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Observer-based Control of Positive Polynomial Fuzzy Systems with Unknown Time Delay

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Abstract

This paper presents a methodology to investigate the observer-based control for positive nonlinear systems with unknown time delay through analyzing the stability and positivity of positive polynomial fuzzy-model-based (PPFMB) observer-control systems with unknown time delay and unmeasurable premise variables. In order to widen the application of research results and relax the research results, the polynomial fuzzy observer-controller employs a membership function depending on estimated premise variables and the observer system matrix is treated as decision variable. This scenario will lead to non-convex stability and positivity conditions when the stability and positivity are analyzed based on the Lyapunov stability theory. To overcome this issue, some transformation techniques and matrix decoupling techniques (MDT) are adopted to turn the non-convex stability/positivity conditions into convex ones so that they can be handled by convex programming techniques. Finally, two simulation examples are presented to demonstrate the feasibility and validity of the analysis results.

Keywords: Polynomial fuzzy observer, polynomial fuzzy controller, positive polynomial fuzzy-model-based (PPFMB) systems, unmeasurable premise variables, unknown time delay

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1. Introduction

Takagi-Sugeno (T-S) fuzzy model [1, 2, 3] is a powerful mathematical tool to represent the nonlinear systems as a collection of local linear subsystems weighted by membership functions. The merit of this systematic form of nonlinear systems is that it allows the theoretical results of linear system to be employed in the study of T-S fuzzy-model-based (FMB) control systems. In recent years, T-S fuzzy model has been widely applied in various topics, and observer-based control is one of them. According to whether the premise variables and system matrix of model are the same as that of observer, this topic can be divided into three cases. The first case is that the fuzzy system and fuzzy observer have same premise variables and system matrix, and the premise variables must be measurable [4], which is referred to as matched framework type in this paper. The second case is that the premise variables of model are allowed to depend on the unmeasurable system state and the premise variables of observer depend on estimated system state, while the observer system matrix is the same as the model system matrix [5, 6], which is referred to as partially matched framework type. The third case is that the observer system matrix is regarded as a decision variable which can be different from the model system matrix [7] and the premise variables are allowed to be same as in the second case. The third case is referred to as mismatched framework type. Comparing these three cases, the last case not only widens the applicability of the designed system but gives more freedom to the observer design. However, it will lead to more complicated non-convex stability conditions which need be handled by more effective convexification method so that these conditions can be solved by convex programming techniques.

Positive systems are a class of systems whose states and outputs are nonnegative whenever the initial conditions are nonnegative. As positive systems widely exist in industrial and natural environment, such as energy market [8], DC-DC power converters [9], pharmacokinetics [10], the research on positive systems is very meaningful. In recent years, many theoretical results for positive

T-S FMB systems have been reported [11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. For the situation that only the output can be measured, the work [18] investigated observer-based control for positive T-S FMB system.

Different from the general observer-based control systems [4, 5, 6, 21] and positive state feedback control systems [22], when considering both positive constrains and observer-based feedback control, the non-convex conditions will be more difficult to be handled due to the existence of positive constrains, especially for partially matched framework type and mismatched framework type, this issue will become very tricky. Therefor, until now, most references investigated the matched framework type [23, 24] on the positive system, which adopted separation principle to handle non-convex conditions. In order to widen the applicability of the designed system, the work [18] investigated partially matched framework type, two-step procedures are used to handle the non-convex positivity and stability conditions, which means the controller gains are obtained in the first step and the observer gains are then obtained in the second step based on the obtained controller gains. However, this two-step procedures may lead to the conservative result due to the reduced flexibility of the choice of controller gains and observer gains. In the exiting works, there are some methods to reduce the conservativeness of analysis result. The first method is to bring the information of membership functions into the stability conditions [25], which can be implemented by using staircase membership functions [26], piecewise-linear membership functions [27], Taylor series membership functions [28] and other techniques [29]. The second method is to employ other types Lyapunov functional, like copositive Lyapunov functional [20] and membership function dependent Lyapunov-Krasovskii functional [30]. In this paper, we adopt the mismatched framework type to relax the analysis result and improve flexibility of observer design, and handle the non-convex stability conditions and positive conditions simultaneously by using an effective one-step procedure.

Time delays widely exist in various applications such as tele-operation [31], biological embedded systems [32], and chemical reactor systems [33]. The time delay is a source of unstability and poor performance, so it cannot be ignored in stability analysis and control synthesis. At present, some works [30, 34, 35, 36] investigated T-S fuzzy time delay system to enlarge the maximum delay bounds. The work [30] designed a membership function dependent Lyapunov-Krasovskii functional to obtain more relax result. However, it cannot eliminate the effect of time delay on the stability region. In work [34], a global stabilization region is obtained by improving the membership function dependent Lyapunov-Krasovskii functional and innovating a method of dealing with the time derivative of membership function. The works [35, 36] provided more less conservative results based on the analysis of [30, 34]. In this paper, we will give a more relaxed result by defining the observer system matrix as a decision variable.

Recently, T-S fuzzy model has been extended to polynomial fuzzy model [37, 38]. LMI-based analysis approach cannot be used directly for system analysis due to the polynomial variables. Instead, sum of squares (SOS) based analysis approach [39] was then utilized to derive conditions in terms of SOS, whose feasible solution can be found numerically, e.g., using the third-party MATLAB toolbox SOSTOOLS [40]. Some papers [41, 42, 43, 44] investigated the design of polynomial fuzzy observer or filter and observer-based control for polynomial fuzzy system. However, to the best of our knowledge, there is no similar work in the literature regarding this topic, that is the consideration of polynomial fuzzy system, positive system, observer-based control, unmeasurable premise variables and time delay.

Motivated by the aforementioned discussion, in this paper, we present a methodology to investigate the polynomial fuzzy observer-based control for positive polynomial fuzzy systems with unknown time delay and unmeasurable premise variables. When design the polynomial fuzzy observer and polynomial fuzzy controller, the premise variables of them can be different from the those of the polynomial fuzzy model, and the observer system matrix is designed as a decision variable which can be different from the model system matrix, this is referred as mismatched framework type, which demonstrate potential to relax the stability analysis results. Moreover, compared with the matched framework type [23, 24], it can handle more complicated case that full states and time delay both are not available. In order to handle the complex non-convex stability conditions and positivity conditions caused by considering positive constraints, observer, controller and unmeasurable premise variables simultaneously, some transformation techniques are combined with matrix decoupling technique (MDT) [45]. As a result, the controller gains and observer gains can be obtained simultaneously rather than separately in two steps.

The rest of this paper is organized as follows. In Section 2, the details of polynomial fuzzy model, polynomial fuzzy observer and polynomial fuzzy controller are described. Then, stability analysis and positivity analysis for PPFMB observer-control system with unknown time delay are carried out in Section 3. In section 4, simulation examples are provided to illustrate the viability and feasibility of analysis results. In section 5, a conclusion is drawn.

The following notations are adopted. A monomial in $\mathbf{x}(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$ is a function of form $x_1^{d_1}(t)x_2^{d_2}(t)\ldots x_n^{d_n}(t)$, where $d_i \geq 0, i \in \{1, 2, \ldots, n\}$ are nonnegative integers. The degree of a monomial is $d = \sum\limits_{i=1}^{n} d_i$. A polynomial $\mathbf{f}(\mathbf{x}(t))$ is a SOS if there exist polynomials $\mathbf{f}_1(\mathbf{x}(t)), \mathbf{f}_2(\mathbf{x}(t)), \ldots, \mathbf{f}_m(\mathbf{x}(t))$ such that $\mathbf{f}(\mathbf{x}(t)) = \sum\limits_{i=1}^{m} \mathbf{f}_i^2(\mathbf{x}(t))$, where $\mathbf{f}_i(\mathbf{x}(t))$ is a polynomial and m is a nonnegative integer. It is clear that $\mathbf{f}(\mathbf{x}(t))$ being a SOS naturally implies $\mathbf{f}(\mathbf{x}(t)) \geq 0$ for all $\mathbf{x}(t) \in \Re^n$. The expressions of $\mathbf{A} \prec 0$ and $\mathbf{A} \succ 0$ mean that all elements of $\mathbf{A}$ are negative and positive, respectively; $\mathbf{B} < 0$ and $\mathbf{B} > 0$ mean that $\mathbf{B}$ is negative definite and positive definite, respectively. $\mathbf{A}^{(\hat{\alpha}, \hat{\beta})}$ is the $\hat{\alpha}$-th row, $\hat{\beta}$-th column element of $\mathbf{A}$. Matrix $\mathbf{Q}$ is called Metzler matrix [46], if its off diagonal elements are all nonnegative. The notation $*$ in a matrix represents the transposed element in the symmetric position. $\mathrm{diag}\{\cdot\}$ denotes a square diagonal matrix with the elements of argument in the diagonal.

2. Preliminaries

In this section, the formulation of polynomial fuzzy model with unknown time delay, polynomial fuzzy observer and polynomial fuzzy controller are described.

2.1. Polynomial Fuzzy Model with Unknown Time Delay

The nonlinear system with unknown time delay is described by a polynomial fuzzy model with unknown time delay of p rules. The i^{th} rule is of the following format:

Rule i: IF $f_1(\mathbf{x}(t))$ is M_1^i AND$\cdots$AND $f_\varphi(\mathbf{x}(t))$ is M_φ^i,

$$\text{THEN}\begin{cases}\dot{\mathbf{x}}(t)=\mathbf{A}_{i0}(\mathbf{x}(t))\mathbf{x}(t)+\mathbf{A}_{i\tau}(\mathbf{x}(t))\mathbf{x}(t-\tau)+\mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t)\\ \mathbf{y}(t)=\mathbf{C}_i(\mathbf{x}(t))\mathbf{x}(t)\\ \mathbf{x}(t)=\boldsymbol{\phi}(t),t\in[-\tau,0]\end{cases}, \quad (1)$$

where $\mathbf{x}(t)\in\Re^n$, $\mathbf{u}(t)\in\Re^m$ and $\mathbf{y}(t)\in\Re^l$ are the state vector, control input vector and output vector of the system, respectively; n,m and l are their dimensions; $f_\eta(\mathbf{x}(t))$ is the premise variable and M_η^i is the fuzzy set corresponding to its premise variable in rule i, $i\in\{1,2,\ldots,p\}$, $\eta\in\{1,2,\ldots\varphi\}$, and φ is a positive integer; $\mathbf{A}_{i0}(\mathbf{x}(t))\in\Re^{n\times n}$, $\mathbf{A}_{i\tau}(\mathbf{x}(t))\in\Re^{n\times n}$, $\mathbf{B}_i(\mathbf{x}(t))\in\Re^{n\times m}$, $\mathbf{C}_i(\mathbf{x}(t))\in\Re^{l\times n}$ are the known polynomial system, time delay, input and output matrices, respectively. $\boldsymbol{\phi}(t)\succ 0$ is the initial states. τ is the unknown constant time delay which satisfies $0<\tau\leq\bar{\tau}$ where $\bar{\tau}$ is the estimate of the unknown time delay.

The dynamics of the nonlinear system is defined as follows:

$$\begin{cases}\dot{\mathbf{x}}(t)=\sum_{i=1}^{p}w_i(\mathbf{x}(t))(\mathbf{A}_{i0}(\mathbf{x}(t))\mathbf{x}(t)+\mathbf{A}_{i\tau}(\mathbf{x}(t))\mathbf{x}(t-\tau)+\mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t))\\ \mathbf{y}(t)=\sum_{i=1}^{p}w_i(\mathbf{x}(t))\mathbf{C}_i(\mathbf{x}(t))\mathbf{x}(t)\\ \mathbf{x}(t)=\boldsymbol{\phi}(t),t\in[-\tau,0]\end{cases}, \quad (2)$$

where $w_i(\mathbf{x}(t))$ is the normalized grade of membership, $w_i(\mathbf{x}(t))\geq 0$, $\sum\limits_{i=1}^{p}w_i(\mathbf{x}(t))=1$, and $w_i(\mathbf{x}(t))=(\prod\limits_{\eta=1}^{\varphi}\mu_{M_\eta^i}(f_\eta(\mathbf{x}(t))))/(\sum\limits_{k=1}^{p}\prod\limits_{\eta=1}^{\varphi}\mu_{M_\eta^k}(f_\eta(\mathbf{x}(t))))$; $\mu_{M_\eta^i}(f_\eta(\mathbf{x}(t)))$ is the grade of membership corresponding to the fuzzy term M_η^i.

Definition 1. *The polynomial fuzzy system* (2) *is said to be positive only if for every nonnegative initial states, its states and outputs are nonnegative [46].*

Lemma 1. *A polynomial fuzzy system with unknown time delay* (2) *is guaranteed to be positive if* $\sum_{i=1}^{p} w_i(\mathbf{x}(t))\mathbf{A}_{i0}(\mathbf{x}(t))$ *is a Metzler matrix; time delay, input and output matrices satisfy the conditions that* $\sum_{i=1}^{p} w_i(\mathbf{x}(t))\mathbf{A}_{i\tau}(\mathbf{x}(t)) \succ 0$, $\sum_{i=1}^{p} w_i(\mathbf{x}(t))\mathbf{B}_i(\mathbf{x}(t)) \succ 0$ *and* $\sum_{i=1}^{p} w_i(\mathbf{x}(t))\mathbf{C}_i(\mathbf{x}(t)) \succ 0$ *when* $\mathbf{u}(t)$ *is nonnegative.*

2.2. Polynomial Fuzzy Observer with Unknown Time Delay

In this paper, we consider the case that all system states are immeasurable. The j^{th} rule of the polynomial fuzzy observer with unknown time delay is as follows:

Rule j: IF $f_1(\hat{\mathbf{x}}(t))$ is M_1^j AND$\cdots$AND $f_\varphi(\hat{\mathbf{x}}(t))$ is M_φ^j,

$$\text{THEN} \begin{cases} \dot{\hat{\mathbf{x}}}(t) = \mathbf{F}_{j0}(\hat{\mathbf{x}}(t))\hat{\mathbf{x}}(t) + \mathbf{B}_j(\hat{\mathbf{x}}(t))\mathbf{u}(t) + \mathbf{L}_j(\hat{\mathbf{x}}(t))(\mathbf{y}(t) - \hat{\mathbf{y}}(t)) \\ \hat{\mathbf{y}}(t) = \mathbf{C}_j(\hat{\mathbf{x}}(t))\hat{\mathbf{x}}(t) \\ \hat{\mathbf{x}}(t) = \hat{\boldsymbol{\phi}}(t), t \in [-\tau, 0] \end{cases}, \quad (3)$$

where $\hat{\mathbf{x}}(t) \in \Re^n$ is the estimated state vector; $\hat{\mathbf{y}}(t) \in \Re^l$ is the estimated output vector; $\mathbf{F}_{j0}(\hat{\mathbf{x}}(t)) \in \Re^{n\times n}$ is polynomial observer system matrix to be determined, $j \in \{1, 2, \ldots, p\}$, $\mathbf{B}_j(\hat{\mathbf{x}}(t)) \in \Re^{n\times m}$ and $\mathbf{C}_j(\hat{\mathbf{x}}(t)) \in \Re^{l\times n}$ are the known input and output matrices of observer, respectively, from the polynomial fuzzy model but are a function of estimated states $\hat{\mathbf{x}}$. $\mathbf{L}_j(\hat{\mathbf{x}}(t)) \in \Re^{n\times l}$ is the polynomial observer gain to be determined. The polynomial fuzzy observer is defined as follows:

$$\begin{cases} \dot{\hat{\mathbf{x}}}(t) = \sum_{j=1}^{p} w_j(\hat{\mathbf{x}}(t))(\mathbf{F}_{j0}(\hat{\mathbf{x}}(t))\hat{\mathbf{x}}(t) + \mathbf{B}_j(\hat{\mathbf{x}}(t))\mathbf{u}(t) + \mathbf{L}_j(\hat{\mathbf{x}}(t))(\mathbf{y}(t) - \hat{\mathbf{y}}(t))) \\ \hat{\mathbf{y}}(t) = \sum_{j=1}^{p} w_j(\hat{\mathbf{x}}(t))\mathbf{C}_j(\hat{\mathbf{x}}(t))\hat{\mathbf{x}}(t) \\ \hat{\mathbf{x}}(t) = \hat{\boldsymbol{\phi}}(t), t \in [-\tau, 0] \end{cases}, \quad (4)$$

where the membership function $w_j(\hat{\mathbf{x}}(t))$ from the polynomial fuzzy model but are function of estimate states $\hat{\mathbf{x}}$.

Remark 1. *Different from the existing literature in which the observer system matrix is predefined and the coefficients of observer are the same as that of model, polynomial observer system matrix* $\mathbf{F}_{j0}(\hat{\mathbf{x}}(t))$ *are designed in* (4) *as decision variables but not predefined variables in this paper, which provides more freedom to the observer design.*

2.3. Polynomial Fuzzy Controller

We adopt the PDC design approach for the design of the polynomial fuzzy controller as follow:

Rule k: IF $f_1(\hat{\mathbf{x}}(t))$ is M_1^k AND$\cdots$AND $f_\varphi(\hat{\mathbf{x}}(t))$ is M_φ^k,

THEN $\mathbf{u}(t) = \mathbf{G}_k(\hat{\mathbf{x}}(t))\hat{\mathbf{x}}(t)$,

where $\mathbf{G}_k(\hat{\mathbf{x}}(t)) \in \Re^{m\times n}$ is the polynomial controller gain to be determined, $k \in \{1, 2, \ldots, p\}$. The polynomial fuzzy controller is given by

$$\mathbf{u}(t) = \sum_{k=1}^{p} w_k(\hat{\mathbf{x}}(t))\mathbf{G}_k(\hat{\mathbf{x}}(t))\hat{\mathbf{x}}(t). \tag{5}$$

where the membership function $w_k(\hat{\mathbf{x}}(t))$ is the same as the membership function of the polynomial fuzzy observer.

3. Stability Analysis and Positivity Analysis

In this section, we investigate the stability and positivity of PPFMB observer-control system with unknown time delay. First, the augmented PPFMB observer-control system with unknown time delay is obtained in Subsection 3.1. Then, the stability analysis and positivity analysis are carried out in Subsection 3.2 and Subsection 3.3, respectively. Finally, the stability and positivity analysis results are summarized in a theorem in Subsection 3.4.

3.1. Augmented PPFMB Observer-Control Systems with Unknown Time Delay

In this paper, we define $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$, $\mathbf{z}(t) = [\hat{\mathbf{x}}^T(t)\ \mathbf{e}^T(t)]^T$ and $\mathbf{e}(t-\tau) = \mathbf{x}(t-\tau) - \hat{\mathbf{x}}(t-\tau)$, $\mathbf{z}(t-\tau) = [\hat{\mathbf{x}}^T(t-\tau)\ \mathbf{e}^T(t-\tau)]^T$. In the following analysis, for simplicity, the time t is dropped for the situation without ambiguity, e.g.,

$\mathbf{x}(t)$, $\hat{\mathbf{x}}(t)$, $\mathbf{e}(t)$ and $\mathbf{z}(t)$ are denoted by $\mathbf{x}$, $\hat{\mathbf{x}}$, $\mathbf{e}$ and $\mathbf{z}$, respectively. In addition, $\hat{\mathbf{x}}(t-\tau)$, $\mathbf{e}(t-\tau)$ and $\mathbf{z}(t-\tau)$ are denoted by $\hat{\mathbf{x}}_\tau$, $\mathbf{e}_\tau$ and $\mathbf{z}_\tau$, respectively. From (2), (4) and (5), the PPFMB observer-control system with unknown time delay is obtained which contains the polynomial fuzzy model, polynomial fuzzy observer and polynomial fuzzy controller as follows:

$$\dot{\mathbf{x}} = \sum_{i=1}^{p}\sum_{k=1}^{p} w_i(\mathbf{x})w_k(\hat{\mathbf{x}})((\mathbf{A}_{i0}(\mathbf{x}) + \mathbf{B}_i(\mathbf{x})\mathbf{G}_k(\hat{\mathbf{x}}))\hat{\mathbf{x}} + \mathbf{A}_{i0}(\mathbf{x})\mathbf{e} + \mathbf{A}_{i\tau}(\hat{\mathbf{x}}_\tau + \mathbf{e}_\tau)), \tag{6}$$

$$\begin{aligned}\dot{\hat{\mathbf{x}}} = &\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p} w_i(\mathbf{x})w_j(\hat{\mathbf{x}})w_k(\hat{\mathbf{x}})((\mathbf{F}_{j0}(\hat{\mathbf{x}}) + \mathbf{B}_j(\hat{\mathbf{x}})\mathbf{G}_k(\hat{\mathbf{x}}) + \mathbf{L}_j(\hat{\mathbf{x}})(\mathbf{C}_i(\mathbf{x}) - \mathbf{C}_j(\hat{\mathbf{x}})))\hat{\mathbf{x}} \\ &+ \mathbf{L}_j(\hat{\mathbf{x}})\mathbf{C}_i(\mathbf{x})\mathbf{e}),\end{aligned} \tag{7}$$

$$\begin{aligned}\dot{\mathbf{e}} = &\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p} w_i(\mathbf{x})w_j(\hat{\mathbf{x}})w_k(\hat{\mathbf{x}})(((\mathbf{A}_{i0}(\mathbf{x}) - \mathbf{F}_{j0}(\hat{\mathbf{x}})) + (\mathbf{B}_i(\mathbf{x}) - \mathbf{B}_j(\hat{\mathbf{x}}))\mathbf{G}_k(\hat{\mathbf{x}}) \\ &- \mathbf{L}_j(\hat{\mathbf{x}})(\mathbf{C}_i(\mathbf{x}) - \mathbf{C}_j(\hat{\mathbf{x}})))\hat{\mathbf{x}} + (\mathbf{A}_{i0}(\mathbf{x}) - \mathbf{L}_j(\hat{\mathbf{x}})\mathbf{C}_i(\mathbf{x}))\mathbf{e} + \mathbf{A}_{i\tau}(\mathbf{x})(\hat{\mathbf{x}}_\tau + \mathbf{e}_\tau)).\end{aligned} \tag{8}$$

To lighten the notations, we define $\sum_{i,j=1}^{p} w_{ij}^{\mathbf{x},\hat{\mathbf{x}}} \equiv \sum_{i=1}^{p}\sum_{j=1}^{p} w_i(\mathbf{x})w_j(\hat{\mathbf{x}})$, $\sum_{j,k=1}^{p} w_{jk}^{\hat{\mathbf{x}}} \equiv \sum_{j=1}^{p}\sum_{k=1}^{p} w_j(\hat{\mathbf{x}})w_k(\hat{\mathbf{x}})$, $\sum_{i,j,k=1}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}} \equiv \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p} w_i(\mathbf{x})w_j(\hat{\mathbf{x}})w_k(\hat{\mathbf{x}})$. The augmented PPFMB observer-control system ((7) and (8)) is given as follows:

$$\dot{\mathbf{z}} = \begin{bmatrix} \dot{\hat{\mathbf{x}}} \\ \dot{\mathbf{e}} \end{bmatrix} = \sum_{i,j,k=1}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}} \left(\boldsymbol{\Xi}_{ijk}(\mathbf{x},\hat{\mathbf{x}})\mathbf{z} + \breve{\boldsymbol{\Xi}}_i(\mathbf{x})\mathbf{z}_\tau\right), \tag{9}$$

where

$$\boldsymbol{\Xi}_{ijk}(\mathbf{x},\hat{\mathbf{x}}) = \begin{bmatrix} \boldsymbol{\Xi}_{ijk}^{1,1}(\mathbf{x},\hat{\mathbf{x}}) & \mathbf{L}_j(\hat{\mathbf{x}})\mathbf{C}_i(\mathbf{x}) \\ \boldsymbol{\Xi}_{ijk}^{2,1}(\mathbf{x},\hat{\mathbf{x}}) & \mathbf{A}_{i0}(\mathbf{x}) - \mathbf{L}_j(\hat{\mathbf{x}})\mathbf{C}_i(\mathbf{x}) \end{bmatrix}, \tag{10}$$

$$\boldsymbol{\Xi}_{ijk}^{1,1}(\mathbf{x},\hat{\mathbf{x}}) = \mathbf{F}_{j0}(\hat{\mathbf{x}}) + \mathbf{B}_j(\hat{\mathbf{x}})\mathbf{G}_k(\hat{\mathbf{x}}) + \mathbf{L}_j(\hat{\mathbf{x}})(\mathbf{C}_i(\mathbf{x}) - \mathbf{C}_j(\hat{\mathbf{x}})), \tag{11}$$

$$\boldsymbol{\Xi}_{ijk}^{2,1}(\mathbf{x},\hat{\mathbf{x}}) = (\mathbf{A}_{i0}(\mathbf{x}) - \mathbf{F}_{j0}(\hat{\mathbf{x}})) + (\mathbf{B}_i(\mathbf{x}) - \mathbf{B}_j(\hat{\mathbf{x}}))\mathbf{G}_k(\hat{\mathbf{x}}) - \mathbf{L}_j(\hat{\mathbf{x}})(\mathbf{C}_i(\mathbf{x}) - \mathbf{C}_j(\hat{\mathbf{x}})), \tag{12}$$

$$\breve{\boldsymbol{\Xi}}_i(\mathbf{x}) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{A}_{i\tau}(\mathbf{x}) & \mathbf{A}_{i\tau}(\mathbf{x}) \end{bmatrix}. \tag{13}$$

In this paper, the objective is to stabilize the PPFMB observer-control system ((6), (7) and (8)), and guarantee its positivity, simultaneously. Due to the definition that $\mathbf{x} = \mathbf{e} + \hat{\mathbf{x}}$, system state $\mathbf{x}$ is positive and asymptotically stable, i.e., $\mathbf{x} \succ 0$ and $\mathbf{x} \to \mathbf{0}$ as time $t \to \infty$, as long as augmented PPFMB observer-control system (9) is asymptotically stable and positive. Thus, the objective can be achieved by designing a polynomial observer and a polynomial controller to stabilize the augmented PPFMB observer-control system (9), and guarantee its positivity, simultaneously.

3.2. Stability Analysis of Augmented PPFMB Observer-Control Systems with Unknown Time Delay

In this part, we use the Lyapunov stability theory to investigate the stability of (9), which leads to non-convex stability conditions. Then, MDT and some transformation techniques are employed to obtain convex stability conditions with the support of the following lemmas.

Lemma 2. *With* $\mathbf{X}$ *and* $\mathbf{Y}$ *of appropriate dimensions and a scalar* $\beta > 0$*, the following inequality holds [47]:*

$$\mathbf{X}^T\mathbf{Y} + \mathbf{Y}^T\mathbf{X} \leq \beta\mathbf{X}^T\mathbf{X} + \beta^{-1}\mathbf{Y}^T\mathbf{Y}.$$

Lemma 3. *With symmetric matrices* $\mathbf{P}$ *and* $\mathbf{R}$ *of appropriate dimensions,* $\mathbf{R} > 0$ *and a scalar* γ*, the following inequality holds [47]:*

$$-\mathbf{P}\mathbf{R}^{-1}\mathbf{P} \leq \gamma^2\mathbf{R} - 2\gamma\mathbf{P}.$$

To proceed with the stability analysis, we choose the Lyapunov-Krasovskii functional as:

$$V = V_1 + V_2 + V_3, \tag{14}$$

where $V_1 = \mathbf{z}^T\mathbf{P}\mathbf{z}, V_2 = \int_{t-\tau}^{t} \mathbf{z}(\alpha)^T\mathbf{S}\mathbf{z}(\alpha)d\alpha, V_3 = \int_{-\tau}^{0}\int_{t+\beta}^{t} \dot{\mathbf{z}}(\alpha)^T\mathbf{R}\dot{\mathbf{z}}(\alpha)d\alpha d\beta, \mathbf{P} =$ $\mathrm{diag}(\mathbf{P}_1^{-1}, \mathbf{P}_2)$, diagonal matrices $\mathbf{P}_1^{-1} > 0$, $\mathbf{P}_2 > 0$; symmetric matrices $\mathbf{S} > 0$,

$\mathbf{R} > 0$. Thus, Lyapunov functional $V > 0$. If it can be shown that $\dot{V}$ is negative, the augmented PPFMB observer-control system with unknown time delay (9) is asymptotically stable.

We define $\boldsymbol{\xi}(t,\tau,\alpha) = \begin{bmatrix}\mathbf{z}^T & \mathbf{z}_\tau^T & \dot{\mathbf{z}}(\alpha)^T\end{bmatrix}^T$ and use the fact that $\sum_{i,j,k=1}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}} = \sum_{r,s,l=1}^{p} w_{rsl}^{\mathbf{x},\hat{\mathbf{x}}} = \sum_{i,j,k=1}^{p}\sum_{r,s,l=1}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}} w_{rsl}^{\mathbf{x},\hat{\mathbf{x}}} = 1$. The Leibniz-Newton formula provides $\mathbf{z} - \mathbf{z}_\tau = \int_{t-\tau}^{t} \dot{\mathbf{z}}(\alpha)d\alpha$. Introducing slack symmetric matrices $\mathbf{W}_1$ and $\mathbf{W}_2$, and applying Leibniz-Newton formula to $\dot{V}$, we can obtain (15) and (17) as follows:

$$
\begin{aligned}
\dot{V}_1 + \dot{V}_2 &= \dot{\mathbf{z}}^T\mathbf{P}\mathbf{z} + \mathbf{z}^T\mathbf{P}\dot{\mathbf{z}} + \mathbf{z}^T\mathbf{S}\mathbf{z} - \mathbf{z}_\tau^T\mathbf{S}\mathbf{z}_\tau \\
&= \sum_{i,j,k=1}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}}\Big((\boldsymbol{\Xi}_{ijk}(\mathbf{x},\hat{\mathbf{x}})\mathbf{z} + \breve{\boldsymbol{\Xi}}_i(\mathbf{x})\mathbf{z}_\tau)^T\mathbf{P}\mathbf{z} + \mathbf{z}^T\mathbf{P}(\boldsymbol{\Xi}_{ijk}(\mathbf{x},\hat{\mathbf{x}})\mathbf{z} + \breve{\boldsymbol{\Xi}}_i(\mathbf{x})\mathbf{z}_\tau)\Big) \\
&\quad + \mathbf{z}^T\mathbf{W}_1(\mathbf{z}-\mathbf{z}_\tau) + (\mathbf{z}-\mathbf{z}_\tau)^T\mathbf{W}_1^T\mathbf{z} - \mathbf{z}^T\mathbf{W}_1\int_{t-\tau}^{t}\dot{\mathbf{z}}(\alpha)d\alpha \\
&\quad - (\int_{t-\tau}^{t}\dot{\mathbf{z}}(\alpha)d\alpha)^T\mathbf{W}_1^T\mathbf{z} + \mathbf{z}_\tau^T\mathbf{W}_2(\mathbf{z}-\mathbf{z}_\tau) + (\mathbf{z}-\mathbf{z}_\tau)^T\mathbf{W}_2^T\mathbf{z}_\tau \\
&\quad - \mathbf{z}_\tau^T\mathbf{W}_2\int_{t-\tau}^{t}\dot{\mathbf{z}}(\alpha)d\alpha - (\int_{t-\tau}^{t}\dot{\mathbf{z}}(\alpha)d\alpha)^T\mathbf{W}_2^T\mathbf{z}_\tau + \frac{1}{\tau}\int_{t-\tau}^{t}\left(\mathbf{z}^T\mathbf{S}\mathbf{z} - \mathbf{z}_\tau^T\mathbf{S}\mathbf{z}_\tau\right)d\alpha \\
&= \frac{1}{\tau}\int_{t-\tau}^{t}\boldsymbol{\xi}^T(t,\tau,\alpha)\sum_{i,j,k=1}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}}\boldsymbol{\Psi}_{ijk}(\mathbf{x},\hat{\mathbf{x}})\boldsymbol{\xi}(t,\tau,\alpha)d\alpha, \qquad (15)
\end{aligned}
$$

where

$$
\boldsymbol{\Psi}_{ijk}(\mathbf{x},\hat{\mathbf{x}}) = \begin{bmatrix}
\begin{matrix}\boldsymbol{\Xi}_{ijk}^T(\mathbf{x},\hat{\mathbf{x}})\mathbf{P} + \mathbf{P}\boldsymbol{\Xi}_{ijk}(\mathbf{x},\hat{\mathbf{x}}) \\ +\mathbf{W}_1 + \mathbf{W}_1^T + \mathbf{S}\end{matrix} & \mathbf{P}\breve{\boldsymbol{\Xi}}_i(\mathbf{x}) - \mathbf{W}_1 + \mathbf{W}_2^T & -\tau\mathbf{W}_1 \\
* & -\mathbf{W}_2 - \mathbf{W}_2^T - \mathbf{S} & -\tau\mathbf{W}_2 \\
* & * & \mathbf{0}
\end{bmatrix}. \qquad (16)
$$

$$
\begin{aligned}
\dot{V}_3 &= \int_{-\tau}^{0}\left(\dot{\mathbf{z}}^T\mathbf{R}\dot{\mathbf{z}} - \dot{\mathbf{z}}(t+\beta)^T\mathbf{R}\dot{\mathbf{z}}(t+\beta)\right)d\beta \\
&= \frac{1}{\tau}\int_{t-\tau}^{t}\tau\left(\begin{matrix}\sum_{i,j,k=1}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}}\left(\boldsymbol{\Xi}_{ijk}(\mathbf{x},\hat{\mathbf{x}})\mathbf{z} + \breve{\boldsymbol{\Xi}}_i(\mathbf{x})\mathbf{z}_\tau\right)^T\mathbf{R}\times \\ \sum_{r,s,l=1}^{p} w_{rsl}^{\mathbf{x},\hat{\mathbf{x}}}\left(\boldsymbol{\Xi}_{rsl}(\mathbf{x},\hat{\mathbf{x}})\mathbf{z} + \breve{\boldsymbol{\Xi}}_r(\mathbf{x})\mathbf{z}_\tau\right) - \dot{\mathbf{z}}(\alpha)^T\mathbf{R}\dot{\mathbf{z}}(\alpha)\end{matrix}\right)d\alpha. \qquad (17)
\end{aligned}
$$

Remark 2. *For simplicity, we will remove* $(\mathbf{x})$, $(\hat{\mathbf{x}})$ *and* $(\mathbf{x},\hat{\mathbf{x}})$ *e.g.,* $\boldsymbol{\Psi}_i(\mathbf{x})$, $\boldsymbol{\Psi}_j(\hat{\mathbf{x}})$ *and* $\boldsymbol{\Psi}_{ijk}(\mathbf{x},\hat{\mathbf{x}})$ *are denoted as* $\boldsymbol{\Psi}_i,\boldsymbol{\Psi}_j$ *and* $\boldsymbol{\Psi}_{ijk}$*, respectively. For situation without ambiguity, the subscript* i *implies that the variable is function of* $\mathbf{x}$*; the subscript* j*,* k *or* jk *implies that the variable is function of* $\hat{\mathbf{x}}$*; the subscript* ijk*,* ij*,* ik *or* rsl*,* rs *implies that the variable is function of both* $\mathbf{x}$ *and* $\hat{\mathbf{x}}$*; no subscript or the subscript is a number implies that the variable is a constant variable.*

According to (15) and (17), $\dot{V}$ can be obtained as follows:

$$\dot{V} = \frac{1}{\tau}\int_{t-\tau}^{t} \boldsymbol{\xi}^T(t,\tau,\alpha)\Big(\sum_{i,j,k=1}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}}\boldsymbol{\Psi}_{1ijk} + \tau\Big(\sum_{i,j,k=1}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}}\boldsymbol{\Psi}_{2ijk}\Big)\mathbf{R}\times$$

$$\Big(\sum_{r,s,l=1}^{p} w_{rsl}^{\mathbf{x},\hat{\mathbf{x}}}\boldsymbol{\Psi}_{2rsl}^T\Big)\Big)\boldsymbol{\xi}(t,\tau,\alpha)d\alpha, \tag{18}$$

where $\boldsymbol{\Psi}_{1ijk} = \begin{bmatrix} \boldsymbol{\Psi}_{ijk}^{1,1} & \boldsymbol{\Psi}_{i}^{1,2} & -\tau\mathbf{W}_1 \\ * & \boldsymbol{\Psi}^{2,2} & -\tau\mathbf{W}_2 \\ * & * & -\tau\mathbf{R} \end{bmatrix}$, $\boldsymbol{\Psi}_{2ijk} = [\boldsymbol{\Xi}_{ijk} \quad \breve{\boldsymbol{\Xi}}_i \quad \mathbf{0}_{2n\times 2n}]^T$, $\boldsymbol{\Psi}_{ijk}^{1,1} = \boldsymbol{\Xi}_{ijk}^T\mathbf{P} + \mathbf{P}\boldsymbol{\Xi}_{ijk} + \mathbf{W}_1 + \mathbf{W}_1^T + \mathbf{S}$, $\boldsymbol{\Psi}_{i}^{1,2} = \mathbf{P}\breve{\boldsymbol{\Xi}}_i - \mathbf{W}_1 + \mathbf{W}_2^T$, $\boldsymbol{\Psi}^{2,2} = -\mathbf{W}_2 - \mathbf{W}_2^T - \mathbf{S}$, $\boldsymbol{\Xi}_{ijk}$ and $\breve{\boldsymbol{\Xi}}_i$ are defined in (10) and (13).

Then, from (18), $\dot{V} < 0$ holds if:

$$\sum_{i,j,k=1}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}}\boldsymbol{\Psi}_{1ijk} + \tau\Big(\sum_{i,j,k=1}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}}\boldsymbol{\Psi}_{2ijk}\mathbf{P}\Big)\mathbf{P}^{-1}\mathbf{R}\mathbf{P}^{-1}\Big(\sum_{r,s,l=1}^{p} w_{rsl}^{\mathbf{x},\hat{\mathbf{x}}}\mathbf{P}\boldsymbol{\Psi}_{2rsl}^T\Big) < 0. \tag{19}$$

In order to avoid the existence of unknown time delay τ in (19), Schur complement is applied to replace unknown time delay τ with the estimated time delay $\bar{\tau}$.

Firstly, applying Schur complement to (19), we have:

$$\sum_{i,j,k=1}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}}\bar{\boldsymbol{\Psi}}_{ijk} < 0, \tag{20}$$

where $\bar{\boldsymbol{\Psi}}_{ijk} = \begin{bmatrix} \boldsymbol{\Psi}_{ijk}^{1,1} & \boldsymbol{\Psi}_{i}^{1,2} & -\tau\mathbf{W}_1 & \tau\boldsymbol{\Xi}_{ijk}^T(\mathbf{x},\hat{\mathbf{x}})\mathbf{P} \\ * & \boldsymbol{\Psi}^{2,2} & -\tau\mathbf{W}_2 & \tau\breve{\boldsymbol{\Xi}}_i^T\mathbf{P} \\ * & * & -\tau\mathbf{R} & \mathbf{0}_{2n\times 2n} \\ * & * & * & -\tau\mathbf{P}\mathbf{R}^{-1}\mathbf{P} \end{bmatrix}$.

Then, applying the Schur complement reversely to (20) meanwhile extracting all the time delay τ, we have:

$$\begin{aligned} &\sum_{i,j,k=1}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}}\bar{\boldsymbol{\Psi}}_{1ijk} \\ &+\tau(\sum_{i,j,k=1}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}}\bar{\boldsymbol{\Psi}}_{2ijk}) \begin{bmatrix} \mathbf{R}^{-1} & \mathbf{0}_{2n\times 2n} \\ \mathbf{0}_{2n\times 2n} & \mathbf{P}^{-1}\mathbf{R}\mathbf{P}^{-1} \end{bmatrix} (\sum_{r,s,l=1}^{p} w_{rsl}^{\mathbf{x},\hat{\mathbf{x}}}\bar{\boldsymbol{\Psi}}_{2rsl}^T) < 0, \end{aligned} \tag{21}$$

where $\bar{\boldsymbol{\Psi}}_{1ijk} = \begin{bmatrix} \boldsymbol{\Psi}_{ijk}^{1,1} & \boldsymbol{\Psi}_{i}^{1,2} \\ * & \boldsymbol{\Psi}^{2,2} \end{bmatrix}$, $\bar{\boldsymbol{\Psi}}_{2ijk} = \begin{bmatrix} -\mathbf{W}_1^T & -\mathbf{W}_2^T \\ \mathbf{P}\boldsymbol{\Xi}_{ijk} & \mathbf{P}\breve{\boldsymbol{\Xi}}_i \end{bmatrix}^T$.

Using the fact that $0 < \tau \leq \bar{\tau}$, the holding of (21) is implied by the following inequality:

$$\begin{aligned} &\sum_{i,j,k=1}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}}\bar{\boldsymbol{\Psi}}_{1ijk} \\ &+\bar{\tau}(\sum_{i,j,k=1}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}}\bar{\boldsymbol{\Psi}}_{2ijk}) \begin{bmatrix} \mathbf{R}^{-1} & \mathbf{0}_{2n\times 2n} \\ \mathbf{0}_{2n\times 2n} & \mathbf{P}^{-1}\mathbf{R}\mathbf{P}^{-1} \end{bmatrix} (\sum_{r,s,l=1}^{p} w_{rsl}^{\mathbf{x},\hat{\mathbf{x}}}\bar{\boldsymbol{\Psi}}_{2rsl}^T) < 0. \end{aligned} \tag{22}$$

Finally, applying Schur complement to (22), $\dot{V} < 0$ holds if

$$\sum_{i,j,k=1}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}}\tilde{\boldsymbol{\Psi}}_{ijk} < 0, \tag{23}$$

where $\tilde{\boldsymbol{\Psi}}_{ijk} = \begin{bmatrix} \boldsymbol{\Psi}_{ijk}^{1,1} & \boldsymbol{\Psi}_{i}^{1,2} & -\mathbf{W}_1 & \boldsymbol{\Xi}_{ijk}^T\mathbf{P} \\ * & \boldsymbol{\Psi}^{2,2} & -\mathbf{W}_2 & \breve{\boldsymbol{\Xi}}_i^T\mathbf{P} \\ * & * & -\frac{1}{\bar{\tau}}\mathbf{R} & \mathbf{0}_{2n\times 2n} \\ * & * & * & -\frac{1}{\bar{\tau}}\mathbf{P}\mathbf{R}^{-1}\mathbf{P} \end{bmatrix}$. $\boldsymbol{\Psi}_{ijk}^{1,1}$, $\boldsymbol{\Psi}_{i}^{1,2}$ and $\boldsymbol{\Psi}^{2,2}$ have been defined below (18).

After the above transformations, the unknown time delay τ has been replaced by the estimated time delay $\bar{\tau}$ in stability condition (23). However, there is another crucial problem that the non-convex terms of (23) should be dealt with, because the non-convex terms hinder the use of convex programming techniques to find the decision variables $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{S}$, $\mathbf{R}$, $\mathbf{W}_1$, $\mathbf{W}_2$, observer system matrices $\mathbf{F}_{j0}$, controller gains $\mathbf{G}_k$ and observer gains $\mathbf{L}_j$ simultaneously.

In order to handle the non-convex terms $\mathbf{P}_1^{-1}\mathbf{B}_j\mathbf{G}_k$ and $\mathbf{P}_1^{-1}(\mathbf{B}_i - \mathbf{B}_j)\mathbf{G}_k$, congruence transformation is applied to $\tilde{\mathbf{\Psi}}_{ijk}$ by pre- and post-multiplying $\mathbf{E} = \text{diag}(\mathbf{P}^{-1}, \mathbf{P}^{-1}, \mathbf{P}^{-1}, \mathbf{P}^{-1})$ with the definition $\mathbf{M}_k = \mathbf{G}_k\mathbf{P}_1$ and $\mathbf{T}_{j0} = \mathbf{F}_{j0}\mathbf{P}_1$. Also, in order to avoid $\mathbf{R}^{-1}$ appearing in the stability conditions, applying Lemma 3 to $\mathbf{P}\mathbf{R}^{-1}\mathbf{P}$ before pre-multiplying and post-multiplying $\tilde{\mathbf{\Psi}}_{ijk}$ by $\mathbf{E}$.

After performing the above transformations on (23), we have the following expression:

$$\hat{\mathbf{\Psi}}_{ijk} = \begin{bmatrix} \hat{\mathbf{\Psi}}_{ijk}^{1,1} & \hat{\mathbf{\Psi}}_{i}^{1,2} & -\hat{\mathbf{W}}_1 & (\mathbf{\Xi}_{ijk}\mathbf{P}^{-1})^T \\ * & -\hat{\mathbf{W}}_2 - \hat{\mathbf{W}}_2^T - \hat{\mathbf{S}} & -\hat{\mathbf{W}}_2 & (\breve{\mathbf{\Xi}}_i\mathbf{P}^{-1})^T \\ * & * & -\frac{1}{\bar{\tau}}\hat{\mathbf{R}} & \mathbf{0}_{2n\times 2n} \\ * & * & * & \hat{\mathbf{\Psi}}_{i}^{4,4} \end{bmatrix}, \tag{24}$$

where

$$\hat{\mathbf{\Psi}}_{ijk}^{1,1} = (\mathbf{\Xi}_{ijk}\mathbf{P}^{-1})^T + \mathbf{\Xi}_{ijk}\mathbf{P}^{-1} + \hat{\mathbf{W}}_1 + \hat{\mathbf{W}}_1^T + \hat{\mathbf{S}}, \tag{25}$$

$$\hat{\mathbf{\Psi}}_{i}^{1,2} = \breve{\mathbf{\Xi}}_i\mathbf{P}^{-1} - \hat{\mathbf{W}}_1 + \hat{\mathbf{W}}_2^T, \qquad \hat{\mathbf{\Psi}}_{i}^{4,4} = \frac{1}{\bar{\tau}}(\lambda^2\hat{\mathbf{R}} - 2\lambda\mathbf{P}^{-1}), \tag{26}$$

$$\hat{\mathbf{W}}_1 = \mathbf{P}^{-1}\mathbf{W}_1\mathbf{P}^{-1}, \quad \hat{\mathbf{W}}_2 = \mathbf{P}^{-1}\mathbf{W}_2\mathbf{P}^{-1}, \quad \hat{\mathbf{R}} = \mathbf{P}^{-1}\mathbf{R}\mathbf{P}^{-1}, \quad \hat{\mathbf{S}} = \mathbf{P}^{-1}\mathbf{S}\mathbf{P}^{-1}, \tag{27}$$

$$\mathbf{\Xi}_{ijk}\mathbf{P}^{-1} = \begin{bmatrix} \mathbf{\Theta}_{jk}^{1,1} + \mathbf{O}_{ij}\mathbf{P}_1 & \mathbf{\Theta}_{ij}^{1,2} \\ \mathbf{\Theta}_{ijk}^{2,1} - \mathbf{O}_{ij}\mathbf{P}_1 & \mathbf{\Theta}_{ij}^{2,2} \end{bmatrix}, \quad \breve{\mathbf{\Xi}}_i\mathbf{P}^{-1} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{A}_{i\tau}\mathbf{P}_1 & \mathbf{A}_{i\tau}\mathbf{P}_2^{-1} \end{bmatrix}, \tag{28}$$

$$\mathbf{\Theta}_{jk}^{1,1} = \mathbf{T}_{j0} + \mathbf{B}_j\mathbf{M}_k, \quad \mathbf{\Theta}_{ij}^{1,2} = \mathbf{L}_j\mathbf{C}_i\mathbf{P}_2^{-1}, \quad \mathbf{O}_{ij} = \mathbf{L}_j(\mathbf{C}_i - \mathbf{C}_j), \tag{29}$$

$$\mathbf{\Theta}_{ijk}^{2,1} = \mathbf{A}_{i0}\mathbf{P}_1 - \mathbf{T}_{j0} + (\mathbf{B}_i - \mathbf{B}_j)\mathbf{M}_k, \quad \mathbf{\Theta}_{ij}^{2,2} = \mathbf{A}_{i0}\mathbf{P}_2^{-1} - \mathbf{L}_j\mathbf{C}_i\mathbf{P}_2^{-1}. \tag{30}$$

As a result, $\sum_{i,j,k=1}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}}\hat{\mathbf{\Psi}}_{ijk} < 0$ implies $\sum_{i,j,k=1}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}}\tilde{\mathbf{\Psi}}_{ijk} < 0$.

In (24), the polynomial fuzzy controller gains $\mathbf{G}_k$ appear in convex terms $\mathbf{B}_j\mathbf{M}_k$ and $(\mathbf{B}_i - \mathbf{B}_j)\mathbf{M}_k$ with $\mathbf{M}_k = \mathbf{G}_k\mathbf{P}_1$, but the polynomial fuzzy observer gains $\mathbf{L}_j$ appear in non-convex terms $\mathbf{L}_j\mathbf{C}_i\mathbf{P}_2^{-1}$ and $\mathbf{L}_j(\mathbf{C}_i - \mathbf{C}_j)\mathbf{P}_1$. In order to handle the non-convex term $\mathbf{L}_j(\mathbf{C}_i - \mathbf{C}_j)\mathbf{P}_1$, applying Lemma 2 on (24) to separate decision variables $\mathbf{L}_j$ and $\mathbf{P}_1$, we have

$$\begin{aligned}\sum_{i,j,k=1}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}}\hat{\mathbf{\Psi}}_{ijk} &= \sum_{i,j,k=1}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}}\breve{\mathbf{\Psi}}_{ijk} + \mathbf{X}^T(\sum_{r,s=1}^{p} w_{rs}^{\mathbf{x},\hat{\mathbf{x}}}\mathbf{Y}_{rs}) + (\sum_{i,j=1}^{p} w_{ij}^{\mathbf{x},\hat{\mathbf{x}}}\mathbf{Y}_{ij}^T)\mathbf{X} \\ &\leq \sum_{i,j,k=1}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}}\breve{\mathbf{\Psi}}_{ijk} + \beta\mathbf{X}^T\mathbf{X} + \beta^{-1}(\sum_{i,j=1}^{p} w_{ij}^{\mathbf{x},\hat{\mathbf{x}}}\mathbf{Y}_{ij}^T)(\sum_{r,s=1}^{p} w_{rs}^{\mathbf{x},\hat{\mathbf{x}}}\mathbf{Y}_{rs}),\end{aligned} \tag{31}$$

where

$$\breve{\mathbf{\Psi}}_{ijk} = \begin{bmatrix} \breve{\mathbf{\Psi}}_{ijk}^{1,1} & \hat{\mathbf{\Psi}}_{i}^{1,2} & -\hat{\mathbf{W}}_1 & (\mathbf{\Pi}_{ijk})^T \\ * & -\hat{\mathbf{W}}_2 - \hat{\mathbf{W}}_2^T - \hat{\mathbf{S}} & -\hat{\mathbf{W}}_2 & (\breve{\mathbf{\Xi}}_i\mathbf{P}^{-1})^T \\ * & * & -\frac{1}{\bar{\tau}}\hat{\mathbf{R}} & \mathbf{0}_{2n\times 2n} \\ * & * & * & \hat{\mathbf{\Psi}}_{i}^{4,4} \end{bmatrix}, \tag{32}$$

$$\breve{\mathbf{\Psi}}_{ijk}^{1,1} = (\mathbf{\Pi}_{ijk})^T + \mathbf{\Pi}_{ijk} + \hat{\mathbf{W}}_1 + \hat{\mathbf{W}}_1^T + \hat{\mathbf{S}}, \qquad \mathbf{\Pi}_{ijk} = \begin{bmatrix} \mathbf{\Theta}_{jk}^{1,1} & \mathbf{\Theta}_{ij}^{1,2} \\ \mathbf{\Theta}_{ijk}^{2,1} & \mathbf{\Theta}_{ij}^{2,2} \end{bmatrix}, \tag{33}$$

$$\mathbf{X} = [[\mathbf{P}_1, \mathbf{0}]\ \mathbf{0}_{n\times 2n}\ \mathbf{0}_{n\times 2n}\ \mathbf{0}_{n\times 2n}], \tag{34}$$

$$\mathbf{Y}_{ij} = \left[\left[\mathbf{O}_{ij}^T, -\mathbf{O}_{ij}^T\right]\ \mathbf{0}_{n\times 2n}\ \mathbf{0}_{n\times 2n}\ \left[\mathbf{O}_{ij}^T, -\mathbf{O}_{ij}^T\right]\right], \tag{35}$$

$\hat{\mathbf{\Psi}}_i^{1,2}$, $\hat{\mathbf{\Psi}}_i^{4,4}$, $\hat{\mathbf{W}}_1$, $\hat{\mathbf{W}}_2$, $\hat{\mathbf{R}}$, $\hat{\mathbf{S}}$ and $\breve{\mathbf{\Xi}}_i\mathbf{P}^{-1}$ are defined in (26)-(27) and (28); $\mathbf{\Theta}_{jk}^{1,1}$, $\mathbf{O}_{ij}$, $\mathbf{\Theta}_{ij}^{1,2}$, $\mathbf{\Theta}_{ijk}^{2,1}$, $\mathbf{\Theta}_{ij}^{2,2}$ are defined in (29)-(30).

In (31), the non-convex term $\mathbf{L}_j\mathbf{C}_i\mathbf{P}_2^{-1}$ still needs to be handled. To avoid affecting the convex terms $\mathbf{T}_{j0}$, $\mathbf{B}_j\mathbf{M}_k$ and $(\mathbf{B}_i - \mathbf{B}_j)\mathbf{M}_k$ when transforming the non-convex term $\mathbf{L}_j\mathbf{C}_i\mathbf{P}_2^{-1}$, MDT [45] is applied to separate (31) into two parts, one part is related to polynomial controller gains and observer system matrices, another part is related to polynomial observer gains.

Then, from (31), $\dot{V} < 0$ holds if

$$\sum_{i,j=1}^{p} w_{ij}^{\mathbf{x},\hat{\mathbf{x}}}\mathbf{\Lambda}_{ij} + \beta^{-1}(\sum_{i,j=1}^{p} w_{ij}^{\mathbf{x},\hat{\mathbf{x}}}\mathbf{Y}_{ij}^T)(\sum_{r,s=1}^{p} w_{rs}^{\mathbf{x},\hat{\mathbf{x}}}\mathbf{Y}_{rs}) < 0, \tag{36}$$

$$\sum_{i,j,k=1}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}}(\mathbf{\Upsilon}_{ijk} + \beta\mathbf{X}^T\mathbf{X}) < 0. \tag{37}$$

where $\breve{\boldsymbol{\Psi}}_{ijk} = \boldsymbol{\Lambda}_{ij} + \boldsymbol{\Upsilon}_{ijk}$; $\mathbf{X}$, $\mathbf{Y}_{ij}$ are defined in (34) and (35),

$$\boldsymbol{\Lambda}_{ij} = \begin{bmatrix} \boldsymbol{\Lambda}_{ij}^{1,1} & \boldsymbol{\Lambda}_{i}^{1,2} & -\hat{\mathbf{W}}_1 & \boldsymbol{\Lambda}_{ij}^{1,4} \\ * & \boldsymbol{\Lambda}^{2,2} & -\hat{\mathbf{W}}_2 & \boldsymbol{\Lambda}_{i}^{2,4} \\ * & * & -\frac{1}{\bar{\tau}}\hat{\mathbf{R}} & \mathbf{0}_{2n\times 2n} \\ * & * & * & \boldsymbol{\Lambda}^{4,4} \end{bmatrix}, \tag{38}$$

$$\boldsymbol{\Upsilon}_{ijk} = \begin{bmatrix} \boldsymbol{\Upsilon}_{ijk}^{1,1} & \boldsymbol{\Upsilon}_{i}^{1,2} & \mathbf{0}_{2n\times 2n} & \boldsymbol{\Upsilon}_{ijk}^{1,4} \\ * & \boldsymbol{\Upsilon}^{2,2} & \mathbf{0}_{2n\times 2n} & \boldsymbol{\Upsilon}_{i}^{2,4} \\ * & * & \mathbf{0}_{2n\times 2n} & \mathbf{0}_{2n\times 2n} \\ * & * & * & \boldsymbol{\Upsilon}^{4,4} \end{bmatrix}, \tag{39}$$

$$\boldsymbol{\Lambda}_{ij}^{1,1} = \hat{\mathbf{W}}_1 + \hat{\mathbf{W}}_1^T + \hat{\mathbf{S}} + \begin{bmatrix} \mathbf{0} & \boldsymbol{\Theta}_{ij}^{1,2} \\ * & \boldsymbol{\Theta}_{ij}^{2,2} + (\boldsymbol{\Theta}_{ij}^{2,2})^T \end{bmatrix} + \begin{bmatrix} -\kappa_5 \mathbf{P}_2^{-1} & \mathbf{0} \\ \mathbf{0} & \kappa_1 \mathbf{I} \end{bmatrix}, \tag{40}$$

$$\boldsymbol{\Upsilon}_{ijk}^{1,1} = \begin{bmatrix} \boldsymbol{\Theta}_{jk}^{1,1} + (\boldsymbol{\Theta}_{jk}^{1,1})^T & (\boldsymbol{\Theta}_{ijk}^{2,1})^T \\ * & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \kappa_5 \mathbf{P}_2^{-1} & \mathbf{0} \\ \mathbf{0} & -\kappa_1 \mathbf{I} \end{bmatrix}, \tag{41}$$

$$\boldsymbol{\Lambda}_{i}^{1,2} = -\hat{\mathbf{W}}_1 + \hat{\mathbf{W}}_2^T + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{i\tau}\mathbf{P}_2^{-1} \end{bmatrix}, \qquad \boldsymbol{\Upsilon}_{i}^{1,2} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{A}_{i\tau}\mathbf{P}_1 & \mathbf{0} \end{bmatrix}, \tag{42}$$

$$\boldsymbol{\Lambda}^{2,2} = -\hat{\mathbf{W}}_2 - \hat{\mathbf{W}}_2^T - \hat{\mathbf{S}} + \begin{bmatrix} \kappa_2 \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \qquad \boldsymbol{\Upsilon}^{2,2} = \begin{bmatrix} -\kappa_2 \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{43}$$

$$\boldsymbol{\Lambda}_{ij}^{1,4} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ (\boldsymbol{\Theta}_{ij}^{1,2})^T & (\boldsymbol{\Theta}_{ij}^{2,2})^T \end{bmatrix}, \qquad \boldsymbol{\Upsilon}_{ijk}^{1,4} = \begin{bmatrix} (\boldsymbol{\Theta}_{jk}^{1,1})^T & (\boldsymbol{\Theta}_{ijk}^{2,1})^T \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{44}$$

$$\boldsymbol{\Lambda}_{i}^{2,4} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (\mathbf{A}_{i\tau}\mathbf{P}_2^{-1})^T \end{bmatrix}, \qquad \boldsymbol{\Upsilon}_{i}^{2,4} = \begin{bmatrix} \mathbf{0} & (\mathbf{A}_{i\tau}\mathbf{P}_1)^T \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{45}$$

$$\boldsymbol{\Lambda}_{i}^{4,4} = \frac{1}{\bar{\tau}}\lambda^2\hat{\mathbf{R}} + \frac{1}{\bar{\tau}} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -2\lambda\mathbf{P}_2^{-1} \end{bmatrix} + \begin{bmatrix} -\kappa_3 \mathbf{P}_2^{-1} & \mathbf{0} \\ \mathbf{0} & \kappa_4 \mathbf{I} \end{bmatrix}, \tag{46}$$

$$\boldsymbol{\Upsilon}_{i}^{4,4} = \frac{1}{\bar{\tau}} \begin{bmatrix} -2\lambda\mathbf{P}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \kappa_3 \mathbf{P}_2^{-1} & \mathbf{0} \\ \mathbf{0} & -\kappa_4 \mathbf{I} \end{bmatrix}, \tag{47}$$

$\boldsymbol{\Theta}_{jk}^{1,1}$, $\boldsymbol{\Theta}_{ij}^{1,2}$, $\boldsymbol{\Theta}_{ijk}^{2,1}$, $\boldsymbol{\Theta}_{ij}^{2,2}$ are defined in (29)-(30); κ_1, κ_2, κ_3, κ_4 and κ_5 all are positive definite, which are introduced into (36) and (37) to relax the stability conditions.

Congruence transformation is applied by pre- and post-multiplying diag$(\mathbf{P}_2, \mathbf{P}_2, \mathbf{P}_2, \mathbf{P}_2, \mathbf{P}_2, \mathbf{P}_2, \mathbf{P}_2, \mathbf{P}_2)$ to both sides of (36), meanwhile $\mathbf{N}_j = \mathbf{P}_2\mathbf{L}_j$ is defined, we have

$$\sum_{i,j=1}^{p} w_{ij}^{\mathbf{x},\hat{\mathbf{x}}}\breve{\mathbf{\Lambda}}_{ij} + \beta^{-1}(\sum_{i,j=1}^{p} w_{ij}^{\mathbf{x},\hat{\mathbf{x}}}\breve{\mathbf{Y}}_{ij}^T)(\sum_{r,s=1}^{p} w_{rs}^{\mathbf{x},\hat{\mathbf{x}}}\breve{\mathbf{Y}}_{rs}) < 0, \tag{48}$$

where

$$\breve{\mathbf{\Lambda}}_{ij} = \begin{bmatrix} \breve{\mathbf{\Lambda}}_{ij}^{1,1} & \breve{\mathbf{\Lambda}}_{i}^{1,2} & -\breve{\mathbf{W}}_1 & \breve{\mathbf{\Lambda}}_{ij}^{1,4} \\ * & \breve{\mathbf{\Lambda}}^{2,2} & -\breve{\mathbf{W}}_2 & \breve{\mathbf{\Lambda}}_{i}^{2,4} \\ * & * & -\frac{1}{\bar{\tau}}\breve{\mathbf{R}} & \mathbf{0}_{2n\times 2n} \\ * & * & * & \breve{\mathbf{\Lambda}}^{4,4} \end{bmatrix}, \tag{49}$$

$$\breve{\mathbf{Y}}_{ij} = \left[\left[\breve{\mathbf{O}}_{ij}^T, -\breve{\mathbf{O}}_{ij}^T\right] \ \mathbf{0}_{n\times 2n} \ \mathbf{0}_{n\times 2n} \ \left[\breve{\mathbf{O}}_{ij}^T, -\breve{\mathbf{O}}_{ij}^T\right]\right], \tag{50}$$

$$\breve{\mathbf{\Lambda}}_{ij}^{1,1} = \breve{\mathbf{W}}_1 + \breve{\mathbf{W}}_1^T + \breve{\mathbf{S}} + \begin{bmatrix} \mathbf{0} & \breve{\mathbf{\Theta}}_{ij}^{1,2} \\ * & \breve{\mathbf{\Theta}}_{ij}^{2,2} + (\breve{\mathbf{\Theta}}_{ij}^{2,2})^T \end{bmatrix} + \begin{bmatrix} -\kappa_5\mathbf{P}_2 & \mathbf{0} \\ * & \kappa_1\mathbf{P}_2^2 \end{bmatrix}, \tag{51}$$

$$\breve{\mathbf{\Lambda}}_{i}^{1,2} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_2\mathbf{A}_{i\tau} \end{bmatrix} - \breve{\mathbf{W}}_1 + \breve{\mathbf{W}}_2^T, \tag{52}$$

$$\breve{\mathbf{\Lambda}}^{2,2} = \begin{bmatrix} \kappa_2\mathbf{P}_2^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} - \breve{\mathbf{W}}_2 - \breve{\mathbf{W}}_2^T - \breve{\mathbf{S}}, \tag{53}$$

$$\breve{\mathbf{\Lambda}}_{ij}^{1,4} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ (\breve{\mathbf{\Theta}}_{ij}^{1,2})^T & (\breve{\mathbf{\Theta}}_{ij}^{2,2})^T \end{bmatrix}, \qquad \breve{\mathbf{\Lambda}}_{i}^{2,4} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (\mathbf{P}_2\mathbf{A}_{i\tau})^T \end{bmatrix}, \tag{54}$$

$$\breve{\mathbf{\Lambda}}_{i}^{4,4} = \frac{1}{\bar{\tau}}\lambda^2\breve{\mathbf{R}} + \frac{1}{\bar{\tau}}\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -2\lambda\mathbf{P}_2 \end{bmatrix} + \begin{bmatrix} -\kappa_3\mathbf{P}_2 & \mathbf{0} \\ \mathbf{0} & \kappa_4\mathbf{P}_2^2 \end{bmatrix}, \tag{55}$$

$$\breve{\mathbf{\Theta}}_{ij}^{1,2} = \mathbf{N}_j\mathbf{C}_i, \quad \breve{\mathbf{\Theta}}_{ij}^{2,2} = \mathbf{P}_2\mathbf{A}_{i0} - \mathbf{N}_j\mathbf{C}_i, \quad \breve{\mathbf{O}}_{ij} = \mathbf{N}_j(\mathbf{C}_i - \mathbf{C}_j), \tag{56}$$

$$\breve{\mathbf{W}}_1 = \mathrm{diag}(\mathbf{P}_2, \mathbf{P}_2)\hat{\mathbf{W}}_1\mathrm{diag}(\mathbf{P}_2, \mathbf{P}_2), \quad \breve{\mathbf{W}}_2 = \mathrm{diag}(\mathbf{P}_2, \mathbf{P}_2)\hat{\mathbf{W}}_2\mathrm{diag}(\mathbf{P}_2, \mathbf{P}_2), \tag{57}$$

$$\breve{\mathbf{R}} = \mathrm{diag}(\mathbf{P}_2, \mathbf{P}_2)\hat{\mathbf{R}}\mathrm{diag}(\mathbf{P}_2, \mathbf{P}_2), \quad \breve{\mathbf{S}} = \mathrm{diag}(\mathbf{P}_2, \mathbf{P}_2)\hat{\mathbf{S}}\mathrm{diag}(\mathbf{P}_2, \mathbf{P}_2). \tag{58}$$

Applying Schur Complement on both (37) and (48) to handle the squared terms $\beta\mathbf{X}^T\mathbf{X}$, $\beta^{-1}(\sum_{i,j=1}^{p} w_{ij}^{\mathbf{x},\hat{\mathbf{x}}}\breve{\mathbf{Y}}_{ij}^T)(\sum_{r,s=1}^{p} w_{rs}^{\mathbf{x},\hat{\mathbf{x}}}\breve{\mathbf{Y}}_{rs})$. Also, the non-convex terms

$\mathbf{P}_2^2$ and $\mathbf{P}_2^{-1}$ are transformed into $\mathbf{P}_2$ by using Schur Complement. Then, $\dot{V} < 0$ holds if

$$\sum_{i,j=1}^{p} w_{ij}^{\mathbf{x},\hat{\mathbf{x}}}\mathbf{\Omega}_{1ij} < 0; \quad \sum_{i,j,k=1}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}}\mathbf{\Omega}_{2ijk} < 0, \tag{59}$$

where

$$\mathbf{\Omega}_{1ij} = \begin{bmatrix} \tilde{\mathbf{\Lambda}}_{ij} & \mathbf{\Omega}_1^{1,2} & \mathbf{\Omega}_1^{1,3} & \mathbf{\Omega}_1^{1,4} & \breve{\mathbf{Y}}_{ij}^T \\ * & -\kappa_1^{-1}\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & -\kappa_2^{-1}\mathbf{I} & \mathbf{0} & \mathbf{0} \\ * & * & * & -\kappa_4^{-1}\mathbf{I} & \mathbf{0} \\ * & * & * & * & -\beta\mathbf{I} \end{bmatrix}, \tag{60}$$

$$\mathbf{\Omega}_{2ijk} = \begin{bmatrix} \tilde{\mathbf{\Upsilon}}_{ijk} & \mathbf{\Omega}_2^{1,2} & \mathbf{\Omega}_2^{1,3} & \breve{\mathbf{X}}^T \\ * & -\kappa_5^{-1}\mathbf{P}_2 & \mathbf{0} & \mathbf{0} \\ * & * & -\kappa_3^{-1}\mathbf{P}_2 & \mathbf{0} \\ * & * & * & -\beta^{-1}\mathbf{I} \end{bmatrix}, \tag{61}$$

$$\tilde{\mathbf{\Lambda}}_{ij} = \begin{bmatrix} \tilde{\mathbf{\Lambda}}_{ij}^{1,1} & \breve{\mathbf{\Lambda}}_i^{1,2} & -\breve{\mathbf{W}}_1 & \breve{\mathbf{\Lambda}}_{ij}^{1,4} \\ * & \tilde{\mathbf{\Lambda}}^{2,2} & -\breve{\mathbf{W}}_2 & \breve{\mathbf{\Lambda}}_i^{2,4} \\ * & * & -\frac{1}{\bar{\tau}}\breve{\mathbf{R}} & \mathbf{0}_{2n\times 2n} \\ * & * & * & \tilde{\mathbf{\Lambda}}^{4,4} \end{bmatrix}, \quad \tilde{\mathbf{\Upsilon}}_{ijk} = \begin{bmatrix} \tilde{\mathbf{\Upsilon}}_{ijk}^{1,1} & \tilde{\mathbf{\Upsilon}}_i^{1,2} & \mathbf{\Upsilon}_{ijk}^{1,4} \\ * & -\kappa_2\mathbf{I} & \tilde{\mathbf{\Upsilon}}_i^{2,4} \\ * & * & \tilde{\mathbf{\Upsilon}}^{4,4} \end{bmatrix}, \tag{62}$$

$$\tilde{\mathbf{\Lambda}}_{ij}^{1,1} = \breve{\mathbf{W}}_1 + \breve{\mathbf{W}}_1^T + \breve{\mathbf{S}} + \begin{bmatrix} \mathbf{0} & \breve{\mathbf{\Theta}}_{ij}^{1,2} \\ * & \breve{\mathbf{\Theta}}_{ij}^{2,2} + (\breve{\mathbf{\Theta}}_{ij}^{2,2})^T \end{bmatrix} + \begin{bmatrix} -\kappa_5\mathbf{P}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{63}$$

$$\tilde{\mathbf{\Lambda}}^{2,2} = -\breve{\mathbf{W}}_2 - \breve{\mathbf{W}}_2^T - \breve{\mathbf{S}}, \tag{64}$$

$$\tilde{\mathbf{\Lambda}}^{4,4} = \frac{1}{\bar{\tau}}\lambda^2\breve{\mathbf{R}} + \frac{1}{\bar{\tau}}\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -2\lambda\mathbf{P}_2 \end{bmatrix} + \begin{bmatrix} -\kappa_3\mathbf{P}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{65}$$

$$\tilde{\mathbf{\Upsilon}}_{ijk}^{1,1} = \begin{bmatrix} \mathbf{\Theta}_{jk}^{1,1} + (\mathbf{\Theta}_{jk}^{1,1})^T & (\mathbf{\Theta}_{ijk}^{2,1})^T \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\kappa_1\mathbf{I} \end{bmatrix}, \tag{66}$$

$$\tilde{\mathbf{\Upsilon}}^{1,2} = (\tilde{\mathbf{\Upsilon}}^{2,4})^T = \begin{bmatrix} \mathbf{0} & (\mathbf{A}_{i\tau}\mathbf{P}_1)^T \end{bmatrix}^T, \tag{67}$$

$$\tilde{\mathbf{\Upsilon}}^{4,4} = \frac{1}{\bar{\tau}}\begin{bmatrix} (-2\lambda\mathbf{P}_1) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\kappa_4\mathbf{I} \end{bmatrix}, \tag{68}$$

$$\mathbf{\Omega}_1^{1,2} = [[\mathbf{0}, \mathbf{P}_2]\ \mathbf{0}_{n\times 6n}]^T, \mathbf{\Omega}_1^{1,3} = [\mathbf{0}_{n\times 2n}\ [\mathbf{P}_2, \mathbf{0}]\ \mathbf{0}_{n\times 4n}]^T, \tag{69}$$

$$\mathbf{\Omega}_1^{1,4} = [\mathbf{0}_{n\times 6n}\ [\mathbf{0}, \mathbf{P}_2]]^T, \mathbf{\Omega}_2^{1,2} = [[\mathbf{I}, \mathbf{0}]\ \mathbf{0}_{n\times 3n}]^T, \tag{70}$$

$$\mathbf{\Omega}_2^{1,3} = [\mathbf{0}_{n\times 3n}\ [\mathbf{I}, \mathbf{0}]]^T, \tilde{\mathbf{X}}^T = [[\mathbf{P}_1, \mathbf{0}]\ \mathbf{0}_{n\times 3n}]^T, \tag{71}$$

$\breve{\mathbf{Y}}_{ij}$ is defined in (50); $\breve{\mathbf{\Lambda}}_i^{1,2}$, $\breve{\mathbf{\Lambda}}_{ij}^{1,4}$, $\breve{\mathbf{\Lambda}}_i^{2,4}$ and $\mathbf{\Upsilon}_{ijk}^{1,4}$ are defined in (52), (54) and (44), respectively; $\breve{\mathbf{\Theta}}_{ij}^{1,2}$, $\breve{\mathbf{\Theta}}_{ij}^{2,2}$, $\breve{\mathbf{O}}_{ij}$, $\breve{\mathbf{W}}_1$, $\breve{\mathbf{W}}_2$, $\breve{\mathbf{R}}$ and $\breve{\mathbf{S}}$ are defined in (56)-(58); $\mathbf{\Theta}_{jk}^{1,1}$ and $\mathbf{\Theta}_{ijk}^{2,1}$ are defined in (29) and (30).

In this paper, polynomial fuzzy observer shares the same premise variables and membership functions with the polynomial fuzzy controller. By grouping terms with the same membership functions for (59), we can obtain more relaxed stability conditions as follow:

$$\sum_{i,j=1}^{p} w_{ij}^{\mathbf{x},\hat{\mathbf{x}}} \mathbf{\Omega}_{1ij} < 0; \sum_{i,k=1; j\le k}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}} (\mathbf{\Omega}_{2ijk} + \mathbf{\Omega}_{2ikj}) < 0, \tag{72}$$

where $\mathbf{\Omega}_{1ij}$ and $\mathbf{\Omega}_{2ijk}$ are defined in (60) and (61).

3.3. Positivity Analysis of Augmented PPFMB Observer-Control Systems with Unknown Time Delay

In this part, we shall conduct positivity analysis for the augmented PPFMB observer-control systems with unknown time delay (9). First, the non-convex positivity conditions are obtained, which is expressed by the following lemma. Then, we use the MDT [45] to obtain convex positivity conditions.

Lemma 4. *The system* (9) *is a positive system if and only if* $\sum_{i,j,k=1}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}} \mathbf{\Xi}_{ijk}$ *is Metzler matrix and* $\sum_{i=1}^{p} w_i \breve{\mathbf{\Xi}}_i \succ 0$*, where* $\mathbf{\Xi}_{ijk}$ *and* $\breve{\mathbf{\Xi}}_i$ *are defined in* (10) *and* (13).

Proof 1. *In order to simplify the analysis, the system* (9) *can be regarded as a polynomial fuzzy time delay system without control input matrix and output matrix, where* $\sum_{i,j,k=1}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}} \mathbf{\Xi}_{ijk}$ *is system matrix and* $\sum_{i=1}^{p} w_i \breve{\mathbf{\Xi}}_i \succ 0$ *is time delay matrix. Thus, the system* (9) *can be regarded as the specific case of the system* (2)*. According to the same line of Lemma 1, Lemma 4 is obtained.*

First, the positivity condition $\sum_{i=1}^{p} w_i \breve{\mathbf{\Xi}}_i \succ 0$ in Lemma 4 is guaranteed if

$$\sum_{i=1}^{p} w_i \mathbf{A}_{i\tau} \succ 0. \tag{73}$$

Second, we should constrain that $\sum_{i,j,k=1}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}} \mathbf{\Xi}_{ijk}$ is Metzler matrix. However, this positivity condition cannot be solved by using convex programming techniques because it is a non-convex condition when putting it together with the stability conditions (72), thus, some transformations need to be imposed on $\sum_{i,j,k=1}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}} \mathbf{\Xi}_{ijk}$ to handle this issue.

In order to transform polynomial observer gains $\mathbf{L}_j$ and polynomial controller gains $\mathbf{G}_k$ without affecting each other, MDT [45] is employed to separate the matrix $\sum_{i,j,k=1}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}} \mathbf{\Xi}_{ijk}$ into two, namely, $\sum_{i,j=1}^{p} w_{ij}^{\mathbf{x},\hat{\mathbf{x}}} \mathbf{\Xi}_{1ij}$ and $\sum_{i,j,k=1}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}} \mathbf{\Xi}_{2ijk}$ through the introduction of predefined matrix $\boldsymbol{\varpi}_1 \succ 0$ and predefined Metzler matrix $\boldsymbol{\varpi}_2$. Then, we have

$$\sum_{i,j,k=1}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}} \mathbf{\Xi}_{ijk} = \sum_{i,j=1}^{p} w_{ij}^{\mathbf{x},\hat{\mathbf{x}}} \mathbf{\Xi}_{1ij} + \sum_{i,j,k=1}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}} \mathbf{\Xi}_{2ijk}, \tag{74}$$

where

$$\mathbf{\Xi}_{1ij} = \begin{bmatrix} \mathbf{L}_j(\mathbf{C}_i - \mathbf{C}_j) + \boldsymbol{\varpi}_2 & \mathbf{L}_j\mathbf{C}_i - \boldsymbol{\varpi}_1 \\ -\mathbf{L}_j(\mathbf{C}_i - \mathbf{C}_j) + \boldsymbol{\varpi}_1 & -\mathbf{L}_j\mathbf{C}_i + \boldsymbol{\varpi}_2 \end{bmatrix},$$

$$\mathbf{\Xi}_{2ijk} = \begin{bmatrix} \mathbf{F}_{j0} + \mathbf{B}_j\mathbf{G}_k - \boldsymbol{\varpi}_2 & \boldsymbol{\varpi}_1 \\ \begin{array}{c} \mathbf{A}_{i0} - \mathbf{F}_{j0} \\ +(\mathbf{B}_i - \mathbf{B}_j)\mathbf{G}_k - \boldsymbol{\varpi}_1 \end{array} & \mathbf{A}_{i0} - \boldsymbol{\varpi}_2 \end{bmatrix}.$$

If both $\sum_{i,j=1}^{p} w_{ij}^{\mathbf{x},\hat{\mathbf{x}}} \mathbf{\Xi}_{1ij}$ and $\sum_{i,j,k=1}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}} \mathbf{\Xi}_{2ijk}$ are Metzler matrices, then $\sum_{i,j,k=1}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}} \mathbf{\Xi}_{ijk}$ is a Metzler matrix.

Pre-multiplying $\mathrm{diag}(\mathbf{P}_2, \mathbf{P}_2)$ to $\sum_{i,j=1}^{p} w_{ij}^{\mathbf{x},\hat{\mathbf{x}}} \mathbf{\Xi}_{1ij}$ and post-multiplying $\mathrm{diag}(\mathbf{P}_1, \mathbf{P}_1)$ to $\sum_{i,j,k=1}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}} \mathbf{\Xi}_{2ijk}$, we obtain:

$$\sum_{i,j=1}^{p} w_{ij}^{\mathbf{x},\hat{\mathbf{x}}} \tilde{\mathbf{\Xi}}_{1ij}, \ \sum_{i,j,k=1}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}} \tilde{\mathbf{\Xi}}_{2ijk}, \tag{75}$$

where

$$\tilde{\mathbf{\Xi}}_{1ij} = \begin{bmatrix} \mathbf{P}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_2 \end{bmatrix} \mathbf{\Xi}_{1ij} = \begin{bmatrix} \mathbf{N}_j(\mathbf{C}_i - \mathbf{C}_j) + \mathbf{P}_2\boldsymbol{\varpi}_2 & \mathbf{N}_j\mathbf{C}_i - \mathbf{P}_2\boldsymbol{\varpi}_1 \\ -\mathbf{N}_j(\mathbf{C}_i - \mathbf{C}_j) + \mathbf{P}_2\boldsymbol{\varpi}_1 & -\mathbf{N}_j\mathbf{C}_i + \mathbf{P}_2\boldsymbol{\varpi}_2 \end{bmatrix},$$

$$\tilde{\mathbf{\Xi}}_{2ijk} = \mathbf{\Xi}_{2ijk} \begin{bmatrix} \mathbf{P}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{T}_{j0} + \mathbf{B}_j\mathbf{M}_k - \boldsymbol{\varpi}_2\mathbf{P}_1 & \boldsymbol{\varpi}_1\mathbf{P}_1 \\ \begin{array}{c}\mathbf{A}_{i0}\mathbf{P}_1 - \mathbf{T}_{j0} \\ +(\mathbf{B}_i - \mathbf{B}_j)\mathbf{M}_k - \boldsymbol{\varpi}_1\mathbf{P}_1\end{array} & \mathbf{A}_{i0}\mathbf{P}_1 - \boldsymbol{\varpi}_2\mathbf{P}_1 \end{bmatrix}.$$

Then, the following conditions can guarantee that $\sum_{i,j=1}^{p} w_{ij}^{\mathbf{x},\hat{\mathbf{x}}}\tilde{\mathbf{\Xi}}_{1ij}$, $\sum_{i,j,k=1}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}}\tilde{\mathbf{\Xi}}_{2ijk}$ are Metzler matrices,

$$\sum_{i,j=1}^{p} w_{ij}^{\mathbf{x},\hat{\mathbf{x}}}\mathbf{\Delta}_{qij}^{(\hat{\alpha},\hat{\beta})} \succeq 0; \quad \forall q = 1, 2, \forall 1 \le \hat{\alpha} \neq \hat{\beta} \le n; \tag{76}$$

$$\sum_{i,j=1}^{p} w_{ij}^{\mathbf{x},\hat{\mathbf{x}}}\mathbf{\Delta}_{qij}^{(\hat{\alpha},\hat{\beta})} \succeq 0; \quad \forall q = 3, 4, \forall 1 \le \hat{\alpha}, \hat{\beta} \le n; \tag{77}$$

$$\sum_{j,k}^{p} w_{jk}^{\hat{\mathbf{x}}}\mathbf{\Delta}_{5jk}^{(\hat{\alpha},\hat{\beta})} \succeq 0; \quad \forall 1 \le \hat{\alpha} \neq \hat{\beta} \le n; \tag{78}$$

$$\sum_{i,j,k=1}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}}\mathbf{\Delta}_{6ijk}^{(\hat{\alpha},\hat{\beta})} \succeq 0; \quad \forall 1 \le \hat{\alpha}, \hat{\beta} \le n; \tag{79}$$

$$\sum_{i}^{p} w_{i}^{\mathbf{x}}\mathbf{\Delta}_{7i}^{(\hat{\alpha},\hat{\beta})} \succeq 0; \quad \forall 1 \le \hat{\alpha} \neq \hat{\beta} \le n \tag{80}$$

$$\mathbf{\Delta}_{8}^{(\hat{\alpha},\hat{\beta})} \succeq 0; \quad \forall 1 \le \hat{\alpha}, \hat{\beta} \le n; \tag{81}$$

where

$$\begin{aligned}
\mathbf{\Delta}_{1ij}^{(\hat{\alpha},\hat{\beta})} &= \mathbf{N}_j^{(\hat{\alpha},s)}(\mathbf{C}_i^{(s,\hat{\beta})} - \mathbf{C}_j^{(s,\hat{\beta})}) + \mathbf{P}_2^{(\hat{\alpha},\hat{\alpha})}\boldsymbol{\varpi}_2^{(\hat{\alpha},\hat{\beta})}, \\
\mathbf{\Delta}_{2ij}^{(\hat{\alpha},\hat{\beta})} &= -\mathbf{N}_j^{(\hat{\alpha},s)}\mathbf{C}_i^{(s,\hat{\beta})} + \mathbf{P}_2^{(\hat{\alpha},\hat{\alpha})}\boldsymbol{\varpi}_2^{(\hat{\alpha},\hat{\beta})}, \\
\mathbf{\Delta}_{3ij}^{(\hat{\alpha},\hat{\beta})} &= \mathbf{N}_j^{(\hat{\alpha},s)}\mathbf{C}_i^{(s,\hat{\beta})} - \mathbf{P}_2^{(\hat{\alpha},\hat{\alpha})}\boldsymbol{\varpi}_1^{(\hat{\alpha},\hat{\beta})}, \\
\mathbf{\Delta}_{4ij}^{(\hat{\alpha},\hat{\beta})} &= -\mathbf{N}_j^{(\hat{\alpha},s)}(\mathbf{C}_i^{(s,\hat{\beta})} - \mathbf{C}_j^{(s,\hat{\beta})}) + \mathbf{P}_2^{(\hat{\alpha},\hat{\alpha})}\boldsymbol{\varpi}_1^{(\hat{\alpha},\hat{\beta})}, \\
\mathbf{\Delta}_{5jk}^{(\hat{\alpha},\hat{\beta})} &= \mathbf{T}_{j0}^{(\hat{\alpha},\hat{\beta})} + \mathbf{B}_j^{(\hat{\alpha},r)}\mathbf{M}_k^{(r,\hat{\beta})} - \boldsymbol{\varpi}_2^{(\hat{\alpha},\hat{\beta})}\breve{\mathbf{P}}_1^{(\hat{\beta},\hat{\beta})}, \\
\mathbf{\Delta}_{6ijk}^{(\hat{\alpha},\hat{\beta})} &= \mathbf{A}_{i0}^{(\hat{\alpha},\hat{\beta})}\mathbf{P}_1^{(\hat{\beta},\hat{\beta})} - \mathbf{T}_{j0}^{(\hat{\alpha},\hat{\beta})} + \left(\mathbf{B}_i^{(\hat{\alpha},r)} - \mathbf{B}_j^{(\hat{\alpha},r)}\right)\mathbf{M}_k^{(r,\hat{\beta})} - \boldsymbol{\varpi}_1^{(\hat{\alpha},\hat{\beta})}\mathbf{P}_1^{(\hat{\beta},\hat{\beta})}, \\
\mathbf{\Delta}_{7i}^{(\hat{\alpha},\hat{\beta})} &= \mathbf{A}_{i0}^{(\hat{\alpha},\hat{\beta})}\mathbf{P}_1^{(\hat{\beta},\hat{\beta})} - \boldsymbol{\varpi}_2^{(\hat{\alpha},\hat{\beta})}\mathbf{P}_1^{(\hat{\beta},\hat{\beta})}, \qquad \mathbf{\Delta}_{8}^{(\hat{\alpha},\hat{\beta})} = \boldsymbol{\varpi}_1^{(\hat{\alpha},\hat{\beta})}\mathbf{P}_1^{(\hat{\beta},\hat{\beta})}.
\end{aligned}$$

The conditions (76) and (77) are to make sure that $\sum_{i,j=1}^{p} w_{ij}^{\mathbf{x},\hat{\mathbf{x}}} \tilde{\boldsymbol{\Xi}}_{1ij}$ is a Metzler matrix meanwhile the conditions from (78) to (81) are to make sure that $\sum_{i,j,k=1}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}} \tilde{\boldsymbol{\Xi}}_{2ijk}$ is a Metzler matrix. As all elements of diagonal positive definite matrices $\mathbf{P}_1$ and $\mathbf{P}_2$ are positive, pre-multiplying diag$(\mathbf{P}_2, \mathbf{P}_2)$ to $\sum_{i,j=1}^{p} w_{ij}^{\mathbf{x},\hat{\mathbf{x}}} \boldsymbol{\Xi}_{1ij}$ and post-multiplying diag$(\mathbf{P}_1, \mathbf{P}_1)$ to $\sum_{i,j,k=1}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}} \boldsymbol{\Xi}_{2ijk}$ does not alter the Metzler property of $\sum_{i,j=1}^{p} w_{ij}^{\mathbf{x},\hat{\mathbf{x}}} \boldsymbol{\Xi}_{1ij}$ and $\sum_{i,j,k=1}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}} \boldsymbol{\Xi}_{2ijk}$. Therefore, conditions from (76) to (81) imply that $\sum_{i,j=1}^{p} w_{ij}^{\mathbf{x},\hat{\mathbf{x}}} \boldsymbol{\Xi}_{1ij}$ and $\sum_{i,j,k=1}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}} \boldsymbol{\Xi}_{2ijk}$ are Metzler matrices. By grouping terms with the same membership functions, more relaxed positivity conditions for (78) and (79) are obtained as follow:

$$\sum_{k=1; j \leq k}^{p} w_{jk}^{\hat{\mathbf{x}}} (\boldsymbol{\Delta}_{5jk}^{(\hat{\alpha},\hat{\beta})} + \boldsymbol{\Delta}_{5kj}^{(\hat{\alpha},\hat{\beta})}) \succeq 0; \forall 1 \leq \hat{\alpha} \neq \hat{\beta} \leq n; \tag{82}$$

$$\sum_{i,k=1; j \leq k}^{p} w_{ijk}^{\mathbf{x},\hat{\mathbf{x}}} (\boldsymbol{\Delta}_{6ijk}^{(\hat{\alpha},\hat{\beta})} + \boldsymbol{\Delta}_{6ijk}^{(\hat{\alpha},\hat{\beta})}) \succeq 0; \forall 1 \leq \hat{\alpha}, \hat{\beta} \leq n. \tag{83}$$

Remark 3. *For the system* (2)*, when* $\boldsymbol{\phi}(\cdot) \succ 0$ *cannot be satisfied,* $\mathbf{x}$ *may not stay in the positive orthant even if the conditions of Lemma 1 holds. Similar to the constrain of system* (2)*, for the augmented system* (9)*, the initial condition must satisfy* $\hat{\boldsymbol{\phi}}(\cdot) \succ 0$ *and* $\boldsymbol{\phi}(\cdot) \succeq \hat{\boldsymbol{\phi}}(\cdot)$*, then the augmented system* (9) *are guaranteed to be positive when Lemma 4 is satisfied.*

3.4. Stability and Positivity Conditions of Closed-Loop PPFMB Observer-Control Systems with Unknown Time Delay

In this subsection, the results of stability analysis presented in Subsection 3.2 and the results of positivity analysis presented in Subsection 3.3 are summarized below.

Theorem 1. *Given the open-loop positive system* (2)*, which is guaranteed to be positive by satisfying the conditions in Lemma 1, the polynomial fuzzy observer* (4) *and the polynomial fuzzy controller* (5) *can make sure that the augmented PPFMB observer-control system with unknown time delay* (9) *is asymptotically* *stable and positive for any* $\hat{\boldsymbol{\phi}}(\cdot) \succ 0$ *and the initial condition* $\boldsymbol{\phi}(\cdot) \succeq \hat{\boldsymbol{\phi}}(\cdot)$ *if there exist diagonal matrices* $\mathbf{P}_1 \in \Re^{n\times n}$, $\mathbf{P}_2 \in \Re^{n\times n}$; *symmetric matrices* $\breve{\mathbf{S}} \in \Re^{2n\times 2n}$, $\breve{\mathbf{R}} \in \Re^{2n\times 2n}$, $\breve{\mathbf{W}}_1 \in \Re^{2n\times 2n}$, $\breve{\mathbf{W}}_2 \in \Re^{2n\times 2n}$; *polynomial matrices* $\mathbf{M}_k \in \Re^{m\times n}$, $\mathbf{N}_j \in \Re^{n\times l}$, $\mathbf{T}_{j0} \in \Re^{n\times n}$, *for* $\forall (j,k) \in \{1,2,\ldots,p\}$ *such that the following SOS-based conditions are satisfied:*

$$\nu^T(\mathbf{P}_1 - \varepsilon_1\mathbf{I})\nu \text{ is SOS;} \quad \nu^T(\mathbf{P}_2 - \varepsilon_2\mathbf{I})\nu \text{ is SOS;} \tag{84}$$

$$\nu^T(\breve{\mathbf{S}} - \varepsilon_3\mathbf{I})\nu \text{ is SOS;} \quad \nu^T(\breve{\mathbf{R}} - \varepsilon_4\mathbf{I})\nu \text{ is SOS;} \tag{85}$$

$$\begin{cases} -\nu^T(\boldsymbol{\Omega}_{1ij} + \varepsilon_{5ij}\mathbf{I})\nu \text{ is SOS}, \forall i,j; \\ -\nu^T\left(\boldsymbol{\Omega}_{2ijk} + \boldsymbol{\Omega}_{2ikj} + \varepsilon_{6ijk}\mathbf{I}\right)\nu \text{ is SOS}, \forall i, j \le k; \end{cases} \tag{86}$$

$$\begin{cases} \nu^T\left(\boldsymbol{\Delta}_{1ij}^{(\hat{\alpha},\hat{\beta})} - \varepsilon_{7ij}\right)\nu \text{ is SOS}, \ \forall i,j, \forall 1 \le \hat{\alpha} \ne \hat{\beta} \le n; \\ \nu^T\left(\boldsymbol{\Delta}_{2ij}^{(\hat{\alpha},\hat{\beta})} - \varepsilon_{8ij}\right)\nu \text{ is SOS}, \ \forall i,j; \forall 1 \le \hat{\alpha} \ne \hat{\beta} \le n; \\ \nu^T\left(\boldsymbol{\Delta}_{3ij}^{(\hat{\alpha},\hat{\beta})} - \varepsilon_{9ij}\right)\nu \text{ is SOS}, \ \forall i,j; \forall 1 \le \hat{\alpha}, \hat{\beta} \le n; \\ \nu^T\left(\boldsymbol{\Delta}_{4ij}^{(\hat{\alpha},\hat{\beta})} - \varepsilon_{10ij}\right)\nu \text{ is SOS}, \ \forall i,j; \forall 1 \le \hat{\alpha}, \hat{\beta} \le n; \\ \nu^T\left(\boldsymbol{\Delta}_{5jk}^{(\hat{\alpha},\hat{\beta})} + \boldsymbol{\Delta}_{5kj}^{(\hat{\alpha},\hat{\beta})} - \varepsilon_{11jk}\right)\nu \text{ is SOS}, \forall j \le k, \forall 1 \le \hat{\alpha} \ne \hat{\beta} \le n; \\ \nu^T\left(\boldsymbol{\Delta}_{6ijk}^{(\hat{\alpha},\hat{\beta})} + \boldsymbol{\Delta}_{6ikj}^{(\hat{\alpha},\hat{\beta})} - \varepsilon_{12ijk}\right)\nu \text{ is SOS}, \forall i, j \le k, \ \forall 1 \le \hat{\alpha}, \hat{\beta} \le n; \end{cases} \tag{87}$$

where ν *is an arbitrary vector independent of* $\mathbf{x}$ *with appropriate dimension;* $\bar{\tau} \ge \tau > 0$ *is the estimated time delay;* $\kappa_1 > 0$, $\kappa_2 > 0$, $\kappa_3 > 0$, $\kappa_4 > 0$, $\kappa_5 > 0$, $\lambda > 0$, $\beta > 0$, $\varepsilon_1 > 0, \ldots, \varepsilon_4 > 0$ *are arbitrary predefined scalars and* $\varepsilon_{5ij} > 0, \ldots, \varepsilon_{12ijk} > 0$ *are predefined scalar polynomials;* $\boldsymbol{\varpi}_1 \succ 0$ *and Metzler matrix* $\boldsymbol{\varpi}_2$ *are predefined ; the decision polynomial observer system matrix is given by* $\mathbf{F}_{j0} = \mathbf{T}_{j0}\mathbf{P}_1^{-1}$; *the polynomial controller and observer gains are given by* $\mathbf{G}_k = \mathbf{M}_k\mathbf{P}_1^{-1}$, $\mathbf{L}_j = \mathbf{P}_2^{-1}\mathbf{N}_j$.

Remark 4. *In Theorem 1, the predefined scalars* $\kappa_1 > 0$, $\kappa_2 > 0$, $\kappa_3 > 0$, $\kappa_4 > 0$, $\kappa_5 > 0$, $\lambda > 0$, $\beta > 0$, $\boldsymbol{\varpi}_1$ *and* $\boldsymbol{\varpi}_2$ *are the relax matrices introduced in the process of convexification, they can be searched by using, e.g., grid search, iterative search or genetic algorithms. Since the diagonal elements of* $\boldsymbol{\varpi}_2$ *do not affect the analysis result, they are fixed to 1 to reduce the numbers of pre-defined* *variables. The purpose of the introduce of predefined scalar or predefined scalar polynomials $\varepsilon_1 > 0, \ldots, \varepsilon_4 > 0$, $\varepsilon_{5ij} > 0, \ldots, \varepsilon_{12ijk} > 0$ are to make sure that the SOS-based conditions in Theorem 1 are strictly positive definite. Usually, they are small scalars or scalar polynomials chosen by the user .*

Remark 5. *The convex stability conditions of system* (9) *are* (72) *in which diagonal matrices $\mathbf{P}_1$, $\mathbf{P}_2$ and symmetric matrices $\breve{\mathbf{S}}$, $\breve{\mathbf{R}}$ all are positive definite matrices. The convex positivity conditions for system* (9) *are* (73)*,* (76)*,* (77)*,* (80)*,* (81)*,* (82) *and* (83) *in which diagonal matrices $\mathbf{P}_1$, $\mathbf{P}_2$ from conditions* (72)*. Conditions* (72)*,* (76)*,* (77)*,* (82) *and* (83) *are summarized as SOS conditions* (84)*-*(87) *in Theorem 1. In Theorem 1, the SOS-based conditions* (84)*-*(86) *are to make sure that system* (9) *is asymptotically stable meanwhile* (84) *and* (87) *are to make sure that system* (9) *is positive. It should be noted that the positivity conditions* (73)*,* (80) *and* (81) *are not summarized in the form of SOS to Theorem 1. Because* (80) *and* (81) *can be guaranteed by predefined matrix $\boldsymbol{\varpi}_1 \succ 0$, predefined Metzler matrix $\boldsymbol{\varpi}_2$ and predefined Metzler matrix $\sum_{i=1}^{p} w_i \mathbf{A}_{i0}$ with any value of diagonal positive definite matrices $\mathbf{P}_1$, positivity condition* (73) *is guaranteed by predefined matrix $\sum_{i=1}^{p} w_i \mathbf{A}_{i\tau} \succ 0$. According to the Lemma 1, it is required that $\sum_{i=1}^{p} w_i \mathbf{A}_{i0}$ is a Metzler matrix and $\sum_{i=1}^{p} w_i \mathbf{A}_{i\tau} \succ 0$ for open-loop positive system.*

4. Simulation Examples

In this section, two examples are provided to demonstrate the effectiveness and applicability of the analysis results.

4.1. Example 1: Numerical Example

A two-rule polynomial fuzzy system with unknown time delay is considered, which is a numerical example. The system matrices, time delay matrices, input and output matrices in each rule are given as follows:

$$\mathbf{A}_{10}(x_2) = \begin{bmatrix} (-0.2454 + a)x_2^2 - 3.7002 & 2.0504 \\ 2.2673 & -0.1428x_2^2 - 1.8012 \end{bmatrix},$$

$$\mathbf{A}_{20}(x_2) = \begin{bmatrix} -0.1602x_2^2 - 3.0854 & 2.0689 \\ 2.2926 & -0.0619x_2^2 - 1.4132 \end{bmatrix},$$
$$\mathbf{A}_{1\tau}(x_2) = \begin{bmatrix} 0.0542 & 0.0136 \\ 0.1521x_2^2 & 0.0141x_2^2 + 0.0659 \end{bmatrix},$$
$$\mathbf{A}_{2\tau}(x_2) = \begin{bmatrix} 0.0483 & 0.0076 \\ 0.0514x_2^2 & 0.0084x_2^2 + 0.0587 \end{bmatrix},$$
$$\mathbf{B}_1 = \mathbf{B}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$\mathbf{C}_1(x_2) = [2.1653 + 0.1451x_2^2 + b \quad 0], \qquad \mathbf{C}_2(x_2) = [1.9740 + 0.0686x_2^2 \quad 0].$$

The observer system matrices $\mathbf{F}_{10}(\hat{x}_2)$ and $\mathbf{F}_{20}(\hat{x}_2)$ are decision variables to be determined. The input and output matrices in each rule are given as follows:

$$\mathbf{B}_1 = \mathbf{B}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$\mathbf{C}_1(\hat{x}_2) = [2.1653 + 0.1451\hat{x}_2^2 + b \quad 0], \qquad \mathbf{C}_2(\hat{x}_2) = [1.9740 + 0.0686\hat{x}_2^2 \quad 0].$$

The observer matrices depend on the estimate premise variable and the
305 model matrices depend on the unmeasurable premise variable, but the coefficients of input and output matrices between the model and observer are

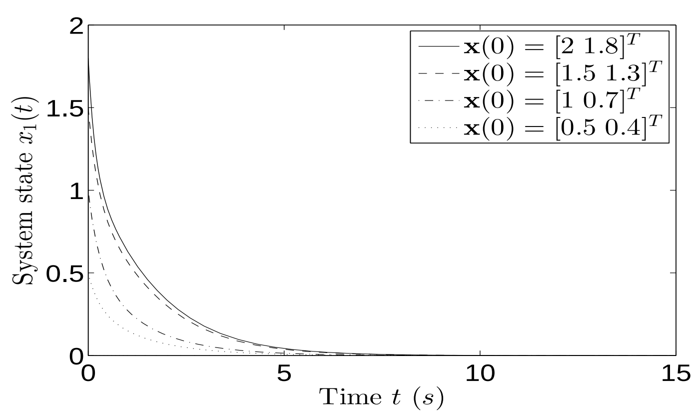

Figure 1: Time responses of system state $x_1(t)$ with different initial conditions

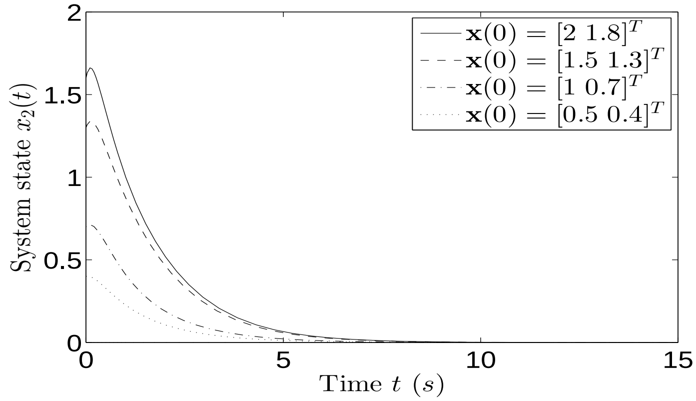

Figure 2: Time responses of system state $x_2(t)$ with 4 different initial conditions

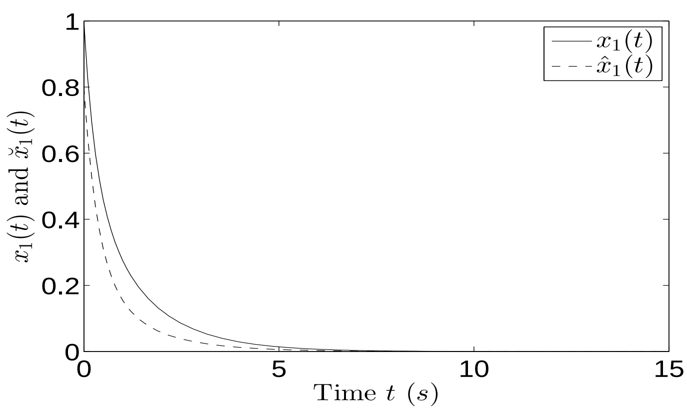

Figure 3: Time responses of system state $x_1(t)$ and estimated state $\hat{x}_1(t)$ for $\mathbf{x}(0) = [1\ 0.7]^T$

same. Meanwhile, the membership functions of polynomial fuzzy system are chosen as $w_1(x_2) = 1 - sin(x_2)^2$ and $w_2(x_2) = 1 - w_1(x_2)$. The membership functions of observer and controller are chosen as $w_1(\hat{x}_2) = 1 - sin(\hat{x}_2)^2$ and $w_2(\hat{x}_2) = 1 - w_1(\hat{x}_2)$.

According to Lemma 1, $\mathbf{A}_{10}, \mathbf{A}_{20}$ are Metzler matrices; $\mathbf{A}_{1\tau} \succ 0, \mathbf{A}_{2\tau} \succ 0$; $\mathbf{B}_1 \succ 0, \mathbf{B}_2 \succ 0$; $\mathbf{C}_1 \succ 0, \mathbf{C}_2 \succ 0$ which suggests that the open-loop polynomial fuzzy system is positive when u is nonnegative.

To demonstrate the effectiveness of our method, Theorem 1 is adopted to de-

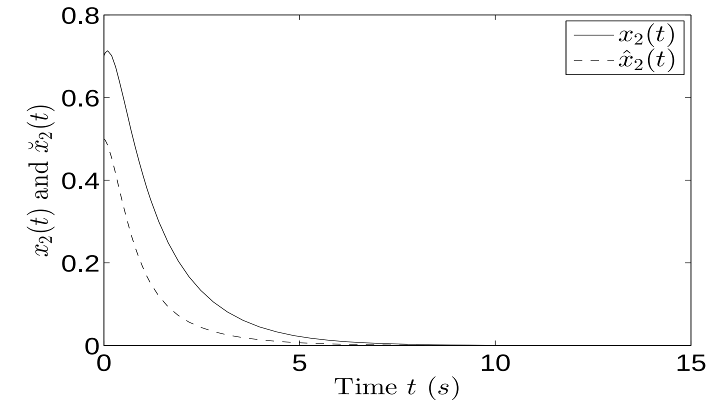

Figure 4: Time responses of system state $x_2(t)$ and estimated state $\hat{x}_2(t)$ for $\mathbf{x}(0) = [1\ 0.7]^T$

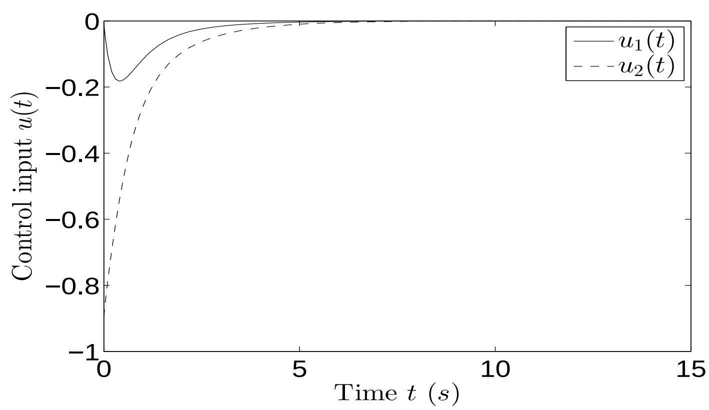

Figure 5: Time responses of control input $\mathbf{u}(t)$ for $\mathbf{x}(0) = [1\ 0.7]^T$

sign polynomial observer and controller to stabilize the unstable positive system with unknown time delay and guarantee the system states always in positive orthant. We choose $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_{12ijk} = 1 \times 10^{-3}$, $\kappa_1 = \kappa_2 = \kappa_4 = 1 \times 10^{-3}$, $\kappa_3 = \kappa_5 = 1 \times 10^{-4}$, $\beta = 1 \times 10^{-5}$, $\lambda = 1 \times 10^{-1}$, $\boldsymbol{\varpi}_1 = \begin{bmatrix} 0.2 & 0.01 \\ 0.01 & 0.2 \end{bmatrix}$, $\boldsymbol{\varpi}_2 = \begin{bmatrix} 1 & 0.01 \\ 0.01 & 1 \end{bmatrix}$, $\mathbf{T}_{j0}(\hat{x}_2)$ of degree 0 and 2, $\mathbf{M}_k(\hat{x}_2)$ of degree 0 and 2, $\mathbf{N}_j(\hat{x}_2)$ of degree 0 and 2; $\bar{\tau} = 0.1$, $\tau = 0.1$.

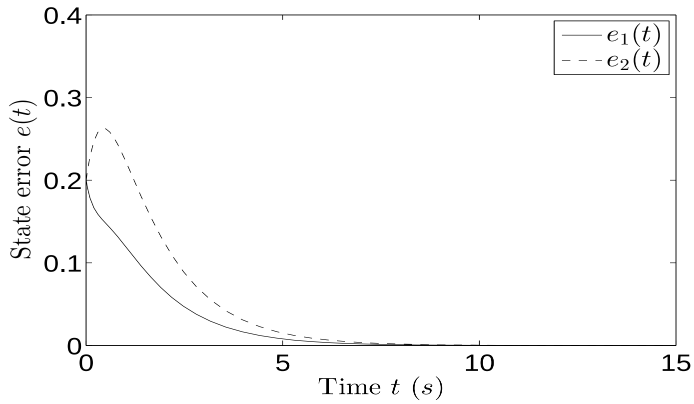

Figure 6: Time responses of estimation error $\mathbf{e}(t)$ of system states for $\mathbf{x}(0) = [1\ 0.7]^T$

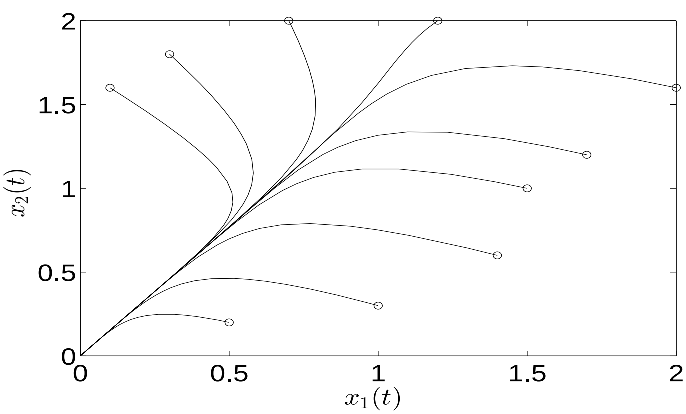

Figure 7: Phase plots of $x_1(t)$ and $x_2(t)$, the initial system states are indicated by "$\circ$"

The Lyapunov function matrices are obtained as follows:

$$\mathbf{P}_1 = \begin{bmatrix} 0.0354 & 0 \\ 0 & 0.0644 \end{bmatrix}, \mathbf{P}_2 = \begin{bmatrix} 0.0113 & 0 \\ 0 & 0.0135 \end{bmatrix},$$

$$\mathbf{R} = \begin{bmatrix} 0.2926 & -0.0034 & 0.0222 & -0.0102 \\ -0.0034 & 0.2906 & 0.0168 & -0.0068 \\ 0.0222 & 0.0168 & 0.3665 & 0.0454 \\ -0.0102 & -0.0068 & 0.0454 & 0.4483 \end{bmatrix},$$

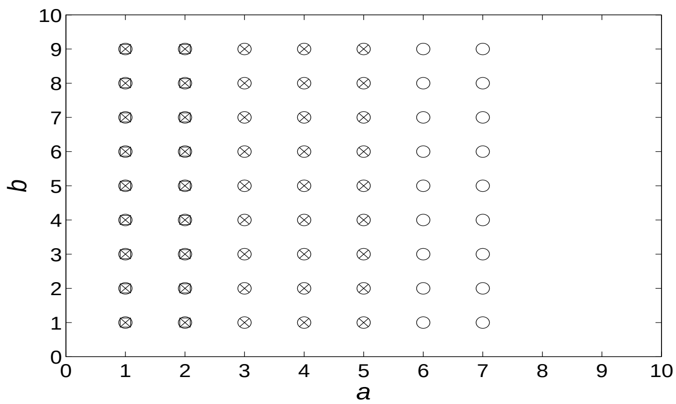

Figure 8: Stability regions for Case I("∘"), Case II("×") and Case III("□") in Example 1.

$$\mathbf{S} = \begin{bmatrix} 0.1521 & -0.0017 & -0.0089 & -0.0044 \\ -0.0017 & 0.1526 & -0.0052 & -0.0028 \\ -0.0089 & -0.0052 & 0.2182 & 0.0315 \\ -0.0044 & -0.0028 & 0.0315 & 0.2727 \end{bmatrix}.$$

The observer system matrices in each rules are obtained as follows:

$$\mathbf{F}_{10}(\hat{x}_2) = \begin{bmatrix} -0.2735\hat{x}_2^2 - 3.7649 & 1.9299 \\ 2.1130 & -0.1504\hat{x}_2^2 - 1.9004 \end{bmatrix},$$

$$\mathbf{F}_{20}(\hat{x}_2) = \begin{bmatrix} -0.1200\hat{x}_2^2 - 3.6025 & 1.9199 \\ 2.1201 & -0.6594\hat{x}_2^2 - 1.8037 \end{bmatrix}.$$

The controller gains and observer gains are obtained as follows:

$$\mathbf{G}_1(\hat{x}_2) = \begin{bmatrix} -0.3932\hat{x}_2^2 + 0.8751 & 0.1239\hat{x}_2^2 - 1.3369 \\ 0.2254\hat{x}_2^2 - 0.9083 & -0.2161\hat{x}_2^2 - 0.6498 \end{bmatrix},$$

$\mathbf{G}_2(\hat{x}_2) = \begin{bmatrix} -0.5499\hat{x}_2^2 + 0.7811 & 0.2445\hat{x}_2^2 - 0.9415 \\ 0.4449\hat{x}_2^2 - 0.1656 & -0.3023\hat{x}_2^2 - 0.6951 \end{bmatrix}$, $\mathbf{L}_1(\hat{x}_2) = [1.0297\hat{x}_2^2 + 0.5406 \quad 3.0533 \times 10^{-9}\hat{x}_2^2 + 0.2776]^T$, $\mathbf{L}_2(\hat{x}_2) = [0.9850\hat{x}_2^2 + 0.4945 \quad 3.0412 \times 10^{-9}\hat{x}_2^2 + 0.2516]^T$.

The above controller and observer gains are applied to control the system. Considering four different initial conditions, the system is stabilized and its system states are remained in the positive quadrant as shown in Figs. 1 and 2. To illustrate that the designed observer can estimate the system states effectively, from Remark 3, we choose the initial conditions $\mathbf{x}(0) = [1\ 0.7]^T$ and $\hat{\mathbf{x}}(0) = [0.8\ 0.5]^T$, the time response of the system states and the estimated system states are shown in Figs. 3 and 4 meanwhile the control signal is shown in Fig. 5. In Fig. 6, it can be seen that the estimation error of system states are positive and asymptotically stable. Furthermore, in order to further demonstrate the stability and positivity results, with the various initial conditions indicated by "$\circ$", the phase plots of x_1 and x_2 are shown in Fig. 7.

In this paper, mismatched framework type is employed, which means the premise variables of model are different from the premise variables of observer-controller and the observer system matrix is designed as a decision variable which can be different from the model system matrix. The difference between the premise variables of the model and the premise variables of the observer-controller widens the application of research results, because it can handle the case that the premise variables are unmeasurable. In order to demonstrate that defining the observer system matrix as a decision variable can relax the stability region and the large delay causes a small stability region, we consider four cases: (I) $\bar{\tau} = 0.1$, the observer system matrix is designed as a decision variable which is allowed to be different from the model system matix; (II) $\bar{\tau} = 0.1$, the coefficients of observer system matrix are set to be the same as the coefficients of model system matrix; (III) $\bar{\tau} = 0.2$, the observer system matrix is designed as a decision variable; (IV) $\bar{\tau} = 0.2$, the coefficients of observer system matrix are set to be the same as the coefficients of model system matrix. When Theorem 1 is applied to this example in these four different cases, different stability regions are obtained and are shown in Fig. 8 indicated by "$\times$"(Case I), "$\circ$"(Case II) and "$\square$"(Case III), where $1 \leq a \leq 10$ and $1 \leq b \leq 9$ (both at the interval of 1) characterizing the floating range of system parameters. Since the conditions of Case IV are too conservative, no stability region is obtained.

4.2. Example 2: Application to a Real Pest Population System

In the following, we consider a real-world example to show the validity of the design results. Firstly, we establish a polynomial model for this real-world positive nonlinear system by using the sector nonlinearity technique. Then, the Theorem 1 is applied on this PPFMB system.

An epidemic model of real pest population system with a control variable [48] is described as follows:

$$\begin{aligned} \dot{x}_1(t) &= -\gamma x_1(t)x_2(t), \\ \dot{x}_2(t) &= \gamma x_1(t)x_2(t) - \omega x_2(t) + u(t), \end{aligned} \tag{88}$$

where $x_1(t)$ represents the number of susceptible insects and $x_2(t)$ represents the number of infective insects at time t, respectively; $\gamma > 0$ is the transmission coefficient from susceptible insects to infective ones; $\omega > 0$ is called the death coefficient of $x_2(t)$; $u(t)$ is a control variable which is used to control the pest population.

Assuming that natural enemies only feed on mature pests, and the time spent by immature insects growing to maturity cannot be ignored, an epidemic model of real pest population system with time delay is represented as follows:

$$\begin{aligned} \dot{x}_1(t) &= -\gamma x_1(t)x_2(t) + \alpha x_1(t-\tau), \\ \dot{x}_2(t) &= \gamma x_1(t)x_2(t) - \omega x_2(t) + u(t), \end{aligned} \tag{89}$$

where $\alpha > 0$ is a coefficient which relates to the proportionality coefficient of existing mature pest population and immature pest population.

Defining the region of interest as $x_2 \in [0, 20]$, the above nonlinear system is modeled by sector nonlinearity technique [49], $f_1(x_2) = x_2 = u_{M_1^1}(x_2)f_{1max} + u_{M_1^2}(x_2)f_{1min}$, where $u_{M_1^1}(x_2) = (f_1(x_2) - f_{1min})/(f_{1max} - f_{1min})$, $u_{M_1^2}(x_2) = 1 - u_{M_1^1}(x_2)$, $f_{1max} = 20$, $f_{1min} = 0$. Then, the nonlinear positive system (89) can be modeled as a 2-rule T-S fuzzy model which is the special polynomial fuzzy model with degree of 0. The positive polynomial fuzzy model is described as follows:

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{2} w_i(\mathbf{x}(t)) \left(\mathbf{A}_{i0}\mathbf{x}(t) + \mathbf{A}_{i\tau}\mathbf{x}(t-\tau) + \mathbf{B}_i\mathbf{u}(t) \right), \tag{90}$$

where $w_1(\mathbf{x}(t)) = x_2/20$ and $w_2(\mathbf{x}(t)) = 1 - w_1(\mathbf{x}(t))$,

$$\mathbf{A}_{10} = \begin{bmatrix} -\gamma f_{1max} & 0 \\ \gamma f_{1max} & -\omega \end{bmatrix}, \mathbf{A}_{20} = \begin{bmatrix} -\gamma f_{1min} & 0 \\ \gamma f_{1min} & -\omega \end{bmatrix},$$
$$\mathbf{A}_{1\tau} = \mathbf{A}_{2\tau} = \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix}, \mathbf{B}_1 = \mathbf{B}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Assume that the output is the function of system states: $\mathbf{y} = x_1$. Then the output matrices are $\mathbf{C}_1 = \mathbf{C}_2 = [1 \quad 0]$. The membership functions are $w_1(\mathbf{x}(t)) = x_2/20$ and $w_2(\mathbf{x}(t)) = 1 - w_1(\mathbf{x}(t))$. Then the output is described as follows:

$$\mathbf{y}(t) = \sum_{i=1}^{2} w_i(\mathbf{x}(t))\mathbf{C}_i\mathbf{x}(t). \tag{91}$$

The polynomial fuzzy observer is described as follows:

$$\begin{aligned} \dot{\hat{\mathbf{x}}}(t) &= \sum_{j=1}^{2} w_j(\hat{\mathbf{x}}(t)) \left(\mathbf{F}_{j0}\hat{\mathbf{x}}(t) + \mathbf{B}_j\mathbf{u}(t) + \mathbf{L}_j(\hat{\mathbf{x}}(t))(\mathbf{y}(t) - \hat{\mathbf{y}}(t))\right), \\ \hat{\mathbf{y}}(t) &= \sum_{j=1}^{2} w_j(\hat{\mathbf{x}}(t))\mathbf{C}_j\hat{\mathbf{x}}(t), \end{aligned} \tag{92}$$

where the membership functions are chosen as $w_1(\hat{\mathbf{x}}(t)) = \hat{x}_2/20$ and $w_2(\hat{\mathbf{x}}(t)) = 1 - w_1(\hat{\mathbf{x}}(t))$, $\mathbf{F}_{j0}$ is the observer system matrix to be determined, the coefficients of observer input matrix $\mathbf{B}_j$ and observer output matrix $\mathbf{C}_j$ are the same as that of the system input matrix $\mathbf{B}_i$ and the system output matrix $\mathbf{C}_i$, respectively. So the observer input matrices $\mathbf{B}_1 = \mathbf{B}_2 = [0 \quad 1]^T$ and observer output matrices $\mathbf{C}_1 = \mathbf{C}_2 = [1 \quad 0]$ are obtained.

The polynomial fuzzy controller is described as follows:

$$\mathbf{u}(t) = \sum_{k=1}^{2} w_k(\hat{\mathbf{x}}(t))\mathbf{G}_k(\hat{\mathbf{x}}(t))\hat{\mathbf{x}}(t), \tag{93}$$

where the membership functions are chosen as $w_1(\hat{\mathbf{x}}(t)) = \hat{x}_2/20$ and $w_2(\hat{\mathbf{x}}(t)) = 1 - w_1(\hat{\mathbf{x}}(t))$ which are the same as the membership functions of polynomial fuzzy observer.

The system state variable x_2 is unmeasurable. $\gamma = 2.5$, $\omega = 6$, $\alpha = 0.5$, $\mathbf{A}_{10}, \mathbf{A}_{20}$ are Metzler matrices; $\mathbf{A}_{1\tau} \succ 0, \mathbf{A}_{2\tau} \succ 0$; $\mathbf{B}_1 \succ 0, \mathbf{B}_2 \succ 0$; $\mathbf{C}_1 \succ$

$0, \mathbf{C}_2 \succ 0$ which mean that the pest population system is positive when u is nonnegative. Also, it can be seen that the membership functions of model depend on the unmeasurable premise variable x_2 and the membership functions of fuzzy observer and fuzzy controller depend on the estimated premise variable $\hat{x}_2$. So the PPFMB observer-control pest population system can be an example to verify the effectiveness of the design results. Now, Theorem 1 is applied to design polynomial fuzzy controller and observer for the pest population system ((90) and (91)). We choose $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 1\times10^{-3}$, $\varepsilon_{5ij} = \varepsilon_{6ijk} = 1\times10^{-4}$, $\varepsilon_{7ij} = \cdots = \varepsilon_{12ijk} = 1\times10^{-3}$, $\kappa_1 = \kappa_2 = \kappa_4 = 1\times10^{-4}$, $\kappa_3 = \kappa_5 = \beta = 1\times10^{-5}$, $\lambda = 1\times10^{-2}$, $\boldsymbol{\varpi}_1 = \begin{bmatrix} 8 & 0 \\ 8 & 0.1 \end{bmatrix}$, $\boldsymbol{\varpi}_2 = \begin{bmatrix} 1 & 0 \\ 8 & 1 \end{bmatrix}$, $\mathbf{M}_k(\hat{x}_2)$ of degree 0; $\mathbf{N}_j(\hat{x}_2)$ of degree 0 and 2; $\bar{\tau} = 1$, $\tau = 1$.

The Lyapunov function matrices are obtained as follows: $\mathbf{P}_1 = \begin{bmatrix} 0.007608 & 0 \\ 0 & 2.236 \end{bmatrix}$, $\mathbf{P}_2 = \begin{bmatrix} 0.0419 & 0 \\ 0 & 0.04501 \end{bmatrix}$, $\mathbf{S} = \begin{bmatrix} 118.6 & 0.2695 & -23.2 & 0.5833 \\ 0.2695 & 120.0 & -9.141 & -0.1305 \\ -23.2 & -9.141 & 116.9 & -11.29 \\ 0.5833 & -0.1305 & -11.29 & 127.3 \end{bmatrix}$,

$$\mathbf{R} = \begin{bmatrix} 1.097\times10^5 & 93.96 & 4828.0 & 74.95 \\ 93.96 & 1.084\times10^5 & 468.3 & 27.74 \\ 4828.0 & 468.3 & 1.135\times10^5 & 481.1 \\ 74.95 & 27.74 & 481.1 & 1.081\times10^5 \end{bmatrix}.$$

The observer system matrices in each rules are obtained as follows: $\mathbf{F}_{10}(\hat{x}_2) = \begin{bmatrix} -68.8862 & 0 \\ 68.8862 & -6.2800 \end{bmatrix}$, $\mathbf{F}_{20}(\hat{x}_2) = \begin{bmatrix} -60.5518 & 0 \\ 60.5518 & -6.2827 \end{bmatrix}$. The controller gains and observer gains are obtained as follows: $\mathbf{G}_1(\hat{x}_2) = [30.0448 \quad 5.5308]$, $\mathbf{G}_2(\hat{x}_2) = [38.1042 \quad 5.5272]$, $\mathbf{L}_1(\hat{x}_2) = [12.3801\hat{x}_2^2 + 1.1392 \quad -4.7596\times10^{-10}\hat{x}_2^2 + 30.3335]^T$, $\mathbf{L}_2(\hat{x}_2) = [12.3801\hat{x}_2^2 + 1.1392 \quad 2.9984\times10^{-11}\hat{x}_2^2 + 30.3335]^T$.

These fuzzy controller and fuzzy observer are applied to the pest population system ((90) and (91)). Considering two different initial conditions, the sys-

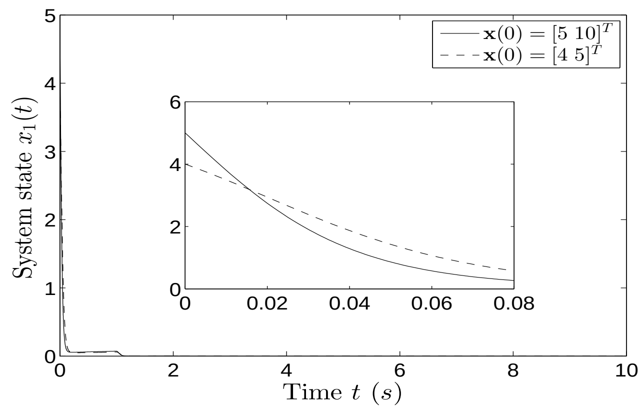

Figure 9: Time responses of system state $x_1(t)$ with two different initial conditions. Solid lines represents the time response of system state $x_1(t)$ with initial conditions $x_1(0) = 5$; dashed lines represents the time response of system state $x_1(t)$ with initial conditions $x_1(0) = 4$.

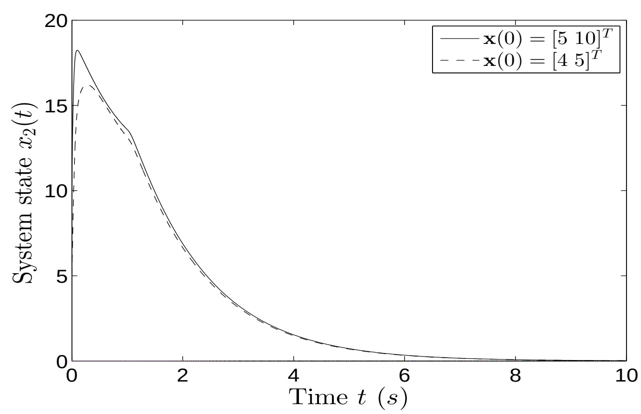

Figure 10: Time responses of system state $x_2(t)$ with two different initial conditions. Solid lines represents the time response of system state $x_2(t)$ with initial conditions $x_2(0) = 10$; dashed lines represents the time response of system state $x_2(t)$ with initial conditions $x_2(0) = 6$.

tem is stabilized and its state variables x_1 and x_2 are remained in the positive quadrant, the time response of system states x_1 and x_2 are shown in Fig. 9 and Fig. 10. To prove that the designed observer can estimate the system states effectively, we choose the initial conditions $\mathbf{x}(0) = [5\ 10]^T$ and $\hat{\mathbf{x}}(0) = [4\ 5]^T$. The time response of system states x_1 and the corresponding estimated system

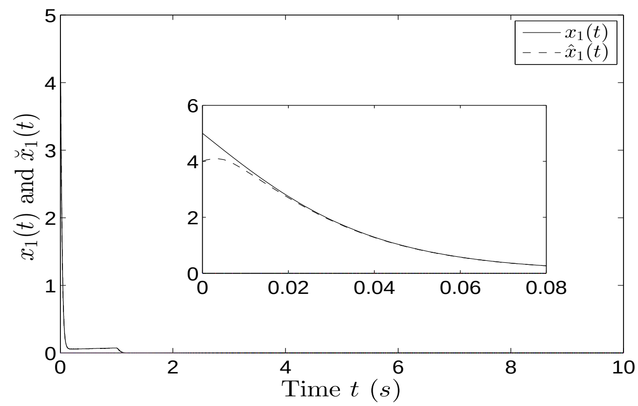

Figure 11: Time responses of system state $x_1(t)$ and estimated state $\hat{x}_1(t)$ with initial conditions $x_1(0) = 5$ and $\hat{x}_1(0) = 4$.

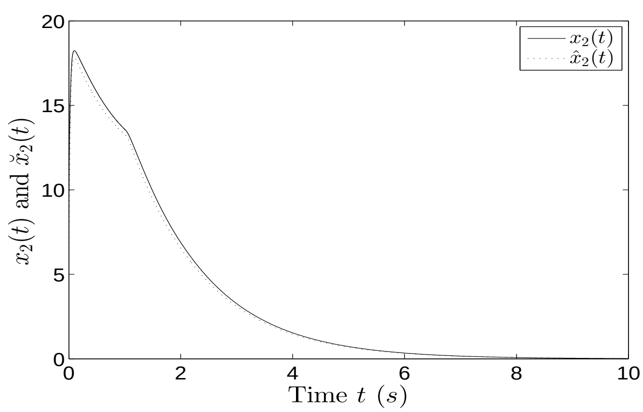

Figure 12: Time responses of system state $x_2(t)$ and estimated state $\hat{x}_2(t)$ with initial conditions $x_2(0) = 10$ and $\hat{x}_2(0) = 5$.

states $\hat{x}_1$ are shown in Fig. 11; the time response of system states x_2 and the corresponding estimated system states $\hat{x}_2$ are shown in Fig. 12. Meanwhile, the estimation error of system states are positive and asymptotically converges to zero as shown in Fig. 13. The control input signal is shown in Fig. 17. In Fig. 9, Fig. 11 and Fig. 13, in order to provide a clearer result, the time response curve for a certain period of time is enlarged and is placed in the corresponding small rectangle windows.

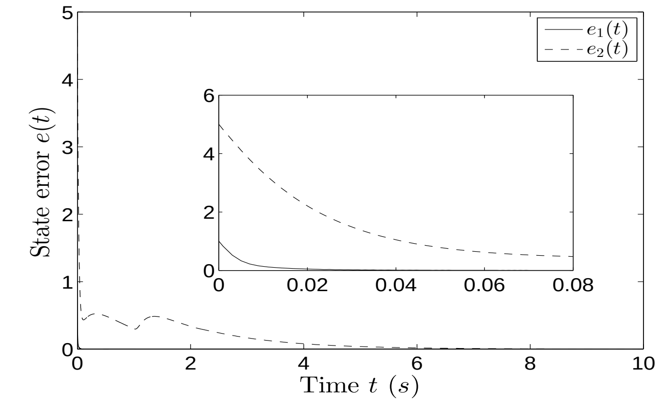

Figure 13: Time responses of estimated error $e(t)$ with initial conditions $\mathbf{x}(0) = [5\ 10]^T$ and $\hat{\mathbf{x}}(0) = [4\ 5]^T$.

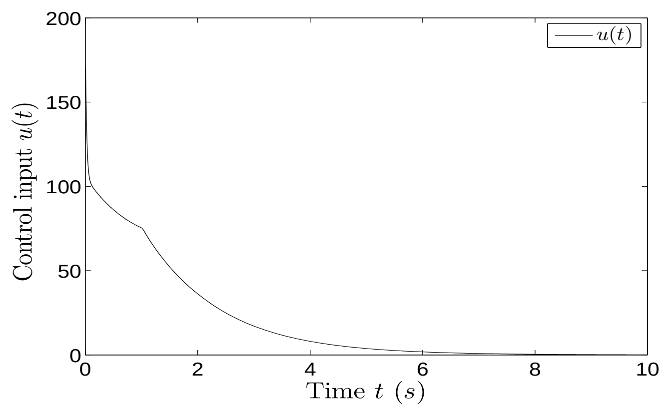

Figure 14: Time response of control input $u(t)$ with initial conditions $\mathbf{x}(0) = [5\ 10]^T$ and $\hat{\mathbf{x}}(0) = [4\ 5]^T$.

Remark 6. *In this example, the observer system matrix* $\mathbf{F}_{j0}$ *is defined as a decision variable, and the above solution and figures are obtained by using Theorem 1. If the coefficient of observer system matrix* $\mathbf{F}_{j0}$ *is defined to be the same as the system matrix* $\mathbf{A}_{i0}$*, no feasible solution can be obtained, which indicates that the observer system matrix is defined as a decision variable rather than a predefined variable can relax the conditions of PPFMB observer-control system.*

5. Conclusion

This paper investigated the stability and positivity of PPFMB observer-control system with unknown time delay based on the Lyapunov stability theory. In the stability and positivity analysis, the scenario that mismatched framework and unknown time delay have been considered, which widens the applicability of the research results and improves the flexibility of observer design, although it will lead to more complicated analysis and non-convex conditions. MDT [45] has been employed to turn the non-convex positivity and stability conditions into convex ones. Two simulation examples have been given to verify the effectiveness of the proposed PPFMB observer-control scheme. In this paper, we only provide an effective methodology to control the nonlinear system at the complicated situation that both full system states and time delay are not available and there is positive constraint on the system states, but not explore novel Lyapunov function candidates to relax results. In the future, some other Lyapunov functions such as fuzzy Lyapunov function, quadratic copositive lyapunov function and linear copositive lyapunov function, can be applied to offer more relaxed stability conditions and positivity conditions for PPFMB observer-control system. In addition, in order to reduce the computational cost, copositive Lyapunov functional can be applied in the future to simplify the stability conditions. Also, if some system states are measurable, reduced order observer can replace full order observer to reduce the computational cost.

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