately gives

$$0 = \frac{b_1[b_3^2 - 2b_3p(b_1 + b_2(q-1)) - p(p-q)b_2(b_1(1+p-q) + b_2(q-1))]}{b_2b_3[b_3 + (-2b_1 + b_2)p][b_3 - (b_1 - b_2)(p-q)][b_1(1+p-q) + b_2(q-1) - b_3]} \quad (4.51)$$

This may be solved for b_3 , giving

$$\begin{aligned} b_3 &= p(b_1 - b_2) + b_2pq \\ &+ \sqrt{p[b_1^2p + b_2^2(p-1)(q-1)q + b_1b_2(p(p-1) + q(q-1))]} \end{aligned} \quad (4.52)$$

There are five toric divisors, which map to the facets $\{(\vec{y}, \vec{v}_a) = 0\}$ in the moment map cone \mathcal{C} , $a = 1, \dots, 5$. These toric divisors are in turn cones over torus-invariant three-dimensional submanifolds S_a . On the other hand, since $b_2(X^{p,q}) = 2$ there are two generating three-cycles σ_1, σ_2 . The homology relations in $H_3(X^{p,q}, \mathbb{Z})$ are correspondingly

$$\begin{aligned} [S_1] &= \sigma_1, & [S_2] &= \sigma_2, & [S_3] &= -\sigma_1 - 2\sigma_2, & [S_4] &= p\sigma_1 + (p+q)\sigma_2, \\ [S_5] &= -p\sigma_1 + (1-p-q)\sigma_2. \end{aligned} \quad (4.53)$$

Field	Multiplicity	R_0 -charge	$U(1)_{B_1}$	$U(1)_{B_2}$	$U(1)_{F_1}$	$U(1)_{F_2}$
X_{12}	pN^2	0	0	-1	1	0
X_{23}	$(p+q-1)N^2$	2	1	2	0	0
X_{34}	N^2	0	$-p$	$-p-q$	0	1
X_{45}	N^2	0	p	$p+q-1$	0	-1
X_{51}	$(p-q)N^2$	0	-1	0	-1	0
X_{24}	N^2	2	$1-p$	$2-p-q$	0	1
X_{31}	$(q-1)N^2$	0	-1	-1	-1	0
X_{35}	$(p-1)N^2$	0	0	-1	0	0
X_{41}	N^2	0	$p-1$	$p+q-1$	-1	-1
X_{52}	qN^2	0	-1	-1	0	0
λ	$(2p+1)(N^2-1)$	1	0	0	0	0

Table 3. The field content of the $X^{p,q}$ quiver theories.

Flux quantization (3.9) imposes

$$\int_{\sigma_1} \eta \wedge \rho = \frac{2(2\pi\ell_s)^4 g_s}{AL^4} M_1, \quad \int_{\sigma_2} \eta \wedge \rho = \frac{2(2\pi\ell_s)^4 g_s}{AL^4} M_2, \quad (4.54)$$

which implies

$$\frac{\int_{\sigma_1} \eta \wedge \rho}{\int_{\sigma_2} \eta \wedge \rho} = \frac{M_1}{M_2}, \quad (4.55)$$

as in (3.11). On the other hand, we may compute the ratio on the left hand side explicitly in terms of toric data using (4.53) and the general formula (3.31). This leads to the following expression for b_2 in terms of the flux integers M_1 and M_2 :

$$b_2 = -\frac{b_1 M_1 p (M_1 + 2M_2)}{M_1^2 (p-1)p + 2M_1 M_2 p (p-1) + M_2^2 [p(p-1) - q(q-1)]}. \quad (4.56)$$

Finally, setting $b_1 = 2$ so that the holomorphic volume form has charge 2, the geometric formula for the central charge is

$$\begin{aligned} & \mathcal{Z}|_{\text{on-shell}} \quad (4.57) \\ &= \frac{24NM_1M_2(M_1 + 2M_2)p(p-q)(p+q-1)[M_1p + M_2(p+q-1)][M_1p + M_2(p+q)]}{[(M_1 + M_2)^2(p-1)p + M_2^2q(1-q)]^2}. \end{aligned}$$

Note that in this case there is no extremization to do: flux quantization uniquely fixes the R-symmetry vector.

We may now consider the four-dimensional $X^{p,q}$ quiver gauge theories on T^2 . The theory has $SU(N)^{2p+1}$ gauge group and the field content is summarized in table 3, where the λ are the gauginos and the remaining fields are bifundamental matter fields. Here $U(1)_{B_I}$, $I = 1, 2$, correspond to baryonic symmetries associated to the three-cycles σ_1, σ_2 ,

respectively, while $U(1)_{F_i}$, $i = 1, 2$, are flavour symmetries corresponding to $U(1)$ isometries under which the holomorphic volume form is uncharged.

We wrap these theories on T^2 , introducing only baryonic flux in the topological twist

$$T_{\text{top}} = B_1 T_{B_1} + B_2 T_{B_2}. \tag{4.58}$$

Geometrically this corresponds to a product $T^2 \times Y_5$. The trial R-charge is a linear combination

$$T_{\text{trial}} = T_{R_0} + \zeta_1 T_{B_1} + \zeta_2 T_{B_2} + \epsilon_1 T_{F_1} + \epsilon_2 T_{F_2}, \tag{4.59}$$

where ζ_I , ϵ_i are parameters. Substituting this data into the trial central charge of [1] and extremizing over the parameters ζ_I , ϵ_i , we find

$$c_{\text{c-ext}} = \frac{24N^2 B_1 B_2 (B_1 + 2B_2) p(p-q)(p+q-1) ([B_1 p + B_2(p+q-1)][B_1 p + B_2(p+q)])}{[(B_1 + B_2)^2 (p-1)p + B_2^2 q(1-q)]^2}. \tag{4.60}$$

This agrees with the geometric result (4.57) on making the identification

$$M_a = N B_a, \tag{4.61}$$

of geometric and field theory baryonic flux parameters M_a , B_a , respectively. As for the quiver gauge theories for $Y^{p,q}$ and $L^{a,b,c}$ that we discussed in previous subsections, we can also determine the R-charges of various fields, and a numerical investigation with $q < p \leq 500$ shows that there is always a chiral operator with negative R-charge. Thus, we can again conclude that the associated superconformal fixed point with central charge as in (4.60) with $p > q > 0$ does not exist. It again remains an open problem as to the fate of these quiver gauge theories compactified on T^2 with baryonic flux only, a point we return to in section 5.

Notice, however, that (4.57) is positive if $q > p > 0$ and $M_1, M_2 > 0$. In fact we find that all the R-charges of fields are also positive in this range, at least for $p < q \leq 500$. Given the similarities with the explicit $\mathcal{Y}^{p,q}$ and $\mathcal{L}^{a,b,c}$ solutions in section 4.1 and 4.2, we are thus naturally led to conjecture that the corresponding complex cones admit compatible $\text{AdS}_3 \times T^2 \times \mathcal{X}^{p,q}$ supergravity solutions with $q > p > 0$. These complex cones have the same toric data as (4.50), but with $q > p > 0$; they are non-convex, much like the $\mathcal{Y}^{p,q}$ and $\mathcal{L}^{a,b,c}$ examples. Unlike those examples, since we do not have explicit metrics, here we need to assume that our conjecture that (3.31) and (3.33) are valid in order to calculate the central charge and the R-charges. Assuming these full solutions exist they will necessarily be cohomogeneity two, with no expectation that the equation of motion can be solved by separation of variables, so finding these solutions explicitly would involve solving a non-linear PDE in two variables. Again, we have no proposal for the dual field theory description.

4.4 $Y_5 = Z^{p,q}$

In the previous section we have shown that for $b_3(Y_5) = 2$ it is not necessary to extremize the trial central charge in the geometric computation as this is determined uniquely by flux

quantization. Instead consider the $Z^{p,q}$ quiver theories of [54]. These have $b_3(Z^{p,q}) = 3$ and we shall see that the gravity result not only fixes the central charge uniquely by flux quantization, but also fixes one of the flux quantum numbers M_I . These field theories blow down to the $X^{p,q}$ theories considered in the previous section, and contain the dP_3 theory as a special limit ($p = 2, q = 1$).

As before the Calabi-Yau cone over the $Z^{p,q}$ manifold is toric, where $p \geq q > 0$ are integers. The inward-pointing normal vectors are

$$\begin{aligned} \vec{v}_1 &= (1, 1, p), & \vec{v}_2 &= (1, 0, p - q + 1), & \vec{v}_3 &= (1, 0, p - q), \\ \vec{v}_4 &= (1, 1, 0), & \vec{v}_5 &= (1, 2, 0), & \vec{v}_6 &= (1, 2, 1). \end{aligned} \quad (4.62)$$

Using the toric data (4.62) we may apply our general formulae from section 3. To prevent expressions becoming completely unwieldy, below we shall only present explicit results for dP_3 , that is $p = 2, q = 1$. However, it is straightforward to compute for general p and q using a computer algebra package. Thus setting $p = 2, q = 1$, using (3.33) the constraint equation (3.3) implies

$$0 = -\frac{2b_1^2(b_2^2 + b_2b_3 + b_3^2 - 3b_1(b_2 + b_3))}{b_2b_3(2b_1 - b_2)(2b_1 - b_3)(b_1 - b_2 - b_3)(b_2 + b_3 - 3b_1)}, \quad (4.63)$$

which admits the solution

$$b_3 = \frac{1}{2} \left(3b_1 - b_2 + \sqrt{9b_1^2 + 6b_1b_2 - 3b_2^2} \right). \quad (4.64)$$

There are six toric divisors which map to the facets $\{(\vec{y}, \vec{v}_a) = 0\}$ in the moment map cone \mathcal{C} , $a = 1, \dots, 6$. These give rise to cones over torus-invariant three-dimensional submanifolds S_a . Since $b_3(Z^{p,q}) = 3$ there are three generating three-cycles which we call $\sigma_1, \sigma_2, \sigma_3$. The homology relations in $H_3(Z^{p,q}, \mathbb{Z})$ are (temporarily restoring general p and q)

$$\begin{aligned} [S_1] &= (p - q)\sigma_1, & [S_2] &= (p - q)\sigma_2, & [S_3] &= -p\sigma_1 + (q - p - 1)\sigma_2 - \sigma_3, \\ [S_4] &= (p + q)\sigma_1 + 2\sigma_2 + 2\sigma_3, & [S_5] &= -p\sigma_1 - \sigma_2 + (q - p - 1)\sigma_3, \\ [S_6] &= (p - q)\sigma_3. \end{aligned} \quad (4.65)$$

Analogously to the previous section, flux quantization imposes

$$\begin{aligned} \int_{\sigma_1} \eta \wedge \rho &= \frac{(2\pi\ell_s)^4 g_s}{AL^4} M_1, & \int_{\sigma_2} \eta \wedge \rho &= \frac{(2\pi\ell_s)^4 g_s}{AL^4} M_2, \\ \int_{\sigma_3} \eta \wedge \rho &= \frac{(2\pi\ell_s)^4 g_s}{AL^4} M_3, \end{aligned} \quad (4.66)$$

which implies the *two* conditions

$$\frac{\int_{\sigma_1} \eta \wedge \rho}{\int_{\sigma_2} \eta \wedge \rho} = \frac{M_1}{M_2}, \quad \frac{\int_{\sigma_1} \eta \wedge \rho}{\int_{\sigma_3} \eta \wedge \rho} = \frac{M_1}{M_3}. \quad (4.67)$$

We may compute the ratio on the left hand side of each expression explicitly in terms of toric data by using (4.65) and formula (3.31). The first condition fixes b_2 in terms of the flux integers M_1, M_2 as in the $X^{p,q}$ case

$$b_2 = \frac{3b_1 M_1 (M_1 + M_2)}{M_1^2 + M_1 M_2 + M_2^2}, \tag{4.68}$$

whilst the second places a restriction on the possible flux integers

$$M_3 = -\frac{M_1(2M_1 + M_2)}{M_1 - M_2}. \tag{4.69}$$

Setting $b_1 = 2$ the geometric central charge for $p = 2, q = 1$ is

$$\mathcal{Z}|_{\text{on-shell}} = \frac{72NM_1M_2(M_1 + M_2)(2M_1 + M_2)(M_1 + 2M_2)}{(M_1^2 + M_1M_2 + M_2^2)^2}. \tag{4.70}$$

We have seen that not only is there no extremization to do, but also that only certain twists are possible, (4.69). Moreover, flux quantization implies that the M_I are integer and therefore only M_1 and M_2 such that (4.69) is integer are permissible. Note that there are an infinite number of choices of M_1 and M_2 which give an integer result for (4.69).

Let us now consider the four-dimensional $Z^{p,q}$ quiver theory reduced on T^2 with baryonic flux. The theory is a $SU(N)^{2(p+1)}$ gauge theory with gauginos, λ and bifundamental fields transforming under the global symmetries as summarized in table 4. As before $U(1)_{B_I}, I = 1, 2, 3$ are the baryonic symmetries associated to the three three-cycles σ_I respectively, while $U(1)_{F_i}, i = 1, 2$ are the flavour symmetries.

We wrap these theories on T^2 by introducing only baryonic flux in the topological twist

$$T_{\text{top}} = B_1 T_{B_1} + B_2 T_{B_2} + B_3 T_{B_3}. \tag{4.71}$$

The trial R-charge is the linear combination

$$T_{\text{trial}} = T_{R_0} + \zeta_1 T_{B_1} + \zeta_2 T_{B_2} + \zeta_3 T_{B_3} + \epsilon_1 T_{F_1} + \epsilon_2 T_{F_2}, \tag{4.72}$$

with ζ_I and ϵ_i parameters over which we extremize. Inserting this data into the trial central charge and extremizing we find that a non-zero solution exists only if

$$B_3 = -\frac{B_1(2B_1 + B_2)}{B_1 - B_2}, \tag{4.73}$$

cf. (4.69). The central charge is then

$$c_{\text{c-ext}} = \frac{72N^2 B_1 B_2 (B_1 + B_2)(2B_1 + B_2)(B_1 + 2B_2)}{(B_1^2 + B_1 B_2 + B_2^2)^2}. \tag{4.74}$$

To compare with the gravity result one should set

$$M_I = N B_I, \tag{4.75}$$

which gives perfect agreement.

Field	Multiplicity	R_0 -charge	$U(1)_{B_1}$	$U(1)_{B_2}$	$U(1)_{B_3}$	$U(1)_{F_1}$	$U(1)_{F_2}$
X_{61}	$(p+q-2)N^2$	2	$q-p$	0	0	0	0
X_{12}	N^2	0	0	$q-p$	0	0	0
X_{23}	N^2	0	p	$p-q+1$	1	0	0
X_{34}	$(p-q)N^2$	0	$-p-q$	-2	-2	-1	0
X_{45}	N^2	0	p	1	$p-q+1$	1	-1
X_{56}	N^2	0	0	0	$q-p$	0	1
X_{13}	$(p-1)N^2$	0	p	1	1	0	0
X_{14}	$(q-1)N^2$	0	$-q$	-1	-1	-1	0
X_{24}	N^2	0	$-q$	$p-q-1$	-1	-1	0
X_{35}	N^2	0	$-q$	-1	$p-q-1$	0	-1
X_{36}	$(q-1)N^2$	0	$-q$	-1	-1	0	0
X_{46}	$(p-1)N^2$	0	p	1	1	1	0
X_{51}	N^2	2	$q-p$	0	$q-p$	0	1
X_{62}	N^2	2	$q-p$	$q-p$	0	0	0
λ	$2(p+1)(N^2-1)$	1	0	0	0	0	0

Table 4. The field content of the $Z^{p,q}$ quiver theories.

Field theoretically one can understand the necessity for the relation between the twist parameters by using the fact that the cubic 't Hooft anomalies for mixed baryonic symmetries vanish for these theories, $k_{B_{I_1}B_{I_2}B_{I_3}} = 0$. This implies that the mixing parameters of the baryonic symmetries in the trial R-symmetry, the ζ_I 's, appear linearly and not quadratically in the trial central charge. Extremizing over these parameters first necessarily implies $b_3(Y_5)$ conditions, where $b_3(Y_5)$ is the third Betti number of the associated Sasaki-Einstein five-manifold. In the case when the number of baryonic symmetries exceeds the number of flavour symmetries n_F , consistency implies that there are $b_3(Y_5) - n_F$ relations between the baryonic twist parameters B_I . This is the field theoretic analogue of the discussion of the latter part of section 3.1. Note that this is a peculiarity of the type of topological twist being performed here, and that a more general topological twist including flavour and R-symmetry (that is not on a T^2) will not in general have linear ζ_I 's in the trial central charge.

Some numerical investigation shows that the putative superconformal field theory cannot exist, since with $p > q$ we always seem to find chiral operators with negative R-charge. Thus, once again it remains an open question to determine the fate of these quiver gauge theories compactified on a T^2 with baryonic flux only. We discuss this below in section 5.

Finally, it is natural to conjecture that the complex cones $\mathcal{Z}^{p,q}$ defined by (4.62) with $q > p > 0$ admit compatible supergravity $\text{AdS}_3 \times T^2 \times \mathcal{Z}^{p,q}$ solutions. These complex cones have the same toric data as (4.62), but with $q > p > 0$, and are consequently non-convex. Again, one can check that the central and R-charges are positive in this case, assuming the validity of our conjectured formulae (3.31) and (3.33) for the case of non-convex toric cones.

5 Discussion

Inspired by the geometric dual of a -maximization put forward in [6], and elaborated upon in [7, 55], in this paper we have formulated a geometric problem that allows one to determine various properties of a class of odd-dimensional “GK geometries” Y_{2n+1} that arise in certain AdS supergravity solutions [9]. In particular, assuming a solution exists, the R-symmetry Killing vector on Y_{2n+1} may be determined by extremizing a function that depends only on certain global, topological data, without knowing the explicit form of the solution. In seven dimensions these backgrounds characterize AdS₃ × Y_7 solutions of type IIB supergravity [5], that are holographically dual to two-dimensional (0,2) SCFTs with a $U(1)_R$ -symmetry, and therefore our geometric problem may be interpreted as a dual to the c -extremization principle [1] in these theories. In nine dimensions instead the backgrounds characterize AdS₂ × Y_9 solutions of eleven-dimensional supergravity that are dual to one-dimensional SCFTs with two supercharges [10] and we have shown that our new variational principle allows one to obtain the two-dimensional Newton constant which is naturally associated with the partition function of the dual superconformal quantum mechanics. For a sub-class of AdS₂ × Y_9 solutions we showed that the variational principle governs the entropy of a certain class of AdS₄ black hole solutions, as well as giving the twisted topological index of certain $\mathcal{N} = 2$ SCFTs compactified on a Riemann surface with a universal twist.

The class of geometries studied in this paper is of independent mathematical interest. The work here, extending that of [9], can be viewed as initial steps in developing a programme analogous to that for Sasakian geometry [41]. Similarly to the latter, in our case the geometry is foliated by a canonical R-symmetry Killing vector with constant norm, with the leaf space being locally conformally Kähler (cf. (2.4)–(2.6)), while the associated metric cone in one dimension higher is here complex, but *not* Kähler. This crucial difference with respect to Sasakian geometry implies that the supersymmetric geometry, as defined in this paper, has some distinctive features which will be interesting to explore further.

In this paper we specialized in sections 3 and 4 to a sub-class of AdS₃ × Y_7 type IIB solutions where $Y_7 = T^2 \times Y_5$. In our examples the five-dimensional geometry of Y_5 was toric, in the sense of possessing a $U(1)^3$ isometry that lifts to a corresponding $(\mathbb{C}^*)^3$ holomorphic action on the complex cone $C(Y_5) \cong \mathbb{R}_{>0} \times Y_5$. However, we have also explained that these are not toric in the sense of symplectic geometry, nor in the usual sense of algebraic geometry. For this reason this class of geometries, which we called non-convex toric geometries, cannot be studied with the standard tools of toric geometry (convex polytopes, *etc*), at least not without appropriate modification. We believe that developing a mathematical framework for these non-convex toric geometries would be worthwhile. In particular, it would be interesting to prove our conjecture that (3.31)–(3.33) hold not only in the toric case, but also in the non-convex toric case. Significant evidence for the validity of this conjecture is that they give the correct gravitational central charge as well as the R-symmetry charges of certain baryonic operators for the specific examples of $Y_5 = \mathcal{Y}^{p,q}$ and $\mathcal{L}^{a,b,c}$, for which explicit solutions are known. In sections 4.3, 4.4 we have conjectured the existence of new classes of AdS₃ × $T^2 \times Y_5$ solutions, where $Y_5 = \mathcal{X}^{p,q}$, $Y_5 = \mathcal{Z}^{p,q}$, respectively, are non-convex toric. It is clearly important to understand the necessary and

sufficient conditions for the existence of metrics satisfying (2.9), analogous to the general existence theorem for toric Sasaki-Einstein metrics in [56].

It is still an interesting open problem to identify the dual $(0, 2)$ SCFTs for *any* of the $\text{AdS}_3 \times T^2 \times Y_5$ solutions discussed in section 4 which, in addition to the above examples, also included $Y_5 = \mathcal{Y}^{p,q}$ [24] and a new class of explicit solutions $\mathcal{L}^{a,b,c}$ that generalise the local construction of [29]. Conversely, we have argued that taking any four-dimensional quiver gauge theory, dual to an $\text{AdS}_5 \times SE_5$ Sasaki-Einstein solution, and compactifying on T^2 , does not flow to a corresponding $\text{AdS}_3 \times T^2 \times Y_5$ solution where the complex structures of $C(Y_5)$ and $C(SE_5)$ are the same, specifically because this latter solution doesn't exist.

Furthermore, we have shown that there are fundamental problems in carrying out the c -extremization procedure for various quiver gauge theories associated with Sasaki-Einstein manifolds, when reduced on T^2 with a baryonic twist. For example, for the quiver gauge theories associated with the $Y^{p,q}$ Sasaki-Einstein geometries (with $p > q > 0$), demanding that the resulting central charge (4.23) is positive leads to negative R-charges for certain chiral fields — see (4.25). Following the discussion of [57] in the context of a -maximization, it is possible that these gauge theories do indeed flow to a $(0, 2)$ SCFT in the IR, but that certain operators are becoming free along the RG flow and, in addition, that the true R-symmetry also involves an “accidental” global symmetry that only appears at the IR fixed point. In some cases when this occurs for $d = 4$, $\mathcal{N} = 1$ SCFTs one can successfully implement a refined version of a -maximization where one eliminates the decoupled gauge-invariant operators [57] (see also e.g. [58–60]). Moreover, an analogous approach has been used in the context of c -extremization for certain $d = 2$, $\mathcal{N} = (0, 2)$ Landau-Ginzburg theories [61, 62]. In the $Y^{p,q}$ examples considered here, from (4.25) we see that an infinite tower of gauge-invariant chiral operators, built from bifundamental chiral fields Y and Z , need to be decoupled. We do not know of any examples where such fields have been decoupled, either in the context of a -maximization or c -extremization, making this a particularly interesting avenue to investigate further. We emphasize that successfully carrying out such a refined c -extremization is unlikely to modify our conclusion that the resulting $\mathcal{N} = (0, 2)$ SCFT is not dual to the relevant $\text{AdS}_3 \times T^2 \times Y_5$ solutions, as we have mentioned several times. Specifically, in the $Y^{p,q}$ quiver gauge theory we have $p > q > 0$ while we have $q > p > 0$ in the $\text{AdS}_3 \times T^2 \times \mathcal{Y}^{p,q}$ solutions.

We conclude by mentioning some other directions for future work. Based on the examples that we have analysed, it is natural to conjecture that the holographic central charge \mathcal{Z} , as a function of the geometric data, agrees with the field theory central charge c_R . Of course, in some sense this is just a restatement of what the AdS/CFT correspondence conjectures, but we have seen that this relation holds for classes of four-dimensional field theories compactified on T^2 , even when there is no corresponding supergravity solution. There hence seems to be a stronger mathematical identity at work, possibly holding off-shell similarly to [63], that will imply the expected AdS/CFT relation when solutions to the supergravity equations do exist. It would be interesting to try to prove this claim, perhaps using the observations made in [64]. We also note that since the trial c -function of [1] is quadratic, the superconformal R-symmetry and central charge c_R of a superconformal $(0, 2)$ theory should be rational. This fact is not immediate from our general geometric

extremal problem summarized in section 2.5, although we note that it is true in all of the examples we have analysed. Understanding the precise relation between our off-shell geometric central charge \mathcal{Z} and the trial c -function, perhaps along the lines of [65] relating Z -minimization and a -maximization, might also help to clarify this issue.

In a different direction, an obvious continuation of our work is to analyse cases where the seven-dimensional manifolds Y_7 are not of the form $T^2 \times Y_5$. It would be interesting to further develop our understanding of the geometry, including the toric case where it is known that the toric examples in the class [21] also have cones with non-convex toric geometries [66]. Finally, and similarly to how the results of [6] predicted \mathcal{F} -extremization in $d = 3$, $\mathcal{N} = 2$ SCFTs, our results also strongly suggest that there exists a general extremization principle for $\mathcal{N} = 2$ superconformal quantum mechanics with a $U(1)$ R-symmetry, extending the proposal of [12].

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A Black hole entropy and the two-dimensional Newton constant

The $D = 4$ magnetically charged black hole of interest is given by [46]

$$\begin{aligned}
 ds_4^2 &= - \left(\rho - \frac{1}{2\rho} \right)^2 dt^2 + \left(\rho - \frac{1}{2\rho} \right)^{-2} d\rho^2 + \rho^2 ds^2(H_2), \\
 F &= \text{vol}(H_2).
 \end{aligned}
 \tag{A.1}$$

It is a solution of minimal $D = 4$ gauged supergravity with bosonic action

$$I = \frac{1}{16\pi G_4} \int d^4x \sqrt{-g} \left(R + 6 - \frac{1}{4} F^2 \right),
 \tag{A.2}$$

and has Bekenstein-Hawking entropy given by

$$S_{BH} = \frac{\text{vol}(H_2)}{8G_4}.
 \tag{A.3}$$

We can uplift this on an arbitrary seven-dimensional Sasaki-Einstein manifold, SE_7 , to obtain a solution of $D = 11$ supergravity using (2.1) of [45]. We can write the metric

on SE_7 as $ds^2(SE_7) = (d\psi + \sigma)^2 + ds^2(KE_6)$ with $d\sigma = 2J_{KE}$ and $\rho(KE_6) = 8J_{KE_6}$. Choosing a convenient overall length normalization, the $D = 11$ metric is given by

$$ds_{11}^2 = \frac{4L^2}{2^{2/3}} \left[ds_4^2 + 4 \left(\left(d\psi + \sigma + \frac{1}{4}A \right)^2 + ds^2(KE_6) \right) \right], \quad (\text{A.4})$$

where $F = dA$. With this length normalization we see that as a $D = 11$ metric the black hole horizon metric is given by

$$ds_{11}^2 = \frac{L^2}{2^{2/3}} \left[ds^2(\text{AdS}_2) + 2ds^2(H^2) + (dz + P)^2 + 16ds^2(KE_6) \right], \quad (\text{A.5})$$

where we have rescaled $z = 4\psi$ and $P = 4\sigma + A$. This is precisely the metric of the AdS_2 solution constructed directly as in (6.15) of [23], and with length normalization as in this paper. Now, using the $D = 11$ metric (A.4), we can reduce to $D = 4$ and deduce that

$$\frac{1}{G_4} = \frac{1}{G_{11}} 2^{13} L^9 \text{vol}(SE_7). \quad (\text{A.6})$$

On the other hand, using (A.5) we can reduce to two dimensions to obtain

$$\frac{1}{G_2} = \frac{1}{G_{11}} 2^{12} L^9 \text{vol}(SE_7) \text{vol}(H_2). \quad (\text{A.7})$$

Combined with (A.3) we conclude that

$$S_{BH} = \frac{1}{4G_2}. \quad (\text{A.8})$$

B Relating $\text{AdS}_2 \times T^2 \times Y_7$ and $\text{AdS}_3 \times Y_7$ solutions

Consider an $\text{AdS}_2 \times Y_9$ solution in $D = 11$ with $Y_9 = T^2 \times Y_7$. This is related by dimensional reduction on one leg of the T^2 followed by T-duality on the other leg to an $\text{AdS}_3 \times Y_7$ solution of type IIB, as shown in appendix C of [22]. Here we make a precise connection between the two-dimensional Newton constant, G_2 , and the central charge of the $d = 2$ SCFT, c_{sugra} , which were defined in section 2.5.

We first recall some well-known results. We assume that the torus metric in $D = 11$ is given by $(dx^9)^2 + (dx^{10})^2$, with periodic coordinates $(x^9, x^{10}) = (x^9 + 2\pi R_2, x^{10} + 2\pi R_1)$. We first reduce along x^{10} , where we have $R_1 \equiv \ell_s g_s^{IIA}$. By integrating the $D = 11$ supergravity action over x^{10} and relating it to the type IIA action we deduce $\frac{R_1}{\ell_p^3} = \frac{1}{\ell_s^8 (g_s^{IIA})^2}$. We next carry out a T-duality over the x^9 direction and we note that R_2 is the type IIA radius of this circle. We then have the T-duality formulae for the type IIB radius, $R_{IIB} = \frac{\ell_s^2}{R_2}$, and the type IIB string coupling, $g_s = \frac{g_s^{IIA} \ell_s}{R_2}$. From this we conclude the following relations between the type IIB quantities and the $D = 11$ quantities:

$$l_s^2 = \frac{l_p^3}{R_1}, \quad g_s = \frac{R_1}{R_2}, \quad R_{IIB} = \frac{\ell_p^3}{R_1 R_2}. \quad (\text{B.1})$$

The $D = 11$ solution of interest has the form

$$\begin{aligned} ds_{11}^2 &= L_{11}^2 e^{-2B_{11}/3} [ds^2(\text{AdS}_2) + (dz + P)^2 + e^{B_{11}} ds_6^2] + e^{B_{11}/3} L_{11}^2 ds^2(T^2), \\ *_{11}G &= L_{11}^6 \left[(dz + P) \wedge \rho_6 \wedge \left(\frac{J_6^2}{2} + J_6 \wedge \text{vol}_{T^2} \right) + \frac{1}{2} * dR_6 \right], \end{aligned} \quad (\text{B.2})$$

where we have added some subscripts for clarity. We make the identification

$$L_{11}^2 \text{vol}(T^2) = (2\pi)^2 R_1 R_2. \quad (\text{B.3})$$

We now use the dimensional reduction and T-duality formula given, for example, in appendix C of [22] to obtain a type IIB solution whose metric is given by

$$ds_{10}^2 = L_{11}^2 e^{-B_{11}/2} [ds^2(\text{AdS}_2) + (dz + P)^2 + e^{B_{11}} ds_6^2] + e^{-B_{11}/2} L_{11}^2 (d\phi + a_1)^2, \quad (\text{B.4})$$

with $da_1 = \text{vol}(\text{AdS}_2)$. In the type IIB solution we now have

$$\Delta(L_{11}\phi) = 2\pi R_{IIB}. \quad (\text{B.5})$$

Recalling that we also have $e^{B_{11}} = R_6/2 = 4e^{B_{10}}$, we can rewrite the metric in the form

$$\begin{aligned} ds_{10}^2 &= (2L_{11}^2) e^{-B_{10}/2} \left[\frac{1}{4} ds^2(\text{AdS}_2) + \frac{1}{4} (d\phi + a_1)^2 + \frac{1}{4} (dz + P)^2 + e^{B_{10}} ds_6^2 \right] \\ &= (2L_{11}^2) e^{-B_{10}/2} \left[ds^2(\text{AdS}_3) + \frac{1}{4} (dz + P)^2 + e^{B_{10}} ds_6^2 \right], \end{aligned} \quad (\text{B.6})$$

which is now precisely in the form of the type IIB AdS₃ solutions, in the notation of this paper, provided that we take

$$L_{10} = 2^{1/2} L_{11}. \quad (\text{B.7})$$

With these results in hand, we can now relate the two-dimensional Newton constant, G_2 , to the one in three dimensions, G_3 , and hence the central charge, c_{sugra} , of the $d = 2$ SCFT. Starting with (2.40) and using (2.36) we deduce that

$$\frac{1}{G_2} = 8 \frac{L_{11}^9 G_{10}}{L_{10}^7 G_{11}} \text{vol}(T^2) \frac{1}{G_3}. \quad (\text{B.8})$$

Next, using (2.37), (2.41), (B.1), (B.3) and (B.5) we can conclude that

$$\frac{1}{G_2} = \frac{2\pi}{3} \frac{\Delta\phi}{2\pi} c_{\text{sugra}}. \quad (\text{B.9})$$

We now return to the AdS₂ × T² × Y₇ solution in $D = 11$ and consider the quantization of $*_{11}G$. There are two types of seven-cycles to analyse: the product of a five-cycle $\Sigma_A^{(5)}$ in Y₇ and the T², and Y₇ itself. For the former, we have

$$\begin{aligned} N_A^{(11)} &\equiv \frac{1}{(2\pi\ell_p)^6} \int_{\Sigma_A^{(5)} \times T^2} *_{11}G = \frac{2L_{11}^6 \text{vol}(T^2)}{(2\pi\ell_p)^6} \int_{\Sigma_A^{(5)}} \frac{1}{2} (dz + P) \wedge \rho_6 \wedge J_6 \\ &= 4 \frac{L_{11}^4}{L_{10}^4} \frac{L_{11}^2 \text{vol}(T^2)}{(2\pi)^2} \frac{\ell_s^4 g_s}{\ell_p^6} N_A^{IIB} \\ &= N_A^{(IIB)}, \end{aligned} \quad (\text{B.10})$$

where we have used the definition of the type IIB flux quantization condition (2.33) in the second line and then (B.1), (B.3) and (B.7) to get to the third. In other words, the four-form flux quantization in $D = 11$ for these seven-cycles is equivalent to the five-form flux quantization condition in type IIB for the five-cycles in Y_7 .

We now turn to the quantization condition for the second type of seven-cycle. We calculate as follows:

$$\begin{aligned}
 N^{(11)} &\equiv \frac{1}{(2\pi\ell_p)^6} \int_{Y_7} *_{11}G = \frac{L_{11}^6}{(2\pi\ell_p)^6} \int_{Y_7} (dz + P) \wedge \rho_6 \wedge \frac{J_6^2}{2} \\
 &= \frac{2L_{11}^6}{(2\pi\ell_p)^6} \frac{(2\pi)^6 \ell_s^8 g_s^2}{3L_{10}^8} c_{\text{sugra}} \\
 &= \frac{c_{\text{sugra}}}{24} \left(\frac{\Delta\phi}{2\pi} \right)^2 \in \mathbb{Z}, \tag{B.11}
 \end{aligned}$$

where we used (2.19), (2.38) and (2.39) to get the second line.

One way to read these relations is as follows. Start with a *bona fide* $\text{AdS}_2 \times T^2 \times Y_7$ solution, *i.e.* with properly quantized four-form flux. In particular it has specific values of $\frac{1}{G_2}$ and $N^{(11)}$. Then, after eliminating c_{sugra} from (B.9) and (B.11), we see that after dimensional reduction and T-duality we obtain a *bona fide* $\text{AdS}_3 \times Y_7$ solution, with AdS_3 metric as in (B.6), with ϕ having specific period given by

$$\frac{\Delta\phi}{2\pi} = 16\pi G_2 N^{(11)}. \tag{B.12}$$

Alternatively, start with a *bona fide* $\text{AdS}_3 \times Y_7$ solution with, in particular, a specific value of c_{sugra} . Then, writing the AdS_3 metric as in (B.6) and demanding that ϕ has period $\Delta\phi$, then after T-duality and uplifting we obtain a *bona fide* $\text{AdS}_2 \times T^2 \times Y_7$ solution only if (B.11) is satisfied.

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