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**E theory
its algebra, gauge fixing, and 7D equations of motion**

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KING'S COLLEGE LONDON

DOCTORAL THESIS

**E theory: its algebra, gauge fixing,
and 7D equations of motion**

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in the

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Abstract

It has been proposed that all strings and branes are contained in the non-linear realisation of the Kac-Moody algebra E_{11} with the semi-direct product of its fundamental representation l_1 , denoted $E_{11} \ltimes l_1$. In the process of building the non-linear realisation, the dynamics of strings and branes are naturally invariant under symmetries which are determined by the relevant decomposition of the E_{11} algebra. At low levels, these dynamics are exactly those of D dimensional maximal supergravity, constructed from the subgroup $GL(D) \otimes E_{11-D}$, with the fields and coordinates of the theory appearing as parameters of the generators of the algebra.

In this thesis, we first construct relevant decompositions of the $E_{11} \ltimes l_1$ algebra. We begin by extending previous calculations of the algebra of the 11D decomposition up to levels 5 and 6 in the adjoint representation. We then calculate the algebra of the 7D decomposition in the adjoint representation, vector representation, and Cartan involution invariant subalgebra up to level 5. The final algebra that we calculate is the 10 dimensional IIB Cartan involution invariant subalgebra up to level 4.

We then build a tangent space metric in the 11D, 5D, and 4D decompositions of the E_{11} algebra, and additionally for the A_1^{+++} algebra, which leads to a description of 4 dimensional gravity. These tangent space metrics are then used to build a set of gauge-fixing conditions which allow gauge symmetries to be fixed in an E_{11} covariant way.

The final part of the thesis constructs the non-linear realisation in the 7 dimensional decomposition of E_{11} at low levels, using the algebra derived in an earlier chapter. This is then used to find a set of dynamical equations, which, when truncated, agree exactly with the equations of motion for the graviton, the scalar, the one-form, and the two-form in 7D maximal supergravity. Finally, we derive the duality relations of the scalar and the graviton with their corresponding dual fields, from the duality relations of the 1-form and 2-form found in the previous section.

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Chapter 1

Introduction

1.1 History

The concept of gravity has intrigued and troubled physicists for centuries, yet it was not until the 1600s that Newton, with the help of discussions with Hook, achieved the first somewhat accurate description of gravity - commonly known as Newton's law of Gravitation. While the theory provided a much sought after theory of celestial motion, there were some inconsistencies between the theory and observations discovered throughout the centuries. Although some of these issues were resolved, there was still an unresolved problem that persisted until the 1900s: the precession of the perihelion of Mercury's orbit. The answer was in Einstein's classical theory of General Relativity developed at the beginning of the 20th century, which postulated that the gravitational force was in fact an artefact of the curvature of spacetime in the presence of mass, and that the presence of a curvature in spacetime naturally impacts the motion of particles.

Parallel to research on gravity, progress was being made on the understanding of the other three fundamental forces of nature. Classical electromagnetism was first correctly described by Maxwell in 1865, by the well-known Maxwell's equations, together with the equation for the Lorentz force. This theory was quantised by Heisenberg and Pauli around 1930, and is known as quantum electrodynamics. The weak force was described by Fermi in 1932, and was successfully unified with the electromagnetic force in 1967 by Weinberg and Salam, in what we call the electroweak theory. Finally, the strong force was described by Yukawa in 1935, and, with the electroweak force, is described by the Standard Model: a

gauge quantum field theory with composite gauge group, $SU(3) \otimes SU(2) \otimes U(1)$. The Standard Model has been successfully observed, ending with the observation of the last of the 17 Standard Model particles, the Higgs Boson, at the LHC in 2012.

With this unification of the other three forces, it is natural to attempt to include gravity, and so the question of finding a quantum theory of Gravity was established. However, in the process of quantising General Relativity, the resultant theory was found to be non-renormalisable. In practical terms, this means that the infinities that arise in the scattering amplitudes of the Feynman diagrams cannot be cancelled using the usual Quantum Field Theory methods, which absorb infinities into parameters of a theory.

A popular solution (and generally considered to be the most likely) is supersymmetric string theory, which essentially assumes that point particles are just different oscillations of a string - a much more elegant solution than requiring at least 17 different types of fundamental particles. String theory indeed contains gravity and other gauge bosons which look like those of the Standard Model (assuming that String theory reduces to a supersymmetric Grand Unified Theory). Additionally, it concisely cancels the divergence of quantum gravity effects, and therefore could be a suitable Quantum Gravity solution.

While there are five 10D superstring theories, Witten proposed in the 1990s that these are all contained in an unique 11D theory, which he called M theory [1], proposing that these 10D theories appear as certain limits of the 11 dimensional theory. However it is not possible to provide a dynamical description of M theory, due to the difficulty of quantising branes. Fortunately, maximal supergravity theories actually provide a low energy description of corresponding superstring theories. For example, 11D M theory, and the IIA and IIB superstrings have low energy descriptions in 11D supergravity [2] and the maximally symmetric IIA [3-5] and IIB [6-8] supergravity theories, respectively. With these effective theories, it is possible to observe some of the features of superstrings and it is for these string theories that we hope to find a more concise description.

We saw that Lie algebras occur in the Standard Model, and so the appearance of a Lie algebra in the unified theory would not be surprising. In fact, Lie algebras additionally arise in the maximal supergravities. In particular, the scalars of maximal supergravities

Dimension	Symmetry group	Coset space
10D IIA	$O(1,1)$	—
10D IIB	$SL(2)$	$SL(2)/SO(2)$
9D	$GL(2)$	$GL(2)/SO(2)$
8D	$SL(2) \otimes SL(3)$	$(SL(2) \otimes SL(3))/(SO(2) \otimes SO(3))$
7D	$SL(5)$	$SL(5)/SO(5)$
6D	$SO(5,5)$	$SO(5,5)/(SO(5) \otimes SO(5))$
5D	E_6	$E_6/USp(8)$
4D	E_7	$E_7/SU(8)$
3D	E_8	$E_8/SO(16)$

TABLE 1.1: Coset symmetries of scalars in maximal supergravity theories.

arise from coset spaces [9, 10]; the 2 dimensional maximal supergravity theory possess an E_9 symmetry [11, 12], the 3 dimensional maximal supergravity has an E_8 symmetry [13], the 4 dimensional theory contains an E_7 symmetry [14, 15], the dynamics of scalars in type IIB supergravity have a $SL(2, \mathcal{R})/SO(2)$ symmetry [7], and it was then established that in dimensions less than 10, there was E_{11-D} coset symmetry [9, 16, 17]. The coset symmetries of the scalars of the maximal supergravity theories are given in table 1.1.

It was initially believed that these symmetries were simply a consequence of dimensional reduction. In addition to these coset symmetries, it was found that the gravity sector of 11 dimensional supergravity arose from a non-linear realisation of the Lie group A_{10} [18]; in this paper, it was shown that the bosonic sector of 11D and IIA supergravity can be found from such a non-linear realisation. This indicated that if we were to find a symmetry group of strings and branes, it must contain A_{10} and the symmetries in table 1.1 as subgroups [19]. After looking for the smallest Kac-Moody algebra with these subgroups, West proposed the E_{11} conjecture [19, 20], which we study in detail in this thesis. This conjecture will be explained in the next section.

The remainder of this chapter contains the E_{11} conjecture, and some recent results. We then explain some general properties of Lie algebras, and the process of non-linear realisation in the context of E theory, which shall be used throughout this thesis. Additionally, the example of deriving gravity in D dimensions from the non-linear realisation shall be given at the end of this chapter, to see the non-linear realisation in action. In chapter 2, the algebra of E_{11} is computed in various decompositions for the 11D, 7D, and the 10D IIB theories. In chapter 3, the tangent space metric and a gauge-fixing multiplet are found

in 11D, 5D, 4D decompositions of E_{11} , and the for Lie algebra A_1^{+++} , which leads to a description of 4 dimensional gravity [21]. Finally, in chapter 4 the equations of motion corresponding to 7 dimensional supergravity are derived directly from E theory. The results in chapter 2 through to 4 are based on published (or soon to be published) results of E theory by the author [22–24].

1.2 E_{11} conjecture

In 2001, Peter West conjectured that the non-linear realisation of the Kac-Moody algebra E_{11} leads to the low energy effective action of strings and branes [19]. At this time, the spacetime was inserted into the theory by hand as a translation operator. However, it was realised in 2003 that it actually corresponded to a generator appearing in the l_1 representation of E_{11} . The conjecture was then uplifted such that it now reads that the non-linear realisation of E_{11} with the semi-direct product of its vector representation l_1 , denoted $E_{11} \ltimes l_1$, contains all of strings and branes [20]. As mentioned, the spacetime arises as parameters of the vector representation, and the fields of the theory, that depend on the spacetime, appear from the adjoint representation of E_{11} in the non-linear realisation. It is for this theory that we obtain more results. The conjecture was essentially proved in 2016 [25], where the 11D and 5D equations of motions were computed from the E_{11} viewpoint, and so from now on in this thesis, we refer to the conjecture as ‘E theory’. We next give some recent results with respect to E theory, to give an overview of the work which has been completed up to this point.

1.2.1 Subsequent results from E theory

E_{11} is the very extended algebra of the finite dimensional semi-simple Lie algebra E_8 , equivalently written E_8^{+++} . To get to the Dynkin diagram of E_{11} from that of E_8 , one adds nodes one by one to the longest ‘arm’ of the Dynkin diagram, each connected with a single line. This process results in an infinite dimensional Kac-Moody algebra. The Dynkin diagram of E_{11} is given in figure 1.1.

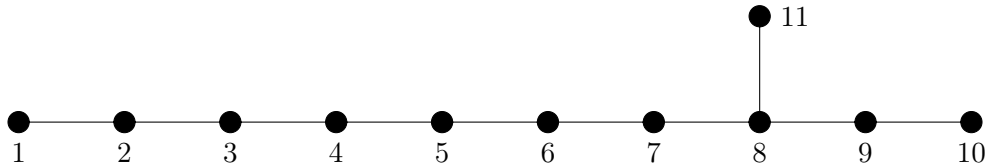


FIGURE 1.1: Dynkin diagram of E_{11} , which is the very extended algebra of the finite dimensional semi-simple Lie algebra, E_8^{+++} .

The maximal supergravity theories arise by taking different decompositions of this E_{11} algebra [26–32]. The dimension of the theory directly corresponds to the number labelling the deleted node for the Dynkin diagram in the 11D theory, 10D IIB theory, and also for theories in less than 9 dimensions. To find the 9D theory, one deletes both node 9 and node 11, and for the 10D IIA theory, one deletes node 9 only. The labelling in our convention is shown in figure 1.1, and the Dynkin diagram showing the deletion of node D is shown in figure 1.2. The deletion of node D results in the subgroup $GL(D) \otimes E_{11-D}$, where the $GL(D)$ gives rise to gravity in the theory, and E_{11-D} is the internal symmetry group of the theory. The $GL(D)$ part of the remaining subgroup corresponds to the Dynkin diagram A_{D-1} . Usually, A_{D-1} is the Dynkin diagram resulting in the $SL(D)$ algebra. However, when A_{D-1} is found as a result of the deletion of a node, as in this case, the resulting algebra is in fact $GL(D)$, rather than $SL(D)$.

As a general property of Kac-Moody algebras, E_{11} has an infinite number of generators, each of which is classified by a parameter we call level. Therefore, there are an infinite number of corresponding fields in the adjoint representation and infinite number of corresponding coordinates in the l_1 representation. Indeed, to get the correct results, it is believed that there is some mechanism which allows one to truncate the theory to low levels, although this mechanism is not yet known. This level at which the theory is truncated depends on in which dimension we are working. However, this truncation is simply a result of the fact that the higher level fields and coordinates are not yet well understood, and could lead to some interesting new phenomena. It has been proposed that the infinite tower of additional fields do not add anymore degrees of freedom. Some of the fields have been found to be related by dualities to the lower level fields [33], and it is thought that others are related to gauging of the theory.

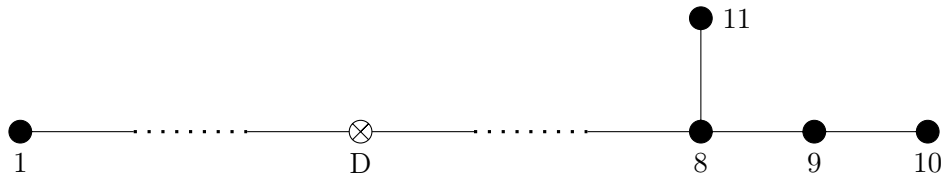


FIGURE 1.2: Dynkin diagram showing the deletion of node D , which gives a description of maximal supergravity in D dimensions from an E_{11} perspective.

The equations of motion of supergravity in 11 and 5 dimensions were completely derived from E theory in 2016 [25], and this essentially proved that it is correct. The calculation for equations of motion in 11D from an E_{11} viewpoint were then published with more details soon after [34]. The equations of motion for the form fields of maximal supergravity in 4 dimensions have been found [35], and the gravity sector is semi-complete, but is expected to agree with known results once it is completed.

In 2016, it was found using E theory that there is an origin of a cosmological constant in supergravity theories [36]. The origin turned out to be a higher level field in 11D, that one truncates to find the maximal supergravity theory in 11D. This indicates that these higher level fields are indeed important, and that truncating them at this time is simply a result of the fact that they are not yet well understood. The cosmological constant does not arise in 11D or IIB supergravity, but there is a unique 10D IIA theory containing a cosmological constant, known as Roman's theory [37]. In lower dimensions, there are many theories containing a cosmological constant. It was found that the origin of the cosmological constant in some of the theories was a result of dimensional reduction on a sphere from the unique IIA theory. However, the origin of the cosmological constants in the other theories was unknown up to this point, and would have been difficult to find without E theory. It has also been shown that exceptional field theory is a truncation of E theory [38].

In addition to this, there is progress being made with a dynamical description of branes from the perspective of E theory. Due to branes being difficult to quantise, it has historically been problematic to derive their dynamics. It turns out that the l_1 representation of E_{11} contains all the brane charges, and in recent papers, the dynamics of M2 and M5 branes in 11D, branes in 8D, in 7D, and the D3 brane in IIB string theory, and finally the

IIA and IIB strings have been constructed [39, 40].

Now we have an idea about the history of E theory and its success, we can give more specific details of what exactly it means and how to apply it. We begin by giving an overview of Kac-Moody algebras.

1.3 Kac-Moody algebras

For a more detailed review of Kac-Moody algebras in this context, the reader is directed to chapter 16 in [41], which this section shall closely follow. We begin with a review of the general features of Lie algebras; their algebra, Cartan matrix, and Dynkin diagram.

A Lie algebra G with rank r contains the generators

$$G = \{H_i, E_\alpha\}, \quad (1.3.1)$$

where α correspond to the roots of the algebra, which are vectors with r components, and H_i , with $i = 1, \dots, r$, is the subalgebra of commuting generators

$$[H_i, H_j] = 0. \quad (1.3.2)$$

The E_α are the remaining generators in the coset G/H , which are diagonalised by the Cartan subalgebra

$$[H_i, E_\alpha] = \alpha_i E_\alpha. \quad (1.3.3)$$

We can always choose a basis for the root space, called the simple roots. Simple roots are defined as positive roots which cannot be written as the sum of two positive roots, where a positive root is one where the first non-zero element is positive. We denote this basis

$$\alpha_a, \quad a = 1, \dots, r. \quad (1.3.4)$$

Recall that r is the rank of the algebra. This basis can then be used to define the Cartan matrix of a Lie algebra

$$A_{ab} = \frac{2(\alpha_a, \alpha_b)}{(\alpha_a, \alpha_a)}, \quad (1.3.5)$$

where $(,)$ is the usual scalar product.

Killing and Cartan recognised that a Cartan matrix of a Lie algebra has the following properties

- A_{ab} is negative or zero if $a \neq b$,
- $A_{ab} = 0 \Leftrightarrow A_{ba} = 0$,
- $v^a A_{ab} v^b > 0$ for any real vector v^a ,

and we can see from equation (1.3.5) that $A_{aa} = 2$. However, we are interested specifically in Kac-Moody algebras, so we choose to drop the final property of positive-definiteness, and ask that the Cartan matrix is symmetrisable. Symmetrisable means that there must exist an invertible diagonal matrix D and a symmetric matrix S , such that the Cartan matrix can be written as DS . In fact, we will decompose the Kac-Moody algebra E_{11} in terms of its Lie subalgebras, and so we are interested in studying the properties of these Lie algebras, rather than the more general Kac-Moody algebras.

From the Cartan matrix, one can construct the unique Dynkin diagram of the algebra. A Dynkin diagram is composed of r nodes, with node a and b being connected with $A_{ab}A_{ba}$ lines. Then there is an arrow on the line(s) in the direction of the shorter root, or no arrow if the roots are the same length. Serre showed that the reverse is also possible; that if one begins with a Dynkin diagram, then you can find the unique Cartan matrix, and from this, derive the algebra. We now will outline the method of how to return to the algebra from the Dynkin diagram.

To find the Cartan matrix A_{ab} from the Dynkin diagram, where $a, b = 1, \dots, r$, we introduce the Chevalley generators H_a, E_a, F_a , which are simply a rescaling of the generators, depending on the new basis of simple roots. These generators, respectively, are the Cartan

FIGURE 1.3: Dynkin diagram of the Lie algebra A_2 .

subalgebra generators, the generators corresponding to the simple roots, and the generators corresponding to the negative of the simple roots. The explicit definition of these Chevalley generators is not given here but the reader is referred to chapter 16.1 in [41] if they are interested. The Chevalley generators have the following commutators

$$\begin{aligned}
 [H_a, H_b] &= 0, \\
 [H_a, E_b] &= A_{ab}E_b, \\
 [H_a, F_b] &= -A_{ab}F_b, \\
 [E_a, F_b] &= \delta_{ab}H_a.
 \end{aligned} \tag{1.3.6}$$

The generators satisfy the Serre relations

$$\begin{aligned}
 [E_a, [E_a, \dots [E_a, E_b] \dots]] &= 0, \\
 [F_a, [F_a, \dots [F_a, F_b] \dots]] &= 0,
 \end{aligned} \tag{1.3.7}$$

with $(1 - A_{ab})$ factors of E_a and F_a , respectively. With these commutators and relations, one can generate the whole Kac-Moody algebra. Let us illustrate with a simple example.

The Dynkin diagram of A_2 is shown in figure 1.3 and it corresponds to the following Cartan matrix

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \tag{1.3.8}$$

It contains the generators

$$H_1, H_2, E_1, E_2, F_1, F_2, [E_1, E_2], [F_1, F_2], \tag{1.3.9}$$

and this is the complete list of generators of A_2 . We can see this is the case if we attempt

to construct more generators from the multiple commutator; we find $[E_1, [E_1, E_2]] = 0$ and $[F_1, [F_1, F_2]] = 0$, due to the Serre relation in equation (1.3.7) containing $(1 - A_{12}) = 2$ factors of E_1 and F_1 . If we try to build any other commutators, we discover that we only find generators that have already been derived. Therefore, there are 8 generators in this Lie algebra. This algebra turns out to be $SU(3)$.

Since it is difficult to study Kac-Moody algebras, we are interested in studying a subclass of these algebras, called Lorentzian algebras. Lorentzian Kac-Moody algebras can be characterised by their Dynkin diagram upon deletion of a node. Their analysis is then based on their decomposition with respect to the subalgebra corresponding to the remaining Dynkin diagram after the deletion of a node [42]. They contain at least one node, which upon deletion, has remaining connected components of the Dynkin diagram which must be of finite type, and at most one affine type. We note here that the definition of the Cartan matrix of an affine Lie algebra is that it must be positive semi-definite, and contain only one zero eigenvalue. One may decompose the Lorentzian Kac-Moody algebra into generators of the subalgebra, which correspond to this new ‘reduced’ Dynkin diagram - ‘reduced’ refers to the fact that we have deleted a node. If the deletion of the node results in multiple disconnected Dynkin diagrams, then the result is an algebra which is a direct product of the algebras described by each of the Dynkin diagrams. With the reduced diagram, the generators of the algebra are parametrised by an integer, l , which we shall refer to as the level. The level of a generator corresponds to the number of times the generator corresponding to the deleted node occurs in the multiple commutator of generators leading to the relevant generator. We note that for the Kac-Moody algebra E_{11} the number of generators of the reduced Dynkin diagram will be infinite. However, they can be built in a useful manner level by level. The representation we recover in this process described above will be the adjoint representation, which shall be one of the two main representations in which we are interested.

We now look at how to find other representations from a Dynkin diagram, more specifically the fundamental representation.

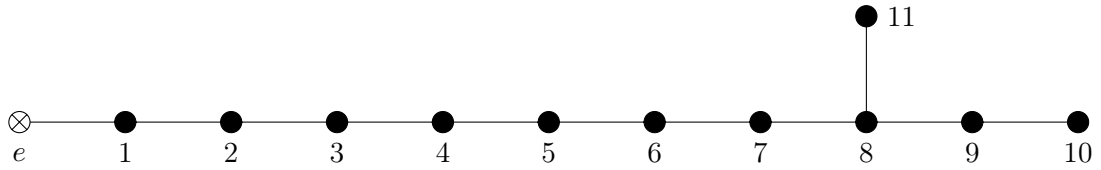


FIGURE 1.4: Diagram showing the deletion of node e in the enlarged E_{11} Dynkin diagram resulting in the vector representation of the algebra.

1.3.1 Representations of Kac-Moody algebras

As we shall be interested in the l_1 (fundamental) representation of E_{11} , we will discuss the process of finding this representation from a Dynkin diagram. We begin by constructing the ‘enlarged’ Dynkin diagram, which is done by connecting a new node e to a node in the Dynkin diagram with a single line. In our case, we shall add the corresponding node to the longest ‘arm’ of the E_{11} Dynkin diagram, which in our convention we label as node 1, as shown in figure 1.4. We then delete this newly added node e and decompose the algebra in terms of it, and apply the same analysis as above for Lorentzian algebras. In this way, we create another parameter m representing the level corresponding to the extra deleted node e . At level $m = 0$, we just find the adjoint representation, but at level $m = 1$, we find the fundamental representation, which shall be denoted by l_1 . We note that the fundamental representation l_1 is a lowest weight representation of E_{11} .

This completes the process of deriving the vector representation from a Dynkin diagram, and so we now move on to introducing the Cartan involution, which shall help us to construct a subgroup of E_{11} .

1.3.2 Cartan involution

We now discuss a useful tool to enable us to build a subalgebra of a Kac-Moody algebra, G . This subalgebra is called the Cartan involution subalgebra, denoted $I_c(G)$, and we will use this in the construction of the non-linear realisation. We define the action of the Cartan involution I_c on the Chevalley generators by

$$I_c(E_a) = -F_a ,$$

$$\begin{aligned} I_c(F_a) &= -E_a, \\ I_c(H_a) &= -H_a, \end{aligned} \tag{1.3.10}$$

and it respects Lie brackets such that $I_c([X, Y]) = [I_c(X), I_c(Y)]$. Note that it takes positive root generators to negative root generators, and vice-versa. Hence, we find a subalgebra of G , which is invariant under the action of the Cartan involution, is constructed from the generators

$$E_a - F_a, \tag{1.3.11}$$

and their multiple commutators. We note that for finite-dimensional semi-simple Lie groups, the Cartan involution invariant subalgebra is just the maximal compact subalgebra.

This subalgebra shall be crucial in the construction of supergravity theories, including (but not limited to) the tangent space metric and the equations of motion. This completes the theory we need regarding Kac-Moody algebras. We now construct the algebra of E_{11} before introducing the theory of the non-linear realisation.

1.3.3 Algebra of $E_{11} \times l_1$

The general Dynkin diagram of E_{11} for the corresponding theory in D dimensions is given in figure 1.2.

We denote the E_{11} and l_1 generators by R^α and l_A , respectively. The commutators of $E_{11} \times l_1$ can in general be written

$$[R^\alpha, R^\beta] = f^{\alpha\beta}_\gamma R^\gamma, \tag{1.3.12}$$

$$[R^\alpha, l_A] = -(D^\alpha)_A{}^B l_B, \tag{1.3.13}$$

where $(D^\alpha)_A{}^B$ is a matrix representation of the first fundamental representation of E_{11} which satisfies

$$[D^\alpha, D^\beta] = f^{\alpha\beta}{}_\gamma D^\gamma . \quad (1.3.14)$$

The indices $\underline{\alpha}, \underline{\beta}, \dots$ denote the index structure of the generators of the adjoint representation, and the A, B, \dots denote the index structure of the generators of the fundamental representation, and hence they run over the generators corresponding to the relevant representation. Note that the Greek indices are underlined to differentiate from Greek indices used in the 10D IIB decomposition of E_{11} .

We note that the notion of the level is additive under the action of a commutator, so a generator appearing in the result of a commutator of level m generator with level n generator must be a level $(m + n)$ generator. We shall derive this algebra corresponding to the 11D decomposition of E_{11} at level 5 and 6, and also for the 7D decomposition up to level 5 in chapter 2.

Additionally, commutators are preserved under the action of the Cartan involution, which acts on the E_{11} and l_1 generators as

$$I_c(R^\alpha) = \begin{cases} -R_{\underline{\alpha}} , & \text{if level} = 0 , \\ (-1)^{\text{level}} R_{\underline{\alpha}} , & \text{if level} \neq 0 , \end{cases}$$

$$I_c(l_A) = - J_{AB}^{-1} \bar{l}^B , \quad (1.3.15)$$

where $R_{\underline{\alpha}} = R^{-\alpha}$. The matrix J_{AB} is a constant matrix, and usually just δ_{AB} . If it is non-trivial, it shall be specified. We also see that the action of Cartan involution takes the lowest weight representation l_1 into a highest weight representation \bar{l}_1 . As shown in section 1.3.2, the Cartan involution invariant generators are

$$S^\alpha = \begin{cases} R^\alpha - R_{\underline{\alpha}} , & \text{if level} = 0 , \\ R^\alpha + (-1)^{\text{level}} R_{\underline{\alpha}} , & \text{if level} \neq 0 . \end{cases} \quad (1.3.16)$$

It is useful to note here that the Cartan involution invariant subalgebra of $\text{GL}(D)$ is $\text{SO}(D)$.

The commutators of this Cartan involution invariant subalgebra shall be derived for the 7D decomposition and the 10D IIB decomposition of E_{11} in chapter 2.

As we will need it in the derivation of the tangent space metric in chapter 3, we include the algebra of the E_{11} generators with the \bar{l}_1 representation

$$[R^\alpha, \bar{l}_A] = \bar{l}^B (\bar{D}^\alpha)_B{}^A, \quad (1.3.17)$$

where the matrices $(\bar{D}^\alpha)_B{}^A$ satisfy

$$[\bar{D}^\alpha, \bar{D}^\beta] = f^{\alpha\beta}{}_{\underline{\gamma}} \bar{D}^\gamma. \quad (1.3.18)$$

The \bar{l}_1 representation is a highest weight representation, which, when acted on with the Cartan involution, becomes the l_1 representation. We shall derive this algebra for the 11D, 5D, and 4D decomposition of E_{11} , and for the A_1^{+++} algebra, as we shall need it in the derivation of the tangent space metric.

Note that we can find a relation of the matrix representations of E_{11} that appear in the commutators with the highest and lowest weight representations. If we act on equation (1.3.13) with the Cartan involution and use the action of the Cartan involution in (1.3.15) and the commutator in (1.3.17), we find the relation

$$\bar{D}^\alpha = (JD^{-\alpha}J^{-1})^T. \quad (1.3.19)$$

This relation shall be useful when we verify that the definition of the tangent space metric that we propose is indeed correct.

Now that we know the algebra of E_{11} and the action of the Cartan involution of E_{11} , we can finally introduce the non-linear realisation of a group, specifically in the case of $(E_{11} \times l_1)/(I_c(E_{11}))$.

1.4 Non-linear realisation

In this section, we study the non-linear realisation of $E_{11} \ltimes l_1$ with the local subgroup being the Cartan involution invariant subalgebra, as this is what we shall be interested in, but the same steps can be taken for any group G with local subgroup H . For a more general introduction of non-linear realisations, the reader is referred to chapter 13.2 of [41].

1.4.1 Non-linear realisation of $E_{11} \ltimes l_1$

The non-linear realisation can be used in any theory where there is suspected to be spontaneous symmetry breaking. Goldstone's theorem states that if the rigid symmetry group G is spontaneously broken to subgroup H in a quantum field theory, then the theory contains $\text{Dim}(G) - \text{Dim}(H)$ massless particles, also called Goldstone particles [43, 44]. The low energy effective theory of these particles is often described as a non-linear realisation of group G with subgroup H [45, 46]. In fact, the symmetries of the non-linear realisation often completely determine the dynamics. It is this property that we hope to exploit - that the dynamics constructed out of a non-linear realisation are intrinsically invariant under the correct symmetries without any prior knowledge of the dynamics needed.

We now recall the non-linear realisation, specifically with respect to the algebra we are interested in, $E_{11} \ltimes l_1$. The non-linear realisation is constructed using the group element $g \in E_{11} \ltimes l_1$ with

$$g = g_l g_E , \tag{1.4.1}$$

where g_l is the group element containing generators of the l_1 representation of the E_{11} algebra, and g_E is a group element containing the adjoint representation generators of E_{11} . Explicitly, these are

$$\begin{aligned} g_l &= e^{z^A l_A} , \\ g_E &= e^{A_\alpha R^\alpha} . \end{aligned} \tag{1.4.2}$$

The R^α represent the generators of the E_{11} algebra, and l_A are the generators of the vector representation of E_{11} , as before. The $A_{\underline{\alpha}}$ will turn out to be the fields of our theory, and the z^A are the generalised spacetime coordinates that the fields will depend on.

Then the non-linear realisation is, by definition, invariant under transformations

$$\begin{aligned} g &\rightarrow g_0 g , & g_0 &\in E_{11} \ltimes l_1 , \\ g &\rightarrow g h , & h &\in I_c(E_{11}) , \end{aligned} \quad (1.4.3)$$

where g_0 represents a rigid transformation, and h is an element of the Cartan involution invariant subalgebra $I_c(E_{11})$, and is a local transformation.

These transformations in equation (1.4.3) can equivalently be written explicitly in terms of g_E and g_l . The rigid transformation becomes

$$\begin{aligned} g_l &\rightarrow g_0 g_l g_0^{-1} , \\ g_E &\rightarrow g_0 g_E , \end{aligned} \quad (1.4.4)$$

and the local transformation is equivalent to

$$g_E \rightarrow g_E h . \quad (1.4.5)$$

Then the dynamics of the non-linear realisation are described by an action which is invariant under these transformations. The Cartan form is constructed in the following way

$$\mathcal{V} = g^{-1} dg = \mathcal{V}_E + \mathcal{V}_l , \quad (1.4.6)$$

which we can split into terms containing the E_{11} generators and terms containing the l_1 representation, or explicitly

$$\begin{aligned} \mathcal{V}_E &= g_E^{-1} dg_E = dz^\Pi G_{\Pi,\alpha} R^\alpha , \\ \mathcal{V}_l &= g_E^{-1} (g_l^{-1} dg_l) g_E = dz^\Pi E_\Pi^A l_A , \end{aligned} \quad (1.4.7)$$

where $E_{\Pi}^A = (e^{A\alpha} D^\alpha)_{\Pi}^A$ will turn out to be the vielbein on the generalised spacetime, with which one can transform tangent space indices into curved indices. We shall use the $G_{\Pi,\alpha}$, which we shall also call Cartan forms, in the derivation of the gauge-fixing conditions in chapter 3, and additionally to find the equations of motion in the 7D decomposition of the E_{11} algebra in chapter 4.

Both \mathcal{V}_E and \mathcal{V}_l are invariant under the rigid transformations, and transform in the following way under the local transformations

$$\begin{aligned}\mathcal{V}_E &\rightarrow h^{-1}\mathcal{V}_E h + h^{-1}dh, \\ \mathcal{V}_l &\rightarrow h^{-1}\mathcal{V}_l h.\end{aligned}\tag{1.4.8}$$

We note that the generalised vielbein transforms on its Π index under the rigid transformation and on its A index under the local transformation of (1.4.3)

$$\begin{aligned}E_{\Pi}^A &\rightarrow D(g_0)_{\Pi}^{\Lambda} E_{\Lambda}^B D(h)_B^A, \\ (E^{-1})_A^{\Pi} &\rightarrow D(h^{-1})_A^B (E^{-1})_B^{\Lambda} D(g_0^{-1})_{\Lambda}^{\Pi},\end{aligned}\tag{1.4.9}$$

and we can conclude that indeed E_A^{Π} is a generalised vielbein on a spacetime which possesses a tangent space, and that the tangent space group is $I_c(E_{11})$. Now we look at how to construct the tangent space metric of E_{11} .

1.4.2 Tangent space metric

We have noted that the spacetime contained in $E_{11} \times l_1$ possesses a tangent space metric. The tangent group will be $I_c(E_{11})$, and so we introduce tangent space objects V^A which have the transformation

$$V^A l_A = h^{-1} V^A l_A h, \quad h \in I_c(E_{11}),\tag{1.4.10}$$

where $h = 1 + a_{\underline{\alpha}}(R^{\alpha} - R^{-\alpha})$ is the infinitesimal transformation. This transformation may also be written explicitly as

$$V^{A'} = V^B D(h)_B{}^A = V^A + V^B (D^{\alpha} - D^{-\alpha})_B{}^A a_{\underline{\alpha}} + \dots, \quad (1.4.11)$$

where the \dots represent higher orders in $a_{\underline{\alpha}}$.

We can now derive the tangent space metric that the theory possesses. Using the l_1 and \bar{l}_1 representations, we can construct a map which is invariant under E_{11} . We define the map as

$$N_A{}^B = (l_A, \bar{l}^B), \quad (1.4.12)$$

and being E_{11} invariant implies that

$$([R^{\alpha}, l_A], \bar{l}^B) + (l_A, [R^{\alpha}, \bar{l}^B]) = 0. \quad (1.4.13)$$

If we insert the commutators given in equation (1.3.13) and (1.3.17), we find

$$D^{\alpha} N = N \bar{D}^{\alpha}. \quad (1.4.14)$$

We can then use the relation (1.3.19) to remove the \bar{D} matrix, and we additionally define

$$K \equiv N(J^{-1})^T, \quad (1.4.15)$$

to give

$$D^{\alpha} K = K(D^{-\alpha})^T. \quad (1.4.16)$$

It turns out that the object K is the invariant metric in which we are interested. This is the invariant metric that we will derive for 11D, 5D, and 4D decompositions of E_{11} and for the A_1^{+++} algebra. Let's show this is in fact an E_{11} invariant metric. We can manipulate

(1.4.16) in the following way to find a relation on the matrix K

$$\begin{aligned} D^\alpha &= ((D^\alpha)^T)^T = (K^{-1}(D^{-\alpha})K)^T \\ &= K^T(D^{-\alpha})^T(K^{-1})^T = K^T K^{-1} D^\alpha K (K^{-1})^T, \end{aligned} \quad (1.4.17)$$

so that we get

$$K(K^{-1})^T D^\alpha = D^\alpha K (K^{-1})^T. \quad (1.4.18)$$

Then using Schur's lemma, we find that

$$K = K^T. \quad (1.4.19)$$

We can insert this back into (1.4.16) to get

$$D^\alpha K = (D^{-\alpha} K)^T. \quad (1.4.20)$$

If we add $\pm D^{-\alpha} K = \pm (D^\alpha K)^T$ to this, we get

$$(D^\alpha \pm D^{-\alpha})K = \pm ((D^\alpha \pm D^{-\alpha})K)^T, \quad (1.4.21)$$

so we see that $(D^\alpha - D^{-\alpha})K$ is some antisymmetric matrix, and we have that

$$(D^\alpha - D^{-\alpha})K + K(D^\alpha - D^{-\alpha})^T = 0. \quad (1.4.22)$$

We saw in equation (1.4.11) that $(D^\alpha - D^{-\alpha})$ is a $I_c(E_{11})$ transformation, and hence we realise that K is an invariant tensor of $I_c(E_{11})$ and we have indeed found the invariant metric.

We saw how the matter representation V^A transforms as in equation (1.4.11), and using this and equation (1.4.21), we can define the invariant quantity

$$\Delta = V^A K_{AB} V^B = V^T K V. \quad (1.4.23)$$

It is this quantity that we shall derive in 11D, 5D, and 4D decompositions of E_{11} and additionally for the A_1^{+++} algebra in chapter 3 of this thesis, such that we can find the metric K_{AB} .

We note that if we take the tangent objects to be infinitesimal $V^A = dz^\Pi E_\Pi^A$, we can write an invariant infinitesimal distance as

$$ds^2 = dz^\Pi g_{\Pi\Lambda} dz^\Lambda, \quad (1.4.24)$$

where $g_{\Pi\Lambda} = E_\Pi^A K_{AB} E_\Lambda^B$, which is a more general form of the well-known relation between a curved metric and the flatspace metric in general relativity, where E_Π^A is behaving like the usual vielbein. We have seen that this is indeed a generalised vielbein, similar to e_μ^a . So we have found that the invariant tangent space metric is a more general form of the well-known flatspace metric $\eta_{\mu\nu}$.

Now we have constructed the tangent space metric, we now see how to use it to construct E_{11} invariant gauge fixing conditions.

1.4.3 Gauge fixing conditions

We can now use the invariant tangent space metric to build an E_{11} invariant set of equations, which are gauge fixing conditions. One may find the Cartan forms in terms of the l_1 representation, and hence in terms of the vielbein, using equations (1.3.13) and (1.4.7) to find

$$-[\mathcal{V}_E, l_A] = dx^\Pi G_{\Pi,\underline{\alpha}} (D^{\underline{\alpha}})_A{}^B l_B = E_A^\Lambda dE_\Lambda^B l_B. \quad (1.4.25)$$

With this, we find the Cartan forms can be written

$$G_{A,B}{}^C = E_A^\Pi G_{\Pi,B}{}^C = E_A^\Pi G_{\Pi,\underline{\alpha}} (D^{\underline{\alpha}})_B{}^C = E_A^\Pi E_B^\Lambda \partial_\Pi E_\Lambda^C, \quad (1.4.26)$$

where we have flattened the first index on the Cartan form using the vielbein, in such a way that the Cartan form is invariant under rigid transformations, and then transforms

under local transformations as in equation (1.4.9).

We can then contract the first two indices with the invariant metric we found in the previous section 1.4.2 to get

$$G^C = K^{AB} G_{A,B}{}^C, \quad (1.4.27)$$

which transforms exactly as V^A in equation (1.4.11). We can set this vector to zero, while still preserving the symmetries under the local $I_c(E_{11})$ transformation, and this gives a set of gauge-fixing conditions. These gauge-fixing conditions can be used to reduce the number of redundant degrees of freedom. We shall find the gauge fixing conditions in chapter 3 for 11D, 5D, and 4D decompositions of E_{11} and also for the A_1^{+++} algebra.

This completes the theory we shall need for the results in this thesis. We shall now derive a toy model of gravity in D dimensions as a non-linear realisation, in order to see the theory described so far in practise.

1.5 A toy model: Gravity as a non-linear realisation

We do a simple example of deriving gravity as a non-linear realisation, which is done using the group $\text{IGL}(D) = \text{GL}(D) \ltimes T^D$. This follows closely chapter 13.2 in [41], but was first done in 4D in [47], and then for general dimension D in [18]. As before, the T^D is the vector representation of $\text{GL}(D)$, and its generators are the translation generators P_a . The generators of $\text{GL}(D)$ are $K^a{}_b$, $a, b = 1, \dots, D$. The algebra is then

$$\begin{aligned} [K^a{}_b, K^c{}_d] &= \delta_b^c K^a{}_d - \delta_d^a K^c{}_b, \\ [K^a{}_b, P_c] &= -\delta_c^a P_b. \end{aligned} \quad (1.5.1)$$

The action of the Cartan involution acts as

$$I_c(K^a{}_b) = -K^b{}_a, \quad (1.5.2)$$

and so the subalgebra is generated by

$$S_{ab} = \delta_{ae}K^e{}_b - \delta_{be}K^e{}_a , \quad (1.5.3)$$

which is the $SO(D)$ algebra, corresponding to Euclidean gravity. As we will need it later, the coset generator is then

$$T_{ab} = \delta_{ae}K^e{}_b + \delta_{be}K^e{}_a . \quad (1.5.4)$$

The action of the Cartan involution on the vector representation takes it to \bar{l}_1 representation

$$I_c(P_a) = -\bar{P}^a , \quad (1.5.5)$$

so that the commutator with the \bar{l}_1 representation is

$$[K^a{}_b, \bar{P}^c] = \delta_b^c \bar{P}^a . \quad (1.5.6)$$

In order to find the scalar product between representations, which will give us the tangent space metric, we note the relation

$$([K^a{}_b, P_c], \bar{P}^d) + (P_c, [K^a{}_b, \bar{P}^d]) = 0 , \quad (1.5.7)$$

which ensures the metric is invariant under $SL(D)$. We find the scalar product to be $N_a{}^b = \delta_a^b$, and so the invariant metric is

$$\mathcal{K}_{ab} = \delta_{ab} , \quad (1.5.8)$$

where the calligraphic font used here is simply to differentiate the metric from the $K^a{}_b$ generator. The group element in this case is

$$g = e^{x^a P_a} e^{h_a{}^b(x) K^a{}_b} , \quad (1.5.9)$$

which results in the Cartan form

$$\mathcal{V} = g^{-1}dg = dx^\mu e_\mu^a (P_a + G_{a,b}{}^c K^b{}_c), \quad (1.5.10)$$

where $e_\mu^a = (e^h)_\mu^a$ transforms under local transformations on the a index, and under rigid transformations on its μ index, and so corresponds to the vielbein of the theory.

Then the Cartan form in terms of the vielbein is

$$G_{a,b}{}^c = e_a^\mu e_b^\nu \partial_\mu (e_\nu^c), \quad (1.5.11)$$

such that we find a gauge fixing condition

$$\partial^a h_a^c = 0. \quad (1.5.12)$$

With the aim of finding the Ricci scalar, we can decompose the $GL(D)$ generator into a generator of the Cartan involution invariant subgroup and its coset, given in equations (1.5.3) and (1.5.4). The Cartan form from equation (1.5.10) is then

$$\mathcal{V} = dx^\mu e_\mu^a (P_a + \frac{1}{2} S_{a,b}{}^c T^b{}_c + \frac{1}{2} Q_{a,b}{}^c J^b{}_c). \quad (1.5.13)$$

The action which is invariant under the local subgroup transformations and second order in derivatives is

$$\begin{aligned} S &= \int d^d x \frac{1}{2} \det(e) (-D_a S_{a,bb} + D_a S_{b,ab} - S_{a,ac} S_{b,bc} + S_{a,bc} S_{b,ac} \\ &\quad - \frac{1}{2} S_{a,bc} S_{a,bc} + S_{a,ab} S_{b,cc} - \frac{1}{2} S_{a,bb} S_{a,cc}), \\ &\equiv \int d^d x \sqrt{\det(g)} R, \end{aligned} \quad (1.5.14)$$

where the covariant derivative is

$$D_a S_{b,cd} = \partial_a S_{b,cd} + Q_{a,be} S_{e,cd} + Q_{a,ce} S_{a,ed} + Q_{a,de} S_{a,ce}, \quad (1.5.15)$$

and one then finds the Ricci scalar, as shown in equation (1.5.14), using that $g_{\mu\nu} =$

$e_\mu^a e_\nu^b \delta_{ab}$ is the metric, and $\det(g)$ is its determinant. We note here that the choice of coefficients in (1.5.14) has been made in order to make the action invariant under diffeomorphisms. It is possible to build a unique action without needing to impose these conditions if one includes the conformal group in the non-linear realisation [47]. It is important to note that, while we have had to make a choice of coefficients in this case, we shall see that this is not necessary for the non-linear realisation of $E_{11} \times l_1$. Indeed, the coefficients are uniquely fixed in the equations of motion derived directly from the non-linear realisation of $E_{11} \times l_1$ without needing to impose any additional conditions.

This completes the introduction to E theory. Now we can now begin to study E theory in specific dimensions. We begin by deriving the algebra of various dimensions.

Chapter 2

E_{11} Algebra

This chapter gives the derivation of the algebra of $E_{11} \times l_1$ in various dimensions. Firstly, the algebra is derived for the adjoint representation at level 5 and 6 in 11 dimensions. Then in a similar way to the 11D algebra, the 7 dimensional algebra shall be derived up to level ± 5 for the adjoint representation, then its l_1 representation, the Cartan involution invariant subalgebra, $I_c(E_{11})$, and finally the coset algebra of the $I_c(E_{11})$ algebra. Then for the 10 dimensional IIB theory, the algebra that is invariant under the action of the Cartan involution is derived, that is the $I_c(E_{11})$ subalgebra, following the same derivation as in the 7D case.

2.1 E_{11} algebra in 11 dimensions at higher levels

The results in this section are to be published [23]. The algebra in 11D was constructed up to level 3 in [19, 48], and the results of level 4 are to be published [23], but this algebra up to level 4 has been given in appendix A for ease of reference. We begin by giving the Dynkin diagram of E_{11} when decomposed into representations of $GL(11)$ as shown in figure 2.1.

We obtain the relevant algebra $GL(11)$ from the E_{11} Dynkin diagram, by deleting node 11. The low level generators of E_{11} were computed initially by hand in the early studies of E_{11} [19, 48], and are given to level 8 in [41], but as a check, one can use the programme

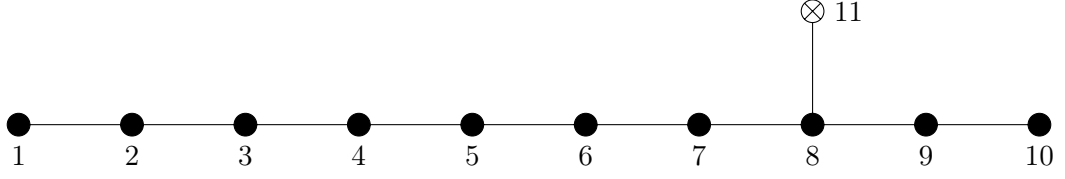


FIGURE 2.1: Dynkin diagram of E_{11} when decomposed into the algebra resulting in the 11 dimensional maximal supergravity.

SimpLie [49]. The generators up to level six are

$$\begin{aligned}
& K^a_b ; R_1^{a_1 a_2 a_3} ; R_2^{a_1 \dots a_6} ; R_3^{a_1 \dots a_8, b} ; \\
& R_{41}^{a_1 \dots a_{11}, b} , R_{42}^{a_1 \dots a_{10}, (b_1 b_2)} , R_{43}^{a_1 \dots a_9, b_1 b_2 b_3} ; \\
& R_{51}^{a_1 a_2 a_3 a_4} , R_{52}^{a_1 a_2 a_3, b} , R_{53}^{a_1 \dots a_4, b} , R_{54}^{a_1 \dots a_6} ; \\
& R_{61}^{a_1 a_2 a_3 a_4} , R_{62}^{a_1 a_2 a_3, b} , R_{63}^a_{b_1 \dots b_5} , R_{69}^a_{b_1 \dots b_5} , R_{64}^a_{b_1 \dots b_4, c} , \\
& R_{65}^{a_1 \dots a_5, (b_1 b_2)} , R_{66}^a_{b_1 b_2 b_3, c_1 c_2} , R_{67}^{a_1 \dots a_4, b_1 b_2 b_3} , R_{68}^{a_1 a_2}_{b_1 \dots b_5, c} ; \dots , \quad (2.1.1)
\end{aligned}$$

where $a_1, \dots, b_1, \dots = 1, \dots, 11$. The semi-colon between certain generators represents an increase in level. The numerical subscripts in orange provide a labelling of the generators; the first subscript denotes the level while the second is a labelling of the generators at a given level. Up to level three there is only one generator at every level and so for these we do not need the second subscript. The orange subscripts were introduced primarily to keep track of generators at level 5 and 6, due to the large number of them, but were extended in this section to low levels in order to keep consistent notation. The blocks of indices are totally antisymmetric except when round brackets are given around a block of indices and then the indices in that block are symmetric.

For completeness, the negative level generators are

$$\begin{aligned}
& R_{-1}^{a_1 a_2 a_3} ; R_{-2}^{a_1 \dots a_6} ; R_{-3}^{a_1 \dots a_8, b} ; \\
& R_{-41}^{a_1 \dots a_{11}, b} , R_{-42}^{a_1 \dots a_{10}, (b_1 b_2)} , R_{-43}^{a_1 \dots a_9, b_1 b_2 b_3} ; \\
& R_{-51}^{a_1 a_2 a_3 a_4} , R_{-52}^{a_1 a_2 a_3, b} , R_{-53}^c_{a_1 \dots a_4, b} , R_{-54}^{b_1 b_2}_{a_1 \dots a_6} ; \\
& R_{-61}^{a_1 a_2 a_3 a_4} , R_{-62}^{a_1 a_2 a_3, b} , R_{-63}^{b_1 \dots b_5}_a , R_{-69}^{b_1 \dots b_5}_a , R_{-64}^{b_1 \dots b_4, c}_a , \\
& R_{-65}^{a_1 \dots a_5, (b_1 b_2)} , R_{-66}^{b_1 b_2 b_3, c_1 c_2}_a , R_{-67}^{a_1 \dots a_4, b_1 b_2 b_3} , R_{-68}^{b_1 \dots b_5, c}_{a_1 a_2} ; \dots , \quad (2.1.2)
\end{aligned}$$

where the indices again run from 1 to 11.

Usually in the 11D decomposition, a generator at level l has $3l$ upper indices. However, using the epsilon symbol to lower indices on a generator, resulting in the Hodge dual of the generator, means this correspondence between the number of indices on a generator and its level no longer holds. A result of this is that we can no longer determine the level of a generator given just its number of indices; one can see, for example, that the $R_{62a_1a_2a_3,b}$ has exactly the same index structure as $R_{-52a_1a_2a_3,b}$. While for our purposes in the derivation of the algebra, the specific generator we are referring to should be clear, it would be problematic to distinguish between the two generators during a derivation of the equations of motion, for example. We use these Hodge duals in cases where the number of indices in an antisymmetric block on the relevant generator would be reduced, and hence the total number of up plus down indices would be reduced. For example, the $R_{53c}^{a_1\dots a_4,b}$ generator is the Hodge dual of the generator $R_{53}^{10,4,1}$, where the numbers separated by commas represent sets of that number of antisymmetric indices. In other words, generator $R_{53c}^{a_1\dots a_4,b}$ is actually

$$R_{53c}^{a_1\dots a_4,b} = \epsilon_{cd_1\dots d_{10}} R_{53}^{d_1\dots d_{10},a_1\dots a_4,b} . \quad (2.1.3)$$

This was done in order to reduce the computing power needed to derive the following commutators, but it is trivial to reinstate the usual generators, using the relation in equation (2.1.3) and analogous relations for other generators.

Now we know which generators appear up to level 6, we begin the derivation of the algebra up to this level.

2.1.1 Low level example the algebra derivation

Before beginning the derivation of the level 5 and 6 algebra, we shall do some examples at lower level, in order to describe our process of derivation. We begin by giving the algebra of the generators of the $GL(11)$ algebra K^a_b where $a, b, \dots = 1, \dots, 11$ and they satisfy the

commutators

$$[K^a_c, K^b_d] = \delta_c^b K^a_d - \delta_d^a K^b_c . \quad (2.1.4)$$

The remaining generators are chosen, by construction, to be irreducible representations of $SL(11)$, and as a result, they satisfy constraints. The generators at level one and two do not require additional constraints while the generator at level three obeys the constraint $R^{[a_1 \dots a_8, b]} = 0$. The constraints for the higher level generators are given in tables 2.1, 2.2, and 2.3. Constraints can be found by first calculating what we have called the ‘naive’ dimension, which is the dimension which follows from their index structure when no constraints are applied. The actual dimension can be calculated in 3 ways: from the Young Tableaux, by using the programme `SimpLie` [49], or using an extension to Mathematica called ‘LieArt’ [50]. The difference of these two ‘dimensions’ gives the number of constraints and this was used to check that the proposed constraints were indeed correct in the derivation.

Since the generators are representations of $SL(11)$, their commutators with K^a_b are determined. For example, at level one we find that

$$[K^b_d, R_1^{a_1 a_2 a_3}] = 3\delta_d^{[a_1} R_1^{b] a_2 a_3} , \quad (2.1.5)$$

while at level minus one we have that

$$[K^b_d, R_{-1 a_1 a_2 a_3}] = -3\delta_{[a_1}^b R_{-1 d] a_2 a_3} . \quad (2.1.6)$$

The higher level commutators with the $SL(11)$ generator follow the same pattern in the way these generators act on upper and lower indices on the generator. The commutators up to level 4 are given in the appendix A, and level 5 and 6 will be given in the following. Note that the ‘G’ part in $GL(11)$ corresponds to the generator $\mathbb{D} = \sum_{a=1}^{11} K^a_a$ whose action is readily computed from the commutators. The remaining generators are part of $SL(11)$, as we would like.

Our process of the derivation of the higher level algebra is as follows. Given two generators

we used the LieArt extension [50] to Mathematica to compute the $SL(11)$ representations that could occur in their commutator. This extension simply takes the tensor product of their representations and gives equivalent irreducible representations. We then discard generators that do not occur in E_{11} at the required level and finally use the constraints and Jacobi identities to fix the coefficients of the generators that occur. Of course, one could alternatively write all possible generators and terms at the relevant level without using the representation theory of the tensor product of two generators, and then use constraints and Jacobi relations to find the result of the commutators. However, we shall see that there are a large number of generators appearing in the result of the commutators, and that calculating the representations first, allows the derivation to be much more tractable. We now begin a simple example of deriving a lower level commutator, which is known, by first looking at the representations that may appear in a commutator.

For example, let us consider the commutator of the level one generator with itself, where the result must contain level 2 generators, as we mentioned that the level is additive under the action of a commutator. The list of possible irreducible $SL(11)$ representations that can arise in the tensor product of the level 1 generator with itself are given by

$$\begin{aligned} \{R_1^{a_1 a_2 a_3}\} \otimes \{R_1^{b_1 b_2 b_3}\} &= 165 \otimes 165 \\ &= \mathbf{462} \oplus 4620 \oplus 9075 \oplus 13068 . \end{aligned} \quad (2.1.7)$$

where the curly brackets around the generators represent the full set of the possible indices on the level 1 generator, which has dimension 165. Since we are interested in the representations which occur at level 2 in the 11D decomposition of E_{11} , we note that the dimension of the level 2 generator, $R_2^{a_1 \dots a_6}$, is 462, which we have suggestively written in bold font in equation (2.1.7). In what follows, the representations in bold shall correspond to the generators in E_{11} at the relevant level. Since the commutator transforms in the tensor product of the representations, we can use the result of the tensor product to discover which generators occur in the result of a commutator. In this example, we can see then that the level 2 generator does indeed occur in the commutator of the level 1 generator

Generator	Dimension	Naive dimension	Number of constraints	Constraints
$R_{41}^{a_1 \dots a_{11}, b}$	11	11	0	—
$R_{42}^{a_1 \dots a_{10}, (b_1 b_2)}$	715	726	11	$R_{42}^{[a_1 \dots a_{10}, b_1] b_2} = 0$
$R_{43}^{a_1 \dots a_9, b_1 b_2 b_3}$	8470	9075	605	$R_{43}^{[a_1 \dots a_9, b_1] b_2 b_3} = 0$ $R_{43}^{[a_1 \dots a_9, b_1 b_2] b_3} = 0$

TABLE 2.1: Level 4 generators of E_{11} with the dimension of the generator with constraints, dimension without constraints, number of constraints, and the equations giving the constraints.

with itself. So we have found that the commutator of the level 1 generator with itself is

$$[R_1^{a_1 a_2 a_3}, R_1^{b_1 b_2 b_3}] = 2R_2^{a_1 a_2 a_3 b_1 b_2 b_3} . \quad (2.1.8)$$

The normalisation of the level two generators is fixed by the factor 2 in this equation. The normalisation up to level 4 has been chosen in [26, 27]. In this case, there was no need to use the Jacobi identity, so let's do a more complicated example which requires its use.

We will consider the commutator of a level five generator with the level minus one generator which must result in generators of level 4. Using LieArt, we find that the $SL(11)$ representations that can occur in the tensor product of these two generators are

$$\{R_{-1}^{a_1 a_2 a_3}\} \otimes \{R_{54}^{b_1 \dots b_6}_{c_1 c_2}\} = \mathbf{8470} \oplus 45375 \oplus 60984 \oplus 98010 \oplus 125840 \\ \oplus 424710 \oplus 495495 \oplus 679536 \oplus 1415700 . \quad (2.1.9)$$

Again, the curly brackets represent the full set of these generators. In this case, the only representation that actually occurs in the E_{11} algebra at the correct level is R_{43} as one can see in table 2.1. We then write the general form of the generators which appear on the right hand side, ensuring that the indices are antisymmetrised, or symmetrised, where necessary. Recall that the generators at level three and above satisfy constraints as they belong to irreducible representations of $SL(11)$, and in particular the constraints satisfied by the level five generator implies conditions for the right hand side of the commutator. The generator $R_{54}^{b_1 \dots b_6}_{c_1 c_2}$ satisfies the constraint $R_{54}^{b_1 \dots b_5 d}_{cd} = 0$, as in table 2.2. Once the right hand side of the commutators is written in such a way that constraints are satisfied, we can finally use the Jacobi identities to find the explicit form of the commutator. In our

example of level 5 with level -1, we may choose to use the Jacobi

$$[[R_1, R_{-1}], R_{43}] + \text{cyclic} = 0 . \quad (2.1.10)$$

The final result is given by

$$\begin{aligned} [R_{-1 a_1 a_2 a_3}, R_{54 c_1 c_2}^{b_1 \dots b_6}] &= 6 \delta_{a_1 a_2 a_3}^{[b_1 b_2 b_3]} \varepsilon_{c_1 c_2 d_1 \dots d_9} R_{43}^{d_1 \dots d_9, [b_4 b_5 b_6]} \\ &\quad - \frac{36}{5} \delta_{[c_1][a_1 a_2] \varepsilon_{a_3][c_2] d_1 \dots d_9} R_{43}^{d_1 \dots d_9, [b_4 b_5 b_6]} \\ &\quad + \frac{6}{5} \delta_{c_1 c_2 [a_1] \varepsilon_{a_2 a_3] d_1 \dots d_9} R_{43}^{d_1 \dots d_9, [b_4 b_5 b_6]} . \end{aligned} \quad (2.1.11)$$

We see that the right hand side satisfies any symmetries, or antisymmetries of the generators in the commutator. This is the process that we follow for the derivation of each of the commutators at level 5 and 6.

Now we have seen a simple example, we now begin the derivation by first giving the representation theory of the tensor products involving the level 5 and 6 generators.

2.1.2 Representation theory of the commutators

We will now explicitly begin the construction the E_{11} algebra up to and including level six. In what follows, we will find the constraints that the generators at levels 5 and 6 obey, and we will then find which generators can occur on the right hand side of the commutators using the fact that they must appear in the tensor product of the generators in the commutators. We will derive this first for level five, before calculating level 6. We will need the representations at level 4, such that we know what may appear in the commutator of level 5 with level -1 generators. The generators at level 4 are

$$R_{41}^{a_1 \dots a_{11}, b}, R_{42}^{a_1 \dots a_{10}, (b_1 b_2)}, R_{43}^{a_1 \dots a_9, b_1 b_2 b_3} . \quad (2.1.12)$$

The constraints for these generators are given in table 2.1. For each generator in this table, we list its actual dimension, naive dimension, the number of constraints, and finally the

explicit constraints. The constraints are just those required to ensure that the generators are irreducible representations of $SL(11)$.

Now we can begin to find the relevant commutators beginning at level 5. The generators at level 5 are

$$R_{51}^{a_1 a_2 a_3 a_4}, R_{52}^{a_1 a_2 a_3, b}, R_{53_c}^{a_1 \dots a_4, b}, R_{54_{b_1 b_2}}^{a_1 \dots a_6}. \quad (2.1.13)$$

The constraints for these generators are given in table 2.2. Recall that we have used the epsilon symbol to lower indices.

The tensor product of the level -1 generators with the level 5 generators can contain level four generators with the following the $SL(11)$ representations

$$\begin{aligned} \{R_{-1 a_1 a_2 a_3}\} \otimes \{R_{51}^{b_1 \dots b_4}\} &= \mathbf{11} \oplus 594 \oplus \mathbf{8470} \oplus 45375, \\ \{R_{-1 a_1 a_2 a_3}\} \otimes \{R_{52}^{b_1 b_2 b_3, c}\} &= \mathbf{11} \oplus 594 \oplus \mathbf{715} \oplus \mathbf{8470} \oplus 22880 \oplus 212355, \\ \{R_{-1 a_1 a_2 a_3}\} \otimes \{R_{53_d}^{b_1 \dots b_4, c}\} &= 594 \oplus \mathbf{715} \oplus \mathbf{8470} \oplus 10296 \oplus 22880 \oplus 27720 \oplus 45375 \\ &\quad \oplus 125840 \oplus 212355 \oplus 5825 \oplus 755040 \oplus 3659040, \\ \{R_{-1 a_1 a_2 a_3}\} \otimes \{R_{54_{c_1 c_2}}^{b_1 \dots b_6}\} &= \mathbf{8470} \oplus 45375 \oplus 60984 \oplus 98010 \oplus 125840 \\ &\quad \oplus 424710 \oplus 495495 \oplus 679536 \oplus 1415700. \end{aligned} \quad (2.1.14)$$

Recall that the notation we use is that the representations in bold are the representations of E_{11} in which we are interested.

Generator	Dimension	Naive dimension	Number of constraints	Constraints
$R_{51}^{a_1 a_2 a_3 a_4}$	330	330	0	—
$R_{52}^{a_1 a_2 a_3, b}$	1485	1815	330	$R_{52}^{[a_1 a_2 a_3, b]} = 0$
$R_{53_c}^{a_1 \dots a_4, b}$	33033	39930	6897	$R_{53_c}^{[a_1 \dots a_4, b]} = 0$ $R_{53_c}^{a_1 a_2 a_3 c, b} = 0$ $R_{53_c}^{a_1 \dots a_4, c} = 0$
$R_{54_{b_1 b_2}}^{a_1 \dots a_6}$	20328	25410	5082	$R_{54_{bc}}^{a_1 \dots a_5 c} = 0$

TABLE 2.2: Level 5 generators of E_{11} with the dimension of the generator with constraints, dimension without constraints, number of constraints, and the equations giving the constraints.

The tensor product of the generators at level 1 with those at level 4 result the following SL(11) representations

$$\begin{aligned}
\{R_1^{a_1 a_2 a_3}\} \otimes \{R_{41}^{b_1 \dots b_{11}, c}\} &= \mathbf{330} \oplus \mathbf{1485} , \\
\{R_1^{a_1 a_2 a_3}\} \otimes \{R_{42}^{b_1 \dots b_{10}, (c_1 c_2)}\} &= \mathbf{1485} \oplus 2145 \oplus \mathbf{33033} \oplus 81312 , \\
\{R_1^{a_1 a_2 a_3}\} \otimes \{R_{43}^{b_1 \dots b_9, c_1 c_2 c_3}\} &= \mathbf{330} \oplus \mathbf{1485} \oplus 4752 \oplus \mathbf{20328} \oplus \mathbf{33033} \\
&\oplus 57200 \oplus 214500 \oplus 440440 \oplus 625482 . \quad (2.1.15)
\end{aligned}$$

Finally, the tensor product of level 2 with level 3 gives the following SL(11) representations

$$\begin{aligned}
\{R_2^{a_1 \dots a_6}\} \otimes \{R_3^{b_1 \dots b_8, c}\} &= \mathbf{330} \oplus \mathbf{1485} \oplus 4752 \oplus \mathbf{20328} \\
&\oplus 29040 \oplus \mathbf{33033} \oplus 214500 \oplus 509652 . \quad (2.1.16)
\end{aligned}$$

Next, we carry out the same steps for the generators at level 6. The generators at level 6 are

$$\begin{aligned}
R_{61}^{a_1 a_2 a_3 a_4} , R_{62}^{a_1 a_2 a_3, b} , R_{63}^a{}_{b_1 \dots b_5} , R_{69}^a{}_{b_1 \dots b_5} , R_{64}^a{}_{b_1 \dots b_4, c} , \\
R_{65}^{a_1 \dots a_5, (b_1 b_2)} , R_{66}^a{}_{b_1 b_2 b_3, c_1 c_2} , R_{67}^{a_1 \dots a_4, b_1 b_2 b_3} , R_{68}^{a_1 a_2}{}_{b_1 \dots b_5, c} . \quad (2.1.17)
\end{aligned}$$

Table 2.3 gives the constraints that these generators satisfy.

We now give the SL(11) representations that appear on the right hand side of the tensor products of the generators. The tensor product of the generators of level -1 with those of level 6 result in generators that belong to the following SL(11) representations

$$\begin{aligned}
\{R_{-1}^{a_1 a_2 a_3}\} \otimes \{R_{61}^{b_1 \dots b_4}\} &= \mathbf{330} \oplus 4752 \oplus \mathbf{20328} \oplus 29040 , \\
\{R_{-1}^{a_1 a_2 a_3}\} \otimes \{R_{62}^{b_1 b_2 b_3, c}\} &= 4752 \oplus \mathbf{20328} \oplus 25740 \oplus 29040 \oplus 70785 \oplus 94380 , \\
\{R_{-1}^{a_1 a_2 a_3}\} \otimes \{R_{63}^b{}_{c_1 \dots c_5}\} &= \mathbf{330} \oplus \mathbf{1485} \oplus 4752 \oplus \mathbf{20328} \\
&\oplus \mathbf{33033} \oplus 214500 \oplus 509652 , \\
\{R_{-1}^{a_1 a_2 a_3}\} \otimes \{R_{69}^b{}_{c_1 \dots c_5}\} &= \mathbf{330} \oplus \mathbf{1485} \oplus 4752 \oplus \mathbf{20328}
\end{aligned}$$

Generator	Dimension	Naive dimension	Number of constraints	Constraints
$R_{61a_1a_2a_3a_4}$	330	330	0	–
$R_{62a_1a_2a_3,b}$	1485	1815	330	$R_{62[a_1a_2a_3,b]} = 0$
$R_{63}^a_{b_1\dots b_5}$	4752	5082	330	$R_{63}^a_{ab_2\dots b_5} = 0$
$R_{69}^a_{b_1\dots b_5}$	4752	5082	330	$R_{69}^a_{ab_2\dots b_5} = 0$
$R_{64}^a_{b_1\dots b_4,c}$	33033	39930	6897	$R_{64}^a_{[b_1\dots b_4,c]} = 0$ $R_{64}^a_{ab_2\dots b_4,c} = 0$
$R_{65}^{a_1\dots a_5,(b_1b_2)}$	25740	30492	4752	$R_{65}^{[a_1\dots a_5,b_1]b_2} = 0$
$R_{66}^a_{b_1b_2b_3,c_1c_2}$	57200	99825	42625	$R_{66}^a_{ab_2b_3,c_1c_2} = 0$ $R_{66}^a_{[b_1b_2b_3,c_1]c_2} = 0$
$R_{67}^{a_1\dots a_4,b_1b_2b_3}$	29040	54450	25410	$R_{67}^{[a_1\dots a_4,b_1]b_2b_3} = 0$
$R_{68}^{a_1a_2}_{b_1\dots b_5,c}$	214500	279510	65010	$R_{68}^{a_1a_2}_{[b_1\dots b_5,c]} = 0$ $R_{68}^{a_1a_2}_{a_1b_2\dots b_5,c} = 0$

TABLE 2.3: Level 6 generators of E_{11} with the dimension of the generator with constraints, dimension without constraints, number of constraints, and the equations giving the constraints.

$$\begin{aligned}
& \oplus \mathbf{33033} \oplus 214500 \oplus 509652 , \\
\{R_{-1a_1a_2a_3}\} \otimes \{R_{64}^b_{c_1\dots c_4,c}\} &= 4752 \oplus \mathbf{20328} \oplus 25740 \oplus 29040 \oplus \mathbf{33033} \\
& \oplus 94380 \oplus 214500 \oplus 266805 \oplus 330330 \\
& \oplus 509652 \oplus 1520640 \oplus 2401245 , \\
\{R_{-1a_1a_2a_3}\} \otimes \{R_{65}^{b_1\dots b_5,c_1c_2}\} &= \mathbf{1485} \oplus 2145 \oplus \mathbf{33033} \oplus 81312 \\
& \oplus 214500 \oplus 825825 \oplus 3088800 , \\
\{R_{-1a_1a_2a_3}\} \otimes \{R_{66}^b_{c_1c_2c_3,d_1d_2}\} &= \mathbf{20328} \oplus 29040 \oplus 70785 \oplus 84942 \\
& \oplus 94380 \oplus 214500 \oplus 509652 \oplus 1520640 \\
& + 2180178 \oplus 2312310 \oplus 2401245 , \\
\{R_{-1a_1a_2a_3}\} \otimes \{R_{67}^{b_1\dots b_4,c_1c_2c_3}\} &= \mathbf{330} \oplus \mathbf{1485} \oplus \mathbf{33033} \oplus 57200 \\
& \oplus 440440 \oplus 625482 \oplus 3633630 , \\
\{R_{-1a_1a_2a_3}\} \otimes \{R_{68}^{b_1b_2}_{c_1\dots c_5,c}\} &= \mathbf{4752} \oplus \mathbf{20328} \oplus 25740 \oplus \mathbf{33033} \\
& \oplus 57200 \oplus 214500 \oplus 266805 \oplus 509652 \\
& + 625482 \oplus 770770 \oplus 1520640 \oplus 2416128 \\
& \oplus 3303300 \oplus 6936930 \oplus 18687240 . \tag{2.1.18}
\end{aligned}$$

While the tensor product of the generators at level 5 with the generators of level 1 leads to generators that belong to the following $SL(11)$ representations

$$\begin{aligned}
\{R_1^{a_1 a_2 a_3}\} \otimes \{R_{51}^{b_1 \dots b_4}\} &= \mathbf{330} \oplus \mathbf{4752} \oplus 20328 \oplus \mathbf{29040} , \\
\{R_1^{a_1 a_2 a_3}\} \otimes \{R_{52}^{b_1 b_2 b_3, c}\} &= \mathbf{4752} \oplus 20328 \oplus \mathbf{25740} \\
&\oplus \mathbf{29040} \oplus 70785 \oplus 94380 , \\
\{R_1^{a_1 a_2 a_3}\} \otimes \{R_{53d}^{b_1 \dots b_4, c}\} &= \mathbf{4752} \oplus 20328 \oplus \mathbf{25740} \oplus \mathbf{29040} \\
&\oplus \mathbf{33033} \oplus 94380 \oplus \mathbf{214500} \oplus 266805 \\
&\oplus 330330 \oplus 509652 \oplus 1520640 \oplus 2401245 , \\
\{R_1^{a_1 a_2 a_3}\} \otimes \{R_{54c_1 c_2}^{b_1 \dots b_6}\} &= \mathbf{330} \oplus 1210 \oplus \mathbf{1485} \oplus \mathbf{4752} \oplus \mathbf{33033} \\
&\oplus \mathbf{57200} \oplus \mathbf{214500} \oplus 62548 \oplus 2416128 . \quad (2.1.19)
\end{aligned}$$

So far we have found which generators appear in the tensor products of relevant generators. As discussed above we can now write down all the possible terms on the right hand side of the commutator with arbitrary coefficients and then require that the constraints satisfied by the generators in the commutators are satisfied by the result of the commutator. Finally we use the Jacobi identities to determine all the unknown coefficients and so find the explicit form of the commutators. We begin with the level 5 commutators.

2.1.3 Algebra at level 5

Now we find the level 5 algebra. Recall that this is done by writing the most general form of the right hand side of the commutator, using our knowledge of which representations occur in the result of the commutator as we found in the previous section. Once we have the general form, we apply constraints, and finally use relevant Jacobi identities to find the following results.

First, we find the level -1 with level 5 commutators to be

$$\begin{aligned}
[R_{-1a_1a_2a_3}, R_{51}^{b_1b_2b_3b_4}] &= -\frac{10}{9}\delta_{a_1a_2a_3}^{[b_1b_2b_3]}R_{41}^{1\dots 11,|b_4]} \\
&\quad -\frac{1}{3!3!9!}\delta_{[a_1}^{[b_1]}\varepsilon_{a_2a_3]c_1\dots c_9}R_{43}^{c_1\dots c_9,|b_2b_3b_4]} , \\
[R_{-1a_1a_2a_3}, R_{52}^{b_1b_2b_3,c}] &= \frac{1}{8!8}\delta_{[a_1}^d\varepsilon_{a_2a_3]c_1\dots c_9}R_{43}^{c_1\dots c_9,b_1b_2b_3} \\
&\quad +\frac{1}{8!8}\delta_{[a_1}^{[b_1]}\varepsilon_{a_2a_3]c_1\dots c_9}R_{43}^{c_1\dots c_9,d|b_2b_3]} \\
&\quad +6\delta_{a_1a_2a_3}^{b_1b_2b_3}R_{41}^{1\dots 11,d} \\
&\quad +6\delta_{a_1a_2a_3}^{d[b_1b_2]}R_{41}^{1\dots 11,|b_3]} \\
&\quad +\frac{27}{10!}\delta_{[a_1a_2}^{[b_1b_2]}\varepsilon_{a_3]c_1\dots c_{10}}R_{41}^{c_1\dots c_{10}|b_3],d} , \\
[R_{-1a_1a_2a_3}, R_{53c}^{b_1b_2b_3b_4,d}] &= -\delta_{a_1a_2a_3}^{[b_1b_2b_3]}\varepsilon_{cd_1\dots d_{10}}R_{42}^{d_1\dots d_{10},|b_4]}d \\
&\quad +\frac{3}{8}\delta_{[a_1a_2|c}^{[b_1b_2b_3]}\varepsilon_{|a_3]d_1\dots d_{10}}R_{42}^{d_1\dots d_{10},|b_4]}d \\
&\quad +3\delta_{[a_1a_2}^{[b_1b_2]}\varepsilon_{a_3]cd_1\dots d_9}R_{43}^{d_1\dots d_9,|b_3b_4]}d \\
&\quad +\frac{21}{32}\delta_{[a_1|c}^{[b_1b_2]}\varepsilon_{|a_2a_3]d_1\dots d_9}R_{43}^{d_1\dots d_9,|b_3b_4]}d \\
&\quad +3\delta_{[a_1a_2}^{[b_1]d}\varepsilon_{a_3]cd_1\dots d_9}R_{43}^{d_1\dots d_9,|b_2b_3b_4]} \\
&\quad +\frac{1}{4}\delta_c^d\delta_{[a_1}^{[b_1]}\varepsilon_{a_2a_3]d_1\dots d_9}R_{43}^{d_1\dots d_9,|b_2b_3b_4]} \\
&\quad -\frac{13}{32}\delta_c^{[b_1]}\delta_{[a_1}^d\varepsilon_{a_2a_3]d_1\dots d_9}R_{43}^{d_1\dots d_9,|b_2b_3b_4]} , \\
[R_{-1a_1a_2a_3}, R_{54c_1c_2}^{b_1\dots b_6}] &= 6\delta_{a_1a_2a_3}^{[b_1b_2b_3]}\varepsilon_{c_1c_2d_1\dots d_9}R_{43}^{d_1\dots d_9,|b_4b_5b_6]} \\
&\quad -\frac{36}{5}\delta_{[c_1|[a_1a_2}^{[b_1b_2b_3]}\varepsilon_{a_3]|c_2]d_1\dots d_9}R_{43}^{d_1\dots d_9,|b_4b_5b_6]} \\
&\quad +\frac{6}{5}\delta_{c_1c_2[a_1}^{[b_1b_2b_3]}\varepsilon_{a_2a_3]d_1\dots d_9}R_{43}^{d_1\dots d_9,|b_4b_5b_6]} , \tag{2.1.20}
\end{aligned}$$

and we find the level 0 with level 5 commutators to be

$$\begin{aligned}
[K_b^a, R_{51}^{c_1\dots c_4}] &= 4\delta_b^{[c_1]}R_{51}^{a|c_2c_3c_4]} \\
&\quad +\delta_b^a R_{51}^{c_1\dots c_4} , \\
[K_b^a, R_{52}^{c_1c_2c_3,d}] &= 3\delta_b^{[c_1]}R_{52}^{a|c_2c_3],d} \\
&\quad +\delta_b^d R_{52}^{c_1c_2c_3,a} \\
&\quad +\delta_b^a R_{52}^{c_1c_2c_3,d} ,
\end{aligned}$$

$$\begin{aligned}
[K^a_b, R_{53d}^{c_1 \dots c_4, c}] &= 4\delta_b^{[c_1]} R_{53d}^{a|c_2 c_3 c_4], c} \\
&\quad + \delta_b^c R_{53d}^{c_1 \dots c_4, a} \\
&\quad - \delta_d^a R_{53b}^{c_1 \dots c_4, c} \\
&\quad + \delta_b^a R_{53d}^{c_1 \dots c_4, c} , \\
[K^a_b, R_{54d_1 d_2}^{c_1 \dots c_6}] &= 6\delta_b^{[c_1]} R_{54d_1 d_2}^{a|c_2 \dots c_6]} \\
&\quad - 2\delta_{[d_1]}^a R_{54b|d_2]}^{c_1 \dots c_6} \\
&\quad + \delta_b^a R_{54d_1 d_2}^{c_1 \dots c_6} .
\end{aligned} \tag{2.1.21}$$

We calculate the level 4 with level 1 commutators to be

$$\begin{aligned}
[R_1^{a_1 a_2 a_3}, R_{41}^{b_1 \dots b_{11}, c}] &= \frac{45}{2} \varepsilon^{b_1 \dots b_{11}} R_{51}^{a_1 a_2 a_3 c} \\
&\quad + \frac{1}{12} \varepsilon^{b_1 \dots b_{11}} R_{52}^{a_1 a_2 a_3, c} , \\
[R_1^{a_1 a_2 a_3}, R_{42}^{b_1 \dots b_{10}, (c_1 c_2)}] &= -\frac{1}{4} \varepsilon^{b_1 \dots b_{10}} (c_1 | R_{52}^{a_1 a_2 a_3, |c_2} \\
&\quad + \frac{9}{4} \varepsilon^{b_1 \dots b_{10}} [a_1 R_{52}^{a_2 a_3}](c_1, c_2) \\
&\quad - \frac{4!}{10!} \varepsilon^{b_1 \dots b_{10} c} R_{53c}^{a_1 a_2 a_3}(c_1, c_2) , \\
[R_1^{a_1 a_2 a_3}, R_{43}^{d_1 \dots d_9, c_1 c_2 c_3}] &= 9\varepsilon^{[c_1 c_2 | d_1 \dots d_9} R_{51}^{a_1 a_2 a_3 | c_3]} \\
&\quad + 108\varepsilon^{d_1 \dots d_9} [a_1 a_2 R_{51}^{a_3}]_{c_1 c_2 c_3} \\
&\quad + 81\varepsilon^{d_1 \dots d_9} [a_1 | [c_1 R_{51}^{c_2 c_3}]_{a_2 a_3}] \\
&\quad + 3\varepsilon^{d_1 \dots d_9} [a_1 a_2 | R_{52}^{c_1 c_2 c_3, |a_3}] \\
&\quad + \frac{1}{4} \varepsilon^{d_1 \dots d_9} [c_1 c_2 R_{52}^{c_3}]_{[a_1 a_2, a_3]} \\
&\quad - \frac{9}{4} \varepsilon^{d_1 \dots d_9} [c_1 | [a_1 R_{52}^{a_2 a_3}]_{c_2, c_3}] \\
&\quad + \frac{3}{7!5} \varepsilon^{bd_1 \dots d_9} [a_1 R_{53b}^{a_2 a_3}]_{[c_1 c_2, c_3]} \\
&\quad - \frac{4!3}{10!} \varepsilon^{bd_1 \dots d_9} [c_1 | R_{53b}^{a_1 a_2 a_3}]_{c_2, c_3} \\
&\quad + \frac{10}{9!} \varepsilon^{b_1 b_2 d_1 \dots d_9} R_{54b_1 b_2}^{a_1 a_2 a_3 c_1 c_2 c_3} .
\end{aligned} \tag{2.1.22}$$

Lastly, for the level 2 with level 3 commutator, we find the result

$$\begin{aligned}
[R_2^{a_1 \dots a_6}, R_3^{b_1 \dots b_8, c}] &= \frac{7!3}{20} \varepsilon^{a_1 \dots a_6} [b_1 \dots b_5 R_{51}^{b_6 b_7 b_8}]_c \\
&\quad - \frac{7!3}{20} \varepsilon^{a_1 \dots a_6} [b_1 \dots b_5 R_{51}^{b_6 b_7 b_8 c}] \\
&\quad - 7 \varepsilon^{a_1 \dots a_6} [b_1 \dots b_5 R_{52}^{b_6 b_7 b_8}]_c \\
&\quad + 7 \varepsilon^{a_1 \dots a_6} [b_1 \dots b_5 R_{52}^{b_6 b_7 b_8, c}] \\
&\quad - \frac{1}{6!6} \varepsilon^{a_1 \dots a_6 d} [b_1 \dots b_4 R_{53d}^{b_5 b_6 b_7 b_8}]_c \\
&\quad + \frac{1}{6!6} \varepsilon^{a_1 \dots a_6 d} [b_1 \dots b_4 R_{53d}^{b_5 b_6 b_7 b_8, c}] \\
&\quad + \frac{5}{6!18} \varepsilon^{a_1 \dots a_6 d_1 d_2} [b_1 b_2 b_3 R_{54d_1 d_2}^{b_4 \dots b_8}]_c \\
&\quad - \frac{5}{6!18} \varepsilon^{a_1 \dots a_6 d_1 d_2} [b_1 b_2 b_3 R_{54d_1 d_2}^{b_4 \dots b_8 c}] , \tag{2.1.23}
\end{aligned}$$

and this completes the level 5 algebra. Level -5 commutators may be found using the action of the Cartan involution as given in equation (1.3.15). We now give the level 6 commutators which we calculated.

2.1.4 Algebra at level 6

Here, we give the commutators of level 6 generators. We find the commutators of the level -1 with level 6 generators to be

$$\begin{aligned}
[R_{-1a_1 a_2 a_3}, R_{61b_1 \dots b_4}] &= \frac{2}{3(7!)} \varepsilon_{a_1 a_2 a_3 b_1 \dots b_4 c_1 \dots c_4} R_{51}^{c_1 \dots c_4} \\
&\quad - \frac{10}{(3!)47!9!} \varepsilon_{b_1 \dots b_4 c_1 \dots c_6} [a_1 R_{54}^{c_1 \dots c_6}]_{a_2 a_3} , \\
[R_{-1a_1 a_2 a_3}, R_{62b_1 b_2 b_3, c}] &= - \frac{39}{22(8!)} \varepsilon_{a_1 a_2 a_3 c c_1 \dots c_6} [b_1 R_{54}^{c_1 \dots c_6}]_{b_2 b_3} \\
&\quad - \frac{39}{22(8!)} \varepsilon_{a_1 a_2 a_3 c_1 \dots c_6} [b_1 b_2 R_{54}^{c_1 \dots c_6}]_{b_3 c} , \\
[R_{-1a_1 a_2 a_3}, R_{63c_1 \dots c_5}^b] &= \frac{19}{2(4!)^3} \varepsilon_{a_1 a_2 a_3 c_1 \dots c_5 d_1 d_2 d_3} R_{51}^{d_1 d_2 d_3 b} \\
&\quad - \frac{95}{14(4!)^3} \delta_{[c_1 c_2 \dots c_5]}^b \varepsilon_{a_1 a_2 a_3 d_1 \dots d_4} R_{51}^{d_1 \dots d_4} \\
&\quad - \frac{35}{9!4!4} \varepsilon_{a_1 a_2 a_3 c_1 \dots c_5 d_1 \dots d_3} R_{52}^{d_1 d_2 d_3, b}
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{6!10!32} \varepsilon_{c_1 \dots c_5 d_1 \dots d_4} [a_1 a_2 R_{53}^{d_1 \dots d_4, b}] \\
& - \frac{1}{(3!4!)^2 10!} \varepsilon_{c_1 \dots c_5 d_1 \dots d_5} [a_1 R_{54}^{d_1 \dots d_5 b}] \\
& - \frac{1}{14(3!4!)^2 9!} \delta_{[c_1 \varepsilon_{c_2 \dots c_5]} c d_1 \dots d_5} [a_1 R_{54}^{d_1 \dots d_5, c}], \\
[R_{-1 a_1 a_2 a_3}, R_{69}^b] &= \frac{297}{280} \varepsilon_{a_1 a_2 a_3 c_1 \dots c_5 d_1 d_2 d_3} R_{51}^{d_1 d_2 d_3 b} \\
& + \frac{297}{392} \delta_{[c_1 \varepsilon_{c_2 \dots c_5]} a_1 a_2 a_3 c d_1 d_2 d_3} R_{52}^{d_1 d_2 d_3, c} \\
& + \frac{3(3!)^2}{25(8!)^2} \varepsilon_{c_1 \dots c_5 d_1 \dots d_4} [a_1 a_2 R_{53}^{d_1 \dots d_4, b}] \\
& - \frac{117}{5(8!)^2} \varepsilon_{c_1 \dots c_5 d_1 \dots d_5} [a_1 R_{54}^{d_1 \dots d_5 b}] \\
& - \frac{117}{7(8!)^2} \delta_{[c_1 \varepsilon_{c_2 \dots c_5]} c d_1 \dots d_5} [a_1 R_{54}^{d_1 \dots d_5, c}], \\
[R_{-1 a_1 a_2 a_3}, R_{64}^b] &= \frac{4}{121(5!)^2} \varepsilon_{a_1 a_2 a_3 c_1 \dots c_4 d_1 \dots d_4} R_{53}^{d_1 \dots d_4, b} \\
& + \frac{1}{3(4!)^2} \varepsilon_{a_1 a_2 a_3 c d_1 \dots d_4} [c_1 c_2 c_3 R_{53}^{d_1 \dots d_4, b}] \\
& - \frac{1}{3(4!)^2} \delta_{[a_1 \varepsilon_{a_2 a_3}] d_1 \dots d_6} [c_1 c_2 c_3 R_{64, 54}^{d_1 \dots d_6, c}] \\
& + \frac{1}{3(4!)^2} \delta_{[a_1 \varepsilon_{a_2 a_3}] d_1 \dots d_6} [c_1 c_2 c_3 R_{54}^{d_1 \dots d_6, c}] \\
& + \frac{1}{(4!)^3} \delta_c^b \varepsilon_{d_1 \dots d_6} [a_1 a_2 | [c_1 c_2 c_3 R_{54}^{d_1 \dots d_6, c}] | a_3] \\
& - \frac{1}{(4!)^3} \varepsilon_{d_1 \dots d_6} [a_1 a_2 | [c_1 c_2 c_3 \delta_c^b R_{54}^{d_1 \dots d_6, c}] | a_3], \\
[R_{-1 a_1 a_2 a_3}, R_{65}^b] &= \frac{5!}{7} \delta_{[c_1 | [b_1 b_2 R_{52}^{b_3 b_4 b_5}, | c_2] | a_1 a_2 a_3} \\
& + \frac{90}{7} \delta_{[c_1 | [b_1 b_2 b_3 R_{52}^{b_4 b_5}, | c_2] | a_1 a_2 a_3} \\
& - \frac{5}{7(7!)} \delta_{[a_1 a_2}^{[c_1 | [b_1 R_{53}^{b_2 \dots b_5}, | c_2] | a_3} \\
& + \frac{4}{7(7!)} \delta_{[a_1 a_2}^{[b_1 b_2 R_{53}^{b_3 b_4 b_5}, | c_1, c_2] | a_3}], \\
[R_{-1 a_1 a_2 a_3}, R_{66}^b] &= \frac{2}{6!} \varepsilon_{a_1 a_2 a_3 c_1 c_2 c_3 b_1 \dots b_5} R_{54}^{b_1 \dots b_5 b} \\
& - \frac{3}{6!} \varepsilon_{a_1 a_2 a_3 b_1 \dots b_5} [d_1 | [c_1 c_2 R_{54}^{b_1 \dots b_5 b} | d_2] \\
& - \frac{2}{6!} \varepsilon_{a_1 a_2 a_3 d_1 d_2 b_1 \dots b_5} [c_1 R_{54}^{b_1 \dots b_5 b} | c_2 c_3] \\
& - \frac{31}{66(6!)} \delta_{[c_1 \varepsilon_{c_2 c_3}] a_1 a_2 a_3 b_1 \dots b_6} R_{54}^{b_1 \dots b_6} \\
& - \frac{2}{3(6!)} \delta_{[c_1 \varepsilon_{c_2}] a_1 a_2 a_3 b_1 \dots b_6} [d_1 R_{54}^{b_1 \dots b_6} | c_3]
\end{aligned}$$

$$\begin{aligned}
& + \frac{13}{66(6!)} \varepsilon_{a_1 a_2 a_3 d_1 d_2 b_1 \dots b_6} \delta_{[c_1}^b R_{54}^{b_1 \dots b_6} c_2 c_3] \\
& + \frac{3}{11(6!)} \delta_{[d_1}^b \varepsilon_{a_1 a_2 a_3 b_1 \dots b_6} [c_1 c_2 R_{54}^{b_1 \dots b_6} c_3] d_2] \\
& - \frac{3}{11(6!)} \delta_{[d_1}^b \varepsilon_{d_2] a_1 a_2 a_3 b_1 \dots b_6} [c_1 R_{54}^{b_1 \dots b_6} c_2 c_3] \ , \\
[R_{-1 a_1 a_2 a_3}, R_{67}^{b_1 \dots b_4, c_1 c_2 c_3}] & = \frac{1}{20} \delta_{a_1 a_2 a_3}^{c_1 c_2 c_3} R_{51}^{b_1 \dots b_4} \\
& - \frac{3}{20} \delta_{a_1 a_2 a_3}^{[c_1 c_2 | [b_1 R_{51}^{b_2 b_3 b_4} | c_3] \\
& + \frac{3}{20} \delta_{a_1 a_2 a_3}^{[c_1 | [b_1 b_2 R_{51}^{b_3 b_4} | c_2, c_3] \\
& - \frac{1}{20} \delta_{a_1 a_2 a_3}^{[b_1 b_2 b_3 R_{51}^{b_4} | c_1 c_2 c_3] \\
& + \frac{5}{6!} \delta_{a_1 a_2 a_3}^{[b_1 b_2 b_3 | R_{52}^{c_1 c_2 c_3, | b_4] \\
& + \frac{1}{2(4!)} \delta_{a_1 a_2 a_3}^{[b_1 b_2 | [c_1 R_{52}^{c_2 c_3} | b_3, b_4] \\
& + \frac{1}{2(4!)} \delta_{a_1 a_2 a_3}^{[b_1 | [c_1 c_2 R_{52}^{c_3} | b_2 b_3, b_4] \\
& - \frac{1}{6(10!)} \delta_{[a_1 a_2}^{[c_1 c_2 | R_{53}^{b_1 \dots b_4, | c_3] a_3] \\
& + \frac{4}{9(10!)} \delta_{[a_1 a_2}^{[c_1 | [b_1 R_{53}^{b_2 b_3 b_4} | c_2, c_3] a_3] \\
& - \frac{1}{3(10!)} \delta_{[a_1 a_2}^{[b_1 b_2 R_{53}^{b_3 b_4} | c_1 c_2, c_3] a_3] \ , \\
[R_{-1 a_1 a_2 a_3}, R_{68}^{b_1 b_2 c_1 \dots c_5, c}] & = - \frac{4}{7!} \varepsilon_{a_1 a_2 a_3 c_1 \dots c_5} d_1 d_2 d_3 R_{53}^{d_1 d_2 d_3} [b_1, b_2] \\
& - \frac{27}{11(7!)} \delta_{[c_1}^{[b_1 | \varepsilon_{c_2 \dots c_5] a_1 a_2 a_3 d_1 \dots d_4} R_{53}^{d_1 \dots d_4, | b_2] c} \\
& - \frac{4}{7!} \varepsilon_{a_1 a_2 a_3 c d_1 d_2 d_3} [c_1 \dots c_4 R_{53}^{d_1 d_2 d_3} [b_1, b_2] c_5] \\
& + \frac{4!}{11(7!)} \delta_{[c_1}^{[b_1 | \varepsilon_{c_2 \dots c_4} | a_1 a_2 a_3 c d_1 \dots d_4} R_{53}^{d_1 \dots d_4, | b_2] c_5] \\
& - \frac{3}{7!} \delta_c^{[b_1 | \varepsilon_{a_1 a_2 a_3 d_1 \dots d_4} [c_1 \dots c_4 R_{53}^{d_1 \dots d_4, | b_2] c_5] \\
& - \frac{25}{3(7!)} \varepsilon_{a_1 a_2 a_3 d_1 \dots d_4} [c_1 \dots c_4 R_{54}^{d_1 \dots d_4} b_1 b_2 c_5] c \\
& - \frac{25}{3(7!)} \varepsilon_{a_1 a_2 a_3 c d_1 \dots d_4} [c_1 c_2 c_3 R_{54}^{d_1 \dots d_4} b_1 b_2 c_4 c_5] \\
& - \frac{350}{33(7!)} \delta_{[c_1}^{[b_1 | \varepsilon_{c_2 c_3 c_4} | a_1 a_2 a_3 d_1 \dots d_5} R_{54}^{b_2] d_1 \dots d_5} c_5] c \\
& - \frac{100}{11(7!)} \delta_{[c_1}^{[b_1 | \varepsilon_{c_2 c_3} | a_1 a_2 a_3 c d_1 \dots d_5} R_{54}^{b_2] d_1 \dots d_5} c_4 c_5] \\
& - \frac{50}{33(7!)} \delta_c^{[b_1 | \varepsilon_{a_1 a_2 a_3 d_1 \dots d_5} [c_1 c_2 c_3 R_{54}^{b_2] d_1 \dots d_5} c_4 c_5]
\end{aligned}$$

$$\begin{aligned}
& - \frac{500}{231(7!)} \delta_{[c_1 c_2]}^{b_1 b_2} \varepsilon_{c_3 c_4 | a_1 a_2 a_3 d_1 \dots d_6} R_{54}^{d_1 \dots d_6}_{[c_5] c} \\
& + \frac{325}{231(7!)} \delta_{[c_1 c_2]}^{b_1 b_2} \varepsilon_{c_3 | a_1 a_2 a_3 c d_1 \dots d_6} R_{54}^{d_1 \dots d_6}_{[c_4 c_5]} \\
& + \frac{25}{33(7!)} \delta_{c[c_1]}^{b_1 b_2} \varepsilon_{c_2 c_3 | a_1 a_2 a_3 d_1 \dots d_6} R_{54}^{d_1 \dots d_6}_{[c_4 c_5]} . \tag{2.1.24}
\end{aligned}$$

The commutators of the level 0 generator with the level 6 generators are

$$\begin{aligned}
[K^a_b, R_{61 c_1 \dots c_4}] &= -4 \delta_{[c_1]}^a R_{61 b | c_2 c_3 c_4} \\
&+ \delta_b^a R_{61 c_1 \dots c_4} , \\
[K^a_b, R_{62 c_1 c_2 c_3, d}] &= -3 \delta_{[c_1]}^a R_{62 b | c_2 c_3, d} \\
&- \delta_d^a R_{62 c_1 c_2 c_3, b} \\
&+ \delta_b^a R_{62 c_1 c_2 c_3, d} , \\
[K^a_b, R_{63}^c_{d_1 \dots d_5}] &= \delta_b^c R_{63}^a_{d_1 \dots d_5} \\
&- 5 \delta_{[d_1]}^a R_{63}^c_{b | d_2 \dots d_5} \\
&+ \delta_b^a R_{63}^c_{d_1 \dots d_5} , \\
[K^a_b, R_{69}^c_{d_1 \dots d_5}] &= \delta_b^c R_{69}^a_{d_1 \dots d_5} \\
&- 5 \delta_{[d_1]}^a R_{69}^c_{b | d_2 \dots d_5} \\
&+ \delta_b^a R_{69}^c_{d_1 \dots d_5} , \\
[K^a_b, R_{64}^c_{d_1 \dots d_4, e}] &= \delta_b^c R_{64}^a_{d_1 \dots d_4, e} \\
&- 4 \delta_{[d_1]}^a R_{64}^c_{b | d_2 \dots d_4, e} \\
&- \delta_e^a R_{64}^c_{d_1 \dots d_4, b} \\
&+ \delta_b^a R_{64}^c_{d_1 \dots d_4, e} , \\
[K^a_b, R_{65}^{c_1 \dots c_5, (d_1 d_2)}] &= 5 \delta_b^{[c_1]} R_{65}^{a | c_2 \dots c_5], (d_1 d_2)} \\
&+ 2 \delta_b^{(d_1]} R_{65}^{c_1 \dots c_5, a | d_2)} \\
&+ \delta_b^a R_{65}^{c_1 \dots c_5, (d_1 d_2)} , \\
[K^a_b, R_{66}^c_{d_1 d_2 d_3, e_1 e_2}] &= \delta_b^c R_{66}^a_{d_1 d_2 d_3, e_1 e_2} \\
&- 3 \delta_{[d_1]}^a R_{66}^c_{b | d_2 d_3, e_1 e_2} \\
&- 2 \delta_{[e_1]}^a R_{66}^c_{d_1 d_2 d_3, b | e_2}
\end{aligned}$$

$$\begin{aligned}
& + \delta_b^a R_{66}^{c d_1 d_2 d_3, e_1 e_2} , \\
[K^a_b, R_{67}^{c_1 \dots c_4, d_1 d_2 d_3}] & = 4\delta_b^{[c_1} R_{67}^{a|c_2 c_3 c_4], d_1 d_2 d_3} \\
& + 3\delta_b^{[d_1} R_{67}^{c_1 \dots c_4, a|d_2 d_3]} \\
& + \delta_b^a R_{67}^{c_1 \dots c_4, d_1 d_2 d_3} , \\
[K^a_b, R_{68}^{c_1 c_2}_{d_1 \dots d_5, e}] & = 2\delta_b^{[c_1} R_{68}^{a|c_2]}_{d_1 \dots d_5, e} \\
& - 5\delta_{[d_1}^a R_{68}^{c_1 c_2}_{b|d_2 \dots d_5], e} \\
& - \delta_e^a R_{68}^{c_1 c_2}_{d_1 \dots d_5, b} \\
& + \delta_b^a R_{68}^{c_1 c_2}_{d_1 \dots d_5, e} . \tag{2.1.25}
\end{aligned}$$

Finally, we calculate the level 5 with level 1 commutators to be

$$\begin{aligned}
[R_1^{a_1 a_2 a_3}, R_{51}^{b_1 \dots b_4}] & = 3\varepsilon^{a_1 a_2 a_3 b_1 \dots b_4 c_1 \dots c_4} R_{61}^{c_1 \dots c_4} \\
& + 6\varepsilon^{b_1 \dots b_4 c_1 \dots c_5} [a_1 a_2 R_{63}^{a_3}]_{c_1 \dots c_5} \\
& + 9R_{69}^{b_1 \dots b_4, a_1 a_2 a_3} , \\
[R_1^{a_1 a_2 a_3}, R_{52}^{b_1 b_2 b_3, c}] & = -\frac{339(3!)^4}{140} \varepsilon^{b_1 b_2 b_3 c c_1 \dots c_5} [a_1 a_2 R_{63}^{a_3}]_{c_1 \dots c_5} \\
& - \frac{113(3!)^4}{35} \varepsilon^{a_1 a_2 a_3 c_1 \dots c_5} c [b_1 b_2 R_{63}^{b_3}]_{c_1 \dots c_5} \\
& + 2\varepsilon^{b_1 b_2 b_3 c c_1 \dots c_5} [a_1 a_2 R_{69}^{a_3}]_{c_1 \dots c_5} \\
& + \frac{8}{3} \varepsilon^{a_1 a_2 a_3 c_1 \dots c_5} c [b_1 b_2 R_{69}^{b_3}]_{c_1 \dots c_5} \\
& + 3R_{65}^{b_1 b_2 b_3} [a_1 a_2, a_3] c \\
& + 1458R_{67}^{a_1 a_2 a_3 c, b_1 b_2 b_3} \\
& + 1458R_{67}^{a_1 a_2 a_3} [b_1, b_2 b_3] c , \\
[R_1^{a_1 a_2 a_3}, R_{53}^{b_1 \dots b_4, c}] & = -2(3!)^2 6! \varepsilon^{b_1 \dots b_4 c c_1 \dots c_4} [a_1 a_2 R_{63}^{a_3}]_{d c_1 \dots c_4} \\
& + 3!(5!)^2 \varepsilon^{a_1 a_2 a_3 c c_1 \dots c_4} [b_1 b_2 b_3 R_{63}^{b_4}]_{d c_1 \dots c_4} \\
& + 1269(3!)^4 6! \delta_d^{[b_1} \varepsilon^{b_2 b_3 b_4] c c_1 \dots c_5} [a_1 a_2 R_{63}^{a_3}]_{c_1 \dots c_5} \\
& - \frac{1}{36} \delta_d^c \varepsilon^{a_1 a_2 a_3 c_1 \dots c_5} [b_1 b_2 b_3 R_{63}^{b_4}]_{c_1 \dots c_5} \\
& - 81(6!)^2 3! \delta_d^{[b_1} \varepsilon^{b_2 b_3} | a_1 a_2 a_3 c c_1 \dots c_5} R_{63}^{b_4]}_{c_1 \dots c_5} \\
& + \frac{5! 8! 2}{33} \varepsilon^{b_1 \dots b_4 c c_1 \dots c_4} [a_1 a_2 R_{69}^{a_3}]_{d c_1 \dots c_4}
\end{aligned}$$

$$\begin{aligned}
& - \frac{5!10!}{891} \varepsilon^{a_1 a_2 a_3 c c_1 \dots c_4} [b_1 b_2 b_3 R_{69}^{b_4}]_{d c_1 \dots c_4} \\
& - \frac{5!7!47}{3!33} \delta_d^{[b_1} \varepsilon^{b_2 b_3 b_4] c c_1 \dots c_5} [a_1 a_2 R_{69}^{a_3}]_{c_1 \dots c_5} \\
& + \frac{5!8!}{3!99} \delta_d^c \varepsilon^{a_1 a_2 a_3 c_1 \dots c_5} [b_1 b_2 b_3 R_{69}^{b_4}]_{c_1 \dots c_5} \\
& + \frac{5!7!10}{33} \delta_d^{[b_1} \varepsilon^{b_2 b_3] a_1 a_2 a_3 c c_1 \dots c_5} R_{69}^{[b_4]}_{c_1 \dots c_5} \\
& + 2 \varepsilon^{a_1 a_2 a_3 b_1 \dots b_4 c_1 \dots c_4} R_{64}^c_{c_1 \dots c_4, d} \\
& - 2 \varepsilon^{a_1 a_2 a_3 c c_1 \dots c_4} [b_1 b_2 b_3 R_{64}^{b_4}]_{c_1 \dots c_4, d} \\
& + \frac{10!}{2} \delta_d^{[a_1} R_{65}^{a_2 a_3] [b_1 b_2 b_3, b_4] c} \\
& + \frac{3(10!)}{8} \delta_d^{[b_1} R_{65}^{b_2 b_3] c [a_1 a_2, a_3] | b_4]} \\
& + \frac{27(10!)}{2} \delta_d^c R_{67}^{b_1 \dots b_4, a_1 a_2 a_3} \\
& + \frac{27(10!)}{4} \delta_d^{[b_1} R_{67}^{b_2 b_3 b_4] c, a_1 a_2 a_3} \\
& - 4!10!27 \delta_d^{[a_1} R_{67}^{b_1 b_2 b_3 b_4, | a_2 a_3] c} \\
& - 81(10!) \delta_d^{[b_1} R_{67}^{b_2 b_3 b_4] [a_1, a_2 a_3] c} \\
& - 7 \varepsilon^{a_1 a_2 a_3 c_1 \dots c_5} [b_1 b_2 b_3 R_{68}^{b_4}]_{c_1 \dots c_5, d}^c \\
& - 7 \varepsilon^{a_1 a_2 a_3 c c_1 \dots c_5} [b_1 b_2 R_{68}^{b_3 b_4}]_{c_1 \dots c_5, d} , \\
[R_1^{a_1 a_2 a_3}, R_{54}^{b_1 \dots b_6}] = & - \frac{10!81}{5} \varepsilon^{a_1 a_2 a_3 b_1 \dots b_6 d_1 d_2} R_{61}^{d_1 d_2 c_1 c_2} \\
& + \frac{(3!)^3 9!4}{5} \delta_{[c_1}^{[b_1} \varepsilon^{b_2 \dots b_6] a_1 a_2 a_3 d_1 d_2 d_3} R_{61}^{c_2]}_{d_1 d_2 d_3} \\
& - \frac{10!81}{5} \delta_{c_1 c_2}^{[b_1 b_2} \varepsilon^{b_3 \dots b_6] a_1 a_2 a_3 d_1 \dots d_4} R_{61}^{d_1 \dots d_4} \\
& + 8 \varepsilon^{a_1 a_2 a_3 b_1 \dots b_6 d_1 d_2} R_{62}^{d_1 d_2 [c_1, c_2]} \\
& + \frac{32}{5} \delta_{[c_1}^{[b_1} \varepsilon^{b_2 \dots b_6] d_1 d_2 d_3 a_1 a_2 a_3} R_{62}^{d_1 d_2 d_3, | c_2]} \\
& + \frac{2(9!)^2}{25} \varepsilon^{b_1 \dots b_6 d_1 d_2 d_3} [a_1 a_2 R_{63}^{a_3}]_{d_1 d_2 d_3 c_1 c_2} \\
& - \frac{18(4!)^3 9!}{25} \delta_{[c_1}^{[b_1} \varepsilon^{b_2 \dots b_6] d_1 \dots d_4} [a_1 a_2 R_{63}^{a_3}]_{c_2]}_{d_1 \dots d_4} \\
& + \frac{2(9!)^2}{25} \delta_{c_1 c_2}^{[b_1 b_2} \varepsilon^{b_3 \dots b_6] d_1 \dots d_5} [a_1 a_2 R_{63}^{a_3}]_{d_1 \dots d_5} \\
& - \frac{2(7!)^2}{55} \varepsilon^{b_1 \dots b_6 d_1 d_2 d_3} [a_1 a_2 R_{69}^{a_3}]_{d_1 d_2 d_3 c_1 c_2} \\
& + \frac{7!8!3}{275} \delta_{[c_1}^{[b_1} \varepsilon^{b_2 \dots b_6] d_1 \dots d_4} [a_1 a_2 R_{69}^{a_3}]_{c_2]}_{d_1 \dots d_4} \\
& - \frac{2(7!)^2}{55} \delta_{c_1 c_2}^{[b_1 b_2} \varepsilon^{b_3 \dots b_6] d_1 \dots d_5} [a_1 a_2 R_{69}^{a_3}]_{d_1 \dots d_5}
\end{aligned}$$

$$\begin{aligned}
& - \frac{3(4!)}{5} \varepsilon^{b_1 \dots b_6 d_1 d_2 d_3} [a_1 a_2 R_{d_1 d_2 d_3}^{64}{}^{a_3}]_{[c_1, c_2]} \\
& + \frac{9(3!)^2}{25} \delta_{[c_1]}^{[b_1} \varepsilon^{b_2 \dots b_6] d_1 \dots d_4} [a_1 a_2 R_{d_1 \dots d_4}^{64}{}^{a_3}] \\
& + \varepsilon^{b_1 \dots b_6 d_1 d_2 d_3} [a_1 a_2 R_{d_1 d_2 d_3, c_1 c_2}^{66}{}^{a_3}] \\
& - 21 \varepsilon^{b_1 \dots b_6 d_1 \dots d_4} [a_1 R_{d_1 \dots d_4}^{68}{}^{a_2 a_3}]_{[c_1, c_2]} \\
& + \frac{21(4!)}{275} \delta_{[c_1]}^{[b_1} \varepsilon^{b_2 \dots b_6] d_1 \dots d_5} [a_1 R_{d_1 \dots d_5, [c_2]}^{68}{}^{a_2 a_3}]. \tag{2.1.26}
\end{aligned}$$

Similar to level 5, the negative level commutators may be found using the action of the Cartan involution as given in equation (1.3.15), and we have completed the level 6 algebra. We note that we have found commutators for level 5 and 6 with level ± 1 , but it is easy to find the commutators with the higher level generators simply by using multiple commutators.

In the introduction, it was mentioned that the supergravity theories occur at low level (specifically, up to level 3 in 11D) from the E_{11} viewpoint, so one may ask why we have calculated the higher level commutators. It was additionally mentioned that the reason for the higher level fields appearing in E theory is not yet clear, but it is believed that they are simply gauge choices, or that they actually are just dual fields to the known fields of supergravity. It is with this in mind that the higher level commutators have been derived; the author is hopeful that the commutators may be used to explore the possible reasons for higher level generators appearing.

This completes the derivation of the level 5 and 6 algebra in the 11D decomposition of E_{11} . We now calculate the Cartan-Killing metric in 11D up to level 4.

2.1.5 Cartan-Killing metric

In this section we find the Cartan-Killing metric of E_{11} in 11 dimensions up to level 4, which may be used to raise and lower indices in the 11D theory. In order to calculate the metric, we begin by writing a general form of terms that may appear in the metric which are compatible with $SL(11)$, in a similar way as discussed in the derivation of the level 5 and 6 algebra of the 11D decomposition of E_{11} . During this process, we must

check the representation theory of the metric and that the constraints are satisfied. Once these properties were satisfied, the following invariant relation was applied to find the Cartan-Killing metric

$$([R^\alpha, R^\beta], R^\kappa) - (R^\alpha, [R^\beta, R^\kappa]) = 0, \quad (2.1.27)$$

where the $[,]$ denote the usual commutators, the $(,)$ denote the action of the metric, and R^α represents the generators of the adjoint representation of E_{11} . We then use the constraints of the generators and equation (2.1.27) to find the explicit form of the metric.

At level 0, we find

$$(K^a_b, K^c_d) = \delta_b^c \delta_d^a - \frac{1}{2} \delta_b^a \delta_d^c. \quad (2.1.28)$$

Then at level 1, we get

$$(R_1^{a_1 a_2 a_3}, R_{-1 b_1 b_2 b_3}) = 3! \delta_{b_1 b_2 b_3}^{a_1 a_2 a_3}. \quad (2.1.29)$$

At level 2, we find the result

$$(R_2^{a_1 \dots a_6}, R_{-2 b_1 \dots b_6}) = -180 \delta_{b_1 \dots b_6}^{a_1 \dots a_6}. \quad (2.1.30)$$

At level 3, we must apply constraints. Using these and (2.1.27), we find

$$(R_3^{a_1 \dots a_8, a}, R_{-3 b_1 \dots b_8, b}) = 2 \times 7! \delta_{b_1 \dots b_8, b}^{a_1 \dots a_8, a}, \quad (2.1.31)$$

where $\delta_{b_1 \dots b_8, b}^{a_1 \dots a_8, a} = \delta_{b_1 \dots b_8}^{a_1 \dots a_8} \delta_b^a - \delta_{b_1 \dots b_8 b}^{a_1 \dots a_8 a}$. Finally, at level 4, after applying the constraints and using (2.1.27), we get

$$\begin{aligned} (R_{41}^{a_1 \dots a_{11}, b}, R_{-41 b_1 \dots b_{11}, b}) &= \frac{11!}{2^5} \delta_{b_1 \dots b_{11}}^{a_1 \dots a_{11}} \delta_b^a, \\ (R_{42}^{a_1 \dots a_{10}, (b_1 b_2)}, R_{-42 c_1 \dots c_{10}, (d_1 d_2)}) &= -\frac{11! 5}{4!} \delta_{c_1 \dots c_{10} (d_1}^{a_1 \dots a_{10} (b_1} \delta_{d_2)}^{b_2)} \\ &\quad - 5^2 \times 10! \delta_{[c_1 \dots c_9 (d_1}^{[a_1 \dots a_9 (b_1} \delta_{c_{10}] (d_2)}^{b_2)} \delta_{d_2]}^{a_{10}]}, \\ (R_{43}^{a_1 \dots a_9, b_1 b_2 b_3}, R_{-43 c_1 \dots c_9, d_1 d_2 d_3}) &= \frac{10! 5}{2^4} \delta_{c_1 \dots c_9}^{a_1 \dots a_9} \delta_{d_1 d_2 d_3}^{b_1 b_2 b_3} \\ &\quad + \frac{10! 9}{2^2} \delta_{c_1 \dots c_9}^{[a_1 \dots a_8 [b_1} \delta_{d_1 d_2 d_3}^{b_2 b_3] a_9]} \end{aligned}$$

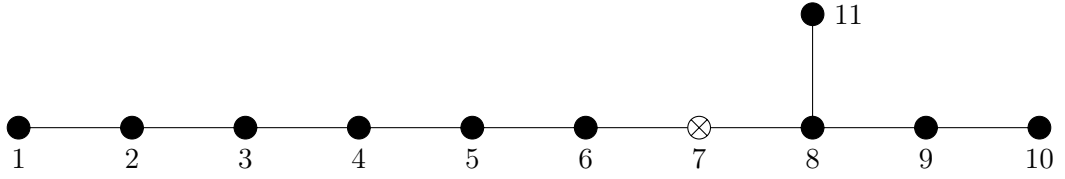


FIGURE 2.2: Dynkin diagram of E_{11} corresponding to the 7 dimensional maximal supergravity theory.

$$+ 7! \times 3^7 \delta_{[c_1 \dots c_8][d_1}^{[a_1 \dots a_8][b_1} \delta_{d_2 d_3]}^{b_2 b_3] a_9] c_9]}, \quad (2.1.32)$$

and we have found the Cartan-Killing metric up to level 4.

This completes the derivation of higher level algebra for the 11D decomposition of E_{11} . In the next section, we find the algebra of E_{11} in 7 dimensions up to level 5, beginning in the same way as in this section.

2.2 E_{11} algebra in 7 dimensions

We now derive the algebra of E_{11} in 7 dimensions. The results of this section are to be published [24]. The commutators of the positive level generators were given in [32]. These commutators were used to derive the following algebra, beginning with the negative level E_{11} generators. Then the known and the derived commutators will be used to find the commutators with the l_1 representation. Furthermore, we will then construct the $I_c(E_{11})$ algebra and the algebra of its coset, both for the adjoint representation and the algebra of the adjoint representation with the l_1 representation. The derivation is similar to the derivation of the higher level 11 dimensional algebra. We begin by giving the generators of the theory.

Deleting node seven in the E_{11} Dynkin diagram, we find the algebra $GL(7) \otimes SL(5)$. The Dynkin diagram is shown in figure 2.2, where the cross on node 7 represents the fact that this is the deleted node.

Decomposing the E_{11} algebra into representations of this algebra, the generators of level 0 up to level 6 are

$$\begin{aligned} K^a_b, R^M_N; R^{aMN}; R^{a_1a_2}_M; R^{a_1a_2a_3M}; R^{a_1\dots a_4}_{MN}; R^{a_1\dots a_5M}_N, \\ R^{a_1\dots a_4,b}; R^{a_1\dots a_6}_{MN,P}, R^{a_1\dots a_6(MN)}, R^{a_1\dots a_5,bMN}; \dots, \end{aligned} \quad (2.2.1)$$

where the lower case indices represent the $SL(7)$, such that $a, b = 1, \dots, 7$, and the upper case indices are the $SL(5)$ indices, where $M, N = 1, \dots, 5$. The indices surrounded by round brackets are symmetric in their permutation. The indices in the remaining blocks are totally antisymmetric (where a comma indicates a new antisymmetric block), and they belong to irreducible representations of $SL(7) \otimes SL(5)$ and so satisfy irreducibility constraints

$$\sum_N R^N_N = 0; \sum_N R^{a_1\dots a_5N}_N = 0, R^{[a_1\dots a_4,b]} = 0; R^{a_1\dots a_6}_{[MN,P]} = 0. \quad (2.2.2)$$

Similarly, the negative level generators are

$$\begin{aligned} R_{aMN}; R_{a_1a_2}^M; R_{a_1a_2a_3M}; R_{a_1\dots a_4MN}; R_{a_1\dots a_5M}^N, \\ R_{a_1\dots a_4,b}; R_{a_1\dots a_6}^{MN,P}, R_{a_1\dots a_6(MN)}, R_{a_1\dots a_5,bMN}; \dots, \end{aligned} \quad (2.2.3)$$

where again the lower case indices represent the $SL(7)$, such that $a, b = 1, \dots, 7$, and the upper case indices are the $SL(5)$ indices, where $M, N = 1, \dots, 5$. Additionally, the generators satisfy analogous constraints to those given in equation (2.2.2).

The l_1 generators when decomposed into $SL(7) \otimes SL(5)$ are given by

$$\begin{aligned} P_a; Z^{MN}; Z^a_M; Z^{a_1a_2M}; Z^{a_1a_2a_3}_{MN}; Z^{a_1a_2a_3,b}, Z^{a_1\dots a_4}, Z^{a_1\dots a_4M}_N; \\ Z^{a_1\dots a_5MN}, Z^{a_1\dots a_5(MN)}, Z^{a_1\dots a_5}_{MN,P}, Z^{a_1\dots a_4,bMN}; \dots, \end{aligned} \quad (2.2.4)$$

where the $SL(7)$ indices are $a, b = 1, \dots, 7$, and the $SL(5)$ indices are $M, N = 1, \dots, 5$. The generators can be given a level, which in 7 dimensions, is the number of up $SL(7)$ indices minus the number of down $SL(7)$ indices for the adjoint representation. To find the level

of the generators of the l_1 representation, one must add 1 to the number of the up minus down $SL(7)$ indices, which then gives the level of a generator. The semi-colons between the generators in equations (2.2.1), (2.2.3), and (2.2.4) represent an increase in this level.

Now we know the generators of the 7D algebra, we can derive the algebra beginning with the commutators of the adjoint representation in 7D.

2.2.1 Adjoint representation

The generators of the $GL(7)$ algebra are denoted by K^a_b and satisfy the usual commutator

$$[K^a_b, K^c_d] = \delta_b^c K^a_d - \delta_d^a K^c_b, \quad (2.2.5)$$

and similarly, the generators of the $SL(5)$ algebra, R^M_N satisfy

$$[R^M_N, R^P_Q] = \delta_N^P R^M_Q - \delta_Q^M R^P_N. \quad (2.2.6)$$

We choose the remaining generators to be irreducible representations of $SL(7) \otimes SL(5)$ and hence they satisfy the constraints in equation (2.2.2). Since the generators are representations of $SL(7)$ and $SL(5)$, the commutators with the K^a_b and R^M_N generators are determined. For example, we have at level 1

$$[K^a_b, R^{cMN}] = \delta_b^c R^{aMN}, \quad (2.2.7)$$

and at level -1, we have

$$[K^a_b, R_{cMN}] = -\delta_c^a R_{bMN}. \quad (2.2.8)$$

The other level generators follow a similar pattern in terms of how the spacetime generator acts on the upper and lower indices. Similar to the 11D case, the notion of level is additive under the commutator, so that the sum of the levels of the generators in the commutator is equivalent to the level of the generator appearing in the result of the commutator.

The action of the level 0 $GL(7)$ generator is

$$\begin{aligned}
[K^a_b, K^c_d] &= \delta_b^c K^a_d - \delta_d^a K^c_b , \\
[K^a_b, R^M_N] &= 0 , \\
[K^a_b, R^{cMN}] &= \delta_b^c R^{aMN} , \\
[K^a_b, R^{c_1 c_2}_M] &= 2\delta_b^{[c_1} R^{a|c_2]}_M , \\
[K^a_b, R^{c_1 c_2 c_3}_M] &= 3\delta_b^{[c_1} R^{a|c_2 c_3]}_M , \\
[K^a_b, R^{c_1 \dots c_4}_{PQ}] &= 4\delta_b^{[c_1} R^{a|c_2 c_3 c_4]}_{PQ} , \\
[K^a_b, R^{c_1 \dots c_5}_M] &= 5\delta_b^{[c_1} R^{a|c_2 \dots c_5]}_M , \\
[K^a_b, R^{c_1 \dots c_4, d}] &= 4\delta_b^{[c_1} R^{a|c_2 c_3 c_4, d]} + \delta_b^d R^{c_1 c_2 c_3 c_4, a} .
\end{aligned} \tag{2.2.9}$$

The action of the $SL(5)$ generator is

$$\begin{aligned}
[R^M_N, R^P_Q] &= \delta_N^P R^M_Q - \delta_Q^M R^P_N , \\
[R^M_N, R^{aPQ}] &= 2\delta_N^{[P} R^{a|M|Q]} - \frac{2}{5}\delta_N^M R^{aPQ} , \\
[R^M_N, R^{ab}_P] &= -\delta_P^M R^{ab}_N + \frac{1}{5}\delta_N^M R^{ab}_P , \\
[R^M_N, R^{a_1 a_2 a_3}_P] &= \delta_N^P R^{a_1 a_2 a_3}_M - \frac{1}{5}\delta_N^M R^{a_1 a_2 a_3}_P , \\
[R^M_N, R^{a_1 \dots a_4}_{PQ}] &= -2\delta_{[P}^M R^{a_1 \dots a_4}_{N|Q]} + \frac{2}{5}\delta_N^M R^{a_1 \dots a_4}_{PQ} , \\
[R^M_N, R^{a_1 \dots a_5}_Q] &= \delta_N^P R^{a_1 \dots a_5}_M - \delta_Q^M R^{a_1 \dots a_5}_N , \\
[R^M_N, R^{a_1 \dots a_4, b}] &= 0 .
\end{aligned} \tag{2.2.10}$$

We now begin calculating the higher level algebra up to level ± 5 . In addition, we also calculate the Cartan involution invariant algebra, and the algebra of the remaining coset generators. Similar to the 11D case, we can use the LieArt extension [50] to Mathematica to check the representation of the tensor product of two generators. With this, we can see which representations can occur in our E_{11} algebra at the correct level, and then write the most general form of the commutator with the representations that we found to appear. We then apply constraints of the generators in the commutator, which ensures that the

constraints are satisfied in the result of the commutator. Lastly, we use Jacobi identities to fix the coefficient of the generators to get the final result.

As an example, we find that tensor product of the level 1 generator with itself has the following representations in $SL(7)$

$$\{R^{aMN}\} \otimes \{R^{bPQ}\} = \mathbf{21} + 28 . \quad (2.2.11)$$

where the curly brackets represent the set of these generators in $SL(7)$, which has dimension 7. The $\mathbf{21}$ corresponds to the level 2 generator $R^{a_1 a_2}_M$, and we can discard the 28 representation, as this is not a representation of E_{11} at the relevant level. In this equation, and what follows, we write the relevant E_{11} representations in bold, and discard the remaining representations. In this example, it is not necessary to use the Jacobi to find the coefficient, as it is chosen by normalisation. The normalisation was chosen to be [32]

$$[R^{aMN}, R^{bPQ}] = \varepsilon^{MNPQR} R^{ab}_R . \quad (2.2.12)$$

We notice that up to level 4, there is only one generator at each level, so the fact that the level is additive under the action of the commutator suggests that we know what will appear, but we may still use this process as a check. However, it is useful in the derivation of the level 5 algebra, where there are two generators. For example, if we look at the commutator of the level 2 with the level 3 generator, we find

$$\{R^{a_1 a_2}_M\} \otimes \{R^{a_3 a_4 a_5 N}\} = \mathbf{21} + \mathbf{224} + 490 , \quad (2.2.13)$$

and we find that we have both level 5 generators appearing in the commutator. Once we know what appears, we write the general form of the commutator. We then apply any constraints from equation (2.2.2), before using the Jacobi to find the explicit coefficients. We use the following Jacobi to find the coefficients of the commutator of the level 2 generator with the level 3 generator

$$[[R^{a_1 a_2}_M, R^{a_3 a_4 a_5 N}], R_{a_1 P Q}] + \text{cyclic} = 0 . \quad (2.2.14)$$

With this, we find that the commutator is

$$[R^{a_1 a_2}_M, R^{a_3 a_4 a_5 N}] = R^{a_1 \dots a_5 N}_M + 2\delta_M^N R^{a_3 a_4 a_5 [a_1, a_2]} . \quad (2.2.15)$$

With the process described above, we can derive the full algebra, which we have done up to level 5. We begin giving the commutators which were already known.

In [32], the positive level commutators involving the level 1 generator were used to define the higher level generators, and the rest of the positive level commutators were found using the Jacobi identities. They are given by

$$\begin{aligned} [R^{aMN}, R^{bPQ}] &= \varepsilon^{MNPQR} R^{ab}_R , \\ [R^{aMN}, R^{b_1 b_2 P}] &= \delta_P^M R^{ab_1 b_2 N} , \\ [R^{a_1 a_2}_M, R^{a_3 a_4}_N] &= R^{a_1 \dots a_4}_{MN} , \\ [R^{a_1 MN}, R^{a_2 a_3 a_4 P}] &= \varepsilon^{MNPQR} R^{a_1 \dots a_4}_Q , \\ [R^{a_1 a_2}_M, R^{a_3 a_4 a_5 N}] &= R^{a_1 \dots a_5 N}_M + 2\delta_R^N R^{a_3 a_4 a_5 [a_1, a_2]} , \\ [R^{a_1 MN}, R^{a_2 \dots a_5 PQ}] &= -2\delta_{[P}^M R^{a_1 \dots a_5 N]}_{Q]} + \delta_{PQ}^{MN} R^{a_2 \dots a_5, a_1} . \end{aligned} \quad (2.2.16)$$

Recall the action of the Cartan involution from equation (1.3.15), and so we have that

$$\begin{aligned} I_c(K^a_b) &= -K^b_a , \\ I_c(R^M_N) &= -R^N_M , \\ I_c(R^{aMN}) &= -R^{aMN} , \\ I_c(R^{a_1 a_2}_M) &= +R_{a_1 a_2}^M , \\ I_c(R^{a_1 a_2 a_3 M}) &= -R_{a_1 a_2 a_3 M} , \\ I_c(R^{a_1 \dots a_4}_{MN}) &= +R_{a_1 \dots a_4}^{MN} , \\ I_c(R^{a_1 \dots a_5 M}_N) &= -R_{a_1 \dots a_5 M}^N , \\ I_c(R^{a_1 \dots a_4, b}) &= -R_{a_1 \dots a_4, b} . \end{aligned} \quad (2.2.17)$$

As a result the commutators of the negative level generators with themselves can be found

from those above using the Cartan involution. We calculate that the negative level generators have the commutators

$$\begin{aligned}
[R_{aMN}, R_{bPQ}] &= \varepsilon_{MNPQR} R_{ab}{}^R, \\
[R_{aMN}, R_{b_1 b_2}{}^P] &= \delta_{[M}^P R_{ab_1 b_2 N]}, \\
[R_{a_1 a_2}{}^M, R_{a_3 a_4}{}^N] &= R_{a_1 \dots a_4}{}^{MN}, \\
[R_{a_1 MN}, R_{a_2 a_3 a_4 P}] &= \varepsilon_{MNPQR} R_{a_1 \dots a_4}{}^{QR}, \\
[R_{a_1 a_2}{}^M, R_{a_3 a_4 a_5 N}] &= R_{a_1 \dots a_5 N}{}^M + 2\delta_N^R R_{a_3 a_4 a_5 [a_1, a_2]}, \\
[R_{a_1 MN}, R_{a_2 \dots a_5}{}^{PQ}] &= -2\delta_{[M}^P R_{a_1 \dots a_5 N]}{}^Q + \delta_{MN}^{PQ} R_{a_2 \dots a_5, a_1}.
\end{aligned} \tag{2.2.18}$$

We found the action of $GL(7)$ on the negative level generators to be

$$\begin{aligned}
[K^a{}_b, R_{cMN}] &= -\delta_c^a R_{bMN}, \\
[K^a{}_b, R_{cd}{}^M] &= -2\delta_{[c}^a R_{|b|d]}{}^M, \\
[K^a{}_b, R_{a_1 a_2 a_3 M}] &= -3\delta_{[a_1} R_{|b|a_2 a_3] M}, \\
[K^a{}_b, R_{a_1 \dots a_4}{}^{PQ}] &= -4\delta_{[c_1}^a R_{b|c_2 c_3 c_4]}{}^{PQ}, \\
[K^a{}_b, R_{a_1 \dots a_5 M}{}^N] &= -5\delta_{[a_1}^a R_{b|a_2 \dots a_5] M}{}^N, \\
[K^a{}_b, R_{c_1 \dots c_4, d}] &= -4\delta_{[c_1}^a R_{b|c_2 c_3 c_4], d} - \delta_d^a R_{c_1 c_2 c_3 c_4, b}.
\end{aligned} \tag{2.2.19}$$

While we find the action of $SL(5)$ on the negative level generators to be

$$\begin{aligned}
[R^M{}_N, R_{aPQ}] &= -2\delta_{[P}^M R_{a|N|Q]} + \frac{2}{5}\delta_N^M R_{aPQ}, \\
[R^M{}_N, R_{a_1 a_2}{}^P] &= \delta_N^P R_{a_1 a_2}{}^M - \frac{1}{5}\delta_N^M R_{a_1 a_2}{}^P, \\
[R^M{}_N, R_{a_1 a_2 a_3 P}] &= -\delta_P^M R_{a_1 a_2 a_3 N} + \frac{1}{5}\delta_N^M R_{a_1 a_2 a_3 P}, \\
[R^M{}_N, R_{a_1 a_2 a_3 P}] &= -\delta_P^M R_{a_1 a_2 a_3 N} + \frac{1}{5}\delta_N^M R_{a_1 a_2 a_3 P}, \\
[R^M{}_N, R_{a_1 \dots a_4}{}^{PQ}] &= 2\delta_N^{[P} R_{a_1 \dots a_4}{}^{N|Q]} - \frac{2}{5}\delta_N^M R_{a_1 \dots a_4}{}^{PQ}, \\
[R^M{}_N, R_{a_1 \dots a_5 P}{}^Q] &= -\delta_P^M R_{a_1 \dots a_5 N}{}^Q + \delta_N^Q R_{a_1 \dots a_5 P}{}^N, \\
[R^M{}_N, R_{a_1 \dots a_4, b}] &= 0.
\end{aligned} \tag{2.2.20}$$

We calculate the level one with level minus one and minus two to be

$$\begin{aligned} [R^{aMN}, R_{bPQ}] &= 4\delta_b^a \delta_{[P}^{[M} R^{N]}_{Q]} + \delta_{PQ}^{MN} (2K^a{}_b - \frac{2}{5}\delta_b^a \sum_c K^c{}_c), \\ [R^{a_1 a_2}{}_M, R_{aPQ}] &= -\varepsilon_{MPQRS} \delta_b^{[a_1} R^{a_2]RS}. \end{aligned} \quad (2.2.21)$$

The coefficients are fixed by taking arbitrary coefficients in these two equations and applying the Jacobi identity involving these generators, as done in the 11D case. Applying the Cartan involution to equation (2.2.21) we also find that

$$[R^{aMN}, R_{a_1 a_2}{}^P] = \varepsilon^{MNPQR} \delta_{[a_1}^a R_{a_2]QR}. \quad (2.2.22)$$

Using the Jacobi identity, we find

$$[R^{a_1 a_2}{}_M, R_{b_1 b_2}{}^N] = 2\delta_{b_1 b_2}^{a_1 a_2} R^N{}_M - 4\delta_M^N \delta_{[b_1}^{[a_1} K^{a_2]}_{b_2]} + \frac{4}{5}\delta_M^N \delta_{b_1 b_2}^{a_1 a_2} K^e{}_e. \quad (2.2.23)$$

For those involving level ± 3 generators, we calculate the following

$$\begin{aligned} [R^{a_1 a_2 a_3}{}_M, R_{bRS}] &= 12\delta_b^{[a_1} \delta_{[R}^M R^{a_2 a_3]}_{S]}, \\ [R_{a_1 a_2 a_3}{}_M, R^{bRS}] &= 12\delta_{[a_1}^b \delta_{[M}^R R^{a_2 a_3]}_{S]}, \\ [R^{a_1 a_2 a_3}{}_M, R_{b_1 b_2}{}^N] &= 12\delta_{b_1 b_2}^{[a_1 a_2} R^{a_3]MN}, \\ [R_{a_1 a_2 a_3}{}_N, R^{b_1 b_2}{}_M] &= 12\delta_{[a_1 a_2}^{b_1 b_2} R_{a_3]MN}, \\ [R^{a_1 a_2 a_3}{}_M, R_{b_1 b_2 b_3}{}_N] &= 4!\delta_{b_1 b_2 b_3}^{a_1 a_2 a_3} R^M{}_N + 72\delta_N^M \delta_{[b_1 b_2}^{[a_1 a_2} K^{a_3]}_{b_3]} \\ &\quad - \frac{72}{5}\delta_{b_1 b_2 b_3}^{a_1 a_2 a_3} \sum_e K^e{}_e. \end{aligned} \quad (2.2.24)$$

We get the commutators with level ± 4 generators to be

$$\begin{aligned} [R^{a_1 \dots a_4}{}_{MN}, R_{bPQ}] &= -2\varepsilon_{MNPQR} \delta_b^{[a_1} R^{a_2 a_3 a_4]R}, \\ [R_{a_1 \dots a_4}{}^{MN}, R^{bPQ}] &= -2\varepsilon^{MNPQR} \delta_{[a_1}^b R_{a_2 a_3 a_4]R}, \\ [R^{a_1 \dots a_4}{}_{MN}, R_{b_1 b_2}{}^P] &= 4!\delta_M^P \delta_{b_1 b_2}^{[a_1 a_2} R^{a_3 a_4]}{}_N, \end{aligned}$$

$$\begin{aligned}
[R_{a_1 \dots a_4}{}^{MN}, R^{b_1 b_2}{}_P] &= 4! \delta_P^M \delta_{[a_1 a_2}^{b_1 b_2} R_{a_3 a_4]}{}^N, \\
[R^{a_1 \dots a_4}{}_{MN}, R_{b_1 b_2 b_3} P] &= -4! \varepsilon_{MNPQR} \delta_{b_1 b_2 b_3}^{[a_1 a_2 a_3} R^{a_4]}{}^{PQ}, \\
[R_{a_1 \dots a_4}{}^{MN}, R^{b_1 b_2 b_3} P] &= -4! \varepsilon^{MNPQR} \delta_{[a_1 a_2 a_3}^{b_1 b_2 b_3} R_{a_4]}{}^{PQ}, \\
[R^{a_1 \dots a_4}{}_{MN}, R_{b_1 \dots b_4}{}^{PQ}] &= 4! 4 \delta_{b_1 \dots b_4}^{a_1 \dots a_4} \delta_{[M}^{[P} R^{Q]}{}_{N]} - 4! 8 \delta_{MN}^{PQ} \delta_{[b_1 b_2 b_3}^{[a_1 a_2 a_3} K^{a_4]}{}_{b_4]} \\
&\quad + \frac{4! 8}{5} \delta_{MN}^{PQ} \delta_{b_1 \dots b_4}^{a_1 \dots a_4} K^e{}_e. \tag{2.2.25}
\end{aligned}$$

Finally, the commutators at level ± 5 we computed to be

$$\begin{aligned}
[R^{a_1 \dots a_5}{}_M, R_{bPQ}] &= 20 \delta_{[P}^M \delta_b^{[a_1} R^{a_2 \dots a_5]}{}_{N|Q]} - 4 \delta_N^M \delta_b^{[a_1} R^{a_2 \dots a_5]}{}_{PQ}, \\
[R_{a_1 \dots a_5} M^N, R^{bPQ}] &= 20 \delta_M^{[P} \delta_{[a_1}^{b} R_{a_2 \dots a_5]}{}^{N|Q]} - 4 \delta_M^N \delta_{[a_1}^b R_{a_2 \dots a_5]}{}^{PQ}, \\
[R^{a_1 \dots a_5}{}_M, R_{b_1 b_2}{}^P] &= 20 \delta_N^P \delta_{b_1 b_2}^{[a_1 a_2} R^{a_3 a_4 a_5]}{}_M - 4 \delta_N^M \delta_{b_1 b_2}^{[a_1 a_2} R^{a_3 a_4 a_5]}{}_P, \\
[R_{a_1 \dots a_5} M^N, R^{b_1 b_2}{}_P] &= 20 \delta_P^N \delta_{[a_1 a_2}^{b_1 b_2} R_{a_3 a_4 a_5]}{}_M - 4 \delta_M^N \delta_{[a_1 a_2}^{b_1 b_2} R_{a_3 a_4 a_5]}{}_P, \\
[R^{a_1 \dots a_5}{}_M, R_{b_1 b_2 b_3} P] &= 5! 2 \delta_P^M \delta_{b_1 b_2 b_3}^{[a_1 a_2 a_3} R^{a_4 a_5]}{}_N - 4! 2 \delta_N^M \delta_{b_1 b_2 b_3}^{[a_1 a_2 a_3} R^{a_4 a_5]}{}_P, \\
[R_{a_1 \dots a_5} M^N, R^{b_1 b_2 b_3} P] &= 5! 2 \delta_M^P \delta_{[a_1 a_2 a_3}^{b_1 b_2 b_3} R_{a_4 a_5]}{}^N - 4! 2 \delta_M^N \delta_{[a_1 a_2 a_3}^{b_1 b_2 b_3} R_{a_4 a_5]}{}^P, \\
[R^{a_1 \dots a_5}{}_M, R_{b_1 \dots b_4}{}^{PQ}] &= 5! 4 \delta_N^{[P} \delta_{b_1 \dots b_4}^{a_1 \dots a_4} R^{a_5]}{}_{M|Q]} - 4! 4 \delta_N^M \delta_{b_1 \dots b_4}^{[a_1 \dots a_4} R^{a_5]}{}^{PQ}, \\
[R_{a_1 \dots a_5} M^N, R^{b_1 \dots b_4}{}_P] &= 5! 4 \delta_{[P}^N \delta_{[a_1 \dots a_4}^{b_1 \dots b_4} R_{a_5]}{}_{M|Q]} - 4! 4 \delta_M^N \delta_{[a_1 \dots a_4}^{b_1 \dots b_4} R_{a_5]}{}^{PQ}, \\
[R^{a_1 \dots a_5}{}_M, R_{b_1 \dots b_5}{}^P] &= -5! 4 \delta_{b_1 \dots b_5}^{a_1 \dots a_5} (\delta_Q^M R^P{}_N - \delta_N^P R^M{}_Q) \\
&\quad + 5! 4 (5 \delta_N^P \delta_Q^M - \delta_Q^P \delta_N^M) \delta_{[b_1 \dots b_4}^{[a_1 \dots a_4} K^{a_5]}{}_{b_5]} \\
&\quad - 4! 4 (5 \delta_N^P \delta_Q^M - \delta_Q^P \delta_N^M) \delta_{b_1 \dots b_5}^{a_1 \dots a_5} K^e{}_e, \\
[R^{a_1 \dots a_4, b}, R_{eMN}] &= -\frac{8}{5} \delta_c^b R^{a_1 \dots a_4}{}_{MN} + \frac{8}{5} \delta_c^{[a_1} R^{a_2 a_3 a_4]}{}_{b] MN}, \\
[R_{a_1 \dots a_4, b}, R^{cMN}] &= -\frac{8}{5} \delta_b^c R_{a_1 \dots a_4}{}^{MN} + \frac{8}{5} \delta_{[a_1}^c R_{a_2 a_3 a_4]}{}_{b] MN}, \\
[R^{a_1 \dots a_4, b}, R_{c_1 c_2}{}^M] &= -\frac{4!}{5} \delta_{c_1 c_2}^{[a_1 | b} R^{a_2 a_3 a_4]}{}_M - \frac{4!}{5} \delta_{b_1 b_2}^{[a_1 a_2} R^{a_3 a_4]}{}_{b] M}, \\
[R_{a_1 \dots a_4, b}, R^{c_1 c_2}{}_M] &= -\frac{4!}{5} \delta_{[a_1 | b}^{c_1 c_2} R_{a_2 a_3 a_4]}{}_M - \frac{4!}{5} \delta_{[a_1 a_2}^{b_1 b_2} R_{a_3 a_4]}{}_{b] M}, \\
[R^{a_1 \dots a_4, b}, R_{c_1 c_2 c_3} M] &= -\frac{(4!)^2}{10} \delta_{c_1 c_2 c_3}^{[a_1 a_2 | b} R^{a_3 a_4]}{}_M + \frac{(4!)^2}{10} \delta_{c_1 c_2 c_3}^{[a_1 a_2 a_3} R^{a_4]}{}_{b] M}, \\
[R_{a_1 \dots a_4, b}, R^{c_1 c_2 c_3} M] &= -\frac{(4!)^2}{10} \delta_{[a_1 a_2 | b}^{c_1 c_2 c_3} R_{a_3 a_4]}{}_M + \frac{(4!)^2}{10} \delta_{[a_1 a_2 a_3}^{c_1 c_2 c_3} R_{a_4]}{}_{b] M}, \\
[R^{a_1 \dots a_4, b}, R_{c_1 \dots c_4}{}^{MN}] &= -\frac{4! 8}{5} \delta_{c_1 \dots c_4}^{[a_1 a_2 a_3 | b} R^{a_4]}{}_{MN} - \frac{4! 8}{5} \delta_{c_1 \dots c_4}^{a_1 \dots a_4} R^{bMN},
\end{aligned}$$

$$\begin{aligned}
[R_{a_1 \dots a_4, b}, R^{c_1 \dots c_4}{}_{MN}] &= -\frac{4!8}{5} \delta_{[a_1 a_2 a_3] b}^{c_1 \dots c_4} R_{|a_4] MN} + -\frac{4!8}{5} \delta_{a_1 \dots a_4}^{c_1 \dots c_4} R_{bMN} , \\
[R^{a_1 \dots a_4, b}, R_{c_1 \dots c_4, d}] &= \frac{384}{5} (\delta_{c_1 \dots c_4}^{a_1 \dots a_4} K^b{}_d + \delta_{c_1 \dots c_4}^{[a_1 a_2 a_3] b} K^{[a_4]}{}_d + \delta_{[c_1 c_2 c_3] d}^{a_1 \dots a_4} K^b{}_{|c_4]} \\
&\quad + 5\delta_d^b \delta_{[c_1 c_2 c_3}^{[a_1 a_2 a_3} K^{a_4]}{}_{c_4]} - 4\delta_{[c_1 c_2 c_3] d}^{[a_1 a_2 a_3] b} K^{[a_4]}{}_{|c_4]}) \\
&\quad - \frac{4!16}{5} (\delta_{c_1 \dots c_4}^{a_1 \dots a_4} \delta_d^b + \delta_{c_1 \dots c_4}^{[a_1 a_2 a_3] b} \delta_d^{[a_4]}) K^e{}_e . \tag{2.2.26}
\end{aligned}$$

Now we have completed the algebra of the adjoint representation in the 7D decomposition of E_{11} , we can use these results and Jacobis to find the algebra of the l_1 representation.

2.2.2 l_1 representation

We now calculate the commutators of the adjoint representation with the generators of the l_1 representation. We calculate the commutators of these generators with those of $GL(7)$ to be

$$\begin{aligned}
[K^a{}_b, P_c] &= -\delta_c^a P_b + \frac{1}{2} \delta_b^a P_c , \\
[K^a{}_b, Z^{MN}] &= \frac{1}{2} \delta_b^a Z^{MN} , \\
[K^a{}_b, Z^c{}_M] &= \delta_b^c Z^a{}_M + \frac{1}{2} \delta_b^a Z^c{}_M , \\
[K^a{}_b, Z^{c_1 c_2 M}] &= 2\delta_b^{[c_1} Z^{a|c_2]M} + \frac{1}{2} \delta_b^a Z^{c_1 c_2 M} , \\
[K^a{}_b, Z^{c_1 c_2 c_3}{}_{PQ}] &= 3\delta_b^{[c_1} Z^{a|c_2 c_3]PQ} + \frac{1}{2} \delta_b^a Z^{c_1 c_2 c_3}{}_{PQ} , \\
[K^a{}_b, Z^{c_1 \dots c_4 M}{}_N] &= 4\delta_b^{[c_1} Z^{a|c_2 c_3 c_4]M}{}_N + \frac{1}{2} \delta_b^a Z^{c_1 \dots c_4 M}{}_N , \\
[K^a{}_b, Z^{c_1 c_2 c_3, d}] &= 3\delta_b^{[c_1} Z^{a|c_2 c_3], d} + \delta_b^d Z^{c_1 c_2 c_3, a} + \frac{1}{2} \delta_b^a Z^{c_1 c_2 c_3, d} , \\
[K^a{}_b, Z^{c_1 \dots c_4}] &= 4\delta_b^{[c_1} Z^{a|c_2 c_3 c_4]} + \frac{1}{2} \delta_b^a Z^{c_1 \dots c_4} . \tag{2.2.27}
\end{aligned}$$

While with the generators of $SL(5)$, we find

$$\begin{aligned}
[R^M{}_N, P_a] &= 0 , \\
[R^M{}_N, Z^{PQ}] &= 2\delta_N^P Z^{[M|Q]} - \frac{2}{5} \delta_N^M Z^{PQ} , \\
[R^M{}_N, Z^a{}_P] &= -\delta_P^M Z^a{}_N + \frac{1}{5} \delta_N^M Z^a{}_P ,
\end{aligned}$$

$$\begin{aligned}
[R^M_N, Z^{a_1 a_2 P}] &= \delta_N^P Z^{a_1 a_2 M} - \frac{1}{5} \delta_N^M Z^{a_1 a_2 P}, \\
[R^M_N, Z^{a_1 a_2 a_3 PQ}] &= -2\delta_{[P}^M Z^{a_1 a_2 a_3 N|Q]} + \frac{2}{5} \delta_N^M Z^{a_1 a_2 a_3 PQ}, \\
[R^M_N, Z^{c_1 \dots c_4 P Q}] &= \delta_N^P Z^{c_1 \dots c_4 M Q} - \delta_Q^M Z^{c_1 \dots c_4 P N}, \\
[R^M_N, Z^{c_1 c_2 c_3, d}] &= 0, \\
[R^M_N, Z^{c_1 \dots c_4}] &= 0.
\end{aligned} \tag{2.2.28}$$

The commutators of the level one E_{11} generator is used to define the normalisation of the l_1 generators such that we get the following result

$$\begin{aligned}
[R^{aMN}, P_b] &= \delta_b^a Z^{MN}, \\
[R^{aMN}, Z^{PQ}] &= -\varepsilon^{MNPQR} Z^a_R, \\
[R^{aMN}, Z^b_P] &= 2\delta_P^{[M} Z^{abN]}, \\
[R^{aMN}, Z^{b_1 b_2 P}] &= \varepsilon^{MNP RS} Z^{ab_1 b_2}_{RS}, \\
[R^{aMN}, Z^{b_1 b_2 b_3 PQ}] &= \delta_{PQ}^{MN} (Z^{b_1 b_2 b_3, a} + Z^{b_1 b_2 b_3 a}) + \delta_{[P}^{[M} Z^{ab_1 b_2 b_3 N]}_{Q]}.
\end{aligned} \tag{2.2.29}$$

Using the Jacobi identities, we find that the commutators of the level two E_{11} generators with those of l_1 are

$$\begin{aligned}
[R^{a_1 a_2 P}, P_b] &= 2\delta_b^{[a_1} Z^{a_2]}_P, \\
[R^{a_1 a_2 P}, Z^{RS}] &= 2\delta_P^{[R} Z^{a_1 a_2 S]}, \\
[R^{a_1 a_2 P}, Z^b_S] &= 2Z^{a_1 a_2 b}_{PS}, \\
[R^{a_1 a_2 P}, Z^{b_1 b_2 R}] &= -2\delta_P^R (Z^{a_1 a_2 b_1 b_2} + Z^{b_1 b_2 [a_1, a_2]}) - \frac{1}{2} Z^{a_1 a_2 b_1 b_2 R}_P.
\end{aligned} \tag{2.2.30}$$

For the commutators with the level three E_{11} generators, we get

$$\begin{aligned}
[R^{a_1 a_2 a_3 M}, P_b] &= -6\delta_b^{[a_1} Z^{a_2 a_3] M}, \\
[R^{a_1 a_2 a_3 M}, Z^{PQ}] &= -2\varepsilon^{MPQRS} Z^{a_1 a_2 a_3}_{RS}, \\
[R^{a_1 a_2 a_3 M}, Z^b_Q] &= Z^{a_1 a_2 a_3 b M}_Q + 6\delta_Q^M Z^{b[a_1 a_2, a_3]} - 6\delta_Q^M Z^{a_1 a_2 a_3 b},
\end{aligned} \tag{2.2.31}$$

and at level four, we calculate

$$\begin{aligned} [R^{a_1 \dots a_4}{}_{MN}, P_b] &= 8\delta_b^{[a_1} Z^{a_2 a_3 a_4]}{}_{MN} , \\ [R^{a_1 \dots a_4}{}_{MN}, Z^{PQ}] &= 8\delta_{MN}^{PQ} Z^{a_1 \dots a_4} + 2\delta_{[M}^{[P} Z^{a_1 \dots a_4]Q]}{}_{|N]} . \end{aligned} \quad (2.2.32)$$

At level five, we get the result

$$\begin{aligned} [R^{a_1 \dots a_5}{}_{MN}, P_c] &= 5\delta_c^{[a_1} Z^{a_2 \dots a_5]}{}_{MN} , \\ [R^{a_1 \dots a_4, b}{}_{MN}, P_c] &= 8\delta_c^{[a_1} Z^{a_2 a_3 a_4], b} + 8(\delta_c^{[a_1} Z^{a_2 a_3 a_4]b} - \delta_c^b Z^{a_1 \dots a_4}) . \end{aligned} \quad (2.2.33)$$

Using the Jacobi identities, we find the commutators of the level minus one generators with those of the l_1 representation to be

$$\begin{aligned} [R_{aMN}, P_b] &= 0 , \\ [R_{aMN}, Z^{PQ}] &= 2\delta_{MN}^{PQ} P_a , \\ [R_{aMN}, Z^b{}_P] &= -\frac{1}{2}\delta_b^a \varepsilon_{MNPQR} Z^{QR} , \\ [R_{aMN}, Z^{b_1 b_2}{}_P] &= -4\delta_{[M}^P \delta_a^{[b_1} Z^{b_2]}{}_{N]} , \\ [R_{aMN}, Z^{b_1 b_2 b_3}{}_{PQ}] &= \frac{3}{2}\varepsilon_{MNPQR} \delta_a^{[b_1} Z^{b_2 b_3]}{}_{R]} , \\ [R_{aMN}, Z^{c_1 c_2 c_3, b}] &= \frac{3}{2}(\delta_a^{[c_1} Z^{c_2 c_3], b}{}_{MN} + \delta_a^b Z^{c_1 c_2 c_3}{}_{MN}) , \\ [R_{aMN}, Z^{c_1 \dots c_4}] &= -\frac{2}{5}\delta_a^{[c_1} Z^{c_2 c_3 c_4]}{}_{MN} , \\ [R_{aMN}, Z^{c_1 \dots c_4} Q_S] &= -32(\delta_{[M}^Q \delta_a^{[c_1} Z^{c_2 c_3 c_4]}{}_{|N]} S + \frac{1}{5}\delta_S^Q \delta_a^{[c_1} Z^{c_2 c_3 c_4]}{}_{MN}) . \end{aligned} \quad (2.2.34)$$

We find the commutators of the level minus two generators to be given by

$$\begin{aligned} [R_{a_1 a_2}{}^P, P_b] &= 0 , \\ [R_{a_1 a_2}{}^P, Z^{RS}] &= 0 , \\ [R_{a_1 a_2}{}^P, Z^b{}_R] &= 2\delta_R^P \delta_{[a_1}^b P_{a_2]} , \\ [R_{a_1 a_2}{}^P, Z^{b_1 b_2}{}_R] &= -2\delta_{a_1 a_2}^{b_1 b_2} Z^{PR} , \\ [R_{a_1 a_2}{}^P, Z^{b_1 b_2 b_3}{}_{RS}] &= -6\delta_{a_1 a_2}^{[b_1 b_2]c} \delta_{[R}^P \delta_{|S]}^c Z^{b_3]} , \end{aligned}$$

$$\begin{aligned}
[R_{a_1 a_2}{}^L, Z^{b_1 \dots b_4}] &= \frac{6}{5} \delta_{a_1 a_2}^{[b_1 b_2} Z^{b_3 b_4]L}, \\
[R_{a_1 a_2}{}^L, Z^{b_1 b_2 b_3, c}] &= -3(\delta_{a_1 a_2}^{[b_1 b_2} Z^{b_3]cL} + \delta_{a_1 a_2}^{c[b_1} Z^{b_2 b_3]L}), \\
[R_{a_1 a_2}{}^L, Z^{b_1 \dots b_4 R_S}] &= 4! \delta_{a_1 a_2}^{[b_1 b_2]} (\delta_L^S Z^{[b_3 b_4]R} - \frac{1}{5} \delta_S^R Z^{[b_3 b_4]L}), \tag{2.2.35}
\end{aligned}$$

and we calculate those at level minus three to be

$$\begin{aligned}
[R_{a_1 a_2 a_3 P}, P_b] &= 0, \\
[R_{a_1 a_2 a_3 P}, Z^{RS}] &= 0, \\
[R_{a_1 a_2 a_3 P}, Z^b{}_S] &= 0, \\
[R_{a_1 a_2 a_3 M}, Z^{b_1 b_2 N}] &= -12 \delta_M^N \delta_{[a_1 a_2}^{b_1 b_2} P_{a_3]}, \\
[R_{a_1 a_2 a_3 M}, Z^{b_1 b_2 b_3}{}_{NP}] &= -3 \varepsilon_{MNPQR} \delta_{a_1 a_2 a_3}^{b_1 b_2 b_3} Z^{QR}, \\
[R_{a_1 a_2 a_3 M}, Z^{b_1 \dots b_4}] &= -\frac{(3!)^2}{5} \delta_{a_1 a_2 a_3}^{[c_1 c_2 c_3} Z^{c_4]}{}_M, \\
[R_{a_1 a_2 a_3 M}, Z^{b_1 b_2 b_3, c}] &= 9(\delta_{a_1 a_2 a_3}^{b_1 b_2 b_3} Z^c{}_M + \delta_{a_1 a_2 a_3}^{c[b_1 b_2} Z^{b_3]}{}_M), \\
[R_{a_1 a_2 a_3 M}, Z^{b_1 \dots b_4 Q_T}] &= 4 \times 4! (\delta_M^Q \delta_{a_1 a_2 a_3}^{[b_1 b_2 b_3} Z^{b_4]}{}_T - \frac{1}{5} \delta_T^Q \delta_{a_1 a_2 a_3}^{[b_1 b_2 b_3} Z^{b_4]}{}_M). \tag{2.2.36}
\end{aligned}$$

At level minus four, we calculate

$$\begin{aligned}
[R_{a_1 \dots a_4}{}^{S_1 S_2}, P_b] &= 0, \\
[R_{a_1 \dots a_4}{}^{S_1 S_2}, Z^{RS}] &= 0, \\
[R_{a_1 \dots a_4}{}^{S_1 S_2}, Z^b{}_S] &= 0, \\
[R_{a_1 \dots a_4}{}^{S_1 S_2}, Z^{b_1 b_2 N}] &= 0, \\
[R_{a_1 \dots a_4}{}^{S_1 S_2}, Z^{b_1 b_2 b_3}{}_{L_1 L_2}] &= -4! \delta_{L_1 L_2}^{S_1 S_2} P_{[a_1} \delta_{a_2 a_3 a_4]}^{b_1 b_2 b_3}, \\
[R_{a_1 \dots a_4}{}^{S_1 S_2}, Z^{b_1 \dots b_4}] &= -\frac{4!}{5} \delta_{a_1 \dots a_4}^{b_1 \dots b_4} Z^{S_1 S_2}, \\
[R_{a_1 \dots a_4}{}^{S_1 S_2}, Z^{b_1 b_2 b_3, c}] &= 0, \\
[R_{a_1 \dots a_4}{}^{S_1 S_2}, Z^{b_1 \dots b_4 R_T}] &= 4 \times 4! \delta_{a_1 \dots a_4}^{b_1 \dots b_4} (\delta_T^{[S_1} Z^{S_2]R} + \frac{1}{5} \delta_T^R Z^{S_1 S_2}). \tag{2.2.37}
\end{aligned}$$

Finally at level minus five, we find

$$\begin{aligned}
[R_{a_1 \dots a_5 R^S}, P_b] &= 0, \\
[R_{a_1 \dots a_5 R^S}, Z^{MN}] &= 0, \\
[R_{a_1 \dots a_5 R^S}, Z^b_M] &= 0, \\
[R_{a_1 \dots a_5 R^S}, Z^{b_1 b_2 N}] &= 0, \\
[R_{a_1 \dots a_5 R^S}, Z^{b_1 b_2 b_3}_{MN}] &= 0, \\
[R_{a_1 \dots a_5 M^N}, Z^{b_1 \dots b_4 R_T}] &= -4!4(\delta_M^N \delta_T^R - 5\delta_T^N \delta_M^R) \delta_{[a_1 \dots a_4}^{b_1 \dots b_4} P_{a_5]}, \\
[R_{a_1 \dots a_5 R^S}, Z^{b_1 \dots b_4}] &= 0, \\
[R_{a_1 \dots a_5 R^S}, Z^{b_1 b_2 b_3, c}] &= 0, \\
[R_{a_1 \dots a_4, b}, P_c] &= 0, \\
[R_{a_1 \dots a_4, b}, Z^{RS}] &= 0, \\
[R_{a_1 \dots a_4, b}, Z^c_S] &= 0, \\
[R_{a_1 \dots a_4, b}, Z^{b_1 b_2 N}] &= 0, \\
[R_{a_1 \dots a_4, b}, Z^{b_1 b_2 b_3}_{MN}] &= 0, \\
[R_{a_1 \dots a_4, b}, Z^{c_1 \dots c_4 M}_N] &= 0, \\
[R_{a_1 \dots a_4, b}, Z^{c_1 \dots c_4}] &= -\frac{4!2}{5}(P_b \delta_{a_1 \dots a_4}^{c_1 \dots c_4} + P_{[a_1} \delta_{|a_2 a_3 a_4]}^{c_1 \dots c_4}), \\
[R_{a_1 \dots a_4, b}, Z^{c_1 c_2 c_3, d}] &= 36(P_{[a_1} \delta_b^{[c_1} \delta_{|a_2 a_3 a_4]}^{c_2 c_3]d} + \delta_b^d P_{[a_1} \delta_{a_2 a_3 a_4]}^{c_1 c_2 c_3}). \tag{2.2.38}
\end{aligned}$$

We have calculated the algebra of the adjoint and fundamental representation of $GL(7) \otimes SL(5)$, we can now use these results to derive the algebra of the Cartan involution invariant subalgebra.

2.2.3 $I_c(E_{11})$ subalgebra

In this section, we derive the Cartan involution invariant subalgebra $I_c(E_{11})$. At level 0, this is $SO(7) \otimes SO(5)$. The Cartan involution invariant generators can be found from the

action of the Cartan involution given in equation (2.2.17) and we find them to be

$$\begin{aligned}
J^a_b &= K^a_b - K^b_a , \\
S^M_N &= R^M_N - R^N_M ; \\
S^{aMN} &= R^{aMN} - R_{aMN} ; \\
S^{a_1 a_2}_M &= R^{a_1 a_2}_M + R_{a_1 a_2}^M ; \\
S^{a_1 a_2 a_3 M} &= R^{a_1 a_2 a_3 M} - R_{a_1 a_2 a_3 M} ; \\
S^{a_1 \dots a_4}_{MN} &= R^{a_1 \dots a_4}_{MN} + R_{a_1 \dots a_4}^{MN} ; \\
S^{a_1 \dots a_5 M}_N &= R^{a_1 \dots a_5 M}_N - R_{a_1 \dots a_5 M}^N , \\
S^{a_1 \dots a_4, b} &= R^{a_1 \dots a_4, b} - R_{a_1 \dots a_4, b} .
\end{aligned} \tag{2.2.39}$$

The following commutators are found by simply inserting the commutators of the adjoint representation, as found in the previous sections.

Let's show this with a simple example. If we want to find the commutator of the level 1 generator with itself, we have

$$\begin{aligned}
[S^{aMN}, S^{bPQ}] &= [R^{aMN} - R_{aMN}, R^{bPQ} - R_{bPQ}] \\
&= [R^{aMN}, R^{bPQ}] - [R_{aMN}, R^{bPQ}] \\
&\quad - [R^{aMN}, R_{bPQ}] + [R_{aMN}, R_{bPQ}] ,
\end{aligned} \tag{2.2.40}$$

and if we then insert the relevant commutators found in the previous section, we get the result

$$[S^{aMN}, S^{bPQ}] = \varepsilon^{MNPQR} S^{ab}_R - 4\delta_b^a \delta_{[P}^{[M} S^{N]}_Q - 2\delta_{PQ}^{MN} J^a_b . \tag{2.2.41}$$

We use this process to derive the following commutators of the Cartan involution invariant subalgebra. We derived that the generators at level zero obey the commutators

$$\begin{aligned}
[J^a_b, J^c_d] &= \delta_b^c J^a_d - \delta_a^c J^b_d - \delta_a^d J^c_b + \delta_d^b J^c_a , \\
[S^M_N, S^P_Q] &= \delta_N^P S^M_Q - \delta_M^P S^N_Q - \delta_Q^M S^P_N + \delta_Q^N S^P_M ,
\end{aligned}$$

$$[J^a_b, S^M_N] = 0 . \quad (2.2.42)$$

Then we find that the level zero $I_c(E_{11})$ generators with the positive level subalgebra generators are given by

$$\begin{aligned} [J^a_b, S^{cMN}] &= \delta_b^c S^{aMN} - \delta_a^c S^{bMN} , \\ [J^a_b, S^{c_1 c_2}_M] &= 2\delta_b^{[c_1} S^{a|c_2]}_M - 2\delta_a^{[c_1} S^{b|c_2]}_M , \\ [J^a_b, S^{c_1 c_2 c_3}_M] &= 3\delta_b^{[c_1} S^{a|c_2 c_3]}_M - 3\delta_a^{[c_1} S^{b|c_2 c_3]}_M , \\ [S^M_N, S^{aPQ}] &= 2\delta_N^P S^{a|M|Q} - 2\delta_M^P S^{a|N|Q} , \\ [S^M_N, S^{ab}_P] &= -\delta_P^M S^{ab}_N + \delta_P^N S^{ab}_M , \\ [S^M_N, S^{a_1 a_2 a_3}_P] &= \delta_N^P S^{a_1 a_2 a_3}_M - \delta_M^P S^{a_1 a_2 a_3}_N . \end{aligned} \quad (2.2.43)$$

At positive level, we find that

$$\begin{aligned} [S^{aMN}, S^{bPQ}] &= \varepsilon^{MNPQR} S^{ab}_R - 4\delta_b^a \delta_{[P}^{[M} S^N]_{Q]} - 2\delta_{PQ}^{MN} J^a_b , \\ [S^{aMN}, S^{b_1 b_2}_P] &= \delta_P^{[M} S^{ab_1 b_2]N} - \varepsilon^{MNPQR} \delta_a^{[b_1} S^{b_2]QR} , \\ [S^{aMN}, S^{b_1 b_2 b_3}_P] &= 12\delta_a^{[b_1} \delta_{[M}^P S^{b_2 b_3]}_{N]} + \varepsilon^{MNP L_1 L_2} S^{ab_1 b_2 b_3}_{L_1 L_2} , \\ [S^{a_1 a_2}_M, S^{b_1 b_2}_N] &= S^{a_1 a_2 b_1 b_2}_{MN} - 2\delta_{b_1 b_2}^{a_1 a_2} S^M_N - 4\delta_N^M \delta_{[b_1}^{[a_1} J^{a_2]}_{b_2]} , \\ [S^{a_1 a_2}_M, S^{b_1 b_2 b_3}_N] &= S^{a_1 a_2 b_1 b_2 b_3}_M - 12\delta_{a_1 a_2}^{[b_1 b_2} S^{b_3]MN} , \\ [S^{a_1 a_2 a_3}_M, S^{b_1 b_2 b_3}_N] &= -4!\delta_{b_1 b_2 b_3}^{a_1 a_2 a_3} S^M_N - 4!3\delta_N^M \delta_{[b_1 b_2}^{[a_1 a_2} J^{a_3]}_{b_3]} . \end{aligned} \quad (2.2.44)$$

The completes the algebra of the Cartan involution invariant subalgebra in the 7D decomposition of E_{11} up to level 3 in the generators. We now calculate the commutators of the l_1 representation with the Cartan involution invariant subalgebra.

2.2.4 $I_c(E_{11})$ with the l_1 representation

In this section, we give the commutators with the $I_c(E_{11})$ generators with those of the l_1 representation. At level zero, we calculated that

$$\begin{aligned}
[J^a{}_b, P_c] &= -\delta_c^a P_b + \delta_c^b P_a, \\
[J^a{}_b, Z^{MN}] &= 0, \\
[J^a{}_b, Z^c{}_M] &= \delta_b^c Z^a{}_M - \delta_a^c Z^b{}_M, \\
[J^a{}_b, Z^{c_1 c_2 M}] &= 2\delta_b^{[c_1} Z^{a|c_2]M} - 2\delta_a^{[c_1} Z^{b|c_2]M}, \\
[S^M{}_N, P_a] &= 0, \\
[S^M{}_N, Z^{PQ}] &= 2\delta_N^{[P} Z^{M|Q]} - 2\delta_M^{[P} Z^{N|Q]}. \tag{2.2.45}
\end{aligned}$$

We find the commutators of the level 1 $I_c(E_{11})$ invariant subalgebra with the l_1 representation to be

$$\begin{aligned}
[S^{aMN}, P_b] &= \delta_b^a Z^{MN}, \\
[S^{aMN}, Z^{PQ}] &= -\varepsilon^{MNPQR} Z^a{}_R - 2\delta_{MN}^{PQ} P_a, \\
[S^{aMN}, Z^b{}_P] &= 2\delta_P^{[M} Z^{abN]} + \frac{1}{2}\delta_b^a \varepsilon_{MNPQR} Z^{QR}, \\
[S^{aMN}, Z^{b_1 b_2 P}] &= \varepsilon^{MNP RS} Z^{ab_1 b_2}{}_{RS} + 4\delta_{[M}^P \delta_a^{[b_1} Z^{b_2]}{}_N], \\
[S^{aMN}, Z^{b_1 b_2 b_3}{}_{PQ}] &= \delta_{PQ}^{MN} (Z^{b_1 b_2 b_3, a} + Z^{b_1 b_2 b_3 a}) + \delta_{[P}^{[M} Z^{ab_1 b_2 b_3|N]}{}_Q] \\
&\quad - \frac{3}{2}\varepsilon_{MNPQR} \delta_a^{[b_1} Z^{b_2 b_3]R}, \\
[S^{aMN}, Z^{c_1 c_2 c_3, b}] &= -\frac{3}{2}(\delta_a^{[c_1} Z^{c_2 c_3]b}{}_{MN} + \delta_a^b Z^{c_1 c_2 c_3}{}_{MN}), \\
[S^{aMN}, Z^{c_1 \dots c_4}] &= \frac{2}{5}\delta_a^{[c_1} Z^{c_2 c_3 c_4]}{}_{MN}, \\
[S^{aMN}, Z^{c_1 \dots c_4} Q_S] &= 32(\delta_{[M}^Q \delta_a^{[c_1} Z^{c_2 c_3 c_4]}{}_{|N]S} \\
&\quad + \frac{1}{5}\delta_S^Q \delta_a^{[c_1} Z^{c_2 c_3 c_4]}{}_{MN}). \tag{2.2.46}
\end{aligned}$$

We find that the commutators of the level 2 $I_c(E_{11})$ subalgebra with those of the l_1 representation are

$$\begin{aligned}
[S^{a_1 a_2}_P, P_b] &= 2\delta_b^{[a_1} Z^{a_2]}_P, \\
[S^{a_1 a_2}_P, Z^{RS}] &= 2\delta_P^{[R} Z^{a_1 a_2]S}], \\
[S^{a_1 a_2}_P, Z^b_S] &= 2Z^{a_1 a_2 b}_{PS} + 2\delta_S^P \delta_{[a_1}^b P_{a_2]}, \\
[S^{a_1 a_2}_P, Z^{b_1 b_2 R}] &= -2\delta_P^R (Z^{a_1 a_2 b_1 b_2} + Z^{b_1 b_2 [a_1, a_2]}) \\
&\quad - \frac{1}{2} Z^{a_1 a_2 b_1 b_2 R}_P - 2\delta_{a_1 a_2}^{b_1 b_2} Z^{PR}, \\
[S_{a_1 a_2}^P, Z^{b_1 b_2 b_3}_{RS}] &= -6\delta_{a_1 a_2}^{[b_1 b_2] \delta_{[R}^P Z^{b_3]}_S}], \\
[S^{a_1 a_2}_L, Z^{b_1 \dots b_4}] &= \frac{6}{5} \delta_{a_1 a_2}^{[b_1 b_2} Z^{b_3 b_4]L}, \\
[S^{a_1 a_2}_L, Z^{b_1 b_2 b_3, c}] &= -3(\delta_{a_1 a_2}^{[b_1 b_2} Z^{b_3]cL} + \delta_{a_1 a_2}^{c[b_1} Z^{b_2 b_3]L}), \\
[S^{a_1 a_2}_L, Z^{b_1 \dots b_4 R}_S] &= 4! \delta_{a_1 a_2}^{[b_1 b_2] (\delta_S^L Z^{b_3 b_4]R} - \frac{1}{5} \delta_S^R Z^{b_3 b_4]L}). \tag{2.2.47}
\end{aligned}$$

We calculate the commutators of the level 3 $I_c(E_{11})$ algebra with those of l_1 representation to be

$$\begin{aligned}
[S^{a_1 a_2 a_3 M}, P_b] &= -6\delta_b^{[a_1} Z^{a_2 a_3]M}, \\
[S^{a_1 a_2 a_3 M}, Z^{PQ}] &= -2\varepsilon^{MPQRS} Z^{a_1 a_2 a_3}_{RS}, \\
[S^{a_1 a_2 a_3 M}, Z^b_N] &= Z^{a_1 a_2 a_3 b M}_N + 6\delta_N^M Z^{b[a_1 a_2, a_3]} - 6\delta_N^M Z^{a_1 a_2 a_3 b}, \\
[S^{a_1 a_2 a_3 M}, Z^{b_1 b_2 N}] &= 12\delta_M^N \delta_{[a_1 a_2}^{b_1 b_2} P_{a_3]}, \\
[S^{a_1 a_2 a_3 M}, Z^{b_1 b_2 b_3}_{NP}] &= 3\varepsilon_{MNPQR} \delta_{a_1 a_2 a_3}^{b_1 b_2 b_3} Z^{QR}, \\
[S^{a_1 a_2 a_3 M}, Z^{b_1 \dots b_4}] &= \frac{(3!)^2}{5} \delta_{a_1 a_2 a_3}^{[c_1 c_2 c_3} Z^{c_4]M}, \\
[S^{a_1 a_2 a_3 M}, Z^{b_1 b_2 b_3, c}] &= -9(\delta_{a_1 a_2 a_3}^{b_1 b_2 b_3} Z^c_M + \delta_{a_1 a_2 a_3}^{c[b_1 b_2} Z^{b_3]M}), \\
[S^{a_1 a_2 a_3 M}, Z^{b_1 \dots b_4 Q}_T] &= -4! 4(\delta_M^Q \delta_{a_1 a_2 a_3}^{[b_1 b_2 b_3} Z^{b_4]_T} \\
&\quad - \frac{1}{5} \delta_T^Q \delta_{a_1 a_2 a_3}^{[b_1 b_2 b_3} Z^{b_4]_M}). \tag{2.2.48}
\end{aligned}$$

We calculate the commutators of the level 4 $I_c(E_{11})$ algebra with those of l_1 representation to be

$$\begin{aligned}
[S^{a_1 \dots a_4}_{MN}, P_b] &= 8\delta_b^{[a_1} Z^{a_2 a_3 a_4]}_{MN} , \\
[S^{a_1 \dots a_4}_{MN}, Z^{PQ}] &= 8\delta_{MN}^{PQ} Z^{a_1 \dots a_4} + 2\delta_{[M}^{[P} Z^{a_1 \dots a_4]Q]}_{|N]} , \\
[S^{a_1 \dots a_4}_{S_1 S_2}, Z^{b_1 b_2 b_3}_{L_1 L_2}] &= -4! \delta_{L_1 L_2}^{S_1 S_2} P_{[a_1} \delta_{a_2 a_3 a_4]}^{b_1 b_2 b_3} , \\
[S^{a_1 \dots a_4}_{S_1 S_2}, Z^{b_1 \dots b_4}] &= -\frac{4!}{5} \delta_{a_1 \dots a_4}^{b_1 \dots b_4} Z^{S_1 S_2} , \\
[S^{a_1 \dots a_4}_{S_1 S_2}, Z^{b_1 b_2 b_3, c}] &= 0 , \\
[S^{a_1 \dots a_4}_{S_1 S_2}, Z^{b_1 \dots b_4 R}_T] &= 4! 4 \delta_{a_1 \dots a_4}^{b_1 \dots b_4} (\delta_T^{[S_1} Z^{S_2]R} + \frac{1}{5} \delta_T^R Z^{S_1 S_2}) . \tag{2.2.49}
\end{aligned}$$

Finally, for the generators at level 5, we derive the following commutators

$$\begin{aligned}
[S^{a_1 \dots a_4, b}, P_c] &= 8\delta_c^{[a_1} Z^{a_2 a_3 a_4], b} + 8(\delta_c^{[a_1} Z^{a_2 a_3 a_4]b} - \delta_c^b Z^{a_1 \dots a_4}) , \\
[S^{a_1 \dots a_5 M}_N, P_c] &= 5\delta_c^{[a_1} Z^{a_2 \dots a_5]M}_N , \\
[S^{a_1 \dots a_5 R}_S, Z^{b_1 \dots b_4}] &= 0 , \\
[S^{a_1 \dots a_5 R}_S, Z^{b_1 b_2 b_3, c}] &= 0 , \\
[S^{a_1 \dots a_5 R}_S, Z^{b_1 \dots b_4 M}_N] &= -4! 4 (\delta_S^R \delta_N^M - 5\delta_N^R \delta_S^M) \delta_{[a_1 \dots a_4}^{b_1 \dots b_4} P_{a_5]} , \\
[S^{a_1 \dots a_4, b}, Z^{b_1 \dots b_4}] &= \frac{4! 2}{5} (P_b \delta_{a_1 \dots a_4}^{b_1 \dots b_4} + P_{[a_1} \delta_{b|a_2 a_3 a_4]}^{b_1 \dots b_4}) , \\
[S^{a_1 \dots a_4, b}, Z^{c_1 c_2 c_3, d}] &= -(3!)^2 (P_{[a_1} \delta_b^{[c_1} \delta_{a_2 a_3 a_4]}^{c_2 c_3]d} + \delta_b^d P_{[a_1} \delta_{a_2 a_3 a_4]}^{c_1 c_2 c_3}) , \\
[S^{a_1 \dots a_4, b}, Z^{b_1 \dots b_4 M}_N] &= 0 . \tag{2.2.50}
\end{aligned}$$

This completes the algebra of the l_1 representation with the $I_c(E_{11})$ representation. We can now find the remaining commutators which are those of the coset generators.

2.2.5 Algebra of the coset of $I_c(E_{11})$

We now find the algebra of the remaining generators of E_{11} , that is, the generators that behave as $I_c(T^\bullet) = -T^\bullet$ under the action of the Cartan involution. These generators are

$$\begin{aligned}
T^a_b &= K^a_b + K^b_a, \\
T^M_N &= R^M_N + R^N_M; \\
T^{aMN} &= R^{aMN} + R_{aMN}; \\
T^{a_1 a_2}_M &= R^{a_1 a_2}_M - R_{a_1 a_2}^M; \\
T^{a_1 a_2 a_3 M} &= R^{a_1 a_2 a_3 M} + R_{a_1 a_2 a_3 M}; \\
T^{a_1 \dots a_4}_{MN} &= R^{a_1 \dots a_4}_{MN} - R_{a_1 \dots a_4}^{MN}; \\
T^{a_1 \dots a_5 M}_N &= R^{a_1 \dots a_5 M}_N + R_{a_1 \dots a_5 M}^N, \\
T^{a_1 \dots a_4, b} &= R^{a_1 \dots a_4, b} + R_{a_1 \dots a_4, b}.
\end{aligned} \tag{2.2.51}$$

We now derive the algebra of these generators.

Firstly, we find at level 0

$$\begin{aligned}
[T^a_b, T^c_d] &= \delta_b^c J^a_d + \delta_a^c J^b_d - \delta_d^a J^c_b - \delta_d^b J^c_a, \\
[T^M_N, T^P_Q] &= \delta_N^P S^M_Q + \delta_M^P S^N_Q - \delta_Q^M S^P_N - \delta_Q^N S^P_M, \\
[T^a_b, T^M_N] &= 0.
\end{aligned} \tag{2.2.52}$$

Then we calculated the level 0 $I_c(E_{11})$ generators with the positive level generators to be

$$\begin{aligned}
[T^a_b, T^{cMN}] &= \delta_b^c S^{aMN} + \delta_a^c S^{bMN}, \\
[T^a_b, T^{cd}_M] &= 2\delta_b^c S^{[a|d]}_M + 2\delta_a^c S^{[b|d]}_M, \\
[T^a_b, T^{c_1 c_2 c_3 M}] &= 3\delta_b^{[c_1} S^{a|c_2 c_3]M} + 3\delta_a^{[c_1} S^{b|c_2 c_3]M}, \\
[T^M_N, T^{aPQ}] &= 2\delta_N^P S^{a|M|Q]} + 2\delta_M^P S^{a|N|Q]} - \frac{4}{5}\delta_N^M S^{aPQ}, \\
[T^M_N, T^{ab}_P] &= -\delta_P^M S^{ab}_N - \delta_P^N S^{ab}_M + \frac{2}{5}\delta_N^M S^{ab}_P, \\
[T^M_N, T^{a_1 a_2 a_3 P}] &= \delta_N^P S^{a_1 a_2 a_3 M} + \delta_M^P S^{a_1 a_2 a_3 N} - \frac{2}{5}\delta_N^M S^{a_1 a_2 a_3 P},
\end{aligned} \tag{2.2.53}$$

and for the positive level coset generators with themselves we find

$$\begin{aligned}
[T^{aMN}, T^{bPQ}] &= \varepsilon^{MNPQR} S^{ab}{}_R + \delta_b^a \delta_{[P}^{[M} S^{N]}_{Q]} + 2\delta_{PQ}^{MN} J^a{}_b, \\
[T^{aMN}, T^{b_1 b_2 P}] &= \delta_P^{[M} S^{ab_1 b_2 N]} - \varepsilon^{MNPQR} \delta_a^{[b_1} S^{b_2]QR}, \\
[T^{aMN}, T^{b_1 b_2 b_3 P}] &= -12\delta_a^{[b_1} \delta_{[M}^P S^{b_2 b_3]N]}, \\
[T^{a_1 a_2 M}, T^{b_1 b_2 N}] &= -\frac{1}{2}\delta_{b_1 b_2}^{a_1 a_2} S^M{}_N - 4\delta_N^M \delta_{[b_1}^{[a_1} J^{a_2] b_2]}, \\
[T^{a_1 a_2 M}, T^{b_1 b_2 b_3 N}] &= 12\delta_{a_1 a_2}^{[b_1 b_2} S^{b_3]MN}, \\
[T^{a_1 a_2 a_3 M}, T^{b_1 b_2 b_3 N}] &= -4!\delta_{b_1 b_2 b_3}^{a_1 a_2 a_3} S^M{}_N - 4!3\delta_N^M \delta_{[b_1 b_2}^{[a_1 a_2} J^{a_3] b_3]}. \tag{2.2.54}
\end{aligned}$$

Finally, we look at the coset generators with the l_1 representation. At level 0, we find

$$\begin{aligned}
[T^a{}_b, P_c] &= -\delta_c^a P_b - \delta_c^b P_a + \delta_b^a P_c, \\
[T^a{}_b, Z^{MN}] &= \delta_b^a Z^{MN}, \\
[T^a{}_b, Z^c{}_M] &= \delta_b^c Z^a{}_M + \delta_a^c Z^b{}_M + \delta_b^a Z^c{}_M, \\
[T^a{}_b, Z^{a_1 a_2 M}] &= 2\delta_b^{[a_1} Z^{a] a_2 M} + 2\delta_a^{[a_1} Z^{b] a_2 M} + \delta_b^a Z^{a_1 a_2 M}, \\
[T^M{}_N, P_a] &= 0, \\
[T^M{}_N, Z^{PQ}] &= 2\delta_N^{[P} Z^{M]Q} + 2\delta_M^{[P} Z^{N]Q} - \frac{4}{5}\delta_N^M Z^{PQ}, \\
[T^M{}_N, Z^a{}_P] &= -\delta_P^M Z^a{}_N - \delta_P^N Z^a{}_M + \frac{2}{5}\delta_N^M Z^a{}_P, \\
[T^M{}_N, Z^{a_1 a_2 P}] &= \delta_N^P Z^{a_1 a_2 M} + \delta_M^P Z^{a_1 a_2 N} - \frac{2}{5}\delta_N^M Z^{a_1 a_2 P}. \tag{2.2.55}
\end{aligned}$$

We calculate the commutators of the level 1 coset generators with the l_1 representation to be

$$\begin{aligned}
[T^{aMN}, P_b] &= \delta_b^a Z^{MN}, \\
[T^{aMN}, Z^{PQ}] &= -\varepsilon^{MNPQR} Z^a{}_R + 2\delta_{MN}^{PQ} P_a, \\
[T^{aMN}, Z^b{}_P] &= 2\delta_P^{[M} Z^{abN]} - \frac{1}{2}\delta_b^a \varepsilon_{MNPQR} Z^{QR}, \\
[T^{aMN}, Z^{b_1 b_2 P}] &= -4\delta_{[M}^P \delta_a^{[b_1} Z^{b_2]N]}. \tag{2.2.56}
\end{aligned}$$

Then we found the commutators of the level 2 coset generators with the l_1 representation to be

$$\begin{aligned}
[T^{a_1 a_2}_M, P_b] &= 2\delta_b^{[a_1} Z^{a_2]}_M, \\
[T^{a_1 a_2}_M, Z^{NP}] &= 2\delta_M^{[N} Z^{a_1 a_2 P]}, \\
[T^{a_1 a_2}_M, Z^b_N] &= -2\delta_N^M \delta_{[a_1}^b P_{a_2]}, \\
[T^{a_1 a_2}_M, Z^{b_1 b_2 N}] &= 2\delta_{a_1 a_2}^{b_1 b_2} Z^{MN}.
\end{aligned} \tag{2.2.57}$$

We calculate the commutators of the level 3 coset generators with the l_1 representation to be

$$\begin{aligned}
[T^{a_1 a_2 a_3 M}, P_b] &= -6\delta_b^{[a_1} Z^{a_2 a_3]M}, \\
[T^{a_1 a_2 a_3 M}, Z^{NP}] &= 0, \\
[T^{a_1 a_2 a_3 M}, Z^b_N] &= 0, \\
[T^{a_1 a_2 a_3 M}, Z^{b_1 b_2 N}] &= 12\delta_M^N \delta_{[a_1 a_2}^{b_1 b_2} P_{a_3]}.
\end{aligned} \tag{2.2.58}$$

Now we derive the commutators of the subalgebra generators with the coset generator. Note that in the following we refer to $I_c(E_{11})$ invariant generators as even, and the coset generators as odd, due to their sign under the action of the Cartan involution.

At level 0, we found

$$\begin{aligned}
[J^a_b, T^c_d] &= \delta_b^c T^a_b + \delta_d^b T^c_a - \delta_a^c T^b_d - \delta_d^a T^c_b, \\
[J^a_b, T^M_N] &= 0, \\
[S^M_N, T^c_d] &= 0, \\
[S^M_N, T^P_Q] &= \delta_N^P T^M_Q + \delta_Q^N T^P_M - \delta_M^P T^N_Q - \delta_Q^M T^P_N.
\end{aligned} \tag{2.2.59}$$

We calculated the level 0 even with the odd generators to be

$$[J^a_b, T^{cMN}] = \delta_b^c T^{aMN} - \delta_a^c T^{bMN},$$

$$\begin{aligned}
[J^a{}_b, T^{cd}{}_M] &= 2\delta_b^{[c} T^{a|d]}{}_M - 2\delta_a^{[c} T^{b|d]}{}_M, \\
[J^a{}_b, T^{c_1 c_2 c_3 M}] &= 3\delta_b^{[c_1} T^{a|c_2 c_3]M} - 3\delta_a^{[c_1} T^{b|c_2 c_3]M}, \\
[S^M{}_N, T^{aPQ}] &= 2\delta_N^{[P} T^{a|M|Q]} - 2\delta_M^{[P} T^{a|N|Q]}, \\
[S^M{}_N, T^{ab}{}_P] &= -\delta_P^M T^{ab}{}_N + \delta_P^N T^{ab}{}_M, \\
[S^M{}_N, T^{a_1 a_2 a_3 P}] &= \delta_N^P T^{a_1 a_2 a_3 M} - \delta_M^P T^{a_1 a_2 a_3 N}, \tag{2.2.60}
\end{aligned}$$

and we found that the level 1 even generators with the odd generators are

$$\begin{aligned}
[S^{aMN}, T^b{}_c] &= -\delta_c^a T^{bMN} - \delta_b^a T^{cMN}, \\
[S^{aMN}, T^P{}_Q] &= -2\delta_Q^{[M} T^{a|P|N]} - 2\delta_P^{[M} T^{a|Q|N]}, \\
[S^{aMN}, T^{bPQ}] &= \varepsilon^{MNPQR} T^{ab}{}_R + \delta_b^a \delta_P^{[M} T^{N]Q]} \\
&\quad + 2\delta_{PQ}^{MN} T^a{}_b - \frac{2}{5} \delta_{PQ}^{MN} \delta_b^a T^c{}_c, \\
[S^{aMN}, T^{b_1 b_2}{}_P] &= \delta_P^{[M} T^{ab_1 b_2 N]} + \varepsilon^{MNPQR} \delta_a^{[b_1} T^{b_2]QR}, \\
[S^{aMN}, T^{b_1 b_2 b_3 P}] &= 12\delta_a^{[b_1} \delta_{[M}^P T^{b_2 b_3]N]} . \tag{2.2.61}
\end{aligned}$$

For level 2 even generators with the odd generators, we get the result

$$\begin{aligned}
[S^{a_1 a_2}{}_M, T^b{}_c] &= -2\delta_c^{[a_1} T^{b|a_2]}{}_M - 2\delta_b^{[a_1} T^{c|a_2]}{}_M, \\
[S^{a_1 a_2}{}_M, T^N{}_P] &= -\delta_M^N T^{a_1 a_2}{}_P - \delta_M^P T^{a_1 a_2}{}_N + \frac{2}{5} \delta_P^N T^{a_1 a_2}{}_M, \\
[S^{a_1 a_2}{}_M, T^{bNP}] &= -\delta_M^{[N} T^{ba_1 a_2 P]} + \varepsilon_{MNPQR} \delta_b^{[a_1} T^{a_2]QR}, \\
[S^{a_1 a_2}{}_M, T^{b_1 b_2}{}_N] &= -\frac{1}{2} \delta_{b_1 b_2}^{a_1 a_2} T^M{}_N - 4\delta_N^M \delta_{[b_1}^{[a_1} T^{a_2] b_2]} + \frac{4}{5} \delta_M^N \delta_{b_1 b_2}^{a_1 a_2} T^c{}_c, \\
[S^{a_1 a_2}{}_M, T^{b_1 b_2 b_3 N}] &= -12\delta_{a_1 a_2}^{[b_1 b_2} T^{b_3]MN}. \tag{2.2.62}
\end{aligned}$$

Finally, we find the level 3 even generators with the odd generators to be

$$\begin{aligned}
[S^{a_1 a_2 a_3 M}, T^b{}_c] &= -\delta_c^{[a_1} T^{b|a_2 a_3]M} - \delta_b^{[a_1} T^{c|a_2 a_3]M}, \\
[S^{a_1 a_2 a_3 M}, T^N{}_P] &= -\delta_P^M T^{a_1 a_2 a_3 N} - \delta_N^M T^{a_1 a_2 a_3 P} - \frac{2}{5} \delta_P^N T^{a_1 a_2 a_3 M}, \\
[S^{a_1 a_2 a_3 M}, T^{bNP}] &= 12\delta_b^{[a_1} \delta_{[N}^M T^{a_2 a_3]P]},
\end{aligned}$$

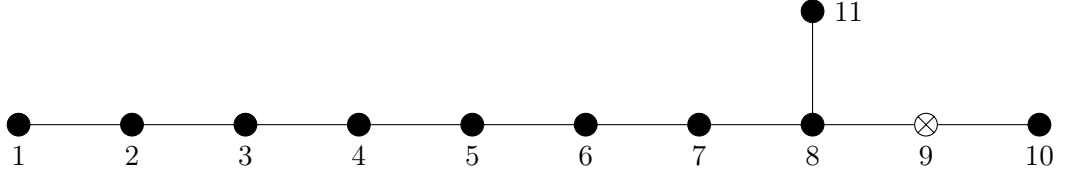


FIGURE 2.3: Dynkin diagram of E_{11} with deleted node 9 resulting in the IIB maximal supergravity theory.

$$\begin{aligned}
[S^{a_1 a_2 a_3 M}, T^{b_1 b_2 N}] &= -12 \delta_{b_1 b_2}^{[a_1 a_2} T^{a_3] MN} , \\
[S^{a_1 a_2 a_3 M}, T^{b_1 b_2 b_3 N}] &= -4! \delta_{b_1 b_2 b_3}^{a_1 a_2 a_3} T^M{}_N - 3 \times 4! \delta_N^M \delta_{[b_1 b_2}^{[a_1 a_2} T^{a_3]}_{b_3]} \\
&\quad + \frac{3!2}{5} \delta_N^M \delta_{b_1 b_2 b_3}^{a_1 a_2 a_3} T^c{}_c ,
\end{aligned} \tag{2.2.63}$$

and we are finished deriving the algebra of the 7D decomposition of E_{11} .

We now derive the Cartan involution invariant subalgebra for 10D IIB decomposition of E_{11} .

2.3 10 dimensional IIB algebra

In this section, we give the $I_c(E_{11})$ algebra of the 10 dimensional IIB theory. The algebra of the IIB theory in the $GL(10) \otimes SL(2)$ adjoint representation was given in multiple papers [26, 27], but the full algebra including the l_1 representation is given in the appendix of [51], and here we use these commutators to derive the algebra in the $I_c(E_{11})$ representation. The algebra has been given in appendix B for ease of reference. The derivation is similar to the derivation of the $I_c(E_{11})$ algebra in the 7D decomposition of E_{11} .

2.3.1 $I_c(E_{11})$ subalgebra and its coset

The Dynkin diagram of E_{11} when decomposed into the $GL(10) \otimes SL(2)$ representation is given in figure 2.3, where the cross denotes the deleted node 9.

The positive level generators up to level 4, of the 10D IIB decomposition in the $GL(10) \otimes SL(2)$ representation are

$$K^a{}_b, R_{\alpha\beta}; R_{\alpha}^{a_1 a_2}; R^{a_1 \dots a_4}; R_{\alpha}^{a_1 \dots a_6}; R_{\alpha_1 \alpha_2}^{a_1 \dots a_8}, R^{a_1 \dots a_7, b}; \dots, \quad (2.3.1)$$

where $a, b, \dots = 1, \dots, 10$ are the $SL(10)$ indices, $\alpha, \beta, \dots = 1, 2$ are the $SL(2)$ indices, and the blocks of lower case Latin indices are antisymmetric. The semi-colons between generators represent a change in level. Then the negative level generators are

$$R_{a_1 a_2}^{\alpha}; R_{a_1 \dots a_4}; R_{a_1 \dots a_6}^{\alpha}; R_{a_1 \dots a_8}^{\alpha_1 \alpha_2}, R_{a_1 \dots a_7, b}; \dots, \quad (2.3.2)$$

and similarly the $a, b, \dots = 1, \dots, 10$ are the $SL(10)$ indices, $\alpha, \beta, \dots = 1, 2$ are the $SL(2)$ indices, and the blocks of lower case Latin indices are antisymmetric. The action of the Cartan Involution given in equation (1.3.15) acts on these generators in the following way

$$\begin{aligned} I_c(K^a{}_b) &= -K^b{}_a, \\ I_c(R_{\alpha\beta}) &= \varepsilon_{\alpha}{}^{\gamma} \varepsilon_{\beta}{}^{\delta} R_{\gamma\delta}, \\ I_c(R_{\alpha}^{a_1 a_2}) &= -R_{a_1 a_2}^{\alpha}, \\ I_c(R^{a_1 \dots a_4}) &= R_{a_1 \dots a_4}, \\ I_c(R_{\alpha}^{a_1 \dots a_6}) &= -R_{a_1 \dots a_6}^{\alpha}, \\ I_c(R_{\alpha_1 \alpha_2}^{a_1 \dots a_8}) &= R_{a_1 \dots a_8}^{\alpha_1 \alpha_2}, \\ I_c(R^{a_1 \dots a_7, b}) &= R_{a_1 \dots a_7, b}. \end{aligned} \quad (2.3.3)$$

Note that $\varepsilon_{\alpha\beta}$ is an invariant tensor of $SL(2)$ and has the properties $\varepsilon_{\alpha\beta} = \varepsilon^{\alpha\beta}$ and $\varepsilon_{12} = 1$.

We then find that the Cartan involution invariant generators are

$$\begin{aligned} J^a{}_b &= K^a{}_b - K^b{}_a, \\ S &= \frac{1}{2}(R_{11} + R_{22}), \\ S_{\alpha}^{a_1 a_2} &= R_{\alpha}^{a_1 a_2} - R_{a_1 a_2}^{\alpha}, \\ S^{a_1 \dots a_4} &= R^{a_1 \dots a_4} + R_{a_1 \dots a_4}, \end{aligned}$$

$$\begin{aligned}
S_\alpha^{a_1 \dots a_6} &= R_\alpha^{a_1 \dots a_6} - R_{a_1 \dots a_6}^\alpha, \\
S_{\alpha_1 \alpha_2}^{a_1 \dots a_8} &= R_{\alpha_1 \alpha_2}^{a_1 \dots a_8} + R_{a_1 \dots a_8}^{\alpha_1 \alpha_2}, \\
S^{a_1 \dots a_7, b} &= R^{a_1 \dots a_7, b} + R_{a_1 \dots a_7, b}.
\end{aligned} \tag{2.3.4}$$

Then the generators in the coset are given by generators which transform under the action of the Cartan involution as $I_c(T^\bullet) = -T^\bullet$. We find these to be

$$\begin{aligned}
T^a_b &= K^a_b + K^b_a, \\
T_1 &= \frac{1}{2}(R_{11} - R_{22}), \\
T_2 &= R_{12}, \\
T_\alpha^{a_1 a_2} &= R_\alpha^{a_1 a_2} + R_{a_1 a_2}^\alpha, \\
T^{a_1 \dots a_4} &= R^{a_1 \dots a_4} - R_{a_1 \dots a_4}, \\
T_\alpha^{a_1 \dots a_6} &= R_\alpha^{a_1 \dots a_6} + R_{a_1 \dots a_6}^\alpha, \\
T_{\alpha_1 \alpha_2}^{a_1 \dots a_8} &= R_{\alpha_1 \alpha_2}^{a_1 \dots a_8} - R_{a_1 \dots a_8}^{\alpha_1 \alpha_2}, \\
T^{a_1 \dots a_7, b} &= R^{a_1 \dots a_7, b} - R_{a_1 \dots a_7, b}.
\end{aligned} \tag{2.3.5}$$

We shall refer to the generators in (2.3.4) as even generators, and those in (2.3.5) as odd generators.

The generators that follow can simply be found by inserting the commutators of the $GL(10) \otimes SL(2)$ into the commutators of $I_c(E_{11})$. Let's give an example of the calculation for commutator of the level 0 $GL(10)$ generator with the level 1 generator. The relevant commutators given in appendix B are

$$\begin{aligned}
[K^a_b, R_\beta^{a_1, a_2}] &= 2\delta_b^{[a_1} R_\alpha^{a]a_2}, \\
[K^a_b, R_{a_1, a_2}^\beta] &= -2\delta_{[a_1}^a R_{b]a_2}^\alpha.
\end{aligned} \tag{2.3.6}$$

So in order to find the Cartan involution invariant subalgebra, we simply use the known commutators in the following way

$$\begin{aligned} [J^a_b, S^{a_1 a_2}_\alpha] &= [K^a_b - K^b_a, R^{a_1 a_2}_\alpha - R^{a_1 a_2}_{\alpha}] \\ &= [K^a_b, R^{a_1 a_2}_\alpha] - [K^b_a, R^{a_1 a_2}_\alpha] - [K^a_b, R^{a_1 a_2}_\alpha] + [K^b_a, R^{a_1 a_2}_\alpha], \end{aligned} \quad (2.3.7)$$

and once we insert the commutators of (2.3.6), we find

$$[J^a_b, S^{a_1 a_2}_\alpha] = 2\delta_b^{[a_1} S^{a]a_2}_\alpha - 2\delta_a^{[a_1} S^{b]a_2}_\alpha. \quad (2.3.8)$$

This process is used to find the commutators of the IIB Cartan involution invariant subalgebra and its coset up to level 4. We now give the commutators of the Cartan involution invariant subalgebra.

The level 0 generators follow the usual SO(10) Lorentz algebra, and were found to be

$$\begin{aligned} [J^a_b, J^c_d] &= \delta_b^c J^a_d - \delta_a^c J^b_d - \delta_d^a J^c_b + \delta_d^b J^c_a, \\ [J^a_b, S] &= 0, \end{aligned} \quad (2.3.9)$$

and the SO(10) generator acts on the Lorentz indices as expected

$$\begin{aligned} [J^a_b, S^{a_1 a_2}_\alpha] &= 2\delta_b^{[a_1} S^{a]a_2}_\alpha - 2\delta_a^{[a_1} S^{b]a_2}_\alpha, \\ [J^a_b, S^{a_1 \dots a_4}] &= 4\delta_b^{[a_1} S^{a]a_2 a_3 a_4} - 4\delta_a^{[a_1} S^{b]a_2 a_3 a_4}, \\ [J^a_b, S^{a_1 \dots a_6}_\alpha] &= 6\delta_b^{[a_1} S^{a]a_2 \dots a_6}_\alpha - 6\delta_a^{[a_1} S^{b]a_2 \dots a_6}_\alpha, \\ [J^a_b, S^{a_1 \dots a_8}_{\alpha\beta}] &= 8\delta_b^{[a_1} S^{a]a_2 \dots a_8}_{\alpha\beta} - 8\delta_a^{[a_1} S^{b]a_2 \dots a_8}_{\alpha\beta}, \\ [J^a_b, S^{a_1 \dots a_7, c}] &= 7\delta_b^{[a_1} S^{a]a_2 \dots a_7, c} + \delta_b^c S^{a_1 \dots a_7, a} \\ &\quad - 7\delta_a^{[a_1} S^{b]a_2 \dots a_7, c} - \delta_a^c S^{a_1 \dots a_7, b}. \end{aligned} \quad (2.3.10)$$

We find the SO(2) generator S acts on the positive levels as

$$[S, S^{a_1 a_2}_\alpha] = -\frac{1}{2}\varepsilon_{\alpha\beta} S^{a_1 a_2}_\beta,$$

$$\begin{aligned}
[S, S^{a_1 \dots a_4}] &= 0, \\
[S, S_\alpha^{a_1 \dots a_6}] &= -\frac{1}{2} \varepsilon^{\alpha\beta} S_\beta^{a_1 \dots a_6}, \\
[S, S_{\alpha_1 \alpha_2}^{a_1 \dots a_8}] &= -\frac{1}{2} (\varepsilon^{\alpha_1 \beta} S_{\beta \alpha_2}^{a_1 \dots a_8} + \varepsilon^{\alpha_2 \beta} S_{\alpha_1 \beta}^{a_1 \dots a_8}), \\
[S, S^{a_1 \dots a_7, b}] &= 0.
\end{aligned} \tag{2.3.11}$$

Then we derived the algebra with the level 1 Cartan involution invariant generator to be

$$\begin{aligned}
[S_\alpha^{a_1 a_2}, S_\beta^{b_1 b_2}] &= -\varepsilon_{\alpha\beta} S^{a_1 a_2 b_1 b_2} - 4\delta_\alpha^\beta \delta_{[b_1}^{[a_1} J^{a_2]}_{b_2]} - 4\varepsilon_{\alpha\beta} \delta_{b_1 b_2}^{a_1 a_2} S, \\
[S_\alpha^{a_1 a_2}, S_\beta^{b_1 \dots b_4}] &= 4S_\alpha^{a_1 a_2 b_1 \dots b_4} + 12\varepsilon^{\alpha\beta} \delta_{a_1 a_2}^{[b_1 b_2} S_\beta^{b_3 b_4]}, \\
[S_\alpha^{a_1 a_2}, S_\beta^{b_1 \dots b_6}] &= -S_{\alpha\beta}^{a_1 a_2 b_1 \dots b_6} - \varepsilon^{\alpha\beta} S^{a_1 a_2 [b_1 \dots b_5, b_6]} - \frac{15}{2} \delta_\beta^\alpha \delta_{a_1 a_2}^{[b_1 b_2} S^{b_3 \dots b_6]}, \\
[S_\alpha^{a_1 a_2}, S_{\beta_1 \beta_2}^{b_1 \dots b_8}] &= 56\delta_{(\beta_1}^\alpha \delta_{a_1 a_2}^{[b_1 b_2} S_{\beta_2)}^{b_3 \dots b_8]}, \\
[S_\alpha^{a_1 a_2}, S^{b_1 \dots b_7, b}] &= 252\varepsilon^{\alpha\beta} \delta_{a_1 a_2}^{[b_1 b_2} S_\beta^{b_3 \dots b_7] b} - 252\varepsilon^{\alpha\beta} \delta_{a_1 a_2}^{[b_1 b_2} S_\beta^{b_3 \dots b_7 b]}.
\end{aligned} \tag{2.3.12}$$

We found the remaining commutators for the higher level even generators to be

$$\begin{aligned}
[S^{a_1 \dots a_4}, S^{b_1 \dots b_4}] &= \frac{8}{3} S^{a_1 \dots a_4 b_1 \dots b_4} - 96\delta_{[b_1 b_2 b_3}^{[a_1 a_2 a_3} J^{a_4]}_{b_4]}, \\
[S^{a_1 \dots a_4}, S_\beta^{b_1 \dots b_6}] &= 90\delta_{a_1 \dots a_4}^{[b_1 \dots b_4} S_\beta^{b_5 b_6]}, \\
[S^{a_1 \dots a_4}, S_{\alpha_1 \alpha_2}^{b_1 \dots b_8}] &= 0, \\
[S^{a_1 \dots a_4}, S^{b_1 \dots b_7, b}] &= -\frac{7!}{4} \delta_{a_1 \dots a_4}^{[b_1 \dots b_4} S^{b_5 b_6 b_7] b} + \frac{7!}{4} \delta_{a_1 \dots a_4}^{[b_1 \dots b_4} S^{b_5 b_6 b_7 b]}, \\
[S_\alpha^{a_1 \dots a_6}, S_\beta^{b_1 \dots b_6}] &= -\frac{6!3}{8} \delta_\beta^\alpha \delta_{b_1 \dots b_5}^{[a_1 \dots a_5} J^{a_6]}_{b_6]} - \frac{6!}{8} \delta_{b_1 \dots b_6}^{a_1 \dots a_6} \varepsilon^{\alpha\beta} S, \\
[S_\alpha^{a_1 \dots a_6}, S_{\beta\gamma}^{b_1 \dots b_8}] &= -1260\delta_\alpha^{(\beta} \delta_{[b_1 \dots b_6}^{a_1 \dots a_6} S_{b_7 b_8]}^{\gamma)}, \\
[S_\alpha^{a_1 \dots a_6}, S^{b_1 \dots b_7, b}] &= -1890\varepsilon_{\alpha\beta} \delta_{[b_1 \dots b_6}^{a_1 \dots a_6} S_{b_7] b}^\beta - 1890\varepsilon_{\alpha\beta} \delta_{[b_1 \dots b_6}^{a_1 \dots a_6} S_{b_7 b]}^\beta, \\
[S_{\alpha_1 \alpha_2}^{a_1 \dots a_8}, S_{\beta_1 \beta_2}^{b_1 \dots b_8}] &= -20160\delta_{\alpha_1 \alpha_2}^{(\beta_1 \beta_2)} \delta_{[b_1 \dots b_7}^{[a_1 \dots a_7} J^{a_8]}_{b_8]} \\
&\quad + 5040\delta_{b_1 \dots b_8}^{a_1 \dots a_8} \delta_{(\alpha_1 \alpha_2)}^{\beta_1 \beta_2} S, \\
[S_{\alpha\beta}^{a_1 \dots a_8}, S^{b_1 \dots b_7, b}] &= 0, \\
[S^{a_1 \dots a_7, a}, S_{b_1 \dots b_7, b}] &= -11340\delta_{b_1 \dots b_7}^{a_1 \dots a_7} J^a_b + 11340\delta_{[b_1 \dots b_7}^{a_1 \dots a_7} J^a_b] \\
&\quad + 11340\delta_{b_1 \dots b_7}^{[a_1 \dots a_7} J^a]_b - 79380\delta_b^a \delta_{[b_1 \dots b_6}^{[a_1 \dots a_6} J^{a_7]}_{b_7]}
\end{aligned}$$

$$\begin{aligned}
& + 79380\delta_{[bb_1\dots b_6]}^{a[a_1\dots a_7]J^{a_7}]_{b_7]} + 79380\delta_{b[b_1\dots b_6]}^{[aa_1\dots a_6]J^{a_7}]_{b_7]} \\
& - 90720\delta_{[b_1\dots b_7]}^{[a_1\dots a_7]J^a]_b} .
\end{aligned} \tag{2.3.13}$$

We now give the commutators of the even generators with the odd generators. These commutators result in odd generators. For the level zero $SO(10)$ commutators we found

$$\begin{aligned}
[J^a_b, T^c_d] &= \delta_b^c T^a_d + \delta_d^b T^c_a - \delta_a^c T^b_d - \delta_d^a T^c_b , \\
[J^a_b, T_1] &= 0 , \\
[J^a_b, T_2] &= 0 , \\
[J^a_b, T_\alpha^{a_1 a_2}] &= 2\delta_b^{[a_1} T_\alpha^{a]a_2} - 2\delta_a^{[a_1} T_\alpha^{b]a_2} , \\
[J^a_b, T^{a_1\dots a_4}] &= 4\delta_b^{[a_1} T^{a|a_2 a_3 a_4]} - 4\delta_a^{[a_1} T^{b|a_2 a_3 a_4]} , \\
[J^a_b, T_\alpha^{a_1\dots a_6}] &= 6\delta_b^{[a_1} T_\alpha^{a|a_2\dots a_6]} - 6\delta_a^{[a_1} T_\alpha^{b|a_2\dots a_6]} , \\
[J^a_b, T_{\alpha\beta}^{a_1\dots a_8}] &= 8\delta_b^{[a_1} T_{\alpha\beta}^{a|a_2\dots a_8]} - 8\delta_a^{[a_1} T_{\alpha\beta}^{b|a_2\dots a_8]} , \\
[J^a_b, T^{a_1\dots a_7, c}] &= 7\delta_b^{[a_1} T^{a|a_2\dots a_7], c} + \delta_b^c T^{a_1\dots a_7, a} \\
& \quad - 7\delta_a^{[a_1} T^{b|a_2\dots a_7], c} - \delta_a^c T^{a_1\dots a_7, b} .
\end{aligned} \tag{2.3.14}$$

While the commutators with level zero $SO(2)$, we calculated to be

$$\begin{aligned}
[S, T^a_b] &= 0 , \\
[S, T_1] &= -T_2 , \\
[S, T_2] &= T_1 , \\
[S, T_\alpha^{a_2 a_2}] &= -\frac{1}{2}\varepsilon_{\alpha\beta} T_\beta^{a_1 a_2} , \\
[S, T^{a_1\dots a_4}] &= 0 , \\
[S, T_\alpha^{a_1\dots a_6}] &= -\frac{1}{2}\varepsilon^{\alpha\beta} T_\beta^{a_1\dots a_6} , \\
[S, T_{\alpha_1\alpha_2}^{a_1\dots a_8}] &= -\frac{1}{2}(\varepsilon^{\alpha_1\beta} T_{\alpha_2\beta}^{a_1\dots a_8} + \varepsilon^{\alpha_2\beta} T_{\alpha_1\beta}^{a_1\dots a_8}) , \\
[S, T^{a_1\dots a_7, c}] &= 0 .
\end{aligned} \tag{2.3.15}$$

We get the result the commutators with the level one even generator to be

$$\begin{aligned}
[S_\alpha^{a_1 a_2}, T^a_b] &= -2\delta_b^{[a_1} T_\alpha^{a|a_2]} - 2\delta_a^{[a_1} T_\alpha^{b|a_2]} , \\
[S_\alpha^{a_1 a_2}, T_1] &= \frac{1}{2}(-1)^\alpha \varepsilon_{\alpha\beta} T_\beta^{a_1 a_2} , \\
[S_\alpha^{a_1 a_2}, T_2] &= -\frac{1}{2}(-1)^\alpha T_\alpha^{a_1 a_2} , \\
[S_\alpha^{a_1 a_2}, T_\beta^{b_1 b_2}] &= -\varepsilon_{\alpha\beta} T^{a_1 a_2 b_1 b_2} + 4\delta_\alpha^\beta \delta_{[b_1}^{[a_1} T^{a_2]}_{b_2]} - \frac{1}{2}\delta^{\alpha\beta} \delta_{b_1 b_2}^{a_1 a_2} T^d_d \\
&\quad + 4\delta_{b_1 b_2}^{a_1 a_2} (-1)^\alpha (-\varepsilon_{\alpha\beta} T_1 + \delta_\beta^\alpha T_2) , \\
[S_\alpha^{a_1 a_2}, T^{b_1 \dots b_4}] &= 4T_\alpha^{a_1 a_2 b_1 \dots b_4} + 12\varepsilon^{\alpha\beta} \delta_{a_1 a_2}^{[b_1 b_2} T_\beta^{b_3 b_4]} , \\
[S_\alpha^{a_1 a_2}, T_\beta^{b_1 \dots b_6}] &= -T_{\alpha\beta}^{a_1 a_2 b_1 \dots b_5, b_6} - \varepsilon^{\alpha\beta} T^{a_1 a_2 [b_1 \dots b_5, b_6]} - \frac{15}{2}\delta_\beta^\alpha \delta_{a_1 a_2}^{[b_1 b_2} T^{b_3 \dots b_6]} , \\
[S_\alpha^{a_1 a_2}, T_{\beta_1 \beta_2}^{b_1 \dots b_8}] &= 56\delta_{(\beta_1}^\alpha \delta_{a_1 a_2}^{[b_1 b_2} T_{\beta_2)}^{b_3 \dots b_8]} , \\
[S_\alpha^{a_1 a_2}, T^{b_1 \dots b_7, c}] &= 252\varepsilon^{\alpha\beta} (\delta_{a_1 a_2}^{[b_1 b_2} T_\beta^{b_3 \dots b_7]b} - \delta_{a_1 a_2}^{[b_1 b_2} T_\beta^{b_3 \dots b_7 b]}) . \tag{2.3.16}
\end{aligned}$$

For the commutators of the even level 2 generators, we find

$$\begin{aligned}
[S^{a_1 \dots a_4}, T^a_b] &= -4\delta_b^{[a_1} T^{a|a_2 a_3 a_4]} - 4\delta_a^{[a_1} T^{b|a_2 a_3 a_4]} , \\
[S^{a_1 \dots a_4}, T_1] &= 0 , \\
[S^{a_1 \dots a_4}, T_2] &= 0 , \\
[S^{a_1 \dots a_4}, T_\beta^{b_1 b_2}] &= -4\varepsilon_{\alpha\beta} T_\beta^{a_1 \dots a_4 b_1 b_2} - 12\varepsilon^{\alpha\beta} \delta_{b_1 b_2}^{[a_1 a_2} S_\beta^{a_3 a_4]} , \\
[S^{a_1 \dots a_4}, T^{b_1 \dots b_4}] &= \frac{8}{3} T^{a_1 \dots a_4 b_1 \dots b_4} + 96\delta_{[b_1 b_2 b_3}^{[a_1 a_2 a_3} T^{a_4]}_{b_4]} - 12\delta_{b_1 \dots b_4}^{a_1 \dots a_4} T^d_d , \\
[S^{a_1 \dots a_4}, T_\alpha^{b_1 \dots b_6}] &= 90\delta_{a_1 \dots a_4}^{[b_1 \dots b_4} T_\alpha^{b_5 b_6]} , \\
[S^{a_1 \dots a_4}, T_{\alpha\beta}^{b_1 \dots b_8}] &= 0 , \\
[S^{a_1 \dots a_4}, T^{b_1 \dots b_7, c}] &= -\frac{7!}{4} (\delta_{a_1 \dots a_4}^{[b_1 \dots b_4} T^{b_5 b_6 b_7]b} - \delta_{a_1 \dots a_4}^{[b_1 \dots b_4} T^{b_5 b_6 b_7 b]}) . \tag{2.3.17}
\end{aligned}$$

We calculated the commutators with the level 3 even generators to be

$$\begin{aligned}
[S_\alpha^{a_1 \dots a_6}, T^a_b] &= -6\delta_b^{[a_1} T_\alpha^{a|a_2 \dots a_6]} - 6\delta_a^{[a_1} T_\alpha^{b|a_2 \dots a_6]} , \\
[S_\alpha^{a_1 \dots a_6}, T_1] &= \frac{1}{2}(-1)^\alpha \varepsilon_{\alpha\beta} T_\beta^{a_1 \dots a_6} , \\
[S_\alpha^{a_1 \dots a_6}, T_2] &= -\frac{1}{2}(-1)^\alpha T_\alpha^{a_1 \dots a_6} ,
\end{aligned}$$

$$\begin{aligned}
[S_{\alpha}^{a_1 \dots a_6}, T_{\beta}^{b_1 b_2}] &= T_{\alpha\beta}^{a_1 \dots a_6 b_1 b_2} - \varepsilon^{\alpha\beta} T^{b_1 b_2 [a_1 \dots a_5, a_6]} - \frac{15}{2} \delta_{\beta}^{\alpha} \delta_{b_1 b_2}^{[a_1 a_2} T^{a_3 \dots a_6]}, \\
[S_{\alpha}^{a_1 \dots a_6}, T^{b_1 \dots b_4}] &= 90 \delta_{a_1 \dots a_4}^{[b_1 \dots b_4} T_{\alpha}^{b_5 b_6]}, \\
[S_{\alpha}^{a_1 \dots a_6}, T_{\beta}^{b_1 \dots b_6}] &= \frac{6!3}{8} \delta_{\beta}^{\alpha} \delta_{b_1 \dots b_5}^{[a_1 \dots a_5} T_{b_6}^{a_6]} - \frac{6!}{8} (-1)^{\alpha} \delta_{b_1 \dots b_6}^{a_1 \dots a_6} (\varepsilon_{\alpha\beta} T_1 - \delta_{\beta}^{\alpha} T_2) \\
&\quad - \frac{6!3}{2^6} \delta_{\beta}^{\alpha} \delta_{b_1 \dots b_6} T^d, \\
[S_{\alpha}^{a_1 \dots a_6}, T_{\beta\gamma}^{b_1 \dots b_8}] &= -1260 \delta_{\alpha}^{(\beta} \delta_{[b_1 \dots b_6}^{a_1 \dots a_6} T_{b_7 b_8]}^{\gamma)}, \\
[S_{\alpha}^{a_1 \dots a_6}, T^{b_1 \dots b_7, b}] &= -1890 \varepsilon_{\alpha\beta} \delta_{[b_1 \dots b_6}^{a_1 \dots a_6} T_{b_7] b}^{\beta} - 1890 \varepsilon_{\alpha\beta} \delta_{[b_1 \dots b_6}^{a_1 \dots a_6} T_{b_7 b]}^{\beta}, \\
[S_{\alpha_1 \alpha_2}^{a_1 \dots a_8}, T_{\beta_1 \beta_2}^{b_1 \dots b_8}] &= 20160 \delta_{\alpha_1 \alpha_2}^{(\beta_1 \beta_2)} \delta_{[b_1 \dots b_7}^{[a_1 \dots a_7} T_{b_8]}^{a_8]} + 2520 \delta_{\alpha_1 \alpha_2}^{(\beta_1 \beta_2)} \delta_{b_1 \dots b_8}^{a_1 \dots a_8} T^d \\
&\quad + 5040 \delta_{b_1 \dots b_8}^{a_1 \dots a_8} \delta_{(\beta_1}^{(\alpha_1} (\varepsilon_{\beta_2)}^{\alpha_2) T_1 - \delta_{\beta_2}^{\alpha_2) T_2)}, \\
[S_{\alpha\beta}^{a_1 \dots a_8}, T^{b_1 \dots b_7, b}] &= 0, \\
[S_{\alpha}^{a_1 \dots a_7, a}, T_{b_1 \dots b_7, b}] &= 11340 \delta_{b_1 \dots b_7}^{a_1 \dots a_7} T^a_b - 11340 \delta_{[b_1 \dots b_7}^{a_1 \dots a_7} T^a_b] \\
&\quad - 11340 \delta_{b_1 \dots b_7}^{[a_1 \dots a_7} T^a]_b + 79380 \delta_b^a \delta_{[b_1 \dots b_6}^{[a_1 \dots a_6} T^{a_7]}_{b_7]} \\
&\quad - 79380 \delta_{[b b_1 \dots b_6}^{[a_1 \dots a_7} T^{a_7]}_{b_7]} - 79380 \delta_b^{[a a_1 \dots a_6} T^{a_7]}_{[b_1 \dots b_6] b_7]} \\
&\quad + 90720 \delta_{[b_1 \dots b_7}^{[a_1 \dots a_7} T^a]_b} + 11340 \delta_{b_1 \dots b_7}^{a_1 \dots a_7} \delta_b^a T^d \\
&\quad - 11340 \delta_{b_1 \dots b_7 b}^{a_1 \dots a_7 a} T^d. \tag{2.3.18}
\end{aligned}$$

Finally, we give the commutators of the odd generators with themselves, beginning with the level 0 commutators, we get the result

$$\begin{aligned}
[T^a_b, T^c_d] &= \delta_b^c J^a_d - \delta_d^b J^c_a + \delta_a^c J^b_d - \delta_d^a J^c_b, \\
[T^a_b, T_1] &= 0, \\
[T^a_b, T_2] &= 0, \\
[T_1, T_2] &= S. \tag{2.3.19}
\end{aligned}$$

The action of the coset generator T^a_b acts in the same way on the Lorentz indices, so that we find

$$\begin{aligned}
[T^a_b, T_{\alpha}^{a_1 a_2}] &= 2 \delta_b^{[a_1} S_{\alpha}^{a] a_2]} + 2 \delta_a^{[a_1} S_{\alpha}^{b] a_2]}, \\
[T^a_b, T^{a_1 \dots a_4}] &= 4 \delta_b^{[a_1} S^{a] a_2 a_3 a_4]} + 4 \delta_a^{[a_1} S^{b] a_2 a_3 a_4]},
\end{aligned}$$

$$\begin{aligned}
[T_b^a, T_\alpha^{a_1 \dots a_6}] &= 6\delta_b^{[a_1} S_\alpha^{a|a_2 \dots a_6]} + 6\delta_a^{[a_1} S_\alpha^{b|a_2 \dots a_6]} , \\
[T_b^a, T_{\alpha\beta}^{a_1 \dots a_8}] &= 8\delta_b^{[a_1} S_{\alpha\beta}^{a|a_2 \dots a_8]} - 8\delta_a^{[a_1} S_{\alpha\beta}^{b|a_2 \dots a_8]} , \\
[T_b^a, T^{a_1 \dots a_7, c}] &= 7\delta_b^{[a_1} S^{a|a_2 \dots a_7], c} + \delta_b^c S^{a_1 \dots a_7, a} \\
&\quad + 7\delta_a^{[a_1} S^{b|a_2 \dots a_7], c} + \delta_a^c S^{a_1 \dots a_7, b} .
\end{aligned} \tag{2.3.20}$$

We then calculated the commutators with the T_1 and T_2 generators to be

$$\begin{aligned}
[T_1, T_\alpha^{a_1 a_2}] &= -\frac{1}{2}(-1)^\alpha \varepsilon_{\alpha\beta} S_\beta^{a_1 a_2} , \\
[T_1, T^{a_1 \dots a_4}] &= 0 , \\
[T_1, T_\alpha^{a_1 \dots a_6}] &= -\frac{1}{2}(-1)^\alpha \varepsilon_{\alpha\beta} S_\beta^{a_1 \dots a_6} , \\
[T_1, T_{\alpha_1 \alpha_2}^{a_1 \dots a_8}] &= -\frac{1}{2}((-1)^{\alpha_1} \varepsilon_{\alpha_1 \beta} S_{\beta \alpha_2}^{a_1 \dots a_8} + (-1)^{\alpha_2} \varepsilon_{\alpha_2 \beta} S_{\alpha_1 \beta}^{a_1 \dots a_8}) , \\
[T_1, T^{a_1 \dots a_7, b}] &= 0 , \\
[T_2, T_\alpha^{a_1 a_2}] &= \frac{1}{2}(-1)^\alpha S_\alpha^{a_1 a_2} , \\
[T_2, T^{a_1 \dots a_4}] &= 0 , \\
[T_2, T_\alpha^{a_1 \dots a_6}] &= \frac{1}{2}(-1)^\alpha S_\alpha^{a_1 \dots a_6} , \\
[T_2, T_{\alpha\beta}^{a_1 \dots a_8}] &= \frac{1}{2}(-1)^\alpha S_{\alpha\beta}^{a_1 \dots a_8} + \frac{1}{2}(-1)^\beta S_{\alpha\beta}^{a_1 \dots a_8} , \\
[T_2, T^{a_1 \dots a_7, b}] &= 0 .
\end{aligned} \tag{2.3.21}$$

For the commutators with the level one coset generators, we find the result

$$\begin{aligned}
[T_\alpha^{a_1 a_2}, T_\beta^{b_1 b_2}] &= -\varepsilon_{\alpha\beta} S^{a_1 a_2 b_1 b_2} + 4\delta_\alpha^\beta \delta_{[b_1}^{[a_1} J^{a_2]}_{b_2]} + 4\delta_{b_1 b_2}^{a_1 a_2} \varepsilon_{\alpha\beta} S , \\
[T_\alpha^{a_1 a_2}, T^{b_1 \dots b_4}] &= 4\varepsilon_{\alpha\beta} S_\beta^{a_1 a_2 b_1 b_2} - 12\varepsilon^{\alpha\beta} \delta_{a_1 a_2}^{[b_1 b_2} S_\beta^{b_3 b_4]} , \\
[T_\alpha^{a_1 a_2}, T_\beta^{b_1 \dots b_6}] &= -S_{\alpha\beta}^{a_1 a_2 b_1 \dots b_6} - \varepsilon^{\alpha\beta} S^{a_1 a_2 [b_1 \dots b_5, b_6]} + \frac{15}{2} \delta_\beta^\alpha \delta_{a_1 a_2}^{[b_1 b_2} S^{b_3 \dots b_6]} , \\
[T_\alpha^{a_1 a_2}, T_{\beta_1 \beta_2}^{b_1 \dots b_8}] &= -56\delta_{(\beta_1}^\alpha \delta_{a_1 a_2}^{[b_1 b_2} S_{\beta_2)}^{b_3 \dots b_8]} , \\
[T_\alpha^{a_1 a_2}, T^{b_1 \dots b_7, b}] &= -252\varepsilon^{\alpha\beta} \delta_{a_1 a_2}^{[b_1 b_2} S_\beta^{b_3 \dots b_7] b} + 252\varepsilon^{\alpha\beta} \delta_{a_1 a_2}^{[b_1 b_2} S_\beta^{b_3 \dots b_7 b]} .
\end{aligned} \tag{2.3.22}$$

Finally, we found the commutators of the higher level odd generators to be

$$\begin{aligned}
[T^{a_1 \dots a_4}, T^{b_1 \dots b_4}] &= \frac{8}{3} S^{a_1 \dots a_4 b_1 \dots b_4} + 96 \delta_{[b_1 b_2 b_3}^{[a_1 a_2 a_3} J^{a_4]}_{b_4]}, \\
[T^{a_1 \dots a_4}, T_{\beta}^{b_1 \dots b_6}] &= -90 \delta_{a_1 \dots a_4}^{[b_1 \dots b_4} S_{\beta}^{b_5 b_6]}, \\
[T^{a_1 \dots a_4}, T_{\alpha_1 \alpha_2}^{b_1 \dots b_8}] &= 0, \\
[T^{a_1 \dots a_4}, T^{b_1 \dots b_7, b}] &= \frac{7!}{4} \delta_{a_1 \dots a_4}^{[b_1 \dots b_4} S^{b_5 b_6 b_7] b} - \frac{7!}{4} \delta_{a_1 \dots a_4}^{[b_1 \dots b_4} S^{b_5 b_6 b_7 b]}, \\
[T_{\alpha}^{a_1 \dots a_6}, T_{\beta}^{b_1 \dots b_6}] &= 270 \delta_{\beta}^{\alpha} \delta_{b_1 \dots b_5}^{[a_1 \dots a_5} J^{a_6]}_{b_6}] + 90 \delta_{b_1 \dots b_6}^{a_1 \dots a_6} \varepsilon^{\alpha \beta} S, \\
[T_{\alpha}^{a_1 \dots a_6}, T_{\beta \gamma}^{b_1 \dots b_8}] &= 1260 \delta_{\alpha}^{(\beta} \delta_{[b_1 \dots b_6}^{a_1 \dots a_6} S_{b_7 b_8]}^{\gamma)}, \\
[T_{\alpha}^{a_1 \dots a_6}, T^{b_1 \dots b_7, b}] &= 1890 \varepsilon_{\alpha \beta} \delta_{[b_1 \dots b_6}^{a_1 \dots a_6} S_{b_7] b}^{\beta} + 1890 \varepsilon_{\alpha \beta} \delta_{[b_1 \dots b_6}^{a_1 \dots a_6} S_{b_7 b]}^{\beta}, \\
[T_{\alpha_1 \alpha_2}^{a_1 \dots a_8}, T_{\beta_1 \beta_2}^{b_1 \dots b_8}] &= 20160 \delta_{\alpha_1 \alpha_2}^{(\beta_1 \beta_2)} \delta_{[b_1 \dots b_7}^{[a_1 \dots a_7} J^{a_8]}_{b_8]} \\
&\quad - 5040 \delta_{b_1 \dots b_8}^{a_1 \dots a_8} \delta_{(\alpha_1 \alpha_2)}^{\beta_1 \beta_2} S, \\
[T_{\alpha \beta}^{a_1 \dots a_8}, T^{b_1 \dots b_7, b}] &= 0, \\
[T^{a_1 \dots a_7, a}, T_{b_1 \dots b_7, b}] &= 11340 \delta_{b_1 \dots b_7}^{a_1 \dots a_7} J^a_b - 11340 \delta_{[b_1 \dots b_7}^{a_1 \dots a_7} J^a_{b]} \\
&\quad - 11340 \delta_{b_1 \dots b_7}^{[a_1 \dots a_7} J^a]_b + 79380 \delta_b^a \delta_{[b_1 \dots b_6}^{[a_1 \dots a_6} J^{a_7]}_{b_7]} \\
&\quad - 79380 \delta_{[b b_1 \dots b_6}^{[a_1 \dots a_7} J^{a_7]}_{b_7]} - 79380 \delta_{b [b_1 \dots b_6}^{[a a_1 \dots a_6} J^{a_7]}_{b_7]} \\
&\quad + 90720 \delta_{[b_1 \dots b_7}^{[a_1 \dots a_7} J^a]_b}. \tag{2.3.23}
\end{aligned}$$

This completes the algebra of the $I_c(E_{11})$ algebra with itself and its coset. We now derive the algebra of $I_c(E_{11})$ with the l_1 generators.

2.3.2 $I_c(E_{11})$ with the l_1 representation

We now give the algebra of the $I_c(E_{11})$ with the l_1 representation for the IIB theory. The generators of the l_1 representation of the 10D IIB algebra are

$$P_a; Z_{\alpha}^a; Z^{a_1 a_2 a_3}; Z_{\alpha}^{a_1 \dots a_5}; Z_{\alpha \beta}^{a_1 \dots a_7}, Z^{a_1 \dots a_7}, Z^{a_1 \dots a_6, b} \dots, \tag{2.3.24}$$

where $a, b, a_1, \dots = 1, \dots, 10$ are the $SL(10)$ indices, $\alpha, \beta, \dots = 1, 2$ are the $SL(2)$ indices, and the blocks of lower case Latin indices are antisymmetric. The action of the $SO(10)$

generator acts in the usual way as we found

$$\begin{aligned}
[J^a_b, P_c] &= -\delta_c^a P_b - \delta_c^b P_a , \\
[J^a_b, Z^c_\alpha] &= \delta_b^c Z^a_\alpha - \delta_a^c Z^b_\alpha , \\
[J^a_b, Z^{a_1 a_2 a_3}] &= 3\delta_b^{[a_1} Z^{a|a_2 a_3]} - 3\delta_a^{[a_1} Z^{b|a_2 a_3]} , \\
[J^a_b, Z^{a_1 \dots a_5}] &= 5\delta_b^{[a_1} Z^{a|a_2 \dots a_5]} - 5\delta_a^{[a_1} Z^{b|a_2 \dots a_5]} , \\
[J^a_b, Z^{a_1 \dots a_7}_\alpha] &= 7\delta_b^{[a_1} Z^{a_1 \dots a_7}_\alpha - 7\delta_a^{[a_1} Z^{b|a_2 \dots a_7}_\alpha] , \\
[J^a_b, Z^{a_1 \dots a_7}] &= 7\delta_b^{[a_1} Z^{a|a_2 \dots a_7]} - 7\delta_a^{[a_1} Z^{b|a_2 \dots a_7]} , \\
[J^a_b, Z^{a_1 \dots a_6, c}] &= 6\delta_b^{[a_1} Z^{a|a_2 \dots a_6], c} + \delta_b^c Z^{a_1 \dots a_6, a} \\
&\quad - 6\delta_a^{[a_1} Z^{b|a_2 \dots a_7], c} - \delta_a^c Z^{a_1 \dots a_6, b} , \tag{2.3.25}
\end{aligned}$$

and the SO(2) generator S

$$\begin{aligned}
[S, P_a] &= 0 , \\
[S, Z^a_\alpha] &= -\frac{1}{2}\varepsilon_{\alpha\beta} Z^a_\beta , \\
[S, Z^{a_1 a_2 a_3}] &= 0 , \\
[S, Z^{a_1 \dots a_5}_\alpha] &= -\frac{1}{2}\varepsilon_{\alpha\beta} Z^{a_1 \dots a_5}_\alpha , \\
[S, Z^{a_1 \dots a_7}_\alpha] &= -\frac{1}{2}(\varepsilon_{\alpha\kappa} Z^{a_1 \dots a_7}_{\kappa\beta} + \varepsilon_{\beta\kappa} Z^{a_1 \dots a_7}_{\alpha\kappa}) , \\
[S, Z^{a_1 \dots a_7}] &= 0 , \\
[S, Z^{a_1 \dots a_6, b}] &= 0 . \tag{2.3.26}
\end{aligned}$$

We found the commutators of level one even generators with l_1 to be

$$\begin{aligned}
[S^{a_1 a_2}_\alpha, P_b] &= \delta_b^{[a_1} Z^{a_2]}_\alpha , \\
[S^{a_1 a_2}_\alpha, Z^b_\beta] &= -\varepsilon_{\alpha\beta} Z^{a_1 a_2 b} + 4\delta_\beta^\alpha \delta_{[a_1}^b P_{a_2]} , \\
[S^{a_1 a_2}_\alpha, Z^{b_1 b_2 b_3}] &= Z^{a_1 a_2 b_1 b_2 b_3} + 6\varepsilon^{\alpha\beta} \delta_{a_1 a_2}^{[b_1 b_2} Z^{b_3]}_\beta , \\
[S^{a_1 a_2}_\alpha, Z^{b_1 \dots b_5}_\beta] &= Z^{a_1 a_2 b_1 \dots b_5}_\alpha - \varepsilon_{\alpha\beta} Z^{a_1 a_2 b_1 \dots b_5} \\
&\quad - \varepsilon_{\alpha\beta} Z^{a_1 a_2 [b_1 \dots b_4, b_5]} - 20\delta_\beta^\alpha \delta_{a_1 a_2}^{[b_1 b_2} Z^{b_3 b_4 b_5]} ,
\end{aligned}$$

$$\begin{aligned}
[S_\alpha^{a_1 a_2}, Z_{\beta_1 \beta_2}^{b_1 \dots b_7}] &= -42 \delta_{(\beta_1}^\alpha \delta_{a_1 a_2}^{[b_1 b_2} Z_{\beta_1}^{b_3 \dots b_7]} , \\
[S_\alpha^{a_1 a_2}, Z^{b_1 \dots b_7}] &= 3 \varepsilon^{\alpha \beta} \delta_{a_1 a_2}^{[b_1 b_2} Z_\beta^{b_3 \dots b_7]} , \\
[S_\alpha^{a_1 a_2}, Z^{b_1 \dots b_6, b}] &= 150 \varepsilon^{\alpha \beta} \delta_{a_1 a_2}^{[b_1 b_2} Z_\beta^{b_3 \dots b_6] b} - 150 \varepsilon^{\alpha \beta} \delta_{a_1 a_2}^{[b_1 b_2} Z_\beta^{b_3 \dots b_7 b]} ,
\end{aligned} \tag{2.3.27}$$

and we calculated the level two commutators to be

$$\begin{aligned}
[S^{a_1 \dots a_4}, P_b] &= 2 \delta_b^{[a_1} Z^{a_2 a_3 a_4]} , \\
[S^{a_1 \dots a_4}, Z_\beta^b] &= -Z_\beta^{a_1 \dots a_4 b} , \\
[S^{a_1 \dots a_4}, Z^{b_1 b_2 b_3}] &= 2 Z^{a_1 \dots a_4 b_1 b_2 b_3} + \frac{3}{5} Z^{a_1 \dots a_4 [b_1 b_2, b_3]} + 48 \delta_{[a_1 a_2 a_3}^{b_1 b_2 b_3} P_{a_4]} , \\
[S^{a_1 \dots a_4}, Z_\alpha^{b_1 \dots b_5}] &= 5! \delta_{a_1 \dots a_4}^{[b_1 \dots b_4} Z_\alpha^{b_5]} , \\
[S^{a_1 \dots a_4}, Z_{\alpha \beta}^{b_1 \dots b_7}] &= 0 , \\
[S^{a_1 \dots a_4}, Z^{b_1 \dots b_7}] &= -5! \delta_{a_1 \dots a_4}^{[b_1 \dots b_4} Z^{b_5 b_6 b_7]} , \\
[S^{a_1 \dots a_4}, Z^{b_1 \dots b_6, b}] &= -\frac{6! 5}{2} (\delta_{a_1 \dots a_4}^{[b_1 \dots b_4} Z^{b_5 b_6] b} - \delta_{a_1 \dots a_4}^{[b_1 \dots b_4} Z^{b_5 b_6 b]}) .
\end{aligned} \tag{2.3.28}$$

At level three, we found

$$\begin{aligned}
[S_\alpha^{a_1 \dots a_6}, P_b] &= \frac{3}{4} \delta_a^{[a_1} Z_\alpha^{a_2 \dots a_6]} , \\
[S_\alpha^{a_1 \dots a_6}, Z_\beta^b] &= -\frac{1}{4} Z_{\alpha \beta}^{a_1 \dots a_6 b} + \frac{3}{4} \varepsilon_{\alpha \beta} Z^{a_1 \dots a_6 b} + \frac{1}{20} \varepsilon_{\alpha \beta} Z^{a_1 \dots a_6, b} , \\
[S_\alpha^{a_1 \dots a_6}, Z^{b_1 b_2 b_3}] &= 0 , \\
[S_\alpha^{a_1 \dots a_6}, Z_\beta^{b_1 \dots b_5}] &= \frac{6!}{2} \delta_\beta^\alpha \delta_{[a_1 \dots a_5}^{b_1 \dots b_5} P_{a_6]} , \\
[S_\alpha^{a_1 \dots a_6}, Z_{\beta_1 \beta_2}^{b_1 \dots b_7}] &= \frac{7!}{4} \delta_{(\beta_1}^\alpha \delta_{a_1 \dots a_6}^{[b_1 \dots b_7} Z_{\beta_2]}^{b_7]} , \\
[S_\alpha^{a_1 \dots a_6}, Z^{b_1 \dots b_7}] &= -270 \varepsilon^{\alpha \beta} \delta_{a_1 \dots a_6}^{[b_1 \dots b_6} Z_\beta^{b_7]} , \\
[S_\alpha^{a_1 \dots a_6}, Z^{b_1 \dots b_6, b}] &= -\frac{6! 5}{4} \varepsilon^{\alpha \beta} (\delta_{a_1 \dots a_6}^{b_1 \dots b_6} Z_\beta^b - \delta_{a_1 \dots a_6}^{[b_1 \dots b_6} Z_\beta^b]) .
\end{aligned} \tag{2.3.29}$$

At level four, we found

$$\begin{aligned}
[S_{\alpha \beta}^{a_1 \dots a_8}, P_a] &= -\delta_a^{[a_1} Z_{\alpha \beta}^{a_2 \dots a_8]} , \\
[S_{\alpha \beta}^{a_1 \dots a_8}, Z_\gamma^b] &= 0 , \\
[S_{\alpha \beta}^{a_1 \dots a_8}, Z^{b_1 b_2 b_3}] &= 0 ,
\end{aligned}$$

$$\begin{aligned}
[S_{\alpha\beta}^{a_1\dots a_8}, Z_\gamma^{b_1\dots b_5}] &= 0, \\
[S_{\alpha_1\alpha_2}^{a_1\dots a_8}, Z_{\beta_1\beta_2}^{b_1\dots b_7}] &= 20160\delta_{\beta_1\beta_2}^{(\alpha_1\alpha_2)}\delta_{[a_1\dots a_7]}^{b_1\dots b_7}P_{a_8}, \\
[S_{\alpha_1\alpha_2}^{a_1\dots a_8}, Z^{b_1\dots b_7}] &= 0, \\
[S_{\alpha_1\alpha_2}^{a_1\dots a_8}, Z^{b_1\dots b_6,b}] &= 0, \\
[S^{a_1\dots a_7,b}, P_a] &= -3\delta_a^b Z^{a_1\dots a_7} + 3\delta_a^{[b} Z^{a_1\dots a_7]} + \frac{21}{20}\delta_a^{[a_1} Z^{a_2\dots a_7],b}, \\
[S^{a_1\dots a_7,b}, Z_\gamma^c] &= 0, \\
[S^{a_1\dots a_7,b}, Z^{b_1b_2b_3}] &= 0, \\
[S^{a_1\dots a_7,b}, Z_\gamma^{b_1\dots b_5}] &= 0, \\
[S^{a_1\dots a_7,a}, Z_{\alpha_1\alpha_2}^{b_1\dots b_7}] &= 0, \\
[S^{a_1\dots a_7,a}, Z^{b_1\dots b_7}] &= 4320\delta_{a_1\dots a_7}^{b_1\dots b_7}P_a - 4320\delta_{[a_1\dots a_7]}^{b_1\dots b_7}P_a, \\
[S^{a_1\dots a_7,a}, Z^{b_1\dots b_6,b}] &= -75600\delta_a^b\delta_{[a_1\dots a_6]}^{b_1\dots b_6}P_{a_7} + 75600\delta_{a[a_1\dots a_6]}^{bb_1\dots b_6}P_{a_7}. \tag{2.3.30}
\end{aligned}$$

Finally, we give the commutators of the odd generators with the l_1 representation. We found the generators at level 0 act as

$$\begin{aligned}
[T_b^a, P_c] &= -\delta_c^a P_b - \delta_c^b P_a + \delta_b^a P_c, \\
[T_b^a, Z_\alpha^c] &= \delta_b^c Z_\alpha^a + \delta_a^c Z_\alpha^b + \delta_b^a Z_\alpha^c, \\
[T_b^a, Z^{a_1a_2a_3}] &= 3\delta_b^{[a_1} Z^{a|a_2a_3]} + 3\delta_a^{[a_1} Z^{b|a_2a_3]} + \delta_b^a Z^{a_1a_2a_3}, \\
[T_b^a, Z_\alpha^{a_1\dots a_5}] &= 5\delta_b^{[a_1} Z_\alpha^{a|a_2\dots a_5]} + 5\delta_a^{[a_1} Z_\alpha^{b|a_2\dots a_5]} + \delta_b^a Z_\alpha^{a_1\dots a_5}, \\
[T_b^a, Z_{\alpha\beta}^{a_1\dots a_7}] &= 7\delta_b^{[a_1} Z_{\alpha\beta}^{a|a_2\dots a_7]} + 7\delta_a^{[a_1} Z_{\alpha\beta}^{b|a_2\dots a_7]} + \delta_b^a Z_{\alpha\beta}^{a_1\dots a_7}, \\
[T_b^a, Z^{a_1\dots a_7}] &= 7\delta_b^{[a_1} Z^{a|a_2\dots a_7]} + 7\delta_a^{[a_1} Z^{b|a_2\dots a_7]} + \delta_b^a Z^{a_1\dots a_7}, \\
[T_b^a, Z^{a_1\dots a_6,c}] &= 6\delta_b^{[a_1} Z^{a|a_2\dots a_6],c} + \delta_b^c Z^{a_1\dots a_6,a} \\
&\quad + 6\delta_a^{[a_1} Z^{b|a_2\dots a_6],c} + \delta_a^c Z^{a_1\dots a_6,b}, \tag{2.3.31}
\end{aligned}$$

and

$$\begin{aligned}
[T_1, P_a] &= 0, \\
[T_1, Z_\alpha^a] &= -\frac{1}{2}(-1)^\alpha \varepsilon_{\alpha\beta} Z_\beta^a,
\end{aligned}$$

$$\begin{aligned}
[T_1, Z^{a_1 a_2 a_3}] &= 0, \\
[T_1, Z_\alpha^{a_1 \dots a_5}] &= -\frac{1}{2}(-1)^\alpha \varepsilon_{\alpha\beta} Z_\beta^{a_1 \dots a_5}, \\
[T_1, Z_{\alpha\beta}^{a_1 \dots a_7}] &= -\frac{1}{2}((-1)^\alpha \varepsilon_{\alpha\gamma} Z_{\gamma\beta}^{a_1 \dots a_7} + (-1)^\beta \varepsilon_{\beta\gamma} Z_{\alpha\gamma}^{a_1 \dots a_7}), \\
[T_1, Z^{a_1 \dots a_7}] &= 0, \\
[T_1, Z^{a_1 \dots a_6, b}] &= 0, \\
[T_2, P_a] &= 0, \\
[T_2, Z_\alpha^a] &= \frac{1}{2}(-1)^\alpha Z_\alpha^a, \\
[T_2, Z^{a_1 a_2 a_3}] &= 0, \\
[T_2, Z_\alpha^{a_1 \dots a_5}] &= \frac{1}{2}(-1)^\alpha Z_\alpha^{a_1 \dots a_5}, \\
[T_2, Z_{\alpha\beta}^{a_1 \dots a_7}] &= \frac{1}{2}((-1)^\alpha Z_{\alpha\beta}^{a_1 \dots a_7} + (-1)^\beta Z_{\alpha\beta}^{a_1 \dots a_7}), \\
[T_2, Z^{a_1 \dots a_7}] &= 0, \\
[T_2, Z^{a_1 \dots a_6, b}] &= 0.
\end{aligned} \tag{2.3.32}$$

Then we calculated the commutators with the level 1 odd generator to be

$$\begin{aligned}
[T_\alpha^{a_1 a_2}, P_b] &= \delta_b^{[a_1} Z_\alpha^{a_2]}, \\
[T_\alpha^{a_1 a_2}, Z_\beta^b] &= -\varepsilon_{\alpha\beta} Z^{a_1 a_2 b} - 4\delta_\beta^\alpha \delta_{[a_1}^b P_{a_2]}, \\
[T_\alpha^{a_1 a_2}, Z_\beta^{b_1 b_2 b_3}] &= Z_\alpha^{a_1 a_2 b_1 b_2 b_3} - 6\varepsilon^{\alpha\beta} \delta_{a_1 a_2}^{[b_1 b_2} Z_\beta^{b_3]}, \\
[T_\alpha^{a_1 a_2}, Z_\beta^{b_1 \dots b_5}] &= Z_{\alpha\beta}^{a_1 a_2 b_1 \dots b_5} - \varepsilon_{\alpha\beta} Z^{a_1 a_2 b_1 \dots b_5} \\
&\quad - \varepsilon_{\alpha\beta} Z^{a_1 a_2 [b_1 \dots b_4, b_5]} + 20\delta_\beta^\alpha \delta_{a_1 a_2}^{[b_1 b_2} Z^{b_3 b_4 b_5]}, \\
[T_\alpha^{a_1 a_2}, Z_{\beta_1 \beta_2}^{b_1 \dots b_7}] &= 42\delta_{(\beta_1}^\alpha \delta_{a_1 a_2}^{[b_1 b_2} Z_{\beta_2}^{b_3 \dots b_7]}, \\
[T_\alpha^{a_1 a_2}, Z_\beta^{b_1 \dots b_7}] &= -3\varepsilon^{\alpha\beta} \delta_{a_1 a_2}^{[b_1 b_2} Z_\beta^{b_3 \dots b_7]}, \\
[T_\alpha^{a_1 a_2}, Z_\beta^{b_1 \dots b_6, b}] &= -150\varepsilon^{\alpha\beta} \delta_{a_1 a_2}^{[b_1 b_2} Z_\beta^{b_3 \dots b_6] b} + 150\varepsilon^{\alpha\beta} \delta_{a_1 a_2}^{[b_1 b_2} Z_\beta^{b_3 \dots b_7 b]}.
\end{aligned} \tag{2.3.33}$$

At level 2, we found

$$[T^{a_1 \dots a_4}, P_b] = 2\delta_b^{[a_1} Z^{a_2 a_3 a_4]},$$

$$\begin{aligned}
[T^{a_1 \dots a_4}, Z_\beta^b] &= -Z_\beta^{a_1 \dots a_4 b}, \\
[T^{a_1 \dots a_4}, Z^{b_1 b_2 b_3}] &= 2Z^{a_1 \dots a_4 b_1 b_2 b_3} + \frac{3}{5}Z^{a_1 \dots a_4 [b_1 b_2, b_3]} - 48\delta_{[a_1 a_2 a_3}^{b_1 b_2 b_3} P_{a_4]}, \\
[T^{a_1 \dots a_4}, Z_\alpha^{b_1 \dots b_5}] &= -5!\delta_{a_1 \dots a_4}^{[b_1 \dots b_4} Z_\alpha^{b_5]}, \\
[T^{a_1 \dots a_4}, Z_{\alpha\beta}^{b_1 \dots b_7}] &= 0, \\
[T^{a_1 \dots a_4}, Z^{b_1 \dots b_7}] &= 5!\delta_{a_1 \dots a_4}^{[b_1 \dots b_4} Z^{b_5 b_6 b_7]}, \\
[T^{a_1 \dots a_4}, Z^{b_1 \dots b_6, b}] &= \frac{6!5}{2}(\delta_{a_1 \dots a_4}^{[b_1 \dots b_4} Z^{b_5 b_6]b} - \delta_{a_1 \dots a_4}^{[b_1 \dots b_4} Z^{b_5 b_6 b]}), \tag{2.3.34}
\end{aligned}$$

and finally, we found the level 3 commutators to be

$$\begin{aligned}
[T_\alpha^{a_1 \dots a_6}, P_b] &= \frac{3}{4}\delta_b^{[a_1} Z_\alpha^{a_2 \dots a_6]}, \\
[T_\alpha^{a_1 \dots a_6}, Z_\beta^b] &= -\frac{1}{4}Z_{\alpha\beta}^{a_1 \dots a_6 b} + \frac{3}{4}\varepsilon_{\alpha\beta} Z^{a_1 \dots a_6 b} + \frac{1}{20}\varepsilon_{\alpha\beta} Z^{a_1 \dots a_6, b}, \\
[T_\alpha^{a_1 \dots a_6}, Z^{b_1 b_2 b_3}] &= 0, \\
[T_\alpha^{a_1 \dots a_6}, Z_\beta^{b_1 \dots b_5}] &= -\frac{6!}{2}\delta_\beta^\alpha \delta_{[a_1 \dots a_5}^{b_1 \dots b_5} P_{a_6]}, \\
[T_\alpha^{a_1 \dots a_6}, Z_{\beta_1 \beta_2}^{b_1 \dots b_7}] &= -\frac{7!}{4}\delta_{(\beta_1}^\alpha \delta_{a_1 \dots a_6}^{[b_1 \dots b_7} Z_{\beta_2]}^{b_7]}, \\
[T_\alpha^{a_1 \dots a_6}, Z^{b_1 \dots b_7}] &= 270\varepsilon^{\alpha\beta} \delta_{a_1 \dots a_6}^{[b_1 \dots b_6} Z_\beta^{b_7]}, \\
[T_\alpha^{a_1 \dots a_6}, Z^{b_1 \dots b_6, b}] &= \frac{6!5}{4}\varepsilon^{\alpha\beta} (\delta_{a_1 \dots a_6}^{b_1 \dots b_6} Z_\beta^b - \delta_{a_1 \dots a_6}^{[b_1 \dots b_6} Z_\beta^b]). \tag{2.3.35}
\end{aligned}$$

Finally at level four, we find

$$\begin{aligned}
[T_{\alpha\beta}^{a_1 \dots a_8}, P_a] &= -\delta_a^{[a_1} Z_{\alpha\beta}^{a_2 \dots a_8]}, \\
[T_{\alpha\beta}^{a_1 \dots a_8}, Z_\gamma^b] &= 0, \\
[T_{\alpha\beta}^{a_1 \dots a_8}, Z^{b_1 b_2 b_3}] &= 0, \\
[T_{\alpha\beta}^{a_1 \dots a_8}, Z_\gamma^{b_1 \dots b_5}] &= 0, \\
[T_{\alpha_1 \alpha_2}^{a_1 \dots a_8}, Z_{\beta_1 \beta_2}^{b_1 \dots b_7}] &= -20160\delta_{\beta_1 \beta_2}^{(\alpha_1 \alpha_2)} \delta_{[a_1 \dots a_7}^{b_1 \dots b_7} P_{a_8]}, \\
[T_{\alpha_1 \alpha_2}^{a_1 \dots a_8}, Z^{b_1 \dots b_7}] &= 0, \\
[T_{\alpha_1 \alpha_2}^{a_1 \dots a_8}, Z^{b_1 \dots b_6, b}] &= 0, \\
[T^{a_1 \dots a_7, b}, P_a] &= -3\delta_a^b Z^{a_1 \dots a_7} + 3\delta_a^b Z^{a_1 \dots a_7] + \frac{21}{20}\delta_a^{[a_1} Z^{a_2 \dots a_7], b}, \\
[T^{a_1 \dots a_7, b}, Z_\gamma^c] &= 0,
\end{aligned}$$

$$\begin{aligned}
[T^{a_1 \dots a_7, b}, Z^{b_1 b_2 b_3}] &= 0, \\
[T^{a_1 \dots a_7, b}, Z_\gamma^{b_1 \dots b_5}] &= 0, \\
[T^{a_1 \dots a_7, a}, Z_{\alpha_1 \alpha_2}^{b_1 \dots b_7}] &= 0, \\
[T^{a_1 \dots a_7, a}, Z^{b_1 \dots b_7}] &= -4320 \delta_{a_1 \dots a_7}^{b_1 \dots b_7} P_a + 4320 \delta_{[a_1 \dots a_7}^{b_1 \dots b_7} P_{a]}, \\
[T^{a_1 \dots a_7, a}, Z^{b_1 \dots b_6, b}] &= 75600 \delta_a^b \delta_{[a_1 \dots a_6}^{b_1 \dots b_6} P_{a_7]} - 75600 \delta_a^{bb_1 \dots b_6} P_{a_7]}. \tag{2.3.36}
\end{aligned}$$

This completes the computation of the $I_c(E_{11})$ of IIB up to level 4. These results are to be published [23], but have subsequently been used to calculate the dynamics of branes in [39].

This completes the chapter on the derivation of E_{11} algebra in various dimensions. In the next chapter, we derive the gauge fixing multiplet in dimensions 11D, 5D, and 4D in E_{11} , and additionally for the A_1^{+++} algebra.

Chapter 3

Gauge fixing multiplet

We showed how to derive the tangent space metric in section 1.4.2, and then the gauge-fixed multiplet in section 1.4.3 for a general decomposition of $E_{11} \times l_1$. In this chapter, we shall carry out this calculation explicitly in 11D, 5D, 4D, and for the algebra A_1^{+++} , which leads to a description of 4D gravity. The results of this chapter were published in 2018 [22].

Following the research of Tumanov and West [51], where the vielbeins were found for the 11D, 5D, 4D, and the algebra A_1^{+++} , we calculate the explicit form of the tangent space metric and the gauge-fixing multiplet in each of these cases.

3.1 11 dimensions

We shall repeat the 11D theory here from section 2.1, for ease of reference, but we choose only to go up to level 3 in the generators as we will only require these. Recall that we can find the generators of the 11 dimensional theory by deleting node 11 from the Dynkin diagram as shown in figure 3.1.

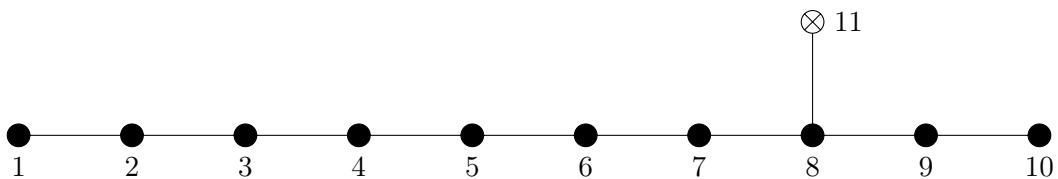


FIGURE 3.1: Dynkin diagram of E_{11} which leads to the 11 dimensional theory of maximal supergravity.

We can then decompose the algebra of $E_{11} \ltimes l_1$ in terms of the remaining $GL(11)$ algebra.

Up to level 3, the generators are

$$K^a_b ; R^{a_1 a_2 a_3} ; R^{a_1 \dots a_6} ; R^{a_1 \dots a_8, b} ; \dots , \quad (3.1.1)$$

where $a, b, \dots = 1, \dots, 11$, and the semi-colons separate the generators by level. We also give the negative level generators, up to level -3

$$R_{a_1 a_2 a_3} ; R_{a_1 \dots a_6} ; R_{a_1 \dots a_8, b} ; \dots , \quad (3.1.2)$$

with $a, b, \dots = 1, \dots, 11$. The Cartan involution acts in the following way

$$\begin{aligned} I_c(K^a_b) &= -K^b_a , \\ I_c(R^{a_1 a_2 a_3}) &= -R_{a_1 a_2 a_3} , \\ I_c(R^{a_1 \dots a_6}) &= -R_{a_1 \dots a_6} , \\ I_c(R^{a_1 \dots a_8, b}) &= -R_{a_1 \dots a_8, b} , \end{aligned} \quad (3.1.3)$$

and recall that the l_1 representation, up to level 3, is

$$P_a ; Z^{a_1 a_2} ; Z^{a_1 \dots a_5} ; Z^{a_1 \dots a_8} , Z^{a_1 \dots a_7, b} ; \dots , \quad (3.1.4)$$

and again $a, b, a_1, \dots = 1, \dots, 11$. The Cartan involution acts on the l_1 representation to give another representation. In fact, the l_1 we use is the lowest representation and the highest weight representation, denoted \bar{l}_1 , is the result we get from acting on l_1 with the Cartan involution. The \bar{l}_1 representation is

$$\bar{P}^a ; \bar{Z}_{a_1 a_2} ; \bar{Z}_{a_1 \dots a_5} ; \bar{Z}_{a_1 \dots a_8} , \bar{Z}_{a_1 \dots a_7, b} ; \dots , \quad (3.1.5)$$

with $a, b, a_1, \dots = 1, \dots, 11$. The commutators of the adjoint representation with itself, and the commutators of the adjoint representation with the fundamental representation are given up to level 4 in Appendix A, but as mentioned, we only require up to level 3. We shall use these commutators to derive the \bar{l}_1 commutators in the following way.

The action of the Cartan involution on the l_1 generators is

$$\begin{aligned}
I_c(P_a) &= -\bar{P}^a, \\
I_c(Z^{a_1 a_2}) &= -\bar{Z}_{a_1 a_2}, \\
I_c(Z^{a_1 \dots a_5}) &= -\bar{Z}_{a_1 \dots a_5}, \\
I_c(Z^{a_1 \dots a_8}) &= -\bar{Z}_{a_1 \dots a_8}, \\
I_c(Z^{a_1 \dots a_7, b}) &= -\bar{Z}_{a_1 \dots a_7, b}.
\end{aligned} \tag{3.1.6}$$

We see that J_{AB} from equation (1.3.15) is δ_{AB} in this case. The commutators of the \bar{l}_1 representation can be derived by acting on the l_1 commutators given in appendix A with the Cartan involution. We find that the commutators of the \bar{l}_1 generators with the level 0 E_{11} generators are

$$\begin{aligned}
[K^a_b, \bar{P}^c] &= \delta_b^c \bar{P}^a - \frac{1}{2} \delta_b^a \bar{P}^c, \\
[K^a_b, \bar{Z}_{a_1 a_2}] &= -2\delta_{[a_1}^a \bar{Z}_{b|a_2]} - \frac{1}{2} \delta_b^a \bar{Z}_{a_1 a_2}, \\
[K^a_b, \bar{Z}_{a_1 \dots a_5}] &= -5\delta_{[a_1}^a \bar{Z}_{b|a_2 \dots a_5]} - \frac{1}{2} \delta_b^a \bar{Z}_{a_1 \dots a_5}, \\
[K^a_b, \bar{Z}_{a_1 \dots a_8}] &= -8\delta_{[a_1}^a \bar{Z}_{b|a_2 \dots a_8]} - \frac{1}{2} \delta_b^a \bar{Z}_{a_1 \dots a_8}, \\
[K^a_b, \bar{Z}_{a_1 \dots a_7, c}] &= -7\delta_{[a_1}^a \bar{Z}_{b|a_2 \dots a_7], c} - \delta_c^a \bar{Z}_{a_1 \dots a_7, b} - \frac{1}{2} \delta_b^a \bar{Z}_{a_1 \dots a_7, c}.
\end{aligned} \tag{3.1.7}$$

We find the commutators with the level 1 E_{11} generator are

$$\begin{aligned}
[R^{a_1 a_2 a_3}, \bar{P}^a] &= 0, \\
[R^{a_1 a_2 a_3}, \bar{Z}_{b_1 b_2}] &= -6\delta_{[b_1 b_2}^{[a_1 a_2} \bar{P}^{a_3]}], \\
[R^{a_1 a_2 a_3}, \bar{Z}_{b_1 \dots b_5}] &= -60\delta_{[b_1 b_2 b_3}^{a_1 a_2 a_3} \bar{Z}_{b_4 b_5}], \\
[R^{a_1 a_2 a_3}, \bar{Z}_{b_1 \dots b_8}] &= 42\delta_{[b_1 b_2 b_3}^{a_1 a_2 a_3} \bar{Z}_{b_4 \dots b_8}], \\
[R^{a_1 a_2 a_3}, \bar{Z}_{b_1 \dots b_7, b}] &= -\frac{945}{4} \delta_{[b_1 b_2 b_3}^{a_1 a_2 a_3} \bar{Z}_{b_4 \dots b_7] b} - \frac{945}{4} \delta_{[b_1 b_2 | b]}^{a_1 a_2 a_3} \bar{Z}_{b_3 \dots b_7}.
\end{aligned} \tag{3.1.8}$$

Finally, we derive the commutators with the negative level E_{11} generators to be

$$\begin{aligned}
[R_{a_1 a_2 a_3}, \bar{P}^a] &= -3\delta_{[a_1}^a \bar{Z}_{a_2 a_3]} , \\
[R_{a_1 a_2 a_3}, \bar{Z}_{a_4 a_5}] &= -\bar{Z}_{a_1 \dots a_5} , \\
[R_{a_1 a_2 a_3}, \bar{Z}_{b_1 \dots b_8}] &= -\bar{Z}_{b_1 \dots b_5 a_1 a_2 a_3} - \bar{Z}_{b_1 \dots b_5 [a_1 a_2, a_3]} , \\
[R_{a_1 \dots a_6}, \bar{P}^a] &= -3\delta_{[a_1}^a \bar{Z}_{a_2 \dots a_6]} , \\
[R_{a_1 \dots a_6}, \bar{Z}_{b_1 b_2}] &= -\bar{Z}_{b_1 b_2 a_1 \dots a_6} - \bar{Z}_{b_1 b_2 [a_1 \dots a_5, a_6]} , \\
[R_{a_1 \dots a_8, a}, \bar{P}^b] &= \frac{4}{3}\delta_a^b \bar{Z}_{a_1 \dots a_8} - \frac{4}{3}\delta_{[a_1}^b \bar{Z}_{a_2 \dots a_8]a} - \frac{4}{3}\delta_{[a_1}^b \bar{Z}_{a_2 \dots a_8], a} .
\end{aligned} \tag{3.1.9}$$

Now we have all the commutators that we require to find the tangent space metric, we begin the derivation of the group element in 11D. Explicitly, the group element $g = g_l g_A$ is written as

$$\begin{aligned}
g_l &= \exp(x^a P_a + x_{a_1 a_2} Z^{a_1 a_2} + x_{a_1 \dots a_5} Z^{a_1 \dots a_5} \\
&\quad + x_{a_1 \dots a_8} Z^{a_1 \dots a_8} + x_{a_1 \dots a_7, b} Z^{a_1 \dots a_7, b}) \dots ,
\end{aligned} \tag{3.1.10}$$

$$\begin{aligned}
g_A &= \exp(h_a{}^b K^a{}_b) \exp(A_{a_1 \dots a_8, b} R^{a_1 \dots a_8, b}) \\
&\quad \times \exp(A_{a_1 \dots a_6} R^{a_1 \dots a_6}) \exp(A_{a_1 a_2 a_3} R^{a_1 a_2 a_3}) \dots ,
\end{aligned} \tag{3.1.11}$$

where we have used the local Cartan involution subalgebra to set the negative level E_{11} generators to zero and the dots represent higher level generators in which we are not interested. We find the parameters

$$x^a ; \quad x_{a_1 a_2} ; \quad x_{a_1 \dots a_5} ; \quad x_{a_1 \dots a_8} , \quad x_{a_1 \dots a_7, b} ; \quad \dots , \tag{3.1.12}$$

to be the coordinates of generalised space-time and the parameters

$$h_a{}^b ; \quad A_{a_1 a_2 a_3} ; \quad A_{a_1 \dots a_6} ; \quad A_{a_1 \dots a_8, b} ; \quad \dots , \tag{3.1.13}$$

to be the gauge fields, which depend on the space-time. Now that we have the relevant algebra and group element, we can derive the tangent space metric.

3.1.1 Tangent space metric

The first step is to find the matter representation. From equation (1.4.10), we find that it is

$$V^A = (T^a, T_{a_1 a_2}, T_{a_1 \dots a_5}, T_{a_1 \dots a_8}, T_{a_1 \dots a_7, b}, \dots) . \quad (3.1.14)$$

The local transformation group $I_c(E_{11})$ at level zero is the local Lorentz transformation, SO(11). At level 1, the local transformation is

$$h = 1 - \Lambda_{a_1 a_2 a_3} (R^{a_1 a_2 a_3} - \eta^{a_1 b_1} \eta^{a_2 b_2} \eta^{a_3 b_3} R_{b_1 b_2 b_3}) . \quad (3.1.15)$$

Since all generators can be found from multiple commutators of the generators $R^{a_1 a_2 a_3}$ and $R_{a_1 a_2 a_3}$, we notice that if the dynamics are invariant under the transformation h (which we see contains the level plus and minus one generators), then they are invariant under all $I_c(E_{11})$ transformations, as higher level generators can be constructed from multiple commutators of the level ± 1 generators.

Using equation (1.4.10), we find that the components of the matter representation must transform under h in the following way

$$\begin{aligned} \delta T^a &= -6\Lambda^{a_1 a_2 a} T_{a_1 a_2} , \\ \delta T_{a_1 a_2} &= 3\Lambda_{a_1 a_2 b} T^b - 60\Lambda^{b_1 b_2 b_3} T_{b_1 b_2 b_3 a_1 a_2} , \\ \delta T_{a_1 \dots a_5} &= \Lambda_{[a_1 a_2 a_3} T_{a_4 a_5]} + 42\Lambda^{b_1 b_2 b_3} T_{b_1 b_2 b_3 a_1 \dots a_5} - 378\Lambda^{b_1 b_2 b_3} T_{b_1 b_2 b_3 [a_1 \dots a_4, a_5]} , \\ \delta T_{a_1 \dots a_8} &= -\Lambda_{[a_1 a_2 a_3} T_{a_4 \dots a_8]} , \\ \delta T_{a_1 \dots a_7, b} &= T_{[a_1 \dots a_5} \Lambda_{a_6 a_7] b} - T_{[a_1 \dots a_5} \Lambda_{a_6 a_7 b]} . \end{aligned} \quad (3.1.16)$$

We now find the invariant metric in two ways and will see that they are equivalent. The first method will use the definition of the invariant map N_A^B introduced in equation (1.4.12) and then use the definition $K = N(J^{-1})^T$ to find the metric. In the second method, we explicitly construct the metric by writing the most general form of the invariant object in equation (1.4.23) and then applying the transformations in equation (3.1.16) to find the coefficients which ensure invariance.

Firstly, we construct the invariant map $N_A^B = (l_A, \bar{l}^B)$. We find the first component is $N_a^b = \delta_a^b = (P_a, \bar{P}^b)$ since N_a^b is invariant under $SL(11)$ transformations. To find the level 2 component $N^{a_1 a_2}_{b_1 b_2}$, we use the invariance of the scalar product

$$([R^{a_1 a_2 a_3}, P_b], \bar{Z}_{c_1 c_2}) + (P_b, [R^{a_1 a_2 a_3}, \bar{Z}_{c_1 c_2}]) = 0, \quad (3.1.17)$$

and using the commutators in appendix A and (3.1.8), we find

$$3\delta_b^{[a_1} (Z^{a_2 a_3}], \bar{Z}_{c_1 c_2}) - 6\delta_{c_1 c_2}^{[a_1 a_2} (P_b, \bar{P}^{a_3}]) = 0. \quad (3.1.18)$$

We therefore get the result

$$N^{a_1 a_2}_{b_1 b_2} = (Z^{a_1 a_2}, \bar{Z}_{b_1 b_2}) = 2\delta_{b_1 b_2}^{a_1 a_2}. \quad (3.1.19)$$

Repeating this process for the higher level components, and we find that the map is

$$N_A^B = \begin{pmatrix} \delta_a^b & 0 & 0 & 0 & 0 \\ 0 & 2\delta_{b_1 b_2}^{a_1 a_2} & 0 & 0 & 0 \\ 0 & 0 & 5!\delta_{b_1 \dots b_5}^{a_1 \dots a_5} & 0 & 0 \\ 0 & 0 & 0 & 7!\delta_{b_1 \dots b_8}^{a_1 \dots a_8} & 0 \\ 0 & 0 & 0 & 0 & \frac{9!}{8}\delta_{b_1 \dots b_7, b}^{a_1 \dots a_7, a} \end{pmatrix}, \quad (3.1.20)$$

where $\delta_{b_1 \dots b_7, b}^{a_1 \dots a_7, a} = \delta_{b_1 \dots b_7}^{a_1 \dots a_7} \delta_b^a - \delta_{b_1 \dots b_7 b}^{a_1 \dots a_7 a}$ and $\delta_{b_1 \dots b_n}^{a_1 \dots a_n} = \delta_{b_1}^{a_1} \dots \delta_{b_n}^{a_n}$. The metric is then given by $K = N(J^{-1})^T$, and since in the eleven dimensional theory, we have $J_{AB} = \delta_{AB}$, we find

the $I_c(E_{11})$ invariant metric up to level 4 is

$$K_{AB} = \begin{pmatrix} \eta_{ab} & 0 & 0 & 0 & 0 \\ 0 & 2\delta^{a_1 a_2, b_1 b_2} & 0 & 0 & 0 \\ 0 & 0 & 5!\delta^{a_1 \dots a_5, b_1 \dots b_5} & 0 & 0 \\ 0 & 0 & 0 & 7!\delta^{a_1 \dots a_8, b_1 \dots b_8} & 0 \\ 0 & 0 & 0 & 0 & \frac{9!}{8}\delta^{a_1 \dots a_7, a, b_1 \dots b_7, b} \end{pmatrix}, \quad (3.1.21)$$

where $\delta^{a_1 \dots a_7, a, b_1 \dots b_7, b} = \delta^{a_1 \dots a_7, b_1 \dots b_8} \delta^{ab} - \delta^{a_1 \dots a_7 a, b_1 \dots b_7 b}$ and $\delta^{a_1 \dots a_n, b_1 \dots b_n} = \delta^{[a_1 | b_1 | \delta^{a_2 | b_2 | \dots \delta^{a_n | b_n]}$.

For the second method, we write down the most general metric containing the matter representation components and use the transformations of equation (3.1.16) to find the coefficients of the metric. We find the invariant quantity from equation (1.4.23) to be

$$\begin{aligned} \Delta = & T^a T_a + 2T^{a_1 a_2} T_{a_1 a_2} + 5!T^{a_1 \dots a_5} T_{a_1 \dots a_5} + 7!T^{a_1 \dots a_8} T_{a_1 \dots a_8} \\ & + \frac{9!}{8} T^{a_1 \dots a_7, b} T_{a_1 \dots a_7, b} + \dots, \end{aligned} \quad (3.1.22)$$

and we see that it agrees with the metric found in the first method. Using this metric, we now find the gauge fixed multiplet in 11 dimensions.

3.1.2 Gauge fixing conditions

The generalised space-time derivatives, from equation (3.1.12) are

$$\partial_A = (\partial_a, \partial^{a_1 a_2}, \partial^{a_1 \dots a_5}, \partial^{a_1 \dots a_8}, \partial^{a_1 \dots a_7, b}, \dots). \quad (3.1.23)$$

Then we find the each component of equation (1.4.27) at first order in derivatives is

$$K^{AB} G_{A,B}{}^c = (\det e)^{\frac{1}{2}} (\partial^a h_a{}^c - \frac{1}{2} \partial^c h_a{}^a), \quad (3.1.24)$$

$$K^{AB} G_{A, B c_1 c_2} = (\det e)^{\frac{1}{2}} (-3\partial^a A_{a c_1 c_2} - \partial_{[c_1 | a | h_{c_2]}^a - \frac{1}{4} \partial_{c_1 c_2} h_a{}^a), \quad (3.1.25)$$

$$K^{AB} G_{A, B c_1 \dots c_5} = -(\det e)^{\frac{1}{2}} (3\partial^a A_{a c_1 \dots c_5} - \frac{1}{2} \partial_{[c_1 c_2} A_{c_3 c_4 c_5]})$$

$$- \frac{1}{4!} \partial_{[c_1 \dots c_4 | a]} h_{c_5]}^a - \frac{1}{5!2} \partial_{c_1 \dots c_5} h_a^a), \quad (3.1.26)$$

$$\begin{aligned} K^{AB} G_{A, B c_1 \dots c_8} = & - (\det e)^{\frac{1}{2}} \left(\frac{3}{2} \partial^a A_{c_1 \dots c_8, a} + \frac{1}{2} \partial_{[c_1 c_2} A_{c_3 \dots c_8]} \right. \\ & + \frac{1}{5!} \partial_{[c_1 \dots c_5} A_{c_6 c_7 c_8]} \\ & \left. + \frac{8}{7!} (\partial_{[c_1 \dots c_7 | a]} h_{c_8]}^a + \frac{1}{7!2} \partial_{c_1 \dots c_8} h_d^d) \right), \quad (3.1.27) \end{aligned}$$

$$\begin{aligned} K^{AB} G_{A, B c_1 \dots c_7, c} = & - (\det e)^{\frac{1}{2}} \left(\partial^a \left(\frac{4}{3} A_{a [c_1 \dots c_7, c]} - \frac{4}{3} A_{a [c_1 \dots c_7], c} \right) \right. \\ & + \frac{1}{2} \partial_{[c_1 c_2} (-A_{c_3 \dots c_7 c]} + A_{c_3 \dots c_7] c}) \\ & - \frac{1}{5!} \partial_{[c_1 \dots c_5} (A_{c_6 c_7 c]} - A_{c_6 c_7] c}) \\ & + \frac{2^7}{7!6!} (7 \partial_{[c_1 \dots c_6 | a, c]} h_{c_7]}^a - 8 \partial_{[c_1 \dots c_7, | a]} h_c]^a \\ & \left. + \frac{1}{2} \partial_{c_1 \dots c_7, c} h_d^d) \right), \quad (3.1.28) \end{aligned}$$

where $\delta_{a_1 \dots a_n, b_1 \dots b_n} = \delta_{[a_1 | b_1]} \delta_{a_2 | b_2]} \dots \delta_{a_n | b_n]}$. We note that $(e^h)_\mu^a$ is the vielbein of the usual spacetime and $\det e$ is its determinant.

If we choose to set $G^C = 0$, we see that the first equation is the de Donder gauge fixing condition. Similarly, the next equation is the gauge fixing condition for the three form field, and so we have found a set of E_{11} invariant gauge fixing conditions.

This completes the calculation of the tangent space metric and gauge fixing conditions in the 11D decomposition of E_{11} . We now find the tangent space metric and gauge fixing conditions in 5 dimensions.

3.2 5 dimensions

To find the five dimensional case, we delete node 5 from the E_{11} Dynkin diagram as shown in figure 3.2 and decompose the $E_{11} \times l_1$ algebra into the remaining $GL(5) \otimes E_6$ algebra. We choose to decompose it in terms of the tangent space group $I_c(GL(5) \otimes E_6) = SO(5) \otimes USp(8)$ as was first done in [17], and additionally in [25].

The generators in terms of the $SO(5) \otimes USp(8)$ representation are

$$K^a_b, \quad R^{\alpha_1 \alpha_2}, \quad R^{\alpha_1 \dots \alpha_4}; \quad R^{a \alpha_1 \alpha_2}; \quad R^{a_1 a_2}_{\alpha_1 \alpha_2};$$

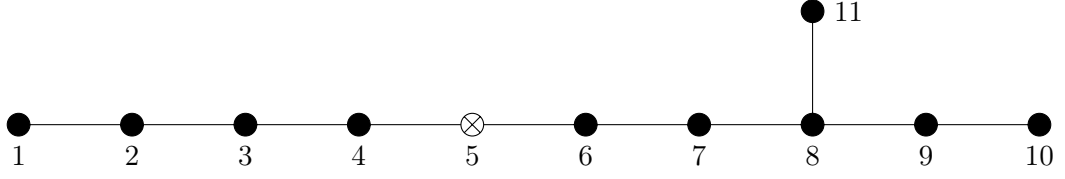


FIGURE 3.2: Dynkin diagram of E_{11} which leads to the 5 dimensional theory of maximal supergravity.

$$R^{a_1 a_2 a_3 \alpha_1 \alpha_2} , \quad R^{a_1 a_2 a_3 \alpha_1 \dots \alpha_4} , \quad R^{a_1 a_2, b} ; \dots , \quad (3.2.1)$$

where $a, b, \dots = 1, \dots, 5$ and $\alpha_1, \alpha_2, \dots = 1, \dots, 8$. The $R^{\alpha_1 \alpha_2}$ and $R^{\alpha_1 \dots \alpha_4}$ correspond to the 36 adjoint representation and the 42 representation of $\text{USp}(8)$, respectively, which make the 78 representation of E_6 . $R^{a \alpha_1 \alpha_2}$ and $R^{a_1 a_2}_{\alpha_1 \alpha_2}$ are the 27 and $\bar{27}$ -dimensional representations of E_6 respectively. $\Omega_{\alpha_1 \alpha_2}$ is the $\text{USp}(8)$ invariant antisymmetric metric, which we use to raise and lower indices. Similar to the 11D case, the semi-colons represent a change of level. The generators obey

$$\begin{aligned} R^{\alpha_1 \alpha_2} &= R^{(\alpha_1 \alpha_2)} , \\ R^{a_1 a_2 a_3 \alpha_1 \alpha_2} &= R^{a_1 a_2 a_3 (\alpha_1 \alpha_2)} , \\ R^{[a_1 a_2, b]} &= 0 . \end{aligned} \quad (3.2.2)$$

The indices on the remaining generators are antisymmetric and Ω is traceless.

The negative level generators of the E_{11} algebra, up to level -3, are

$$R_{a \alpha_1 \alpha_2} ; \quad R_{a_1 a_2}^{\alpha_1 \alpha_2} ; \quad R_{a_1 a_2 a_3 \alpha_1 \alpha_2} , \quad R_{a_1 a_2 a_3 \alpha_1 \dots \alpha_4} , \quad R_{a_1 a_2, b} ; \quad \dots , \quad (3.2.3)$$

where $a, b, \dots = 1, \dots, 5$ and $\alpha_1, \alpha_2, \dots = 1, \dots, 8$ and with analogous constraints as in equation (3.2.2).

Up to level 3, the l_1 representation has the generators

$$P_a ; \quad Z_{\alpha_1 \alpha_2} ; \quad Z^a_{\alpha_1 \alpha_2} ; \quad Z^{a_1 a_2 \alpha_1 \alpha_2} , \quad Z^{a_1 a_2 \alpha_1 \dots \alpha_4} , \quad Z^{ab} , \quad \dots , \quad (3.2.4)$$

where $Z^{a_1 a_2 \alpha_1 \alpha_2} = Z^{a_1 a_2 (\alpha_1 \alpha_2)}$ and there are no symmetries on the indices of Z^{ab} . Additionally, $a, b, \dots = 1, \dots, 5$ and $\alpha_1, \alpha_2, \dots = 1, \dots, 8$. Then the \bar{l}_1 representation has the generators

$$\bar{P}_a ; \quad \bar{Z}_{\alpha_1 \alpha_2} ; \quad \bar{Z}^a_{\alpha_1 \alpha_2} ; \quad \bar{Z}^{a_1 a_2 \alpha_1 \alpha_2} , \quad \bar{Z}^{a_1 a_2 \alpha_1 \dots \alpha_4} , \quad \bar{Z}^{ab} , \quad \dots , \quad (3.2.5)$$

where $a, b, \dots = 1, \dots, 5$ and $\alpha_1, \alpha_2, \dots = 1, \dots, 8$ and they satisfy analogous constraints as those of the l_1 representation.

The commutators of the generators are given in appendix C, except those with the \bar{l}_1 representation which we shall find here. The Cartan involution acts on the generators of E_{11} and l_1 as

$$\begin{aligned} I_c(K^a_b) &= -K^b_a , \\ I_c(R^{(\alpha_1 \alpha_2)}) &= R^{(\alpha_1 \alpha_2)} , \\ I_c(R^{\alpha_1 \dots \alpha_4}) &= -R^{\alpha_1 \dots \alpha_4} , \\ I_c(R^{a \alpha_1 \alpha_2}) &= -\Omega^{\alpha_1 \beta_1} \Omega^{\alpha_2 \beta_2} R_{a \beta_1 \beta_2} , \\ I_c(P_c) &= -\bar{P}^c , \\ I_c(Z^{\alpha_1 \alpha_2}) &= -\Omega^{\alpha_1 \beta_1} \Omega^{\alpha_2 \beta_2} \bar{Z}_{\beta_1 \beta_2} , \\ I_c(Z^{a \alpha_1 \alpha_2}) &= -\Omega^{\alpha_1 \beta_1} \Omega^{\alpha_2 \beta_2} \bar{Z}_{a \beta_1 \beta_2} . \end{aligned} \quad (3.2.6)$$

Note that in this case, the J_{AB} of equation (1.3.15) is non-trivial.

Thus, we find that the commutators of the level zero E_{11} generators with the \bar{l}_1 algebra are

$$\begin{aligned} [K^a_b, \bar{P}^c] &= \delta_b^c \bar{P}^a - \frac{1}{2} \delta_b^a \bar{P}^c , \\ [K^a_b, \bar{Z}_{\alpha_1 \alpha_2}] &= -\frac{1}{2} \delta_b^a \bar{Z}_{\alpha_1 \alpha_2} , \\ [K^a_b, \bar{Z}_{c \alpha_1 \alpha_2}] &= -\delta_b^c \bar{Z}_{a \alpha_1 \alpha_2} - \frac{1}{2} \delta_b^a \bar{Z}_{c \alpha_1 \alpha_2} , \\ [R^{(\alpha_1 \alpha_2)}, \bar{P}^a] &= 0 , \\ [R^{(\alpha_1 \alpha_2)}, \bar{Z}_{\gamma_1 \gamma_2}] &= 2 \delta_{[\gamma_1}^{(\alpha_1} \Omega^{\alpha_2) \kappa} \bar{Z}_{\kappa \gamma_2]} , \end{aligned}$$

$$\begin{aligned}
[R^{(\alpha_1\alpha_2)}, \bar{Z}_{a\gamma_1\gamma_2}] &= 2\delta_{[\gamma_1}^{(\alpha_1}\Omega^{\alpha_2)\kappa}\bar{Z}_{a\kappa\gamma_2}], \\
[R^{\alpha_1\dots\alpha_4}, \bar{P}^a] &= 0, \\
[R^{\alpha_1\dots\alpha_4}, \bar{Z}_{\kappa_1\kappa_2}] &= -(\Omega^{[\alpha_1\alpha_2}\delta_{[\kappa_1}^{\alpha_3}\Omega^{\alpha_4]\gamma}\bar{Z}_{\gamma\kappa_2}] + \delta_{[\kappa_1}^{[\alpha_1}\delta_{\kappa_2]}^{\alpha_2]\Omega^{\alpha_3\gamma_1}\Omega^{\alpha_4]\gamma_2}\bar{Z}_{\gamma_1\gamma_2} \\
&\quad - \frac{1}{4}\Omega_{\kappa_1\kappa_2}\Omega^{[\alpha_1\alpha_2}\Omega^{\alpha_3\gamma_1}\Omega^{\alpha_4]\gamma_2}\bar{Z}_{\gamma_1\gamma_2} - \frac{1}{12}\Omega^{[\alpha_1\alpha_2}\Omega^{\alpha_3\alpha_4]}\bar{Z}_{\kappa_1\kappa_2}), \\
[R^{\alpha_1\dots\alpha_4}, \bar{Z}_{a\kappa_1\kappa_2}] &= (\Omega^{[\alpha_1\alpha_2}\delta_{[\kappa_1}^{\alpha_3}\Omega^{\alpha_4]\gamma}\bar{Z}_{a\gamma\kappa_2}] + \delta_{[\kappa_1}^{[\alpha_1}\delta_{\kappa_2]}^{\alpha_2]\Omega^{\alpha_3\gamma_1}\Omega^{\alpha_4]\gamma_2}\bar{Z}_{a\gamma_1\gamma_2} \\
&\quad - \frac{1}{4}\Omega_{\kappa_1\kappa_2}\Omega^{[\alpha_1\alpha_2}\Omega^{\alpha_3\gamma_1}\Omega^{\alpha_4]\gamma_2}\bar{Z}_{a\gamma_1\gamma_2} - \frac{1}{12}\Omega^{[\alpha_1\alpha_2}\Omega^{\alpha_3\alpha_4]}\bar{Z}_{a\kappa_1\kappa_2}). \quad (3.2.7)
\end{aligned}$$

We find that the commutators of the positive level E_{11} generators with the \bar{l}_1 representation are

$$\begin{aligned}
[R^{a\alpha_1\alpha_2}, \bar{P}^b] &= 0, \\
[R^{a\alpha_1\alpha_2}, \bar{Z}_{\beta_1\beta_2}] &= -2(\delta_{\beta_1\beta_2}^{a\alpha_1\alpha_2} + \frac{1}{8}\Omega^{a\alpha_1\alpha_2}\Omega_{\beta_1\beta_2})\bar{P}^a, \\
[R^{a\alpha_1\alpha_2}, \bar{Z}_{b\beta_1\beta_2}] &= 4\delta_b^a(\Omega^{[\alpha_1\gamma}\delta_{[\beta_2]}^{\alpha_2]}\bar{Z}_{\beta_2]\gamma} + \frac{1}{8}\Omega^{a\alpha_1\alpha_2}\bar{Z}_{\beta_1\beta_2} - \frac{1}{8}\Omega_{\beta_1\beta_2}\Omega^{a\alpha_1\gamma_1}\Omega^{\alpha_2\gamma_2}\bar{Z}_{\gamma_1\gamma_2}), \\
[R^{a_1a_2\alpha_1\alpha_2}, \bar{Z}_{b\beta_1\beta_2}] &= 4(\delta_{\beta_1\beta_2}^{a_1\alpha_1\alpha_2} + \frac{1}{8}\Omega^{a_1\alpha_1\alpha_2}\Omega_{\beta_1\beta_2})\delta_b^{[a_1}\bar{P}^{a_2]}. \quad (3.2.8)
\end{aligned}$$

Finally, we derive the commutators of the \bar{l}_1 algebra with the negative level E_{11} generators

$$\begin{aligned}
[R_{a\alpha_1\alpha_2}, \bar{P}^b] &= -\delta_a^b\bar{Z}_{\alpha_1\alpha_2}, \\
[R_{a_1a_2\alpha_1\alpha_2}, \bar{P}^b] &= 2\delta_{[a_1}^b\bar{Z}_{a_1]\alpha_1\alpha_2}, \\
[R_{a\alpha_1\alpha_2}, \bar{Z}_{\beta_1\beta_2}] &= -(4\Omega_{\alpha_1[\beta_1}\bar{Z}_{a\alpha_2]\beta_2} - \frac{1}{2}\Omega_{\beta_1\beta_2}\bar{Z}_{a\alpha_1\alpha_2} - \frac{1}{2}\Omega_{\alpha_1\alpha_2}\bar{Z}_{a\beta_1\beta_2}). \quad (3.2.9)
\end{aligned}$$

The group element $g = g_l g_A$ is

$$\begin{aligned}
g_l &= \exp(x^a P_a + x_{\alpha_1\alpha_2} Z^{\alpha_1\alpha_2} + x_a^{\alpha_1\alpha_2} Z^a_{\alpha_1\alpha_2} + x_{a_1a_2\alpha_1\alpha_2} Z^{a_1a_2\alpha_1\alpha_2} \\
&\quad + x_{a_1a_2\alpha_1\dots\alpha_4} Z^{a_1a_2\alpha_1\dots\alpha_4} + x_{ab} Z^{ab}) \dots, \\
g_A &= e^{h_a{}^b K^a} e^{(\varphi_{\alpha_1\alpha_2} R^{\alpha_1\alpha_2} + \varphi_{\alpha_1\dots\alpha_4} R^{\alpha_1\dots\alpha_4})} e^{A_{a_1a_2}{}^{\alpha_1\alpha_2} R^{a_1a_2}{}_{\alpha_1\alpha_2}} e^{A_{a\alpha_1\alpha_2} R^{a\alpha_1\alpha_2}} \dots \quad (3.2.10)
\end{aligned}$$

We see that g_l is parametrised by

$$x^a ; x_{\alpha_1\alpha_2} ; x_a^{\alpha_1\alpha_2} ; \dots , \quad (3.2.11)$$

which are the coordinates of the generalised space-time, and g_A is parametrised by

$$h_a^b , \varphi_{\alpha_1\alpha_2} , \varphi_{\alpha_1\dots\alpha_4} ; A_{a\alpha_1\alpha_2} ; A_{a_1a_2}^{\alpha_1\alpha_2} ; \dots , \quad (3.2.12)$$

which are the fields living on the space-time and we have set the coefficients of the negative level generators in the group element to zero using the local symmetry $I_c(E_{11})$ as before. Now we have the relevant algebra and the group element, we can now find the tangent space metric in 5D.

3.2.1 Tangent space metric

The matter representation is given by

$$V^A = (T^a, T_{\alpha_1\alpha_2}, T_{a\alpha_1\alpha_2}, \dots) . \quad (3.2.13)$$

We want to find the metric which is invariant under the transformations of the subgroup $I_c(E_{11})$. At level zero, we have the local Lorentz transformations and $\text{USp}(8)$. Then at level one, the transformation is given by

$$h = 1 - \Lambda_{a\alpha_1\alpha_2} (R^{a\alpha_1\alpha_2} - \eta^{ab}\Omega^{\alpha_1\beta_1}\Omega^{\alpha_2\beta_2}R_{b\beta_1\beta_2}) . \quad (3.2.14)$$

Again, we have that if the dynamics are invariant under the above h , they are invariant under transformations at all levels of the $I_c(E_{11})$ subalgebra.

Similar to the eleven dimensional case, we find the transformation of the matter representation components under $I_c(E_{11})$ to be

$$\delta T^a = -2\Lambda^{a\alpha_1\alpha_2} T_{\alpha_1\alpha_2} ,$$

$$\begin{aligned}\delta T_{\alpha_1\alpha_2} &= \Lambda_{a\alpha_1\alpha_2} T^a + 4\Lambda^a_{[\alpha_1|\gamma} T_{a|\alpha_2]\gamma} - \frac{1}{2}\Omega_{\alpha_1\alpha_2}\Lambda^{a\beta_1\beta_2} T_{a\beta_1\beta_2}, \\ \delta T_{a\alpha_1\alpha_2} &= -4\Lambda_{a[\alpha_1\gamma} T_{\alpha_2]\gamma} + \frac{1}{2}\Omega_{\alpha_1\alpha_2}\Lambda_a^{\beta_1\beta_2} T_{\beta_1\beta_2}.\end{aligned}\quad (3.2.15)$$

Then we find the tangent space metric which is invariant under these transformations to be

$$\Delta = K_{AB}V^AV^B = T^aT_a + T^{\alpha_1\alpha_2}T_{\alpha_1\alpha_2} - 2T^{a\alpha_1\alpha_2}T_{a\alpha_1\alpha_2} + \dots, \quad (3.2.16)$$

and we find the metric to be

$$K_{AB} = \begin{pmatrix} \eta_{ab} & 0 & 0 \\ 0 & (\Omega^{\alpha_1[\beta_1}\Omega^{|\alpha_2|\beta_2]} - \frac{1}{8}\Omega^{\alpha_1\alpha_2}\Omega^{\beta_1\beta_2}) & 0 \\ 0 & 0 & 2\eta_{ab}(\Omega^{\alpha_1[\beta_1}\Omega^{|\alpha_2|\beta_2]} - \frac{1}{8}\Omega^{\alpha_1\alpha_2}\Omega^{\beta_1\beta_2}) \end{pmatrix}, \quad (3.2.17)$$

and we have calculated the tangent space metric in 5D. We now use this metric to find the gauge fixing conditions.

3.2.2 Gauge fixing conditions

We find the space-time derivatives from the coordinates in equation (3.2.11) to be

$$\partial_A = (\partial_a, \partial^{\alpha_1\alpha_2}, \partial^a_{\alpha_1\alpha_2}, \dots). \quad (3.2.18)$$

Then we calculate in five dimensions equation (1.4.27) to be

$$K^{AB}G_{A,B}{}^c = (\det e)^{\frac{1}{2}}(\partial^a h_a{}^c - \frac{1}{2}\partial^c h_a{}^a), \quad (3.2.19)$$

$$K^{AB}G_{A,B\kappa_1\kappa_2} = -(\det e)^{\frac{1}{2}}(\partial^a A_{a\kappa_1\kappa_2} - \frac{1}{2}(f^{-1})^{\alpha_1\alpha_2}_{\nu_1\nu_2}\partial_{\alpha_1\alpha_2}f^{\nu_1\nu_2}_{\kappa_1\kappa_2} + \frac{1}{2}\partial_{\kappa_1\kappa_2}h_a{}^a), \quad (3.2.20)$$

$$\begin{aligned}K^{AB}G_{A,Bc\kappa_1\kappa_2} &= -(\det e)^{\frac{1}{2}}(2\partial^a A_{a\kappa_1\kappa_2} + 2\Omega^{[\gamma_1[\alpha_1}\delta_{\kappa_1\kappa_2}^{\gamma_2]\alpha_2]}\partial_{\alpha_1\alpha_2}A_{a\gamma_1\gamma_2} \\ &\quad + \frac{1}{2}(f^{-1})^{\nu_1\nu_2}_{\kappa_1\kappa_2}\partial_{c\alpha_1\alpha_2}(f^{\alpha_1\alpha_2}_{\nu_1\nu_2}) - \partial_{a\kappa_1\kappa_2}h_c{}^a - \frac{1}{2}\partial_{c\kappa_1\kappa_2}h_a{}^a),\end{aligned}\quad (3.2.21)$$

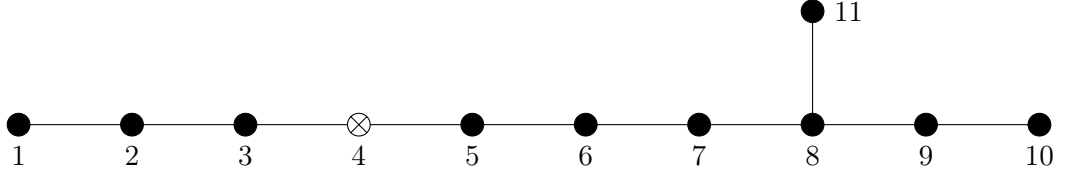


FIGURE 3.3: Dynkin diagram of E_{11} which leads to the 4 dimensional theory of maximal supergravity.

where $f^{\alpha_1}_{\nu_1} = (f^\phi)^{\alpha_1}_{\nu_1}$, $f^{\nu_1}_{\alpha_1} = (f^{-\phi})^{\nu_1}_{\alpha_1}$ and $f^{\alpha_1\alpha_2}_{\nu_1\nu_2} = f^{\alpha_1}_{[\nu_1} f^{\alpha_2}_{\nu_2]}$ is a vielbein with respect to the $\text{USp}(8)$ generator $R^{\alpha_1\alpha_2}$, and $\det e$ is the determinant of the vielbein (e^h).

We choose $G^C = 0$ and still preserve the symmetries of the non-linear realisation and we can see the result as a gauge fixing condition. Therefore, we have found gauge-fixing conditions in 5D.

We now do the same derivation in 4 dimensions.

3.3 4 dimensions

In this section, we derive the tangent space metric, and gauge fixing conditions in the 4D decomposition of E_{11} . For the four dimensional theory, we delete node 4 from the E_{11} Dynkin diagram as shown in figure 3.3. We choose to work with $\text{GL}(4) \otimes \text{SL}(8)$ rather than $\text{GL}(4) \otimes E_7$, but one can reconstruct the E_7 representation if necessary.

We decompose the $E_{11} \times l_1$ algebra in terms of $\text{GL}(4) \otimes \text{SL}(8)$ to find the positive level generators up to level 2 [35]

$$\begin{aligned} K^a_b, \quad R^I_J, \quad R^{I_1 \dots I_4}; \quad R^{a I_1 I_2}, \quad R^a_{I_1 I_2}; \\ \hat{K}^{ab}, \quad R^{a_1 a_2 I}_J, \quad R^{a_1 a_2 I_1 \dots I_4}; \dots, \end{aligned} \quad (3.3.1)$$

where $a, b, \dots = 1, \dots, 4$ and $I, J, \dots = 1, \dots, 8$. The negative level generators to level -2 are

$$R_{a I_1 I_2}, \quad R_a^{I_1 I_2}; \quad R_{a_1 a_2}^I{}_J, \quad R_{a_1 a_2 I_1 \dots I_4}, \dots, \quad (3.3.2)$$

with $a, b, \dots = 1, \dots, 4$ and $I, J, \dots = 1, \dots, 8$. All sets of similar indices are antisymmetric, and there are no constraints.

The l_1 representation is given by

$$P_a ; Z^{I_1 I_2} , Z_{I_1 I_2} ; Z^a , Z^a{}_J , Z^{a I_1 \dots I_4} ; \dots , \quad (3.3.3)$$

where $a, b, \dots = 1, \dots, 4$ and $I, J, \dots = 1, \dots, 8$, and the \bar{l}_1 representation is

$$\bar{P}_a ; \bar{Z}^{I_1 I_2} , \bar{Z}_{I_1 I_2} ; \bar{Z}^a , \bar{Z}^a{}_J , \bar{Z}^{a I_1 \dots I_4} ; \dots , \quad (3.3.4)$$

where $a, b, \dots = 1, \dots, 4$ and $I, J, \dots = 1, \dots, 8$. The Cartan involution acts on the E_{11} generators as

$$\begin{aligned} I_c(K^a{}_b) &= -K^b{}_a , \\ I_c(R^I{}_J) &= -R^J{}_I , \\ I_c(R^{I_1 \dots I_4}) &= -\star R^{I_1 \dots I_4} \equiv -\frac{1}{4!} \epsilon^{I_1 \dots I_4 J_1 \dots J_4} R^{J_1 \dots J_4} , \\ I_c(R^{a I_1 I_2}) &= -\tilde{R}_{a I_1 I_2} , \\ I_c(R^a{}_{I_1 I_2}) &= -\tilde{R}_a{}^{I_1 I_2} , \end{aligned} \quad (3.3.5)$$

and on the l_1 representation as

$$\begin{aligned} I_c(P_c) &= -\bar{P}^c , \\ I_c(Z^{I_1 I_2}) &= -\bar{Z}_{J_1 J_2} , \\ I_c(Z_{I_1 I_2}) &= -\bar{Z}^{J_1 J_2} , \\ I_c(Z^c) &= -\bar{Z}_c , \\ I_c(Z^c{}_J) &= -\bar{Z}_{c I}{}^J , \\ I_c(Z^{a I_1 \dots I_4}) &= -\bar{Z}_{c I_1 \dots I_4} . \end{aligned} \quad (3.3.6)$$

Again, J_{AB} is trivial in this case.

We then can use the commutators given in appendix D to find that the commutators of the level zero E_{11} algebra with the \bar{l}_1 representation are

$$\begin{aligned}
[K^b{}_a, \bar{P}^c] &= -\delta_a^c \bar{P}^b + \frac{1}{2} \delta_a^b \bar{P}^c, \\
[K^b{}_a, \bar{Z}_{I_1 I_2}] &= -\frac{1}{2} \delta_a^b \bar{Z}_{I_1 I_2}, \\
[K^b{}_a, \bar{Z}^{I_1 I_2}] &= -\frac{1}{2} \delta_a^b \bar{Z}^{I_1 I_2}, \\
[K^b{}_a, \bar{Z}_c] &= -\delta_c^b \bar{Z}_a - \frac{1}{2} \delta_a^b \bar{Z}_c, \\
[R^I{}_J, \bar{P}^c] &= 0, \\
[R^I{}_J, \bar{Z}_{I_1 I_2}] &= -2\delta_{[I_1}^J \bar{Z}_{|I|I_2]} + \frac{1}{4} \delta_I^J \bar{Z}_{I_1 I_2}, \\
[R^I{}_J, \bar{Z}^{I_1 I_2}] &= 2\delta_I^{[I_1} \bar{Z}^{J|I_2]} - \frac{1}{4} \delta_I^J \bar{Z}^{I_1 I_2}, \\
[R^I{}_J, \bar{Z}_a] &= 0, \\
[R^{I_1 \dots I_4}, \bar{P}^a] &= 0, \\
[R^{I_1 \dots I_4}, \bar{Z}_{J_1 J_2}] &= -\delta_{J_1 J_2}^{[I_1 I_2} \bar{Z}^{I_3 I_4]}, \\
[R^{I_1 \dots I_4}, \bar{Z}^{J_1 J_2}] &= -\frac{1}{24} \varepsilon^{I_1 \dots I_4 J_1 \dots J_4} \bar{Z}_{J_3 J_4}, \\
[R^{I_1 \dots I_4}, \bar{Z}_a] &= 0.
\end{aligned} \tag{3.3.7}$$

We calculate that the commutators of the level 1 E_{11} generators with the l_1 generators are

$$\begin{aligned}
[R^{a I_1 I_2}, \bar{P}^b] &= 0, \\
[R^{a I_1 I_2}, \bar{Z}_{J_1 J_2}] &= -2\delta_{J_1 J_2}^{I_1 I_2} \bar{P}^a, \\
[R^{a I_1 I_2}, \bar{Z}^{J_1 J_2}] &= 0, \\
[R^a{}_{I_1 I_2}, \bar{P}^b] &= 0, \\
[R^a{}_{I_1 I_2}, \bar{Z}_{J_1 J_2}] &= 0, \\
[R^a{}_{I_1 I_2}, \bar{Z}^{J_1 J_2}] &= -2\delta_{I_1 I_2}^{J_1 J_2} \bar{P}^a, \\
[R^{a I_1 I_2}, \bar{Z}_b] &= 2\delta_b^a \bar{Z}^{I_1 I_2}, \\
[R^a{}_{I_1 I_2}, \bar{Z}_b] &= -2\delta_b^a \bar{Z}_{I_1 I_2}.
\end{aligned} \tag{3.3.8}$$

We find that the commutators with the level -1 generators are

$$\begin{aligned}
[\tilde{R}_{aI_1I_2}, \bar{P}^b] &= -\delta_a^b \bar{Z}_{I_1I_2}, \\
[\tilde{R}_a{}^{I_1I_2}, \bar{P}^b] &= \delta_a^b \bar{Z}^{I_1I_2}, \\
[\tilde{R}_a{}^{I_1I_2}, \bar{Z}_{J_1J_2}] &= -\delta_{J_1J_2}^{I_1I_2} \bar{Z}_a, \\
[\tilde{R}_{aI_1I_2}, \bar{Z}^{J_1J_2}] &= -\delta_{I_1I_2}^{J_1J_2} \bar{Z}_a.
\end{aligned} \tag{3.3.9}$$

We choose to work with the algebra which is invariant under the tangent space group $I_c(E_{11})$ as we did in the 5D case. At level zero, this algebra is $SO(1,3) \otimes SU(8) = I_c(GL(4) \otimes E_7)$. We find that the generators which are invariant under $I_c(E_{11})$ are

$$\begin{aligned}
J^a{}_b &= K^a{}_b - K^b{}_a, \\
J^I{}_J &= K^I{}_J - K^J{}_I, \\
S^{I_1 \dots I_4} &= R^{I_1 \dots I_4} - \star R^{I_1 \dots I_4}, \\
S_{\pm}^{aI_1I_2} &= R^{aI_1I_2} - \tilde{R}_{aI_1I_2} \pm i(R^a{}_{I_1I_2} + \tilde{R}_a{}^{I_1I_2}), \\
S^{a_1a_2I}{}_J &= R^{a_1a_2I}{}_J - \tilde{R}_{a_1a_2}{}^J{}_I, \\
S^{a_1a_2I_1 \dots I_4} &= R^{a_1a_2I_1 \dots I_4} + \star \tilde{R}_{a_1a_2}{}^{I_1 \dots I_4}, \\
S^{ab} &= \hat{K}^{ab} - \tilde{K}_{ab},
\end{aligned} \tag{3.3.10}$$

where the \star represents the dual $\star R^{I_1 \dots I_4} \equiv \epsilon^{I_1 \dots I_4}{}_{J_1 \dots J_4} R^{J_1 \dots J_4}$.

The remaining coset generators in E_{11} are given by

$$\begin{aligned}
T^a{}_b &= K^a{}_b + K^b{}_a, \\
T^I{}_J &= K^I{}_J + K^J{}_I, \\
T^{I_1 \dots I_4} &= R^{I_1 \dots I_4} + \star R^{I_1 \dots I_4}, \\
T_{\pm}^{aI_1I_2} &= R^{aI_1I_2} + \tilde{R}_{aI_1I_2} \pm i(R^a{}_{I_1I_2} - \tilde{R}_a{}^{I_1I_2}), \\
T^{a_1a_2I}{}_J &= R^{a_1a_2I}{}_J + \tilde{R}_{a_1a_2}{}^J{}_I, \\
T^{a_1a_2I_1 \dots I_4} &= R^{a_1a_2I_1 \dots I_4} - \star \tilde{R}_{a_1a_2}{}^{I_1 \dots I_4}, \\
S^{ab} &= \hat{K}^{ab} + \tilde{K}_{ab}.
\end{aligned} \tag{3.3.11}$$

The l_1 representation decomposed into the $\text{SO}(1,3) \otimes \text{SU}(8)$ representation is

$$P_a ; Z_{\pm}^{I_1 I_2} ; Z^a , Z_{+}^{a I_1 \dots I_4} , Z_{-}^{a I_1 \dots I_4} , Z_A^{a I J} , Z_S^{a I J} ; \dots , \quad (3.3.12)$$

where

$$\begin{aligned} Z_{\pm}^{I_1 I_2} &= Z^{I_1 I_2} \pm i Z_{I_1 I_2} , \\ Z_{+}^{a I_1 \dots I_4} &= Z^{a I_1 \dots I_4} + \frac{1}{4!} \epsilon^{I_1 \dots I_4 K_1 \dots K_4} Z^{a K_1 \dots K_4} , \\ Z_{-}^{a I_1 \dots I_4} &= Z^{a I_1 \dots I_4} - \frac{1}{4!} \epsilon^{I_1 \dots I_4 K_1 \dots K_4} Z^{a K_1 \dots K_4} , \\ Z_A^{a I J} &= \frac{1}{2} (Z^{a I J} - Z^{a J I}) , \\ Z_S^{a I J} &= \frac{1}{2} (Z^{a I J} + Z^{a J I}) . \end{aligned} \quad (3.3.13)$$

The commutators of the $E_{11} \ltimes l_1$ algebra when decomposed into representations of $\text{GL}(4) \otimes \text{SU}(8)$ can be found in the appendix D. We then find the commutators of the $I_c(E_{11})$ generators $S_{\pm}^{a I_1 I_2}$ with the generators of the l_1 representation to be

$$\begin{aligned} [S_{\pm}^{a I_1 I_2}, P_b] &= \delta_b^a Z_{\pm}^{I_1 I_2} , \\ [S_{\pm}^{a I_1 I_2}, Z_{\pm J_1 J_2}] &= -Z_{+}^{a I_1 I_2 J_1 J_2} \pm 2i \delta_{[J_1}^{[I_1} Z_S^{a J_2] I_2]} , \\ [S_{\pm}^{a I_1 I_2}, Z_{\mp J_1 J_2}] &= -4\delta_{J_1 J_2}^{I_1 I_2} P_a \pm 2i \delta_{J_1 J_2}^{I_1 I_2} Z^a - Z_{-}^{a I_1 I_2 J_1 J_2} \pm 2i \delta_{[J_1}^{[I_1} Z_A^{a J_2] I_2]} , \\ [S_{\pm}^{a I_1 I_2}, Z^b] &= \mp 2i \delta_a^b Z_{\pm}^{I_1 I_2} , \\ [S_{\pm a}^{I_1 I_2}, Z_{S J K}^b] &= \pm 8i \delta_a^b \delta_{[J}^{[I_1} Z_{\mp}^{I_2] K]} , \\ [S_{\pm a}^{I_1 I_2}, Z_{A J K}^b] &= \mp 8i \delta_a^b \delta_{[J}^{[I_1} Z_{\pm}^{I_2] K]} , \\ [S_{\pm}^{a I_1 I_2}, Z_{+}^{b J_1 \dots J_4}] &= 12\delta_a^b (\delta_{[J_1 J_2}^{I_1 I_2} Z_{\mp J_3 J_4}] + \frac{1}{4!} \epsilon^{J_1 \dots J_4 I_1 I_2 K_1 K_2} Z_{\mp K_1 K_2}) , \\ [S_{\pm}^{a I_1 I_2}, Z_{-}^{b J_1 \dots J_4}] &= 12\delta_a^b (\delta_{[J_1 J_2}^{I_1 I_2} Z_{\pm J_3 J_4}] - \frac{1}{4!} \epsilon^{J_1 \dots J_4 I_1 I_2 K_1 K_2} Z_{\pm K_1 K_2}) . \end{aligned} \quad (3.3.14)$$

We write the group element in terms of the above $I_c(E_{11})$ generators, finding

$$\begin{aligned} g_l &= \exp(x^a P_a + x_{\pm I_1 I_2} Z^{\pm I_1 I_2} + \hat{x}_a Z^a + x_{a I_1 \dots I_4}^{+} Z_{+}^{a I_1 \dots I_4} \\ &\quad + x_{a I_1 \dots I_4}^{-} Z_{-}^{a I_1 \dots I_4} + x_a^A Z_A^{a I J} + x_a^S Z_S^{a I J} + \dots) , \end{aligned} \quad (3.3.15)$$

$$\begin{aligned}
g_A = & \exp(h_a{}^b K^a{}_b) \exp(\varphi_I{}^J R^I{}_J) \exp(\varphi_{I_1\dots I_4} R^{I_1\dots I_4}) \exp(\hat{h}_{ab} \hat{K}^{(ab)}) \\
& \times \exp(A_{a_1 a_2}{}^J{}_I R^{a_1 a_2 I}{}_J) \exp(A_{a_1 a_2 I_1 \dots I_4} R^{a_1 a_2 I_1 \dots I_4}) \\
& \times \exp(A_{a I_1 I_2} R^{a I_2 I_1} + A_a{}^{I_1 I_2} R^a{}_{I_1 I_2}), \dots,
\end{aligned} \tag{3.3.16}$$

where g_l is parametrised by

$$x^a; \quad x_{\pm I_1 I_2}; \quad \hat{x}_a, \quad x_{a I_1 \dots I_4}^-, \quad x_{a I_1 \dots I_4}^+, \quad x_a^S{}^J, \quad x_a^A{}^J \dots, \tag{3.3.17}$$

which again are the coordinates of generalised space-time and g_A is parametrised by the fields living on the generalised space-time

$$h_a{}^b, \quad \varphi_I{}^J; \quad A_{a I_1 I_2}, \quad A_a{}^{I_1 I_2}; \quad \hat{h}_{ab}, \quad A_{a_1 a_2}{}^I{}_J, \quad A_{a_1 a_2 I_1 \dots I_4}, \dots \tag{3.3.18}$$

The local $I_c(E_{11})$ symmetry has been used to set to zero all the coefficients of the negative level generators in the group element. Indices are raised and lowered with the delta functions δ^{ab} and δ^{IJ} for Lorentz indices and internal indices, respectively.

We now find the tangent space metric in 4 dimensions.

3.3.1 Tangent space metric

The matter representation is given by

$$V^A = (T^a, T_{\pm I_1 I_2}, \hat{T}_a, T_{a I_1 \dots I_4}^+, T_{a I_1 \dots I_4}^-, T_a^S{}^J, T_a^A{}^J, \dots). \tag{3.3.19}$$

We want to find dynamics which are invariant under the level zero transformation which is local Lorentz and $SU(8)$, and at level one, the infinitesimal transformation

$$h = 1 - \sum_{\pm} \Lambda_{a \pm I_1 I_2} S_{\pm}^{a I_1 I_2}. \tag{3.3.20}$$

Under this transformation, we find the components of the matter representation transform as

$$\begin{aligned}
\delta T^a &= -4 \sum_{\pm} \Lambda_{a\pm J_1 J_2} T_{\mp}^{I_1 I_2}, \\
\delta T_{\pm I_1 I_2} &= \Lambda_{a\pm I_1 I_2} T^a \mp 2i \Lambda_{a\pm I_1 I_2} \hat{T}^a + 24 \Lambda_{a\mp J_1 J_2} T_a^{+J_1 J_2}{}_{I_1 I_2} \\
&\quad + 24 \Lambda_{a\pm J_1 J_2} T_a^{-J_1 J_2}{}_{I_1 I_2} \mp 8i \Lambda_{a\pm J[I_1} T_a^{AJ}{}_{|I_2]} \mp 8i \Lambda_{a\mp J[I_1} T_a^{SJ}{}_{|I_2]}, \\
\delta \hat{T}_a &= 2i \sum_{\pm} \pm \Lambda_{a\pm I_1 I_2} T_{\mp}^{I_1 I_2}, \\
\delta T_a^+{}_{I_1 \dots I_4} &= -\frac{1}{2} \sum_{\pm} (\Lambda_{a\pm[I_1 I_2} T_{\pm I_3 I_4]} + \frac{1}{4!} \epsilon_{I_1 \dots I_4}{}^{K_1 \dots K_4} \Lambda_{a\pm K_1 K_2} T_{\pm K_3 K_4}), \\
\delta T_a^-{}_{I_1 \dots I_4} &= -\frac{1}{2} \sum_{\pm} (\Lambda_{a\pm[I_1 I_2} T_{\mp I_3 I_4]} - \frac{1}{4!} \epsilon_{I_1 \dots I_4}{}^{K_1 \dots K_4} \Lambda_{a\pm K_1 K_2} T_{\mp K_3 K_4}), \\
\delta T_{S I J}^a &= -\sum_{\pm} \pm 2i \Lambda_{a\pm L} (J T_{\pm I})^L, \\
\delta T_{A I J}^a &= -\sum_{\pm} \pm 2i \Lambda_{a\pm L} [J T_{\mp I}]^L,
\end{aligned} \tag{3.3.21}$$

and we find the invariant quantity from equation (1.4.23) up to level 2 to be

$$\begin{aligned}
\Delta &= K_{AB} V^A V^B = T^a T_a + 4 \sum_{\pm} T_{\pm I_1 I_2} T_{\mp}^{I_1 I_2} + 4 \hat{T}_a \hat{T}^a \\
&\quad + 24 T_{a I_1 \dots I_4}^+ T^{+a I_1 \dots I_4} + 24 T_{a I_1 \dots I_4}^- T^{-a I_1 \dots I_4} \\
&\quad + 4 T_a^{S I}{}_J T^{S a}{}_I{}^J + 4 T_a^{A I}{}_J T^{A a}{}_I{}^J + \dots
\end{aligned} \tag{3.3.22}$$

Hence, we find that the invariant metric is

$$K_{AB} = \begin{pmatrix} \delta_{ab} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4\delta^{I_1 I_2, J_1 J_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 4\delta^{I_1 I_2, J_1 J_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4\delta^{ab} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 24\delta^{ab} \delta^{I_1 \dots I_4, J_1 \dots J_4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 24\delta^{ab} \delta^{I_1 \dots I_4, J_1 \dots J_4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4\delta^{ab} \delta_{IK} \delta^{JL} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4\delta^{ab} \delta_{IK} \delta^{JL} \end{pmatrix}. \tag{3.3.23}$$

We can now use this metric to derive the gauge fixing conditions in 4 dimensions.

3.3.2 Gauge fixing conditions

The space-time derivatives with respect to the generalised space-time coordinates in equation (3.3.17) are

$$\partial_A = (\partial_a, \partial_{\pm}^{I_1 I_2}, \hat{\partial}^a, \dots), \quad (3.3.24)$$

and hence we find that G^C componentwise is

$$K^{AB}G_{A,B}{}^c = (\det e)^{\frac{1}{2}} (\partial^a h_a{}^c - \frac{1}{2} \partial^c h_a{}^a), \quad (3.3.25)$$

$$\begin{aligned} K^{AB}G_{A,B+K_1 K_2} &= -(\det e)^{\frac{1}{2}} \left(\frac{1}{\det \beta} \partial^a A_{a K_1 K_2} + \frac{1}{4 \det \beta} \partial^{I_1 I_2} \varphi_{K_1 K_2 I_1 I_2} \right. \\ &\quad \left. + \frac{1}{4} \partial_{I_1 I_2} \varphi_{K_1 K_2}^{I_1 I_2} + \frac{1}{8} \partial_{K_1 K_2} h_a{}^a \right), \end{aligned} \quad (3.3.26)$$

$$\begin{aligned} K^{AB}G_{A,B-K_1 K_2} &= -(\det e)^{\frac{1}{2}} \delta_{I_1 I_2, K_1 K_2} \left(\frac{1}{\det \beta} \partial^a A_a{}^{I_1 I_2} \right. \\ &\quad \left. + \frac{1}{4 \det \beta} \partial_{J_1 J_2} \star \varphi^{I_1 I_2 J_1 J_2} \right. \\ &\quad \left. + \frac{1}{4} \partial^{J_1 J_2} \varphi_{J_1 J_2}^{I_1 I_2} + \frac{1}{8} \partial^{I_1 I_2} h_a{}^a \right), \end{aligned} \quad (3.3.27)$$

$$\begin{aligned} K^{AB}G_{A,Bc} &= -(\det e)^{\frac{1}{2}} \left(\partial^a \hat{h}_{(ac)} + \frac{1}{4 \det \beta} \partial^{I_1 I_2} A_{c I_1 I_2} \right. \\ &\quad \left. - \frac{1}{4 \det \beta} \partial_{I_1 I_2} A_c{}^{I_1 I_2} + \frac{1}{4} \hat{\partial}_a h_c{}^a - \frac{1}{8} \hat{\partial}_c h_a{}^a \right). \end{aligned} \quad (3.3.28)$$

where $\det \beta$ is the determinant of the two by two sub-matrix of the vielbein, composed only of the components corresponding to the level two generalised coordinates, as defined in [51]. The derivatives $\hat{\partial}$ represent the level 2 coordinates, with the hat notation differentiating it from the usual spacetime coordinates. We again choose $G^C = 0$ and see the result is a set of gauge fixing conditions for E_{11} in 4 dimensions.

Lastly, we find the gauge fixed multiplet for the A_1^{+++} Lie algebra, which is the algebra from which one can find gravity in 4 dimensions in the non-linear realisation.



FIGURE 3.4: Dynkin diagram of A_1^{+++} which leads to a description of 4 dimensional gravity.

3.4 A_1^{+++}

Finally, we find the gauge fixed multiplet for the Lie algebra A_1^{+++} , using the non-linear realisation of A_1^{+++} with its first fundamental representation, which is the Lie algebra leading to a low energy effective action for 4-dimensional gravity. Its Dynkin diagram is shown in 3.4 where node 4 has been deleted and we decompose the $A_1^{+++} \times l_1$ algebra in terms of the remaining $GL(4)$ algebra. Up to level 2, the generators of A_1^{+++} in the $GL(4)$ representation are

$$K^a{}_b ; R^{ab} ; R^{ab,cd} ; \dots , \quad (3.4.1)$$

with $a, b, \dots = 1, \dots, 4$, and which satisfy

$$\begin{aligned} R^{ab} &= R^{(ab)} , \\ R^{ab,cd} &= R^{[ab],[cd]} . \end{aligned} \quad (3.4.2)$$

The negative level generators up to level -2 are

$$R_{ab} ; R_{ab,cd} ; \dots , \quad (3.4.3)$$

and satisfy analogous conditions, and where $a, b, \dots = 1, \dots, 4$. Then the l_1 representation has the generators

$$P_a ; Z^a ; Z^{abc} , Z^{ab,c} ; \dots , \quad (3.4.4)$$

where $a, b, \dots = 1, \dots, 4$, and with the constraints

$$Z^{abc} = Z^{(abc)} , \quad Z^{ab,c} = Z^{[ab],c} , \quad Z^{[ab,c]} = 0 . \quad (3.4.5)$$

The \bar{l}_1 representation is

$$\bar{P}_a ; \bar{Z}^a ; \bar{Z}^{abc} , \bar{Z}^{ab,c} ; \dots , \quad (3.4.6)$$

where $a, b, \dots = 1, \dots, 4$, and with analogous constraints as in equation (3.4.5). The commutators of the adjoint and fundamental representation generators are given in the appendix E. The Cartan involution acts on the A_1^{+++} generators as

$$\begin{aligned} I_c(K^a_b) &= -K^b_a , \\ I_c(R_{ab}) &= -R^{ab} , \\ I_c(R^{ab,cd}) &= R_{ab,cd} , \end{aligned} \quad (3.4.7)$$

and on the l_1 representation, resulting in the \bar{l}_1 representation, as

$$\begin{aligned} I_c(P_c) &= -\bar{P}^c , \\ I_c(Z^c) &= -\bar{Z}_c , \\ I_c(Z^{cde}) &= -\bar{Z}_{cde} , \\ I_c(Z^{cd,e}) &= -\bar{Z}_{cd,e} . \end{aligned} \quad (3.4.8)$$

Notice that J_{AB} is once again trivial in this case. The using the commutators of the l_1 representation, we find that the commutators of the level zero generators with the \bar{l}_1 algebra are

$$\begin{aligned} [K^b_a, \bar{P}^c] &= \delta_a^c \bar{P}^b - \frac{1}{2} \delta_a^b \bar{P}^c , \\ [K^b_a, \bar{Z}^c] &= -\delta_c^b \bar{Z}^a - \frac{1}{2} \delta_a^b \bar{Z}^c , \\ [K^b_a, \bar{Z}_{cde}] &= -\delta_c^b \bar{Z}_{ade} - \delta_d^b \bar{Z}_{cae} - \delta_e^b \bar{Z}_{cda} - \frac{1}{2} \delta_a^b \bar{Z}_{cde} , \\ [K^b_a, \bar{Z}_{cd,e}] &= -\delta_c^b \bar{Z}_{ad,e} - \delta_d^b \bar{Z}_{ca,e} - \delta_e^b \bar{Z}_{cd,a} - \frac{1}{2} \delta_a^b \bar{Z}_{cd,e} . \end{aligned} \quad (3.4.9)$$

Then we find that the commutators with the positive level generators of $GL(4)$ and \bar{l}_1 are

$$[R^{ab}, \bar{P}^c] = 0 ,$$

$$\begin{aligned}
[R^{ab}, \bar{Z}_c] &= -2\delta_c^{(a}\bar{P}^{b)}, \\
[R^{ab}, \bar{Z}_{cde}] &= -\frac{2}{3}(\delta_{cd}^{(ab)}\bar{Z}_e + \delta_{de}^{(ab)}\bar{Z}_c + \delta_{ec}^{(ab)}\bar{Z}_d), \\
[R^{ab}, \bar{Z}_{cd,e}] &= -\frac{4}{3}(\delta_{de}^{(ab)}\bar{Z}_c - \delta_{ce}^{(ab)}\bar{Z}_d).
\end{aligned} \tag{3.4.10}$$

Finally, we calculate the commutators with the level 1 generators of $GL(4)$ and \bar{l}_1 to be

$$\begin{aligned}
[R_{ab}, \bar{P}^c] &= -\delta_{(a}^c\bar{Z}_{b)}, \\
[R_{ab}, \bar{Z}^c] &= -\bar{Z}_{abc} - \bar{Z}_{c(a,b)}, \\
[R_{ab,cd}, \bar{P}^e] &= -\delta_{[a}^e\bar{Z}_{b]cd} + \frac{1}{4}(\delta_a^e\bar{Z}_{b(c,d)} - \delta_b^e\bar{Z}_{a(c,d)}) - \frac{3}{8}(\delta_c^e\bar{Z}_{ab,d} + \delta_a^e\bar{Z}_{ab,c}).
\end{aligned} \tag{3.4.11}$$

This completes the algebra that we need to calculate the tangent space metric. We can now begin its construction.

The group element $g = g_l g_A$ is

$$g_l = \exp(x^a P_a + y_a Z^a + x_{abc} Z^{abc} + x_{ab,c} Z^{ab,c}) \dots, \tag{3.4.12}$$

$$g_A = \exp(h_a{}^b K^a{}_b) \exp(A_{ab,cd} R^{ab,cd}) \exp(A_{ab} R^{ab}) \dots, \tag{3.4.13}$$

where g_l is parametrised by the space-time coordinates

$$x^a; y_a; x_{abc}, x_{ab,c}, \tag{3.4.14}$$

and g_A is parametrised by the fields living on the space-time

$$h_a{}^b; A_{ab}; A_{ab,cd}. \tag{3.4.15}$$

We now build the tangent space metric for A_1^{+++} .

3.4.1 Tangent space metric

We find that the Tangent object from equation (1.4.10) is then

$$V^A = (T^a, \hat{T}_a, T_{abc}, T_{ab,c}, \dots). \quad (3.4.16)$$

The dynamics should be invariant under the Cartan Involution invariant subgroup. At level zero, this is $SO(4)$. To be invariant under all remaining generators of the subgroup, it must be invariant under

$$h = 1 - \Lambda_{ab}(R^{ab} - \eta^{ab}\eta^{cd}R_{cd}). \quad (3.4.17)$$

We find that the components of the matter representation transform as

$$\begin{aligned} \delta T^d &= -2\Lambda^{cd}\hat{T}_c, \\ \delta \hat{T}_d &= \Lambda_{cd}T^c - 2\Lambda^{ce}T_{ced} - \frac{8}{3}\Lambda^{ce}T_{dc,e}, \\ \delta T_{abc} &= \Lambda_{(ab}\hat{T}_{c)}, \\ \delta T_{ab,c} &= \Lambda_{c[a}\hat{T}_{b]}, \end{aligned} \quad (3.4.18)$$

under h , where indices are raised and lowered with the Kronecker delta, δ_{ab} . Thus, we find that the invariant quantity from equation (1.4.23) is

$$K_{AB}V^AV^B = T^aT_a + 2\hat{T}_a\hat{T}^a + 4T_{abc}T^{abc} + \frac{16}{3}T_{ab,c}T^{ab,c} + \dots, \quad (3.4.19)$$

and hence we find that the invariant metric is

$$K_{AB} = \begin{pmatrix} \delta_{ab} & 0 & 0 & 0 \\ 0 & 2\delta^{ab} & 0 & 0 \\ 0 & 0 & 4\delta^{abc,def} & 0 \\ 0 & 0 & 0 & \frac{16}{3}(\delta^{ab,de}\delta^{cf} - \delta^{abc,def}) \end{pmatrix}. \quad (3.4.20)$$

This ends the calculation of the tangent space metric for A_1^{+++} . We finally can use these

results to calculate gauge-fixing conditions which arise as an $I_c(A_1^{+++})$ invariant multiplet in A_1^{+++} .

3.4.2 Gauge fixing conditions

From equation (3.4.14), we see that the generalised space-time derivatives are given by

$$\partial_A = (\partial_a, \partial^{\hat{a}}, \partial_{abc}, \partial_{ab,c}, \dots), \quad (3.4.21)$$

and we find that the multiplet G^C of equation (1.4.27) componentwise is

$$K^{AB}G_{A,B}{}^c = (\det e)^{\frac{1}{2}}(\partial^a h_a{}^c - \frac{1}{2}\partial^c h_a{}^a), \quad (3.4.22)$$

$$K^{AB}G_{A,Bc} = -(\det e)^{\frac{1}{2}}(\partial^a A_{ac} + \frac{1}{2}\partial_{\hat{a}} h_c{}^a + \frac{1}{2}\partial_{\hat{c}} h_a{}^a), \quad (3.4.23)$$

$$\begin{aligned} K^{AB}G_{A,Bc_1c_2c_3} &= (\det e)^{\frac{1}{2}}(\partial^a(A_{a(c_1,c_2c_3)}) - \frac{1}{2}\partial_{[\hat{c}_1} A_{c_2c_3]} \\ &\quad - \frac{3}{4}\partial_{(c_1c_2|a|} h_{c_3)}{}^a) - \frac{1}{8}\partial_{c_1c_2c_3} h_a{}^a), \end{aligned} \quad (3.4.24)$$

$$\begin{aligned} K^{AB}G_{A,B[c_1c_2],c_3} &= (\det e)^{\frac{1}{2}}(\partial^a(\frac{3}{4}A_{[c_1c_2],c_3a} - \frac{1}{2}A_{a[c_1,c_2]c_3}) \\ &\quad - \frac{1}{2}\partial_{[\hat{c}_1} A_{c_2]c_3} - \frac{3}{16}(2\partial_{[c_1|a,c_3|} h_{c_2]}{}^a + \partial_{[c_1c_2],a} h_{c_3}{}^a \\ &\quad - 3\partial_{[c_1c_2],a|} h_{c_3]}{}^a + \frac{1}{2}\partial_{[c_1c_2],c_3} h_a{}^a)). \end{aligned} \quad (3.4.25)$$

Setting $G^C = 0$ gives us the gauge fixing conditions.

This completes the calculation of the gauge-fixing conditions in 11D, 5D, 4D, and for A_1^{+++} . In the next chapter, we study the 7D theory in more detail and find the equation of motions of the 7D supergravity theory from an E_{11} viewpoint.

Chapter 4

Equations of motion in 7 dimensions

In this chapter, we find the equations of motion in the 7 dimensional decomposition of E_{11} . The work is to be published [24]. Similar calculations have been done in 11D and 5D [25,34], and partially completed in 4D [35]. The chapter will follow a similar structure to the derivation of the 11D equations of motion in [34]. We begin by calculating the non-linear realisation in 7D.

4.1 Cartan forms

We derived the algebra of the 7D theory in section 2.2, and shall repeat the generators here for ease of reference. Recall that deleting node 7 on the E_{11} Dynkin diagram, repeated in figure 4.1, results in a $GL(7) \otimes SL(5)$ algebra.

Then the generators up to level 6 are

$$\begin{aligned}
 &K^a{}_b, R^M{}_N; R^{aMN}; R^{a_1 a_2}{}_M; R^{a_1 a_2 a_3}{}_M; R^{a_1 \dots a_4}{}_{MN}; R^{a_1 \dots a_5}{}_M{}_N, \\
 &R^{a_1 \dots a_4, b}; R^{a_1 \dots a_6}{}_{MN, P}, R^{a_1 \dots a_6}{}^{(MN)}, R^{a_1 \dots a_5, b}{}^{MN}; \dots, \quad (4.1.1)
 \end{aligned}$$

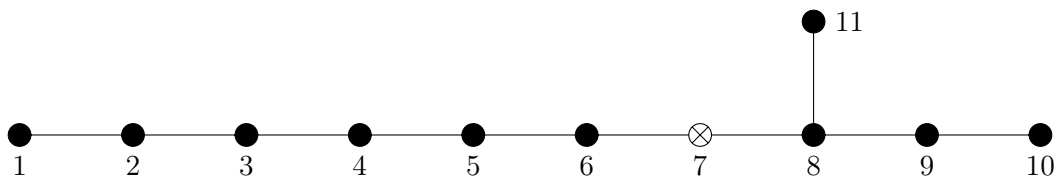


FIGURE 4.1: Dynkin diagram of E_{11} corresponding to the 7 dimensional maximal supergravity theory.

where the lower case Latin indices correspond to $SL(7)$, such that $a, b, \dots = 1, \dots, 7$, and the upper case Latin indices correspond to $SL(5)$, where $M, N, \dots = 1, \dots, 5$. Recall that the indices surrounded by round brackets are symmetric in their permutation, and the blocks with no brackets, separated by commas, are totally antisymmetric. The generators satisfy irreducibility constraints

$$\sum_N R^N{}_N = 0; \quad \sum_N R^{a_1 \dots a_5 N}{}_N = 0, \quad R^{[a_1 \dots a_4, b]} = 0; \quad R^{a_1 \dots a_6}{}_{[MN, P]} = 0. \quad (4.1.2)$$

Similarly, the negative level generators are

$$\begin{aligned} & R_{aMN}; \quad R_{a_1 a_2}{}^M; \quad R_{a_1 a_2 a_3 M}; \quad R_{a_1 \dots a_4 MN}; \quad R_{a_1 \dots a_5 M}{}^N, \\ & R_{a_1 \dots a_4, b}; \quad R_{a_1 \dots a_6}{}^{MN, P}, \quad R_{a_1 \dots a_6 (MN)}, \quad R_{a_1 \dots a_5, b MN}; \dots, \end{aligned} \quad (4.1.3)$$

where $a, b, a_1, \dots = 1, \dots, 7$, and $M, N, \dots = 1, \dots, 5$, and they satisfy analogous constraints as those given in equation (4.1.2).

Recall that the l_1 generators when decomposed into $SL(7) \otimes SL(5)$ are

$$\begin{aligned} & P_a; \quad Z^{MN}; \quad Z^a{}_M; \quad Z^{a_1 a_2 M}; \quad Z^{a_1 a_2 a_3}{}_{MN}; \quad Z^{a_1 a_2 a_3, b}, \quad Z^{a_1 \dots a_4}, \quad Z^{a_1 \dots a_4 M}{}_N; \\ & Z^{a_1 \dots a_5 MN}, \quad Z^{a_1 \dots a_5 (MN)}, \quad Z^{a_1 \dots a_5}{}_{MN, P}, \quad Z^{a_1 \dots a_4, b MN}; \dots, \end{aligned} \quad (4.1.4)$$

with $a, b, a_1, \dots = 1, \dots, 7$ and $M, N, \dots = 1, \dots, 5$.

We can now implement the non-linear realisation as described in section 1.4.1. We write the group element of $E_{11} \ltimes l_1$ as $g = g_l g_E$ where

$$\begin{aligned} g_E = & \dots e^{A_{a_1 \dots a_5, b MN} R^{a_1 \dots a_5, b MN}} e^{A_{a_1 \dots a_6 (MN)} R^{a_1 \dots a_6 (MN)}} e^{A_{a_1 \dots a_6}{}^{MN, P} R^{a_1 \dots a_6}{}_{MN, P}} \\ & \times e^{h_{a_1 \dots a_4, b} R^{a_1 \dots a_4, b}} e^{\varphi_{a_1 \dots a_5 M}{}^N R^{a_1 \dots a_5 M}{}_N} e^{A_{a_1 \dots a_4}{}^{MN} R^{a_1 \dots a_4}{}_{MN}} \\ & \times e^{A_{a_1 a_2 a_3 M} R^{a_1 a_2 a_3 M}} e^{A_{a_1 a_2}{}^M R^{a_1 a_2}{}_M} e^{A_{a MN} R^{a MN}} e^{\varphi_M{}^N R^M{}_N} e^{h_a{}^b K^a{}_b}, \end{aligned} \quad (4.1.5)$$

and

$$g_l = e^{x^a P_a} e^{x_{MN} Z^{MN}} e^{x_a{}^M Z^a{}_M} e^{x_{a_1 a_2 M} Z^{a_1 a_2 M}} e^{x_{a_1 a_2 a_3}{}^{MN} Z^{a_1 a_2 a_3}{}_{MN}}$$

$$\times e^{x_{a_1 a_2 a_3, b} Z^{a_1 a_2 a_3, b}} e^{x_{a_1 \dots a_4} Z^{a_1 \dots a_4}} e^{x_{a_1 \dots a_4 M^N} Z^{a_1 \dots a_4 M^N}} + \dots \quad (4.1.6)$$

The parameters of g_E are

$$\begin{aligned} h_a^b, \quad \varphi_M^N; \quad A_{aMN}; \quad A_{a_1 a_2}^M; \quad A_{a_1 a_2 a_3 M}; \\ A_{a_1 \dots a_4}^{MN}; \quad h_{a_1 \dots a_4, b}, \quad \varphi_{a_1 \dots a_5 M^N}; \dots, \end{aligned} \quad (4.1.7)$$

and will turn out to be the fields of the theory, and these will depend on the parameters of g_l , which are

$$\begin{aligned} x^a; \quad x_{MN}; \quad x_a^M; \quad x_{a_1 a_2 M}; \quad x_{a_1 a_2 a_3}^{MN}; \\ x_{a_1 a_2 a_3, b}, \quad x_{a_1 \dots a_4}, \quad x_{a_1 \dots a_4 M^N}; \dots, \end{aligned} \quad (4.1.8)$$

and correspond to the generalised spacetime coordinates. In addition, we notice that we have gauged away the negative level generators in our group element.

The Cartan form from equation (1.4.6) is then

$$\begin{aligned} \mathcal{V}_E = G_a^b K^a_b + G_M^N R^M_N + G_{aMN} R^{aMN} + G_{a_1 a_2}^M R^{a_1 a_2}_M \\ + G_{a_1 a_2 a_3 M} R^{a_1 a_2 a_3 M} + G_{a_1 \dots a_4}^{MN} R^{a_1 \dots a_4}_{MN} \\ + G_{a_1 \dots a_4, b} R^{a_1 \dots a_4, b} + G_{a_1 \dots a_5 M^N} R^{a_1 \dots a_5 M^N}_N \\ + G_{a_1 \dots a_6}^{MN, P} R^{a_1 \dots a_6}_{MN, P} + G_{a_1 \dots a_6 (MN)} R^{a_1 \dots a_6 (MN)} \\ + G_{a_1 \dots a_5, b MN} R^{a_1 \dots a_5, b MN} + \dots \end{aligned} \quad (4.1.9)$$

The level 5 Cartan forms contain the dual graviton and the dual scalar. Notice that we have truncated the Cartan form at level 6, ignoring level 7 and above, as this is all we will need in this section.

We calculate the explicit form of the Cartan forms in terms of the fields of equation (4.1.5) up to level 4, finding

$$G_a^b = (e^{-1} de)_a^b,$$

$$\begin{aligned}
G_M^N &= (f^{-1}df)_M^N, \\
G_{aMN} &= e_a^\mu f_M^{\dot{M}} f_N^{\dot{N}} dA_{\mu\dot{M}\dot{N}}, \\
G_{a_1 a_2}^M &= e_{a_1}^{\mu_1} e_{a_2}^{\mu_2} f_M^{\dot{M}} (dA_{\mu_1 \mu_2 \dot{M}} - \frac{1}{2} \varepsilon^{\dot{M}\dot{N}\dot{P}\dot{Q}\dot{R}} A_{[\mu_1 \dot{N} \dot{P} \dot{Q} \dot{R}] \dot{M}} dA_{\mu_2 \dot{Q} \dot{R}}), \\
G_{a_1 a_2 a_3 M} &= e_{a_1}^{\mu_1} e_{a_2}^{\mu_2} e_{a_3}^{\mu_3} f_M^{\dot{M}} (dA_{\mu_1 \mu_2 \mu_3 \dot{M}} - A_{[\mu_1 \dot{N} \dot{M} \dot{P} \dot{Q} \dot{R}] \dot{N}} dA_{\mu_2 \mu_3 \dot{Q} \dot{R}} \\
&\quad + \frac{1}{3!} A_{[\mu_1 \dot{N} \dot{M} \dot{P} \dot{Q} \dot{R}] \dot{N}} A_{\mu_2 \dot{R} \dot{S}} dA_{\mu_3 \dot{P} \dot{Q}} \varepsilon^{\dot{P} \dot{Q} \dot{R} \dot{S} \dot{N}}), \\
G_{a_1 \dots a_4}^{MN} &= e_{a_1}^{\mu_1} \dots e_{a_4}^{\mu_4} f_M^{\dot{M}} f_N^{\dot{N}} (dA_{\mu_1 \dots \mu_4 \dot{M} \dot{N}} \\
&\quad - \varepsilon^{\dot{M}\dot{N}\dot{P}\dot{Q}\dot{R}} A_{[\mu_1 \dot{P} \dot{Q} \dot{R}] \dot{M}} dA_{\mu_2 \mu_3 \mu_4 \dot{N}} \\
&\quad - \frac{1}{2} \varepsilon^{\dot{M}\dot{N}\dot{P}\dot{Q}\dot{R}} A_{[\mu_1 \dot{P} \dot{Q} \dot{R}] \dot{M}} A_{\mu_2 \dot{R} \dot{S}} dA_{\mu_3 \mu_4 \dot{N}}^{\dot{S}} \\
&\quad - 5 A_{[\mu_1 \dot{P} \dot{Q} \dot{R}] \dot{M}} A_{\mu_2 \dot{R} \dot{S}} A_{\mu_3 \dot{P}} dA_{\mu_4 \dot{N}}^{\dot{S}}). \tag{4.1.10}
\end{aligned}$$

where $e_\mu^a = (e^h)_\mu^a$ and $f_M^{\dot{M}} = (e^\varphi)_{\dot{M}}^M$ are the vielbeins corresponding to the graviton and the scalar, respectively. The dots on the internal indices \dot{M}, \dot{N} represent the fact that they are world indices, and M, N represent flat indices in the internal space. We only go up to level 4 in the explicit forms of the Cartan forms as this is the up to the level we use in the derivation of the equations of motion. For equations of motion of the higher level fields, we do not use this explicit form, and instead find them from the transformations of the equations of motion of the 1-form and 2-form. However, one may confirm that the equations of motion are indeed correct, by use of the explicit forms at higher level.

We now want to find the transformations of these Cartan forms.

4.2 Transformations of the Cartan forms

We will now find the transformations of the Cartan forms under the local transformations as in equation (1.4.3). Recall that these local transformations act on the Cartan forms as in equation (1.4.8) and that they are inert under the rigid $E_{11} \ltimes l_1$ transformations. The local transformation $h \in I_c(E_{11})$ at level ± 1 in seven dimensions is

$$h = 1 - \Lambda_{aMN} S^{aMN}, \tag{4.2.1}$$

where $S^{aMN} = R^{aMN} - R_{aMN}$, which we call the level 1 generator of the Cartan subalgebra. From the construction of the non-linear realisation, the equations of motion are intrinsically invariant under the rigid transformations of E_{11} , so we must ensure that the transformations are invariant under the local $I_c(E_{11})$ transformations. If the equations of motion are invariant under the level 0 $I_c(E_{11})$ transformation, which is the $SO(7) \otimes SO(5)$ transformation, this implies that the equations are invariant under Lorentz transformations. Then if the equations of motion are invariant under the level 0 $I_c(E_{11})$ generator and additionally the generators at level 1, then the equations must be invariant under $I_c(E_{11})$ at all levels, as higher levels can be constructed from multiple commutators of the level 1 generators. So our aim is to find equations of motion which are invariant under these level 1 transformations. We find that under this transformation, the Cartan form transforms as

$$\delta\mathcal{V}_E = [\Lambda_{aMN}S^{aMN}, \mathcal{V}_E] - S^{aMN}d\Lambda_{aMN}. \quad (4.2.2)$$

Explicitly, we find the Cartan form transformations up to level 5 to be

$$\begin{aligned} \delta G_a{}^b &= 2\Lambda^{bMN}G_{aMN} - \frac{2}{5}\delta_a^b\Lambda^{cMN}G_{cMN}, \\ \delta G_M{}^N &= 4\Lambda^{cPN}G_{cPM} - \frac{4}{5}\delta_M^N\Lambda^{cPQ}G_{cPQ}, \\ \delta G_{aMN} &= -\Lambda_{bMN}G_a{}^b - 2\Lambda_{aP[N}G_M]{}^P \\ &\quad - \varepsilon_{MNPQR}\Lambda^{bQR}G_{ba}{}^P - d\Lambda_{aMN}, \\ \delta G_{a_1a_2}{}^M &= \varepsilon^{MNPQR}\Lambda_{[a_1NP}G_{a_2]QR} + 12\Lambda^{bNM}G_{ba_1a_2N}, \\ \delta G_{a_1a_2a_3M} &= \Lambda_{[a_1NM}G_{a_2a_3]}{}^N - 2\varepsilon_{MPQRS}\Lambda^{bPQ}G_{ba_1a_2a_3}{}^{RS}, \\ \delta G_{a_1\dots a_4}{}^{MN} &= \varepsilon^{MNPQR}\Lambda_{[a_1PQ}G_{a_2a_3a_4]R} + 20\Lambda^{bP[N}G_{ba_1\dots a_4}P^M] \\ &\quad - 2\Lambda_{bMN}G_{a_1\dots a_4,b}, \\ \delta G_{a_1\dots a_5M}{}^N &= -2\Lambda_{[a_1PM}G_{a_2\dots a_5]}{}^{PN} + \frac{2}{5}\delta_M^N\Lambda_{[a_1}G_{a_2\dots a_5]}{}^{PQ}, \\ \delta G_{a_1\dots a_4,b} &= \frac{4}{5}\Lambda^{bMN}G_{a_1\dots a_4}{}^{MN} - \frac{4}{5}\Lambda_{[a_1MN}G_{a_2a_3a_4]b}{}^{MN}. \end{aligned} \quad (4.2.3)$$

The Cartan form will not be preserved under the transformations of equation (4.2.3), as we

get some contributions to level -1 which is clear from equation (4.2.2); we see that S^{aMN} contains a level -1 generator, and \mathcal{V}_E contains a level 0 generator, and it is this commutator which will give negative level contributions, as well as a negative level term arising from the second term in equation (4.2.2). We choose to set the level minus one contribution to zero in order to preserve our gauge choice and find the condition

$$[\Lambda \cdot R^{(-1)}, \mathcal{V}^{(0)}] - d\Lambda \cdot R^{(-1)} = 0, \quad (4.2.4)$$

where the superscripts give the relevant level contribution.

We find a constraint on the parameter Λ^{aMN}

$$d\Lambda^{aMN} = G_b^a \Lambda^{bMN} + 2G_P^{[M} \Lambda^{a|P|N]}, \quad (4.2.5)$$

which is solved by

$$\Lambda_c^{\mu\dot{K}\dot{L}} e_\mu^a f_{\dot{K}}^M f_{\dot{L}}^N = \Lambda^{aMN}, \quad (4.2.6)$$

where the c subscript represents the fact that the $\Lambda^{\mu\dot{K}\dot{L}}$ is a constant. We plug this back into the transformation of the level 1 Cartan form in equation (4.2.3) to get

$$\delta G_{aMN} = -2\Lambda_{bMN} G_{(ba)} - 4\Lambda_{aP[N} G_{(M]P)} - \varepsilon_{MNPQR} \Lambda^{bQR} G_{ba}^P. \quad (4.2.7)$$

We note at this point, that in the following calculations, we come across the derivative of the parameter Λ^{aMN} , and that using equation (4.2.6), we notice that $\Lambda^{\mu\dot{M}\dot{N}}$ is independent of the generalised space-time coordinates, and this property can be used to move $\Lambda^{\mu\dot{M}\dot{N}}$ around derivatives.

So far we have written the Cartan forms as forms and so what we actually have is

$$G_\alpha = dz^\Pi G_{\Pi,\alpha}, \quad (4.2.8)$$

where the index Π represents the l_1 representation, and index α is the adjoint representation index. We notice that once the Cartan forms are written as $G_{\Pi,\alpha}$, they are no longer invariant under the rigid transformations. We can correct this by changing the first index into a tangent index using the inverse vielbein

$$G_{A,\alpha} = (E^{-1})_A{}^\Pi G_{\Pi,\alpha} , \quad (4.2.9)$$

and the Cartan forms are inert under the rigid transformations again. We shall refer to this first index as the l_1 index. We find that the variation on the l_1 tangent index is

$$\begin{aligned} \delta G_{a,\alpha} &= -2\Lambda_{aMN} G^{MN}{}_{,\alpha} , \\ \delta G^{MN}{}_{,\alpha} &= \Lambda^{aMN} G_{a,\alpha} . \end{aligned} \quad (4.2.10)$$

We note that both the l_1 index and the α index transform under $I_c(E_{11})$ and so the full transformation of the relevant Cartan form is the sum of (4.2.3) and (4.2.10).

We note that later we shall be interested in ensuring that the transformations of the Cartan forms are antisymmetric in all Lorentz indices, including this l_1 index, for the positive level Cartan forms. We can use the transformation of the l_1 index to ensure that this is the case in the following way. Beginning at level 1, if we redefine the Cartan form to be

$$\mathcal{G}_{[a_1,a_2]MN} = G_{[a_1,a_2]MN} + \varepsilon_{MNPQR} G^{QR}{}_{,a_1a_2}{}^P , \quad (4.2.11)$$

it then transforms as

$$\begin{aligned} \delta \mathcal{G}_{[a_1,a_2]MN} &= -2\Lambda^{bMN} G_{[a_1,(a_2)b]} - 4\Lambda_{[a_2P[N} G_{a_1],(M)P]} \\ &\quad - \frac{3}{2}\varepsilon_{MNPQR} \Lambda_{bQR} G_{[a_1,ba_2]}{}^P , \end{aligned} \quad (4.2.12)$$

and so we have achieved our aim of keeping all the indices on the positive level Cartan forms antisymmetric in the transformation.

At level two, we define the form, leading to a transformation with totally antisymmetric indices on the Cartan forms, to be

$$\mathcal{G}_{[a_1, a_2 a_3]}^M = G_{[a_1, a_2 a_3]}^M - 4G^{NM},_{a_1 a_2 a_3 N} , \quad (4.2.13)$$

while the transformation is

$$\delta\mathcal{G}_{[a_1, a_2 a_3]}^M = \varepsilon^{PQRS} \Lambda_{[a_2 PQ} G_{a_1, a_3]RS} + 16\Lambda^{bNM} G_{[a_1, b a_2 a_3]N} . \quad (4.2.14)$$

At level 3, we choose

$$\mathcal{G}_{[a_1, a_2 a_3 a_4]M} = G_{[a_1, a_2 a_3 a_4]M} + \frac{1}{2}\varepsilon^{PQRS} G^{PQ},_{a_1 a_2 a_3 a_4}{}^{RS} , \quad (4.2.15)$$

which has the transformation

$$\delta\mathcal{G}_{[a_1, a_2 a_3 a_4]M} = \Lambda_{[a_2 NM} G_{a_1, a_3 a_4]}^N - \frac{5}{2}\Lambda^{bPQ} \varepsilon_{PQRS} G_{[a_1, b a_2 a_3 a_4]}^{RS} . \quad (4.2.16)$$

At level 4, we define

$$\mathcal{G}_{[a_1, \dots a_5]}^{MN} = G_{[a_1, \dots a_5]}^{MN} + 4G^{P[N},_{a_1 \dots a_5 P}{}^{M]} , \quad (4.2.17)$$

which has the transformation

$$\begin{aligned} \delta\mathcal{G}_{[a_1, a_2 \dots a_5]}^{MN} &= \varepsilon^{PQRMN} \Lambda_{[a_2 PQ} G_{a_1, a_3 a_4 a_5]R} \\ &\quad + 24\Lambda^{bP[N} G_{[a_1, b a_2 \dots a_5]P}{}^{M]} - 2\Lambda^{bMN} G_{[a_1, \dots a_5], b} . \end{aligned} \quad (4.2.18)$$

Finally, the transformations at level 5 are

$$\delta G_{[a_1, \dots a_6]M}^N = -2\Lambda_{[a_2 PM} G_{a_1, a_3 \dots a_6]}^{PN} + \frac{2}{5}\delta_M^N \Lambda_{[a_2 PQ} G_{a_1, a_3 \dots a_6]}^{PQ} + \dots , \quad (4.2.19)$$

and

$$\delta G_{[a_1, \dots, a_5], b} = \frac{4}{5} \Lambda_{bMN} G_{[a_1, \dots, a_5]}^{MN} - \frac{4}{5} \Lambda_{[a_2 MN} G_{a_1, a_3 a_4 a_5] b}^{MN} + \dots, \quad (4.2.20)$$

where the \dots represent higher level Cartan forms. We note that we have not defined new objects at level 5 as we are only concerned with the explicit transformation into the lower level Cartan forms in this section. We do in fact find a level 6 form in the last section, which is a term arising from this transformation. However, we give it an arbitrary coefficient, as the algebra is not yet confirmed at that level.

One can think about the above definitions in terms of a gauge symmetry which requires the indices to be antisymmetric. As a result one sees that the extra coordinates beyond those of spacetime are required to ensure gauge symmetry. In the E_{11} approach, we do not require gauge symmetries - only the symmetries of the non-linear realisation, and it turns out that the results are indeed gauge invariant.

Now we have completed the calculation of the transformations of the Cartan forms in 7D, we can begin the derivation of the equations of motion in the 7D decomposition.

4.3 Equations of motion

We now look for the duality relations of the form fields in the seven dimensional theory. We want the most general equations that are invariant under local Lorentz transformations. We expect that the level 1 form is dual to the level 4 form and so we find the most general expression to be

$$D_{a_1 a_2 MN} \equiv \mathcal{G}_{[a_1, a_2] MN} + e_2 \varepsilon_{a_1 a_2}{}^{b_1 \dots b_5} \mathcal{G}_{[b_1, \dots, b_5] MN} = 0, \quad (4.3.1)$$

and the level 2 form is dual to the level 3 form giving

$$D_{a_1 a_2 a_3}{}^M \equiv \mathcal{G}_{[a_1, a_2 a_3]}{}^M + e_3 \varepsilon_{a_1 a_2 a_3}{}^{b_1 \dots b_4} \mathcal{G}_{[b_1, \dots, b_4]}{}^M = 0, \quad (4.3.2)$$

where e_2 and e_3 are constants. To find these constants, we transform the dualities under the local symmetry using the transformations of the Cartan forms we derived in the last section. We find that they transform into each other if $e_2 = \mp \frac{i}{2}$ and that $e_3 = \pm \frac{i}{3}$. In the following, we arbitrarily choose to work with the first sign, i.e. $e_2 = -\frac{i}{2}$ and $e_3 = \frac{i}{3}$, but one can easily recover the other case simply by changing the sign whenever an i appears.

We calculate that the variations of the dualities under the local transformation is

$$\delta D_{a_1 a_2 M N} = -\frac{3}{2} \varepsilon_{M N P Q R} \Lambda^{b Q R} D_{a_1 b a_2}{}^P + \dots, \quad (4.3.3)$$

and

$$\begin{aligned} \delta D_{a_1 a_2 a_3 M} &= \varepsilon_{P Q R S M} \Lambda_{[a_2 P Q} D_{a_1 a_3] R S} \\ &\quad + \frac{i}{3} \varepsilon_{a_1 a_2 a_3}{}^{b_1 \dots b_4} \Lambda_{b_2 N M} D_{b_1 b_3 b_4}{}^N, \end{aligned} \quad (4.3.4)$$

where the \dots represent the terms which involve gravity and scalar Cartan forms and their duals. The fact that, up to graviton and scalar terms, the two equations vary into each other indicate that we have found the correct equations. We anticipate that the remaining graviton and scalar terms are simply those of the corresponding dualities, and we shall study these terms in detail in section 4.4. Additionally, we anticipated in the previous section that the invariance of the duality equations required objects which are totally antisymmetrised in their indices. We note that we could have simply made the transformations without antisymmetrising first, and we would have found that we would need to add terms which are equivalent to using the objects as defined in the previous section.

We choose to work with equations that are second order in derivatives, and which contain the level 0, 1, or 2 form fields. We begin with taking derivatives of equation (4.3.1) and (4.3.2), such that the 3 and 4 form fields drop out, by use of the explicit forms of the Cartan forms.

We find that $D_{a_1 a_2 M N}$ from equation (4.3.1) becomes

$$\begin{aligned} \partial_{\mu_1}(\det(e)^{\frac{1}{2}} G^{[\mu_1, \mu_2] \dot{M} \dot{N}}) + \frac{2}{3} G_{[\mu_1, \mu_3] \dot{Q} \dot{R}} G^{[\mu_1, \mu_2 \mu_3] \dot{P}} \varepsilon^{\dot{M} \dot{N} \dot{P} \dot{Q} \dot{R}} \\ + \frac{i}{4} \varepsilon^{\mu_1 \mu_2 \nu_1 \dots \nu_5} G_{[\mu_1, \nu_2 \nu_3] [\dot{M} G_{[\nu_1, \nu_4 \nu_5] \dot{N}]} = 0, \end{aligned} \quad (4.3.5)$$

and $D_{a_1 a_2 a_3 M}$ from equation (4.3.2) becomes

$$\partial_{\mu_1}(\det(e)^{\frac{1}{2}} G^{[\mu_1, \mu_2 \mu_3] \dot{P}}) - \frac{i}{3} \varepsilon^{\mu_1 \mu_2 \mu_3 \nu_1 \dots \nu_4} G_{[\mu_1, \nu_2] \dot{N} \dot{P}} G_{[\nu_1, \nu_3 \nu_4] \dot{N}} = 0. \quad (4.3.6)$$

We notice that the factors of the $\det(e)^{\frac{1}{2}}$ appear due to their appearance in the inverse vielbein, similar to the 11D case [25, 34].

We want to find the equations (4.3.1) and (4.3.2) in terms of the totally antisymmetric Cartan forms in equations (4.2.12), (4.2.14), (4.2.16), and (4.2.18), in order to apply these transformations. Therefore, we flatten the indices corresponding to both the spacetime and internal indices, which leads to

$$\begin{aligned} E^{aQR} \equiv & \frac{1}{2} G_{c,d}{}^d G^{[c,a]QR} - G_{c,d}{}^c G^{[d,a]QR} \\ & - G_{c,d}{}^a G^{[c,d]QR} - 2G_{c,P}{}^{[Q} G^{c,a]P|R]} \\ & + \det(e)^{\frac{1}{2}} e_c{}^\mu \partial_\mu (G^{[c,a]QR}) \\ & + \frac{2}{3} G_{[c_1, c_2] MN} G^{[c_1, ac_2] P} \varepsilon^{MNPQR} \\ & + \frac{i}{4} \varepsilon^{cab_1 \dots b_5} G_{[c, b_2 b_3] [Q} G_{[b_1, b_4 b_5] R]} = 0, \end{aligned} \quad (4.3.7)$$

and we also find

$$\begin{aligned} E^{a_1 a_2}{}_M \equiv & \frac{1}{2} G_{c,d}{}^d G^{[c, a_1 a_2]}{}_M - G_{c,d}{}^c G^{[d, a_1 a_2]}{}_M \\ & - 2G_{c,d}{}^{[a_1} G^{c, d|a_2]}{}_M + G_{c,M}{}^P G^{[c, a_1 a_2]}{}_P \\ & + \det(e)^{\frac{1}{2}} e_c{}^\mu \partial_\mu (G^{[c, a_1 a_2]}{}_M) \\ & - \frac{i}{3} \varepsilon^{ca_1 a_2 b_1 \dots b_4} G_{[c, b_2] NM} G_{[b_1, b_3 b_4]}{}^N = 0. \end{aligned} \quad (4.3.8)$$

Our hope is that these equations will turn out to be the usual equations of motion of the 7D supergravity theory. We now vary them to show that they are invariant under $I_c(E_{11})$.

First, we want to find the terms which have derivatives with respect to the level one coordinates. This is done in the following way. If we have a term in the variation that looks like

$$\Lambda^{dMN} G_{d,\alpha} f^\alpha_{MN} , \quad (4.3.9)$$

then we may cancel this term by adding a term of the form

$$-G^{MN}{}_{,\alpha} f^\alpha_{MN} , \quad (4.3.10)$$

to the equation of motion that we are considering, using equation (4.2.10). In the following, we will refer to these terms as l_1 terms. The reason for this is that we are aiming to find all terms in the variation that contain the usual derivatives of spacetime, and we see that these can arise from the transformation of terms with derivatives with respect to level 1 coordinates. The other terms in the variation of the level 1 derivatives are terms with level 2 derivatives, and so are at a higher level than we are interested, due to the fact that they do not directly transform into the usual spacetime derivatives arising at level 0.

We will give an outline of the process of the transformation of equation (4.3.7) under the local $I_c(E_{11})$ transformations. If we vary the terms in the first three lines into the terms that appear in the equation of motion for gravity, we find

$$\begin{aligned} & e_{a_2}{}^{\mu_2} \partial_{\mu_1} (\omega_{\tau, \mu_1 \mu_2} \det(e) - \det(e)^{\frac{1}{2}} G_{\tau, [\mu_1 \mu_2]}) \Lambda^{\tau QR} \\ & - 2\Lambda_{a_2 MN} G^{a_1, dMN} G_{[a_1, d]}{}^{QR} - 8\Lambda^{dN[Q} G_{a_1, dNM} G_{[a_1, a_2]}{}^{M|R]} \\ & - 2\Lambda_{a_1 MN} G^{a_1, dMN} G_{[d, a_2]}{}^{QR} + 2\Lambda_{dMN} G^{a_1, dMN} G_{[a_1, a_2]}{}^{QR} , \end{aligned} \quad (4.3.11)$$

where

$$\det(e)^{\frac{1}{2}} w_{c, ab} = -G_{a, (bc)} + G_{b, (ac)} + G_{c, [ab]} , \quad (4.3.12)$$

which turns out to be the spin connection. We note both terms on the second line and the last term in equation (4.3.11) will be cancelled with terms from the transformation of last two lines in equation (4.3.7). We also notice that the 2nd and the 5th terms are terms of the form given in equation (4.3.9), and so we can add corresponding l_1 terms. We are then left with the first term, which we can manipulate in the following way with the aim of finding the Ricci tensor. We derive that

$$\begin{aligned} e_\mu^a \partial_\nu (\det(e) \omega_{\tau, \nu\mu}) = \\ \det(e) (e_b^\nu \partial_\nu \omega_{\tau, ba} + (e_\mu^a \partial_\nu e_c^\mu) \omega_{\tau, \nu c} + (e_c^\lambda \partial_\nu e_\lambda^c) \omega_{\tau, \nu a} + \partial_\nu e_b^\nu \omega_{\tau, ba}). \end{aligned} \quad (4.3.13)$$

The first term looks like something we will need. The second term we can rewrite as

$$\begin{aligned} e_b^\lambda \partial_\nu e_\lambda^a \omega_{\mu, \nu b} &= G_{c, \nu a} \omega_{\mu, c\nu} = (-G_{a, (c\nu)} + G_{c, (a\nu)} + G_{\nu, [ac]}) \omega_{\mu, c\nu} \\ &= \omega_{\nu, ac} \omega_{\mu, c\nu}, \end{aligned} \quad (4.3.14)$$

and the final two terms in equation (4.3.13) look like

$$\omega_{\mu, ab} \partial_\lambda (\det(e) e_b^\lambda) = -\det(e) \omega_{\mu, ab} \omega_{\lambda, b\lambda}. \quad (4.3.15)$$

We note that the Ricci tensor is

$$R_\mu^a = \partial_\mu \omega_{\nu, ab} e_b^\nu - \partial_\nu \omega_{\mu, ab} e_b^\nu + \omega_{\mu, ac} \omega_{\nu, cb} e_b^\nu - \omega_{\nu, ac} \omega_{\mu, cb} e_b^\nu, \quad (4.3.16)$$

and so what we have in equation (4.3.13) is

$$\det(e) (R_\tau^{a2} - \partial_\tau (\omega_{\nu, a2b}) e_b^\nu) \Lambda^{\tau QR}, \quad (4.3.17)$$

and again the 2nd term in this equation is an l_1 term of the form in equation (4.3.9).

Including l_1 terms, we find that the equation of motion for the one form is

$$\mathcal{E}^{aQR} \equiv \frac{1}{2} G_{c,d}^d G^{[c,a]QR} - G_{c,d}^c G^{[d,a]QR} - G_{c,d}^a G^{[c,d]QR}$$

$$\begin{aligned}
& -2G_{c,P}{}^{[Q]}G^{[c,a]P|R]} + \det(e)^{\frac{1}{2}}e_c{}^\mu\partial_\mu(G^{[c,a]QR}) \\
& + \frac{3}{2}G_{[c_1,c_2]MN}G^{[c_1,ac_2]}{}_P\varepsilon^{MNPQR} + \frac{i}{4}\varepsilon^{cab_1\dots b_5}G_{[c,b_2b_3]}{}^{[Q]}G_{[b_1,b_4b_5]}{}^R] \\
& + \frac{2}{5}G_{c,M}{}^MG^{[c,a]QR} + 8G^{P[Q}{}_{,cPM}G^{c,a]M|R]} \\
& + e_{\mu_2}{}^a\partial_{\mu_1}(\det(e)^{\frac{1}{2}}G^{QR,[\mu_1\mu_2]}) + \det(e)\partial^{QR}(\omega_\nu{}^{ab})e_b{}^\nu \\
& - 2\det(e)^{\frac{1}{2}}\partial^{P|R]}G^{a,(P|Q]} - G^{P[R|}{}_{,d}{}^dG^{a,(P|Q]} + 2G^{P[R|}{}_{,N}{}^{]Q]}G^{a,(NP)} \\
& - 2G^{P[R|}{}_{,P}{}^NG^{a,(N|Q]} - 2G^{P[R|}{}_{,d}{}^dG^{d,(P|Q]} .
\end{aligned} \tag{4.3.18}$$

We see that we have changed the notation from E^{aQR} to \mathcal{E}^{aQR} . We will use this calligraphic font to indicate equations where l_1 terms are included, and use the usual font where no l_1 terms are included.

We find that the transformation of the one form equation of motion is

$$\begin{aligned}
\delta\mathcal{E}^{aQR} &= -\frac{3}{2}\Lambda_{dMN}\varepsilon^{MNPQR}E^{da}{}_P + \Lambda^{bQR}E_b{}^a + \Lambda^{aP[Q]}E^{(R]P)} \\
& - \frac{i}{2}\varepsilon^{cab_1\dots b_5}\varepsilon^{MNPQR}G_{[c,b_2]MN}\Lambda_{[b_3TP}D_{b_1,b_4b_5]}{}^T \\
& + 4!\Lambda^{[b_3P[R}D^{b_1ab_2]}{}_PG_{[b_1,b_2b_3]}{}^Q] ,
\end{aligned} \tag{4.3.19}$$

where $E_a{}^b$ is

$$\begin{aligned}
E_a{}^b &\equiv \det(e)R_a{}^b - G_{a,(MN)}G^{b,(MN)} \\
& - 2(2G_{[c,a]MN}G^{[c,b]MN} - \frac{1}{5}\delta_a^b G_{[c_1,c_2]MN}G^{[c_1,c_2]MN}) \\
& - 3(3G_{[c_1,ac_2]MN}G^{[c_1,bc_2]MN} - \frac{2}{5}\delta_a^b G_{[c_1,\dots,c_3]}{}^PG^{[c_1,\dots,c_3]}{}_P) ,
\end{aligned} \tag{4.3.20}$$

and $E_{(QR)}$ is

$$\begin{aligned}
E_{(QR)} &\equiv G_{c,d}{}^dG_{c,(QR)} - 2G_{c,d}{}^cG_{d,(QR)} \\
& + 4G_{c,[QN]}G_{c,(RN)} + 2\det(e)^{\frac{1}{2}}e_c{}^\mu\partial_\mu G_{c,(QR)} \\
& + 8(G_{[c_1,c_2]QP}G_{[c_1,c_2]RP} + \frac{1}{5}\delta_{QR}G_{[c_1,c_2]}{}^{NP}G_{[c_1,c_2]NP}) \\
& - 6(G_{[c_1,c_2c_3]QP}G_{[c_1,c_2c_3]R} + \frac{1}{5}\delta_{QR}G_{[c_1,c_2c_3]}{}^PG_{[c_1,c_2c_3]P}) .
\end{aligned} \tag{4.3.21}$$

Since the variation vanishes, we find that $E_a^b = 0$ and $E_{(QR)} = 0$ and we see that we have found the gravity and scalar equations of motion.

With the same steps, we also find that the equation of motion for the two form is

$$\begin{aligned}
\mathcal{E}^{a_1 a_2}{}_M \equiv & \frac{1}{2} G_{c,d}{}^d G^{[c,a_1 a_2]}{}_M - G_{c,d}{}^c G^{[d,a_1 a_2]}{}_M - 2G_{c,d}{}^{[a_1} G^{c,d]a_2]}{}_M \\
& + G_{c,M}{}^P G^{[c,a_1 a_2]}{}_P + \det(e)^{\frac{1}{2}} e_c{}^\mu \partial_\mu (G^{[c,a_1 a_2]}{}_M) \\
& - \frac{i}{3} \varepsilon^{ca_1 a_2 b_1 \dots b_4} (G_{[c,b_2]NM} G^{[b_1, b_3 b_4]N}) - \frac{1}{5} G_{c,P}{}^P G^{[c,a_1 a_2]}{}_M \\
& + \frac{1}{3} \varepsilon^{PQRS} M \left(\frac{1}{2} G^{PQ}{}_{,d} G^{[a_1, a_2]RS} - 2G^{PQ}{}_{,d}{}^{[a_1} G^{d, a_2]}{}^{RS} \right. \\
& \left. + \det(e)^{\frac{1}{2}} \partial^{PQ} (G^{[a_1, a_2]RS}) \right) - \frac{2}{3} G^{QR}{}_{,S}{}^P G^{[a_1, a_2]ST} \varepsilon_{TMPQR} \\
& + 2G^{PQ}{}_{,d}{}^{PQ} G^{[d, a_1 a_2]}{}_M - 4G^{NP}{}_{,c}{}^{NM} G^{[c, a_1 a_2]}{}_P . \tag{4.3.22}
\end{aligned}$$

Then the transformation of the equation of motion for the two form is calculated to be

$$\begin{aligned}
\delta \mathcal{E}^{a_1 a_2}{}_M = & \frac{2}{3} \Lambda^{[a_1 | PQ} E^{a_2] RS} \varepsilon_{PQRS} M \\
& + \frac{i}{3} \Lambda_{cNM} \left(\frac{1}{2} G_{b,d}{}^d \varepsilon^{bca_1 a_2 d_1 d_2 d_3} - G_{b,d}{}^b \varepsilon^{dca_1 a_2 d_1 d_2 d_3} \right. \\
& \left. - 2G_{b,d}{}^{[a_1} \varepsilon^{bcd] a_2] d_1 d_2 d_3} \right) D_{d_1 d_2 d_3}{}^N \\
& + \frac{i}{3} \varepsilon^{bca_1 a_2 d_1 d_2 d_3} \det(e)^{\frac{1}{2}} e_b{}^\mu \partial_\mu (\Lambda_{cNM} D_{d_1 d_2 d_3}{}^N) . \tag{4.3.23}
\end{aligned}$$

The next step is to do the transformation of the equations of motion for gravity and scalars that we have just found. If we begin with the equation of motion for gravity in equation (4.3.20), we must add l_1 terms to $\omega_{c,ab}$. The reasoning behind this is similar to the antisymmetrisation of the objects in the transformations in 4.2, and so we make the following redefinition of the spin connection in equation (4.3.12)

$$\det(e)^{\frac{1}{2}} \Omega_{c,ab} = \det(e)^{\frac{1}{2}} \omega_{c,ab} - \frac{2}{5} \delta_{bc} G^{MN}{}_{,aMN} + \frac{2}{5} \delta_{ac} G^{MN}{}_{,bMN} , \tag{4.3.24}$$

so that we find the transformation of this object is

$$\delta(\det(e)^{\frac{1}{2}} \Omega_{c,ab}) = -2\Lambda_{cMN} G_{[a,b]MN} - 2\Lambda_{bMN} G_{[a,c]MN} + 2\Lambda_{aMN} G_{[b,c]MN}$$

$$+ \frac{4}{5} \delta_{bc} \Lambda^{dMN} G_{[a,d]MN} - \frac{4}{5} \delta_{ac} \Lambda^{dMN} G_{[b,d]MN} , \quad (4.3.25)$$

and we see that all the Cartan forms in the transformation are now antisymmetric in their Lorentz indices, including the l_1 index. We then replace $\omega_{c,ab}$ with $\Omega_{c,ab}$ in R_a^b to find

$$\mathcal{R}_a^b \equiv e_a^\mu \partial_\mu \Omega_{\nu,}^{bd} e_d^\nu - e_a^\mu \partial_\nu \Omega_{\mu,}^{bd} e_d^\nu + \Omega_{a,}^b \Omega_{d,}^{cd} - \Omega_{d,}^b \Omega_{a,}^{cd} . \quad (4.3.26)$$

We find that the equation of motion for the graviton including l_1 terms

$$\begin{aligned} \mathcal{E}_a^b &\equiv \det(e) \mathcal{R}_a^b - G_{a,(MN)} G^{b,(MN)} \\ &\quad - 2(2G_{[c,a]MN} G^{[c,b]MN} - \frac{1}{5} \delta_a^b G_{[c_1,c_2]MN} G^{[c_1,c_2]MN}) \\ &\quad - 3(3G_{[c_1,ac_2]MN} G^{[c_1,bc_2]MN} - \frac{2}{5} \delta_a^b G_{[c_1,\dots,c_3]P} G^{[c_1,\dots,c_3]P}) \\ &\quad - 2\partial^{MN} G_{[c,a]MN} \delta^{bc} - 4G^{NP}{}_{,P}{}^M G_{[c,a]MN} \delta^{bc} \\ &\quad - \frac{2}{5} G^{MN}{}_{,aMN} \omega_{d,}^{bd} + \frac{2}{5} G^{MN}{}_{,dMN} \omega_{a,}^{bd} \\ &\quad + 2G^{MN,bc} G_{[c,a]MN} + 2G^{MN}{}_{,a}{}^c G^{[c,b]MN} \\ &\quad - 4G^{PN}{}_{,aMN} G^{b,(MP)} - 4G^{PN,bMN} G_{a,(MP)} . \end{aligned} \quad (4.3.27)$$

We note that although the \mathcal{R}_a^b is no longer symmetric in a and b , one can verify that \mathcal{E}_a^b is symmetric in a and b including the l_1 terms as required.

The transformation of the equation of motion of the graviton is calculated to be

$$\begin{aligned} \delta \mathcal{E}_a^b &= -2\Lambda_{aMN} E^{bMN} - 2\Lambda^{bMN} E_{aMN} + \frac{4}{5} \delta_a^b \Lambda_{cMN} E^{cMN} \\ &\quad - 3i\Lambda_{dMN} \varepsilon_{dac_1c_2d_1d_2d_3} G^{[c_1,bc_2]N} D^{d_1d_2d_3M} \\ &\quad - 3i\Lambda_{dMN} \varepsilon^{dbc_1c_2d_1d_2d_3} G_{[c_1,ac_2]}^N D_{d_1d_2d_3}^M \\ &\quad + \frac{6i}{5} \delta_a^b \Lambda_{dMN} \varepsilon_{c_1dc_2c_3d_1d_2d_3} G_{[c_1,c_2c_3]}^N D_{d_1d_2d_3}^M . \end{aligned} \quad (4.3.28)$$

In a similar process, beginning with equation (4.3.21) as the equation of motion for the scalar, we find that the equation of motion for the scalar including l_1 terms is

$$\begin{aligned}
\mathcal{E}_{(QR)} \equiv & G_{c,d}{}^d G_{c,(QR)} - 2G_{c,d}{}^c G_{d,(QR)} \\
& + 4G_{c,[QN]} G_{c,(RN)} + 2 \det(e)^{\frac{1}{2}} e_c{}^\mu \partial_\mu G_{c,(QR)} \\
& + 8(G_{[c_1,c_2]QP} G_{[c_1,c_2]RP} + \frac{1}{5} \delta_{QR} G_{[c_1,c_2]NP} G_{[c_1,c_2]NP}) \\
& - 6(G_{[c_1,c_2,c_3]Q} G_{[c_1,c_2,c_3]R} + \frac{1}{5} \delta_{QR} G_{[c_1,c_2,c_3]P} G_{[c_1,c_2,c_3]P}) \\
& - \frac{8}{5} \delta_{QR} \partial_\mu (\det(e)^{\frac{1}{2}} G^{PN}{}_{,\mu PN}) - 4G_{P(R),cP|Q} G_{c,d}{}^d \\
& + 4G^{MN}{}_{,dMN} G_{d,(QR)} + 8G_{P(R),dP|Q} G_{c,d}{}^c \\
& - 8G^{PN}{}_{,cP|Q} G_{c,|R)}{}^N + 8G_{P(R),}{}^{cPN} G_{c,N|Q)} \\
& - 2 \det(e)^{\frac{1}{2}} e_c{}^\mu \partial_\mu (G_{N(R,cQ)N}) , \tag{4.3.29}
\end{aligned}$$

and its transformation is

$$\begin{aligned}
\delta \mathcal{E}_{(QR)} = & 8\Lambda_{cPR} E_{cPQ} + 8\Lambda_{cPQ} E_{cPR} - \frac{16}{5} \delta_{QR} \Lambda_{cPN} E^{cPN} \\
& + 2i\Lambda_{dPR} \varepsilon^{dd_1 d_2 d_3 c_1 c_2 c_3} G_{[c_1,c_2,c_3]Q} D_{d_1 d_2 d_3}{}^P \\
& + 2i\Lambda_{dPQ} \varepsilon^{dd_1 d_2 d_3 c_1 c_2 c_3} G_{[c_1,c_2,c_3]R} D_{d_1 d_2 d_3}{}^P \\
& - \frac{4i}{5} \delta_{QR} \Lambda_{dPN} \varepsilon^{dd_1 d_2 d_3 c_1 c_2 c_3} G_{[c_1,c_2,c_3]N} D_{d_1 d_2 d_3}{}^P . \tag{4.3.30}
\end{aligned}$$

Using the symmetries of the non-linear realisation, we have found a set of equations that transform into each other. These are the equations of motion for the graviton (4.3.27), the scalar (4.3.29), the one form (4.3.18), and the two form (4.3.22) in seven dimensions. If we truncate the equations so that they only contain derivatives with respect to the usual coordinates of spacetime, then these equations are those of the seven dimensional maximal supergravity as found in [52].

We have found the equations of motion that are second order in derivatives. We now derive the duality relations which are first order in derivatives, and in particular the duality relations of the graviton and the scalar with level 5 fields.

4.4 First order duality relations

In this section we will derive the duality relations which are first order in derivatives and their variations. In addition to those we discussed in section 4.3 we will find the duality relations that relate the graviton to the dual graviton and the scalar field to the dual scalar field. We begin by recalling, from section 4.3, the duality relation which relates the two form to the three form

$$D_{a_1 a_2 a_3}{}^M \equiv \mathcal{G}_{a_1 a_2 a_3}{}^M + \frac{i}{3} \varepsilon_{a_1 a_2 a_3}{}^{b_1 \dots b_4} \mathcal{G}_{b_1 \dots b_4}{}^M = 0, \quad (4.4.1)$$

and its variation was given by

$$\delta D_{a_1 a_2 a_3 M} = \varepsilon_{PQRS} \Lambda_{[a_2 PQ} D_{a_1 a_3] RS} + \frac{i}{3} \varepsilon_{a_1 a_2 a_3}{}^{b_1 \dots b_4} \Lambda_{b_2 NM} D_{b_1 b_3 b_4}{}^N. \quad (4.4.2)$$

We observe that it transforms into itself and the 1-form duality relation.

In section 4.3, we also transformed the duality relation which relates the one form to the four form but we did not include the terms in the variation that contained the scalar fields, graviton, or the fields at level five. When carrying out the variation to include the extra terms we find that the duality relation of equation (4.3.1) becomes modified by an l_1 term following the procedure explained in the previous section. We add the l_1 term to the 1-form duality relation, such that we find

$$\mathcal{D}_{a_1 a_2 MN} \equiv \mathcal{G}_{a_1 a_2 MN} - \frac{i}{2} \varepsilon_{a_1 a_2}{}^{b_1 \dots b_5} \mathcal{G}_{b_1 \dots b_5 MN} + G_{MN, [a_1 a_2]} = 0. \quad (4.4.3)$$

Carrying out the variation of this duality relation including this additional term gives the result

$$\begin{aligned} \delta \mathcal{D}_{a_1 a_2 MN} = & - \Lambda_{[a_2 P [N} D_{a_1], (M) P]} + \Lambda_{b MN} D_{b, [a_1 a_2]} - \frac{3}{2} \varepsilon_{MNPQR} \Lambda^{bQR} D_{a_1 b a_2}{}^P \\ & + 4i \Lambda_{[a_2 P [N | \varepsilon_{a_1]}{}^{b_1 b b_2 \dots b_5} D_{b_1, b b_2 \dots b_5 [P | M]]}, \end{aligned} \quad (4.4.4)$$

where

$$D_{b,[a_1 a_2]} \equiv \omega_{b,[a_1 a_2]} \det(e)^{\frac{1}{2}} + i\varepsilon_{a_1 a_2}{}^{b_1 \dots b_5} \mathcal{G}_{[b_1, \dots, b_5], b} , \quad (4.4.5)$$

and

$$D_{a,(MN)} \equiv 2G_{a,(MN)} - 4i\varepsilon_a{}^{b_1 \dots b_6} \mathcal{G}_{b_1, \dots, b_6(MN)} . \quad (4.4.6)$$

Since we require that the transformation of the 1-form duality relation is zero, we find

$$D_{b,[a_1 a_2]} = 0 , \quad D_{a,(MN)} = 0 . \quad (4.4.7)$$

which are the the duality relation of the scalar field and the graviton respectively, and the last term in equation (4.4.4) indicates that

$$D_{[a_1, a_2 \dots a_6][MN]} \equiv \mathcal{G}_{[a_1, a_2 \dots a_6][MN]} \doteq 0 , \quad (4.4.8)$$

where the \doteq notation, first used in [53], implies that this holds modulo some Lorentz transformation, which is not yet well understood. In order to explore this result more, one should study the transformation of the equation of motion, which is second order in derivatives, of the 4-form into this 5-form, and then integrate the result to find the origin of this result. It is important to note that there was also a similar relation found in the 5D case [25], for a higher level field that is dual to one of the scalars. This suggests that is a phenomenon which is present throughout the lower dimensional decompositions of E_{11} , although the mechanism is not yet well understood.

We now transform the new duality relations we have found in equation (4.4.4) using the variations given in (4.2.3). In a similar process to the previous transformation of the 1-form duality relation, we must add l_1 terms to the scalar duality relation, so that we actually have

$$\begin{aligned} \mathcal{D}_{a,(MN)} \equiv & 2G_{a,(MN)} - 4i\varepsilon_a{}^{b_1 \dots b_6} \mathcal{G}_{b_1, \dots, b_6(MN)} \\ & - 4G_{PN, aPM} - 4G_{PM, aPN} + \frac{8}{5} \delta_{MN} G^{PQ}{}_{, aPQ} , \end{aligned} \quad (4.4.9)$$

which then transforms into

$$\delta\mathcal{D}_{a,(MN)} = 8\Lambda_{cPN}D_{acPM} + 8\Lambda_{cPM}D_{acPN} - \frac{16}{5}\delta_N^M\Lambda_{cPQ}D_{acPQ} + \dots, \quad (4.4.10)$$

where the \dots represent level 6 fields, which we have not considered. While the the gravity duality relation including the l_1 terms is

$$\begin{aligned} \mathcal{D}_{b,[a_1a_2]} &= \omega_{b,[a_1a_2]} \det(e)^{\frac{1}{2}} + i\varepsilon_{a_1a_2}{}^{b_1\dots b_5}\mathcal{G}_{[b_1\dots b_5],b} \\ &\quad - \frac{4}{5}\delta_{ba_2}G^{MN}{}_{,a_1MN} + \frac{4}{5}\delta_{ba_1}G^{MN}{}_{,a_2MN}, \end{aligned} \quad (4.4.11)$$

which is essentially the same process we carried out in the previous section when uplifting the spin-connection $\omega_{b,[a_1a_2]}$ to $\Omega_{b,[a_1a_2]}$ in equation (4.3.24). This then transforms into

$$\begin{aligned} \delta\mathcal{D}_{b,[a_1a_2]} &= -2\Lambda_{bMN}D_{a_1a_2MN} + \frac{4}{5}\Lambda^{dMN}\delta_{b[a_2}D_{a_1]dMN} \\ &\quad + \frac{4}{6!5}\Lambda^{dMN}\delta_{b[a_2}\varepsilon_{a_1]}{}^{d_1\dots d_6}D_{d_1\dots d_6,dMN} \\ &\quad - \frac{4}{6!}\Lambda_{[a_2MN}\varepsilon_{a_1]}{}^{d_1\dots d_6}D_{d_1\dots d_6,bMN} + \partial_b\tilde{\Lambda}_{a_1a_2} + \dots, \end{aligned} \quad (4.4.12)$$

where the \dots represent some level 6 fields which we have not considered, and

$$D_{a_1\dots a_6,bMN} \equiv eG_{[a_1,\dots,a_6],bMN} + \varepsilon_{a_1\dots a_6}{}^d\mathcal{G}_{[d,b]MN} = 0, \quad (4.4.13)$$

is a new duality that we have found, involving a level 6 field which we have not included in our calculations, hence the undetermined constant e . This relation is analogous to the duality relation connecting the $A_{a_1a_2a_3}$ field to the $A_{b_1\dots b_9,a_1a_2a_3}$ in 11 dimensions as discussed in [36, 54]. Finally,

$$\partial_b\tilde{\Lambda}_{a_1a_2} = -\frac{i}{25}\varepsilon_{a_1a_2}{}^{b_1\dots b_5}\Lambda_{b_1MN}G_{b,b_2\dots b_5}{}^{MN} + \dots, \quad (4.4.14)$$

is a Lorentz transformation which should be expected as we are varying a duality relation that only holds modulo Lorentz transformations, also similar to the situation in [36, 54].

One may notice an unusual factor of i that appears in the duality of the graviton and the

five form field. This can be removed by some simple field redefinitions. If we take the parameter $\Lambda_{aMN} \rightarrow i\Lambda_{aMN}$, and change the fields with an odd number of Lorentz indices by a factor of i , and finally by taking the derivative $\partial^{MN} \rightarrow i\partial^{MN}$. These redefinitions remove the factors of i that appear throughout the duality relations.

Indeed we have found that the graviton and the scalar satisfy duality relations at first order in derivatives with level 5 fields, as well as new dualities involving level 6 fields. This completes the derivation of the equations of motion and the duality relations at first order in derivatives of the fields in the 7D decomposition of E_{11} .

Chapter 5

Conclusion

This thesis has focused on deriving new results from the perspective of E theory. We have derived new algebras in various representations of E_{11} . Recall that the decomposition of the algebra, which leads to a maximal supergravity theory in D dimensions, is $GL(D) \otimes E_{11-D}$. We first computed the algebra for the 11 dimensional decomposition of E_{11} , which leads to a description of the 11D supergravity theory at low levels, for the adjoint generators at levels 5 and 6. With the same methods as in the 11D case, we derived the algebra in 7 dimensions for up to the level 5 generators in both the adjoint and fundamental representation, before using the action of the Cartan involution on these commutators to derive the algebra of the local Cartan involution invariant subgroup, $I_c(E_{11})$, and its coset. We finally calculated the IIB Cartan involution invariant algebra by acting on the previously known commutators with the action of the Cartan involution, to find the subalgebra of the adjoint and fundamental representation up to level 4, in a similar way to the 7D derivation.

In the next section, we derived the Cartan involution invariant tangent space metric from the vielbeins in the relevant dimension in 11, 5, and 4 dimensional decompositions of E_{11} , and additionally for A_1^{+++} , which is a Lie algebra whose non-linear realisation leads to 4 dimensional gravity. These metrics were found by first constructing the relevant group element, matter representation, and its transformation under the local subgroup. Then we constructed a map from the highest and lowest weight representation which is invariant under the $I_c(E_{11})$ subgroup. This map was then used to derive the invariant tangent space metrics. By using this metric and the explicit Cartan forms of the relevant theory, we

finally derived gauge-fixing conditions which are E_{11} invariant for the 11D, 5D, and 4D decompositions of E_{11} and for the A_1^{+++} algebra, whose non-linear realisation leads to the theory of 4D gravity.

In the last section of this thesis, we derived the 7 dimensional supergravity theory. We began by calculating the Cartan forms and their transformations using the algebra found in an earlier section, before using these results to derive equations of motion of the 7 dimensional maximal supergravity theory from E theory. The equations of motion were those for the graviton, the scalar, the one-form, and the two-form of the theory. The transformations of the equations of motion under the local transformations were found, and these all transformed into each other, and hence we have found that the equations of motion form an $I_c(E_{11})$ multiplet. These equations of motion agreed with previous results of the maximal supergravity equations of motion in 7 dimensions [52], and so strongly suggest that E theory is a correct theory. In the final section, we completed the derivation of the first order duality relations of the graviton and scalar with their corresponding dual fields arising at level 5 in the 7D theory, and the transformations of these duality relations. We found that the duality relations at first order all transform into each other (although some modulo transformations arose), forming a multiplet, as well as finding a new duality relation of the one form to a level six form.

Since E theory is relatively new, there are many exciting directions which new research can take. The 10 dimensional IIB algebra [23] derived in section 2.3 of this thesis has already been used in the derivation of dynamics of branes [39], and the topic of branes in various dimensions could lead to some interesting results due to branes being difficult to quantise. They have been studied in 11D, 10D IIB, 8D, and 7D [39, 40], and hence there are still many dimensions to explore.

It was mentioned in the introduction that there are an infinite number of generators in E_{11} . It was found that fields where the number of indices in an antisymmetric block on the fields are less than 10 gives rise to dual fields [55]. For fields with blocks of 10 indices or more, it is known that some of these corresponding to gauging of the theory, however there are still some whose purpose is unknown. Indeed, the level 5 and 6 algebra which

has been derived could be used to derive the dynamics of these higher level fields, and then one could examine the resulting properties of these higher level coordinates. In any case, the study of the higher level fields could lead to new insights into the need for an infinite dimensional algebra describing strings and branes.

Finally, in the 7 dimensional theory, it would be interesting to study the higher level fields in detail. In particular, we hope to find the coefficient of the new duality of the 1-form with a level 6 field in equation (4.4.13), and other other dualities at this level; studying these higher level fields might give an indication of the nature this infinite tower of fields in E_{11} , similar to our proposal in the 11D case. One may also study the second order equations of motion that contain the Cartan forms dual to the one-form and two-form, which were removed using dualities in our studies. In particular, transforming the four-form (which is dual to the one-form) equation of motion to find equations of motion for the level 5 Cartan forms would give more of an insight into the condition that has been found on the antisymmetric part of the dual scalar. In addition, it would be useful to check the derivation of the equations of motion of the graviton and the scalar from the dualities which are first order in derivatives, as was done for the one-form and two-form. This derivation would involve subtleties involving the fact that some of these relations hold modulo certain transformations. Finally, another interesting direction would be to determine the duality relations for the fields which result in the gauged seven dimensional supergravity theories. In general, there are many directions left to explore in E theory, with the most obvious being to begin building the theory and equations of motion in other dimensions.

Appendix A

Algebra in 11D

In this section, we give the 11D algebra up to level 4, both of the adjoint representation and the vector representation. This algebra is given up to level 3 in [19] and at level 4 in [23]. In the following, the lower case latin indices are $a, b, \dots = 1, \dots, 11$.

A.1 Algebra at level 0

At level 0, there is one generator K^a_b , and it has the following commutator with itself

$$[K^a_c, K^b_d] = \delta_c^b K^a_d - \delta_d^a K^b_c. \quad (\text{A.1})$$

A.2 Algebra at level 1

At level 1, the generator is $R_1^{a_1 a_2 a_3}$, and at level -1, the generator is $R_{-1 a_1 a_2 a_3}$. The commutators with level ± 1 are

$$\begin{aligned} [K^b_d, R_1^{a_1 a_2 a_3}] &= 3\delta_d^{[a_1} R_1^{b] a_2 a_3}, \\ [K^b_d, R_{-1 a_1 a_2 a_3}] &= -3\delta_{[a_1}^b R_{-1 d] a_2 a_3}, \\ [R_1^{a_1 a_2 a_3}, R_{-1 b_1 b_2 b_3}] &= 18K^{[a_1}_{[b_1} \delta_{b_2 b_3]}^{a_2 a_3]} - 2\mathbb{D}\delta_{b_1 b_2 b_3}^{a_1 a_2 a_3}, \end{aligned} \quad (\text{A.1})$$

where $\mathbb{D} = \sum_c K^c_c$.

A.3 Algebra at level 2

The generator at level 2 is $R_2^{a_1 \dots a_6}$, and the corresponding level -2 generator is $R_{-2a_1 \dots a_6}$.

The commutators with level ± 2 are

$$\begin{aligned}
[K_d^b, R_2^{a_1 \dots a_6}] &= 6\delta_d^{[a_1} R_2^{b|a_2 \dots a_6]}, \\
[K_d^b, R_{-2a_1 \dots a_6}] &= -6\delta_{[a_1}^b R_{-2d|a_2 \dots a_6]}, \\
[R_1^{a_1 a_2 a_3}, R_1^{b_1 b_2 b_3}] &= 2R_2^{a_1 a_2 a_3 b_1 b_2 b_3}, \\
[R_{-1a_1 a_2 a_3}, R_{-1b_1 b_2 b_3}] &= 2R_{-2a_1 a_2 a_3 b_1 b_2 b_3}, \\
[R_{-1a_1 a_2 a_3}, R_2^{b_1 \dots b_6}] &= 60\delta_{a_1 a_2 a_3}^{[b_1 b_2 b_3} R_1^{b_4 b_5 b_6]}, \\
[R_1^{a_1 a_2 a_3}, R_{-2b_1 \dots b_6}] &= 60\delta_{[b_1 b_2 b_3}^{a_1 a_2 a_3} R_{-1b_4 b_5 b_6]}, \\
[R_2^{a_1 \dots a_6}, R_{-2b_1 \dots b_6}] &= 120\mathbb{D}\delta_{b_1 \dots b_6}^{a_1 \dots a_6} \\
&\quad - 1080K^{[a_1}_{[b_1} \delta_{b_2 \dots b_6]}^{a_2 \dots a_6]}. \tag{A.1}
\end{aligned}$$

A.4 Algebra at level 3

The generator at level 3 is $R_3^{a_1 \dots a_8, c}$, and the level -3 generator is $R_{-3a_1 \dots a_8, c}$. Then the commutators with level ± 3 generators are

$$\begin{aligned}
[K_d^b, R_3^{a_1 \dots a_8, c}] &= 8\delta_d^{[a_1} R_3^{b|a_2 \dots a_8], c} \\
&\quad + \delta_d^c R_3^{a_1 \dots a_8, b}, \\
[K_d^b, R_{-3a_1 \dots a_8, c}] &= -8\delta_{[a_1}^b R_{-3d|a_2 \dots a_8], c} \\
&\quad - \delta_c^b R_{-3a_1 \dots a_8, d}, \\
[R_1^{a_1 a_2 a_3}, R_2^{b_1 \dots b_6}] &= 6R_3^{a_1 a_2 a_3 [b_1 \dots b_5, b_6]}, \\
[R_{-1a_1 a_2 a_3}, R_{-2b_1 \dots b_6}] &= 6R_{-3a_1 a_2 a_3 [b_1 \dots b_5, b_6]}, \\
[R_{-1a_1 a_2 a_3}, R_3^{b_1 \dots b_8, d}] &= -112\delta_{a_1 a_2 a_3}^{[b_1 b_2 | d} R_2^{b_3 \dots b_8]} \\
&\quad + 112\delta_{a_1 a_2 a_3}^{[b_1 b_2 b_3} R_2^{b_4 \dots b_8] d}, \\
[R_1^{a_1 a_2 a_3}, R_{-3b_1 \dots b_8, b}] &= -112\delta_{[b_1 b_2 | b}^{a_1 a_2 a_3} R_{-2|b_3 \dots b_8]}
\end{aligned}$$

$$\begin{aligned}
& + 112\delta_{[b_1 b_2 b_3]}^{a_1 a_2 a_3} R_{-2 b_4 \dots b_8} b , \\
[R_{2 a_1 \dots a_6}, R_{-3 b_1 \dots b_8, b}] & = - 3360\delta_{[b_1 \dots b_5] b}^{a_1 \dots a_6} R_{-1 b_6 b_7 b_8} \\
& - 3360\delta_{[b_1 \dots b_6]}^{a_1 \dots a_6} R_{-1 b_7 b_8} b , \\
[R_{-2 a_1 \dots a_6}, R_3^{b_1 \dots b_8, d}] & = - 3360\delta_{a_1 \dots a_6}^{[b_1 \dots b_5] d} R_1^{[b_6 b_7 b_8]} \\
& - 3360\delta_{a_1 \dots a_6}^{[b_1 \dots b_6]} R_1^{b_7 b_8} d , \\
[R_{-3 a_1 \dots a_8, c}, R_3^{b_1 \dots b_8, d}] & = - 8960\mathbb{D}_c^{[b_1} \delta_{a_1 \dots a_8}^{b_2 \dots b_8] d} \\
& + 8960\mathbb{D}_{a_1 \dots a_8}^{b_1 \dots b_8} \delta_c^d \\
& - 62720K^{[b_1}_{[a_1] \delta_c^{b_2} \delta_{[a_2 \dots a_8]}^{b_3 \dots b_8] d} \\
& - 71680K^{[b_1}_{[a_1} \delta_{a_2 \dots a_8}^{b_2 \dots b_8]} \delta_c^d \\
& + 8960K^d_{[a_1} \delta_{a_2 \dots a_8}^{b_1 \dots b_8]} c \\
& + 8960K^{[b_1}_{c} \delta_{a_1 \dots a_8}^{b_2 \dots b_8] d} \\
& - 8960\delta_{a_1 \dots a_8}^{b_1 \dots b_8} K^d_c . \tag{A.1}
\end{aligned}$$

A.5 Algebra at level 4

The generators at level 4 are

$$R_{41}^{a_1 \dots a_{11}, b}, \quad R_{42}^{a_1 \dots a_{10}, (b_1 b_2)}, \quad R_{43}^{a_1 \dots a_9, b_1 b_2 b_3}, \tag{A.1}$$

and the level -4 generators are

$$R_{-41}^{a_1 \dots a_{11}, b}, \quad R_{-42}^{a_1 \dots a_{10}, (b_1 b_2)}, \quad R_{-43}^{a_1 \dots a_9, b_1 b_2 b_3}. \tag{A.2}$$

The commutators of the level 4 generators with the level 0 generator are

$$\begin{aligned}
[K^b_d, R_{41}^{a_1 \dots a_{11}, c}] & = 11R_{41}^{[a_1 \dots a_{10} | b, c] \delta_d^{[a_{11}]} \\
& + 11R_{41}^{[a_1 \dots a_{10} | c, b] \delta_d^{[a_{11}]}}, \\
[K^b_d, R_{42}^{a_1 \dots a_{10}, (c_1 c_2)}] & = 20R_{42}^{[a_1 \dots a_9 | (c_1 |, | a_{10}] b] \delta_d^{[c_2]} \\
& + 10R_{42}^{[a_1 \dots a_9 | (c_1, c_2) b] \delta_d^{[a_{10}]}
\end{aligned}$$

$$\begin{aligned}
& - 90R_{42}^{[a_1\dots a_8|b(c_1,c_2)|a_9]\delta_d^{a_{10}}}, \\
[K_d^b, R_{43}^{a_1\dots a_9,c_1c_2c_3}] &= 108R_{43}^{[c_1c_2|[a_1\dots a_7,a_8a_9]b]\delta_d^{c_3}} \\
& - 72R_{43}^{[a_1\dots a_7|[c_1c_2,c_3]b]a_8\delta_d^{a_9}} \\
& + 252R_{43}^{[a_1\dots a_6|b[c_1c_2,c_3]]a_7a_8\delta_d^{a_9}}. \tag{A.3}
\end{aligned}$$

For the level 1 with level 3 generators, we find the following commutators

$$\begin{aligned}
[R_1^{a_1a_2a_3}, R_3^{b_1\dots b_8,c}] &= R_{41}^{a_1a_2a_3b_1\dots b_8,c} \\
& + 3R_{42}^{[a_1a_2|b_1\dots b_8,|a_3]c} \\
& - \frac{1}{3}R_{41}^{[a_1a_2|b_1\dots b_8c,|a_3]} \\
& + \frac{3}{2}R_{43}^{[a_1|b_1\dots b_8,|a_2a_3]c} \\
& - \frac{1}{6}R_{43}^{b_1\dots b_8c,a_1a_2a_3}, \tag{A.4}
\end{aligned}$$

The commutator of the level 2 generator with itself gives

$$\begin{aligned}
[R_2^{a_1\dots a_6}, R_2^{b_1\dots b_6}] &= 4R_{41}^{a_1\dots a_6[b_1\dots b_5,b_6]} \\
& + 5R_{43}^{a_1\dots a_6[b_1b_2b_3,b_4b_5b_6]}, \tag{A.5}
\end{aligned}$$

The level 4 generator R_{41} with the negative level generators have the commutators

$$\begin{aligned}
[R_{41}^{a_1\dots a_{11},c}, R_{-1}b_1b_2b_3] &= -\frac{495}{4}R_3^{[a_1\dots a_8],c}\delta_{b_1b_2b_3}^{[a_9a_{10}a_{11}]}, \\
[R_{41}^{a_1\dots a_{11},c}, R_{-2}b_1\dots b_6] &= 27720R_2^{[a_1\dots a_5|c]\delta_{b_1\dots b_6}^{[a_6\dots a_{11}]}, \\
[R_{41}^{a_1\dots a_{11},c}, R_{-3}b_1\dots b_8,d] &= 184800R_1^{[a_1a_2a_3]\delta_{b_1\dots b_8}^{a_4\dots a_{11}}}\delta_d^c \\
& - 184800R_1^{[a_1a_2a_3]\delta_{[b_1\dots b_7|d]}\delta_{[b_8]}^c}, \\
[R_{41}^{a_1\dots a_{11},c}, R_{-41}b_1\dots b_{11},d] &= -137214000K^{[a_1]_{[b_1]}\delta_d^{a_2}\delta_{[b_2\dots b_{11}]}^{a_3\dots a_{11}]^c} \\
& - 9147600K^{[a_1]_{[b_1]}\delta_{[b_2\dots b_{11}]}^{a_2\dots a_{11}}}\delta_d^c, \\
[R_{41}^{a_1\dots a_{11},c}, R_{-42}b_1\dots b_{10},(d_1d_2)] &= 0, \\
[R_{41}^{a_1\dots a_{11},c}, R_{-43}b_1\dots b_9,d_1d_2d_3] &= 0. \tag{A.6}
\end{aligned}$$

The level 4 generator R_{42} with the negative level generators have the commutators

$$\begin{aligned}
[R_{42}^{a_1 \dots a_{10}, (c_1 c_2)}, R_{-1 b_1 b_2 b_3}] &= -\frac{405}{2} R_3^{[a_1 \dots a_8], (c_1 \delta_{b_1 b_2 b_3}^{c_2}) | a_9 a_{10}]} \\
&\quad - 180 R_3^{[a_1 \dots a_7 | (c_1, c_2) \delta_{b_1 b_2 b_3}^{a_8 a_9 a_{10}}]}, \\
[R_{42}^{a_1 \dots a_{10}, (c_1 c_2)}, R_{-2 b_1 \dots b_6}] &= 0, \\
[R_{42}^{a_1 \dots a_{10}, (c_1 c_2)}, R_{-3 b_1 \dots b_8, d}] &= -302400 R_1^{[a_1 a_2 a_3 \delta_{b_1 \dots b_8}^{a_4 \dots a_{10}}] (c_1 \delta_d^{c_2})} \\
&\quad - 302400 R_1^{(c_1 | [a_1 a_2 \delta_d^{a_3} \delta_{b_1 \dots b_8}^{a_4 \dots a_{10}}] | c_2)} \\
&\quad + 302400 R_1^{[a_1 a_2 | (c_1 \delta_d^{c_2}) \delta_{b_1 \dots b_8}^{a_3 \dots a_{10}}]}, \\
[R_{42}^{a_1 \dots a_{10}, (c_1 c_2)}, R_{-41 b_1 \dots b_{11}, d}] &= 0, \\
[R_{42}^{a_1 \dots a_{10}, (c_1 c_2)}, R_{-42 b_1 \dots b_{10}, (d_1 d_2)}] &= -504000 \mathbb{D} \delta_{(d_1)}^{[a_1} \delta_{b_1 \dots b_{10}}^{a_2 \dots a_{10}] (c_1 \delta_{|d_2}^{c_2})} \\
&\quad + 1159200 \mathbb{D} \delta_{b_1 \dots b_{10}}^{a_1 \dots a_{10}} \delta_{(d_1)}^{(c_1 \delta_{d_2}^{c_2})} \\
&\quad - 9072000 K^{[a_1} \delta_{[b_1} \delta_{b_2 \dots b_{10}}^{a_2 \dots a_{10}}] \delta_{(d_1)}^{(c_1 \delta_{d_2}^{c_2})} \\
&\quad + 1512000 \delta_{(d_1)}^{[a_1} \delta_{b_1 \dots b_{10}}^{a_2 \dots a_{10}}] (c_1 K^{c_2})_{|d_2)} \\
&\quad - 1663200 \delta_{b_1 \dots b_{10}}^{a_1 \dots a_{10}} K^{(c_1} (d_1 \delta_{d_2}^{c_2}) , \\
[R_{42}^{a_1 \dots a_{10}, (c_1 c_2)}, R_{-43 b_1 \dots b_9, d_1 d_2 d_3}] &= 0. \tag{A.7}
\end{aligned}$$

The commutators of the level 4 generator R_{43} with the negative level generators are

$$\begin{aligned}
[R_{43}^{a_1 \dots a_9, c_1 c_2 c_3}, R_{-1 b_1 b_2 b_3}] &= -189 R_3^{[a_1 \dots a_8], [c_1 \delta_{b_1 b_2 b_3}^{c_2 c_3}] | a_9]} \\
&\quad - 432 R_3^{[a_1 \dots a_7 | [c_1, c_2 \delta_{b_1 b_2 b_3}^{c_3}] | a_8 a_9]} \\
&\quad - 252 R_3^{[a_1 \dots a_6 | [c_1 c_2, c_3] \delta_{b_1 b_2 b_3}^{a_7 a_8 a_9}]}, \\
[R_{43}^{a_1 \dots a_9, c_1 c_2 c_3}, R_{-2 b_1 \dots b_6}] &= 75600 R_2^{[a_1 \dots a_5 | [c_1 \delta_{b_1 \dots b_6}^{c_2 c_3}] | a_6 \dots a_9]} \\
&\quad - 15120 R_2^{[a_1 \dots a_4 | [c_1 c_2 \delta_{b_1 \dots b_6}^{c_3}] | a_5 \dots a_9]} \\
&\quad + 20160 R_2^{[a_1 a_2 a_3 | c_1 c_2 c_3 \delta_{b_1 \dots b_6}^{a_4 \dots a_9}]}, \\
[R_{43}^{a_1 \dots a_9, c_1 c_2 c_3}, R_{-3 b_1 \dots b_8, d}] &= -282240 R_1^{[a_1 a_2 a_3 \delta_d^{a_4} \delta_{b_1 \dots b_8}^{a_5 \dots a_9}] c_1 c_2 c_3} \\
&\quad - 282240 R_1^{[a_1 a_2 a_3 \delta_{b_1 \dots b_8}^{a_4 \dots a_9}] [c_1 c_2 \delta_d^{c_3}] } \\
&\quad - 564480 R_1^{[c_1 | [a_1 a_2 \delta_d^{a_3} \delta_{b_1 \dots b_8}^{a_4 \dots a_9}] | c_2 c_3]} \\
&\quad + 564480 R_1^{[c_1 | [a_1 a_2 \delta_{b_1 \dots b_8}^{a_3 \dots a_9}] | c_2 \delta_d^{c_3}] }
\end{aligned}$$

$$\begin{aligned}
& - 282240R_1^{[c_1c_2|[a_1\delta_d^{a_2}\delta_{b_1\dots b_8}^{a_3\dots a_9}]|c_3]} \\
& + 282240R_1^{[c_1c_2|[a_1\delta_{b_1\dots b_8}^{a_2\dots a_9}]\delta_d^{c_3}]}, \\
[R_{43}^{a_1\dots a_9,c_1c_2c_3}, R_{-41}b_1\dots b_{11},d] &= 0, \\
[R_{43}^{a_1\dots a_9,c_1c_2c_3}, R_{-42}b_1\dots b_{10},(d_1d_2)] &= 0, \\
[R_{43}^{a_1\dots a_9,c_1c_2c_3}, R_{-43}b_1\dots b_9,d_1d_2d_3] &= 1814400\mathbb{D}\delta_{b_1\dots b_9}^{a_1\dots a_9}\delta_{d_1d_2d_3}^{c_1c_2c_3} \\
& + 9797760\mathbb{D}\delta_{d_1d_2d_3}^{[c_1c_2|[a_1\delta_{b_1\dots b_9}^{a_2\dots a_9}]|c_3]} \\
& - 11702880K^{[a_1]_{[b_1\delta_{b_2\dots b_9}^{a_2\dots a_9}]\delta_{d_1d_2d_3}^{c_1c_2c_3}} \\
& - 8436960K^{[a_1]_{[d_1\delta_{b_1\dots b_9}^{a_2\dots a_9}]|c_1\delta_{[d_2d_3]}^{c_2c_3}} \\
& + 15240960K^{[a_1]_{[b_1\delta_{d_1d_2d_3}^{a_2a_3}[c_1\delta_{[b_2\dots b_9]}^{c_2c_3}]|a_4\dots a_9]} \\
& + 4354560K^{[a_1]_{[d_1\delta_{d_2d_3}^{a_2}[c_1\delta_{b_1\dots b_9}^{c_2c_3}]|a_3\dots a_9]} \\
& - 67495680K^{[a_1]_{[b_1\delta_{d_1d_2d_3}^{a_2}[c_1c_2\delta_{[b_2\dots b_9]}^{c_3}]|a_3\dots a_9]} \\
& - 15240960\delta_{d_1d_2d_3}^{[a_1a_2a_3]\delta_{[b_1\dots b_8]}^{a_4\dots a_9}[c_1c_2K^{c_3}]_{b_9}} \\
& - 3961440\delta_{b_1\dots b_9}^{a_1\dots a_9}K^{[c_1]_{[d_1\delta_{d_2d_3}^{c_2c_3}]} \\
& + 17418240\delta_{d_1d_2d_3}^{[c_1|[a_1a_2\delta_{[b_1\dots b_8]}^{a_3\dots a_9}]|c_2K^{c_3}]_{b_9}} \\
& - 15785280\delta_{[d_1d_2]}^{[c_1|[a_1\delta_{b_1\dots b_9}^{a_2\dots a_9}]|c_2K^{c_3}]_{d_3}} \\
& - 10069920\delta_{d_1d_2d_3}^{[c_1c_2|[a_1\delta_{[b_1\dots b_8]}^{a_2\dots a_9}]K^{c_3}]_{b_9}} \tag{A.8}
\end{aligned}$$

The negative level commutators resulting in the level -4 generators are

$$\begin{aligned}
[R_{-1}a_1a_2a_3, R_{-3}b_1\dots b_8,d] &= R_{-41}a_1a_2a_3b_1\dots b_8,d \\
& + 3R_{-42}b_1\dots b_8[a_1a_2,a_3]d \\
& - \frac{1}{3}R_{-41}b_1\dots b_8d[a_1a_2,a_3] \\
& + \frac{3}{2}R_{-43}b_1\dots b_8[a_1,a_2a_3]d \\
& - \frac{1}{6}R_{-43}b_1\dots b_8d,a_1a_2a_3, \\
[R_{-2}a_1\dots a_6, R_{-2}b_1\dots b_6] &= 4R_{-41}a_1\dots a_6[b_1\dots b_5,b_6] \\
& + 5R_{-43}a_1\dots a_6[b_1b_2b_3,b_4b_5b_6], \tag{A.9}
\end{aligned}$$

The level -4 generator R_{-41} with the positive level generators have the commutators

$$\begin{aligned}
[K^b_d, R_{-41 a_1 \dots a_{11}, c}] &= -11 R_{-41 [a_1 \dots a_{10} | d, c] \delta^b_{| a_{11}]} , \\
[R_{-41 a_1 \dots a_{11}, c}, R_1^{b_1 b_2 b_3}] &= -\frac{495}{4} R_{-3 [a_1 \dots a_8 |, c] \delta^{b_1 b_2 b_3}_{| a_9 a_{10} a_{11}]} , \\
[R_{-41 a_1 \dots a_{11}, c}, R_2^{b_1 \dots b_6}] &= 27720 R_{-2 [a_1 \dots a_5 | c] \delta^{b_1 \dots b_6}_{| a_6 \dots a_{11}]} , \\
[R_{-41 a_1 \dots a_{11}, c}, R_3^{b_1 \dots b_8, d}] &= 184800 R_{-1 [a_1 a_2 a_3 \delta^{b_1 \dots b_8}_{a_4 \dots a_{11}}] \delta^d_c \\
&\quad + 184800 R_{-1 [a_1 a_2 a_3 \delta^{[b_1 \dots b_7 | d] \delta^{b_8]}_c}]} . \tag{A.10}
\end{aligned}$$

The commutators of the level -4 generator R_{-42} with the positive level generators are

$$\begin{aligned}
[K^b_d, R_{-42 a_1 \dots a_{10}, (c_1 c_2)}] &= 20 R_{-42 (c_1 | [a_1 \dots a_9, a_{10}] d \delta^b_{| c_2})} \\
&\quad - 10 R_{-42 [a_1 \dots a_9 | (c_1, c_2) d] \delta^b_{| a_{10}}} \\
&\quad + 90 R_{-42 [a_1 \dots a_8 | d (c_1, c_2) | a_9] \delta^b_{a_{10}}} , \\
[R_{-42 a_1 \dots a_{10}, c_1 c_2}, R_1^{b_1 b_2 b_3}] &= -\frac{405}{2} R_{-3 [a_1 \dots a_8 |, (c_1 \delta^{b_1 b_2 b_3}_{c_2}) | a_9 a_{10}]} \\
&\quad - 180 R_{-3 [a_1 \dots a_7 | (c_1, c_2) \delta^{b_1 b_2 b_3}_{| a_8 a_9 a_{10}}]} , \\
[R_{-42 a_1 \dots a_{10}, (c_1 c_2)}, R_2^{b_1 \dots b_6}] &= 0 , \\
[R_{-42 a_1 \dots a_{10}, c_1 c_2}, R_3^{b_1 \dots b_8, d}] &= -302400 R_{-1 [a_1 a_2 a_3 \delta^{b_1 \dots b_8}_{a_4 \dots a_{10}}] (c_1 \delta^d_{c_2})} \\
&\quad - 302400 R_{-1 (c_1 | [a_1 a_2 \delta^d_{a_3} \delta^{b_1 \dots b_8}_{a_4 \dots a_{10}}] | c_2)} \\
&\quad + 302400 R_{-1 [a_1 a_2 | (c_1 \delta^d_{c_2}) \delta^{b_1 \dots b_8}_{| a_3 \dots a_{10}}]} . \tag{A.11}
\end{aligned}$$

Finally, the commutators of the level -4 generator R_{-43} with the positive level generators are

$$\begin{aligned}
[K^b_d, R_{-43 a_1 \dots a_9, c_1 c_2 c_3}] &= -108 R_{-43 [c_1 c_2 | [a_1 \dots a_7, a_8 a_9] d] \delta^b_{| c_3}} \\
&\quad + 72 R_{-43 [a_1 \dots a_7 | [c_1 c_2, c_3] d] a_8 \delta^b_{a_9}} \\
&\quad - 252 R_{-43 a_1 \dots a_6 | d [c_1 c_2, c_3] | a_7 a_8 \delta^b_{a_9}} , \\
[R_{-43 a_1 \dots a_9, c_1 c_2 c_3}, R_1^{b_1 b_2 b_3}] &= -189 R_{-3 [a_1 \dots a_8 |, [c_1 \delta^{b_1 b_2 b_3}_{c_2 c_3}] | a_9]} \\
&\quad - 432 R_{-3 [a_1 \dots a_7 | [c_1, c_2 \delta^{b_1 b_2 b_3}_{c_3}] | a_8 a_9]}
\end{aligned}$$

$$\begin{aligned}
& - 252R_{-3}[a_1\dots a_6|[c_1c_2,c_3]\delta_{[a_7a_8a_9]}^{b_1b_2b_3}] , \\
[R_{-43}a_1\dots a_9,c_1c_2c_3, R_2^{b_1\dots b_6}] &= 75600R_{-2}[a_1\dots a_5|[c_1\delta_{c_2c_3}^{b_1\dots b_6}]_{a_6\dots a_9}] \\
& - 15120R_{-2}[a_1\dots a_4|[c_1c_2\delta_{c_3}^{b_1\dots b_6}]_{a_5\dots a_9}] \\
& + 20160R_{-2}[a_1a_2a_3|c_1c_2c_3\delta_{[a_4\dots a_9]}^{b_1\dots b_6}] , \\
[R_{-43}a_1\dots a_9,c_1c_2c_3, R_3^{b_1\dots b_8,d}] &= - 282240R_{-1}[a_1a_2a_3\delta_{a_4}^d\delta_{a_5\dots a_9}^{b_1\dots b_8}]_{c_1c_2c_3} \\
& + 282240R_{-1}[a_1a_2a_3\delta_{a_4\dots a_9}^{b_1\dots b_8}]_{[c_1c_2\delta_{c_3}^d]} \\
& - 564480R_{-1}[c_1|[a_1a_2\delta_{a_3}^d\delta_{a_4\dots a_9}^{b_1\dots b_8}]_{c_2c_3}] \\
& + 564480R_{-1}[c_1|[a_1a_2\delta_{a_3\dots a_9}^{b_1\dots b_8}]_{c_2\delta_{c_3}^d}] \\
& - 282240R_{-1}[c_1c_2|[a_1\delta_{a_2}^d\delta_{a_3\dots a_9}^{b_1\dots b_8}]_{c_3}] \\
& + 282240R_{-1}[c_1c_2|[a_1\delta_{a_2\dots a_9}^{b_1\dots b_8}]\delta_{c_3}^d] \tag{A.12}
\end{aligned}$$

This completes the algebra of the adjoint representation up to level 4. We now give the algebra of the l_1 generators up to level 4.

A.6 l_1 algebra

In this section, we give the commutators with the l_1 generators. The generators of the l_1 representation up to level 4 are

$$\begin{aligned}
& P_c ; \quad Z^{c_1c_2} ; \quad Z^{c_1\dots c_5} ; \quad Z^{c_1\dots c_8} , \quad Z^{c_1\dots c_7,d} ; \quad Z^{c_1\dots c_8,d_1d_2d_3} , \\
& Z^{c_1\dots c_9,(d_1d_2)} , \quad Z^{c_1\dots c_9,d_1d_2} , \quad Z_{(1)}^{c_1\dots c_{10},d} , \quad Z_{(2)}^{c_1\dots c_{10},d} , \quad Z^{c_1\dots c_{11}} ; \dots \tag{A.1}
\end{aligned}$$

We begin with the commutators of the level 0 generator of E_{11} with l_1

$$\begin{aligned}
[K^a_b, P_c] &= -\delta_c^a P_b + \frac{1}{2}\delta_b^a P_c , \\
[K^a_b, Z^{c_1c_2}] &= 2\delta^{[c_1}_b Z^{a|b_2]} + \frac{1}{2}\delta_b^a Z^{c_1c_2} , \\
[K^a_b, Z^{c_1\dots c_5}] &= 5\delta_b^{[c_1} Z^{a|c_2\dots c_5]} + \frac{1}{2}\delta_b^a Z^{c_1\dots c_5} , \\
[K^a_b, Z^{c_1\dots c_8}] &= 8\delta_b^{[c_1} Z^{a|c_2\dots c_8]} + \frac{1}{2}\delta_b^a Z^{c_1\dots c_8} ,
\end{aligned}$$

$$\begin{aligned}
[K^a_b, Z^{c_1 \dots c_7, d}] &= 7\delta_b^{[c_1} Z^{a|c_2 \dots c_7], d} + \delta_b^d Z^{c_1 \dots c_7, a} + \frac{1}{2}\delta_b^a Z^{c_1 \dots c_7, d} , \\
[K^a_b, Z^{c_1 \dots c_8, d_1 d_2 d_3}] &= 8\delta_b^{[c_1} Z^{a|c_2 \dots c_8], d_1 d_2 d_3} + 3\delta_b^{[d_1} Z^{c_1 \dots c_8, a|d_2 d_3]} + \frac{1}{2}\delta_b^a Z^{c_1 \dots c_8, d_1 d_2 d_3} , \\
[K^a_b, Z^{c_1 \dots c_9, (d_1 d_2)}] &= 9\delta_b^{[c_1} Z^{a|c_2 \dots c_9], (d_1 d_2)} + 2\delta_b^{(d_1} Z^{c_1 \dots c_9, a|d_2)} + \frac{1}{2}\delta_b^a Z^{c_1 \dots c_9, (d_1 d_2)} , \\
[K^a_b, Z^{c_1 \dots c_9, d_1 d_2}] &= 9\delta_b^{[c_1} Z^{a|c_2 \dots c_9], d_1 d_2} + 2\delta_b^{[d_1} Z^{c_1 \dots c_9, a|d_2]} + \frac{1}{2}\delta_b^a Z^{c_1 \dots c_9, d_1 d_2} , \\
[K^a_b, Z_{(1)}^{c_1 \dots c_{10}, d}] &= 10\delta_b^{[c_1} Z_{(1)}^{a|c_2 \dots c_{10}, d} + \delta_b^d Z_{(1)}^{c_1 \dots c_{10}, a} + \frac{1}{2}\delta_b^a Z_{(1)}^{c_1 \dots c_{10}, d} , \\
[K^a_b, Z_{(2)}^{c_1 \dots c_{10}, d}] &= 10\delta_b^{[c_1} Z_{(2)}^{a|c_2 \dots c_{10}, d} + \delta_b^d Z_{(2)}^{c_1 \dots c_{10}, a} + \frac{1}{2}\delta_b^a Z_{(2)}^{c_1 \dots c_{10}, d} , \\
[K^a_b, Z^{c_1 \dots c_{11}}] &= \frac{3}{2}\delta_b^a Z^{c_1 \dots c_{11}} . \tag{A.2}
\end{aligned}$$

Now we move onto the level 1 generator with l_1

$$\begin{aligned}
[R_1^{a_1 a_2 a_3}, P_b] &= 3\delta_b^{[a_1} Z^{a_2 a_3]} , \\
[R_1^{a_1 a_2 a_3}, Z^{b_1 b_2}] &= Z^{a_1 a_2 a_3 b_1 b_2} , \\
[R_1^{a_1 a_2 a_3}, Z^{b_1 \dots b_5}] &= Z^{b_1 \dots b_5 a_1 a_2 a_3} + Z^{b_1 \dots b_5 [a_1 a_2, a_3]} , \\
[R_1^{a_1 a_2 a_3}, Z^{b_1 \dots b_8}] &= Z^{a_1 a_2 a_3 b_1 \dots b_8} + \frac{4}{135} Z^{a_1 a_2 a_3 [b_1 \dots b_5, b_6 b_7 b_8]} \\
&\quad - \frac{20}{63} Z^{a_1 a_2 a_3 [b_1 \dots b_6, b_7 b_8]} - Z_{(2)}^{a_1 a_2 a_3 [b_1 \dots b_7, b_8]} , \\
[R_1^{a_1 a_2 a_3}, Z^{b_1 \dots b_7, c}] &= \frac{3}{8} Z_{(1)}^{b_1 \dots b_7 c [a_1 a_2, a_3]} - Z^{a_1 a_2 a_3 c [b_1 \dots b_6, b_7]} + Z^{a_1 a_2 a_3 c [b_1 \dots b_4, b_5 b_6 b_7]} \\
&\quad + \frac{1}{2} Z^c [b_1 \dots b_5 | [a_1 a_2, a_3] | b_6 b_7] + Z^c [b_1 \dots b_6 | [a_1 a_2, (a_3) | b_7]] \\
&\quad - \frac{3}{7} Z^c [b_1 \dots b_6 | [a_1 a_2, a_3] | b_7] + Z^{a_1 a_2 a_3 c [b_1 \dots b_5, b_6 b_7]} . \tag{A.3}
\end{aligned}$$

The level 2 generator with l_1 gives the following commutators

$$\begin{aligned}
[R_2^{a_1 \dots a_6}, P_b] &= -3\delta_b^{[a_1} Z^{a_2 \dots a_6]} , \\
[R_2^{a_1 \dots a_6}, Z^{b_1 b_2}] &= -Z^{a_1 \dots a_6 b_1 b_2} + \frac{1}{3} Z^{a_1 \dots a_6 [b_1, b_2]} , \\
[R_2^{a_1 \dots a_6}, Z^{b_1 \dots b_5}] &= -Z^{a_1 \dots a_6 b_1 \dots b_5} + \frac{4}{189} Z^{a_1 \dots a_6 [b_1 b_2, b_3 b_4 b_5]} + \frac{40}{441} Z^{a_1 \dots a_6 [b_1 b_2 b_3, b_4 b_5]} \\
&\quad - \frac{55}{336} Z_{(1)}^{a_1 \dots a_6 [b_1 \dots b_4, b_5]} + \frac{5}{16} Z_{(2)}^{a_1 \dots a_6 [b_1 \dots b_4, b_5]} . \tag{A.4}
\end{aligned}$$

The commutators of the level 3 generator with the l_1 are

$$\begin{aligned}
[R_3^{a_1 \dots a_8, b}, P_c] &= -\frac{4}{3} \delta_c^b Z^{a_1 \dots a_8} + \frac{4}{3} \delta_c^{[a_1} Z^{a_2 \dots a_8] b} + \frac{4}{3} \delta_c^{[a_1} Z^{a_2 \dots a_8], b}, \\
[R_3^{a_1 \dots a_8, b} Z^{c_1 c_2}] &= -\frac{16}{135} Z^{c_1 c_2 b [a_1 \dots a_5, a_6 a_7 a_8]} + \frac{16}{189} Z^{c_1 c_2 b [a_1 \dots a_6, a_7 a_8]} - \frac{16}{189} Z^{c_1 c_2 [a_1 \dots a_7, a_8] b} \\
&\quad + \frac{4}{63} Z^{c_1 c_2 [a_1 \dots a_7, (a_8) b]} + \frac{1}{42} Z_{(1)}^{c_1 c_2 b [a_1 \dots a_7, a_8]} + \frac{1}{6} Z_{(2)}^{c_1 c_2 b [a_1 \dots a_7, a_8]} \\
&\quad - \frac{1}{42} Z_{(1)}^{a_1 \dots a_8 c_1 c_2, b} + \frac{1}{6} Z_{(2)}^{a_1 \dots a_8 c_1 c_2, b}. \tag{A.5}
\end{aligned}$$

The final positive level commutators are those of the level 4 generators with the l_1 generators

$$\begin{aligned}
[R_{41}^{a_1 \dots a_{11}, b}, P_c] &= -\frac{605}{224} \delta_c^{[a_1} Z_{(1)}^{b | a_2 \dots a_{10}, a_{11}]} + \frac{165}{32} \delta_c^{[a_1} Z_{(2)}^{b | a_2 \dots a_{10}, a_{11}]} \\
&\quad - \frac{33}{2} \delta_c^{[a_1} Z^{a_2 \dots a_{11}] b}, \\
[R_{42}^{a_1 \dots a_{10}, (b_1 b_2)}, P_c] &= -\frac{55}{1008} \delta_c^{(b_1} Z_{(1)}^{b_2) [a_1 \dots a_9, a_{10}]} + \frac{55}{16} \delta_c^{(b_1} Z_{(2)}^{b_2) [a_1 \dots a_9, a_{10}]} \\
&\quad - \frac{5}{1008} \delta_c^{[a_1} Z_{(1)}^{a_2 \dots a_{10}] (b_1, b_2)} + \frac{5}{16} \delta_c^{[a_1} Z_{(2)}^{a_2 \dots a_{10}] (b_1, b_2)}, \\
[R_{43}^{a_1 \dots a_9, b_1 b_2 b_3}, P_c] &= -\frac{6}{7} \delta_c^{[c_3} Z^{c_2] [a_2 \dots a_9, a_1] | b_1]} - \frac{24}{49} \delta_c^{[a_9} Z^{a_2 \dots a_8] [b_2 b_3, b_1] | a_1]} \\
&\quad + \frac{32}{105} \delta_c^{[a_9} Z^{a_2 \dots a_8] [b_3, b_1 b_2] | a_1]}. \tag{A.6}
\end{aligned}$$

We can now write the commutators of the negative levels with the l_1 generators. We begin writing those with level -1

$$\begin{aligned}
[R_{-1 a_1 a_2 a_3}, Z^{b_1 b_2}] &= 6 P_{[a_1} \delta_{a_2 a_3]}^{b_1 b_2}, \\
[R_{-1 a_1 a_2 a_3}, Z^{b_1 \dots b_5}] &= 60 Z^{[b_1 b_2} \delta_{a_1 a_2 a_3]}^{b_3 b_4 b_5]}, \\
[R_{-1 a_1 a_2 a_3}, Z^{b_1 \dots b_8}] &= 42 Z^{[b_1 \dots b_5} \delta_{a_1 a_2 a_3]}^{b_6 b_7 b_8]}, \\
[R_{-1 a_1 a_2 a_3}, Z^{b_1 \dots b_7, c}] &= \frac{945}{4} Z^{[b_1 \dots b_5} \delta_{a_1 a_2 a_3]}^{b_6 b_7] c} \\
&\quad + \frac{945}{4} Z^{[b_1 \dots b_4 | c] \delta_{a_1 a_2 a_3]}^{b_5 b_6 b_7]}, \\
[R_{-1 a_1 a_2 a_3}, Z^{b_1 \dots b_8, c_1 c_2 c_3}] &= -945 Z^{b_1 \dots b_8} \delta_{a_1 a_2 a_3}^{c_1 c_2 c_3} \\
&\quad - 2835 Z^{[b_1 \dots b_7 | [c_1 \delta_{a_1 a_2 a_3}^{c_2 c_3}] | b_8]} \\
&\quad - 2835 Z^{[b_1 \dots b_6 | [c_1 c_2 \delta_{a_1 a_2 a_3}^{c_3}] | b_7 b_8]}
\end{aligned}$$

$$\begin{aligned}
& - 945Z^{[b_1\dots b_5|c_1c_2c_3]b_6b_7b_8} \delta_{a_1a_2a_3} \\
& - 42525Z^{[b_1\dots b_6|[c_1|,|b_7\delta_{a_1a_2a_3}^{b_8}]|c_2c_3]} \\
& + 42525Z^{[b_1\dots b_5|[c_1c_2|,|b_6\delta_{a_1a_2a_3}^{b_7b_8}]|c_3]} \\
& - 14175Z^{[b_1\dots b_4|c_1c_2c_3,|b_5\delta_{a_1a_2a_3}^{b_6b_7b_8}]}, \\
[R_{-1a_1a_2a_3}, Z^{b_1\dots b_9,(c_1c_2)}] &= \frac{254016}{11}Z^{[b_1\dots b_6|[c_1|,|b_7\delta_{a_1a_2a_3}^{b_8b_9}]|c_2]} \\
& + \frac{31752}{11}Z^{[b_1\dots b_6|(c_1,c_2)]\delta_{a_1a_2a_3}^{b_7b_8b_9}}, \\
[R_{-1a_1a_2a_3}, Z^{b_1\dots b_9,c_1c_2}] &= -2646Z^{[b_1\dots b_8\delta_{a_1\dots a_3}^{b_9}]c_1c_2} \\
& + 5292Z^{[b_1\dots b_7|[c_1\delta_{a_1a_2a_3}^{c_2}]|b_8b_9]} \\
& - 2646Z^{[b_1\dots b_6|c_1c_2]\delta_{a_1a_2a_3}^{b_7b_8b_9}} \\
& - 5292Z^{[b_1\dots b_6|[c_1|,|b_7\delta_{a_1a_2a_3}^{b_8b_9}]|c_2]} \\
& - 2646Z^{[b_1\dots b_5|c_1c_2,|b_6\delta_{a_1a_2a_3}^{b_7b_8b_9}]}, \\
[R_{-1a_1a_2a_3}, Z_{(1)}^{b_1\dots b_{10},c_1}] &= \frac{18900}{11}Z^{[b_1\dots b_8\delta_{a_1a_2a_3}^{b_9b_{10}}]c} \\
& + \frac{18900}{11}Z^{[b_1\dots b_7|c]\delta_{a_1a_2a_3}^{b_8b_9b_{10}}} \\
& + \frac{61740}{11}Z^{[b_1\dots b_6|c|,b_7\delta_{a_1a_2a_3}^{b_8b_9b_{10}}]}, \\
[R_{-1a_1a_2a_3}, Z_{(2)}^{b_1\dots b_{10},c_1}] &= \frac{8940}{11}Z^{[b_1\dots b_8\delta_{a_1a_2a_3}^{b_9b_{10}}]c} \\
& + \frac{8940}{11}Z^{[b_1\dots b_7|c]\delta_{a_1a_2a_3}^{b_8b_9b_{10}}} \\
& + 700Z^{[b_1\dots b_6|c|,b_7\delta_{a_1a_2a_3}^{b_8b_9b_{10}}]}, \\
[R_{-1a_1a_2a_3}, Z^{b_1\dots b_{11}}] &= 120Z^{[b_1\dots b_8\delta_{a_1a_2a_3}^{b_9b_{10}b_{11}}]}. \tag{A.7}
\end{aligned}$$

The commutators of level -2 with the l_1 generators are

$$\begin{aligned}
[R_{-2a_1\dots a_6}, Z^{b_1\dots b_5}] &= 360P_{[a_1}\delta_{a_2\dots a_6}^{b_1\dots b_5}], \\
[R_{-2a_1\dots a_6}, Z^{b_1\dots b_8}] &= 2520Z^{[b_1b_2\delta_{a_1\dots a_6}^{b_3\dots b_8}]}, \\
[R_{-2a_1\dots a_6}, Z^{b_1\dots b_7,c}] &= 5670Z^{[b_1b_2\delta_{a_1\dots a_6}^{b_3\dots b_7}]c} - 5670Z^{[b_1|c]\delta_{a_1\dots a_6}^{b_2\dots b_7}}, \\
[R_{-2a_1\dots a_6}, Z^{b_1\dots b_8,c_1c_2c_3}] &= 567000(Z^{[b_1\dots b_5\delta_{a_1\dots a_6}^{b_6b_7b_8}]c_1c_2c_3} - 3Z^{[b_1\dots b_4|[c_1\delta_{a_1\dots a_6}^{c_2c_3}]|b_5\dots b_8]} \\
& + 3Z^{[b_1b_2b_3|[c_1c_2\delta_{a_1\dots a_6}^{c_3}]|b_4\dots b_8]} \\
& - Z^{[b_1b_2|c_1c_2c_3]\delta_{a_1\dots a_6}^{b_3\dots b_8}}),
\end{aligned}$$

$$\begin{aligned}
[R_{-2a_1\dots a_6}, Z^{b_1\dots b_9, (d_1 d_2)}] &= 0, \\
[R_{-2a_1\dots a_6}, Z^{b_1\dots b_9, c_1 c_2}] &= -158760 Z^{[b_1\dots b_5 \delta_{a_1\dots a_6}^{b_6\dots b_9}] c_1 c_2} - 317520 Z^{[b_1\dots b_4 | [c_1 \delta_{a_1\dots a_6}^{c_2}] | b_5\dots b_9]} \\
&\quad - 158760 Z^{[b_1 b_2 b_3 | c_1 c_2 \delta_{a_1\dots a_6}^{b_4\dots b_9}]}, \\
[R_{-2a_1\dots a_6}, Z_{(1)}^{b_1\dots b_{10}, c}] &= \frac{1587600}{11} Z^{[b_1\dots b_4 | c] \delta_{a_1\dots a_6}^{b_5\dots b_{10}}} - \frac{1587600}{11} Z^{[b_1\dots b_5 \delta_{a_1\dots a_6}^{b_6\dots b_{10}}] c}, \\
[R_{-2a_1\dots a_6}, Z_{(2)}^{b_1\dots b_{10}, c}] &= \frac{25200}{11} Z^{[b_1\dots b_4 | c] \delta_{a_1\dots a_6}^{b_5\dots b_{10}}} - \frac{25200}{11} Z^{[b_1\dots b_5 \delta_{a_1\dots a_6}^{b_6\dots b_{10}}] c}, \\
[R_{-2a_1\dots a_6}, Z^{b_1\dots b_{11}}] &= 5040 Z^{[b_1\dots b_5 \delta_{a_1\dots a_6}^{b_6\dots b_{11}}]}. \tag{A.8}
\end{aligned}$$

Next, we write the commutators of level -3 with the l_1 generators

$$\begin{aligned}
[R_{-3a_1\dots a_8, b}, Z^{c_1\dots c_8}] &= 6720 P_{[a_1 \delta_{a_2\dots a_9}^{c_1\dots c_8}] b} - 6720 P_b \delta_{a_1\dots a_8}^{c_1\dots c_8}, \\
[R_{-3a_1\dots a_8, b}, Z^{c_1\dots c_7, d}] &= 52920 P_{[a_1 \delta_{a_2\dots a_8}^{[c_2\dots c_7] d}] \delta_b^{c_1}} + 52920 P_{[a_1 \delta_{a_2\dots a_8}^{c_1\dots c_7}] \delta_b^d}, \\
[R_{-3a_1\dots a_8, b}, Z^{c_1\dots c_8, d_1 d_2 d_3}] &= 9525600 (-Z^{[d_1 d_2] \delta_b^{[c_1 \delta_{a_1\dots a_8}^{c_2\dots c_8}] d_3}} - Z^{[c_1 c_2 \delta_b^{c_3} \delta_{a_1\dots a_8}^{c_4\dots c_8}] d_1 d_2 d_3} \\
&\quad + Z^{[c_1 c_2 \delta_{a_1\dots a_8}^{c_3\dots c_8}] [d_2 d_3 \delta_b^{d_1}] + Z^{[d_1 d_2 \delta_b^{d_3}] \delta_{a_1\dots a_8}^{c_1\dots c_8}} \\
&\quad + 2Z^{[d_1 | [c_1 \delta_b^{c_2} \delta_{a_1\dots a_8}^{c_3\dots c_8}] | d_2 d_3]} + 2Z^{[c_1 | [d_1 \delta_b^{d_2} \delta_{a_1\dots a_8}^{d_3}] | c_2\dots c_8]}), \\
[R_{-3a_1\dots a_8, b}, Z^{c_1\dots c_9, (d_1 d_2)}] &= \frac{91445760}{11} (Z^{[c_1 c_2 \delta_{a_1\dots a_8}^{c_3\dots c_9}] (d_2 \delta_b^{d_1})} - Z^{(d_1 | [c_1 \delta_b^{c_2} \delta_{a_1\dots a_8}^{c_3\dots c_9}] | d_2)} \\
&\quad - Z^{[c_1 | (d_1 \delta_b^{d_2}) \delta_{a_1\dots a_8}^{c_2\dots c_9}]}, \\
[R_{-3a_1\dots a_8, b}, Z_{(1)}^{c_1\dots c_{10}, d}] &= \frac{4233600}{11} (Z^{[c_1 c_2 \delta_{[a_1 \delta_{a_2\dots a_8}^{c_3\dots c_{10}}] b}]^d} - Z^{[c_1 c_2 \delta_{a_1\dots a_8}^{c_3\dots c_{10}}] \delta_b^d}), \\
[R_{-3a_1\dots a_8, b}, Z_{(2)}^{c_1\dots c_{10}, d}] &= 873600 Z^{[c_1 c_2 \delta_{a_1\dots a_8}^{c_3\dots c_{10}}] \delta_b^d} - 873600 Z^{[c_1 c_2 \delta_{[a_1 \delta_{a_2\dots a_8}^{c_3\dots c_{10}}] b}]^d}, \\
[R_{-3a_1\dots a_8, b}, Z^{c_1\dots c_{11}}] &= 0. \tag{A.9}
\end{aligned}$$

Finally, we give the commutators of the level -4 generators with the l_1 generators

$$\begin{aligned}
[R_{-41a_1\dots a_{11}, b}, Z^{c_1\dots c_8, d_1 d_2 d_3}] &= 0, \\
[R_{-41a_1\dots a_{11}, b}, Z^{c_1\dots c_8, (d_1 d_2)}] &= 0, \\
[R_{-41a_1\dots a_{11}, b}, Z^{c_1\dots c_9, d_1 d_2}] &= 0, \\
[R_{-41a_1\dots a_{11}, b}, Z^{c_1\dots c_{10}, d}] &= 47628000 (P_{[a_1 \delta_{a_2\dots a_{11}}^{[c_2\dots c_{10}] d}] \delta_b^{c_1}} - P_{[a_1 \delta_{a_2\dots a_{11}}^{c_1\dots c_{10}}] \delta_b^d}), \\
[R_{-41a_1\dots a_{11}, b}, Z^{c_1\dots c_{10}, d}] &= 756000 (P_{[a_1 \delta_{a_2\dots a_{11}}^{[c_2\dots c_{10}] d}] \delta_b^{c_1}} - P_{[a_1 \delta_{a_2\dots a_{11}}^{c_1\dots c_{10}}] \delta_b^d}),
\end{aligned}$$

$$\begin{aligned}
[R_{-41} a_1 \dots a_{11}, b, Z^{c_1 \dots c_{11}}] &= 907200 P_b \delta_{a_1 \dots a_{11}}^{c_1 \dots c_{11}}, \\
[R_{-42} a_1 \dots a_{10}, (b_1 b_2), Z^{c_1 \dots c_8, d_1 d_2 d_3}] &= 0, \\
[R_{-42} a_1 \dots a_{10}, (b_1 b_2), Z^{c_1 \dots c_9, (d_1 d_2)}] &= -\frac{1028764800}{11} (2P_{[a_1 |} \delta_{(b_1}^{(d_1} \delta_{b_2)}^{[c_1} \delta_{|a_2 \dots a_{10}]}}^{c_2 \dots c_9] | d_2}) \\
&\quad + P_{[a_1} \delta_{a_2 \dots a_{10]}^{c_1 \dots c_9}} \delta_{(b_1}^{(d_1} \delta_{b_2)}^{d_2)}), \\
[R_{-42} a_1 \dots a_{10}, (b_1 b_2), Z^{c_1 \dots c_9, d_1 d_2}] &= 0, \\
[R_{-42} a_1 \dots a_{10}, (b_1 b_2), Z_{(1)}^{c_1 \dots c_{10}, d}] &= -\frac{95256000}{11} (\delta_{[a_1}^d \delta_{a_2 \dots a_{10}] (b_2}^{c_1 \dots c_{10}} P_{b_1}) + P_{[a_1} \delta_{a_2 \dots a_{10]}^{c_1 \dots c_{10}} \delta_{(b_2}^d) \\
&\quad - P_{(b_1} \delta_{b_2)}^d \delta_{a_2 \dots a_{10}}^{c_1 \dots c_{10}}), \\
[R_{-42} a_1 \dots a_{10}, (b_1 b_2), Z_{(2)}^{c_1 \dots c_{10}, d}] &= -4536000 (\delta_{[a_1}^d \delta_{a_2 \dots a_{10}] (b_2}^{c_2 \dots c_{10}} P_{b_1}) + P_{[a_1} \delta_{a_2 \dots a_{10]}^{c_1 \dots c_{10}} \delta_{(b_2}^d) \\
&\quad - P_{(b_1} \delta_{b_2)}^d \delta_{a_1 \dots a_{10}}^{c_1 \dots c_{10}}), \\
[R_{-42} a_1 \dots a_{10}, (b_1 b_2), Z^{c_1 \dots c_{11}}] &= 0, \\
[R_{-43} a_1 \dots a_9, b_1 b_2 b_3, Z^{c_1 \dots c_8, d_1 d_2 d_3}] &= -342921600 (P_{[a_1 |} \delta_{b_1 b_2 b_3}^{[c_1 c_2 c_3} \delta_{|a_2 \dots a_9]}^{c_4 \dots c_8] d_1 d_2 d_3} \\
&\quad - P_{[a_1} \delta_{a_2 \dots a_9]}^{c_1 \dots c_8} \delta_{b_1 b_2 b_3}^{d_1 d_2 d_3} - 3P_{[a_1 |} \delta_{b_1 b_2 b_3}^{[d_1 | [c_1 c_2} \delta_{a_2 \dots a_9]}^{c_3 \dots c_3] | d_2 d_3]} \\
&\quad + 3P_{[a_1 |} \delta_{b_1 b_2 b_3}^{[d_1 d_2 | [c_1} \delta_{|a_2 \dots a_9]}^{c_2 \dots c_8] | d_3}]), \\
[R_{-43} a_1 \dots a_9, b_1 b_2 b_3, Z^{c_1 \dots c_9, (d_1 d_2)}] &= 0, \\
[R_{-43} a_1 \dots a_9, b_1 b_2 b_3, Z^{c_1 \dots c_9, d_1 d_2}] &= -42865200 P_{[a_1} \delta_{a_2 \dots a_9]}^{c_1 \dots c_9} \delta_{[b_3}^{d_1 d_2} \delta_{b_1 b_2]} \\
&\quad - 137168640 P_{[a_1} \delta_{a_2 | [b_1}^{d_1 d_2} \delta_{b_2 b_3]}^{c_1 \dots c_9} | a_3 \dots a_9] \\
&\quad - 68584320 P_{[b_1 |} \delta_{[a_1 a_2}^{d_1 d_2} \delta_{a_3 \dots a_9]}^{c_1 \dots c_9} | b_2 b_3] \\
&\quad - 48580560 P_{[b_1} \delta_{b_2 b_3]}^{d_1 d_2} \delta_{a_1 \dots a_9}^{c_1 \dots c_9} \\
&\quad - 85730400 P_{[b_1} \delta_{b_2 | [a_1}^{d_1 d_2} \delta_{a_2 \dots a_9]}^{c_1 \dots c_9} | b_3], \\
[R_{-43} a_1 \dots a_9, b_1 b_2 b_3, Z_{(1)}^{c_1 \dots c_{10}, d}] &= 0, \\
[R_{-43} a_1 \dots a_9, b_1 b_2 b_3, Z_{(2)}^{c_1 \dots c_{10}, d}] &= 0, \\
[R_{-43} a_1 \dots a_9, b_1 b_2 b_3, Z^{c_1 \dots c_{11}}] &= 0. \tag{A.10}
\end{aligned}$$

This completes the 11D algebra up to level 4.

Appendix B

Algebra of 10D IIB theory

In this appendix, we give the commutators of the $E_{11} \ltimes l_1$ algebra decomposed into the 10D IIB representation which is $GL(10) \otimes SL(2)$ up to level 4. The $SL(10)$ indices are $a, b, \dots = 1, \dots, 10$, and the $SL(2)$ indices are $\alpha, \beta, \dots = 1, 2$. The algebra is given originally in [26, 51, 56].

B.1 Algebra of the adjoint representation

At level 0, we have the $GL(10)$ generator, which have the following commutators with the level 0 and the positive level generators

$$\begin{aligned}
[K^a_b, K^c_d] &= \delta^c_d K^a_b - \delta^a_d K^c_b, \\
[K^a_b, R_{\alpha\beta}] &= 0, \\
[K^a_b, R_{\alpha}^{a_1 a_2}] &= 2\delta_b^{[a_1} R_{\alpha}^{a] a_2}, \\
[K^a_b, R^{a_1 \dots a_4}] &= 4\delta_b^{[a_1} R^{a] a_2 a_3 a_4}, \\
[K^a_b, R_{\alpha}^{a_1 \dots a_6}] &= 6\delta_b^{[a_1} R_{\alpha}^{a] a_2 \dots a_6}, \\
[K^a_b, R_{\alpha\beta}^{a_1 \dots a_8}] &= 8\delta_b^{[a_1} R_{\alpha\beta}^{a] a_2 \dots a_8}, \\
[K^a_b, R^{a_1 \dots a_7, c}] &= 7\delta_b^{[a_1} R^{a] a_2 \dots a_7, c} + \delta_b^c R^{a_1 \dots a_7, a},
\end{aligned} \tag{B.1}$$

and with the negative level generators are

$$[K^a_b, R_{a_1 a_2}^{\alpha}] = -2\delta_{[a_1}^b R_{b] a_2}^{\alpha},$$

$$\begin{aligned}
[K^a{}_b, R_{a_1\dots a_4}] &= -4\delta_{[a_1}^a R_{|b|a_2a_3a_4]} , \\
[K^a{}_b, R_{a_1\dots a_6}^\alpha] &= -6\delta_{[a_1}^a R_{|b|a_2\dots a_6]}^\alpha , \\
[K^a{}_b, R_{a_1\dots a_8}^{\alpha\beta}] &= -8\delta_{[a_1}^a R_{|b|a_2\dots a_8]}^{\alpha\beta} , \\
[K^a{}_b, R_{a_1\dots a_7,c}] &= -7\delta_{[a_1}^a R_{|b|a_2\dots a_7],c} - \delta_c^a R_{a_1\dots a_7,b} .
\end{aligned} \tag{B.2}$$

The commutators of the $SL(2, \mathcal{R})$ generator with the positive level generators are

$$\begin{aligned}
[R_{\alpha\beta}, R_{\gamma\delta}] &= \delta_{(\alpha}^\sigma \varepsilon_{\beta)\gamma} R_{\sigma\delta} + \delta_{(\alpha}^\sigma \varepsilon_{\beta)\delta} R_{\gamma\sigma} , \\
[R_{\alpha\beta}, R_\gamma^{a_1 a_2}] &= \delta_{(\alpha}^\delta \varepsilon_{\beta)\gamma} R_\delta^{a_1 a_2} , \\
[R_{\alpha\beta}, R^{a_1\dots a_4}] &= 0 , \\
[R_{\alpha\beta}, R_\gamma^{a_1\dots a_6}] &= \delta_{(\alpha}^\delta \varepsilon_{\beta)\gamma} R_\delta^{a_1\dots a_6} , \\
[R_{\alpha\beta}, R_{\gamma\delta}^{a_1\dots a_8}] &= \delta_{(\alpha}^\sigma \varepsilon_{\beta)\gamma} R_{\sigma\delta}^{a_1\dots a_8} + \delta_{(\alpha}^\sigma \varepsilon_{\beta)\delta} R_{\gamma\sigma}^{a_1\dots a_8} , \\
[R_{\alpha\beta}, R^{a_1\dots a_7,b}] &= 0 ,
\end{aligned} \tag{B.3}$$

and with the negative level generators are

$$\begin{aligned}
[R_{\alpha\beta}, R_{a_1 a_2}^\gamma] &= -\delta_{(\alpha}^\gamma \varepsilon_{\beta)\delta} R_{a_1 a_2}^\delta , \\
[R_{\alpha\beta}, R_{a_1\dots a_4}] &= 0 , \\
[R_{\alpha\beta}, R_{a_1\dots a_6}^\gamma] &= -\delta_{(\alpha}^\gamma \varepsilon_{\beta)\delta} R_{a_1\dots a_6}^\delta , \\
[R_{\alpha\beta}, R_{a_1\dots a_8}^{\gamma\delta}] &= -\delta_{(\alpha}^\gamma \varepsilon_{\beta)\sigma} R_{a_1\dots a_8}^{\sigma\delta} - \delta_{(\alpha}^\delta \varepsilon_{\beta)\sigma} R_{a_1\dots a_8}^{\gamma\sigma} , \\
[R_{\alpha\beta}, R_{a_1\dots a_7,b}] &= 0 .
\end{aligned} \tag{B.4}$$

The commutators of the positive level E_{11} generators with themselves are

$$\begin{aligned}
[R_\alpha^{a_1 a_2}, R_\beta^{a_3 a_4}] &= -\varepsilon_{\alpha\beta} R^{a_1\dots a_4} , \\
[R_\alpha^{a_1 a_2}, R^{a_3\dots a_6}] &= 4R_\alpha^{a_1\dots a_6} , \\
[R^{a_1\dots a_4}, R^{a_5\dots a_8}] &= \frac{8}{3} R^{a_1\dots a_4[a_5 a_6 a_7, a_8]} .
\end{aligned} \tag{B.5}$$

Then the commutators of the negative level E_{11} generators are

$$\begin{aligned}
[R_{a_1 a_2}^\alpha, R_{a_3 a_4}^\beta] &= -\varepsilon^{\alpha\beta} R_{a_1 \dots a_4}, \\
[R_{a_1 a_2}^\alpha, R_{a_3 \dots a_6}] &= 4R_{a_1 \dots a_6}^\alpha, \\
[R_\beta^{a_1 a_2}, R_\beta^{a_3 \dots a_8}] &= -R_{\alpha\beta}^{a_1 \dots a_8} - \varepsilon_{\alpha\beta} R^{a_1 a_2 [a_3 \dots a_7, a_8]}, \\
[R_{a_1 a_2}^\beta, R_{a_3 \dots a_8}^\beta] &= -R_{a_1 \dots a_8}^{\alpha\beta} - \varepsilon^{\alpha\beta} R_{a_1 a_2 [a_3 \dots a_7, a_8]}, \\
[R_{a_1 \dots a_4}, R_{a_5 \dots a_8}] &= \frac{8}{3} R_{a_1 \dots a_4 [a_5 a_6 a_7, a_8]}. \tag{B.6}
\end{aligned}$$

The commutators of the positive level generators with the negative level generators up to level 3 are

$$\begin{aligned}
[R_\alpha^{a_1 a_2}, R_{b_1 b_2}^\beta] &= 4\delta_\alpha^\beta \delta_{[b_1}^{[a_1} K^{a_2]}_{b_2]} - \frac{1}{2} \delta_\alpha^\beta \delta_{b_1 b_2}^{a_1 a_2} K^d - 2\delta_{b_1 b_2}^{a_1 a_2} \varepsilon^{\beta\gamma} R_{\alpha\gamma}, \\
[R_\alpha^{a_1 a_2}, R_{b_1 \dots b_4}] &= -12\varepsilon_{\alpha\beta} \delta_{[b_1 b_2}^{a_1 a_2} R_{b_3 b_4]}^\beta, \\
[R_{a_1 a_2}^\alpha, R^{b_1 \dots b_4}] &= -12\varepsilon^{\alpha\beta} \delta_{a_1 a_2}^{[b_1 b_2} R_\beta^{b_3 b_4]}, \\
[R^{a_1 \dots a_4}, R_{b_1 \dots b_4}] &= 12\delta_{b_1 \dots b_4}^{a_1 \dots a_4} K^d - 96\delta_{[b_1 b_2 b_3}^{[a_1 a_2 a_3} K^{a_4]}_{b_4]}, \\
[R_\alpha^{a_1 a_2}, R_{b_1 \dots b_6}^\beta] &= \frac{15}{2} \delta_\alpha^\beta \delta_{[b_1 b_2}^{a_1 a_2} R_{b_3 \dots b_6]}, \\
[R_{a_1 a_2}^\alpha, R_\beta^{b_1 \dots b_6}] &= \frac{15}{2} \delta_\beta^\alpha \delta_{a_1 a_2}^{[b_1 b_2} R_\beta^{b_3 \dots b_6]}, \\
[R^{a_1 \dots a_4}, R_{b_1 \dots b_6}^\alpha] &= 90\delta_{[b_1 \dots b_4}^{a_1 \dots a_4} R_{b_5 b_6]}^\alpha, \\
[R_{a_1 \dots a_4}, R_\alpha^{b_1 \dots b_6}] &= 90\delta_{a_1 \dots a_4}^{[b_1 \dots b_4} R_\alpha^{b_5 b_6]}, \\
[R_\alpha^{a_1 \dots a_6}, R_{b_1 \dots b_6}^\beta] &= 270\delta_\alpha^\beta \delta_{[b_1 \dots b_5}^{[a_1 \dots a_5} K^{a_6]}_{b_6]} - \frac{135}{4} \delta_\alpha^\beta \delta_{b_1 \dots b_6}^{a_1 \dots a_6} K^d - 45\delta_{b_1 \dots b_6}^{a_1 \dots a_6} \varepsilon^{\beta\gamma} R_{\alpha\gamma}. \tag{B.7}
\end{aligned}$$

The commutators of the level ± 4 generators with the level ± 1 generators are

$$\begin{aligned}
[R_\alpha^{a_1 a_2}, R_{b_1 \dots b_8}^{\beta\gamma}] &= -56\delta_\alpha^{(\beta} \delta_{[b_1 b_2}^{a_1 a_2} R_{b_3 \dots b_8]}^{\gamma)}, \\
[R_{a_1 a_2}^\alpha, R_{\beta\gamma}^{b_1 \dots b_8}] &= -56\delta_{(\beta}^\alpha \delta_{a_1 a_2}^{[b_1 b_2} R_{\gamma]}^{b_3 \dots b_8]}, \\
[R_\alpha^{a_1 a_2}, R_{b_1 \dots b_7, b}] &= -252\varepsilon_{\alpha\beta} \delta_{[b_1 b_2}^{a_1 a_2} R_{b_3 \dots b_7, b]}^\beta + 252\varepsilon_{\alpha\beta} \delta_{[b_1 b_2}^{a_1 a_2} R_{b_3 \dots b_7, b]}^\beta, \\
[R_{a_1 a_2}^\alpha, R^{b_1 \dots b_7, b}] &= -252\varepsilon^{\alpha\beta} \delta_{a_1 a_2}^{[b_1 b_2} R_{\beta}^{b_3 \dots b_7, b]} + 252\varepsilon^{\alpha\beta} \delta_{a_1 a_2}^{[b_1 b_2} R_{\beta}^{b_3 \dots b_7, b]}. \tag{B.8}
\end{aligned}$$

The commutators of the level ± 2 with level ± 4 generators are

$$\begin{aligned}
[R^{a_1 \dots a_4}, R_{b_1 \dots b_8}^{\alpha\beta}] &= 0, \\
[R_{a_1 \dots a_4}, R_{\alpha\beta}^{b_1 \dots b_8}] &= 0, \\
[R^{a_1 \dots a_4}, R_{b_1 \dots b_7, b}] &= -1260 \delta_{[b_1 \dots b_4]}^{\alpha_1 \dots \alpha_4} R_{b_5 b_6 b_7] b} + 1260 \delta_{[b_1 \dots b_4]}^{\alpha_1 \dots \alpha_4} R_{b_5 b_6 b_7 b}], \\
[R_{a_1 \dots a_4}, R^{b_1 \dots b_7, b}] &= -1260 \delta_{a_1 \dots a_4}^{[b_1 \dots b_4]} R^{b_5 b_6 b_7] b} + 1260 \delta_{a_1 \dots a_4}^{[b_1 \dots b_4]} R^{b_5 b_6 b_7 b}]. \tag{B.9}
\end{aligned}$$

The commutators of the level ± 3 with level ± 4 generators are

$$\begin{aligned}
[R_{\alpha}^{a_1 \dots a_6}, R_{b_1 \dots b_8}^{\beta\gamma}] &= 1260 \delta_{\alpha}^{(\beta} \delta_{[b_1 \dots b_6]}^{a_1 \dots a_6} R_{b_7 b_8]}^{\gamma)}, \\
[R_{a_1 \dots a_6}^{\alpha}, R_{\beta\gamma}^{b_1 \dots b_8}] &= 1260 \delta_{(\beta}^{\alpha} \delta_{a_1 \dots a_6}^{[b_1 \dots b_6]} R_{\gamma]}^{b_7 b_8}], \\
[R_{\alpha}^{a_1 \dots a_6}, R_{b_1 \dots b_7, b}] &= 1890 \varepsilon_{\alpha\beta} \delta_{[b_1 \dots b_6]}^{\alpha_1 \dots \alpha_6} R_{b_7] b}^{\beta} - 1890 \varepsilon_{\alpha\beta} \delta_{[b_1 \dots b_6]}^{\alpha_1 \dots \alpha_6} R_{b_7 b}^{\beta}], \\
[R_{a_1 \dots a_6}^{\alpha}, R^{b_1 \dots b_7, b}] &= 1890 \varepsilon^{\alpha\beta} \delta_{a_1 \dots a_6}^{[b_1 \dots b_6]} R_{\beta}^{b_7] b} - 1890 \varepsilon^{\alpha\beta} \delta_{a_1 \dots a_6}^{[b_1 \dots b_6]} R_{\beta}^{b_7 b}]. \tag{B.10}
\end{aligned}$$

The commutators of the level ± 4 generators with themselves are

$$\begin{aligned}
[R_{\alpha_1 \alpha_2}^{a_1 \dots a_8}, R_{b_1 \dots b_8}^{\beta_1 \beta_2}] &= -20160 \delta_{\alpha_1 \alpha_2}^{(\beta_1 \beta_2)} \delta_{[b_1 \dots b_7]}^{[a_1 \dots a_7]} K_{b_8]}^{a_8}] + 25200 \delta_{\alpha_1 \alpha_2}^{(\beta_1 \beta_2)} \delta_{b_1 \dots b_8}^{a_1 \dots a_8} K^d{}_d \\
&\quad + 5040 \delta_{b_1 \dots b_8}^{a_1 \dots a_8} \delta_{(\alpha_1}^{(\beta_1} \varepsilon^{\beta_2) \gamma} R_{\alpha_2) \gamma}], \\
[R_{\alpha\beta}^{a_1 \dots a_8}, R_{b_1 \dots b_7, b}] &= 0, \\
[R_{a_1 \dots a_8}^{\alpha\beta}, R^{b_1 \dots b_7, b}] &= 0, \\
[R^{a_1 \dots a_7, a}, R_{b_1 \dots b_7, b}] &= -11340 \delta_{b_1 \dots b_7}^{a_1 \dots a_7} K^a{}_b + 11340 \delta_{[b_1 \dots b_7]}^{a_1 \dots a_7} K^a{}_b] \\
&\quad + 11340 \delta_{b_1 \dots b_7}^{[a_1 \dots a_7]} K^a]{}_b - 79380 \delta_b^a \delta_{[b_1 \dots b_6]}^{[a_1 \dots a_6]} K^{a_7]}_{b_7]} \\
&\quad + 79380 \delta_{[b_1 \dots b_6]}^{[a_1 \dots a_6]} K^{a_7]}_{b_7]} + 79380 \delta_{[b_1 \dots b_6]}^{[a_1 \dots a_6]} K^{a_7]}_{b_7]} \\
&\quad - 90720 \delta_{[b_1 \dots b_7]}^{[a_1 \dots a_7]} K^a]{}_b + 11340 \delta_{b_1 \dots b_7}^{a_1 \dots a_7} \delta_b^a K^d{}_d \\
&\quad - 11340 \delta_{b_1 \dots b_7 b}^{a_1 \dots a_7 a} K^d{}_d. \tag{B.11}
\end{aligned}$$

This completes the algebra of the adjoint representation of E_{11} in the 10D IIB decomposition. Now we give its algebra with the l_1 representation.

B.2 Algebra with the l_1 representation

Next we give the commutators of the E_{11} generators with the l_1 generators in the 10D IIB decomposition up to level 4. The commutators of the GL(10) E_{11} generator with the l_1 generators are

$$\begin{aligned}
[K^a{}_b, P_c] &= -\delta_c^a P_b + \frac{1}{2}\delta_b^a P_c, \\
[K^a{}_b, Z_\alpha^c] &= \delta_b^c Z_\alpha^a + \frac{1}{2}\delta_b^a Z_\alpha^c, \\
[K^a{}_b, Z^{a_1 a_2 a_3}] &= 3\delta_b^{[a_1} Z^{a|a_2 a_3]} + \frac{1}{2}\delta_b^a Z^{a_1 a_2 a_3}, \\
[K^a{}_b, Z_\alpha^{a_1 \dots a_5}] &= 5\delta_b^{[a_1} Z_\alpha^{a|a_2 \dots a_5]} + \frac{1}{2}\delta_b^a Z_\alpha^{a_1 \dots a_5}, \\
[K^a{}_b, Z_{\alpha\beta}^{a_1 \dots a_7}] &= 7\delta_b^{[a_1} Z_{\alpha\beta}^{a|a_2 \dots a_7]} + \frac{1}{2}\delta_b^a Z_{\alpha\beta}^{a_1 \dots a_7}, \\
[K^a{}_b, Z^{a_1 \dots a_7}] &= 7\delta_b^{[a_1} Z^{a|a_2 \dots a_7]} + \frac{1}{2}\delta_b^a Z^{a_1 \dots a_7}, \\
[K^a{}_b, Z^{a_1 \dots a_6, c}] &= 6\delta_b^{[a_1} Z^{a|a_2 \dots a_6], c} + \delta_b^c Z^{a_1 \dots a_6, a} + \frac{1}{2}\delta_b^a Z^{a_1 \dots a_6, c}, \tag{B.1}
\end{aligned}$$

and the commutators of the l_1 representation with the SL(2, R) generator are

$$\begin{aligned}
[R_{\alpha\beta}, P_a] &= 0, \\
[R_{\alpha\beta}, Z_\gamma^a] &= \delta_{(\alpha}^\delta \varepsilon_{\beta)\gamma} Z_\delta^a, \\
[R_{\alpha\beta}, Z^{a_1 a_2 a_3}] &= 0, \\
[R_{\alpha\beta}, Z_\gamma^{a_1 \dots a_5}] &= \delta_{(\alpha}^\delta \varepsilon_{\beta)\gamma} Z_\delta^{a_1 \dots a_5}, \\
[R_{\alpha\beta}, Z_{\gamma\delta}^{a_1 \dots a_7}] &= \delta_{(\alpha}^\sigma \varepsilon_{\beta)\gamma} Z_{\sigma\delta}^{a_1 \dots a_7} + \delta_{(\alpha}^\sigma \varepsilon_{\beta)\delta} Z_{\gamma\sigma}^{a_1 \dots a_7}, \\
[R_{\alpha\beta}, Z^{a_1 \dots a_7}] &= 0, \\
[R_{\alpha\beta}, Z^{a_1 \dots a_6, b}] &= 0. \tag{B.2}
\end{aligned}$$

The commutators of the l_1 representation with the level 1 E_{11} generators are

$$\begin{aligned}
[R_\alpha^{a_1 a_2}, P_a] &= \delta_a^{[a_1} Z_\alpha^{a_2]}, \\
[R_\alpha^{a_1 a_2}, Z_\beta^{a_3}] &= -\varepsilon_{\alpha\beta} Z^{a_1 a_2 a_3}, \\
[R_\alpha^{a_1 a_2}, Z_\alpha^{a_3 a_4 a_5}] &= Z_\alpha^{a_1 \dots a_5},
\end{aligned}$$

$$[R_\alpha^{a_1 a_2}, Z_\beta^{a_3 \dots a_7}] = Z_{\alpha\beta}^{a_1 \dots a_7} - \varepsilon_{\alpha\beta} Z^{a_1 \dots a_7} - \varepsilon_{\alpha\beta} Z^{a_1 a_2 [a_3 \dots a_6, a_7]} . \quad (\text{B.3})$$

The commutators of the remaining positive level E_{11} generators with the l_1 generators are

$$\begin{aligned} [R^{a_1 \dots a_4}, P_a] &= 2\delta_a^{[a_1} Z^{a_2 a_3 a_4]} , \\ [R^{a_1 \dots a_4}, Z_\alpha^{a_5}] &= -Z_\alpha^{a_1 \dots a_5} , \\ [R^{a_1 \dots a_4}, Z^{a_5 a_6 a_7}] &= 2Z^{a_1 \dots a_7} + \frac{3}{5} Z^{a_1 \dots a_4 [a_5 a_6, a_7]} , \\ [R_\alpha^{a_1 \dots a_6}, P_a] &= \frac{3}{4} \delta_a^{[a_1} Z_\alpha^{a_2 \dots a_6]} , \\ [R_\alpha^{a_1 \dots a_6}, Z_\beta^{a_7}] &= -\frac{1}{4} Z_{\alpha\beta}^{a_1 \dots a_7} + \frac{3}{4} \varepsilon_{\alpha\beta} Z^{a_1 \dots a_7} + \frac{1}{20} \varepsilon_{\alpha\beta} Z^{a_1 \dots a_6, a_7} , \\ [R_{\alpha\beta}^{a_1 \dots a_8}, P_a] &= -\delta_a^{[a_1} Z_{\alpha\beta}^{a_2 \dots a_8]} , \\ [R^{a_1 \dots a_7, b}, P_a] &= -3\delta_a^b Z^{a_1 \dots a_7} + 3\delta_a^{[b} Z^{a_1 \dots a_7]} + \frac{21}{20} \delta_a^{[a_1} Z^{a_2 \dots a_7], b} . \end{aligned} \quad (\text{B.4})$$

The commutators of level -1 E_{11} generator with the l_1 generators are

$$\begin{aligned} [R_{a_1 a_2}^\alpha, Z_\beta^b] &= -4\delta_\beta^\alpha \delta_{[a_1}^b P_{a_2]} , \\ [R_{a_1 a_2}^\alpha, Z^{b_1 b_2 b_3}] &= -6\varepsilon^{\alpha\beta} \delta_{a_1 a_2}^{[b_1 b_2} Z_\beta^{b_3]} , \\ [R_{a_1 a_2}^\alpha, Z_\beta^{b_1 \dots b_5}] &= 20\delta_\beta^\alpha \delta_{a_1 a_2}^{[b_1 b_2} Z^{b_3 b_4 b_5]} , \\ [R_{a_1 a_2}^\alpha, Z_{\alpha_1 \alpha_2}^{b_1 \dots b_7}] &= 42\delta_{(\alpha_1}^\alpha \delta_{a_1 a_2}^{[b_1 b_2} Z_{\alpha_2]}^{b_3 \dots b_7]} , \\ [R_{a_1 a_2}^\alpha, Z^{b_1 \dots b_7}] &= -3\varepsilon^{\alpha\beta} \delta_{a_1 a_2}^{[b_1 b_2} Z_\beta^{b_3 \dots b_7]} , \\ [R_{a_1 a_2}^\alpha, Z^{b_1 \dots b_6, b}] &= -150\varepsilon^{\alpha\beta} \delta_{a_1 a_2}^{[b_1 b_2} Z_\beta^{b_3 \dots b_6] b} + 150\varepsilon^{\alpha\beta} \delta_{a_1 a_2}^{[b_1 b_2} Z_\beta^{b_3 \dots b_6 b]} . \end{aligned} \quad (\text{B.5})$$

The commutators of level -2 generators with the l_1 generators are

$$\begin{aligned} [R_{a_1 \dots a_4}, Z^{b_1 b_2 b_3}] &= 48\delta_{[a_1 a_2 a_3}^{b_1 b_2 b_3} P_{a_4]} , \\ [R_{a_1 \dots a_4}, Z_\alpha^{b_1 \dots b_5}] &= 120\delta_{a_1 \dots a_4}^{[b_1 \dots b_4} Z_\alpha^{b_5]} , \\ [R_{a_1 \dots a_4}, Z_{\alpha_1 \alpha_2}^{b_1 \dots b_7}] &= 0 , \\ [R_{a_1 \dots a_4}, Z^{b_1 \dots b_7}] &= -120\delta_{a_1 \dots a_4}^{[b_1 \dots b_4} Z^{b_5 b_6 b_7]} , \\ [R_{a_1 \dots a_4}, Z^{b_1 \dots b_6, b}] &= -1800\delta_{a_1 \dots a_4}^{[b_1 \dots b_4} Z^{b_5 b_6] b} + 1800\delta_{a_1 \dots a_4}^{[b_1 \dots b_4} Z^{b_5 b_6 b]} . \end{aligned} \quad (\text{B.6})$$

The commutators of the level -3 E_{11} generator with the l_1 generators are

$$\begin{aligned}
[R_{a_1\dots a_6}^\alpha, Z_\beta^{b_1\dots b_5}] &= -360\delta_\beta^\alpha \delta_{[a_1\dots a_5}^{b_1\dots b_5} P_{a_6]} , \\
[R_{a_1\dots a_6}^\alpha, Z_{\alpha_1\alpha_2}^{b_1\dots b_7}] &= -1260\delta_{(\alpha_1}^\alpha \delta_{a_1\dots a_6}^{[b_1\dots b_6} Z_{\alpha_2]}^{b_7]} , \\
[R_{a_1\dots a_6}^\alpha, Z^{b_1\dots b_7}] &= 270\varepsilon^{\alpha\beta} \delta_{a_1\dots a_6}^{[b_1\dots b_6} Z_\beta^{b_7]} , \\
[R_{a_1\dots a_6}^\alpha, Z^{b_1\dots b_7}] &= 900\varepsilon^{\alpha\beta} \delta_{a_1\dots a_6}^{b_1\dots b_6} Z_\beta^b - 900\varepsilon^{\alpha\beta} \delta_{a_1\dots a_6}^{[b_1\dots b_6} Z_\beta^b] . \tag{B.7}
\end{aligned}$$

Finally, the commutators of the level -4 E_{11} generators with the l_1 generators are

$$\begin{aligned}
[R_{a_1\dots a_8}^{\alpha_1\alpha_2}, Z_{\beta_1\beta_2}^{b_1\dots b_7}] &= 20160\delta_{\beta_1\beta_2}^{(\alpha_1\alpha_2)} \delta_{[a_1\dots a_7}^{b_1\dots b_7} P_{a_8]} , \\
[R_{a_1\dots a_8}^{\alpha_1\alpha_2}, Z^{b_1\dots b_7}] &= 0 , \\
[R_{a_1\dots a_8}^{\alpha_1\alpha_2}, Z^{b_1\dots b_6, b}] &= 0 , \\
[R_{a_1\dots a_7, a}, Z_{\alpha_1\alpha_2}^{b_1\dots b_7}] &= 0 , \\
[R_{a_1\dots a_7, a}, Z^{b_1\dots b_7}] &= 4320\delta_{a_1\dots a_7}^{b_1\dots b_7} P_a - 4320\delta_{[a_1\dots a_7}^{b_1\dots b_7} P_a] , \\
[R_{a_1\dots a_7, a}, Z^{b_1\dots b_6, b}] &= -75600\delta_a^b \delta_{[a_1\dots a_6}^{b_1\dots b_6} P_{a_7]} + 75600\delta_a^{bb_1\dots b_6} P_{a_7]} . \tag{B.8}
\end{aligned}$$

This completes the 10D IIB algebra up to level 4.

Appendix C

Algebra in 5D

The E_{11} algebra in its five dimensional decomposition can be found in [51]. In this appendix, we give the commutators of the E_{11} generators with the l_1 generators of the 5 dimensional decomposition of E_{11} which is $GL(5) \otimes USp(8)$. The $SL(5)$ indices are $a, b, \dots = 1, \dots, 5$, and the $USp(8)$ indices are $\alpha, \beta, \dots = 1, \dots, 8$. The commutators of l_1 generators with level 0 E_{11} generators are

$$\begin{aligned}
[K^a{}_b, P_c] &= -\delta_c^a P_b + \frac{1}{2} \delta_b^a P_c, \\
[K^a{}_b, Z^{\alpha_1 \alpha_2}] &= \frac{1}{2} \delta_b^a Z^{\alpha_1 \alpha_2}, \\
[K^a{}_b, Z^{c \alpha_1 \alpha_2}] &= \delta_b^c Z^{a \alpha_1 \alpha_2} + \frac{1}{2} \delta_b^a Z^{c \alpha_1 \alpha_2}, \\
[R^{(\alpha_1 \alpha_1)}, P_a] &= 0, \\
[R^{(\alpha_1 \alpha_2)}, Z^{\beta_1 \beta_2}] &= 2\Omega^{(\alpha_1 [\beta_1 Z^{\alpha_2) \beta_2]}], \\
[R^{(\alpha_1 \alpha_2)}, Z^{a \beta_1 \beta_2}] &= 2\Omega^{(\alpha_1 [\beta_1 Z^{a \alpha_2) \beta_2]}], \\
[R^{\alpha_1 \dots \alpha_4}, P_a] &= 0, \\
[R^{\alpha_1 \dots \alpha_4}, Z^{\beta_1 \beta_2}] &= \Omega^{[\alpha_1 \alpha_2 \Omega^{\alpha_3 [\beta_1 Z^{\alpha_4] \beta_2}]} + \Omega^{[\alpha_1 [\beta_1 \Omega^{\alpha_2 \beta_2}] Z^{\alpha_3 \alpha_4}]} \\
&\quad - \frac{1}{4} \Omega^{\beta_1 \beta_2} \Omega^{[\alpha_1 \alpha_2 Z^{\alpha_3 \alpha_4}]} - \frac{1}{12} \Omega^{[\alpha_1 \alpha_2 \Omega^{\alpha_3 \alpha_4}] Z^{\beta_1 \beta_2}}, \\
[R^{\alpha_1 \dots \alpha_4}, Z^{a \beta_1 \beta_2}] &= -(\Omega^{[\alpha_1 \alpha_2 \Omega^{\alpha_3 [\beta_1 Z^{a \alpha_4] \beta_2}]} + \Omega^{[\alpha_1 [\beta_1 \Omega^{\alpha_2 \beta_2}] Z^{a \alpha_3 \alpha_4}]} \\
&\quad - \frac{1}{4} \Omega^{\beta_1 \beta_2} \Omega^{[\alpha_1 \alpha_2 Z^{a \alpha_3 \alpha_4}]} - \frac{1}{12} \Omega^{[\alpha_1 \alpha_2 \Omega^{\alpha_3 \alpha_4}] Z^{a \beta_1 \beta_2}}). \tag{C.1}
\end{aligned}$$

The commutators with the positive level generators are

$$\begin{aligned}
[R^{a\alpha_1\alpha_2}, P_b] &= \delta_b^a Z^{\alpha_1\alpha_2}, \\
[R^{a_1a_2\alpha_1\alpha_2}, P_a] &= -2\delta_b^{[a_1} Z^{a_2]\alpha_1\alpha_2}, \\
[R^{a\alpha_1\alpha_2}, Z^{\beta_1\beta_2}] &= 4\Omega^{[\alpha_1[\beta_1} Z^{a\alpha_2]\beta_2]} - \frac{1}{2}\Omega^{\beta_1\beta_2} Z^{a\alpha_1\alpha_2} - \frac{1}{2}\Omega^{\alpha_1\alpha_2} Z^{a\beta_1\beta_2}, \quad (C.2)
\end{aligned}$$

and with the negative level generators are

$$\begin{aligned}
[R_{a\alpha_1\alpha_2}, P_b] &= 0, \\
[R_{a\alpha_1\alpha_2}, Z^{\beta_1\beta_2}] &= 2(\delta_{\alpha_1\alpha_2}^{\beta_1\beta_2} + \frac{1}{8}\Omega_{\alpha_1\alpha_2}\Omega^{\beta_1\beta_2})P_a, \\
[R_{a\alpha_1\alpha_2}, P_b] &= 0, \\
[R_{a\alpha_1\alpha_2}, Z^{\beta_1\beta_2}] &= 2(\delta_{\alpha_1\alpha_2}^{\beta_1\beta_2} + \frac{1}{8}\Omega_{\alpha_1\alpha_2}\Omega^{\beta_1\beta_2})P_a, \\
[R_{a\alpha_1\alpha_2}, Z^{b\beta_1\beta_2}] &= 4\delta_a^b(\Omega_{[\alpha_1\gamma}\delta_{\alpha_2]}^{[\beta_1} Z^{\beta_2]\gamma}] + \frac{1}{8}\Omega_{\alpha_1\alpha_2}Z^{\beta_1\beta_2} - \frac{1}{8}\Omega^{\beta_1\beta_2}\Omega_{\alpha_1\gamma_1}\Omega_{\alpha_2\gamma_2}Z^{\gamma_1\gamma_2}, \\
[R_{a_1a_2\alpha_1\alpha_2}, Z^{b\beta_1\beta_2}] &= 4(\delta_{\alpha_1\alpha_2}^{\beta_1\beta_2} + \frac{1}{8}\Omega_{\alpha_1\alpha_2}\Omega^{\beta_1\beta_2})\delta_{[a_1}^b P_{a_2]}. \quad (C.3)
\end{aligned}$$

This completes the 5 dimensional algebra that we require.

Appendix D

Algebra in 4D

We give the commutators of the $E_{11} \times l_1$ algebra decomposed into representations of $GL(4) \otimes SL(8)$ first given in [35, 57]. The $SL(4)$ indices are $a, b, \dots = 1, \dots, 4$, and the $SL(8)$ indices $I, J, \dots = 1, \dots, 8$. The commutators of the level zero E_{11} generators with the l_1 generators are

$$\begin{aligned}
[K^a_b, P_c] &= -\delta_c^a P_b + \frac{1}{2} \delta_b^a P_c, \\
[K^a_b, Z^{I_1 I_2}] &= \frac{1}{2} \delta_b^a Z^{I_1 I_2}, \\
[K^a_b, Z_{I_1 I_2}] &= \frac{1}{2} \delta_b^a Z_{I_1 I_2}, \\
[K^a_b, Z^c] &= \delta_b^c Z^a + \frac{1}{2} \delta_b^a Z^c, \\
[R^I_J, P_c] &= 0, \\
[R^I_J, Z^{I_1 I_2}] &= 2\delta_J^{[I_1} Z^{I] I_2} - \frac{1}{4} \delta_J^I Z^{I_1 I_2}, \\
[R^I_J, P_c] &= 0, \\
[R^I_J, Z_{I_1 I_2}] &= -2\delta_{[I_1}^I Z_{J] I_2} + \frac{1}{4} \delta_J^I Z_{I_1 I_2}, \\
[R^I_J, Z^a] &= 0, \\
[R^{I_1 \dots I_4}, P_a] &= 0, \\
[R^{I_1 \dots I_4}, Z^{J_1 J_2}] &= \frac{1}{24} \varepsilon^{I_1 \dots I_4 J_1 \dots J_4} Z_{J_3 J_4}, \\
[R^{I_1 \dots I_4}, Z_{J_1 J_2}] &= \delta_{J_1 J_2}^{[I_1 I_2} Z^{I_3 I_4]}, \\
[R^{I_1 \dots I_4}, Z^a] &= 0.
\end{aligned} \tag{D.1}$$

The commutators of the positive level E_{11} generators with the l_1 generators are

$$\begin{aligned}
[R^{aI_1I_2}, P_b] &= \delta_b^a Z^{I_1I_2} , \\
[R^a_{I_1I_2}, P_b] &= \delta_b^a Z_{I_1I_2} , \\
[R^a_{I_1I_2}, Z^{J_1J_2}] &= \delta_{I_1I_2}^{J_1J_2} Z^a , \\
[R^{aI_1I_2}, Z_{J_1J_2}] &= -\delta_{J_1J_2}^{I_1I_2} Z^a .
\end{aligned} \tag{D.2}$$

The commutators with the negative level generators are

$$\begin{aligned}
[\tilde{R}_{aI_1I_2}, P_b] &= 0 , \\
[\tilde{R}_{aI_1I_2}, Z^{J_1J_2}] &= 2\delta_{J_1J_2}^{I_1I_2} P_a , \\
[\tilde{R}_{aI_1I_2}, Z_{J_1J_2}] &= 0 , \\
[\tilde{R}_a^{I_1I_2}, P_b] &= 0 , \\
[\tilde{R}_a^{I_1I_2}, Z^{J_1J_2}] &= 0 , \\
[\tilde{R}_a^{I_1I_2}, Z_{J_1J_2}] &= -2\delta_{I_1I_2}^{J_1J_2} P_a , \\
[\tilde{R}_{aI_1I_2}, Z^b] &= -2\delta_a^b Z_{I_1I_2} , \\
[\tilde{R}_a^{I_1I_2}, Z^b] &= -2\delta_a^b Z^{I_1I_2} .
\end{aligned} \tag{D.3}$$

This completes the 4D algebra that we require.

Appendix E

Algebra of A_1^{+++}

The algebra of A_1^{+++} was first given in [51]. We give the commutators we require below.

We decompose A_1^{+++} in terms of $GL(4)$, and the $GL(4)$ indices are $a, b, \dots = 1, \dots, 4$.

The commutators of the level zero generators of $GL(4)$ with the l_1 algebra are

$$\begin{aligned}
[K^a{}_b, P_c] &= -\delta_c^a P_b + \frac{1}{2} \delta_b^a P_c, \\
[K^a{}_b, Z^c] &= \delta_b^c Z^a + \frac{1}{2} \delta_b^a Z^c, \\
[K^a{}_b, Z^{cde}] &= \delta_b^c Z^{ade} + \delta_b^d Z^{cae} + \delta_b^e Z^{cda} + \frac{1}{2} \delta_b^a Z^{cde}, \\
[K^a{}_b, Z^{cd,e}] &= \delta_b^c Z^{ad,e} + \delta_b^d Z^{ca,e} + \delta_b^e Z^{cd,a} + \frac{1}{2} \delta_b^a Z^{cd,e}.
\end{aligned} \tag{E.1}$$

The commutators with the positive level generators are

$$\begin{aligned}
[R^{ab}, P_c] &= \delta_c^{(a} Z^{b)}, \\
[R^{ab}, Z^c] &= Z^{abc} + Z^{c(a,b)}, \\
[R^{ab,cd}, P_e] &= -\delta_e^{[a} Z^{b]cd} + \frac{1}{4} (\delta_e^a Z^{b,(c,d)} - \delta_e^b Z^{a,(c,d)}) - \frac{3}{8} (\delta_e^c Z^{ab,d} + \delta_e^d Z^{ab,c}).
\end{aligned} \tag{E.2}$$

The commutators with the negative level generators are

$$\begin{aligned}
[R_{ab}, P_c] &= 0, \\
[R_{ab}, Z^c] &= 2\delta_{(a}^c P_{b)}, \\
[R_{ab}, Z^{cde}] &= \frac{2}{3} (\delta_{(ab)}^{cd} Z^e + \delta_{(ab)}^{de} Z^c + \delta_{(ab)}^{ec} Z^d),
\end{aligned}$$

$$[R_{ab}, Z^{cd,e}] = \frac{4}{3}(\delta^d e_{(ab)} Z^c - \delta_{(ab)}^{ce} Z^d). \quad (\text{E.3})$$

This completes the algebra of A_1^{+++} that we require.

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